**5.9 特徵基底判別定理** Let $V$ be a vector space with $dim(V) = n$. If $T \in \mathcal{L}(V)$ and $\lambda_1 \ldots, \lambda_k$ are the distinct eigenvalues of $T$, then $T$ is diagonalizable if and only if $dim(E_T(\lambda_1)) + \cdots + dim(E_T(\lambda_k)) = n$.

**5.2 特徵值 ⇔ 特徵根** If $A \in F^{n\times n}$, then $\lambda$ is an eigenvalue of $A$ if and only if $f_A(\lambda) \equiv det(A - \lambda I_n) = 0_F$.

**5.4 特空定理** If $T$ is linear operator on vector space $V$ and $\lambda$ is an eigenvalue of $T$, then $T(x) = \lambda x \Leftrightarrow x \in E_T(\lambda)$.

**5.7 特徵值重數定理** Let $V$ be a finite-dimensional vector space. If $\lambda$ is an eigenvalue of $T \in \mathcal{L}(V)$ with multiplicity $m$, then $1 \le dim(E_T(\lambda)) \le m$.

**5.5 跨特徵空間不冗定理** Let $T$ be a linear operator on an $n$-dimensional vector space $V$. Let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of $T$. If $\varnothing \ne S_i \subseteq E_T(\lambda_i)$ is linearly independent for each $i = 1, \ldots, k$, then $S_1, \ldots, S_k$ are pairwise disjoint and $S_1 \cup \cdots \cup S_k$ is linearly independent.

**5.11 特徵空間直和定理** If $T \in \mathcal{L}(V)$ for a finite-dimensional vector space $V$, then $V$ is the direct sum of the eigenspaces of $T$ is and only if $V$ has an eigenbasis for $T$.

**5.23 Cayley-Hamilton Theorem** If $T \in \mathcal{L}(V)$ for a finite-dimensional vector space $V$ over $F$, $f_T(T) = T_0$.

**5.21 縮水觀察** Let $T \in \mathcal{L}(V)$ for vector space $V$ with $dim(V) < \infty$. If $U$ is a $T$-invariant subspace of $V$, then $f_{T_U}(t)|f_T(t)$.

**5.22 循環定理** Let $T \in \mathcal{L}(V)$ with $dim(V) < \infty$. Let $U = C_T(x) \equiv span\left(\cup_{i\le 0} T^i(x)\right)$ with $x \in V\backslash\{0_V\}$. Let $k = dim(U)$. (a) The ordered set $\beta = \langle x, T(x), \ldots, T^{k-1}(x)\rangle$ is a basis of $U$; (b) If $\sum_{i=0}^{k} a_i T^i(x) = 0_V$ with $a_k = 1_F$, then $f_{T_U}(t) = (-1)^k \sum_{i=0}^{k} a_i t^i$

**Definition 內積函數** 首項線性、共軛對稱、正定

**6.1 內積基本性質**

**6.3 正交定理** Let $x$ be a vector in inner-product space $V$. Let $S$ be a nonempty orthogonal set of nonzero vectors in $V$. Let $R$ be a finite subset of $S$. If $x = \sum_{y\in R} a_y y$ holds for scalars $a_y$, then $a_y = \frac{\langle x|y\rangle}{\langle y|y\rangle}$ holds for each $y \in R$

**正交無零則不冗**

**6.4 正交演算法** For any linearly independent subset $\alpha$ of inner-product space $V$ with $|\alpha| = n$, the set $\beta$ recursively defined below is orthogonal basis of $span(\alpha)$: $\beta_1 = \alpha_1$ and $\beta_j = \alpha_j - \sum_{i=1}^{j-1} \frac{\langle \alpha_j|\beta_i\rangle}{\langle \beta_i|\beta_i\rangle}\beta_i$ for each $j = 2, \ldots, n$.

**6.2 長度的基本性質**

**Definition 正交補集** For any nonempty subset $S$ of inner-product space $W$, the orthogonal complement of $S$ is $S^\perp \equiv \{x \in W : \langle x|y\rangle = 0_F$ holds for all $y \in S\}$.

**正補定理** If $V$ is a subspace of an inner-product space $W$ with $dim(W) < \infty$, then $V \bigoplus V^\perp = W$.

**6.25 特徵值譜定理** Let $T \in \mathcal{L}(V)$ for inner-product space $V$ with $dim(V) < \infty$ that has an orthonormal eigenbasis for $T$. Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of $T$.

1. For $i \in \{1, \ldots, k\}$, $E_T(\lambda_i)^\perp = E_T(\lambda_1) \bigoplus \cdots \bigoplus E_T(\lambda_{i-1}) \bigoplus E_T(\lambda_{i+1}) \bigoplus \cdots \bigoplus E_T(\lambda_k)$
2. If each $T_i$ with $1 \le i \le k$ is the orthogonal projection of $V$ on $E_T(\lambda_i)$, then
   (a) For indices $1 \le i, j \le k$, if $i = j$, then $T_iT_j = T_i$; if $i \ne j$, $T_iT_j = T_0$
   (b) (Resolution of $I_V$) $T_1 + T_2 + \cdots + T_k = I_V$
   (c) (Spectral decomposition of $T$) $\lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k = T$

**正交投影** A projection $T$ of inner-product space $W$ is orthogonal if $T(W)^\perp = N(T)$; $N(T)^\perp = T(W)$.

**6.8 泛函定理** For any functional $f$ on an inner-product space $V$ with $dim(V) < \infty$, there is a unique vector $y \in V$ such that $f(x) = \langle x|y\rangle$ holds for all $x \in V$.

**6.9 翻牆定理** For any linear operator $T$ on any inner-product space $V$ with $dim(V) < \infty$, there is a unique operator $T^*$ on $V$ such that $\langle T(x)|y\rangle = \langle x|T^*(y)\rangle$ holds for all vectors $x, y \in V$. Moreover, this unique $T^*$ is linear.

**6.10 翻牆推論** Let $V$ be a finite-dimensional inner-product space. The following statements hold for any linear operator $T$ on $V$. 1. For any $x, y \in V$, $\langle x|T(y)\rangle = \langle T^*(x)|y\rangle$; 2. For any orthonormal basis $\beta$ of $V$, $[T^*]_\beta^\beta = ([T]_\beta^\beta)^*$

**6.11 伴隨線運基本性質** Let $V$ be an inner-product space over $F$ with $dim(V) < \infty$. The following equations hold for any $T_1, T_2, T \in \mathcal{L}(V)$ and any $a \in F$: 1. $(aT_1 + T_2)^* = \bar{a}T_1^* + T_2^*$; 2. $(T_1T_2)^* = T_2^*T_1^*$; 3. $(T^*)^* = T$; 4. $I_V^* = I_V$

**Observation 伴隨線運的特徵值** If $\lambda$ is an eigenvalue of $T$, then $\bar{\lambda}$ is an eigenvalue of $T^*$

**Definition 常態線運、常態方陣** $TT^* = T^*T$; $AA^* = A^*A$

**常態小觀察** $T$ is normal if and only if $[T]_\beta^\beta$ is normal

**6.16 值譜證明起手式** If $V$ has an orthonormal eigenbasis for $T$, then $T$ is normal

**6.15 常態線運基本性質** 1. For each $x \in V$, $\|T(x)\| = \|T^*(x)\|$; 2. For each $a \in F$, $T + aI_V$ is normal; 3. If $(\lambda, x)$ is an eigenpair of $T$, then $(\bar{\lambda}, x)$ is an eigenpair of $T^*$; 4. If $x$ and $y$ are in distinct eigenspaces of $T$, then $\langle x|y\rangle = 0_F$

**Corollary 4 特徵值譜推論** If $V$ has an orthonormal eigenbasis for $T$, then the othogonal projection $T_i$ of $V$ on $E_T(\lambda_i)$ is a polynomial in $T$, where $\lambda_i$ is the $i$-th distinct eigenvalue of $T$.

**6.14 舒爾定理** If $f_T(t)$ splits, then there is an orthonormal basis $\beta$ of $V$ such that $[T]_\beta^\beta$ is upper triangular