Hypergeometric functions: An exploration of quadratic relations derived from physics

Mikey Whelan
supervised by prof. Ruth Britto

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Abstract

We explore the modern discoveries of quadratic relations derived from scalar Feynman diagrams in terms of hypergeometric functions. We outline how it may be possible to relate these results to intersection theory for twisted homologies and cohomologies. By parameterising the momentum integral, we show how Feynman integrals solve hypergeometric systems. Sunrise diagrams can be expressed as an integral of Bessel functions which leads to quadratic relations. By demonstrating a difference in (co)homology dimensions, we show that the currently known twisted period relations in hypergeometric functions are not reconcilable with the quadratic relations found from physics as of yet.

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1 Introduction

The success of Quantum Electrodynamics in the forties and fifties came in part due to its prediction of the value of the anomalous electron dipole moment. As new experiments were being designed, more Feynman diagrams were being evaluated to fill in coefficients of the Dyson series to verify the power of the theory despite the founders' suspicions around the mathematical validity of the renormalisation technique. After decades of error corrections in computation, theory would correspond to measurement up to fourth-order corrections. In 2017, Laporta computed the fourth-order correction to 1100 decimal places [1] semi-analytically, grouping the results from hundreds of Feynman diagrams into polylogarithms and elliptical integrals with rational coefficients. One such integral that appeared in these calculations was the 2-dimensional equal-mass 4-loop *sunrise diagram*. When solving Feynman integrals, generalised hypergeometric functions often emerge. While solving Feynman integrals for sunrise diagrams, it is possible to factor the integral into modified Bessel functions. This eventually leads non-trivially to the Broadhurst result, a quadratic identity in ${}_4F_3$ hypergeometric functions with all rational coefficients (12). It is difficult to compute Feynman diagrams before introducing numerical methods, but the study of hypergeometric functions may offer insight into solutions of families of Feynman integrals. Conversely, physics may motivate the discovery of new relations between hypergeometric functions by reviving the field of intersection theory which had been producing quadratic identities in hypergeometric functions since the nineties. The starting point for this project will be the scalar multi-loop Feynman integral

$$\mathcal{I} = e^{l\epsilon\gamma_E}(\mu^2)^{\nu - lD/2} \int \prod_{r=1}^{l} \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{n_{\text{int}}} \frac{1}{(-q_j^2 + m_j^2)^{\nu_j}} \tag{1}$$

where μ is a parameter with mass units to make the integral dimensionless, ν is the cumulative sum of all the ν_j exponents in the propagators, q_j are the momenta flowing through internal edges which are a linear combination of the external momenta and independent loop momenta. The dimension D of the integral need not be an integer, expressed as $D = \text{integer} - 2\epsilon$ in dimensional regularisation. We will cast the above momentum-space integral into a parametric integral in the Symanzik graph polynomials. Families of Feynman integrals can be simplified to a linear combination of master integrals and can also be related by dimensional-shift operators. Tensor integrals from quantum field theory can be reduced to scalar integrals, and so we only consider those.

Quadratic identities from Feynman diagrams are rare and not well studied. As far as we are aware, no attempt has yet been made to relate the quadratic identities derived from sunrise diagrams to previous results produced by intersection theory. The Broadhurst result contains 2 pairs of ${}_4F_3$ functions with degeneracies in the entries which are related by integers, whereas the quadratic relations produced by intersection theory without degeneracy are longer. Recently there have been newer results published by Lee and Pomeransky [2] with quadratic identities in integrals of Bessel functions. These have not yet been solved as hypergeometric functions, but they may admit solutions as Lauricella F_C functions which are also known to intersection theory [3]. The dimension of (co)homology plays an important role because it determines the size of the cycle and cocycle intersection matrices and thus the number of hypergeometric functions that contribute to the identities.

The structure of this report will be as follows: first we will introduce some tools in section 2 which will be used in this project to solve or analyse families of integrals derived from quantum field theories. Following Stefan Weinzierl's lecture series [4] we will introduce parametric representations of the scalar Feynman integral including the Feynman and Lee-Pomeransky representations. We will introduce the techniques used to produce contiguous identities between Feynman integrals with different parameters. We will define the hypergeometric functions that will appear later in this report including series and integral representations. Then we will show a proof that Feynman integrals solve $\mathcal{A}$-hypergeometric systems. In section 3, intersection theory for twisted homology and cohomology groups will be introduced for later analysis of Euler-type integral representations. These integrals can be extended to the complex plane and expressed as a pairing of a twisted cycle and cocycle. Due to duality between (co)homology groups, it is possible to define topologically invariant twisted cycle intersection numbers and twisted cocycle intersection numbers. For a basis of (co)homologies, these pairing culminate in quadratic relations. Next we will explore the special cases where quadratic hypergeometric relations were discovered from Feynman diagrams. The independent approaches of Broadhurst and Zhou in section 4 and of Lee and Pomeransky in section 5 are based on the multi-loop 2-point Feynman integral and its uniquely ability to be factoriseded into Bessel functions. In the case of Broadhurst's relations, we will explore a proof of his conjectured relations due to Zhou, highlighting some of the insightful techniques used. In the case of Lee and Pomeransky, we reproduce some of the calculations omitted from their proof and tabulate an expression for the 3-loop 2-point function with general masses. We will then conclude the investigation in section 6 by trying to reconcile the generic quadratic identities from physics to the general quadratic identities found in the literature for ${}_{m+1}F_m$ and F_C hypergeometric functions. We then conclude by offering an outlook on how further progress can be made attempting to understand quadratic hypergeometric identities through intersection theory.

2 Multi-loop Feynman integrals

The tools introduced in this section will be pertinent to the analysis of hypergeometric relations arising from scalar Feynman integrals. Although many quantum field theories have Feynman rules that lead to tensor integrals involving loop momenta in the numerator, these can be reduced to scalar integrals. The most important Feynman rule in scalar theory is the form of the propagator

$$\frac{1}{(-q^2-m^2)^\nu}$$

for an internal edge as these pose the greatest difficulty solving multi-loop diagrams. The exponent ν is kept as a variable. Regularisation schemes were a necessary innovation to be able to handle divergences in Feynman diagrams. Expressed through integral representations involving polynomials raised to exponents, we can solve the diagrams of interest in this investigation exactly in terms of hypergeometric functions. Under a Minkowski metric the scalar product in D-dimensional spacetime is given as $p^2 = (p^0)^2 - (p^1)^2 - \cdots - (p^{D-1})^2$. We will denote Minkowski vectors with lowercase letters and Euclidean vectors after Wick rotation by uppercase letters. We will also switch between using $\mathcal{I}$ as defined before (1) and $\mathcal{J} = \left(e^{l\epsilon\gamma_E}(\mu^2)^{\nu-lD/2}\right)^{-1}\mathcal{I}$ to avoid carrying prefactors which do not play any role.

2.1 Dimensional regularisation

Feynman diagrams often result in infinities when evaluated directly. Dimensional regularisation is one scheme used to handle this whereby a parameter ϵ is introduced into the dimension D of the integral so that the Laurent series may be computed near zero to isolate divergences.

Since the dimension of integration is no longer an integer, we extend the properties of integration to D dimensions. Namely, we enforce:

$$\begin{aligned}
\text{Linearity: } & \int (f(K)+g(K))\, d^DK = \int f(K)\, d^DK + \int g(K)\, d^DK \\
\text{Scaling: } & \int f(\lambda K)\, d^DK = \lambda^{-D}\int f(\lambda K)\, d^DK \\
\text{Translation: } & \int f(K+v)\, d^DK = \int f(K)\, d^DK \quad \text{for all } D\text{-dimensional vectors } v \\
\text{Normalisation: } & \int \frac{d^Dk}{\pi^{\frac{D}{2}}} e^{-K^2} = 1.
\end{aligned}$$

The measure is normalised with the Gaussian function to comply with the integer dimension results. We will take advantage of all of these properties later when parameterising the momentum Feynman integral. The following example illustrates its application in handling an integral which would diverge for $D=4$:

Example 2.1 (Exercise 4 Weinzierl)**.** After Wick rotation, the one-loop master formula integral is

$$\mathcal{I} = e^{\epsilon\gamma_E}(\mu^2)^{\nu-D/2-a}\int \frac{d^DK}{\pi^{D/2}}\frac{(K^2)^a}{(UK^2+V)^\nu}.$$

Using the solid angle formula in arbitrary dimension, we have

$$\int d^Dx f(x^2) = \frac{2\pi^{D/2}}{\Gamma(\frac{D}{2})}\int_0^\infty dx\, x^{D-1}f(x^2) = \frac{\pi^{D/2}}{\Gamma(\frac{D}{2})}\int_0^\infty d(x^2)\,(x^2)^{\frac{D}{2}-1}f(x^2)$$

which relies on the gamma function as an extension of the factorial from positive integers into $\mathbb{C}$. Setting $K^2 = x$, we get

$$\mathcal{I} = e^{\epsilon\gamma_E}(\mu^2)^{\nu-D/2-a}\frac{1}{\Gamma(\frac{D}{2})}\int_0^\infty dx\, x^{\frac{D}{2}-1}\frac{x^a}{(Ux+V)^\nu}.$$

Then after rescaling and factoring out V, this integral is now the form of one of several integral representations of the beta function

$$B(a,b) = \int_0^\infty t^{a-1}(1+t)^{-a-b}dt.$$

The final result is

$$\mathcal{I} = e^{\epsilon\gamma_E}(\mu^2)^{\nu-D/2-a}\frac{\Gamma(\frac{D}{2}-a)\Gamma(\nu-\frac{D}{2}-a)}{\Gamma(\frac{D}{2})\Gamma(\nu)}U^{(-\frac{D}{2}-a)}V^{(-v+\frac{D}{2}+a)}.$$

Setting $a=0$, $U=1$, $V=m^2$, $\nu=1$, then the *tadpole* integral

$$\mathcal{T} = e^{\epsilon\gamma_E}(\frac{m^2}{\mu^2})^{\frac{D}{2}-1}\Gamma(1-\frac{D}{2})$$

is recovered. Clearly this diverges for $D=4$ and so we take the ϵ expansion for $D=4-2\epsilon$ around $\epsilon=0$ given as

$$\frac{m^2}{\mu^2}\left[-\frac{1}{\epsilon}-1+\ln\frac{m^2}{\mu^2}+\left(-\frac{1}{2}\ln^2\frac{m^2}{\mu^2}+\ln\frac{m^2}{\mu^2}+\frac{\pi^2}{6}-1\right)\epsilon\right]+\mathcal{O}(\epsilon^2).$$

There is pole isolated as a single $\frac{1}{\epsilon}$ term which physicists may handle with an appropriate renormalisation scheme, or that may be cancelled out in a broader summation of multiple integrals contributing to a scattering amplitudes.

$\Diamond$

2.2 Symanzik graph polynomials

From a Feynman graph G alone we can derive 2 polynomials, $\mathcal{U}$ and $\mathcal{F}$, which will appear in the parameterised Feynman integral. The *spanning tree* construction is given below as it is the more convenient approach to compute by hand:

A *spanning k-forest* of a graph G is a subgraph $F \in \mathcal{T}^k$ which visits each vertex, contains no loops, and is comprised of k connected components. A spanning tree is a spanning 1-forest. We label each internal edges e_i with an associated a_i. Constructing $\mathcal{U}$ requires finding all spanning trees T_i of G. For each spanning tree T_j, take the product of all the a_i's associated with missing edges. That is, all $e_i \in (G-T_j)$. Summing over all spanning trees, the result can be expressed as

$$\mathcal{U} = \sum_{T_i\in\mathcal{T}^1}\prod_{e_i\notin T_i} a_i.$$

$\mathcal{U}$ is homogeneous of degree l in the a_i parameters which will become known as *Schwinger* parameters. Each term has coefficient 1 also.

Constructing $\mathcal{F}$ requires finding all spanning 2-forests of G, each being a pairing $(T_1,T_2)\in\mathcal{T}^2$ of 2 trees. Each tree T_i has associated to it a sum P_{T_i} of incoming external momenta, and these 2 sums will be multiplied and scaled by μ^2. The expression is given by

$$\mathcal{F} = \sum_{(T_i,T_j)\in\mathcal{T}^2}\left(\prod_{e_k\notin(T_i,T_j)} a_k\right)\frac{P_{T_i}P_{T_j}}{\mu^2}+\mathcal{U}\sum_i a_i\frac{m_i^2}{\mu^2}.$$

For each spanning 2-forest, take a product of all the a_i's associated with the missing edges and then multiply this by the result of multiplying the sums of external momenta entering both trees. Finally add the quantity $\mathcal{U}_a\sum_{i=1}^{n_{\text{int}}} a_i\frac{m_i^2}{\mu^2}$. $\mathcal{F}$ is homogeneous of degree $l+1$ in the Schwinger parameters.

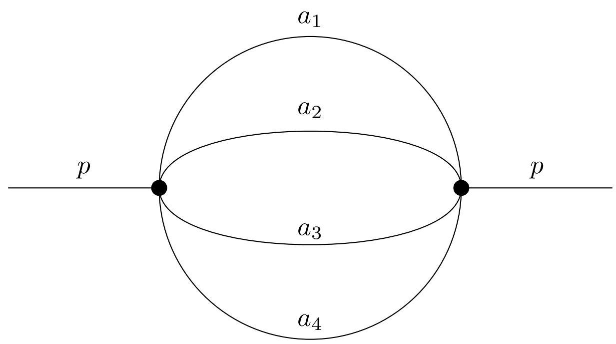

Figure 1: 3-loop 2-point diagram with Schwinger parameters a_i on edges

Example 2.2. The 3-loop 2-point graph in figure 1 has 4 different spanning trees which are obtained by removing all the internal edges except one. These contribute to $\mathcal{U}$ as

$$\mathcal{U} = a_2a_3a_4 + a_1a_3a_4 + a_1a_2a_4 + a_1a_2a_3.$$

The 3-loop 2-point graph has only one spanning 2-forest which is obtained by removing all of the internal edges. The external momenta incident to each tree are p and $-p$ respectively and the second graph polynomial is given by

$$\mathcal{F} = -a_1a_2a_3a_3\frac{p^2}{\mu^2} + \mathcal{U}_a\sum_{i=1}^{4} a_i\frac{m_i^2}{\mu^2}.$$

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In the following section, a different definition for the Symanzik polynomials will arise naturally from the derivation of parametric representations of Feynman integrals. Another example is given in the appendix for the 1-loop 2-point diagram (A) where the polynomials are computed explicitly with these other definitions.

2.3 Parametric representations of Feynman integrals

After Wick rotation and dropping prefactors, the generalised scalar Feynman integral with Euclidean vectors is

$$\mathcal{J} = \int \prod_{r=1}^{l} \frac{d^DK_r}{\pi^{\frac{D}{2}}} \prod_{j=1}^{n_{\text{int}}} \frac{1}{(Q_j^2 + m_j^2)^{\nu_j}}.$$

Schwinger's Trick is the name of the identity

$$\frac{1}{A^\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty da\ a^{\nu-1} e^{-aA}$$

which will be used to promote the momentum representation of the generalised Feynman integral to a parametric one. To this end, we invoke a generalisation of this identity

$$\frac{1}{\prod_i D_i} = \frac{1}{\Gamma(\nu)} \int_{\mathbb{R}_+^{n_{\text{int}}}} da\ a^{\nu-1} e^{-\sum_{i=1}^{n_{\text{int}}} a_i D_i}$$

to include more factors in the denominators. Here the absence of an index on ν and a denotes a multi-index shorthand notation $\Gamma(\nu) = \prod_{i=1}^{n_{\text{int}}} \Gamma(\nu_i)$ and $da\ a^{\nu-1} = \prod_{i=1}^{n_{\text{int}}} da_i\ a_i^{\nu_i-1}$.

2.3.1 Feynman parameter representation

Now performing Schwinger's Trick to the propagators of the general form of the Feynman integral results in

$$\mathcal{J} = \frac{1}{\Gamma(\nu)} \int_{\mathbb{R}_+^{n_{\text{int}}}} da\; a^{\nu-1} \int \prod_{r=1}^{l} \frac{d^D K_r}{\pi^{\frac{D}{2}}} e^{-\sum_{i=1}^{n_{\text{int}}} a_i D_i}.$$

The Schwinger parameter representation contains

$$-\sum_{j=1}^{n_{\text{int}}} a_j (Q_j^2 + m_j^2)$$

in the argument of the exponential which can be expressed as $-(K^\intercal M K + 2\mathcal{Q}^\intercal K + J)$. The terms are sorted by powers of vectors K_i since they are relevant to the loop momentum integration. Note that K and $\mathcal{Q}$ are l-vectors whose entries are themselves D-vectors. J is a scalar and can contain external momenta p_i and masses. M is positive-definite [5]. To proceed we will manipulate the exponent in order to bring it to the form of a Gaussian integral with only a quadratic term.

Upon the substitution $K \to K - M^{-1}\mathcal{Q}$ the integration measure is unchanged and the expression in the exponent becomes

$$-(K^\intercal M K - K^\intercal \mathcal{Q} - \mathcal{Q}^\intercal K - \mathcal{Q}^\intercal M^{-1}\mathcal{Q} + 2\mathcal{Q}^\intercal K + J) = -(K^\intercal M K - \mathcal{Q}^\intercal M^{-1} \mathcal{Q} + J).$$

Factoring out the non-K terms, the integral now is

$$\mathcal{J} = \frac{e^{\mathcal{Q}^\intercal M^{-1}\mathcal{Q} - J}}{\Gamma(\nu)} \int_{\mathbb{R}_+^{n_{\text{int}}}} da\; a^{\nu-1} \int \prod_{r=1}^{l} \frac{d^D K_r}{\pi^{\frac{D}{2}}} e^{-K^\intercal M K}.$$

The remaining exponent can be factorised by diagonalising the positive-definite M to $SRS^\intercal$. S is orthogonal and R has diagonal entries r_i which are the eigenvalues of M. We can perform the next substitution $b = S^\intercal K$ which also leaves the measure unchanged. Since R is diagonal, the argument of the exponent $-b^\intercal R b$ is nothing but $-\sum_i^l b_i^2 r_i$. Thus the exponent factors into Gaussian integrals.

$$\begin{aligned}
\mathcal{J} &= \frac{e^{\mathcal{Q}^\intercal M^{-1}\mathcal{Q} - J}}{\Gamma(\nu)} \int_{\mathbb{R}_+^{n_{\text{int}}}} da\; a^{\nu-1} \int \prod_{r=1}^{l} \frac{d^D b_r}{\pi^{\frac{D}{2}}} e^{-r_1 b_1^2} \cdot e^{-r_2 b_2^2} \cdots e^{-r_l b_l^2} \\
&= \frac{e^{\mathcal{Q}^\intercal M^{-1}\mathcal{Q} - J}}{\Gamma(\nu)} \int_{\mathbb{R}_+^{n_{\text{int}}}} da\; a^{\nu-1} \prod_{r=1}^{l} r_i^{-\frac{D}{2}} \\
&= \frac{e^{\mathcal{Q}^\intercal M^{-1}\mathcal{Q} - J}}{\Gamma(\nu)} \int_{\mathbb{R}_+^{n_{\text{int}}}} da\; a^{\nu-1} \det(M)^{-\frac{D}{2}}
\end{aligned}$$

where the π factors in the measure cancelled the π contributions from the Gaussian integrals due to the normalisation. In the last line we use the property that the determinant of a matrix is the product of its eigenvalues. Then the Symanzik polynomials are defined by $\mathcal{U} = \det(M)$ and $\mathcal{F} = \det(M)(J - \mathcal{Q}^\intercal M^{-1}\mathcal{Q})$ such that we may recombine the mass and momenta data into

$$\mathcal{J} = \frac{1}{\Gamma(\nu)} \int_{\mathbb{R}_+^{n_{\text{int}}}} da\; a^{\nu-1} \mathcal{U}^{-\frac{D}{2}} e^{-\frac{\mathcal{F}}{\mathcal{U}}}.$$

In order to remove the exponent, we will prepare another Schwinger identity. We insert a 1 inside the integral in the form of $1 = \int_0^\infty dt\; t^{-1} \delta(1 - t^{-1} f(a))$ for some function $f(a)$ to be determined.

$$\mathcal{J} = \frac{1}{\Gamma(\nu)} \int_0^\infty dt \int_{\mathbb{R}_+^{n_{\text{int}}}} da\ a^{\nu-1} \frac{\delta\left(1 - \frac{f(a)}{t}\right)}{t} \mathcal{U}^{-\frac{D}{2}} e^{-\frac{\mathcal{F}}{\mathcal{U}}}.$$

Recalling that the homogeneity of $\mathcal{F}$ is 1 degree larger than that of $\mathcal{U}$, the substitution $a_i \to ta_i$ results in a factor of t in the argument of the exponent. With this substitution, we may now fix $f(a)$ to be homogeneous degree 1 in a_i in order to remove the t dependence from the delta function:

$$\mathcal{J} = \frac{1}{\Gamma(\nu)} \int_0^\infty dt \int_{\mathbb{R}_+^{n_{\text{int}}}} da\ a^{\nu-1} t^{\nu} \frac{\delta\left(1 - f(a)\right)}{t} t^{-\frac{Dl}{2}} \mathcal{U}^{-\frac{D}{2}} e^{-t\frac{\mathcal{F}}{\mathcal{U}}}.$$

Then collecting the t's the integral over t can be taken, exchanging the exponential factor with a delta function and a polynomial factor. Including the previously discarded prefactors and choosing $f(a) = \sum_i a_i$, the Feynman parameter representation (without multi-index shorthand) is given by

$$\mathcal{I} = \frac{e^{l\epsilon\gamma_E}\Gamma(\omega)}{\prod_{j=1}^{n_{\text{int}}} \Gamma(\nu_j)} \int_{a_j \geq 0} d^{n_{\text{int}}}a\ \delta\left(1 - \sum_{j=1}^{n_{\text{int}}} a_j\right) \left(\prod_{j=1}^{n_{\text{int}}} a_j^{\nu_j - 1}\right) \frac{\mathcal{U}^{\omega - \frac{D}{2}}}{\mathcal{F}^{\omega}} \tag{2}$$

where $\omega = \sum_i \nu_i - \frac{lD}{2}$ is known as the *superficial degree of divergence*. $f(a)$ can be any hyperplane $\sum_i c_i a_i$ satisfying $a_i \geq 0$ not all equal to zero, and so can be chosen to make integration convenient.

It is possible to reduce tensor integrals derived from quantum field theory of the form

$$\mathcal{J} = \int \prod_{r=1}^{l} \frac{d^D K_r}{\pi^{\frac{D}{2}}} K_{i_1}^{\mu_1} \cdots K_{i_t}^{\mu_t} \prod_{j=1}^{n_{\text{int}}} \frac{1}{(Q_j^2 + m_j^2)^{\nu_j}}$$

to scalar integrals following the same Schwinger parameter procedure. The substitution $K \to K - M^{-1}\mathcal{Q}$ will affect the Schwinger parameter integration as the loop momentum now also appears in the numerator outside the exponent. The loop momenta are instead evaluated as Gaussian integrals of the form

$$\int d^D x\ (x^2)^a e^{-rx^2}.$$

Thus it suffices to study only scalar multi-loop integrals for the purposes of this investigation.

2.3.2 Lee-Pomeransky representation

By rescaling the Euler-type integral representation for the beta function and matching the parameters to the exponents in (2), the identity

$$\left(\frac{\mathcal{F}}{\mathcal{U}}\right)^{-\omega} B(\omega, \frac{D}{2} - \omega) = \int_0^\infty dt\ t^{\omega-1} \left(1 - t\frac{\mathcal{F}}{\mathcal{U}}\right)^{-\frac{D}{2}}$$

allows for the substitution

$$\mathcal{J} = \frac{\Gamma(\frac{D}{2})}{\Gamma(\frac{D}{2} - \omega) \prod_{j=1}^{n_{\text{int}}} \Gamma(\nu_j)} \int_0^\infty dt\ t^{\omega-1} \int_{\mathbb{R}_+^{n_{\text{int}}}} da\ a^{\nu-1} \delta(1 - f(a)) (\mathcal{U} + t\mathcal{F})^{-\frac{D}{2}}.$$

Rescaling $a_i \to u_i/t$ to match the homogeneity of $(\mathcal{U}+t\mathcal{F})$ and collecting t's, the remaining integral over t is nothing but $\int_0^\infty dt\ t^{-1}\delta(1 - t^{-1}f(u)) = 1$ and so we now write the Lee-Pomeransky representation

$$\mathcal{I} = \frac{e^{l\epsilon\gamma_E}\Gamma(\frac{D}{2})}{\Gamma(\frac{D}{2} - \omega) \prod_{j=1}^{n_{\text{int}}} \Gamma(\nu_j)} \int_{u_j \geq 0} d^{n_{\text{int}}} \left(\prod_{j=1}^{n_{\text{int}}} u_j^{\nu_j - 1}\right) \mathcal{G}^{-\frac{D}{2}} \tag{3}$$

which involves the sum $\mathcal{G} = \mathcal{U} + \mathcal{F}$ of both graph polynomials. In the Feynman parameter representation where the parameters are always positive, the delta function with the sum over the parameters restricts the region of integration to a simplex. Therefore the bounds of integration themselves contain Feynman parameters and the computations become more challenging after each evaluation. The LP integral is over the entire positive sector. Several evaluations of Feynman diagrams are in the appendix with both the Feynman and LP representations.

2.4 IBP identities, master integrals and differential equations

In this section we follow along with Weinzierl chapter 6 in constructing contiguous integration-by-parts (IBP) identities in the denominator exponents ν which have a basis of master integrals (MIs). Then one can set up a system of first-order differential equations in the kinematic variables whose solutions will solve the entire family of Feynman integrals with a basis in these MIs.

In the dimensional regularisation regime we still have translational invariance. We also have that the integral of a total derivative vanishes such that

$$e^{l\epsilon\gamma_E}(\mu^2)^{\nu-lD/2}\int\prod_{r=1}^{l}\frac{d^Dk_r}{i\pi^{\frac{D}{2}}}\frac{\partial}{\partial k_i^\mu}\left[q^\mu_{\text{IBP}}\prod_{j=1}^{n_{\text{int}}}\frac{1}{(-q_j^2+m_j^2)^{\nu_j}}\right]=0$$

where q_{IBP} is any choice of the external or loop momenta. After expanding derivatives and absorbing terms into the denominator, the result is still in the form of Feynman integrals with modified parameters. These produce recursion relations in the indices ν which get brought down and incremented by 1 during differentiation. Defining the dimensional shift operator and the raising operator by the action of

$$\mathbb{D}^{\pm}\mathcal{I}(D) = \mathcal{I}(D\pm 2)$$

$$\mathbf{j}^+\mathcal{I}_{v_1\cdots v_j\cdots v_n} = v_j\mathcal{I}_{v_1\cdots(v_j+1)\cdots v_n}$$

respectively, all of these operators commute. These operators allow one to relate integrals whose dimensions differs by 2 via

$$\mathcal{I}_{v_1\cdots v_n}(D-2) = \mathcal{U}(\mathbf{1}^+,\ldots,\mathbf{n}^+)\mathcal{I}_{v_1\cdots v_n}(D)$$

where $\mathcal{U}$ is the Symanzik polynomial whose arguments are themselves raising operators. Next after defining $\mathcal{F}'_{x_j}$ to be the coefficient of x_j in the Symanzik polynomial, we can differentiate with respect to a kinetic variable x_j to find

$$\frac{\partial}{\partial x_j}\mathcal{I}_{\nu_1\cdots\nu_n} = -\mathcal{F}'_{x_j}(\mathbf{1}^+,\ldots,\mathbf{n}^+)\mathbb{D}^+\mathcal{I}_{\nu_1\cdots\nu_n}.$$

Now given some basis $\vec{\mathcal{I}}$ of master integrals $(\mathcal{I}_{\{\nu\}_1},\ldots,\mathcal{I}_{\{\nu\}_N})^T$ with index sets $\{\nu\}_i$, we can differentiate any $\mathcal{I}_{\{\nu\}_i}$ with respect to any kinetic variable x_j and express the result in terms of a linear combination of all the master integrals $\mathcal{I}_{\{\nu\}_k}$. As will be demonstrated by the example below, the coefficients $A_{x_j,i,k}$ are all rational functions in D,ν,x.

Example 2.3. The 1-loop 2-point function with equal masses and arbitrary exponents is given for $x=-\frac{p^2}{m^2}$ by

$$\mathcal{I}_{\nu_1\nu_2}(D,x) = e^{l\epsilon\gamma_E}(m^2)^{\nu-lD/2}\int\frac{d^Dk_r}{i\pi^{\frac{D}{2}}}\frac{1}{(-q_1^2+m^2)^{\nu_1}(-q_2^2+m^2)^{\nu_2}}.$$

Selecting p as q_{IBP} yields the relation

$$0 = (\nu_1-\nu_2)\mathcal{I}_{\nu_1\nu_2} - \nu_1\mathcal{I}_{(\nu_1+1)(\nu_2+1)} - \nu_2\mathcal{I}_{(\nu_1-1)(\nu_2+1)} - \nu_1 x\mathcal{I}_{(\nu_1+1)\nu_2} - \nu_2 x\mathcal{I}_{\nu_1(\nu_2+1)}.$$

Similarly for k,

$$0 = (D - \nu_1 - 2\nu_2)\mathcal{I}_{\nu_1\nu_2} - \nu_1 \mathcal{I}_{(\nu_1+1)(\nu_2-1)} + \nu_1(2+x)\mathcal{I}_{(\nu_1-1)\nu_2} + 2\nu_2 \mathcal{I}_{\nu_1(\nu_2+1)}.$$

This allows any $\mathcal{I}_{\nu_1\nu_2}$ to be reduced down to a linear combination of $\mathcal{I}_{10}$ and $\mathcal{I}_{11}$ which are named the master integrals.

Next we can express this basis of master integrals in terms of another basis with $D+2$ dimensions instead using the dimension shift and raising operators first and then using the previous recursion relations to reduce the indices down to the new set of master integrals. For the 1L2P function we have the first Symanzik polynomial $\mathcal{U} = a_1 + a_2$. Hence in terms of the new basis of master integrals

$$\mathcal{I}_{10}(D,x) = (\mathbf{1}^+ + \mathbf{2}^+)\mathcal{I}_{10}(D+2,x) = \mathcal{I}_{20}(D+2,x) = \left(-\frac{D}{2}\mathcal{I}_{10}(D+2,x) + 0\right),$$
$$\mathcal{I}_{11}(D,x) = (\mathbf{1}^+ + \mathbf{2}^+)\mathcal{I}_{11}(D+2,x) = 2\mathcal{I}_{21}(D+2,x) = \left(-\frac{D}{4-x}\mathcal{I}_{10}(D+2,x) - \frac{2(D-1)}{4+x}\mathcal{I}_{11}(D+2,x)\right).$$

Finally we can set up the first order system of differential equations using the second Symanzik polynomial $\mathcal{F} = a_1 a_2 x + (a_1 + a_2)^2$. We only need the coefficient of our single kinematic variable x. Then our differential equation is

$$\frac{\partial}{\partial x}\mathcal{I}_{\nu_1\nu_2} = -(\mathbf{1}^+)(\mathbf{2}^+)\mathbb{D}^+\mathcal{I}_{\nu_1\nu_2}$$

which gives

$$\frac{\partial}{\partial x}\mathcal{I}_{10} = 0,$$
$$\frac{\partial}{\partial x}\mathcal{I}_{11} = \frac{D-2}{x(4+x)}\mathcal{I}_{10} - \frac{4+(4-D)x}{2x(4+x)}\mathcal{I}_{11}$$

and the coefficients are indeed rational functions. ◊

2.5 Hypergeometric functions

Solutions to these integrals often involve functions such as ${}_pF_q$, Appell or Lauricella hypergeometric function as can be seen by computations in the appendix. Klausen [5] and de la Cruz [6] showed that scalar Feynman integrals are solutions to $\mathcal{A}$-hypergeometric systems. As will be shown in the next section, they solve a certain holomorphic system of homogeneity and toric differential equations derived from D, ν_i and the coefficients of the graph polynomial $\mathcal{G}$. As such, we find that $\mathcal{A}$-hypergeometric functions emerge when evaluating these diagrams. The DLMF [7] lists series definitions, integral representations and differential equations for hypergeometric functions.

The ${}_pF_q$ function is a function of p interchangeable parameters, q more interchangeable parameters and a variable. The series definition is given by

$${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;x) = \sum_{k=0}^{\infty}\frac{(a_1)_k\ldots(a_p)_k}{(b_1)_k\ldots(b_q)_k}\frac{x^k}{k!}$$

where $(a)_k$ is the rising factorial

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

In the study of hypergeometric functions the rising factorial is denoted with the subscript whereas in combinatorics it is preferred with a superscript.

The ${}_2F_1(a,b;c;x)$ function is one that will show up frequently and so we will derive an integral representation of this function as well. The series converges for $|x|<1$. First recall the definition of the beta function which is expressed as an Euler-type integral:

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}(1-t)^{b-1}dt$$

which is used in

$${}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \sum_{k=0}^{\infty} B(a+k,c-a)\frac{(b)_k}{(1)_k}x^k.$$

Expressing the beta function as an integral over t and separating the t^k part, we get

$$\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} \sum_{k=0}^{\infty} t^k \frac{(b)_k}{(1)_k}x^k dt.$$

We can identify the summation as the binomial expansion of $(1-tx)^{-b}$. The final expression, assuming $Re(a), Re(c-a) > 0$, is

$${}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-tx)^{-b}dt. \tag{4}$$

After the application of an Euler transformation

$$t \to \frac{1-t}{1-xt}$$

to the integral, we get

$$\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-tx)^{-b}dt = -(1-z)^{c-a-b}\int_1^0 t^{c-a-1}(1-t)^{a-1}(1-tx)^{b-c}dt.$$

This gives the well-known Euler transformation

$${}_2F_1(a,b;c;x) = (1-x)^{c-a-b}\ {}_2F_1(c-a,c-b;c;x).$$

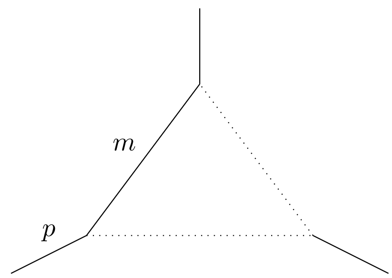
It is also possible to use integration by parts to compute *contiguous relations*, identities in hypergeometric functions where the parameters differ by integers. One such identity was recovered in the appendix by computing the same diagram with both the Feynman and LP representation.

The Lauricella F_C function is another that will emerge in the exploration of quadratic hypergeometric relations. It is one of 4 Lauricella functions F_A, F_B, F_C, F_D. The F_C function with L variables is defined by the series

$$F_C^{(L)}(a,b;c_1,\ldots,c_L;x_1,\ldots,x_L) = \sum_k^{\infty} \frac{(a)_{\sum k_i}(b)_{\sum k_i} x_1^{k_1}\cdots x_L^{k_L}}{(c_1)_{k_1}\cdots(c_L)_{k_L} k_1!\cdots k_L!} \tag{5}$$

where $k = \{k_1,\ldots,k_L\}$ is a multi-index running over all combination of non-negative integers. By setting any of the variables x_i to zero, then k_i can only take the value of zero in the summation or else $x_i^{k_i}$ vanishes. This effectively eliminates one of the summation variables and it also sets $(c_i)_{k_i}$ equal to unity, turning the $F_C^{(L)}$ function into a $F_C^{L-1)}$ function. Some special cases of F_C include $L=1$ which is just the ${}_2F_1$ function and $L=2$ which is called the F_4 Appell hypergeometric function.

Example 2.4 (${}_2F_1$ solution to 3-point function, one external massive leg with adjacent massive propagator)**.** This example demonstrates the evaluation of a Feynman diagram with two different integral representations which recovers an identity for the ${}_2F_1$ function.



Using the spanning tree approach where the legs are labelled a_1, a_2, a_3 clockwise and the propagators similarly, we find for graph polynomials

$$\begin{aligned}
\mathcal{U} &= a_1 + a_2 + a_3, \\
\mathcal{F} &= -a_1 a_2 p_2^2 - a_2 a_3 p_3^2 - a_3 a_1 p_1^2 + \mathcal{U}(a_1 m_1^2 + a_2 m_2^2 + a_3 m_3^2).
\end{aligned}$$

To proceed to the integration we use the Feynman parameter representation (2)

$$\mathcal{I} = e^{\epsilon\gamma_E}\Gamma(3 - \frac{D}{2}) \int d^3 a \delta(1 - (a_1 + a_2 + a_3)) \frac{\mathcal{U}(a)^{3-D}}{\mathcal{F}(a)^{3-\frac{D}{2}}}.$$

As with any 1-loop function, the delta function reduces $\mathcal{U}$ to unity. Keeping only $p_1^2 \neq 0, m_1^2 \neq 0$ and setting $D = 4 - 2\epsilon$, we find

$$\mathcal{I}(p_1^2; m_1) = e^{\epsilon\gamma_E}\Gamma(1-\epsilon) \int d^2 a \left(-a_3 a_1 p^2 + a_1 m^2\right)^{-1-\epsilon}.$$

Due to the convenient choice of massive legs, the initial integral was only a function of a_1 and a_3. We may factor out a_1 and evaluate the innermost integral

$$\int_0^{1-a_3} da_1 (a_1)^{-1-\epsilon} = \frac{(1-a_3)^{-\epsilon}}{-\epsilon}$$

which allows us to express the final integral in terms of the ${}_2F_1$ function

$$\begin{aligned}
\mathcal{I}(p_1^2; m_1) &= e^{\epsilon\gamma_E} \frac{\Gamma(1-\epsilon)}{-\epsilon} \int_0^1 da (1-a)^{-\epsilon} (m_1^2(1 - \frac{p_1^2}{m_1^2} a)^{-1-\epsilon} \\
&= e^{\epsilon\gamma_E} \frac{\Gamma(1-\epsilon)}{-\epsilon} (m_1^2)^{-1-\epsilon} B(1, 1-\epsilon)\, {}_2F_1(1+\epsilon, 1; 2-\epsilon; \frac{p_1^2}{m_1^2}) \\
&= -e^{\epsilon\gamma_E} \frac{\Gamma(1-\epsilon)}{\epsilon(1-\epsilon)} (m_1^2)^{-1-\epsilon}\, {}_2F_1(1+\epsilon, 1; 2-\epsilon; \frac{p_1^2}{m_1^2}).
\end{aligned}$$

Evaluating the same diagram but this time with the Lee-Pomeransky representation (3):

$$\begin{aligned}
\mathcal{I} &= \frac{e^{\epsilon\gamma_E}\Gamma(\frac{D}{2})}{\Gamma(D-3)} \int_0^\infty \int_0^\infty \int_0^\infty da\, db\, dc\, (a + b + c - acp^2 + a^2 m^2 + abm^2 + acm^2)^{-D/2} \\
&= \frac{e^{\epsilon\gamma_E}\Gamma(\frac{D}{2})}{\Gamma(D-3)} \left(\frac{-1}{1-\frac{D}{2}}\right) \int_0^\infty \int_0^\infty da\, db\, \frac{(a + b + a^2 m^2 + abm^2)^{1-D/2}}{1 - a(p^2 - m^2)} \\
&= \frac{e^{\epsilon\gamma_E}\Gamma(\frac{D}{2})}{\Gamma(D-3)} \left(\frac{1}{1-\frac{D}{2}}\right) \left(\frac{1}{2-\frac{D}{2}}\right) \int_0^\infty da\, \frac{(a + a^2 m^2)^{2-D/2}}{(1 - a(p^2 - m^2))(1 + am^2)}.
\end{aligned}$$

Rescaling $a \to x/m^2$, pulling out the a factor in the numerator and cancelling the second factor in the denominator,

$$\begin{aligned}
\mathcal{I} &= \frac{e^{\epsilon\gamma_E}\Gamma(\frac{D}{2})}{\Gamma(D-3)}\left(\frac{1}{1-\frac{D}{2}}\right)\left(\frac{1}{2-\frac{D}{2}}\right)(m^2)^{-3+D/2}\frac{m^2}{p^2-m^2}B(D-3,3-\frac{D}{2})\ {}_2F_1(1,D-3;\frac{D}{2};\frac{p^2}{p^2-m^2}) \\
&= \frac{e^{\epsilon\gamma_E}\Gamma(3-\frac{D}{2})}{(1-\frac{D}{2})(\frac{D}{2}-2)}(m^2)^{-3+D/2}\frac{m^2}{m^2-p^2}\ {}_2F_1(1,D-3;\frac{D}{2};\frac{p^2}{p^2-m^2}) \\
&= \frac{e^{\epsilon\gamma_E}\Gamma(1+\epsilon)}{\epsilon(1-\epsilon)}(m^2)^{-1-\epsilon}\frac{m^2}{m^2-p^2}\ {}_2F_1(1,1-2\epsilon;2-\epsilon;\frac{p^2}{p^2-m^2}).
\end{aligned}$$

Comparing the ${}_2F_1$ function to the previous, the b parameter and the variable are different. To recover the same ${}_2F_1$ parameters as in the Feynman parameters we must infer the identity

$$ {}_2F_1(a,b;c;z) = (1-z)^{-a}\ {}_2F_1(a,c-b;c;\frac{z}{z-1}) $$

which we saw before via an Euler transformation. In our case,

$$ {}_2F_1(1,3-\frac{D}{2};\frac{D}{2};\frac{p^2}{m^2}) = \frac{m^2}{m^2-p^2}\ {}_2F_1(1,D-3;\frac{D}{2};\frac{p^2}{p^2-m^2}). $$

In the limiting case $\epsilon \to 0$, then ${}_2F_1(1,1;2;z) = -\frac{\ln(1-z)}{z}$ which diverges for $\frac{p^2}{m^2} \geq 1$.

A further example, the single mass 1-loop 2-point function is given (A) and results in a contiguous identity.

$\Diamond$

2.6 Feynman integrals as $\mathcal{A}$-hypergeometric systems

Gelfand-Kapranov-Zelevinsky (GKZ) systems can be used to establish a connection between Feynman integrals and hypergeometric functions. Beginning from the LP representation (3) which relies on the polynomial $\mathcal{G}$, we identify a variable x_j to each monomial in the LP variables u_j to form a generalised polynomial $G(z,x)$. The parameters will instead be denoted by z in GKZ systems.

$$ G(z,x) = \sum_j x_j (z_1)^{a_{1j}}(z_2)^{a_{2j}}\cdots(z_n)^{a_{nj}}. $$

$\mathcal{A}$ is an integer matrix whose zeroth row contains all 1's and the remaining entries of each column encode the exponents of each monomial. Then defining the Euler operator $\theta_j = x_j\frac{\partial}{\partial x_j}$ we have the identities

$$ \sum_j \theta_j G = G, \qquad \sum_j a_{ij}\theta_j G = z_j\frac{\partial}{\partial z_j}G. $$

For any LP integral $\mathcal{I}$ (neglecting prefactors) we have

$$
\begin{aligned}
\sum_j a_{0j}\theta_j \mathcal{I} = \sum_j x_j \frac{\partial}{\partial x_j}\mathcal{I} &= -\frac{D}{2}\int_{z_j\geq 0} d^n z \, \left(\prod_{k=1}^{n} z_k^{\nu_k - 1}\right) G^{-1-D/2} \sum_j x_j z_1^{a_{1j}} z_2^{a_{2j}} \\
&= -\frac{D}{2}\int_{z_j\geq 0} d^n z \, \left(\prod_{k=1}^{n} z_k^{\nu_k - 1}\right) G^{-D/2} \\
&= -\frac{D}{2}\mathcal{I}, \\
\sum_j a_{ij}\theta_j \mathcal{I} &= \int_{z_j\geq 0} d^n z \, \left(\prod_{k=1}^{n} z_k^{\nu_k - 1}\right) z_j \frac{\partial}{\partial z_j} G^{-D/2} \\
&\overset{\text{IBP}}{=} -\nu_i \int_{z_j\geq 0} d^n z \, \left(\prod_{k=1}^{n} z_k^{\nu_k - 1}\right) G^{-D/2} \\
&= -\nu_i \mathcal{I}.
\end{aligned}
$$

These equations are summarised in $(\mathcal{A}\theta + \beta) = 0$ for $\beta_0 = D/2, \beta_i = \nu_i$. This is called the **homogeneity operator**. The next type of operator to consider are the **toric operators**

$$
\Box_\ell \equiv \prod_{\ell_i > 0} \partial_j^{\ell_i} - \prod_{\ell_i < 0} \partial_j^{-\ell_i}, \quad \ell \in \ker_{\mathbb{Z}}(\mathcal{A}).
$$

Satisfying $\Box_\ell f = 0$ is equivalent to satisfying $(\partial^u - \partial^v) f = 0$ for any $u, v \in \mathbb{N}_0^n$ such that $\mathcal{A}u = \mathcal{A}v$. Defining $|u| = \sum_i u_i$ just for this section,

$$
\partial^u \mathcal{I} = \frac{\Gamma(1 - \frac{D}{2})}{\Gamma(1 - \frac{D}{2} - |\mathbf{u}|)} \int_{z_j \geq 0} d^n z \, \left(\prod_{k=1}^{n} z_k^{\nu_k - 1 + \mathbf{a_{ki} u_i}}\right) G^{-D/2 - |\mathbf{u}|}
$$

with the summation over i implied. Highlighted in bold above is the u dependence. From $\mathcal{A}u = \mathcal{A}v$ we deduce $|u| = |v|$ and $a_{ki}u_i = a_{ki}v_i$ such that u can be exchanged for v in the above equation. Hence $(\partial^u - \partial^v)\mathcal{I} = 0$ as required.

Since LP representations of Feynman integrals satisfy the system of toric and homogeneity differential equations, scalar Feynman integrals are said to be $\mathcal{A}$-hypergeometric.

3 Intersection theory for hypergeometric functions

The application of Riemann twisted period relations to the hypergeometric functions relevant to this project is based on the integral representations of these functions. Euler-type integral representations of hypergeometric functions are the starting point and consist of nothing more than a product of polynomials raised to exponent such that the integrand is expressed as

$$u(t) = \prod \ell_i(t)^{\alpha_i}.$$

Each ℓ_i defines a hypersurface $D_i = \{t \in \mathbb{C}^n | \ell_i(t) = 0\}$. Set $D = \bigcup D_i$.

Recalling the integral representation (4) of the ${}_2F_1$ function, we extend this integral to the complex plane. One must be careful in handling branching, separating the integrand into a single-valued part and a multi-valued part:

$$\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-tx)^{-b}dt$$
$$= \int_\Delta \underbrace{t^a(1-t)^{c-a}(1-tx)^-b}_{u(t)} \cdot \underbrace{\frac{dt}{t(1-t)}}_{\varphi} = \int_\Delta u(t) \cdot \varphi.$$

The single-valued part of the integrand gets combined with the form dt. The combination is a single-valued form φ called a twisted cocycle. What remains is the integration cycle and the multi-valued $u(t)$ consisting of polynomials raised to powers. Both Δ and a given branch choice of $u(t)$ together constitute a twisted cycle. They are holomorphic on $M = \mathbb{C}^n - D$. With this notation, the ${}_2F_1$ integral should be considered the pairing of a twisted cycle and cocycle.

The *twist* is defined by $\omega = du/u$ and is single-valued. Cocycles are said to be equivalent if their difference is an exterior derivative of an (n-1)-form since

$$\int_\Delta u(t)(\varphi + d\xi) = \int_\Delta u(t)\varphi + \int_\Delta u(t)d\xi = \int_\Delta u(t)\varphi + \int_{\partial\Delta} u(t)\xi = \int_\Delta u(t)\varphi.$$

Here we used that $\partial\Delta = 0$. The set of equivalence classes of cocycles using the exterior derivative comprised the *de Rham* cohomology group $H^n(M, d)$. We incorporate the twist to promote d to a covariant derivative $\nabla_\omega = d + \omega$. Now the set of equivalence classes using the covariant derivative defines the *twisted* de Rham cohomology group $H^n(M, \nabla_\omega)$.

With the pairing between cycle and cocycle defined, we now wish to define twisted cocycle intersection numbers and twisted cycle intersection numbers towards arriving at quadratic identities in hypergeometric functions.

3.1 Cocycle intersection numbers

$H(M, \Delta_\omega)$ and $H_c(M, \Delta_{-\omega})$ are known to be dual to each other, where the subscript c in the second group denotes that the cocycles are mapped onto other cocycles in the same class but with compact support. The representation φ of $H(M, \Delta_\omega)$ and representation ϕ of $H_c(M, \Delta_{-\omega})$ have intersection number

$$I_{ch}(\varphi, \phi) = \int_M \iota(\varphi) \wedge \phi$$

for the isomorphism $\iota\colon H(M, \Delta_\omega) \to H_c(M, \Delta_\omega)$ which is the inverse of the natural isomorphism of the space of smooth forms with compact support into the space of smooth forms. To compute the intersection numbers we find ι and use residues to solve in terms of a rational expression in α_i. A proof due to Matsumoto [8] where the surfaces are all hyperplanes will be shown for the

1-dimensional analog.

For some selections of hyperplanes $\{\ell_0, \ell_1, \ldots, \ell_k\}$, twisted cocycles of the form

$$\varphi = d\ln\frac{\ell_0}{\ell_1} \wedge d\ln\frac{\ell_1}{\ell_2} \cdot d\ln\frac{\ell_{k-1}}{\ell_k} \tag{6}$$

can span $H^k(M, \Delta_\omega)$. The twist

$$\omega = \sum_j \alpha_j \frac{d\ell_j}{\ell_j}$$

has the residue α_j around the hyperplane L_j defined by ℓ_j.

For a hyperplane L_j in projective space, define a small tubular region V_j (shown in red in Fig 2) surrounding L_j and then a larger tubular region U_j. Then let h_j be a smooth function such that

$$h_j(t) = \begin{cases} 1, & t \in V_j, \\ \in [0,1], & t \in U_j - V_j, \\ 0, & \text{else.} \end{cases} \tag{7}$$

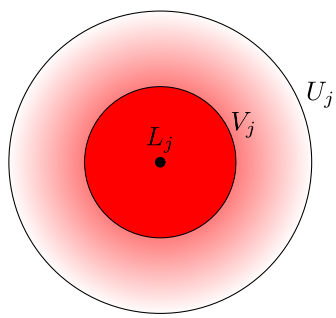


Figure 2: Interpolated value of h_j on tubes surrounding the hyperplane L_j

For any φ_I as constructed above (6), there exists a unique holomorphic function ψ^p such that

$$\nabla_\omega \psi^j = \varphi_I \tag{8}$$

on U_j without L_j. Then

$$\psi^j(L_j) = \frac{\mathrm{Res}_{L_j}(\varphi_I)}{\mathrm{Res}_{L_j}(\omega)}.$$

This can be seen by expanding $\omega = \sum_{n=-1}^{\infty} a_n z^n dz$, $\varphi_I = \sum_{n=-1}^{\infty} b_n z^n dz$ and $\psi^j = \sum_{n=0}^{\infty} c_n z^n$ for a local variable z on U_j. Then we express (8) as

$$\sum_{n=0}^{\infty} \left[n c_n z^{n-1} dz + c_n z^n \sum_{m=-1}^{\infty} a_m z^m dz \right] = \sum_{n=-1}^{\infty} b_n z^n dz.$$

We re-index the right hand size and we use the product of two series to put c_n and a_m under one summation as

$$\sum_{n=0}^{\infty} \left[n c_n dz + \sum_{m=-1}^{n-1} a_m c_{n-m-1} \right] z^{n-1} dz = \sum_{n=0}^{\infty} b_{n-1} z^{n-1} dz$$

$$\Rightarrow n c_n + \sum_{m=-1}^{n-1} a_m c_{n-m-1} = b_{n-1}$$

where we set $n = 0$ to find $c_0 = b_{-1}/a_{-1}$. In other words, at L_p when $z = 0$ we are left with the ratio of the residues of φ_I and ω. The residue of ω is α_j and the residue of φ_I is ± 1 if it has a pole at L_j at all. Next we use h_j to write

$$\nabla_\omega(h_j\psi^j) = \psi^j dh_j + h_j d\psi^j + \omega \wedge h_j\psi^j = \psi^j dh_j + h_j d\psi^j + h_j\omega \wedge \psi^j = \psi^j dh_j + h_j\nabla_\omega(\psi^j).$$

The final term gives by definition $h_j\nabla_\omega(\psi^j) = h_j\varphi_I$. Since $h_j = 1$ and $dh_j = 0$ on V_j, if we subtract these all from φ_I then the result will vanish on all V and we have found our map

$$\iota(\varphi_I) = \varphi_I - \sum_j \nabla_\omega(h_j\psi^j).$$

We may now evaluate

$$I_{ch}(\varphi_I, \varphi_J) = \int \iota(\varphi_I) \wedge \varphi_J = -\sum_j \int_{U_j - V_j} dh_j\psi^j \wedge \varphi_J$$

where dh_j vanishes everywhere except on $U_j - V_j$. Towards Stokes' theorem, we have

$$d(\psi^j h_j\varphi_J) = dh_j\psi^j \wedge \varphi_J + h_j(d\psi^j) \wedge \varphi_J + h_j\psi^j \wedge (d\varphi_J).$$

where only the first term contributes to the integral.

$$I_{ch}(\varphi_I, \varphi_J) = \sum_j \int_{U_j - V_j} d(\psi^j h_j\varphi_J).$$

Using Stokes' theorem to express the contour in terms of the boundaries of U_j and V_j

$$-\sum_j \int_{\partial U_j - \partial V_j} \psi^j h_j\varphi_J = \sum_j \int_{\partial V_j} \psi^j\varphi_J = 2\pi i \sum_j \mathrm{Res}_{L_j}(\psi^j\varphi_J) = 2\pi i \sum_{j \in I\cap J} \frac{1}{\alpha_j}.$$

We are left only with a counter integral of a small circle around each L_j and we get a contribution from the residue of $\psi^j\varphi_I$ where one exists. This generalises to higher dimensions by adding extra residues factors in the denominator and adding another $2\pi i$ factor, however it is convention where or not to include the $2\pi i$ factors in the definition of the cocycle intersection numbers.

Although quadratic relations from Feynman integrals have not been connected with intersection theory yet, there certainly is a connection to Feynman diagrams. Cocycle intersection numbers can be used to project Feynamn integrals onto a certain basis of *maximal cuts* [9].

3.2 Cycle intersection numbers

Recalling the definition of log on the complex plane, $\log = \mathrm{Log}(|z|) + 2\pi i k$. This is used to compute exponents $t^z = e^{z\log(z)}$. By fixing a branch and circling anticlockwise around $t = 0$ back to the starting point, t^z picks up an $e^{2\pi i z}$ factor.

The Pochhammer contour in Fig 3 encircles 2 points, both clockwise and anticlockwise each, to form a closed cycle. The winding number is zero everywhere. The cycle can be deformed so that the 4 lines are on the real axis. Despite overlapping in opposite directions, their contributions to an integral do not cancel due to the extra exponential factors. Consider $t^{\beta_1-1}(1-t)^{\beta_2-1}$ (non-integer parameters) evaluated along this curve starting at the arrow. Encircling the points $t = 0$ and $t = 1$ according to the Pochhammer contour, the total factors picked up by the 4 horizontal lines (in order) are $-e^{2\pi i\beta_2}$, $e^{2\pi i\beta_1+\beta_2}$, $-e^{2\pi i\beta_1}$ and 1. These can be factored

$$(1 - e^{2\pi i\beta_1})(1 - e^{2\pi i\beta_2}).$$

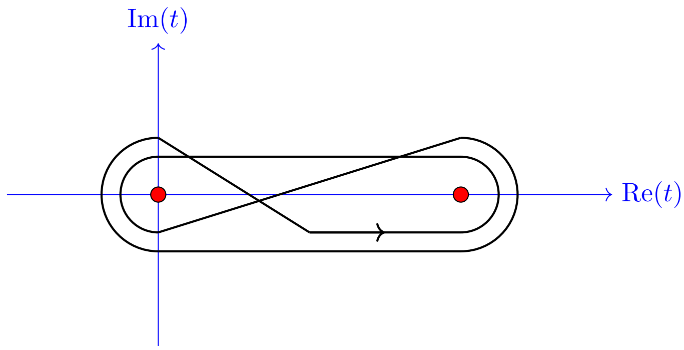


Figure 3: Pochhammer contour γ encircling two punctures in the complex plane

Hence the beta function integral can be analytically continued by relating the Pochhammer contour γ to the open interval $(0,1)$ as

$$(1-e^{2\pi i\beta_1})(1-e^{2\pi i\beta_2})\cdot\int_0^1 t^{\beta_1-1}(1-t)^{\beta_2-1}dt=\int_\gamma t^{\beta_1-1}(1-t)^{\beta_2-1}dt.$$

The same holds true by introducing another function $f(t)$ as a factor in the integrand which is holomorphic on $(0,1)$. Namely, $(1-tx)^c$ recovers the ${}_2F_1$ integral and so it can be analytically continued with the same Pochhammer contour. If instead we scale the contour (i.e the scaling factor is applied to the integrand) by

$$\gamma'=\frac{\gamma}{(1-e^{2\pi i\beta_1})(1-e^{2\pi i\beta_2})}$$

and deform the path into a pair of circles $C_\epsilon(0), C_\epsilon(1)$ with radius ϵ and 4 overlapping horizontal lines like in figure 3, the contour with factors becomes

$$-\frac{C_\epsilon(0)}{1-e^{2\pi i\beta_1}}+[\epsilon,1-\epsilon]+\frac{C_\epsilon(1)}{1-e^{2\pi i\beta_2}}.$$

The *regularisation* of the open interval (p,q) is a twisted cycle constructed from 2 circles with radius ϵ centred at p and q with a path $[p+\epsilon,q-\epsilon]$. Let α_1,α_2 by the exponent parameters and $c_i=\exp(2\pi i\alpha_i)$. The cycle, including factors, is given by

$$\mathrm{Reg}(p,q)=\frac{C_\epsilon(p)}{1-c_1}+[\epsilon,1-\epsilon]-\frac{C_\epsilon(q)}{1-c_2}.$$

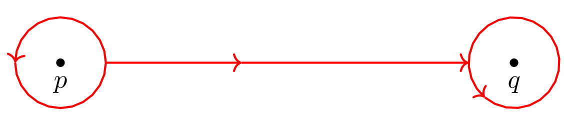


Figure 4: Regularisation of an interval

Note a sign change due to the lack of the -1 in the exponent which was present before in the beta function example. This cycle $\mathrm{Reg}(p,q)$=γ' becomes a *twisted* cycle $\gamma'\otimes u(t)$ when it is loaded with a branch $u(t)$ of the integrand. The factors attached to the circles may also be considered part of the loaded function but since they are independent of $u(t)$ we do not absorb them into $u(t)$. Computing the boundary and picking up a c_i factor when encircling a pole gives

$$\partial(C_\epsilon(p)\otimes u(t))=[p+\epsilon]\otimes u(p+\epsilon)(c_1-1),$$

$$\partial(C_\epsilon(q)\otimes u(t)) = [q-\epsilon]\otimes u(q-\epsilon)(c_2-1),$$
$$\partial([p+\epsilon,q-\epsilon]\otimes u(t)) = [q-\epsilon]\otimes u(q-\epsilon) - [p+\epsilon]\otimes u(p+\epsilon)$$

and taking into account the factors on the circles we have $\partial(\gamma'\otimes u(t)) = 0$. It is indeed a cycle. We may now give an invariant intersection number for twisted cycles. To compute the intersection for 2 intervals, we *regularise* the first and deform the second. For each intersection point, we evaluate the product of both functions evaluated at that point. For it to be an invariant, we must have $u_2(t) = (u_1(t))^{-1}$ so that only the exponential factors attached to the circles contribute to the intersection number.

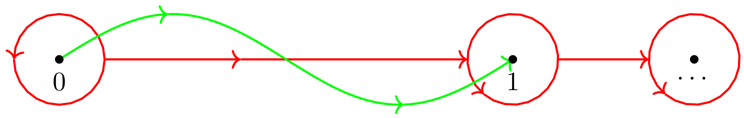


Figure 5: Twisted cocycle intersection with regularisation and deformation

The self-intersection of $(0,1)$ shown in Fig 5 intersects at 3 points, whereas the intersection of $(0,1)$ and $(1,x)$ for $x>1$ intersects at only 1 point. For a sequence of twisted cycles which are the intervals from one pole to the next along the real axis, the cycle intersections are given by

$$I_{jk} = \begin{cases} c_j/(c_j-1), & j=k+1 \\ -\frac{c_jc_{j-1}-1}{(c_j-1)(c_{j+1}-1)}, & j=k \\ 1/c_k, & j+1=k \\ 0, & \text{else.} \end{cases} \tag{9}$$

3.3 Riemann twisted period relations

With the notation of a twisted cycle intersection number defined, we can now write the precise definition for the twisted homology group:

$$H_n(M,\mathcal{L}_\omega^\wedge) = \{\ker\partial : C_n(M,\mathcal{L}_\omega^\wedge)\to C_{n-1}(M,\mathcal{L}_\omega^\wedge)\}/\partial C_{n+1}(M,\mathcal{L}_\omega^\wedge)$$

where $C_n(M,\mathcal{L}_\omega^\wedge)$ is the n-dimensional twisted chain group. The regularisation of an interval is an element of $\ker\partial$ since we computed the boundary of that twisted cycle to be zero due to the twist (the factors associated to the circles which would be absorbed into $u(t)$). Since the hypersurfaces D_i divide $M-D$ into a set of bounded chambers, a convenient choice of these chambers can span the homology group.

Following the notation of Aomoto [10], define $\{\Delta_i^+\}$ and $\{\Delta_i^-\}$ as the basis of the twisted homologies, and define $\{\xi^+\}$ and $\{\eta_i^-\}$ as the basis of the twisted cohomologies. The entries of the period matrices are defined as

$$(\Pi_+)_{ij} = \int_{\Delta_j^+} U\cdot\xi_i, \qquad (\Pi_-)_{ij} = \int_{\delta_j^-} U^{-1}\cdot\eta_i$$

and contain hypergeometric functions with different entries. The integrands for the period matrices are inverses of each other as was prescribed by the condition that the twisted cycle intersection numbers be invariant. This means that the parameters α_i that appear in Π_+ will be the opposite sign in Π_-. The entries of intersection matrices of cycles and cocycles are

$$A_{ij} = \Delta_i^+\cdot\Delta_j^-, \qquad B_{ij} = I_{ch}(\xi_i^+,\eta_j^-).$$

They are full-rank when a basis is chosen that spans the (co)homology groups. Cho [11] uses a Poincaré duality between the homology group and cohomology group to prove the existence of a bilinear relations. This duality also equates the dimension of the homology and cohomology groups. By counting the number of solutions to $\omega = 0$, we can an upper bounder on the dimension of (co)homology [12]. The twisted Riemann period relations [11] are

$$\Pi_+ (A^{-1})^\intercal \Pi_-^\intercal = B \tag{10}$$

$$\Pi_-^\intercal B^{-1} \Pi_+ = A^\intercal . \tag{11}$$

These relations are the source of quadratic relations central to this project. In the following sections, we will explore quadratic relations derived from Feynman diagrams. Then in section (6) we will revisit twisted Riemann period relations and compute some quadratic period relations for single-variable integrals as these are compatible with the definition of intersection numbers developed thus far. Then when more advanced cases arise from physics with multi-variable integrals and degenerate hyperplane arrangements, we will look to quadratic relations computed in the literature.

4 Quadratic relations due to Broadhurst and Zhou

On the basis of numerical experimentation, David Broadhurst conjectured a set of quadratic relations of hypergeometric functions derived from 2-dimensional Feynman diagrams with all equal masses [13]

$$\begin{aligned}7\,{}_4F_3\left(\begin{matrix}1/2,\ 2/3,\ 2/3,\ 5/6\\ 7/6,\ 7/6,\ 4/3\end{matrix}\ \middle|\ 1\right){}_4F_3\left(\begin{matrix}-1/2,\ 1/6,\ 1/3,\ 4/3\\ -1/6,\ 5/6,\ 5/3\end{matrix}\ \middle|\ 1\right)\\ +10\,{}_4F_3\left(\begin{matrix}1/6,\ 1/3,\ 1/3,\ 1/2\\ 2/3,\ 5/6,\ 5/6\end{matrix}\ \middle|\ 1\right){}_4F_3\left(\begin{matrix}-7/6,\ -1/2,\ -1/3,\ 2/3\\ -5/6,\ 1/6,\ 1/3\end{matrix}\ \middle|\ 1\right)=40.\end{aligned} \tag{12}$$

In this section we focus on the integral representation of one family of generic Feynman diagrams, the equal-mass sunrise diagrams shown in figure 6. It is possible to express these integrals as *Bessel moments*, products of modified Bessel functions raised to exponents. Exploiting the differential equations that these products of Bessel functions solve, one can tabulate differential operators to annihilate these integrals up to a rational number [14]. These tools were used to create a recursive formula for computing determinants of matrices containing many variations of sunrise diagrams [15]. While these relations generate quadratic identities in Feynman integrals, the individual integrals must still be solved. As of yet, there is no standardised approach to solving them as hypergeometric functions and one must use a variety of hypergeometric identities and transformations [16].

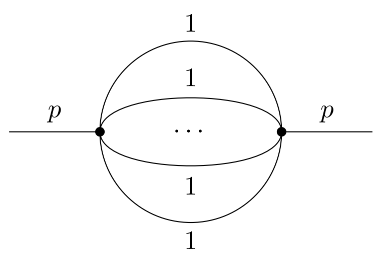


Figure 6: L-loop sunrise diagram with equal masses

4.1 Bessel moment representation

L-loop 2-point functions in 2 dimensions can be expressed as an integral of modified Bessel functions. In the equal mass case, we consider the zeroth-order functions

$$I_0(t)=\frac{1}{\pi}\int_0^\pi e^{t\cos(\theta)}d\theta,\qquad K_0(t)=\int_0^\pi e^{t\cosh(\theta)}d\theta.$$

Recalling example (2.2), we compute the Symanzik polynomials of the L-loop 2-point function using the spanning tree construction.

$$\mathcal{U}=\left(\prod_{i=1}^{L+1}a_i\right)\left(\sum_{i=1}^{L+1}a_i^{-1}\right),\qquad \mathcal{F}=\mathcal{U}\sum_{i=1}^{L+1}a_i-p^2\prod_{i=1}^{L+1}a_i.$$

Due to Vanhove [14], a series expansion reduces the parametric integral to the form

$$\mathcal{I}_L=2^L\int_0^\infty dz\ I_0(\sqrt{p^2}x)K_0^{L+1}(x)x \tag{13}$$

where p is the external momenta and all the masses are equal to unity. The power of the denominators are also equal to unity.

Setting $t = \sqrt{p^2}$, Vanhove found the follow set of polynomial differential operators [14] that annihilate the n-loop integral up to a constant:

$$(t-2)\mathcal{I}_1(t) + t(t-4)D\mathcal{I}_1(t) = 2!$$
$$(t-3)\mathcal{I}_2(t) + (3t^2 - 20t + 9)D\mathcal{I}_2(t) + t(t-1)(t-9)D^2\mathcal{I}_2(t) = -3!$$
$$(t-4)\mathcal{I}_3(t) + (7t^2 - 68t + 64)D\mathcal{I}_3 + (6t^3 - 90t^2 + 192t)D^2\mathcal{I}_3(t) + t^2(t-4)(t-16)D^3\mathcal{I}_3(t) = 4!$$
$$\dots$$

In the case where p equals to unity as well, we get the so-called *Bessel moment*

$$\mathrm{IKM}(a, b, c) = \int_0^\infty [I_0(x)]^a [K_0(x)]^b x^c dx.$$

These Bessel moments were the objects of interest to Broadhurst. The determinant

$$\det \begin{pmatrix} \mathrm{IKM}(1,5,1) & \mathrm{IKM}(1,5,3) \\ \mathrm{IKM}(2,4,1) & \mathrm{IKM}(2,4,3) \end{pmatrix} = \frac{\pi^4}{24^2}$$

is comprised of the Bessel moments associated with the ${}_4F_3$ functions above (12). In particular, these moments arose in Laporta's computation of the 4-loop contributions to the anomalous magnetic dipole moment of the electron. The bottom row didn't appear in the final expression but were used in intermediate steps [17] [1].

This quadratic formula was proven by Yajun Zhou [15] who uses Vanhove operators to factorise Wrońskian matrices containing integrals of the form (13). Differentiating with respect to t and then setting this value equal to unity reduces down to a Bessel moment. As a result, we can recursively solve determinants of Bessel moments $(M_k)_{ab} = \mathrm{IKM}(a, 2k-1+a, 2b-1)$, $(N_k)_{ab} = \mathrm{IKM}(a, 2k-2+a, 2b-1)$ by

$$\det M_{k-1} \det M_k = \frac{2\Gamma^2(k/2)(\det N_{k-1})^2}{2(2k+1)} \prod_{j=1}^{k} \left[\frac{(2j)^2}{(2j)^2-1}\right]^{k-1/2},$$
$$\det N_{k-1} \det N_k = \frac{(2k+1)(\det M_k)^2}{(k+1)(k-1)!} \prod_{j=2}^{k+1} \left[\frac{(2j-1)^2}{(2j-1)^2-1}\right]^{k}.$$

With these recursion relations it suffices to evaluate

$$\det M_1 = \mathrm{IKM}(1,2,1) = \frac{\pi}{\sqrt{3^3}}, \quad \det N_1 = \mathrm{IKM}(1,3,1) = \frac{\pi^2}{2^4}$$

in order to recover Broadhurst' stated conjecture.

4.2 Hilbert transforms of Bessel functions

Even in this most simplified case with all masses equal to unity, these sunrise diagrams as Bessel moments are still difficult to solve. Zhou later made a long series of computations on the 4-loop 2-point function including explicit evaluations of the Bessel moments above [16]. As an intermediate step in evaluating $\mathrm{IKM}(1,5,1)$, for instance, Zhou used Hilbert transforms defined by

$$\mathbb{H}(f(u)) = \frac{1}{\pi}\mathrm{p.v.}\int_\infty^\infty \frac{f(\tau)}{u-\tau} d\tau$$

in order to relate $\mathrm{IKM}(1,5,1)$ to $\mathrm{IKM}(3,3,1)$ before continuing on with further computations. We will explicitly recompute the result to show the use of this approach. Most of the work is done by the following three identities of Hilbert transforms [18]:

$$\int_{-\infty}^{\infty} f(x)\mathbb{H}g(x)dx + \int_{-\infty}^{\infty} g(x)\mathbb{H}f(x)dx = 0, \tag{14}$$

$$\mathbb{H}(f\mathbb{H}g + g\mathbb{H}f) = (\mathbb{H}f)(\mathbb{H}g) - fg, \tag{15}$$

$$\mathbb{H}(u^n f(u)) = \frac{1}{\pi}\text{p.v.}\int_{\infty}^{\infty} \frac{f(\tau)}{u-\tau}d\tau = u^n \frac{1}{\pi}\text{p.v.}\int_{\infty}^{\infty} \frac{f(\tau)}{u-\tau}d\tau - \frac{1}{\pi}\sum_{k=0}^{n-1} u^k \int_{\infty}^{\infty} f(\tau)d\tau. \tag{16}$$

(14) and (15) require that $f \in L^p(\mathbb{R}), g \in L^q(\mathbb{R})$ and $\frac{1}{p}+\frac{1}{q} = 1$ with $L^k(\mathbb{R})$ being the space of k-integrable functions. Then for shorthand Zhou defines

$$\iota = \pi I_0, \quad \iota_+ = \iota e^{-x}\mathbb{I}_{(0,\infty)}, \quad \iota_- = \iota e^{x}\mathbb{I}_{(-\infty,0)}, \qquad \kappa = K_0(|x|), \quad \kappa_+ = \kappa e^{-x}, \quad \kappa_- = \kappa e^{x}$$

such that $\mathbb{H}\iota_+ = -\kappa_+, \quad \mathbb{H}\iota_- = \kappa_-, \quad \mathbb{H}\kappa_+ = \iota_+, \quad \mathbb{H}\kappa_- = -\iota_-$. Here $\mathbb{I}_{(-\infty,0)}$ is the indicator function which equals 1 in the specified range and 0 elsewhere. Filling $(\iota_+, \iota_-\mathbb{I}_{(-L,L)})$ into the second identity (15) and taking the limit as L goes to infinity yields

$$\mathbb{H}\left(\iota_+\mathbb{H}(\iota_-\mathbb{I}_{(-L,L)}) + (\iota_-\mathbb{I}_{(-L,L)})\mathbb{H}(\iota_+)\right) = (\mathbb{H}(\iota_+))(\mathbb{H}(\iota_-\mathbb{I}_{(-L,L)})) - (\iota_+)(\iota_-\mathbb{I}_{(-L,L)})$$

$$\Rightarrow \mathbb{H}\left(\iota_+\kappa_- - \iota_-\kappa_+\right) = -\kappa_+\kappa_- - \iota_+\iota_-$$

$$\Rightarrow \mathbb{H}\left(\iota\kappa\cdot\text{sgn}\right) = -\kappa^2 \quad \text{and} \quad \mathbb{H}(\kappa^2) = \iota\kappa\,\text{sgn}$$

where the last term vanished due to mismatched ranges of indicator functions, and we may apply the operator again on both sides to give negative identity. Filling $(\kappa^2, \iota\kappa\cdot\text{sgn})$ into the first identity (14) immediately gives

$$\pi^2\text{IKM}(2,2,0) = \text{IKM}(0,4,0). \tag{17}$$

Filling $(\kappa^2, \iota\kappa\cdot\text{sgn})$ into the second identity (15) gives

$$\mathbb{H}(\iota^2\kappa^2 - \kappa^4) = -2\iota\kappa^3\cdot\text{sgn}.$$

Next using the third identity (16)

$$\mathbb{H}(t(\iota^2\kappa^2 - \kappa^4)) = t\int_{-\infty}^{\infty}\frac{f(\tau)d\tau}{\pi(t-\tau)} - \frac{1}{\pi}\int_{-\infty}^{\infty}(\iota^2\kappa^2 - \kappa^4)d\tau$$

where the final term vanishes due to the equality of the above IKM moments (17) leaving only

$$\mathbb{H}(t(\iota^2\kappa^2 - \kappa^4)) = t\mathbb{H}(\iota^2\kappa^2 - \kappa^4) = -2t\iota\kappa^3\cdot\text{sgn}.$$

Finally, we fill $(\kappa^2, t(\iota^2\kappa^2 - \kappa^4))$ into the first identity (14) and discover

$$\langle\kappa^2, -2t\iota\kappa^3\cdot\text{sgn}\rangle + \langle\iota\kappa\cdot\text{sgn}, t(\iota^2\kappa^2 - \kappa^4)\rangle = 0$$

$$\Rightarrow -2\int_0^{\infty}\pi I_0(t)K_0^5(t)tdt + \int_0^{\infty}(\pi I_0(t))^3K_0^3(t)tdt - \int_0^{\infty}\pi I_0(t)K_0^5(t)tdt = 0$$

leading to the relation between Bessel moments

$$\text{IKM}(1,5,1) = \frac{\pi^2}{3}\text{IKM}(3,3,1).$$

While this only offers a linear relationship in hypergeometric functions, this approach may be useful in future calculations involving larger determinants of Bessel moments which may be reduced down.

5 Fully general 2-point functions

Building upon the previous representation of the equal mass sunrise diagrams as Bessel Moments, Lee and Pomeransky wrote the L-loop 2-point vacuum diagram again in terms of I and K Bessel functions with parameters in their subscripts and arguments. Even with this added generality, these functions still satisfy similar differential equations. Similar to finding a system of differential equations of master integrals, Lee and Pomeransky find a Pfaffian system $dT = AT$ for the set T of integrals obtained by choosing either the zeroth or first derivative for each Bessel function [2]. They report several generic quadratic identities as well as one identity with a mass parameter, apparently adding an extra degree of generality compared to the Broadhurst result.

5.1 Pfaffian form and bilinear relations

The general L-loop vacuum diagram in momentum space (which before with all the constraints would have been written $\mathrm{IKM}(0, L+1, 1)$ with a particular choice of normalisation factor is of the form after Wick rotation:

$$\mathcal{T} = 2^{D-1}\pi^{D/2}\Gamma\left(\frac{D}{2}\right)\int \delta\left(\sum_{l=0}^{L} P_l\right)\prod_{l=0}^{L}\left[\frac{d^D P_l}{\pi^{D/2}}\frac{m_l\Gamma(a_l)}{(P_l^2+m_l^2)^{\alpha_l}}\right].$$

Representing the delta function in exponential form for $\mu = D - 2\alpha + 1$, it is possible to write each propagator as a Bessel function

$$\mathcal{T} = \int_0^\infty dx\, x^{D-1}\prod_{l=0}^{L} P_0(\mu_l, m_l, x), \tag{18}$$

$$P_0(\mu, m, x) = 2m\left(\frac{x}{2m}\right)^{\frac{1-\mu}{2}} K_{\frac{\mu-1}{2}}(mx).$$

In the Broadhurst case, $\mu = 1$ and $m = 1$ so we recover $2K_0(x)$. Now looking at derivatives of P_0,

$$P_1 = \frac{1}{m}\partial_x P_0 = -2m\left(\frac{x}{2m}\right)^{\frac{1-\mu}{2}} K_{\frac{\mu+1}{2}}(mx)$$

and so we compute differential equations in P_0 and P_1 expressed as a single vector $P = \begin{pmatrix} P_0 \\ P_1 \end{pmatrix}$.

$$m\partial_m P = (mx\sigma + \mu\overline{n})P, \tag{19}$$

$$x\partial_x P = (m\partial_m - \mu)P = (mx\sigma - \mu n)P, \tag{20}$$

where $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\overline{n} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Unique to the 2-point functions is its ability to have its integrand in coordinate space be expressed as a product of the same type of function, each containing separate masses. Crucial to this investigation is that derivatives of these Bessel functions can be reduced to only the zeroth and first derivative, for instance

$$\frac{d^2}{dx^2}K_0(x) = K_0(x) + \frac{K_1(x)}{x}.$$

For each Bessel function in (18), Lee and Pomeransky generalise the integral to have a binary choice of P_0 or P_1 per propagator corresponding to the zeroth or first derivatives of the Bessel function inside. This gives 2^{L+1} choices for the integrand. Of course, the choices where they are all P_0's is the original diagram. The introduction of this set of functions means one can proceed to take derivatives of the masses to obtain results in the form of a linear combination of other functions in this set, reminiscent to the process of setting up master integrals (2.4). Such a choice of P_i's can be characterised by a binary number $\mathbf{a}$

$$\mathcal{T}_{\mathbf{a}} = \int_0^\infty dx\, x^{D-1} \prod_{l=0}^{L} P_{a_l}(\mu_l, m_l, x)$$

with our original $\mathcal{T}$ corresponding to all zeros. If we take $\int_0^\infty dx \partial_x (x^D t) = 0$, then we can use the differential equation in ∂_x to find that

$$\begin{aligned}
0 &= \int_0^\infty dx D x^{D-1} t + \int_0^\infty dx\, x^{D-1}(x\partial_x t) \\
&= DT + \int_0^\infty dx\, x^{D-1} \left(x \sum m_l \sigma_l - \sum \mu_l n_l \right) t \\
&= \int_0^\infty x^{D-1} \left(xM - W \right) t
\end{aligned}$$

where we use (20) to differential t with respect to all the masses. The σ_l operators now act on each individual propagator. Moving the $W = (\sum \mu_l n_l - D)$ integral to the other side,

$$M \int_0^\infty dx\, x^D t = W \int_0^\infty dx\, x^{D-1} t \qquad \Rightarrow \qquad \int_0^\infty dx\, x^D t = M^{-1} W T$$

if M^{-1} exists. Then using the other useful differential equation in ∂_m,

$$\begin{aligned}
\partial_{m_l} T = \int_0^\infty dx\, x^{D-1} \partial_m t &= \int_0^\infty dx\, x^{D-1} (x\sigma_l + \frac{\mu_l \bar{n}_l}{m_l}) t \\
&= \sigma_l \int_0^\infty dx\, x^D t + \frac{\mu_l \bar{n}_l}{m_l} \int_0^\infty dx\, x^{D-1} t \\
&= (\sigma_l M^{-1} W + \frac{\mu_l \bar{n}_l}{m_l}) T \equiv A_l T \\
\Rightarrow dT = AT
\end{aligned}$$

which constitutes a Pfaffian system after it is shown by computation that $dA = 0$ (in appendix of [2]). A and W also satisfy $A^\intercal W = WA$. Suppose $U_\pm$ satisfy $dU_\pm = \pm A U_\pm$. Then

$$d(U_-^\intercal W U_+) = -U_- A^\intercal W U_+ + U_- W A U_+ = 0$$

such that $U_-^\intercal W U_+$ is a bilinear form that is only a function of D and μ. Lee and Pomeransky report a quadratic identity

$$\begin{aligned}
\mathrm{IKM}[(2,1)_m,(0,1)_1,1] \cdot \mathrm{IKM}[(3,0)_m,(0,1)_1,3] - \mathrm{IKM}[(2,1)_m,(0,1)_1,3] \cdot \mathrm{IKM}[(3,0)_m,(0,1)_1,1] \\
= \frac{4(1-5m^2)}{(1-m^2)^2(1-9m^2)^2}
\end{aligned} \tag{21}$$

with a mass parameter m. As of yet, this set of Bessel moments have not been solved in terms of hypergeometric functions. After some numerical experimentation with the *Hyperdire* Mathematica package [19], I have been able to solve 2 out of 4 of them. Namely

$$\mathrm{IKM}[(3,0)_m,(0,1)_1,1] = F_C(1,1;1,1,1;m^2,m^2,m^2),$$

$$\mathrm{IKM}[(3,0)_m,(0,1)_1,3] = F_C(2,2;1,1,1;m^2,m^2,m^2).$$

In order to complete the set in compliance with the form of twisted period relations, we must seek similar F_C functions with $\{m^2, m^2, m^2\}$ as the variables. The remaining 5 parameters should differ by integers. I was unable to find such entries that match the asymptotic behaviour of the pair of remaining Bessel moments which diverge in 2 direction but have not exhausted all possibilities.

5.2 Solution as Lauricella F_C functions

We will now take a series expansion in x of these integral to express them as hypergeometric functions. This can be illustrated by considering the 1-loop general vacuum function

$$\begin{aligned}\mathcal{T} = T_{00} &= \int_0^\infty dx\, x^{D-1} P_0(\mu_A, m_A, x) P_0(\mu_B, m_B, x) \\ &= \int_0^\infty dx\, x^{D-1} P_0(\mu_A, m_A, x) \left[2m_B (\frac{x}{2m_B})^{\frac{1-\mu_B}{2}} K_{\frac{\mu_B-1}{2}}(m_B x) \right].\end{aligned}$$

We may choose to express the K Bessel function in terms of I functions instead using

$$K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \pi\nu}$$

which was not possible before in the quadratic relation for $\nu = 0$. The expression inside the brackets above reads

$$\begin{aligned}&\left[2m_B \left(\frac{x}{2m_B}\right)^{\frac{1-\mu_B}{2}} \frac{\pi}{2} \frac{I_{-\frac{\mu_B-1}{2}}(m_B x) - I_{\frac{\mu_B-1}{2}}(m_B x)}{\sin \pi \frac{\mu_B-1}{2}} \right] \\ &= \left[2m_B \left(\frac{x}{2m_B}\right)^{\frac{1-\mu_B}{2}} \frac{\pi}{2} \frac{I_{\frac{\mu_B-1}{2}}(m_B x) - I_{-\frac{\mu_B-1}{2}}(m_B x)}{\cos \frac{\pi\mu_B}{2}} \right].\end{aligned}$$

Thus when we replace a single $K_\nu(x)$ Bessel function we get 2 terms in this self-contained manner. If we replace L Bessel functions then this expands to 2^L new functions. These functions can again be characterised by a binary number $\boldsymbol{\rho}$ of length L where each digit determines the choice of sign of both the term and the subscript of the single $\pm I_{\mp\nu}(x)$ that replaces the $K_\nu(x)$. Note that the above 2 terms correspond to a single entry T_{00} of T. For a different binary number $\mathbf{a}$ we may perform the same replacement but with P_1. For a given $\mathbf{a}$ (recall that $\mathbf{a} = 00$ for our original vacuum diagram),

$$T_{\mathbf{a}} = \sum_{\boldsymbol{\rho}} V_a^{(\boldsymbol{\rho})}$$

where we now introduce the set of functions

$$V^{(\boldsymbol{\rho})} = \int_0^\infty dx\, x^{D-1} P(\mu_0, m_0, x) \otimes \bigotimes_{l=1}^{L} Q^{(\rho_l)}(\mu_l, m_l, x), \quad T = \sum_{\boldsymbol{\rho}} V^{(\boldsymbol{\rho})},$$

where each $Q^{(\rho_l)}$ vector contains both choices of I Bessel function. Here we kept the zeroth Bessel function as a K and swapped all the rest. When $\mathbf{a}$ is specified, then $V_{\mathbf{a}}^{(\rho)}$ is the result of selecting either the top or bottom entry at each site. These still satisfy the Pfaffian system differential equation. In order to recover some given $T_{\mathbf{a}}$, we select either the top or bottom entry of each Q vector. We have to sum over every possible binary number to account for the both Bessel functions that replaced with each K_l.

We now seek to express this representation in terms of the Lauricella function F_C. To this end, we invoke the series expansion of the I_ν Bessel function

$$I_\nu(x) = \sum_{k=0}^\infty \frac{(x/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)}.$$

We will focus on solving the single V^0_{00} term that belongs to the 1-loop vacuum function $T_{00} = V^0_{00} + V^1_{00}$ above (5.2). We need not keep prefactors along the way as we only aim to illustrate how these integrals may be solved as F_C functions.

$$V^0_{00} \propto \int_0^\infty dx\, x^{D-1} \left[x^{-\nu_A} K_{\nu_A}(m_A x)\right] \left[x^{-\nu_B} I_{\nu_B}(m_B x)\right].$$

Rescaling to eliminate m_A from the argument of K,

$$V^0_{00} \propto \int_0^\infty dx\, x^{D-1-\nu_A-\nu_B} K_{\nu_A}(x) I_{\nu_B}(\frac{m_B}{m_A}x).$$

Now we will expand I and perform the integral over x for each term, and then recombine the results after. With the remaining K function, we use the identity [20]

$$\int_0^\infty dx\, x^{\beta-1} K_\nu(x) = 2^{\beta-2}\Gamma\left(\frac{\beta+\nu}{2}\right)\Gamma\left(\frac{\beta-\nu}{2}\right)$$

to perform the integration. To express the result as a series, we observe the sequence of exponents of x from each term of the expansion to be

$$\beta_k - 1 = \{(D-1-\nu_k), \quad (D-1-\nu_k+2), \quad (D-1-\nu_k+4), \quad \dots\},$$

i.e. increasing by $+2$ each time. Then the first terms of the expansion are

$$\begin{aligned} V^0_{00} \propto\ & \frac{(m_B/2m_A)^{\nu_B}}{0!\Gamma(\nu_B+1)} 2^{D-\nu_A-2}\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{D-2\nu_A}{2}\right) \\ &+ \frac{(m_B/2m_A)^{\nu_B+2}}{1!\Gamma(\nu_B+2)} 2^{D-\nu_A}\Gamma\left(\frac{D}{2}+1\right)\Gamma\left(\frac{D-2\nu_A}{2}+1\right) + \dots . \end{aligned}$$

Each of the kth terms in this expansion contains the factors $2^{-\nu_B-2k}$ due the the I_ν expansion and $2^{D-\nu_A-2+2k}$ from the integral, so there is no k dependence. We may factor this $2^{D-\nu_A-\nu_B-2}$ quantity out of the expansion. We may also factor out $(m_B/m_A)^{\nu_B}$. Then we square what remains inside the bracket such that the exponent increases in increments of only 1 instead of 2. Hence

$$V^0_{00} \propto \left[\frac{(\frac{m_B^2}{m_A^2})^0}{0!\Gamma(\nu_B+1)}\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{D-2\nu_A}{2}\right) + \frac{(\frac{m_B^2}{m_A^2})^1}{1!\Gamma(\nu_B+2)}\Gamma\left(\frac{D}{2}+1\right)\Gamma\left(\frac{D-2\nu_A}{2}+1\right) + \dots\right]$$

$$= \sum_{k=0}^\infty \frac{\Gamma(\frac{D}{2}+k)\Gamma(\frac{D}{2}-\nu_A+k)(\frac{m_B^2}{m_A^2})^k}{\Gamma(\nu_B+1+k)k!} \quad = \quad \frac{\Gamma(1+\nu_B)}{\Gamma(D/2)\Gamma(D/2-\nu_A)}\sum_{k=0}^\infty \frac{(\frac{D}{2})_k(\frac{D}{2}-\nu_A)_k(\frac{m_B^2}{m_A^2})^k}{(\nu_B+1)_k k!}.$$

Now we recall the definition of the F_C Lauricella function (5)

$$F_C^{(L)}(a,b;c_1,\dots,c_L;x_1,\dots,x_L) = \sum_{k=0}^\infty \frac{(b_1)_{\sum k_i}(b_2)_{\sum k_i} x_1^{k_1}\cdots x_L^{k_L}}{(c_1)_{k_1}\cdots(c_L)_{k_L} k_1!\cdots k_L 2}$$

to see that V^0_{00} is proportional to $F_C^{(1)}(D/2, D/2-\nu_B; 1+\nu_B; \frac{m_B^2}{m_A^2})$.

Having discarded many prefactors along the way, we will now quote the full result due to Lee and Pomeransky [2]:

$$\begin{aligned} V_{\mathbf{a}}^{(\rho)} &= 2^{D-1}(-1)^{a_0} m_0^{-D+\sum_{l=1}^L \mu_l}\Gamma(b_1)\Gamma(b-2) \\ &\times \prod_{l=1}^L (-1)^{a_l}\Gamma(1-c_l)\left(\frac{m_l}{m_0}\right)^{c_l+\frac{\mu_l-1}{2}} F_C^{(L)}\left(b_1, b_2; c_1, \dots, c_L; \frac{m_1^2}{m_0^2}, \dots, \frac{m_L^2}{m_0^2}\right). \end{aligned}$$

where b_1, b_2, c_l are constants also given in terms of $\mathbf{a}, \rho$ and μ_l.

For instance, consider the 1-loop vacuum diagram with masses m_0 and m_1, with exponents 1. Then

$$V_{00} = V_{00}^0 + V_{00}^1 = 2^{D-1} m_0^{D-2} \Gamma\left(\frac{D}{2}\right)\Gamma(1)$$

$$\times \left[\Gamma\left(1-\frac{D}{2}\right)\left(\frac{m_1}{m_0}\right)^{D-1} F_C^{(1)}\left(1, \frac{D}{2}; \frac{D}{2}; \frac{m_1^2}{m_0^2}\right) + \Gamma\left(\frac{D}{2}-1\right)\left(\frac{m_1}{m_0}\right)^1 F_C^{(1)}\left(1, 2-\frac{D}{2}; 2-\frac{D}{2}; \frac{m_1^2}{m_0^2}\right)\right]$$

$$= 2^{D-1} m_0^{D-2} \Gamma\left(\frac{D}{2}\right) \frac{m_0^2}{m_0^2 - m_1^2}\left[\Gamma\left(1-\frac{D}{2}\right)\left(\frac{m_1}{m_0}\right)^{D-1} + \Gamma\left(\frac{D}{2}-1\right)\left(\frac{m_1}{m_0}\right)\right].$$

Or the 2-loop vacuum diagram with exponents 1. The 2 loops require a linear combination of four F_C functions:

Entries of $F_C^{(2)}(b_1, b_2; c_1, c_2; z_1^2, z_2^2)$ for 2-loop vacuum diagram				
ρ	b_1	b_2	c_1	c_2
00	D/2	1	D/2	D/2
01	1	2 - D/2	D/2	2 - D/2
10	1	2 - D/2	2 - D/2	D/2
11	2 - D/2	3 - D	2 - D/2	2 - D/2

with $z_i = \frac{m_i^2}{m_0^2}$. Then invoking the results of Berends et al. [20] who computed a closed form for the 2-loop sunrise with general masses and exponents 1, we may take the limit as the outer leg $q^2 \to 0$ in their expression containing four $F_C^{(3)}$ functions. This limit is easy to take due to q^2 only appearing as a variable of the F_C functions in the overall expression. Recalling (2.5), after taking the limit the final variable vanishes from the function along with its corresponding c parameter. By this process, an $F_C^{(3)}(b_1, b_2; c_1, c_2, c_3; x_1, x_2, x_3)$ function becomes an $F_C^{(2)}(b_1, b_2; c_1, c_2; x_1, x_2)$ function and we may read off the equivalent $F_C^{(2)}$ functions to find the entries of the vacuum version of the 2-loop sunrise. The entries are the same as tabulated above and we recover the same expression up to the different normalisation factors used by each set of authors. In particular, one removes a $\Gamma(D/2)$ factor from the LP representation which will cause some sign changes as seen in [20].

Now with one more loop, the table is given with twice as many rows:

Entries of $F_C^{(3)}(b_1, b_2; c_1, c_2, c_3; z_1^2, z_2^2, z_3^2)$ for 3-loop vacuum diagram					
ρ	b_1	b_2	c_1	c_2	c_3
000	D/2	1	D/2	D/2	D/2
001	1	2 - D/2	D/2	D/2	2 - D/2
010	1	2 - D/2	D/2	2 - D/2	D/2
100	1	2 - D/2	2 - D/2	D/2	D/2
011	2 - D/2	3 - D	D/2	2 - D/2	2 - D/2
101	2 - D/2	3 - D	2 - D/2	D/2	2 - D/2
110	2 - D/2	3 - D	2 - D/2	2 - D/2	D/2
111	3 - D	4 - 3D/2	2 - D/2	2 - D/2	2 - D/2

Defining $\frac{D}{2} = 1 + \nu$ for shorthand, we write out the vacuum terms for $T(m_1, m_2, m_3, m_0) \times \left[2^{D-1} m_0^{3D-4}\sqrt{z_1 z_2 z_3}\right]^{-1} =$

$$\begin{aligned}
&\Gamma(1+\nu)\Gamma(1) \times \Gamma^3(-\nu) \times z_1^\nu z_2^\nu z_3^\nu \times F_C^{(3)}(1+\nu, 1; 1+\nu, 1+\nu, 1+\nu; z_1, z_2, z_3) \\
&+\Gamma(1)\Gamma(1-\nu) \times \Gamma^2(-\nu)\Gamma(\nu) \times z_1^\nu z_2^\nu \times F_C^{(3)}(1, 1-\nu; 1+\nu, 1+\nu, 1-\nu; z_1, z_2, z_3) \\
&+\Gamma(1)\Gamma(1-\nu) \times \Gamma^2(-\nu)\Gamma(\nu) \times z_1^\nu z_3^\nu \times F_C^{(3)}(1, 1-\nu; 1+\nu, 1-\nu, 1+\nu; z_1, z_2, z_3) \\
&+\Gamma(1)\Gamma(1-\nu) \times \Gamma^2(-\nu)\Gamma(\nu) \times z_2^\nu z_3^\nu \times F_C^{(3)}(1, 1-\nu; 1-\nu, 1+\nu, 1+\nu; z_1, z_2, z_3) \\
&+\Gamma(1-\nu)\Gamma(1-2\nu) \times \Gamma(-\nu)\Gamma^2(\nu) \times z_1^\nu \times F_C^{(3)}(1-\nu, 1-2\nu; 1+\nu, 1-\nu, 1-\nu; z_1, z_2, z_3) \\
&+\Gamma(1-\nu)\Gamma(1-2\nu) \times \Gamma(-\nu)\Gamma^2(\nu) \times z_2^\nu \times F_C^{(3)}(1-\nu, 1-2\nu; 1-\nu, 1+\nu, 1-\nu; z_1, z_2, z_3) \\
&+\Gamma(1-\nu)\Gamma(1-2\nu) \times \Gamma(-\nu)\Gamma^2(\nu) \times z_3^\nu \times F_C^{(3)}(1-\nu, 1-2\nu; 1-\nu, 1-\nu, 1+\nu; z_1, z_2, z_3) \\
&+\Gamma(1-2\nu)\Gamma(1-3\nu) \times \Gamma^3(\nu) \times F_C^{(3)}(1-2\nu, 1-3\nu; 1-\nu, 1-\nu, 1-\nu; z_1, z_2, z_3).
\end{aligned}$$

Then we introduce $z_4 = p^2/m_0^2$ as a new variable in the Lauricella functions along with $1+\nu$ as the new c_4 parameter, both appended on to the $F_C^{(3)}$ functions to turn them into $F_C^{(3)}$'s. Then an expression for the 3-loop sunrise diagram shown in figure 7 with general masses and dimension is

$$S(p^2, m_1^2, m_2^2, m_3^2, m_0^2) \times \left[2^{D-1} m_0^{3D-4} \sqrt{z_1 z_2 z_3}\right]^{-1} =$$

$$\begin{aligned}
&\Gamma(1+\nu)\Gamma(1) \times \Gamma^3(-\nu) \times z_1^\nu z_2^\nu z_3^\nu \times F_C^{(4)}(1+\nu, 1; 1+\nu, 1+\nu, 1+\nu, 1+\nu; z_1, z_2, z_3, z_4) \\
&+\Gamma(1)\Gamma(1-\nu) \times \Gamma^2(-\nu)\Gamma(\nu) \times z_1^\nu z_2^\nu \times F_C^{(4)}(1, 1-\nu; 1+\nu, 1+\nu, 1-\nu, 1+\nu; z_1, z_2, z_3, z_4) \\
&+\Gamma(1)\Gamma(1-\nu) \times \Gamma^2(-\nu)\Gamma(\nu) \times z_1^\nu z_3^\nu \times F_C^{(4)}(1, 1-\nu; 1+\nu, 1-\nu, 1+\nu, 1+\nu; z_1, z_2, z_3, z_4) \\
&+\Gamma(1)\Gamma(1-\nu) \times \Gamma^2(-\nu)\Gamma(\nu) \times z_2^\nu z_3^\nu \times F_C^{(4)}(1, 1-\nu; 1-\nu, 1+\nu, 1+\nu, 1+\nu; z_1, z_2, z_3, z_4) \\
&+\Gamma(1-\nu)\Gamma(1-2\nu) \times \Gamma(-\nu)\Gamma^2(\nu) \times z_1^\nu \times F_C^{(4)}(1-\nu, 1-2\nu; 1+\nu, 1-\nu, 1-\nu, 1+\nu; z_1, z_2, z_3, z_4) \\
&+\Gamma(1-\nu)\Gamma(1-2\nu) \times \Gamma(-\nu)\Gamma^2(\nu) \times z_2^\nu \times F_C^{(4)}(1-\nu, 1-2\nu; 1-\nu, 1+\nu, 1-\nu, 1+\nu; z_1, z_2, z_3, z_4) \\
&+\Gamma(1-\nu)\Gamma(1-2\nu) \times \Gamma(-\nu)\Gamma^2(\nu) \times z_3^\nu \times F_C^{(4)}(1-\nu, 1-2\nu; 1-\nu, 1-\nu, 1+\nu, 1+\nu; z_1, z_2, z_3, z_4) \\
&+\Gamma(1-2\nu)\Gamma(1-3\nu) \times \Gamma^3(\nu) \times F_C^{(4)}(1-2\nu, 1-3\nu; 1-\nu, 1-\nu, 1-\nu, 1+\nu; z_1, z_2, z_3, z_4).
\end{aligned}$$

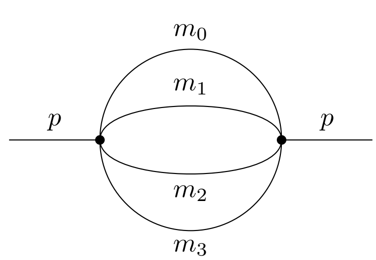


Figure 7: 3 loop sunrise diagram with general masses

6 Results of twisted period relations

Having encountered quadratic hypergeometric identities from sunrise diagrams, the open question that this section addresses is if these relations can be expressed as a quadratic period relation from intersection theory.

We will generate quadratic relations for the beta function and ${}_2F_1$ function using the twisted (co)homology theory developed in section 3. Then we will look to the literature to find results for the ${}_4F_3$ function that appeared in the Broadhurst relation, as well as for the F_C function as they may eventually appear in a Lee-Pomeransky relation.

6.1 Beta function

As seen in section 3, the beta functions admits the integral representation

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt = \int_0^1 t^a(1-t)^b \frac{dt}{t(1-t)}$$

whose poles we identify as $x_1 = 0$, $x_2 = 1$,$x_3 = \infty$. These define the chambers $(0,1)$ and $(1,\infty)$. The dimension of (co)homology is 1 and so we need only select $(0,1)$ to construct the twisted cycle. The self-intersection number is given by (9) as

$$\Delta^+ \cdot \Delta^- = -\frac{c_1c_2 - 1}{(c_1-1)(c_2-1)}.$$

Next we can express the twisted cocycle in the form

$$\varphi = \frac{dt}{t-1} - \frac{dt}{t-0}$$

so that we may apply the cocycle intersection results to write

$$I_c(\varphi,\varphi) = 2\pi i\left(\frac{1}{a} + \frac{1}{b}\right)$$

and arrive at the quadratic relation

$$B(a,b)B(-a,-b) = 2\pi i(\frac{1}{a} + \frac{1}{b})\left(-\frac{\exp(2\pi i(a+b)) - 1}{(\exp(2\pi i a) - 1)(\exp(2\pi i b) - 1)}\right). \tag{22}$$

This result will become useful in the following relations to simplify products of gamma functions.

6.2 ${}_2F_1$ hypergeometric function

Recalling the integral representation

$${}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-tx)^{-b}dt$$

we identify the poles

$$x_1 = 0, x_2 = 1, x_3 = 1/x, x_4 = \infty.$$

The (co)homologies are 2 dimensional and we can select the intervals $(0,1)$, $(\frac{1}{x},\infty)$ as the basis of twisted cycles and

$$\omega_{12} = \frac{dt}{t-x_1} - \frac{dt}{t-x_2}, \quad \omega_{34} = \frac{dt}{t-x_3} - \frac{dt}{t-x_4}$$

as the basis of twisted cocycles. We now apply the period relation (10).

$$\Pi_+ = \begin{bmatrix} \int_0^1 U\omega_{12} & \int_{1/x}^\infty U\omega_{12} \\ \int_0^1 U\omega_{34} & \int_{1/x}^\infty U\omega_{34} \end{bmatrix}, \qquad \Pi_- = \begin{bmatrix} \int_0^1 U^{-1}\omega_{12} & \int_{1/x}^\infty U^{-1}\omega_{12} \\ \int_0^1 U^{-1}\omega_{34} & \int_{1/x}^\infty U^{-1}\omega_{34} \end{bmatrix}.$$

In particular, the intersection matrices A and B are diagonal so they are easy to invert. The first row second column entry leads to

$$(A^{-1})_{11} \int_0^1 U\omega_{12} \cdot \int_0^1 U^{-1}\omega_{34} + (A^{-1})_{22} \int_{1/x}^\infty U\omega_{12} \cdot \int_{1/x}^\infty U^{-1}\omega_{34} = 0.$$

Each of these factors evaluate as

$$\begin{aligned}
(A^{-1})_{11} &= -\frac{(c_1-1)(c_2)-1}{c_1c_2-1} = 2\pi i \left(\frac{1}{a} + \frac{1}{c-a}\right) \frac{1}{B(a,c-a)B(-a,a-c)}, \\
\int_0^1 U\omega_{12} &= \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-tx)^{-b}dt = B(a,c-a)\,{}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix} \middle| x\right), \\
\int_0^1 U^{-1}\omega_{34} &= -x\int_0^1 t^{-a}(1-t)^{a-c}(1-tx)^{b-1}dt = -xB(1-a,1+a-c)\,{}_2F_1\left(\begin{matrix} 1-a,\ 1-b \\ 2-c \end{matrix} \middle| x\right), \\
(A^{-1})_{22} &= -\frac{(c_3-1)(c_4)-1}{c_3c_4-1} = 2\pi i \left(\frac{1}{-b} + \frac{1}{b-c}\right) \frac{1}{B(-b,b-c)B(b,c-b)}, \\
\int_{1/x}^\infty U\omega_{12} &= \int_{1/x}^\infty t^{a-1}(1-t)^{c-a-1}(1-tx)^{-b}dt = -(-1)^{c-a-b}x^{1-c}B(b-c+1,1-b)\,{}_2F_1\left(\begin{matrix} b-c+1,\ a-c+1 \\ 2-c \end{matrix} \middle| x\right), \\
\int_{1/x}^\infty U^{-1}\omega_{34} &= -x\int_{1/x}^\infty t^{-a}(1-t)^{a-c}(1-tx)^{b-1}dt = x(-1)^{-c+a+b}x^{c-1}B(c-b,b)\,{}_2F_1\left(\begin{matrix} c-b,\ c-a \\ c \end{matrix} \middle| x\right).
\end{aligned}$$

We invoked the results above (22) to express the intersection terms in terms of beta functions. These simplify with the other beta function factors and result in an overall $-1/c$ factor in the first time and $1/c$ in the second. The final result is

$$ {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix} \middle| x\right) {}_2F_1\left(\begin{matrix} 1-a,\ 1-b \\ 2-c \end{matrix} \middle| x\right) = {}_2F_1\left(\begin{matrix} a-c+1,\ b-c+1 \\ 2-c \end{matrix} \middle| x\right) {}_2F_1\left(\begin{matrix} c-a,\ c-b \\ c \end{matrix} \middle| x\right). \tag{23}$$

Beginning from the LHS, we use the transformation

$$ {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix} \middle| x\right) = (1-x)^{c-a-b}\,{}_2F_1\left(\begin{matrix} c-a,\ c-b \\ c \end{matrix} \middle| x\right)$$

on both the functions with the overall effect of cancelling the $(1-x)$ prefactor. That is to say, this functional equation is not unique but rather just a combination of the contiguous and transformations we encountered before. The $(1,1)$ entry gives another quadratic relation

$$\begin{aligned}
&{}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix} \middle| x\right) {}_2F_1\left(\begin{matrix} -a,\ -b \\ -c \end{matrix} \middle| x\right) - 1 \qquad (24) \\
&= \frac{ab(c-a)(c-b)}{c^2(c+1)(c-1)}\,{}_2F_1\left(\begin{matrix} a-c+1,\ b-c+1 \\ 2-c \end{matrix} \middle| x\right) {}_2F_1\left(\begin{matrix} c-a+1,\ c-b+1 \\ c+2 \end{matrix} \middle| x\right).
\end{aligned}$$

Relations derived from diagonal entries have a symmetrical property which is that the parameters which appear in a given pair of functions are identical except that only the sign of a, b, c are swapped. The rational prefactor $\frac{ab(c-a)(c-b)}{c^2(c+1)(c-1)}$ is also invariant under swapping of all the signs. The above relation is invariant under $a_i \to -a_i$.

6.3 Generalised ${}_{m+1}F_m$ hypergeometric functions

Up until now we have only considered 1-dimensional Euler-type integrals. However, the Euler-type integral representations of ${}_3F_2$ and beyond are of a higher dimension. Another subtlety not yet encountered is a degenerate hyperplane arrangement where hyperplanes intersect at a point more than pairwise. It is beyond the scope of this project to address their solutions and so we now instead turn to the quadratic period relations for ${}_{m+1}F_m$ functions computed in the last decade.

Goto [21] computed a basis of twisted cycles and cocycles for the generalised hypergeometric function given by the integral representation (neglecting gamma factors) for $b_i - b_j \notin \mathbb{Z}$

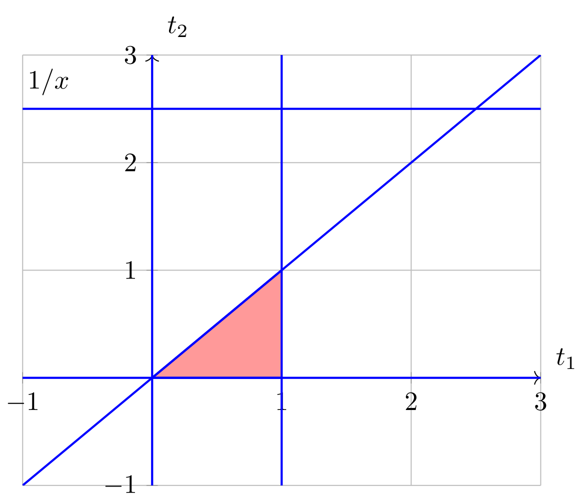
$$\int_D \prod_{j=1}^{m-1} t_j^{a_j - b_{j+1}} (t_j - t_{j+1})^{b-j+1-a_{j+1}-1} \cdot t_m^{a_m-1}(1-t_1)^{b_1-a_1-1}(1-xt_m)^{-a)} dt_1 \wedge \cdots \wedge dt_m \quad (25)$$

with D the region satisfying $0 < t_m < \cdots < t_1 < 1$. This can best be demonstrated with $m = 2$ which is the ${}_3F_2$ function given by

$$ {}_3F_2 \left(\begin{matrix} a_0, \ a_1, \ a_2 \\ b_1, \ b_2 \end{matrix} \middle| \, x \right) = \prod \Gamma(\dots)$$

$$\times \int_\Delta t_1{}^{a_1-b_2} (1-t_1)^{b_2-a_2-1} t_2{}^{a_2-1} (t_1-t_2)^{b_1-a_1-1} (1-xt_2)^{a_0} \, dt_1 \wedge dt_2.$$

Shown below is the hyperplane arrangement with the integration region in red:



This hyperplane arrangement has degeneracy around the origin where the hyperplanes intersect three times.

We are most interested in the ${}_4F_3$ function which appears in the Broadhurst result. The spanning cocycles are

$$\varphi_0 = \frac{dt_1 \wedge dt_2 \wedge dt_3}{t_3(1-t_1)(t_1-t_2)(t_2-t_3)}, \qquad \varphi_1 = \frac{xdt_1 \wedge dt_2 \wedge dt_3}{(1-xt_3)(t_1-t_2)(t_2-t_3)},$$

$$\varphi_2 = \frac{xdt_1 \wedge dt_2 \wedge dt_3}{t_1(1-xt_3)(1-t_1)(t_2-t_3)}, \qquad \varphi_3 = \frac{xdt_1 \wedge dt_2 \wedge dt_3}{t_2(1-xt_3)(1-t_1)(t_1-t_2)}. \quad (26)$$

These cocycles have a diagonal intersection matrix which is full rank.
Goto also computes a diagonal intersection matrix for the twisted cycles which rely on transformations of variables to bring the integral cycle to a product of regularised intervals which can be

computed since we already know how to compute intersection numbers of intervals. Just as we saw for the ${}_2F_1$, the cycle intersection numbers contain complex exponents of the parameters and so are not rational like the cocycle intersection numbers. Nevertheless their function in the final equation is to cancel out the gamma factors, leaving only hypergeometric functions and rational factors in the final equation. The $(1,1)$ entry of the twisted period relation is given by [11]

$$
\begin{aligned}
\frac{b_1b_2b_3}{a_1a_2a_3} = \frac{b_1b_2b_3}{a_1a_2a_3}\,{}_4F_3&\left(\begin{matrix} a_0,\ a_1,\ a_2,\ a_3 \\ b_1,\ b_2,\ b_3 \end{matrix}\,\middle|\, x\right){}_4F_3\left(\begin{matrix} -a_0,\ -a_1,\ -a_2,\ -a_3 \\ -b_1,\ -b_2,\ -b_3 \end{matrix}\,\middle|\, x\right)\\
&+x^2\frac{a_0(a_0-b_1)(b_1-a_1)}{b_1(b_1^2-1)}\frac{(a_2-b_1)(a_3-b_1)}{(b_2-b_1)(b_3-b_1)}\\
&\times{}_4F_3\left(\begin{matrix} 1+a_0-b_1,\ 1+a_1-b_1,\ 1+a_2-b_1,\ 1+a_3-b_1 \\ 2-b_1,\ 1+b_2-b_1,\ 1-b_3+b_1 \end{matrix}\,\middle|\, x\right)\\
&\times{}_4F_3\left(\begin{matrix} 1-a_0+b_1,\ 1-a_1+b_1,\ 1-a_2+b_1,\ 1-a_0+b_1 \\ 2+b_1,\ 1-b_2+b_1,\ 1-b_3+b_1 \end{matrix}\,\middle|\, x\right)\\
&+x^2\frac{a_0(a_0-b_2)(b_2-a_2)}{b_2(b_2^2-1)}\frac{(a_1-b_2)(a_3-b_2)}{(b_1-b_2)(b_3-b_2)}\\
&\times{}_4F_3\left(\begin{matrix} 1+a_0-b_2,\ 1+a_1-b_2,\ 1+a_2-b_2,\ 1+a_3-b_2 \\ 1+b_1-b_2,\ 2-b_2,\ 1-b_3+b_1 \end{matrix}\,\middle|\, x\right)\\
&\times{}_4F_3\left(\begin{matrix} 1-a_0+b_2,\ 1-a_1+b_2,\ 1-a_2+b_2,\ 1-a_0+b_2 \\ 1-b_1+b_2,\ 2+b_2,\ 1-b_3+b_2 \end{matrix}\,\middle|\, x\right)\\
&+x^2\frac{a_0(a_0-b_3)(b_3-a_3)}{b_3(b_3^2-1)}\frac{(a_1-b_3)(a_2-b_3)}{(b_1-b_3)(b_2-b_3)}\\
&\times{}_4F_3\left(\begin{matrix} 1+a_0-b_3,\ 1+a_1-b_3,\ 1+a_2-b_3,\ 1+a_3-b_3 \\ 1+b_1-b_3,\ 1+b_2-b_3,\ 2-b_2 \end{matrix}\,\middle|\, x\right)\\
&\times{}_4F_3\left(\begin{matrix} 1-a_0+b_3,\ 1-a_1+b_3,\ 1-a_2+b_3,\ 1-a_0+b_3 \\ 1-b_1+b_3,\ 1-b_2+b_3,\ 2+b_3 \end{matrix}\,\middle|\, x\right).
\end{aligned} \tag{27}
$$

In particular for $x=0$ then the functions evaluate to 1 and the identity clearly holds. Otherwise a numerical evaluation for an arbitrary selection of parameters confirms that this relation holds.

6.4 Obstacles reconciling Broadhurst with intersection theory

Here we answer the question whether or not the quadratic relations (12) derived from sunrise diagrams can be expressed as twisted period relations from intersection theory of hypergeometric functions.

Under the assumption $b_i - b_j \notin \mathbb{Z}$ then the top left entry of the quadratic period relations for the ${}_4F_3$ function gives 4 pairs of ${}_4F_3$'s. However, the Broadhurst quadratic relation involves only 2 pairs of ${}_4F_3$ functions. Upon inspection, the 4 functions in the Broadhurst relation all feature parameters such that there exists one pair in the top row and one pair in the bottom row which do differ by an integer:

$$
\begin{aligned}
40 = 7\,{}_4F_3&\left(\begin{matrix} 1/2,\ 2/3,\ 2/3,\ 5/6 \\ 7/6,\ 7/6,\ 4/3 \end{matrix}\,\middle|\, 1\right){}_4F_3\left(\begin{matrix} -1/2,\ 1/6,\ 1/3,\ 4/3 \\ -1/6,\ 5/6,\ 5/3 \end{matrix}\,\middle|\, 1\right)\\
+10\,{}_4F_3&\left(\begin{matrix} 1/6,\ 1/3,\ 1/3,\ 1/2 \\ 2/3,\ 5/6,\ 5/6 \end{matrix}\,\middle|\, 1\right){}_4F_3\left(\begin{matrix} -7/6,\ -1/2,\ -1/3,\ 2/3 \\ -5/6,\ 1/6,\ 1/3 \end{matrix}\,\middle|\, 1\right).
\end{aligned}
$$

We could begin by developing an ansatz for the form of the quadratic period relations that would result in an equation involving 2 pairs of the functions and a rational term. This gives information about the dimension of the (co)homology groups. Since the intersection matrices are full rank, we should expect at least 1 pair contributed per row in the final equation. This implies that a (co)homology group dimension of 3 or greater would produce too many pairs of functions, whereas a dimension of 1 would trivially give only 1 pair. We must conclude that the (co)homology groups are of dimension 2 and so the period matrices and intersection matrices would be 2×2. The degeneracy shown above in the parameters brings the possibility of lower dimension for the (co)homology groups.

With this constraint we can revisit the original bases of cycles and cocycles for the 4 dimensional (co)homology groups and attempt to select a subset of these as our new bases for 2-dimensional groups. The middle 2 entries of both the cycles and cocycles vanish in the case of $b_1 = b_2$, so we take the first and last as the bases. Then since the intersection matrices are diagonal and full rank, these would automatically constitute valid bases. The resulting quadratic relation would be precisely the same as the 4-dimensional version (27) without the middle 2 terms. The remaining terms are symmetrical in b_1 and b_2. Unfortunately, numerical experiments disprove this relation. This suggests that the dimension was not 2 after all. Indeed we verify directly the dimension of the groups using critical points.

Counting of Critical Points

For the twist $\omega = d\log u$ the dimension of the groups is the number of solutions [12] to

$$\omega = 0$$

which also satisfy further conditions [22]. In the case of the ${}_4F_3$ representation used by Goto,

$$u = x^{a_1-b_2} y^{a_2-b_3} z^{a_3-1} (x-y)^{b_2-a_2-1} (y-z)^{b_3-a_3-1} (1-x)^{b_1-a_1-1} (1-kz)^{-a_0}.$$

Then using the values in the first Broadhurst function this reduces to

$$u = \frac{z^{5/6}\sqrt{1-x}\sqrt{x-y}\sqrt{y-z}}{y^{2/3}\sqrt{x}\sqrt{1-z}}$$

giving the components of the twist

$$\{\omega_i\} = \left\{ \frac{1}{2}\left(\frac{1}{x(x-1)}\right), \frac{1}{2}\left(\frac{y-4x}{3y(x-y)} + \frac{1}{y-z}\right), \frac{1}{2}\left(\frac{1}{1-z} + \frac{5}{3z} + \frac{1}{z-y}\right) \right\}.$$

Solving $\omega = 0$ numerically admits 3 critical points

$$\alpha_1 = (0.63, 0.40, 0.27), \quad \alpha_2 = (-0.94 + 1.5i, -1.42 - 2.86i, 1.93 + 0.23i), \quad \alpha_3 = \overline{\alpha_2}$$

which are non-vanishing in

$$u(\alpha_1) = 0.096, \quad u(\alpha_2) = 2.18 + 3.72i, \quad u(\alpha_3) = 2.18 - 3.72i.$$

Additionally, the arguments of $u(\alpha_i)$ are all distinct and the Hessian matrices of u evaluated at these points are invertible since the determinants (-2.30, $-0.0394 + 0.0241i$ and $-0.0394 - 0.0241i$ respectively) are non-zero.

With all these conditions met, the dimensionality of the groups is 3. Therefore it does not appear to be low enough to admit a quadratic relation with only 2 pairs of hypergeometric functions. The integral representation (25) does not allow for the Broadhurst results to be expressed as a twisted period relation and so a different representation is required which solves a system with degeneracies in the parameters.

Despite not having the full set of hypergeometric functions for the Lee-Pomeransky result, we can still proceed with the same critical point counting protocol. Like in the Broadhurst case, we seek a dimension of 2 for the (co)homologies. In the case of the F_C representation used by Goto [3],

$$u = x^{-c_1} y^{-c_2} z^{-c_3} (1-x-y-z)^{c_1+c_2+c_3-a-3} (1-\sum \frac{k_i}{x_i})^{-b}.$$

Entering the values from $F_C(1,1;1,1,1;m^2,m^2,m^2)$ yields

$$u = \frac{1}{xyz(1-x-y-z)(1-m^2(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}))}$$

which is symmetric in x, y and z. Then there are 5 critical points

$$\alpha_1 = \left(m^2, m^2, \frac{1}{2}(1-3m^2)\right), \quad \alpha_2 = \left(m^2, \frac{1}{2}(1-3m^2), m^2\right), \quad \alpha_3 = \left(\frac{1}{2}(1-3m^2), (m^2, m^2)\right),$$

$$\alpha_4 = \frac{1}{8}\left(1+9m^2-\sqrt{1-14m^2+81m^4}, 1+9m^2-\sqrt{1-14m^2+81m^4}, 1+9m^2-\sqrt{1-14m^2+81m^4}\right),$$

$$\alpha_5 = \frac{1}{8}\left(1+9m^2+\sqrt{1-14m^2+81m^4}, 1+9m^2+\sqrt{1-14m^2+81m^4}, 1+9m^2+\sqrt{1-14m^2+81m^4}\right).$$

All five of $u(\alpha_i)$ result in real numbers with the quantity in the square root being positive for any m. They do not have a unique argument like in the Broadhurst case, and so we can only assign 5 as an upper limit in the dimension. Also similar to the Broadhurst case, the F_C system used to derive the integral representation in [3] disallows the parameters $c_i = 1$ to be integers and so a different system is required regardless in order to handle the degeneracies.

While the Broadhurst and Lee-Pomeransky results do not admit quadratic period relation within the current knowledge of intersection theory, there is still an inference to make about the form of the quadratic relation that would satisfy both. If we consider the left hand side of either of the matrix equations (10) (11) and a diagonal basis of cycles and cocycles,

$$\begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} F_1' & F_3' \\ F_2' & F_4' \end{pmatrix}$$

then we see know that we must find the relation on a diagonal element in order to recover a constant term that appears in both the Broadhurst and Lee-Pomeransky relation. Here F_i and F_i' are the entries to both the period matrices and $a, 0, 0, d$ are the entries to one of the intersection matrices. We also know that all four of the functions share either the same cocycle or same cycle. When multiplied out, the parameters α_i will have opposite signs within the pair $F_i \cdot F_i'$ of functions like in the $(1,1)$ entry for the ${}_2F_1$ computed above (24). This is another constraint that can only be used to rule out the general ${}_{m+1}F_m$ system as admitting the Broadhurst result as there is not enough information to guess the form of u that would appear in a suitable integral representation.

7 Conclusion

By parameterising a scalar multi-loop Feynman integral in terms of the graph polynomial $\mathcal{G}$, these integrals satisfy the toric and homogeneity operators associated with an $\mathcal{A}$-hypergeometric systems. Using Euler-type integral representations for hypergeometric functions, we computed explicit examples for 1-loop diagrams in terms of the ${}_2F_1$ function. We also computed a basis of master integrals and a system of differential equations for one family of diagrams using dimensional-shift operators.

L-loop sunrise diagrams can be cast as the integral over a product of Bessel functions. In the highly generic 2 dimensional case of all masses being equal to unity, the product of Bessel functions admits Vanhove operators. This leads to recursive formulae that generate quadratic identities in Bessel moments when the external momentum parameter is also set equal to unity. However, the individual integrals must still be solved as hypergeometric functions in order to report more quadratic hypergeometric identities. There is no standard approach to solving such integrals and the approach used to solve the 4-loop case involved a long series of hypergeometric identities.

Intersection theory gives rise to quadratic hypergeometric identities in the form of twisted Riemann period relations. To derive these, one must determine a basis of (co)homologies for a given hypergeometric system. As far as we are aware, the relations in the literature only cover ${}_{m+1}F_m$ systems where the parameters do *not* differ by integers. The existence of degeneracy in the Broadhurst result results in a dimension of (co)homology which matches neither that of the general results from literature nor the number of pairs in the result itself when checked against the general integral representation of the ${}_4F_3$ function. Therefore with only the quadratic relations we have encountered in literature, it is impossible to express the Broadhurst as an identity derived from intersection theory. In order to make further progress, the hypergeometric systems for $bi - bj \notin \mathbb{Z}$ must be revisited to allow for these degeneracies.

Lee and Pomeransky set up a Pfaffian system for vacuum sunrise diagrams and gave an expression for evaluating the general-parameter case as a sum of F_C functions. We re-derived the series expansion for a 1-loop diagram and then tabulated the entries of a 3-loop sunrise diagram with dimensional regularisation. Their quadratic relation with variable mass has not yet been expressed in hypergeometric functions. On the basis of numerical experiment, I have solved 2 out of the 4 of their modified Bessel moments. The entries are highly degenerate and thus the same issue arose as with the Broadhurst result where the hypergeometric systems studied in the intersection theory literature have non-corresponding (co)homology dimension. We await further development from intersection theory, namely integral representations of ${}_4F_3$ and F_C hypergeometric systems constrained by degenerate parameters such that the dimension of (co)homology is precisely 2. Then it may be possible to select a basis of cycles and cocycles that lead to the quadratic relations from physics.

A Appendix

Symanzik polynomials for the 1-loop 2-point function

For this example we compute the Symanzik polynomials using the definition that arises when parameterising the momentum integral rather than the spanning tree construction. We assign the loop momentum

$$q_1 = k - p$$
$$q_2 = k$$

by enforcing conservation of momentum at each vertex. Reverting to Minkowski vectors, $\mathcal{U}$ and $\mathcal{F}$ are defined

$$\mathcal{U} = \det(M)$$
$$\mathcal{F} = \det(M)(J + v^T M^{-1} v)$$

where the matrix M, vector v and scalar J are computed (summing over repeated indices) from

$$a_j(-q_j^2 + m_j^2) = -k_r M_{rs} k_s + 2k_r \cdot v_r + J.$$

In our 1L2P example there is only one loop momentum k to consider. The sum on the left hand side expands to

$$a_1(-k^2 - p^2 + 2kp + m_1^2) + a_2(-k^2 + m_2^2),$$
$$M = [a_1 + a_2], \quad v = (a_1 p), \quad J = a_1(m_1^2 - p^2) + a_2 m_2^2,$$

yielding the Symanzik polynomials

$$\mathcal{U} = a_1 + a_2$$
$$\mathcal{F} = a_1 a_2(-p^2) + (a_1 + a_2)[a_1 m_1^2 + a_2 m_2^2].$$

1-loop 2-point function: one massive propagator

First we use the Feynman parameter representation

$$\mathcal{I}(p^2; m) = e^{\epsilon\gamma_E}\Gamma(2 - \frac{D}{2}) \int_0^1 da \frac{1}{(a(ap^2 - m^2 - p^2))^{2-\frac{D}{2}}}$$
$$= e^{\epsilon\gamma_E}\Gamma(2 - \frac{D}{2}) \int_0^1 da(a(ap^2 - m^2 - p^2))^{-2+\frac{D}{2}}.$$

Now we massage the integrand into the form of the ${}_2F_1$ integral representation as

$$e^{\epsilon\gamma_E}\Gamma(2 - \frac{D}{2}) \int_0^1 da a^{\frac{D}{2}-2}(m^2 - p^2)^{\frac{D}{2}-2}(1 - (\frac{p^2}{p^2 - m^2})a)^{\frac{D}{2}-2}$$
$$= e^{\epsilon\gamma_E}\Gamma(2 - \frac{D}{2})(m^2 - p^2)^{\frac{D}{2}-2} B(\frac{D}{2} - 1, 1)\ {}_2F_1(\frac{-D}{2} + 2, \frac{D}{2} - 1; \frac{D}{2}; \frac{p^2}{p^2 - m^2})$$
$$= e^{\epsilon\gamma_E}\frac{\Gamma(2 - \frac{D}{2})}{\frac{D}{2} - 1}(m^2 - p^2)^{\frac{D}{2}-2}\ {}_2F_1(\frac{-D}{2} + 2, \frac{D}{2} - 1; \frac{D}{2}; \frac{p^2}{p^2 - m^2}).$$

Then in dimension $D = 2 - 2\epsilon$ and swapping the first 2 interchangeable parameters of the ${}_2F_1$ function,

$$\mathcal{I}(p^2; m) = -e^{\epsilon\gamma_E}\frac{\Gamma(1 + \epsilon)}{\epsilon}(m^2 - p^2)^{-1-\epsilon}\ {}_2F_1(-\epsilon, 1 + \epsilon; 1 - \epsilon; \frac{p^2}{p^2 - m^2}).$$

Now with the Lee-Pomeransky representation:

$$\mathcal{I} = \frac{e^{\epsilon\gamma_E}\Gamma(\frac{D}{2})}{\Gamma(D-2)}\int_0^\infty\int_0^\infty dadb(a+b-abp^2+a^2m^2+abm^2)^{-D/2}$$

$$= \frac{e^{\epsilon\gamma_E}\Gamma(\frac{D}{2})}{\Gamma(D-2)}\int_0^\infty\int_0^\infty dadb(b(1-ap^2+am^2)+a(1+am^2))^{-D/2}$$

$$= -\frac{e^{\epsilon\gamma_E}\Gamma(\frac{D}{2})}{\Gamma(D-2)}(1-\frac{D}{2})^{-1}\int_0^\infty daa^{1-D/2}(1+am^2)^{1-D/2}(1-a(p^2-m^2))^{-1}.$$

Rescale $a \to x/m^2$

$$= -\frac{e^{\epsilon\gamma_E}\Gamma(\frac{D}{2})}{\Gamma(D-2)}(1-\frac{D}{2})^{-1}(\frac{1}{m^2})^{2-D/2}\int_0^\infty dxx^{1-D/2}(1+x)^{1-D/2}(1-x\frac{p^2-m^2}{m^2})^{-1}$$

$$= \frac{e^{\epsilon\gamma_E}\Gamma(\frac{D}{2})}{\Gamma(D-2)}(1-\frac{D}{2})^{-1}(\frac{1}{m^2})^{2-D/2}\frac{m^2}{p^2-m^2}\int_0^\infty dxx^{1-D/2}(1+x)^{1-D/2}(x-\frac{m^2}{p^2-m^2})^{-1}$$

$$= \frac{e^{\epsilon\gamma_E}\Gamma(\frac{D}{2})}{\Gamma(D-2)}(\frac{D}{2}-1)^{-1}(\frac{1}{m^2})^{2-D/2}\frac{m^2}{m^2-p^2}B(D-2,2-\frac{D}{2})\ {}_2F_1(1,D-2;\frac{D}{2};\frac{p^2}{p^2-m^2})$$

$$= \frac{e^{\epsilon\gamma_E}\Gamma(2-\frac{D}{2})}{\frac{D}{2}-1}(m^2)^{-1+D/2}(m^2-p^2)^{-1}\ {}_2F_1(1,D-2;\frac{D}{2};\frac{p^2}{p^2-m^2}).$$

Comparing to the Feynman solution we are nearly there, but we have some unwanted prefactors and the 2 first parameters of the ${}_2F_1$ function are wrong. Here we introduce the identity

$${}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\ {}_2F_1(c-a,c-b;c;z)$$

which allows us to recover precisely the solution given by the Feynman parameter.

1L3P - One massive propagator opposite to one massive external leg

If instead of m_1 we choose m_2 to be the massive propagator then $\mathcal{F}$ is now a function of all three parameters. Now we must choose which parameter to integrate using the delta function. In principle both works out the same but if we use a_2 then we keep the expression degree 1 in each parameter and our integral reads

$$\mathcal{I}(p_1^2;m_2) = e^{\epsilon\gamma_E}\Gamma(1-\epsilon)\int d^2a(-a_3a_1p^2+m^2-a_1m^2-a_3m^2)^{-1-\epsilon}.$$

If we focus on a_1 then we may write the expression inside the brackets as

$$(a_1(-p^2a_3-m^2)+m^2(1-a_3))^{-1-\epsilon}$$

which allows us to do the first integral

$$\int_0^{1-a_3} da_1(a_1f(a_3)+g(a_3))^{-1-\epsilon} = \frac{((1-a_3)f+g)^{-\epsilon}-g^{-\epsilon}}{-\epsilon f}.$$

Overall the integral looks like

$$\mathcal{I}(p_1^2;m_2) = e^{\epsilon\gamma_E}\frac{\Gamma(1-\epsilon)}{\epsilon}\int_0^1 da\frac{((1-a)(-ap^2-m^2)+m^2(1-a))^{-\epsilon}-(m^2(1-a))^{-\epsilon}}{ap^2+m^2}$$

$$= e^{\epsilon\gamma_E} \frac{\Gamma(1-\epsilon)}{\epsilon} \int_0^1 da \frac{(ap^2)^{-\epsilon}(a-1)^{-\epsilon} - (m^2(1-a))^{-\epsilon}}{ap^2+m^2}.$$

We are nearly ready to use the hypergeometric integral representation. We first massage the denominator

$$(ap^2+m^2)^{-1} = (m^2)^{-1}(1 - \frac{-p^2}{m^2}a)^{-1}$$

and use the p^2 in the first term to rewrite

$$(ap^2)^{-\epsilon}(a-1)^{-\epsilon} = a^{-\epsilon}(-p^2)^{-\epsilon}(1-a)^{-\epsilon}$$

so that we may arrive out our final answer

$$\begin{aligned}\mathcal{I}(p_1^2; m_2) = \frac{e^{\epsilon\gamma_E}}{\epsilon} \Bigg[&-(m^2)^{-\epsilon} \frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)}{\Gamma(2-\epsilon)}\ {}_2F_1(1,1;2-\epsilon;\frac{-p^2}{m^2}) \\ &+ m^{-2}(-p^2)^{-\epsilon} \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}\ {}_2F_1(1,1-\epsilon;2-2\epsilon;\frac{-p^2}{m^2}) \Bigg].\end{aligned}$$

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