6.16 常態定理 Let $T \in \mathbb{L}(V)$ for inner-product space V over $\mathbb{C}$ with $\dim(V) < \infty$. V has an orthonormal eigenbasis for T iff T is normal, i.e. $TT^* = T^*T$.

Def 自伴線運、方陣 T is self-adjoint if $T^* = T$. Square matrix A is Hermitian if $A^* = A$.

Obs 自伴小觀察 If β is an orthonormal basis of a inner-product space V with $\dim(V) < \infty$, then T is self-adjoint iff $[T]_\beta^\beta$ is self-adjoint.

Obs 自伴線運基本性質 If T is a self-adjoint linear operator on an inner-product space V over $F \in \{\mathbb{R}, \mathbb{C}\}$ with $\dim(V) < \infty$, then 1. each eigenvalue of T is real (even if $F = \mathbb{C}$, and) 2. the characteristic polynomial $f_T(t)$ of T splits (even if $F = \mathbb{R}$)

6.24 投影：正交 $\Leftrightarrow$ 自伴 If T is a projection of inner-product space W, then T is an orthogonal projection of W iff T is self-adjoint

Cor 自伴線運推論 Let $T \in \mathbb{L}(V)$ for inner-product space W over $F = \mathbb{C}$ wth $\dim(V) < \infty$. If T is normal, T is self-adjoint iff every eigenvalue of T is real.

6.17 自伴定理 Let $T \in \mathbb{L}(V)$ for V over $\mathbb{R}$ with $\dim(V) < \infty$. V has an orthonormal eigenbasis for T iff T is self-adjoint.

Cor 常態線運推論 If $T \in \mathbb{L}(V)$ for W over $\mathbb{C}$ with $\dim(W) < \infty$, then T is normal iff $T^* = g(T)$ for some polynomial $g \in \mathbb{P}(\mathbb{C})$

Def 么正、正交方陣 Let $Q \in F^{n\times n}$ with $F \in \{\mathbb{C}, \mathbb{R}\}$, • Q is unitary if $Q^*Q = I_n$ (i.e. $Q^* = Q^{-1}$) • Q is orthogonal if $Q^tQ = I_n$ (i.e. $Q^t = Q^{-1}$)

Def 么正、正交等價 Let $A, B \in F^{n\times n}$ with $F \in \{\mathbb{C}, \mathbb{R}\}$, • A is unitarily equivalent to B if there is a unitary matrix Q with $A = Q^*BQ$ • A is orthogonally equivalent to B if there is an orthogonal matrix Q with $A = Q^tBQ$

6.20 If $A \in \mathbb{R}^{n\times n}$, then A is self-adjoint (i.e. symmetric) iff A is orthogonally (i.e. unitarily) equivalent to a diagonal matrix in $\mathbb{R}^{n\times n}$

6.19 If $A \in \mathbb{C}^{n\times n}$, then A is normal iff A is unitarily equivalent to a diagonal matrix in $\mathbb{C}^{n\times n}$

6.21 方陣舒爾 If $f_A(t)$ splits for $A \in F^{n\times n}$, then A is unitarily equivalent to an upper-triangular matrix in $F^{n\times n}$.

么正、正交線運 Let $T \in \mathbb{L}(V)$, V is an inner-product space over $F \in \{\mathbb{C}, \mathbb{R}\}$. • T is unitary if $T^*T = I_V$ • T is orthogonal if T is unitary and $F = \mathbb{R}$

6.18 If $T \in \mathbb{L}(V)$ for V over $F \in \{\mathbb{C}, \mathbb{R}\}$, with $\dim(V) < \infty$, then the following are equivalent: • $T^*T = I_V$ • $\langle T(x)|T(y)\rangle = \langle x|y\rangle$ holds for all vectors $x, y \in V$ • For any orthogonal eigenbasis β of V, $T(\beta)$ is an orthonormal basis of V • There is a $\beta \subseteq V$ s.t. β and $T(\beta)$ are both orthonormal bases of V • $\|T(x)\| = \|x\|$ holds for all vectors $x \in V$

Cor 么正、正交自伴定理 If $T \in \mathbb{L}(V)$ for V over $\mathbb{C}$ ($\mathbb{R}$) with $\dim(V) < \infty$, then T is unitary (orthogonal and self-adjoint) iff • V has an orthonormal eigenbasis for T • each eigenvalue og T has absolute value 1

6.13 最短解 If $E : Ax = b$ with $A \in F^{m\times n}$ and $b \in F^m$ is a system of linear equations with $S(E) \neq \emptyset$, then there is exactly one vector x in $S(E) \cap L_{A^*}(F^m)$ w.r.t. the standard inner product. Moreover, the vector x is the unique vector in $S(E)$ with minimum $||x||$

6.12 最佳近似解 Let $A \in F^{m\times n}$ and $b \in F^m$. • For any inner-product function of F^m, $\exists x \in F^n$ that minimizes $\|Ax - b\|$ • If $\operatorname{rank}(A) = n$, then $x = (A^*A)^{-1}A^*b$ is the unique minimizer of $\|Ax - b\|$ w.r.t. the standard inner product of F^m

Obs 標準內積觀察 Let $A \in F^{m\times n}$. For any $x \in F^n$ and $y \in F^m$, we have $\langle Ax|y\rangle_m = \langle x|A^*y\rangle_n$

Obs 矩陣位階觀察 For any $A \in F^{m\times n}$, $\operatorname{rank}(A^*A) = \operatorname{rank}(A)$

Obs 伴隨矩陣觀察 For any $A \in F^{m\times n}$, if $\operatorname{rank}(A) = n$, then $A^*A \in F^{n\times n}$ is invertible.

Def 正定 A self-adjoint $T \in \mathbb{L}(V)$ for inner-product space V over F is positive definite if

$$< T(x)|x >$$

6.26 奇異值定理 Let $T \in \mathbb{L}(V, W)$ for inner-product spaces V and W over $F \in \{\mathbb{C}, \mathbb{R}\}$ with $\operatorname{rank}(T) = r$, $\dim(V) = n$, and $\dim(W) = m$.

6.27 SVD For any matrix $A = F^{m\times n}$ with rank r,

$$A = QSR*$$

holds for:

Def Pseudoinverse $T^\dagger \in \mathbb{L}(W, V)$ defined by $T^\dagger = (T')^{-1}T''$

偽反線轉定理