5.9 特徵基底判別定理 Let V be a vector space with $dim(V) = n$. If $T \in \mathcal{L}(V)$ and $\lambda_1 \ldots, \lambda_k$ are the distinct eigenvalues of T, then T is diagonalizable if and only if $dim(E_T(\lambda_1)) + \cdots + dim(E_T(\lambda_k)) = n$.

5.2 特徵值 ⇔ 特徵根 If $A \in F^{n \times n}$, then λ is an eigenvalue of A if and only if $f_A(\lambda) \equiv det(A - \lambda I_n) = 0_F$.

5.4 特空定理 If T is linear operator on vector space V and λ is an eigenvalue of T, then $T(x) = \lambda x \Leftrightarrow x \in E_T(\lambda)$.

5.7 特徵值重數定理 Let V be a finite-dimensional vector space. If λ is an eigenvalue of $T \in \mathcal{L}(V)$ with multiplicity m, then $1 \leq dim(E_T(\lambda)) \leq m$.

5.5 跨特徵空間不冗定理 Let T be a linear operator on an n-dimensional vector space V. Let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of T. If $\varnothing \neq S_i \subseteq E_T(\lambda_i)$ is linearly independent for each $i = 1, \ldots, k$, then $S_1, \ldots, S_k$ are pairwise disjoint and $S_1 \cup \cdots \cup S_k$ is linearly independent.

5.11 特徵空間直和定理 If $T \in \mathcal{L}(V)$ for a finite-dimensional vector space V, then V is the direct sum of the eigenspaces of T is and only if V has an eigenbasis for T.

5.23 Cayley-Hamilton Theorem If $T \in \mathcal{L}(V)$ for a finite-dimensional vector space V over F, $f_T(T) = T_0$.

5.21 縮水觀察 Let $T \in \mathcal{L}(V)$ for vector space V with $dim(V) < \infty$. If U is a T-invariant subspace of V, then $f_{T_U}(t) | f_T(t)$.

5.22 循環定理 Let $T \in \mathcal{L}(V)$ with $dim(V) < \infty$. Let $U = C_T(x) \equiv span\left(\cup_{i \leq 0} T^i(x)\right)$ with $x \in V \setminus \{0_V\}$. Let $k = dim(U)$. (a) The ordered set $\beta = \langle x, T(x), \ldots, T^{k-1}(x) \rangle$ is a basis of U; (b) If $\sum_{i=0}^{k} a_i T^i(x) = 0_V$ with $a_k = 1_F$, then $f_{T_U}(t) = (-1)^k \sum_{i=0}^{k} a_i t^i$

Definition 內積函數 1. $d(ax + y, z) = a \cdot d(x, z) + d(y, z)$; 2. $d(y, x) = d(\bar{x}, y)$; 3. if $x \neq 0_V$, then $d(x, x) \in \mathcal{R}^+$

6.1 內積基本性質 1. $\langle x | ay + z \rangle = \bar{a}\langle x | y \rangle + \langle x | z \rangle$; 2. $\langle x | 0_V \rangle = \langle 0_V | x \rangle = 0_F$; 3. $\langle x | x \rangle = 0_F \iff x = 0_V$; 4. If $\langle x | y \rangle = \langle x | z \rangle$ holds for all x, then $y = z$

內積必定是某種標準內積 If β is an orthonormal, then $d(x, y) = ([y]_\beta)^* [x]_\beta$

6.3 正交定理 Let x be a vector in inner-product space V. Let S be a nonempty orthogonal set of nonzero vectors in V. Let R be a finite subset of S. If $x = \sum_{y \in R} a_y y$ holds for scalars a_y, then $a_y = \frac{\langle x | y \rangle}{\langle y | y \rangle}$ holds for each $y \in R$; (note: 正交無零則不冗)

6.4 正交演算法 For any linearly independent subset α of inner-product space V with $|\alpha| = n$, the set β recursively defined below is orthogonal basis of $span(\alpha)$: $\beta_1 = \alpha_1$ and $\beta_j = \alpha_j - \sum_{i=1}^{j-1} \frac{\langle \alpha_j | \beta_i \rangle}{\langle \beta_i | \beta_i \rangle} \dot{\beta}_i$ for each $j = 2, \ldots, n$.

6.2 長度的基本性質 1. $\|ax\| = |a|\|x\|$; 2. $\|x\| \in \mathcal{R}^+ \cup \{0_R\}$; 3. $\|x\| = 0_R \iff x = 0_V$; 4. Cauchy-Schwarz inequality $|\langle x | y \rangle| \leq \|x\| \cdot \|y\|$; 5. Triangle inequality $|x + y\| \leq \|x\| + \|y\|$

Definition 正交補集 For any nonempty subset S of inner-product space W, the orthogonal complement of S is $S^\perp \equiv \{x \in W : \langle x | y \rangle = 0_F$ holds for all $y \in S\}$.

正補定理 If V is a subspace of an inner-product space W with $dim(W) < \infty$, then $V \bigoplus V^\perp = W$.

正交投影 A projection T of inner-product space W is orthogonal if $T(W)^\perp = N(T)$; $N(T)^\perp = T(W)$.

6.8 泛函定理 For any functional f on an inner-product space V with $dim(V) < \infty$, there is a unique vector $y \in V$ such that $f(x) = \langle x | y \rangle$ holds for all $x \in V$.

6.9 翻牆定理 For any linear operator T on any inner-product space V with $dim(V) < \infty$, there is a unique operator T^* on V such that $\langle T(x) | y \rangle = \langle x | T^*(y) \rangle$ holds for all vectors $x, y \in V$. Moreover, this unique T^* is linear.

6.10 翻牆推論 Let V be a finite-dimensional inner-product space. The following statements hold for any linear operator T on V. 1. For any $x, y \in V$, $\langle x | T(y) \rangle = \langle T^*(x) | y \rangle$; 2. For any orthonormal basis β of V, $[T^*]_\beta^\beta = ([T]_\beta^\beta)^*$

6.11 伴隨線運基本性質 Let V be an inner-product space over F with $dim(V) < \infty$. The following equations hold for any $T_1, T_2, T \in \mathcal{L}(V)$ and any $a \in F$: 1. $(aT_1 + T_2)^* = \bar{a}T_1^* + T_2^*$; 2. $(T_1 T_2)^* = T_2^* T_1^*$; 3. $(T^*)^* = T$; 4. $I_V^* = I_V$

Observation 伴隨線運的特徵值 If λ is an eigenvalue of T, then $\bar{\lambda}$ is an eigenvalue of T^*

Definition 常態線運、常態方陣 $TT^* = T^*T$; $AA^* = A^*A$

常態小觀察 T is normal if and only if $[T]_\beta^\beta$ is normal

6.16 值譜證明起手式 If V has an orthonormal eigenbasis for T, then T is normal

6.15 常態線運基本性質 1. For each $x \in V$, $\|T(x)\| = \|T^*(x)\|$; 2. For each $a \in F$, $T + aI_V$ is normal; 3. If (λ, x) is an eigenpair of T, then $(\bar{\lambda}, x)$ is an eigenpair of T^*; 4. If x and y are in distinct eigenspaces of T, then $\langle x | y \rangle = 0_F$

6.25 特徵值譜定理 Let $T \in \mathcal{L}(V)$ for inner-product space V with $dim(V) < \infty$ that has an orthonormal eigenbasis for T. Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of T.

1. For $i \in \{1, \ldots, k\}$, $E_T(\lambda_i)^\perp = E_T(\lambda_1) \bigoplus \cdots \bigoplus E_T(\lambda_{i-1}) \bigoplus E_T(\lambda_{i+1}) \bigoplus \cdots \bigoplus E_T(\lambda_k)$
2. If each T_i with $1 \leq i \leq k$ is the orthogonal projection of V on $E_T(\lambda_i)$, then
   (a) For indices $1 \leq i, j \leq k$, if $i = j$, then $T_i T_j = T_i$; if $i \neq j$, $T_i T_j = T_0$
   (b) (Resolution of I_V) $T_1 + T_2 + \cdots + T_k = I_V$
   (c) (Spectral decomposition of T) $\lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k = T$

Corollary 4 特徵值譜推論 If V has an orthonormal eigenbasis for T, then the othogonal projection T_i of V on $E_T(\lambda_i)$ is a polynomial in T, where λ_i is the i-th distinct eigenvalue of T.

6.14 舒爾定理 If $f_T(t)$ splits, then there is an orthonormal basis β of V such that $[T]_\beta^\beta$ is upper triangular

2.14 魔法定理 & 2.11 魔杖定理 $[T(y)]_\gamma = [T]_\beta^\gamma \times [y]_\beta$; $[TT']_\alpha^\gamma = [T]_\beta^\gamma \times [T']_\alpha^\beta$, which $[T]_\alpha^\beta = [T(\alpha_j)]_\beta$

5.10 直和等價定理 If V_i is subset of W. $dim(W) < \infty$; 1. $W = V_1 \bigoplus V_1 \bigoplus \cdots \bigoplus V_k$; 2. $W = V_1 + \cdots + V_k$ and if $x_1 + \ldots x_k = 0$ with $x_1 \in V_1, \ldots, x_k \in V_k$, then $x_1 = \cdots = x_k = 0_W$; 3. Any $x \in W$ can be uniquely written as $x = x_1 + \ldots x_k$ for $x_1 \in V_1, \ldots, x_k \in V_k$; 4. If each B_i with $1 \leq i \leq k$ is a basis of V_i, then $B_1 \cup \cdots \cup B_k$ is a basis of W; 5. There is a basis $B_1 \cup \cdots \cup B_k$ of W such that B_i is a basis of V_i