6.16 常態定理 Let $T \in \mathbb{L}(V)$ for inner-product space V over $\mathbb{C}$ with $\dim(V) < \infty$. V has an orthonormal eigenbasis for T iff T is normal, i.e. $TT^* = T^*T$

Def 自伴線運、方陣 T is self-adjoint if $T^* = T$. Square matrix A is Hermitian if $A^* = A$

Obs 自伴小觀察 If β is an orthonormal basis of a inner-product space V with $\dim(V) < \infty$, then T is self-adjoint iff $[T]_\beta^\beta$ is self-adjoint.

Obs 自伴線運基本性質 If T is a self-adjoint linear operator on an inner-product space V over $F \in \{\mathbb{R}, \mathbb{C}\}$ with $\dim(V) < \infty$, then 1. each eigenvalue of T is real (even if $F = \mathbb{C}$, and) 2. the characteristic polynomial $f_T(t)$ of T splits (even if $F = \mathbb{R}$)

6.24 投影：正交 ⇔ 自伴 If T is a projection of W, then T is an orthogonal projection of W iff T is self-adjoint

Cor 自伴線運推論 Let $T \in \mathbb{L}(V)$ for inner-product space W over $F = \mathbb{C}$ wth $\dim(V) < \infty$. If T is normal, T is self-adjoint iff every eigenvalue of T is real.

6.17 自伴定理 Let $T \in \mathbb{L}(V)$ for V over $\mathbb{R}$ with $\dim(V) < \infty$. V has an orthonormal eigenbasis for T iff T is self-adjoint

Cor 常態線運推論 If $T \in \mathbb{L}(V)$ for W over $\mathbb{C}$ with $\dim(W) < \infty$, then T is normal iff $T^* = g(T)$ for some polynomial $g \in \mathbb{P}(\mathbb{C})$

Def 么正、正交方陣 Let $Q \in F^{n\times n}$ with $F \in \{\mathbb{C}, \mathbb{R}\}$, • Q is unitary if $Q^*Q = I_n$ (i.e. $Q^* = Q^{-1}$) • Q is orthogonal if $Q^tQ = I_n$ (i.e. $Q^t = Q^{-1}$)

Def 么正、正交等價 Let $A, B \in F^{n\times n}$ with $F \in \{\mathbb{C}, \mathbb{R}\}$, • A is unitarily equivalent to B if there is a unitary matrix Q with $A = Q^*BQ$ • A is orthogonally equivalent to B if there is an orthogonal matrix Q with $A = Q^tBQ$

6.20 If $A \in \mathbb{R}^{n\times n}$, then A is self-adjoint (i.e. symmetric) iff A is orthogonally (i.e. unitarily) equivalent to a diagonal matrix in $\mathbb{R}^{n\times n}$

6.19 If $A \in \mathbb{C}^{n\times n}$, then A is normal iff A is unitarily equivalent to a diagonal matrix in $\mathbb{C}^{n\times n}$

6.21 方陣舒爾 If $f_A(t)$ splits for $A \in F^{n\times n}$, then A is unitarily equivalent to an upper-triangular matrix in $F^{n\times n}$.

Def 么正、正交線運 Let $T \in \mathbb{L}(V)$, V is an inner-product space over $F \in \{\mathbb{C}, \mathbb{R}\}$. • T is unitary if $T^*T = I_V$ • T is orthogonal if T is unitary and $F = \mathbb{R}$

6.18 If $T \in \mathbb{L}(V)$ for V over $F \in \{\mathbb{C}, \mathbb{R}\}$, with $\dim(V) < \infty$, then the following are equivalent: • $T^*T = I_V$ • $\langle T(x)|T(y)\rangle = \langle x|y\rangle$ holds for all vectors $x, y \in V$ • For any orthogonal eigenbasis β of V, $T(\beta)$ is an orthonormal basis of V • There is a $\beta \subseteq V$ s.t. β and $T(\beta)$ are both orthonormal bases of V • $\|T(x)\| = \|x\|$ holds for all vectors $x \in V$

Cor 么正、正交自伴定理 If $T \in \mathbb{L}(V)$ for V over $\mathbb{C}$ ($\mathbb{R}$) with $\dim(V) < \infty$, then T is unitary (orthogonal and self-adjoint) iff • V has an orthonormal eigenbasis for T • each eigenvalue of T has absolute value 1

6.13 最短解 If $E : Ax = b$ with $A \in F^{m\times n}$ and $b \in F^m$ is a system of linear equations with $S(E) \neq \emptyset$, then there is exactly one vector x in $S(E) \cap L_{A^*}(F^m)$ w.r.t. the standard inner product. Moreover, the vector x is the unique vector in $S(E)$ with minimum $||x||$

6.12 最佳近似解 Let $A \in F^{m\times n}$ and $b \in F^m$. • For any inner-product function of F^m, $\exists x \in F^n$ that minimizes $\|Ax - b\|$ • If $\operatorname{rank}(A) = n$, then $x = (A^*A)^{-1}A^*b$ is the unique minimizer of $\|Ax - b\|$ w.r.t. the standard inner product of F^m

Obs 標準內積觀察 Let $A \in F^{m\times n}$. For any $x \in F^n$ and $y \in F^m$, we have $\langle Ax|y\rangle_m = \langle x|A^*y\rangle_n$

Obs 矩陣位階觀察 For any $A \in F^{m\times n}$, $\operatorname{rank}(A^*A) = \operatorname{rank}(A)$

Obs 伴隨矩陣觀察 For any $A \in F^{m\times n}$, if $\operatorname{rank}(A) = n$, then $A^*A \in F^{n\times n}$ is invertible.

Def 正定 A self-adjoint $T \in \mathbb{L}(V)$ for V over F is positive definite (semidefinite) if $\langle T(x)|x\rangle \in \mathbb{R}^+$ ($\langle T(x)|x\rangle \in \mathbb{R}^+ \cup \{0_\mathbb{R}\}$) holds for all $x \in V\backslash\{0_V\}$ ($x \in V$)

Obs （半）正定線運等價條件 If $T \in \mathbb{L}(V)$ with $\dim(V) < \infty$ is self-adjoint, then T is positive (semi)definite iff all eigenvalues are positive (non-negative) real numbers

Obs 半正定方陣等價條件 For $F \in \{\mathbb{C}, \mathbb{R}\}$, $A \in F^{n\times n}$ is positive semidefinite iff $A = B^*B$ holds for some $B \in F^{n\times n}$

Obs 半正定平方性質 If $T_1, T_2 \in \mathbb{L}(V)$ ($A, B \in F^{n\times n}$) with $T_1^2 = T_2^2$ ($A^2 = B^2$), then $T_1 = T_2$ ($A = B$)

Def 伴隨線轉 $T \in \mathbb{L}(V, W)$. $T^* : W \to V$ is an *adjoint* of T if $\langle T(x)|y\rangle_W = \langle x|T^*(y)\rangle_V$ holds $\forall x \in V, y \in W$

Obs 伴隨線轉觀察 If $T \in \mathbb{L}(V, W)$ with $\dim(V) < \infty, \dim(W) < \infty$ then, • Both T^*T, TT^* are positive semidefinite • $N(T^*T) = N(T)$ • $\operatorname{rank}(T^*T) = \operatorname{rank}(TT^*) = \operatorname{rank}(T) = \operatorname{rank}(T^*)$

6.26 奇異值定理 Let $T \in \mathbb{L}(V, W)$ for V and W over $F \in \{\mathbb{C}, \mathbb{R}\}$ with $\operatorname{rank}(T) = r$, $\dim(V) = n$, and $\dim(W) = m$. (a) There exist a set $\sigma = \langle\sigma_1, \cdots, \sigma_r\rangle$ of positive real numbers and orthonormal bases $\beta = \langle\beta_1, \cdots, \beta_n\rangle$ of V and $\gamma = \langle\gamma_1, \cdots, \gamma_m\rangle$ of W s.t. with $\sigma_k = 0_F$ for $r + 1 \le k \le \max(n, m)$ and $\gamma_k = 0_W$ for $m + 1 \le k \le n$ 1. $T(\beta_j) = \sigma_j\gamma_j$ holds for each $j = 1, \cdots, n$. (b) Moreover, the following hold for any such σ, β, γ: 2. $T^*(\gamma_i) = \sigma_i\beta_i$ holds for $i = 1, \cdots, m$ with $\beta_k = 0_V$ for $n + 1 \le k \le m$ 3. Each (σ_j^2, β_j) with $1 \le j \le n$ is an eigenpair of T^*T 4. Each (σ_i^2, γ_i) with $1 \le i \le m$ is an eigenpair of TT^*

6.27 SVD For any matrix $A = F^{m\times n}$ with rank r, $A = QSR*$ holds for: • unitary matrices $Q \in F^{m\times m}$ and $R \in F^{n\times n}$ • a diagonal matrix $S \in F^{m\times n}$ whose $l = \min(m, n)$ diagonal elements $\sigma_i = S_{i,i}$ with $i \in \{1, \cdots, l\}$ satisfy 1. $\sigma_1, \cdots, \sigma_r \in \mathbb{R}^+$ 2. $\sigma_{r+1} = \cdots = \sigma_l = 0_F$

Def 偽反線轉 $T^\dagger \in \mathbb{L}(W, V)$ defined by $T^\dagger = (T')^{-1}T''$

偽反線轉定理 Let $T \in \mathbb{L}(V, W)$ for V and W over $F \in \{\mathbb{C}, \mathbb{R}\}$ with $\operatorname{rank}(T) = r, \dim(V) = n, \dim(W) = m$. If σ, β, γ are as ensured by **6.26**, then $T^\dagger$ is the unique function in $\mathbb{L}(W, V)$ satisfying $T^\dagger(\gamma_j) = \begin{cases} \frac{1}{\sigma_j}\beta_j & \text{if} 1 \le j \le r \\ 0_V & \text{otherwise} \end{cases}$

偽反矩陣觀察 $S^\dagger$ 對角線上非零的值是原本的倒數，另外 $S^\dagger$ 的行數和列數會互換。

偽反線運等價條件 If $T_1 \in \mathbb{L}(V, W)$ and $T_2 \in \mathbb{L}(W, V)$ for V, W over $F \in \{\mathbb{C}, \mathbb{R}\}$ and $\dim < \infty$, then $T_2 = T_1^\dagger$ iff **all** following hold: (a) $T_1T_2T_1 = T_1$ (b) $T_2T_1T_2 = T_2$ (c) T_1T_2 and T_2T_1 are self-adjoint

偽反矩陣定理 If $A = QSR^*$ is a SVD of matrix $A \in F^{m\times n}$ with $F \in \{\mathbb{C}, \mathbb{R}\}$, then $A^\dagger = RS^\dagger Q^*$ is a SVD of matrix $A^\dagger \in F^{n\times m}$

6.30 For any system of linear equations $E : Ax = b$ with $A \in F^{m\times n}$, $y = A^\dagger b$ is the unique vector in F^n with • $\|Ay - b\| \le \|Ax - b\|$ for any $x \in F^n$ and • $\|y\| < \|x\|$ any $x \in F^n\backslash\{y\}$ with $\|Ay - b\| = \|Ax - b\|$

6.28 極分解定理 For any square matrix $A \in F^{n\times n}$ with $F \in \{\mathbb{C}, \mathbb{R}\}$, there exist a unitary matrix $Q \in F^{n\times n}$ and a positive semidefinite matrix $P \in F^{n\times n}$ s.t. $A = QP$. And, if A is invertible, then the decomposition is unique ($Q = Q_0R^*, P = RSR^*$)