5.9 特徵基底判別定理 Let V be a vector space with $dim(V) = n$. If $T \in \mathcal{L}(V)$ and $\lambda_1 \dots, \lambda_k$ are the distinct eigenvalues of T, then T is diagonalizable if and only if $dim(E_T(\lambda_1)) + \dots + dim(E_T(\lambda_k)) = n$.

5.2 特徵值 $\Leftrightarrow$ 特徵根 If $A \in F^{n\times n}$, then λ is an eigenvalue of A if and only if $f_A(\lambda) \equiv det(A - \lambda I_n) = 0_F$.

5.4 特空定理 If T is linear operator on vector space V and λ is an eigenvalue of T, then $T(x) = \lambda x \Leftrightarrow x \in E_T(\lambda)$.

5.7 特徵值重數定理 Let V be a finite-dimensional vector space. If λ is an eigenvalue of $T \in \mathcal{L}(V)$ with multiplicity m, then $1 \le dim(E_T(\lambda)) \le m$.

5.5 跨特徵空間不冗定理 Let T be a linear operator on an n-dimensional vector space V. Let $\lambda_1, \dots, \lambda_k$ be distinct eigenvalues of T. If $\varnothing \ne S_i \subseteq E_T(\lambda_i)$ is linearly independent for each $i = 1, \dots, k$, then $S_1, \dots, S_k$ are pairwise disjoint and $S_1 \cup \dots \cup S_k$ is linearly independent.

5.11 特徵空間直和定理 If $T \in \mathcal{L}(V)$ for a finite-dimensional vector space V, then V is the direct sum of the eigenspaces of T is and only if V has an eigenbasis for T.

5.23 Cayley-Hamilton Theorem If $T \in \mathcal{L}(V)$ for a finite-dimensional vector space V over F, $f_T(T) = T_0$.

5.21 縮水觀察 Let $T \in \mathcal{L}(V)$ for vector space V with $dim(V) < \infty$. If U is a T-invariant subspace of V, then $f_{T_U}(t) | f_T(t)$.

5.22 循環定理 Let $T \in \mathcal{L}(V)$ with $dim(V) < \infty$. Let $U = C_T(x) \equiv span\left(\cup_{i\le 0} T^i(x)\right)$ with $x \in V \backslash \{0_V\}$. Let $k = dim(U)$. (a) The ordered set $\beta = \langle x, T(x), \dots, T^{k-1}(x)\rangle$ is a basis of U; (b) If $\sum_{i=0}^{k} a_i T^i(x) = 0_V$ with $a_k = 1_F$, then $f_{T_U}(t) = (-1)^k \sum_{i=0}^{k} a_i t^i$

Definition 內積函數 1. $d(ax + y, z) = a \cdot d(x, z) + d(y, z)$; 2. $d(y, x) = \bar{d(x, y)}$; 3. if $x \ne 0_V$, then $d(x, x) \in \mathcal{R}^+$

6.1 內積基本性質 1. $\langle x|ay + z\rangle = \bar{a}\langle x|y\rangle + \langle x|z\rangle$; 2. $\langle x|0_V\rangle = \langle 0_V|x\rangle = 0_F$; 3. $\langle x|x\rangle = 0_F \iff x = 0_V$; 4. If $\langle x|y\rangle = \langle x|z\rangle$ holds for all x, then $y = z$

6.3 正交定理 Let x be a vector in inner-product space V. Let S be a nonempty orthogonal set of nonzero vectors in V. Let R be a finite subset of S. If $x = \sum_{y\in R} a_y y$ holds for scalars a_y, then $a_y = \frac{\langle x|y\rangle}{\langle y|y\rangle}$ holds for each $y \in R$; (note: 正交無零則不冗)

6.4 正交演算法 For any linearly independent subset α of inner-product space V with $|\alpha| = n$, the set β recursively defined below is orthogonal basis of $span(\alpha)$: $\beta_1 = \alpha_1$ and $\beta_j = \alpha_j - \sum_{i=1}^{j-1} \frac{\langle \alpha_j|\beta_i\rangle}{\langle \beta_i|\beta_i\rangle} \dot{\beta}_i$ for each $j = 2, \dots, n$.

6.2 長度的基本性質 1. $\|ax\| = |a|\|x\|$; 2. $\|x\| \in \mathcal{R}^+ \cup \{0_R\}$; 3. $\|x\| = 0_R \iff x = 0_V$; 4. Cauchy-Schwarz inequality $|\langle x|y\rangle| \le \|x\| \cdot \|y\|$; 5. Triangle inequality $|\, x+y\| \le \|x\| + \|y\|$

Definition 正交補集 For any nonempty subset S of inner-product space W, the orthogonal complement of S is $S^\perp \equiv \{x \in W : \langle x|y\rangle = 0_F$ holds for all $y \in S\}$.

正補定理 If V is a subspace of an inner-product space W with $dim(W) < \infty$, then $V \bigoplus V^\perp = W$.

正交投影 A projection T of inner-product space W is orthogonal if $T(W)^\perp = N(T)$; $N(T)^\perp = T(W)$.

6.8 泛函定理 For any functional f on an inner-product space V with $dim(V) < \infty$, there is a unique vector $y \in V$ such that $f(x) = \langle x|y\rangle$ holds for all $x \in V$.

6.9 翻牆定理 For any linear operator T on any inner-product space V with $dim(V) < \infty$, there is a unique operator T^* on V such that $\langle T(x)|y\rangle = \langle x|T^*(y)\rangle$ holds for all vectors $x, y \in V$. Moreover, this unique T^* is linear.

6.10 翻牆推論 Let V be a finite-dimensional inner-product space. The following statements hold for any linear operator T on V. 1. For any $x, y \in V$, $\langle x|T(y)\rangle = \langle T^*(x)|y\rangle$; 2. For any orthonormal basis β of V, $[T^*]_\beta^\beta = ([T]_\beta^\beta)^*$

6.11 伴隨線運基本性質 Let V be an inner-product space over F with $dim(V) < \infty$. The following equations hold for any $T_1, T_2, T \in \mathcal{L}(V)$ and any $a \in F$: 1. $(aT_1 + T_2)^* = \bar{a}T_1^* + T_2^*$; 2. $(T_1T_2)^* = T_2^* T_1^*$; 3. $(T^*)^* = T$; 4. $I_V^* = I_V$

Observation 伴隨線運的特徵值 If λ is an eigenvalue of T, then $\bar{\lambda}$ is an eigenvalue of T^*

Definition 常態線運、常態方陣 $TT^* = T^*T$; $AA^* = A^*A$

常態小觀察 T is normal if and only if $[T]_\beta^\beta$ is normal

6.16 值譜證明起手式 If V has an orthonormal eigenbasis for T, then T is normal

6.15 常態線運基本性質 1. For each $x \in V$, $\|T(x)\| = \|T^*(x)\|$; 2. For each $a \in F$, $T + aI_V$ is normal; 3. If (λ, x) is an eigenpair of T, then $(\bar{\lambda}, x)$ is an eigenpair of T^*; 4. If x and y are in distinct eigenspaces of T, then $\langle x|y\rangle = 0_F$

6.25 特徵值譜定理 Let $T \in \mathcal{L}(V)$ for inner-product space V with $dim(V) < \infty$ that has an orthonormal eigenbasis for T. Let $\lambda_1, \dots, \lambda_k$ be the distinct eigenvalues of T.

1. For $i \in \{1, \dots, k\}$, $E_T(\lambda_i)^\perp = E_T(\lambda_1) \bigoplus \dots \bigoplus E_T(\lambda_{i-1}) \bigoplus E_T(\lambda_{i+1}) \bigoplus \dots \bigoplus E_T(\lambda_k)$
2. If each T_i with $1 \le i \le k$ is the orthogonal projection of V on $E_T(\lambda_i)$, then
   (a) For indices $1 \le i,\ j \le k$, if $i = j$, then $T_iT_j = T_i$; if $i \ne j$, $T_iT_j = T_0$
   (b) (Resolution of I_V) $T_1 + T_2 + \dots + T_k = I_V$
   (c) (Spectral decomposition of T) $\lambda_1T_1 + \lambda_2T_2 + \dots + \lambda_kT_k = T$

Corollary 4 特徵值譜推論 If V has an orthonormal eigenbasis for T, then the othogonal projection T_i of V on $E_T(\lambda_i)$ is a polynomial in T, where λ_i is the i-th distinct eigenvalue of T.

6.14 舒爾定理 If $f_T(t)$ splits, then there is an orthonormal basis β of V such that $[T]_\beta^\beta$ is upper triangular

2.14 魔法定理 & 2.11 魔杖定理 $[T(y)]_\gamma = [T]_\beta^\gamma \times [y]_\beta$; $[TT']_\alpha^\gamma = [T]_\beta^\gamma \times [T']_\alpha^\beta$, which $[T]_\alpha^\beta = [T(\alpha_j)]_\beta$