Thermodynamic Consistency of the Spin-Boson model

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Abstract

This report deals with my ongoing internship at Quandela, Paris.

In this open quantum systems project, we look at the thermodynamics of a single photon source in a cavity interacting with a bath and waveguide at the fluctuating level. The thermodynamics can be studied at the fluctuating level using the two-point measurement scheme with counting fields. Previously, this formalism was used to thoroughly characterise the system-bath interaction, with a system consisting of a quantum dot inside a cavity. Presently, we look at the spin-boson style of qubit-bath interaction, which is more often used in literature, as well as a waveguide which is used to capture photons leaking from the system. I perform a perturbative expansion of the tilted dynamics towards writing the moment generating function for.

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1 Introduction and Theory

Understanding and monitoring energy fluctuations at the quantum scale is a central issue in quantum technologies. Presently, we investigate the thermodynamics of a single-photon source (SPS), consisting of a coherently driven quantum dot inside a cavity. The coherent drive is performed by a laser, acting as a work source, while the environment acts as a heat source. The photons useful for quantum computing leave the cavity through a waveguide. The laser can either be modelled as an external drive (inducing a time dependent term in the system Hamiltonian), or using a time-independent picture, where the coherent field is introduced by applying a displacement operator on a mode of the waveguide. At the microscopic level, the conservation of energy allows us to relate the statistics of the number of photons which exit the cavity (the photons useful for quantum computing) to the work and heat statistics. This would provide a relation between the number of photons we can expect to produce and the cost of running and maintaining the system.

In practice, modelling the full unitary dynamics is too complex, so one resorts to quantum master equations. However, the most common master equations in the literature used to evolve such systems are derived using sets of physical assumptions that may cause thermodynamics to break down. In this internship, the overarching goal is to faithfully model our physical setup and then to find fully thermodynamically consistent master equations. Towards this end, we report on the progress made so far.

We will begin in this section by specifying a microscopic model derived from the physical setup used by Quandela. The model is based off a quantum dot (QD) inside a micropillar cavity 1.1, which informs our choice of Hamiltonian. Then we will introduce the counting field formalism which is used to characterise fluctuations 1.2. We will revisit some recent results which used this formalism to tackle a variation of this setup 1.3.

In the following section 2, we will express the first and second laws of thermodynamics at the unitary level. We will then start computing Dyson series terms in the interaction picture to find the contributions to the generating function $\mathcal{G}$.

We then present the results of the calculations in section 3, and what they imply about fluctuation theorems and generating function symmetries.

The system being analysed is given by

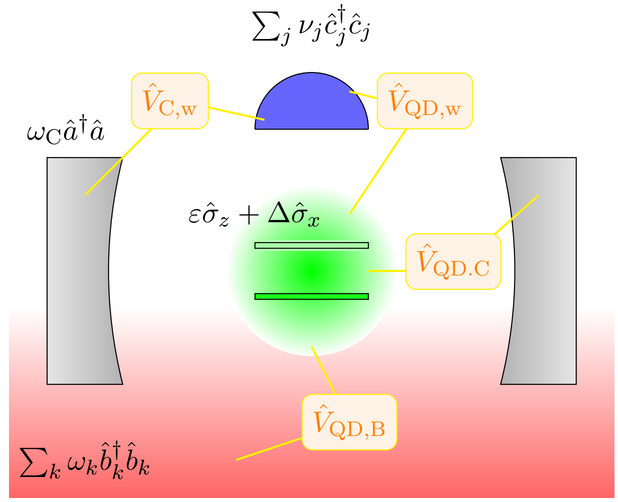
$$\hat{H}_{\text{Total}} = \hat{H}_{\text{QD}} + \hat{H}_{\text{Cavity}} + \hat{H}_{\text{Bath}} + \hat{H}_{\text{Waveguide}} + \hat{V}_{\text{JC}} + \hat{V}_{\text{Phonon}} + \hat{V}_{\text{Waveguide+Cavity}} + \hat{V}_{\text{Waveguide+QD}} \quad (1)$$

which contains free terms and coupling terms. We seek, for the first time with this system, to investigate its thermodynamic consistency using counting fields. The precise form of the interaction terms greatly impacts the analysis. Therefore, we must motivate the form of this Hamiltonian and how to tackle its thermodynamic viability.

1.1 Derivation of physical model

To model the photon source, our Hamiltonian should capture the following elements based off the experimental setup used by Quandela:

- A 2-level system (QD)
- A resonant mode of the cavity (C)
- Photons leaking from the cavity into the waveguide (w)
- A phonon bath (B)
- The atom coupling to environment and cavity
- Laser driving/coherent driving



The above sketch depicts the quantum dot in the centre, surrounded by a cavity and with a waveguide above. The respective energies and various interactions are also shown. Not depicted is the work source, as we may choose to model this either as an external drive or as a coherent state in the waveguide.

1.1.1 Single-photon Source

The single-photon source used in Quandela, depicted in Fig. (1), is based on an InGaAs quantum dot cooled to 4K inside GaAs/$Al_{0.9}Ga_{0.1}As$ distributed Bragg reflectors [1][2]. The laser is injected via the waveguide. The main frequency emitted by the quantum dot is that of the cavity, ω_{C}. There are other frequencies also (such as $\omega_{\mathrm{C}} \pm \Omega$, with the Rabi frequency due to laser driving) which get filtered out.

The SPS is a 2-level system with ground state and excited state $|g\rangle$ and $|e\rangle$ respectively. The SPS interacts with a single cavity mode through a Jaynes-Cummings interaction, and it also interacts with many phonon modes via some operator $\hat{Q}$ through a spin-boson interaction. This operator $\hat{Q}$ is often chosen as $\hat{\sigma}_Z$ or $|e\rangle\langle e|$ [3].

Figure 1: Single-photon source: Quantum dot and micropillar inside central pillar of the circular frame, with waveguide above [1, 2]. Outgoing photons are filtered for frequency.

1.1.2 Spin-Boson model

In order to model the qubit interaction with an environment, we use a spin-boson model. This interaction term can be derived by considering an electron transfer system embedded in a lattice of ions. Expanding the Coulomb force leads to a harmonic oscillator potential. Typically, these interactions do not induce a transition in the SPS. Phonons are treated like harmonic oscillators, with each mode interacting with the dipole through a $\hat{\sigma}_z x_k$ term in the spin-boson Hamiltonian given by [4]

$$\hat{H}_{\text{SB}} = \varepsilon\hat{\sigma}_z + \Delta\hat{\sigma}_x + \frac{1}{2}\sum_k \left(\frac{p_k^2}{m_k} + m_k\omega_k x_k^2 - \hat{\sigma}_z q_0 c_k x_k\right). \tag{2}$$

The displacement operator can be written as a linear combination $\hat{b}^\dagger + \hat{b}$ of bath operators, to arrive at

$$\hat{H}_{\text{SB}} = \varepsilon\hat{\sigma}_z + \Delta\hat{\sigma}_x + \sum_k \hat{b}_k^\dagger\hat{b}_k - \hat{\sigma}_z\sum_k g_k\left(\hat{b}_k^\dagger + \hat{b}_k\right), \tag{3}$$

discarding any constant energy shift. Thus, our coupling term is a product in $\mathcal{H}_{\text{Q}} \otimes \mathcal{H}_{\text{B}}$. This contrasts with a Jaynes-Cummings type of interaction as was studied before in [5].

1.2 Counting Fields

In this project, we work with the two-point measurement technique with counting fields [6, 5]. In this formalism, we identify terms in our Hamiltonian whose fluctuations we would like to investigate. We then attach a counting field λ, giving a tilted Hamiltonian $\hat{H}^{\vec{\lambda}}$. Then by solving for unitary evolution or master equations, we seek symmetries in the counting field which constrain thermodynamic quantities. The strict energy-conservation condition puts a constraint on the tilted master equations.

The probability to observe a fluctuation Δa when measuring an observable $\hat{A}(t)$ at times 0 and t is given by

$$p(\Delta a) \equiv \sum \delta(\Delta a - (a_t - a_o))\, P[a_t, a_0] \tag{4}$$

which is a sum over joint probability distributions. The statistics of $p(\Delta a)$ are characterised using the moment generating function, defined as the Fourier transform

$$\mathcal{G}(\lambda) = \int_{-\infty}^{\infty} \mathrm{d}\Delta a\; e^{i\lambda\Delta a} p(\Delta a) \tag{5}$$

of $p(\Delta a)$. Differentiating with respect to λ will pull down a factor of $i(a_t - a_0)$. Then setting $\lambda = 0$ recovers the nth moment of $p(\Delta a)$. Inserting the definition for the joint probability

$$\begin{aligned}
\mathcal{G}(\lambda) &= \sum e^{i\lambda(a_t - a_0)}\,\mathrm{Tr}\left[P_{a_t}U(t,0)P_{a_0}\rho_0 P_{a_0}U^\dagger(t,0)P_{a_t}\right] \\
&= \sum e^{i\lambda a_t}\,\mathrm{Tr}\left[P_{a_t}U(t,0)e^{-\frac{i\lambda\hat{A}(0)}{2}}\bar{\rho}_0 e^{-\frac{i\lambda\hat{A}(0)}{2}}U^\dagger(t,0)P_{a_t}\right] \\
&= \mathrm{Tr}\left[e^{\frac{i\lambda\hat{A}(t)}{2}}U(t,0)e^{-\frac{i\lambda\hat{A}(0)}{2}}\bar{\rho}_0 e^{-\frac{i\lambda\hat{A}(0)}{2}}U^\dagger(t,0)P_{a_t}e^{\frac{i\lambda\hat{A}(t)}{2}}\right] \\
&= \mathrm{Tr}\left[U_\lambda(t,0)\hat{\rho}_0 U_{-\lambda}^\dagger(t,0)\right] \\
&= \mathrm{Tr}\,\rho^{\vec{\lambda}}(t).
\end{aligned} \tag{6}$$

Here, $\rho^{\vec{\lambda}}$ is called the *tilted* density matrix, and is no longer Hermitian because the tilted propagators acting on the left and right are modified. Let's also define the generating function of the reversed dynamics:

$$\mathcal{G}^R(\lambda) = \mathrm{Tr}\left[U_\lambda^\dagger(t)\rho^R(0)U_{-\lambda}(t)\right]. \tag{7}$$

If at $t = 0$, the density matrix is factorised into Gibbs states, there exists a symmetry (see Appendix (A.3) between the forward and reversed dynamics given by

$$\mathcal{G}(\vec{\lambda}, t) = \mathcal{G}^R(-\vec{\lambda} + i\vec{\nu}, t), \tag{8}$$

where $\vec{\nu}$ is the vector of inverse temperatures β_i defining the Gibbs states. The condition for energy conservation in the absence of external driving is $\partial_\lambda \mathcal{G}(\lambda, t)|_{\lambda=0} = 0$ which follows if the interaction term commutes with the free Hamiltonian terms. A consequence is the condition

$$\hat{\rho}_{\vec{\lambda}}(t) = \hat{\rho}_{\vec{\lambda}+\chi\vec{1}}(t) \tag{9}$$

where $\vec{1} = (1, 1, \ldots, 1)$ and $\chi \in \mathbb{R}$. At the level of master equations, this strict requirement (9) is rendered as the energy balance condition (see appendix of [7])

$$\mathcal{L}(t)_{\vec{\lambda}}[\cdot] = \mathcal{L}(t)_{\vec{\lambda}+\chi\hat{1}}[\cdot]. \tag{10}$$

At the level of master equations, there exists also a *generalised detailed balance* condition from (8) which relates the forward and reversed dynamics $\mathcal{L}^\dagger$ and $\mathcal{L}^R$.

Naturally, the interaction terms will not commute with the free part of the Hamiltonian in our case. Instead, we can work with the weak-coupling limit.

It is significant to note that, in general, commonly used master equation schemes are not thermodynamically consistent according to these criteria. For instance, the Redfield equation, which relies on the Born and Markov approximations, is not consistent [7] without modifications. The offending approximations help significantly to make the equations numerically tractable and to express the equation as local in time. To instead avoid any physical approximations at all, we can expand the Dyson series in the coupling term. This will impact the dynamics, but the calculations used to get there would still be useful in future for deriving master equations.

1.3 Thermodynamic Consistency of Qubit + Laser + Bath via counting fields

As of this year, the thermodynamic consistency of a qubit and laser in a bath has been studied thoroughly [5], where the Hamiltonian considered was

$$\frac{\omega_A}{2}\hat{\sigma}_z + \omega_c \hat{a}^\dagger \hat{a} + \sum_k \omega_k \hat{b}_k^\dagger \hat{b}_k + \frac{g_0}{2}(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a} + \hat{a}^\dagger) + (\hat{\sigma}_+ + \hat{\sigma}_-)\sum_k \frac{g_k}{2}(\hat{b} + \hat{b}^\dagger) \tag{11}$$

which, compared to our model, does not include a waveguide-like term for extracting photons from the cavity.

The authors derive a collection of thermodynamic laws and identities in different regimes. At the average level, they write a first and second law of thermodynamics. They relate the forward and time-reversed work fluctuations in a *work fluctuation theorem*, in both the autonomous and non-autonomous picture.

In particular, the final term of (11) is significant for the choice of qubit coupling. In the literature, a more common choice is $\hat{\sigma}_z$ coupling to this sum of bath operators while using the raising and lowering operators for the Jaynes-Cummings interaction, as in the spin-boson model [3].

1.4 Interaction picture and Dyson series expansion

Since many common master equations involve in their derivation some physical assumptions that may break counting field symmetries like the Born and Markov approximations, in this work I instead make a perturbative expansion of the Dyson series for the tilted density matrix to second order. Such an expansion is non-convergent [8] without correction. This approach limits the scope of the expression to small time/weak coupling. The truncated Dyson series nevertheless serves as an appropriate starting point. It allows to make progress towards solving the evolution before making any physical assumptions or approximations. This approach also avoids needing to integrate the master equations to find the tilted density matrix.

Let $\{\hat{H}_i\}$ denote the N commuting terms in our Hamiltonian to which we apply a counting field. Then clearly

$$e^{\frac{i}{2}\sum_j \lambda_j \hat{H}_j}\hat{H}_i e^{-\frac{i}{2}\sum_j \lambda_j \hat{H}_j} = \hat{H}_i$$

and so a product or linear combination of the $\{\hat{H}_i\}$ terms does not get counting fields. Now consider an interaction term $\hat{V}$. Applying counting fields gives

$$e^{\frac{i}{2}\sum_j \lambda_j \hat{H}_j}\hat{V} e^{-\frac{i}{2}\sum_j \lambda_j \hat{H}_j} = \hat{V}(\vec{\lambda}/2).$$

When we move to the interaction picture also with respect to $\{\hat{H}_i\}$, we have

$$e^{i\sum_j \hat{H}_j}\hat{V}(\vec{\lambda}/2)e^{-i\sum_j \hat{H}_j} = e^{i\sum_j \hat{H}_j}e^{\frac{i}{2}\sum_j \lambda_j \hat{H}_j}\hat{V} e^{-\frac{i}{2}\sum_j \lambda_j \hat{H}_j}e^{-i\sum_j \hat{H}_j} = \hat{V}(\lambda_1/2{+}t, \ldots, \lambda_N/2{+}t) = V(\vec{\lambda}/2{+}t).$$

That is to say, the counting fields λ_i add with the time t in the argument since we exponentiate on both sides with the same operator. Then to compute some $\hat{H}(-\vec{\lambda})$, we flip the sign of the λ_i's in the argument. We still have that $\hat{V}(\vec{\lambda}/2+t)$ is Hermitian. In total, we write the tilted interaction Hamiltonian in the interaction picture as $\hat{H}_I(t, \vec{\lambda}/2) = \sum_j \hat{V}_j(\vec{\lambda}/2+t)$, which is still Hermitian. We evolve the tilted density matrix in the interaction picture according to

$$\rho_I^{\vec{\lambda}}(t) = U^{\vec{\lambda}}(t)\rho_I^{\vec{\lambda}}(0)\left(U^{-\vec{\lambda}}(t)\right)^\dagger = \mathcal{T}\exp\left(-i\int_0^t \mathrm{d}s\; H_I^{\vec{\lambda}}(s)\right)\times\rho_I^{\vec{\lambda}}(0)\times\left[\mathcal{T}\exp\left(-i\int_0^t \mathrm{d}s\; H_I^{-\vec{\lambda}}(s)\right)\right]^\dagger. \tag{12}$$

Expanding the Dyson series

$$\left[\hat{\mathbb{1}} - i\int_0^t \mathrm{d}s\; \hat{H}_I^{\vec{\lambda}}(s) - \int_0^t\int_0^{s_1}\mathrm{d}s_1\mathrm{d}s_2\; \hat{H}_I^{\vec{\lambda}}(s_1)\hat{H}_I^{\vec{\lambda}}(s_2)+\ldots\right]\times\rho_I^{\vec{\lambda}}(0)\times\left[\hat{\mathbb{1}} - i\int_0^t \mathrm{d}s\; \hat{H}_I^{-\vec{\lambda}}(s) - \int_0^t\int_0^{s_1}\mathrm{d}s_1\mathrm{d}s_2\; \hat{H}_I^{-\vec{\lambda}}(s_1)\hat{H}_I^{-\vec{\lambda}}(s_2)+\ldots\right]^\dagger \tag{13}$$

$$= \left[\hat{\mathbb{1}} - i\int_0^t \mathrm{d}s\; \hat{H}_I^{\vec{\lambda}}(s) - \int_0^t\int_0^{s_1}\mathrm{d}s_1\mathrm{d}s_2\; \hat{H}_I^{\vec{\lambda}}(s_1)\hat{H}_I^{\vec{\lambda}}(s_2)+\ldots\right]\times\rho_I^{\vec{\lambda}}(0)\times\left[\hat{\mathbb{1}} + i\int_0^t \mathrm{d}s\; \hat{H}_I^{-\vec{\lambda}}(s) - \int_0^t\int_0^{s_1}\mathrm{d}s_1\mathrm{d}s_2\; \hat{H}_I^{-\vec{\lambda}}(s_2)\hat{H}_I^{-\vec{\lambda}}(s_1)+\ldots\right], \tag{14}$$

we keep terms up to quadratic order. The truncated expansion gives

$$\mathcal{G}(t,\vec{\lambda}) = \mathrm{Tr}\,\rho_I^{\vec{\lambda}}(0) - i\,[\text{Linear terms}] + \mathrm{Tr}\left[\int_0^t\int_0^t \hat{H}_I^{\vec{\lambda}}(s_1)\rho_I^{\vec{\lambda}}(0)\hat{H}_I^{-\vec{\lambda}}(s_2)\mathrm{d}s_1\mathrm{d}s_2\right]$$
$$-\,\mathrm{Tr}\left[\int_0^t\int_0^{s_1} \hat{H}_I^{\vec{\lambda}}(s_1)\hat{H}_I^{\vec{\lambda}}(s_2)\rho_I^{\vec{\lambda}}(0)\mathrm{d}s_1\mathrm{d}s_2\right] - \mathrm{Tr}\left[\int_0^t\int_0^{s_1} \rho_I^{\vec{\lambda}}(0)\hat{H}_I^{-\vec{\lambda}}(s_2)\hat{H}_I^{-\vec{\lambda}}(s_1)\mathrm{d}s_1\mathrm{d}s_2\right].$$

The quadratic part consists of 2 terms which are time-ordered, and one mixed term which is not time-ordered. The Hamiltonians acting on ρ from the left have positive counting fields and the Hamiltonians acting from the right have negative counting fields. Setting all $\lambda_i = 0$ recovers the regular dynamics.

Of course, this is an equation which is not time-convolutional in ρ. In comparison, the Redfield weak-coupling master equation (here without counting fields) between a system X and a bath is [9]

$$\frac{\partial}{\partial t}\rho_X(t) = -\,\mathrm{Tr}\left[\int_0^t \mathrm{d}s\left[\hat{H}(t),\left[\hat{H}(s),\rho(s)\right]\right]\right] \tag{15}$$

which *does* involve a time integration over $\rho(s)$. The Born approximation says that we may write $\rho(s) = \rho_X(s)\otimes\rho_{\text{Baths}}(0)$ factorised for all times s if the bath is big enough. We then approximate $\rho(s)\approx\rho(t)$ to lowest order in $\hat{H}$. This leaves an ordinary time-local differential equation

$$\frac{\partial}{\partial t}\rho_X(t) = -\,\mathrm{Tr}\left[\int_0^t \mathrm{d}s\left[\hat{H}(t),\left[\hat{H}(s),\rho_X(t)\otimes\rho_{\text{Baths}}\right]\right]\right]. \tag{16}$$

Finally, if a weak perturbation decays quickly and the bath becomes fully memoryless, doing away with the finite upper limit of integration, we apply the Markov approximation

$$\frac{\partial}{\partial t}\rho_X(t) = -\operatorname{Tr}\left[\int_0^\infty \mathrm{d}s \left[\hat{H}(t), \left[\hat{H}(s), \rho_X(t) \otimes \rho_{\text{Baths}}\right]\right]\right]. \tag{17}$$

This master equation has some similarities to the truncated Dyson series which means the counting field calculations are transferable.

In [3], Gustin and Hughes write a master equation in the low temperature limit for a Qubit + Laser + Bath system which contains dissipator terms $\kappa\mathcal{D}[a](\rho)$ and $\gamma\mathcal{D}[\hat{\sigma}_-](\rho)$ for cavity decay and exciton decay respectively. It also contains a weak coupling term $\mathbb{L}(\rho)$ given by

$$\mathbb{L}(\rho) = \int_0^\infty d\tau \Gamma_{\mathrm{w}}(\tau)\big(\tilde{N}(t-\tau,t)\rho N - N\tilde{N}(t-\tau,t)\rho\big) + \text{h.c.}. \tag{18}$$

It contains a correlation function in the integrand which itself requires another integration. This master equation is more tedious as one must compute $\mathbb{L}$ at many times. With such master equations adopted to include the waveguide, one could use these dissipators to write an equation for the number of photons leaking from the cavity in terms of work and heat. In the next section, we will consider how to attach counting fields to our Hamiltonian and how to trace out a bath.

2 Methods: Analytic computation for several couplings

In this section, we consider various types of coupling terms that are likely candidates to appear inside the Hamiltonian of a photonic system we are interested in. We aim to compute the generating function $\mathcal{G}$ to analyse fluctuations. Conveniently, by taking the starting conditions to be in thermal or coherent states, we can handle different coupling terms separately, as mixing together different coupling terms destroys the trace.

Before we get there, let us write the first and second laws of thermodynamics for this system, in the cases of external driving and in the case of coherent drive. Whence we learn the thermodynamic significance of each term in the Hamiltonian.

2.1 Laws of thermodynamics at the unitary level

At the level of the total Hamiltonian, we give a thermodynamic interpretation to the expectation value of each term. By looking at the change in energy and entropy, we derive first and second laws of thermodynamics.

2.1.1 Driving applied through waveguide

In this case, we consider the waveguide as offering a work source, with photons leaking from the cavity into the waveguide being of use. These photons can then be put through a filter to collect photons of the same frequency.

For our time-independent Hamiltonian, we write 4 free terms and 4 coupling terms, with the zeroth mode of the waveguide being in a coherent state and the rest thermal:

$$\hat{H}_{\text{total}} = \hat{H}_{\text{QD}} + H_{\text{C}} + H_{\text{B}} + \hat{H}_{\text{w}} + \hat{V}_{\text{QD,C}} + \hat{V}_{\text{QD,B}} + \hat{V}_{\text{C,w}} + \hat{V}_{\text{Q,w}} \tag{19}$$

For a closed system with conservation of total energy under unitary evolution, we can compare energies at 2 different times and group the terms of the Hamiltonian into thermodynamics quantities. Setting for brevity $\Delta\rho(t) = \rho(t) - \rho(0)$, we have

$$\begin{aligned}\text{Tr}\left[\Delta\rho(t)\hat{H}_{\text{total}}\right] &= \text{Tr}\left[\Delta\rho(t)\left(\hat{H}_{\text{QD}} + \hat{H}_{\text{C}} + \sum \hat{V}_i\right)\right] + \text{Tr}\left[\Delta\rho(t)\nu_0\hat{c}_0^\dagger\hat{c}_0\right] + \text{Tr}\left[\Delta\rho(t)\hat{H}_{\text{w}\backslash 0}\right] + \text{Tr}\left[\Delta\rho(t)\hat{H}_{\text{B}}\right] \\ &= \Delta E(t) + (W(t) - Q_{\text{w}}(t)) - Q_{\text{B}}(t) = 0 \\ &\implies \Delta E(t) = \Delta E_{\text{w}}(t) + Q_{\text{B}}(t)\end{aligned} \tag{20}$$

for $\hat{H}_{\text{w}\backslash 0} = \hat{H}_{\text{w}} - \nu_0\hat{c}_0^\dagger\hat{c}_0$. We define the system as the quantum dot, the cavity, and interaction. The thermodynamic interpretation of these terms is as follows: the variation of the quantum dot, cavity and interaction terms is the change in *system* energy $\Delta E(t)$; the total variation of energy in the waveguide is $\Delta_{\text{w}}E(t)$; the variation of energy in the thermal bath is the (negative) heat transferred to the system. In particular, the total change in energy of the waveguide can be split. The term with the mode ν_0 in the waveguide is a coherent state work source. The rest of the terms can be considered a heat bath. This is what we are interested in because photons exit the cavity through the waveguide.

Start with $\rho(0) = \rho_{\text{X}}(0) \otimes \rho_{\text{B}}(0) \otimes \left(|\alpha\rangle\langle\alpha| \otimes \rho_{\text{w}\backslash 0\text{th mode}}(0)\right)$ factorised. The zeroth mode of the waveguide is a coherent state. The bath and other waveguides modes are in thermal states. The Von Neumann entropy is given by

$$S_{\text{X}}(t) = -\,\text{Tr}\left[\rho_{\text{X}}(t)\ln\rho_{\text{X}}(t)\right]. \tag{21}$$

Then enforcing conservation of total entropy, [10] we find (see appendix (47))

$$\begin{aligned}
\Delta S(t) &= -\operatorname{Tr}\left[\rho_{\mathrm{X}}(t)\ln\rho_{\mathrm{X}}(t)\right] + \operatorname{Tr}\left[\rho_{\mathrm{X}}(0)\ln\rho_{\mathrm{X}}(0)\right] \\
&= -\operatorname{Tr}\left[\rho(t)\ln\left(\rho_X(t)\bigotimes_{\text{Baths}}\rho_{\mathrm{B_i}}(0)\right)\right] + \operatorname{Tr}\left[\rho(t)\ln\rho(t)\right] + \sum_{\mathrm{B,w}\backslash 0}\left(\operatorname{Tr}\left[(\rho_{\mathrm{B_i}}(t) - \rho_{\mathrm{B_i}}(0)\ln\rho_{\mathrm{B_i}}(0)\right]\right) \\
&= -\operatorname{Tr}\left[\rho(t)\ln\left(\rho_X(t)\bigotimes_{\mathrm{B,w}\backslash 0}\rho_{\mathrm{B_i}}(0)\right)\right] + \operatorname{Tr}\left[\rho(t)\ln\rho(t)\right] + \beta_{\mathrm{w}}Q_{\mathrm{w}}(t) + \beta_{\mathrm{B}}Q_{\mathrm{B}}(t).
\end{aligned} \tag{22}$$

The first two terms in the last line are a *relative entropy* which is always positive. Subtracting the phonon bath and waveguide terms, we derive a second law of thermodynamics

$$\Delta S(t) - \beta_{\mathrm{w}}Q_{\mathrm{w}}(t) - \beta_{\mathrm{B}}Q_{\mathrm{B}}(t) \geq 0. \tag{23}$$

2.1.2 External driving

In this scenario we have that all modes of the waveguide are now thermal. There's an external drive now acting on the system which contributes work, and so total energy is no longer conserved. Our Hamiltonian is the same as (19) except we add a $\hat{H}_{\mathrm{QD}}^{\mathrm{drive}}(t)$ term to the system Hamiltonian. Instead,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E(t) = \frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Tr}\left[\hat{H}_{\mathrm{S}}(t)\rho(t)\right] &= \frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Tr}\left[\rho(t)\left(\hat{H}_{\mathrm{QD}} + \hat{H}_{\mathrm{C}} + \sum\hat{V}_i + \hat{H}_{\mathrm{QD}}^{\mathrm{drive}}(t)\right)\right] \\
&= \operatorname{Tr}\left[\rho(t)\frac{\mathrm{d}}{\mathrm{d}t}\hat{H}_{\mathrm{QD}}^{\mathrm{drive}}(t)\right] + \operatorname{Tr}\left[\hat{H}_{\mathrm{S}}(t)\frac{\mathrm{d}}{\mathrm{d}t}\rho(t)\right].
\end{aligned} \tag{24}$$

2.2 Starting conditions and tracing: A framework for analysing generic Hamiltonians

We assume that the (tilted) density matrix begins in a factored form. When tracing out everything, we can take the product of the traces of the subspaces.

We will be choosing between thermal states and coherent states as our starting conditions for the system. Tracing over a thermal bath or a coherent state will yield modified expression compared to literature due to the presence of these counting fields. We seek to follow the propagation of the counting fields in our calculations.

In general, when the qubit/system operators acts on the starting state, it does not become traceless. A ladder operator acting on a coherent state also lets the trace survive. Therefore, we have that a Qubit $\otimes$ Coherent State term survives at *first order* and depends on α, i.e. the qubit coupling to the coherent mode of the waveguide.

2.2.1 Linear terms in expansion

The qubit Hamiltonian will be some combination of $\hat{\sigma}_z$ and $\hat{\sigma}_x$ operators, whereas coupling terms are typically either $\hat{\sigma}_z$ or $\hat{\sigma}_x$. Naturally, the trace is invariant under time-evolving $\hat{\sigma}_{x/z}$ by $t + \lambda/2$

$$\operatorname{Tr}\left[e^{i\hat{H}_{\mathrm{QD}}(t+\frac{\lambda}{2})}\hat{O}e^{-i\hat{H}_{\mathrm{QD}}(t+\frac{\lambda}{2})}\frac{e^{-\beta\hat{H}_{\mathrm{QD}}}}{Z_{\mathrm{QD}}}\right] = \operatorname{Tr}\left[\hat{O}\frac{e^{-\beta\hat{H}_{\mathrm{QD}}}}{Z_{\mathrm{QD}}}\right] \tag{25}$$

by cycling the propagators since the thermal state is time independent. Tracing out a coherent state acted on by a linear combination of ladder operators does give a t and λ dependence:

$$\operatorname{Tr}\left[\left(\hat{c}(t,\lambda) + \hat{c}^{\dagger}(t,\lambda)\right)\rho_{\mathrm{coh}}\right] = e^{-i\nu(t+\frac{\lambda}{2})}\alpha + e^{i\nu(t+\frac{\lambda}{2})}\alpha^{*}. \tag{26}$$

All the other cases considered are traceless. Only the coherent state carries some sort of dependence on λ up to linear order.

2.2.2 Quadratic terms in expansion

When tracing over the quadratic terms in the Dyson series, the different types of coupling terms (along with the type of starting condition) will pair off, such that each type of coupling may be treated separately.

$$
\begin{aligned}
\mathcal{O}(H^2\rho) \to \mathrm{Tr}\Bigg[&\Big(\boxed{\text{Qubit} \otimes \text{Thermal}} + \boxed{\text{Qubit} \otimes \text{Coherent}} + \boxed{\text{Coherent} \otimes \text{Thermal}} + \cdots\Big) \\
&\times\ \rho(0)\ \times \Big(\boxed{\text{Qubit} \otimes \text{Thermal}} + \boxed{\text{Qubit} \otimes \text{Coherent}} + \boxed{\text{Coherent} \otimes \text{Thermal}} + \cdots\Big)\Bigg]
\end{aligned}
$$

$$
= \mathrm{Tr}\left[\left(\boxed{\text{Qubit} \otimes \text{Thermal}}^{\,2} + \boxed{\text{Qubit} \otimes \text{Coherent}}^{\,2} + \boxed{\text{Coherent} \otimes \text{Thermal}}^{\,2} + \cdots\right)\rho(0)\right]
$$

The utility of this observation is that one can analyse each class of couplings independently in the Dyson series expansion of the tilted density matrix, and so the results can readily be applied to different microscopic systems.

This relies on how the operators transform into the interaction picture when also dressed with counting fields. For a ladder operator $\hat{a}$ which has a free term $\omega\hat{a}^\dagger\hat{a}$, this transforms as

$$\hat{a}(t,\lambda) = e^{-i\omega(t+\lambda/2)}\hat{a}. \tag{27}$$

2.2.3 Tracing out a thermal bath

Since the thermal states of the form $\omega\hat{a}^\dagger\hat{a}$ is diagonal in the Fock basis, one application of a ladder operator will destroy the trace. Applying an odd number of ladder operators will destroy the trace too. To keep the density matrix diagonal, one applies a balanced number of creation and annihilation operators. Tracing out a quadratic term with a thermal state requires the combination $\hat{a}_k\hat{a}_k^\dagger$ or $\hat{a}_k^\dagger\hat{a}_k$:

$$
\begin{aligned}
&\mathrm{Tr}\left[\left(\hat{a}_k(t_1,\lambda) + \hat{a}_k^\dagger(t_1,\lambda)\right)\left(\hat{a}_k(t_2,\pm\lambda) + \hat{a}_k^\dagger(t_2,\pm\lambda)\right)\rho_{\mathrm{th}}\right] \qquad (28)\\
&= \mathrm{Tr}\left[\left(\hat{a}_k(t_1,\lambda)\hat{a}_k^\dagger(t_2,\pm\lambda) + \hat{a}_k^\dagger(t_1,\lambda)\hat{a}_k(t_2,\pm\lambda)\right)\rho_{\mathrm{th}}\right]
\end{aligned}
$$

Given the form of (27), if there is no relative sign difference between the counting field, then the counting field dependencies in the phases cancel. Thus, only the term $\hat{H}(t_1,\lambda)\rho\hat{H}(t_2,-\lambda)$ keeps a λ dependence.

The bath correlation function in terms of a trace is equivalent to a Fourier transform of the spectral density [11]

$$C_B(t) = \sum_k |g_k|^2\,\mathrm{Tr}\left[\left(\hat{a}_k^\dagger\hat{a}_k e^{i\omega_k t} + \hat{a}_k\hat{a}_k^\dagger e^{-i\omega_k t}\right)\rho_{\mathrm{th}}\right] = \int_{-\infty}^{\infty} \mathrm{d}\omega\,\frac{J(w)}{\pi}\frac{e^{-i\omega t}}{1-e^{-\beta\omega}}. \tag{29}$$

The trace of a quadratic term containing bath couplings can be expressed as

$$
\begin{aligned}
&\sum_{m,n} g_m g_n\,\mathrm{Tr}\left[\left(\hat{a}_m(t_1,\lambda) + \hat{a}_m^\dagger(t_1,\lambda)\right)\left(\hat{a}_n(t_2,\pm\lambda) + \hat{a}_n^\dagger(t_2,\pm\lambda)\right)\rho_{\mathrm{th}}\right] \qquad (30)\\
&= \sum_k |g_k|^2\,\mathrm{Tr}\left[\left(\hat{a}_k^\dagger\hat{a}_k e^{i\omega_k(t_1-t_2+\frac{\lambda}{2}\mp\frac{\lambda}{2})} + \hat{a}_k\hat{a}_k^\dagger e^{-i\omega_k(t_1-t_2+\frac{\lambda}{2}\mp\frac{\lambda}{2}))}\right)\rho_{\mathrm{th}}\right] = C_B\left(t_1 - t_2 + \frac{\lambda}{2} \mp \frac{\lambda}{2}\right) \qquad (31)
\end{aligned}
$$

where we applied $\delta_{m,n}$ to ignore terms linear in a given mode $\hat{a}_m$. The appropriate spectral function is

$$J(\omega) = \alpha\omega^3 e^{-\omega^2/2\omega_{\text{cutoff}}^2} \tag{32}$$

and we will use this for the remainder of the project. α is the coupling strength and ω_{cutoff} is a cutoff frequency that depends on the size of the qubit [12].

If instead one performs the same calculation on the single mode $\hat{H}_C = \omega\hat{a}^\dagger\hat{a}$, one gets

$$\frac{1}{e^{\beta\omega}-1} \cdot 2\cos\left(t_2 - t_1 - \frac{\lambda}{2} \pm \frac{\lambda}{2}\right) + e^{-i\omega(t_2 - t_1 - \frac{\lambda}{2} \pm \frac{\lambda}{2})} \tag{33}$$

The introduction of these bath correlation functions greatly complicates the analytical expressions, preventing a closed-form solution to the time integration. In our model, a thermal bath appears coupled to either the qubit, or the cavity.

2.2.4 Tracing out a coherent state

For now, we consider just the truncated Dyson series in which case we only consider the coherent state at the starting time. Tracing out a quadratic coherent state term gives

$$\begin{aligned} &\operatorname{Tr}\left[\left(\hat{c}(t_1,\lambda) + \hat{c}^\dagger(t_1,\lambda)\right)\left(\hat{c}(t_2,\pm\lambda) + \hat{c}^\dagger(t_2,\pm\lambda)\right)\rho_{\text{coh}}\right] \\ &= \alpha^2 e^{-i(t_1+t_2+\frac{\lambda}{2}\pm\frac{\lambda}{2})} + |\alpha|^2\left(e^{-i(t_1-t_2+\frac{\lambda}{2}\mp\frac{\lambda}{2})}\right) + \text{c.c.} \end{aligned} \tag{34}$$

Instead, if we had already applied a displacement transformation $\mathcal{D}[\alpha]$ before moving to the interaction picture, then the coherent state turns into $|0\rangle\langle 0|$ and we have:

$$\begin{aligned} &\operatorname{Tr}\left[\left(\hat{c}(t_1,\lambda) + \hat{c}^\dagger(t_1,\lambda) + e^{-i\frac{\lambda}{2}}\alpha + e^{i\frac{\lambda}{2}}\alpha^*\right)\left(\hat{c}(t_2,\pm\lambda) + \hat{c}^\dagger(t_2,\pm\lambda) + e^{\mp i\frac{\lambda}{2}}\alpha + e^{\pm i\frac{\lambda}{2}}\alpha^*\right)\rho_{\text{coh}}\right] \\ &= e^{-i(t_1-t_2+\frac{\lambda}{2}\mp\frac{\lambda}{2})} + \left(e^{-i\frac{\lambda}{2}}\alpha + e^{i\frac{\lambda}{2}}\alpha^*\right)\left(e^{-i(\pm\frac{\lambda}{2})}\alpha + e^{i(\mp\frac{\lambda}{2})}\alpha^*\right) \\ &= e^{-i(t_1-t_2+\frac{\lambda}{2}\mp\frac{\lambda}{2})} + \left[\alpha^2 e^{-i(\frac{\lambda}{2}\pm\frac{\lambda}{2})} + |\alpha|^2 e^{-i(\frac{\lambda}{2}\mp\frac{\lambda}{2})} + \text{c.c.}\right] \end{aligned} \tag{35}$$

There is an extra exponential term now. The displacement operator also transforms the free term as

$$\hat{c}^\dagger\hat{c} \to \hat{c}^\dagger\hat{c} + \alpha(t)\hat{c}^\dagger + \alpha^*(t)\hat{c} + \text{const.}$$

2.2.5 Microscopic model

Now we can turn our attention to the system at hand, namely the laser being injected into the waveguide. The qubit, bath, cavity and most of the waveguide begin in a thermal state. The zeroth mode of the waveguide is in a coherent state. Tracing out everything to find $\mathcal{G}$ and solving as much as we can, the coupling terms include

- **qubit** $\otimes$ **bath** which gives integrals over a correlation function,
- **qubit** $\otimes$ **waveguide** which gives integrals over a correlation function,
- **cavity** $\otimes$ **waveguide** which gives integrals over a correlation function,
- **qubit** $\otimes$ **cavity** which can be solved exactly,
- **qubit** $\otimes$ **waveguide 0th mode** which gives (ordered) time integrals over $\alpha(t)$.

With each of these interactions in principle having different coupling strengths, to observe symmetries of $\mathcal{G}(t,\vec{\lambda})$ overall we would need to see symmetries for each of these individual terms.

2.3 Comparison with master equations

To speculate on the potential reach of the perturbative expansion, it would be useful to compare the dynamics with a common master equation. Since the interactions we are interested in contribute independently in section 2.2.2, it suffices to consider only the case of a quantum dot coupled weakly to the phonon bath.

The derivation of the Redfield master equation follows a similar form to what we have considered already. In [9], the weak coupling master equation is given by

$$\begin{aligned} \frac{\partial}{\partial t}\rho_Q(t) = \int_0^t \mathrm{d}s \;\; \mathrm{Tr}\Big[& C_B(t-s)\left(\hat{A}(s)\rho_Q(t)\hat{A}(t) - \hat{A}(t)\hat{A}(s)\rho_Q(t)\right) \\ & + C_B(s-t)\left(\hat{A}(t)\rho_Q(s)\hat{A}(s) - \rho_Q(t)\hat{A}(s)\hat{A}(t)\right)\Big] \end{aligned} \tag{36}$$

for a system-bath coupling of the form $\hat{A} \otimes \hat{B}$. This is a linear ordinary differential equation in $\rho_Q(t)$. The similarities with our approach are that we assume a factored starting state $\rho(0) = \rho_Q(0) \otimes \rho_B$, with the bath state unchanging. Another central assumption is that $\rho_Q(t) \approx \rho_Q(s)$, making the equation local in time.

By expressing the action of matrix multiplication of $\hat{A}$ on the left and on the right of ρ_Q as superoperators, we can incrementally evolve ρ_Q. Since C_B itself requires an integration, we conveniently precompute the results in a lookup table, so that during the time integration we may interpolate the correlation function.

3 Results

We have analytically solved the terms contributing to the tilted generating function for different coupling terms and probed for some of the exact symmetries that arose from the idealised case, namely the reverse dynamics symmetry and strict energy conservation condition.

3.1 Analytic expressions for contributions to $\mathcal{G}$

We now report the expressions obtained from the second order truncated Dyson series which contribute to the generating function. After writing the quadratic terms in the series expansion, tracing over everything, and integrating out the times, we are left with these integrals for each of the contributions to the microscopic model. For brevity, we consider the *unbiased regime* $\varepsilon = 0$. Also set $\Delta = 1$, which only amounts to rescaling the time and $\vec{\lambda}$. Then $\hat{H}_{\mathrm{Q}} = \hat{\sigma}_x$. Finally, set $\beta = 1$.

Qubit $\otimes$ Phonon Bath

In the two time-ordered terms, taking the trace over $\hat{\sigma}_z(t,\vec{\lambda})\hat{\sigma}_z(s,\vec{\lambda})$ gives no λ_Q dependence and we need not go further. The mixed term does have a λ_Q and λ_B dependence. Integrating out the times, we find a complicated integral over ω:

$$\int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{e^{\omega-\omega^2-2it-i\omega t+i\omega\lambda_B-2i\lambda_Q}\omega^3}{(1+e^2)(e^{\omega}-1)(\omega^2-4)} \left(-e^{2+4i\lambda_Q}(e^{i(2+\omega)t}-1)^2(\omega-2)^2 - \left(e^{2it}-e^{i\omega t}\right)^2(2+\omega^2)\right). \tag{37}$$

Qubit $\otimes$ Thermal Cavity Mode

Again, the time-ordered terms keep no counting field dependence when traced out. Looking at the mixed term, we can readily integrate over the times and get

$$-\frac{1}{(1+e^2)(e^{\omega_C}-1)} \cdot \left[\frac{1}{4}e^{-2i(t+\lambda_Q)}(e^{2it}-1)^2(1+e^{2+4i\lambda_Q}) \right. \tag{38}$$

$$\left. + e^{\omega_C-2it-i\omega t+i\omega\lambda_B-2i\lambda_Q}\left(\left(\frac{e^{2it}-e^{i\omega t}}{\omega-2}\right)^2 + \left(e^{2+41\lambda_Q}\frac{e^{2it+i\omega t}}{\omega+2}\right)^2\right)\right] \tag{39}$$

From the form of these terms in $\mathcal{G}(t,\vec{\lambda})$, we see that the λ's appear in separate, independent phases. This means that there is no cancellation when $\vec{\lambda} = (\lambda,\lambda,\lambda,\lambda)$.

Other interactions

The Qubit $\otimes$ Waveguide mode is the same as the interaction with the phonon bath; both are initialised in a thermal state. The only difference would be the choice of parameters, which does not affect the analysis.

3.2 Symmetries

By numerically solving the tilted dynamics for the Qubit $\otimes$ Phonon Bath system starting in a thermal state (A.3), we have shown on the basis of numerical experiments that the time-reversal symmetry $\mathcal{G}(t,\vec{\lambda}) = \mathcal{G}_R(t,-\vec{\lambda}+i\vec{\beta})$ holds.

To verify if $\rho(t,\vec{\lambda}) = \rho(t,\vec{\lambda}+\chi\vec{1})$, we plot the system energy $\mathrm{Tr}\left[\hat{H}_{\mathrm{Q}}\rho(t,\vec{\lambda})\right]$ of this setup while varying the counting fields. All parameters are set to 1, except the system Hamiltonian which is set to $\hat{H}_{\mathrm{Q}} = \frac{1}{\sqrt{2}}\left(\hat{\sigma}_z + \hat{\sigma}_x\right)$.

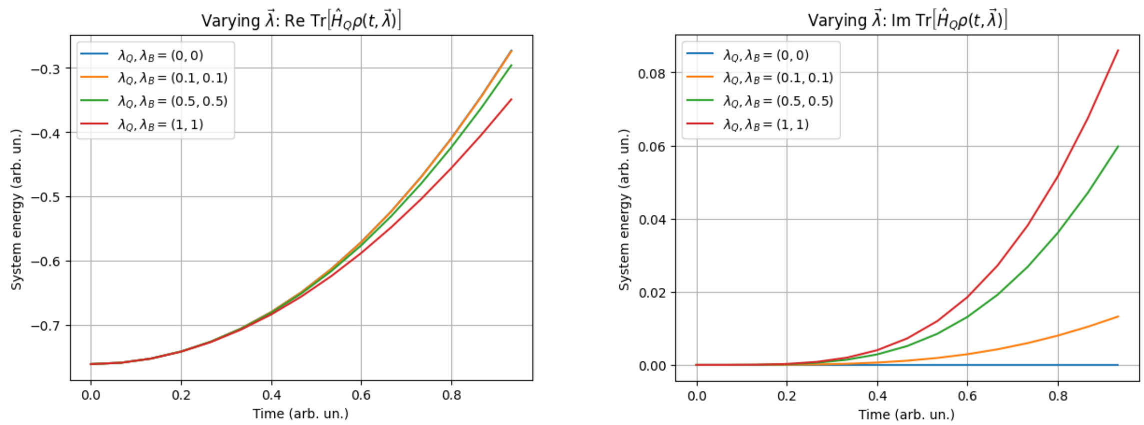


Figure 2: Evolution of tilted density matrix system energy diverges when collectively varying counting fields.

We observe in Fig.(2) that the non-unitarity of the tilted propagators introduces an imaginary component in the energy. Since setting $(\lambda_Q, \lambda_B) = (0, 0)$ returns the regular unitary dynamics with real observables, the potential symmetry $\rho(t, \vec{0}) = \rho(t, \chi\vec{1})$ would need to also return purely real expectation values. Since that is not the case, the symmetry is broken.

3.3 Comparison to master equation

To understand the limitations of this perturbative analysis, we can compare in Fig. (3.3) the behaviour of a Qubit $\otimes$ Phonon Bath setup using our expression for $\rho(t, \vec{\lambda})$ with $\vec{\lambda} = 0$. We can compare the system energy against the evolution according to the weak coupling master equation (36). As before, all parameters are set to 1. The system Hamiltonian is set to $\hat{H}_{\text{QD}} = \frac{1}{\sqrt{2}}(\hat{\sigma}_z + \hat{\sigma}_x)$, and it is coupled to a thermal bath.

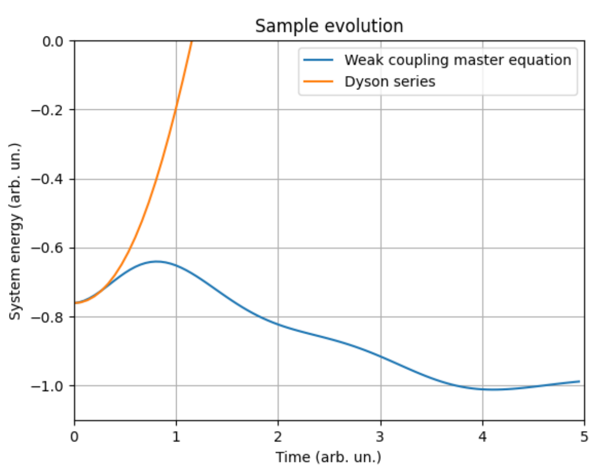


Figure 3: Evolutionary behaviour of weak phonon coupling master equation against the truncated Dyson series

Manifestly, the perturbative expansion only captures the behaviour of the evolution about the onset of evolution, and not towards the steady-state limit. This is the trade-off of discarding the differential equation in favour of expanding about the origin. Therefore, to continue the project and make further analysis with the counting fields, we shall move to a time-local equation. This also contains quadratic terms in the coupling, so the expressions in section 2 will apply.

4 Conclusion

We presented calculations that can be used to analyse a generic microscopic model in the interaction picture. This project considered various classes of coupling terms, and perturbatively expanded the time propagators to evolve from $\rho(0)$ which was factored into thermal or coherent states with respect to the counting field Hamiltonians $\hat{H}_i$.

We investigated the experimental setup consisting of a 2-level quantum dot, a cavity, a waveguide for outgoing photons, and a bath. We derived the Hamiltonian model for this setup and considered both an autonomous and external driving. At the unitary level, we identified terms in the Hamiltonian to thermodynamic quantities like energy, heat, work, and outgoing photon energy. Thus, we identified a first and second law of thermodynamics.

We wrote expressions (one closed, and another in integral form) for the quadratic terms contributing to the generating function $\mathcal{G}$ in the Dyson series expansion. We wrote only the terms with a counting field dependence, as only those matter for investigating symmetries. The explicit dependence on these fields were different, such that the exact energy conservation condition (9) will not hold. For instance, counting fields can appear inside the bath correlation function, or they can appear as phases on the qubit. As such, the functional dependence of these fields means that they do not speak to each other. Indeed, numerical experiments shows that the exact symmetry did not hold. When plotting the system energy from the tilted density matrix, there was a $\chi\hat{1}$ dependence.

To make further progress moving forward, we will switch back to a time-local master equation while carrying over similar calculations. While it is possible to modify the truncated Dyson series to converge in the steady-state limit [8], we would gleam more insight by dressing a more commonly used master equation with the counting fields. In terms of the analytic expressions, we need to handle the bath frequency integral. It will be useful to move into a secular approximation, where it was shown to satisfy the generalised detailed balance condition [7] when the Redfield master equation is modified to preserve the positivity condition on the dynamics. For each combination of interaction terms, we will need to analyse the bath function with time already integrated out to determine how best to approximate the expression in terms of more tractable functions.

A Appendix

A.1 Tracing out a thermal bath

We take the initial state ρ_B of the bath to be in a Gibbs state. This state is diagonal in the Fock basis. The tilted interaction Hamiltonian contains a linear combination of the bath creation and annihilation operators. For any given diagonal matrix element $|n\rangle\langle n|$, both $\hat{b}$ and $\hat{b}^\dagger$ will move this entry to the off-diagonal. Hence, we have that $\mathrm{Tr}[\hat{b}\rho_B] = \mathrm{Tr}[\hat{b}^\dagger \rho_B] = 0$. A term in the nth order expansion of $\mathcal{G}(t, \vec{\lambda})$ will have n creation or annihilation operators applied to it. If n is odd, taking the trace destroys this term. If n is even, the trace will destroy this term also, except in the case where the number of $\hat{b}$ and $\hat{b}^\dagger$ operators is balanced. $C_B(t)$ is the bath spectral function given by

$$C_B(t) = \sum_k |g_k|^2 \,\mathrm{Tr}_B \left[\left(\hat{b}_k^\dagger \hat{b}_k e^{i\omega_k t} + \hat{b}_k \hat{b}_k^\dagger e^{-i\omega_k} \rho_B \right) \right]. \tag{40}$$

We see inside the trace the pairings $\hat{b}^\dagger\hat{b}$ and $\hat{b}\hat{b}^\dagger$ which both preserve diagonal elements. Our integration variables s_i and bath counting field λ_B will enter into the argument of this function. To second order,

$$\begin{aligned}
\mathcal{G}(t,\vec{\lambda}) = \mathrm{Tr}\,\rho_I^{\vec{\lambda}}(0) - \mathrm{Tr}\left[\int_0^t \int_0^{s_1} H_I^{\vec{\lambda}}(s_1) H_I^{\vec{\lambda}}(s_2) \rho_I^{\vec{\lambda}}(0) \mathrm{d}s_1 \mathrm{d}s_2\right] \\
+ \mathrm{Tr}\left[\int_0^t \int_0^{t} H_I^{\vec{\lambda}}(s_1) \rho_I^{\vec{\lambda}}(0) H_I^{-\lambda}(s_2) \mathrm{d}s_1 \mathrm{d}s_2\right] \\
- \mathrm{Tr}\left[\int_0^t \int_0^{s_1} \rho_I^{\vec{\lambda}}(0) H_I^{-\lambda}(s_s) H_I^{-\lambda}(s_1) \mathrm{d}s_1 \mathrm{d}s_2\right].
\end{aligned}$$

The first and third integrals come from the time ordered quadratic terms of the expansion, whereas the middle term is a combination of the linear terms from either side. Now, we can keep only bath terms that are equivalent to the number operator. The middle term will have the bath counting field dependence:

$$\begin{aligned}
\mathrm{Tr}\,\rho_I^{\lambda}(t) = \mathrm{Tr}\,\rho_I^{\lambda}(0) - \mathrm{Tr}\left[\int_0^t \int_0^{s_1} {\sigma_z}_I^{\lambda}(s_1) {\sigma_z}_I^{\lambda}(s_2) \rho_S^{\lambda}(0) C(t-s) \mathrm{d}s_1 \mathrm{d}s_2\right] \\
+ \mathrm{Tr}\left[\int_0^t \int_0^{t} {\sigma_z}_I^{\lambda}(s_1) \rho_S^{\lambda}(0) {\sigma_z}_I^{-\lambda}(s_2) C(s-t-\lambda_B) \mathrm{d}s_1 \mathrm{d}s_2\right] \\
- \mathrm{Tr}\left[\int_0^t \int_0^{s_1} \rho_S^{\lambda}(0) {\sigma_z}_I^{-\lambda}(s_2) {\sigma_z}_I^{-\lambda}(s_1) C(s-t) \mathrm{d}s_1 \mathrm{d}s_2\right].
\end{aligned}$$

In the limiting case of the unbiased regime, the $+\lambda, +\lambda$ and the $-\lambda, -\lambda$ terms lose the λ dependencies entirely and so only the mixed term matters.

A.2 2-point measurement towards and generating function

The joint probability of measuring eigenvalue a_0 at time 0 and eigenvalue a_t at time t is given by

$$P[a_t, a_0] = \mathrm{Tr}\left[P_{a_t} U(t,0) P_{a_0} \rho_0 P_{a_0} U^\dagger(t,0) P_{a_t}\right].$$

To see that they sum to unity: Cycle the final P_{a_t} to the front, then $\sum_a P_a = \hat{1}$. Next, cycle the time evolution operators to be beside each other so they cancel. Then cycle the other projectors next to each other and use the same identity to get $\mathrm{Tr}\,\rho_0 = 1$.

$$p(\Delta a) \equiv \sum \delta(\Delta a - (a_t - a_o))\, P[a_t, a_0] \tag{41}$$

$$\mathcal{G}(\lambda) = \int_{-\infty}^{\infty} \mathrm{d}\Delta a \; e^{i\lambda\Delta a} p(\Delta a) = \int_{-\infty}^{\infty} \mathrm{d}\Delta a \; e^{i\lambda\Delta a} \sum \delta(\Delta a - (a_t - a_o)) \, P[a_t, a_0] = \sum e^{i\lambda(a_t - a_0)} P[a_t, a_0]. \tag{42}$$

If you differentiate with respect to λ, you pull down a factor of $i(a_t - a_0)$. Then setting $\lambda = 0$ recovers the nth moment of $p(\Delta a)$. Inserting the definition for the joint probability gives the expression for the tilted density matrix

$$\begin{aligned}
\mathcal{G}(\lambda) &= \sum e^{i\lambda(a_t - a_0)} \operatorname{Tr}\left[P_{a_t} U(t,0) P_{a_0} \rho_0 P_{a_0} U^\dagger(t,0) P_{a_t}\right] \\
&= \sum e^{i\lambda a_t} \operatorname{Tr}\left[P_{a_t} U(t,0) e^{-\frac{i\lambda \hat{A}(0)}{2}} \bar{\rho}_0 e^{-\frac{i\lambda \hat{A}(0)}{2}} U^\dagger(t,0) P_{a_t}\right] \\
&= \operatorname{Tr}\left[e^{\frac{i\lambda \hat{A}(t)}{2}} U(t,0) e^{-\frac{i\lambda \hat{A}(0)}{2}} \bar{\rho}_0 e^{-\frac{i\lambda \hat{A}(0)}{2}} U^\dagger(t,0) P_{a_t} e^{\frac{i\lambda \hat{A}(t)}{2}}\right] \\
&= \operatorname{Tr}\left[U_\lambda(t,0) \hat{\rho}_0 U^\dagger_{-\lambda}(t,0)\right] \\
&= \operatorname{Tr} \rho^\lambda(t).
\end{aligned} \tag{43}$$

A.3 Time-reversal symmetry

The forward-propagating and time-reversed generating functions defined by

$$\mathcal{G}(t, \vec{\lambda}) = \operatorname{Tr}\left[U^{\vec{\lambda}}(t) \rho_R(0) \left(U^{-\vec{\lambda}}(t)\right)^\dagger\right], \qquad \mathcal{G}_R(t, \vec{\lambda}) = \operatorname{Tr}\left[\left(U^{\vec{\lambda}}(t)\right)^\dagger \rho_R(0) U^{-\vec{\lambda}}(t)\right]. \tag{44}$$

Now consider $\rho(0) = \rho_R(0) = e^{-\vec{\beta}\cdot\vec{H}}/Z$, where $\vec{H}$ is a vector of all the counting fields. Then

$$\begin{aligned}
\mathcal{G}_R(t, -\vec{\lambda}) &= \operatorname{Tr}\left[e^{-\frac{i}{2}\vec{\lambda}\cdot\vec{H}} U^\dagger(t) e^{\frac{i}{2}\vec{\lambda}\cdot\vec{H}} \times \rho_R(0) \times e^{\frac{i}{2}\vec{\lambda}\cdot\vec{H}} U(t) e^{-\frac{i}{2}\vec{\lambda}\cdot\vec{H}}\right] \\
&= \frac{1}{Z} \operatorname{Tr}\left[U(t) e^{-i\vec{\lambda}\cdot\vec{H}} U^\dagger(t) e^{\vec{\lambda}\cdot\vec{H}} \times e^{-\vec{\beta}\cdot\vec{H}}\right].
\end{aligned} \tag{45}$$

This is almost equal to the forward-propagating expression. The substitution $\vec{\lambda} \to \vec{\lambda} - i\vec{\beta}$ recreates a term identical to $\rho(0)$ between the evolution operators $U(t)$ and $U^\dagger(t)$ above, and destroys the rightmost factor. Thus we have the symmetry

$$\mathcal{G}(t, \vec{\lambda}) = \mathcal{G}_R(t, -\vec{\lambda} + i\vec{\beta}). \tag{46}$$

A.4 Second law of thermodynamics

This is the derivation for the second law mentioned in section 2. One mode of the waveguide is in a coherent state and the rest are thermal.

$$
\begin{aligned}
\Delta S(t) &= -\operatorname{Tr}\left[\rho_{\mathrm{X}}(t)\ln\rho_{\mathrm{X}}(t)\right] + \operatorname{Tr}\left[\rho_{\mathrm{X}}(0)\ln\rho_{\mathrm{X}}(0)\right] \\
&= -\operatorname{Tr}\left[\rho(t)\ln\rho_{\mathrm{X}}(t)\right] + \operatorname{Tr}\left[\rho(0)\ln\rho(0)\right] - \sum_{\mathrm{B,w}}\operatorname{Tr}\left[\rho_{\mathrm{B_i}}(0)\ln\rho_{\mathrm{B_i}}(0)\right] \\
&= -\operatorname{Tr}\left[\rho(t)\ln\rho_{\mathrm{X}}(t)\right] + \operatorname{Tr}\left[\rho(t)\ln\rho(t)\right] - \sum_{\mathrm{B,w}\backslash 0}\operatorname{Tr}\left[\rho_{\mathrm{B_i}}(0)\ln\rho_{\mathrm{B_i}}(0)\right] \\
&= -\operatorname{Tr}\left[\rho(t)\ln\rho_{X}(t)\right] + \operatorname{Tr}\left[\rho(t)\ln\rho(t)\right] + \sum_{\mathrm{B,w}\backslash 0}\left(\operatorname{Tr}\left[(\rho_{\mathrm{B_i}}(t)-\rho_{\mathrm{B_i}}(0)\ln\rho_{B}(0)\right] - \operatorname{Tr}\left[\rho_{\mathrm{B_i}}(t)\ln\rho_{\mathrm{B_i}}(0)\right]\right) \\
&= -\operatorname{Tr}\left[\rho(t)\ln\rho_{X}(t)\right] + \operatorname{Tr}\left[\rho(t)\ln\rho(t)\right] + \sum_{\mathrm{B,w}\backslash 0}\left(\operatorname{Tr}\left[(\rho_{\mathrm{B_i}}(t)-\rho_{\mathrm{B_i}}(0)\ln\rho_{\mathrm{B_i}}(0)\right] - \operatorname{Tr}\left[\rho(t)\ln\rho_{\mathrm{B_i}}(0)\right]\right) \\
&= -\operatorname{Tr}\left[\rho(t)\ln\left(\rho_{X}(t)\bigotimes_{\mathrm{B,w}\backslash 0}\rho_{\mathrm{B_i}}(0)\right)\right] + \operatorname{Tr}\left[\rho(t)\ln\rho(t)\right] + \sum_{\mathrm{B,w}\backslash 0}\left(\operatorname{Tr}\left[(\rho_{\mathrm{B_i}}(t)-\rho_{\mathrm{B_i}}(0)\ln\rho_{\mathrm{B_i}}(0)\right]\right) \\
&= -\operatorname{Tr}\left[\rho(t)\ln\left(\rho_{X}(t)\bigotimes_{\mathrm{B,w}\backslash 0}\rho_{\mathrm{B_i}}(0)\right)\right] + \operatorname{Tr}\left[\rho(t)\ln\rho(t)\right] - \beta_{\mathrm{w}}\mathcal{W}_n n(t) + \beta_{\mathrm{B}}Q(t)
\end{aligned}
\tag{47}
$$

In the 3rd line second term, I used conservation of total entropy. I also removed the ν_0 mode from the last term because it starts in a coherent state, and pure states have entropy zero. In the 6th line, I used a property of the logarithm of matrices to combine the first and last term of the previous line into one product inside the logarithm. In the 7th line, the final term is identified as the entropy flow. Using the explicit form of the thermal state $\rho_B(0)$, for the phonon bath and waveguide we have

$$\operatorname{Tr}\left[(\rho_{\mathrm{B}}(t)-\rho_{\mathrm{B}}(0))\cdot -\beta_{\mathrm{B}}\hat{H}_{\mathrm{B}}\right] = \beta_{\mathrm{B}}Q_{\mathrm{B}}(t)$$

$$\operatorname{Tr}\left[(\rho_{\mathrm{w}}(t)-\rho_{\mathrm{w}}(0))\cdot -\beta_{\mathrm{w}}\hat{H}_{\mathrm{w}}\right] = \beta_{\mathrm{w}}E_{\mathrm{w}}(t)$$

The first two terms together give a *relative entropy* which is always positive. We end up with a second law of thermodynamics

$$\Delta S(t) + \beta_{\mathrm{w}}\mathcal{W}_n n(t) - \beta_{\mathrm{B}}Q(t) \geq 0. \tag{48}$$

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