3.5 The general equation of the second degree

The most general expression, which contains terms involving x and y in a degree not higher than the second, must contain terms x^2, xy, y^2, x, y and a constant. The notation which is in general use for this expression is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c \ldots \ldots \ldots (1)$$

The quantity (1) is known as the general expression of the second degree and when equated to zero is called **the general equation of the second degree.**

The most general expression involving terms only of the second degree in x, y and z is $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy \ldots \ldots \ldots (2)$

where the coefficients occur in the order of the alphabet.

If in this expression (2), we put z = 1, we get $ax^2 + by^2 + c + 2fy + 2gx + 2hxy$, which, after rearrangement, gives $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$.

Now in **Solid Geometry** we use three coordinates. Also many formulae in **Plane Geometry** are derived from those of Solid Geometry by putting z = 1.

We therefore, in Plane Geometry, use the notation corresponding to which we have the standard notation in Solid Geometry.

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3.6 Condition for the general equation of the second degree to represent a pair of straight lines.

Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ... ... (1)

represent a pair of straight lines having the point of intersection (α, β). If we transfer the origin to the point (α, β) keeping the directions of the axes unaltered, then the equation (1) reduces to

$$a(x+\alpha)^2 + 2h(x+\alpha)(y+\beta) + b(y+\beta)^2 + 2g(x+\alpha) + 2f(y+\beta) + c = 0$$

$$\text{or, } ax^2 + 2hxy + by^2 + 2(a\alpha + h\beta + g)x + 2(h\alpha + b\beta + f)y + a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c) = 0 \text{ ... ... (2)}$$

The equation (2) may represent a pair of straight lines if it is reduced to a second degree homogeneous equation in x and y. This is possible only when the coefficents of x and y and the constant term in (2) are separately equal to zero. That is,

$a\alpha + h\beta + g = 0$ ... ... (3)

$h\alpha + b\beta + f = 0$ ... ... (4)

and $a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c = 0$ ... ... (5)

The equation (5) may be written as

$\alpha(a\alpha + h\beta + g) + \beta(h\alpha + b\beta + f) + g\alpha + f\beta + c = 0$ ... ... (6)

Using (3) and (4) in (6), we get

$g\alpha + f\beta + c = 0$ ... ... (7)

Now if we eliminate α and β from (3), (4) and (7), we get the required condition which is

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

or, $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$. which is the required condition for a pair of straight lines.

Alternate Method

The given equation is

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ... ... (1)

If $a \neq 0$, we multiply equation (1) by a and then arrange in powers of x, we have

$a^2x^2 + 2ahxy + aby^2 + 2agx + 2afy + ac = 0$

or, $a^2x^2 + 2ax(hy + g) = -aby^2 - 2afy - ac$

or, $(ax)^2 + 2ax(hy + g) + (hy + g)^2 = (hy + g)^2 - aby^2 - 2afy - ac$

or, $(ax + (hy + g))^2 = y^2(h^2 - ab) + 2y(gh - af) + g^2 - ac$

or, $ax + hy + g = \pm\sqrt{y^2(h^2 - ab) + 2y(gh - af) + g^2 - ac}$ ... ... (2)

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The equation (2) may represent a pair of straight lines if the expresson under the radical sign of the right hand side of the equation (2) is a perfect square.

The condition for this is

$(2(gh - af)^2 = 4(h^2 - ab)(g^2 - ac)$

or, $(gh - af)^2 = (h^2 - ab)(g^2 - ac)$

or, $g^2h^2 - 2afgh + a^2f^2 = g^2h^2 - abg^2 - ach^2 + a^2bc$

or, $abc + 2fgh - 2f^2 - bg^2 - ch^2 = 0$ which is the required condition for a pair of straight lines.

Cor. Point of intersection of the two straight lines represented by the equation

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$

Solving (3) and (4), we have

$$\alpha = \frac{hf - bg}{ab - h^2} = \frac{G}{C} \text{ and } \beta = \frac{gh - af}{ab - h^2} = \frac{F}{C}$$

where C, F, G are respectively the cofactors of c, f, g in the determinant

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

3.7 Angle between two straight lines represented by the equation

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Let $lx + my + n = 0$... ... . (1) and $l'x + m'y + n' = 0$ ... ... (2)

be the two separate straight lines represented by the equation

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ... . (3)

$\therefore ax^2 + 2hxy + by^2 + 2gx + 2fy + c = (lx + my + n)(l'x + m'y + n')$

or, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

$= ll'x^2 + (lm' + l'm)xy + mm'y^2 + (nl' + n'l)x + (mn' + m'n)y + nn'$ ... ... (4)

Equating the coefficients of x^2, xy, y^2, x, y and the constant term from both sides of (4), we get $ll' = a$, $mm' = b$, $nn' = c$, $lm' + l'm = 2h$

$mn' + m'n = 2f$ and $nl' + n'l = 2g$.

Now equations (1) and (2) can be written as $y = -\frac{l}{m}x - \frac{n}{m}$ and $y = -\frac{l'}{m'}x - \frac{n'}{m'}$ respectively.

So slopes of the lines represented by (1) and (2) are $-\frac{l}{m}$ and $-\frac{l'}{m'}$ respectively.

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Example 6. Show that the equation $bx^2 - 2hxy + ay^2 = 0$ represents a pair of straight lines which are at right angles to the pair given by the equation $ax^2 + 2hxy + by^2 = 0$.

[D. U. H. 1976; R. U. H. 1978, 1985]

Proof : Let $y - m_1x = 0$ and $y - m_2x = 0$ be the lines represented by the equation

$ax^2 + 2hxy + by^2 = 0$ ... ... (1)

Then we have $m_1 + m_2 = -\frac{2h}{b}$ and $m_1m_2 = \frac{a}{b}$. Now the equations of the lines perpendicular to the above two lines and passing through the origin are $m_1y + x = 0$ and $m_2y + x = 0$.

Therefore, the equation of the perpendicular lines to the given lines are

$(m_1y + x)(m_2y + x) = 0$

or, $m_1m_2y^2 + (m_1 + m_2)xy + x^2 = 0$

or, $\frac{a}{b}y^2 + \left(\frac{-2h}{b}\right)xy + x^2 = 0$

or, $ay^2 - 2hxy + bx^2 = 0$

or, $bx^2 - 2hxy + ay^2 = 0$ **(Proved)**

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CHAPTER FOUR

THE CIRCLE

4.1 Definition of circle

A **circle** is the locus of a point in a plane which moves so that its distance from a fixed point is always equal to a given distance. The fixed point is called the **centre** and the given distance is called the **radius** of the circle.

or, equivalently

a **circle** is the set of all points in a plane whose distance from a fixed point in the plane is a constant.

4.2 The equations of circles

If the axes are rectangular, then the equation of a circle whose centre is at (h, k) and whose radius is r is clearly $(x-h)^2 + (y-k)^2 = r^2$ ... ... (1)

If the axes are inclined at an angle ω then the corresponding equation of the circle will be $(x-h)^2 + (y-k)^2 + 2(x-h)(y-k)\cos\omega = r^2$ ... ... (2)

Again the equation of a circle having centre at the origin (0, 0) and radius r is

$$x^2 + y^2 = r^2 \text{ ... ... (3)}$$

If the axes are rectangular, then the equation $x^2 + y^2 + 2gx + 2fy + c = 0$ ... ... (4) represents a circle having centre at $(-g, -f)$ and its radius equal to $\sqrt{g^2 + f^2 - c}$.

If $g^2 + f^2 > c$, the radius of this circle is **real**.

If $g^2 + f^2 = c$, the radius vanishes, i.e. the circle becomes a point coinciding with the point $(-g, -f)$. Such a circle is called a **point-circle.**

If $g^2 + f^2 < c$, the radius of the circle is **imaginary.**

Also the equation of the circle which is described on the line joining the points (x_1, y_1) and (x_2, y_2) as diameter is $(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$ ... ... (5)

4.2.1 The polar equation of a circle

The polar equation of a circle is $r_1^2 + r^2 - 2rr_1\cos(\theta - \theta_1) = a^2$ ... ... (6)

where (r_1, θ_1) is the centre of the circle, (r, θ) is any point on the circle and a is the radius of the circle.

Cor. 1 If the pole lies on the circle, then $r_1 = a$ and so the equation of the circle is

$r = 2a\cos(\theta - \theta_1)$ ... ... (7)

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Cor. 2 If $\theta_1 = 0$, then the initial line is a diameter and the pole is the extremity of the diameter. So the equation of the circle is $r = 2a\cos\theta$ ... ... (8)

Cor. 3 If the centre of the circle is origin i.e the pole, then the equation of the circle is

$r = a$ ... ... (9)

4.2.2 The conditions for the general equation of second degree to represent a circle

The conditior s for the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

to represent a circle are $a = b$ and $h = 0$ i.e the coefficients of x^2 and y^2 are equal and there is no term involving the product xy.

4.3 Tangent and normal at a point to a circle.

The **tangent** at any point of a circle is a straight line which meet the circle there, but, being produced, does not cut it; this tangent is always perpendicular to the radius drawn from the centre to the point of contact.

The equation of the tangent at any point (x', y') on the circle

$x^2 + y^2 = a^2$ is $xx' + yy' = a^2$ ... ... (1)

The equation of the tangent at any point (x', y') lying on the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ is

$xx' + yy' + g(x + x') + f(y + y') + c = 0$ ... ... (2)

The **normal** at any point P of a curve is the straight line which passes through P and is perpendicular to the tangent at P.

The equations of the normals at any point (x', y') of the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 + 2gx + 2fy + c = 0$ are respectively $x'y - xy' = 0$ and $y(x' + g) - x(y' + f) + fx' - gy' = 0$.

4.3.1 The rule of finding the length of the tangent to a circle.

We have to express the equation of the given circle in such a way that it's right hand side is zero and the coeffcients of both x^2 and y^2 is equal to 1. Then putting the coordinates of the given point of the left hand side of the equation we get the required length of the tangent to the circle.

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CHAPTER FIVE

CONIC SECTIONS

THE GENERAL EQUATION OF SECOND DEGREE

5.1 Definition of conic section

Conic sections are curves in which a plane cuts a cone; i.e the section of a cone by a plane is called a **conic section** or simply a **conic**. The name conic section is derived from the fact that these curves were first obtained by cutting a cone in various ways.

Alternative form of definition

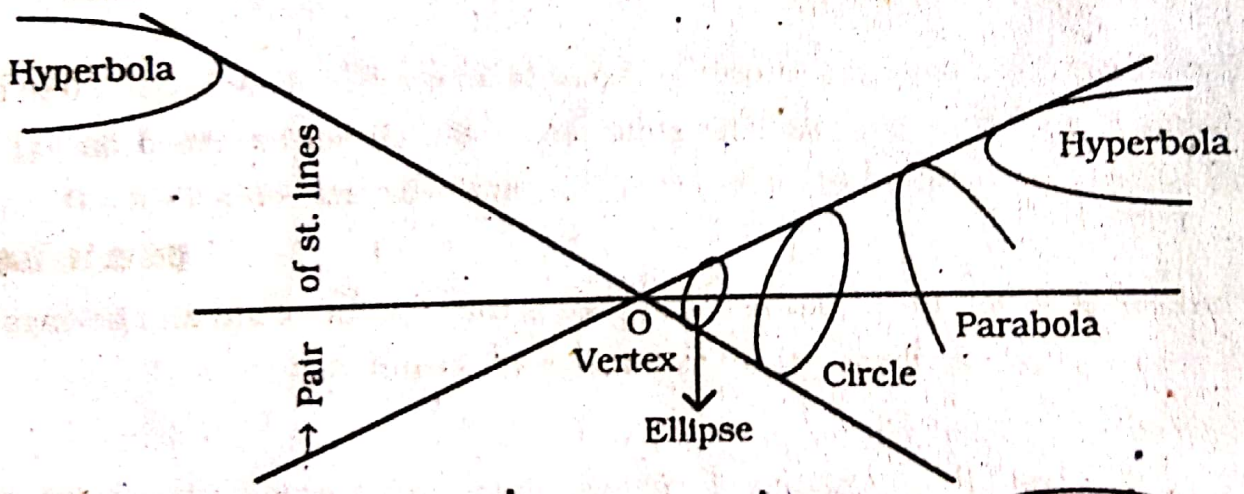
The locus of a point P, which moves in a plane so that its distance from a fixed point is always in a constant ratio to its perpendicular distance from a fixed straight line, is called a **conic section** or simply a **conic.**

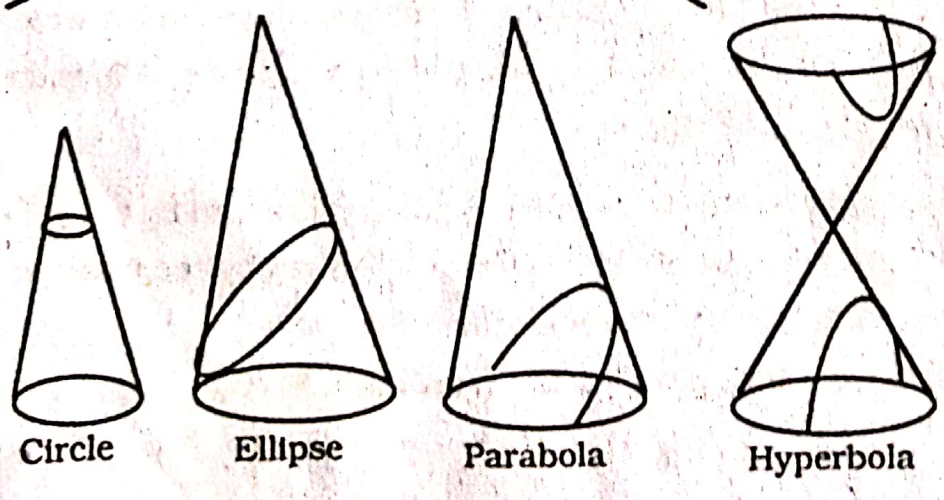
The fixed point is called the **focus** and is usually denoted by S. The constant ratio is called the **eccentricity** and is denoted by e which is a positive number. The fixed straight line is called the **directrix**. The straight line passing through the focus and perpendicular to the directrix is called the **axis.**

When e =1, the conic section is called a **parabola.**

When e < 1, it is called an **ellipse.**

When e > 1, it is called a **hyperbola.**





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If the cutting plane passes through the vertex, the section consists of two straight lines, real, coincident or imaginary.

If the cutting plane does not pass through the vertex of the cone the conic belongs to one of the following four types :

(1) If the cutting plane is parallel to the base, the section is a **circle.**

(2) If the cutting plane is not parallel to the base and cuts entirely accross one nappe of the cone, the conic is an **ellipse.**

(3) If the cutting plane is parallel to the rectilinear element of the cone, the conic is a **parabola.**

(4) If the plane cuts both nappes of the cone, the conic is a **hyperbola.**

Thus a pair of straight lines, the circle, the ellipse, the parabola and the hyperbola all come under the common name of conics.

5.2 The general equation of the second degree

Theorem 1. The general equation of the second degree in x and y reprsents a conic.

Proof : Consider the general equation of the second degree

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ... ... (1)

First suppose that the axes of coordinates are rectangular. Let us keep the origin fixed and rotate the axes through an angle θ such that $\tan 2\theta = \frac{2h}{a-b}$. Then the xy term will be vanished (or absent) in the transformed equation. so our new equation will be (say) $a'x^2 + b'y^2 + 2g'x + 2f'y + c = 0$ ... ... (2)

Then from the theory of invariants, we have

$$a + b = a' + b' \text{ ... ... (3)}$$

$$ab - h^2 = a'b' \text{ ... ... (4)}$$

$$\text{and } \Delta = \Delta'$$

or, $abc + 2fgh - af^2 - bg^2 - ch^2 = a'b'c - a'f'^2 - b'g'^2\}$ ... ... (5)

Case I if $ab - h^2 = 0$ then $a'b' = 0$.

Therefore, either a' or b' is zero. Let $a' = 0$.

Then equation (2) becomes

$b'y^2 + 2g'x + 2f'y + c = 0$ ... ... (6)

or, $b'\left(y^2 + \frac{2f'}{b'}y\right) = -2g'x - c$

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$$\text{or, } y^2 + \frac{2f'}{b'}y = -\frac{2g'}{b'}x - \frac{c}{b'}$$

$$\text{or, } y^2 + 2y\frac{f'}{b'} + \frac{f'^2}{b'^2} = = -\frac{2g'}{b'}x - \frac{c}{b'} + \frac{f'^2}{b'^2}$$

$$\text{or, } \left(y + \frac{f'}{b'}\right)^2 = -\frac{2g'}{b'}x - \frac{c}{b'} + \frac{f'^2}{b'^2} \ldots\ldots\ldots (7)$$

which represents the equation of a parabola having its axis parallel to the x-axis. If $g' = 0$, (7) represents a pair of straight lines (real or imaginary).

It may be noted here that

$\Delta = \Delta' = -b'g'^2 \neq 0.$

Similarly, if $b' = 0$, then equation (2) becomes $a'x^2 + 2g'x + 2f'y + c = 0 \ldots\ldots\ldots (8)$

which represents a parabola having its axis parallel to the y-axis.

If $f' = 0$, (8) represents a pair of straight lines (real or imaginary).

Case II If $ab - h^2 \neq 0$, then $a'b' \neq 0$.

Therefore, neither a' nor b' is zero.

Then equation (2) becomes

$$a'\left(x^2 + \frac{2g'}{a'}x\right) + b'\left(y^2 + \frac{2f'}{b'}y\right) + c = 0$$

$$\text{or, } a'\left(x^2 + 2.x.\frac{g'}{a'} + \frac{g'^2}{a'^2}\right) + b'\left(y^2 + 2y.\frac{f'}{b'} + \frac{f'^2}{b'^2}\right) = \frac{g'^2}{a'} + \frac{f'^2}{b'} - c$$

$$\text{or, } a'\left(x + \frac{g'}{a'}\right)^2 + b'\left(y + \frac{f'}{b'}\right)^2 = \frac{g'^2}{a'} + \frac{f'^2}{b'} - c \ldots\ldots\ldots (9)$$

Now transferring the origin to $\left(-\frac{g'}{a'}, -\frac{f'}{b'}\right)$; the equation (9) reduces to

$$a'x^2 + b'y^2 = \frac{g'^2}{a'} + \frac{f'^2}{b'} - c \ldots\ldots\ldots (10)$$

which represents (i) an **ellipse** if a' and b' have the same sign and (ii) a hyperbola if a' and b' are of opposite sign and in particular if $a' + b' = 0$, the hyperbola will be a rectangular one.

Now a' and b' have the same or opposite sign according as $a'b'$ is positive or negative, that is, according as $ab - h^2$ is positive or negative.

Thus if $ab - h^2$ is positive (1) will represent an **ellipse**, but if $ab - h^2$ is negative (1) will represent a hyperbola which will be rectangular if $a' + b' = 0$, that is, if $a + b = 0$.

For equation (8) to represent an ellipse or a hyperbola $\frac{g'^2}{a'} + \frac{f'^2}{b'} - c \neq 0$.

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Hence $\Delta = \Delta' = a'b'c - a'f'^2 - b'g'^2$

$= -a'b'\left(\frac{g'^2}{a'} + \frac{f'^2}{b'} - c\right) \neq 0.$

Further, if (8) represents a real ellipse, a', b' and $\left(\frac{g'^2}{a'} + \frac{f'^2}{b'} - c\right)$ must have the same sign. Therefore, in this case, we have (i) $\Delta < 0$ if $a' + b' = a + b > 0$,

and (ii) $\Delta > 0$ if $a' + b' = a + b < 0$.

Thus the general equation of the second degree represents a real ellipse if $ab - h^2 > 0$ and (ii) $a + b$ and Δ are of opposite signs. The ellipse is imaginary if $a + b$ and Δ are of the same sign.

Hence the general equation of the second degree represents

(i) a parabola, if $ab - h^2 = 0$.

(ii) an ellipse, if $ab - h^2 > 0$

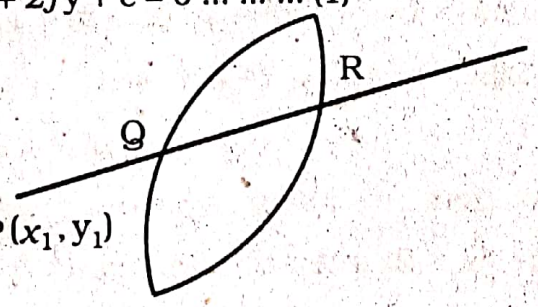
(iii) a hyperbola, if $ab - h^2 < 0$.

(iv) a rectangular hyperbola, if $ab - h^2 < 0$, and $a + b = 0$.

5.3 Intersection of a conic with a line through a given point

Let $P(x_1, y_1)$ be the given point and the equation of the given conic be

$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ... ... (1)



Let the line through (x_1, y_1) make an angle θ with the x-axis. Then the equation of the line is $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$ ... ... (2)

or, $\frac{x - x_1}{l} = \frac{y - y_1}{m} = r$ ... ... (3)

where $l = \cos\theta$, $m = \sin\theta$, $l^2 + m^2 = 1$ and r is the algebraic distance of the point (x, y) on the line from the point (x_1, y_1).

$\therefore x = x_1 + lr$ and $y = y_1 + mr$ ... ... (4)

Therefore, the points of intersection of (1) and (4) are given by

$a(x_1 + lr)^2 + 2h(x_1 + lr)((y_1 + mr) + b(y_1 + mr)^2 + 2g(x_1 + lr) + 2f(y_1 + mr) + c = 0$

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Example 3. Reduce the conic $8x^2 + 4xy + 5y^2 - 24x - 24y = 0$ to its standard form and find its all properties. Also ketch the rough graph of the conic.

[C. U. H. 1980; N. U. H. 1995, 2003]

Solution : The given equation of the conic is

$S \equiv 8x^2 + 4xy + 5y^2 - 24x - 24y = 0 \ldots \ldots \ldots (1)$

Comparing the equation (1) with the second degree general equation

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get

$a = 8, h = 2, b = 5, g = -12, f = -12, c = 0.$

$\therefore \Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 8.5.0 + 2.(-12).(-12).2 - 8.(-12)^2 - 5.(-12)^2 - 0.2^2$

$= 0 + 576 - 1152 - 720 - 0 = -1296 \neq 0$

and $ab - h^2 = 40 - 4 = 36 > 0.$

Hence the given equation (1) represents an **ellipse.**

Again $\dfrac{\partial S}{\partial x} = 16x + 4y - 24 = 0 \Rightarrow 4x + y - 6 = 0 \ldots \ldots \ldots (2)$

$\dfrac{\partial S}{\partial y} = 4x + 10y - 24 = 0 \Rightarrow 2x + 5y - 12 = 0 \ldots \ldots \ldots (3)$

Solving (2) and (3), we get the coordinates of the centre of the conic which are $(1, 2)$.

Shifting the origin to (1, 2) and remaining the direction of the axes unaltered, the equation (1) takes the from

$8x^2 + 4xy + 5y^2 + c' = 0 \ldots \ldots \ldots (4)$

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where $c' = gx_1 + fy_1 + c = (-12).1 + (-12).2 + 0 = -36$

Putting $c' = -36$ in (4), we get

$8x^2 + 4xy + 5y^2 = 36$ ... ... (5)

When the xy term is removed from the equation (5) by the rotation of axes, let the reduced equation be $a_1x^2 + b_1y^2 = 36$ ... ... (6)

The by the invariant properties, we have

$a_1 + b_1 = a + b = 8 + 5 = 13$

and $a_1b_1 = ab - h^2 = 40 - 4 = 36.$

$b_1 = 13 - a_1$

$a_1b_1 = a_1(13 - a_1) = 36$

or, $13a_1 - a_1^2 = 36$

or, $a_1^2 - 13a_1 + 36 = 0$

or, $(a_1 - 4)(a_1 - 9) = 0$

$\therefore \left.\begin{matrix} a_1 = 4 \\ b_1 = 9 \end{matrix}\right\} \left.\begin{matrix} a_1 = 9 \\ b_1 = 4 \end{matrix}\right\}$

Since c' is negative, so we choose $a_1 < b_1$.

$\therefore a_1 = 4, \quad b_1 = 9.$

Putting the values of a_1 and b_1 in (6), we get

$4x^2 + 9y^2 = 36$

or, $\frac{x^2}{9} + \frac{y^2}{4} = 1$

or, $\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1.$

which is the **standard form** of an ellipse.

(i) Length of the major axis = 2.3 = 6

(ii) Length of the minor axis = 2.2 = 4

Alternative method of finding the semi-axes of the conic

Writing the equation (5) in the form

$\frac{8}{36}x^2 + \frac{4}{36}xy + \frac{5}{36}y^2 = 1$

or, $\frac{2}{9}x^2 + \frac{1}{9}xy + \frac{5}{36}y^2 = 1$ ... ... (*)

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Now by formula the semi-axes of the conic are given by $\left(a-\frac{1}{r^2}\right)\left(b-\frac{1}{r^2}\right)=h^2$

$\therefore \left(\frac{2}{9}-\frac{1}{r^2}\right)\left(\frac{5}{36}-\frac{1}{r^2}\right)=\left(\frac{1}{18}\right)^2$

or, $\frac{10}{324}-\frac{2}{9r^2}-\frac{5}{36r^2}+\frac{1}{r^4}=\frac{1}{324}$

or, $\frac{1}{36}-\frac{13}{36r^2}+\frac{1}{r^4}=0$

or, $r^4-13r^2+36=0$

or, $(r^2-9)(r^2-4)=0$

$\therefore \left.\begin{array}{l} r^2=9 \Rightarrow r=3 \\ r^2=4 \Rightarrow r=2 \end{array}\right\}$ (Length is positive)

i.e $r_1=3$ and $r_2=2$.

Length of its major axis $=2r_1=2.3=5$

Length of its minor axis $=2r_2=2.2=4$.

(iii) The equation of its major axis referred to the centre (1, 2) as origin (new axes) is

$\left(a-\frac{1}{r_1^2}\right)x+hy=0$

i.e $\left(\frac{2}{9}-\frac{1}{9}\right)x+\frac{1}{18}y=0$

or, $\frac{1}{9}x+\frac{1}{18}y=0$

or, $2x+y=0$

Again the equation of its major axis referred to the **old axes** is

$2(x-1)+y-2=0$

or, $2x-2+y-2=0$

or, $2x+y-4=0$

(iv) The equation of its minor axis referred to the centre (1, 2) as origin (new axes) is

$\left(a-\frac{1}{r_2^2}\right)x+hy=0$

i.e $\left(\frac{2}{9}-\frac{1}{4}\right)x+\frac{1}{18}y=0$

or, $(8-9)x+2y=0$

or, $-x+2y=0$

or, $x-2y=0$

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[C. U. H. 1983; R. U. H. 1983, 1985]

12.13 Shortest distance between two lines

Definition : Skew lines

By skew lines, we mean lines which do not intersect or equivalenty the lines which do not lie in the same plane.

Defintion : Shortest distance between two skew lines

The straight line which is perpendicular to each of the two skew lines (or non-intersecting lines) is called the **line of shortest distance.** The length of the shortest distance line intercepted between the skew lines is called the **length of the shortest distance or** simply the **shortest distance.** The shortest distance is briefly wirtten as S. D.

.To show that the length of the line intercepted between the two lines which is perpendicular to both is the shortest distance between them.

Let AB and CD be the two given non-intersecting lines. A line is completely determined if it intersects two lines at right angles. Thus there is one and only one line which intersects the two given lines at right angles, say, at G and H.

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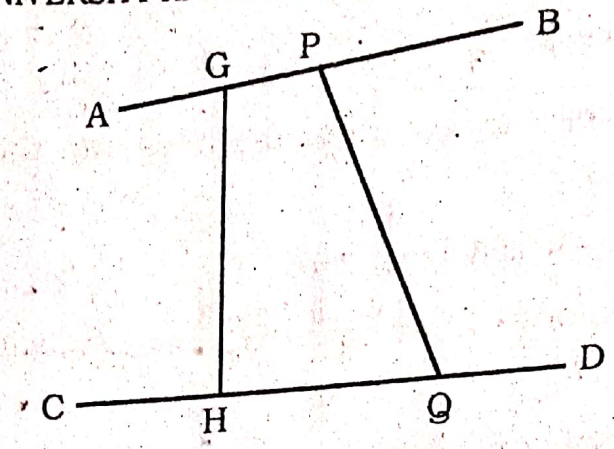

Let P and Q be any two points, one on each of the two given lines.

Then GH = projection of PQ on the line GH = PQcosθ,

where θ is the angle between PQ and GH.

Since cosθ < 1, PQ is always greater than GH.

Thus GH being always less than any distance between two lines is the shortest distance.

12.13.1 Magnitude and equations of S. D.

To find the magnitude and equations of the shortest distance between two lines whose equations are given.

First Method (Method of projection)

Let the equations of the two given lines be $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ ... ... (1)

and $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ ... ... (2)

where (x_1, y_1, z_1) is a point on (1) and (x_2, y_2, z_2) is a point on (2).

Let l, m, n be the direction cosines of the shortest distance which is perpendicular to both the lines (1) and (2).

Then we have $ll_1 + mm_1 + nn_1 = 0$

and $ll_2 + mm_2 + nn_2 = 0$

Therefore, by cross-multiplication, we get

$$\frac{l}{m_1n_2 - m_2n_1}=\frac{m}{n_1l_2 - n_2l_1}=\frac{n}{l_1m_2 - l_2m_1}$$

$$=\frac{\sqrt{(l^2 + m^2 + n^2)}}{\sqrt{\{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2\}}}$$

$$=\frac{1}{\sqrt{\Sigma(m_1n_2 - m_2n_1)^2}}$$

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$$\therefore l = \frac{m_1n_2 - m_2n_1}{\sqrt{\Sigma(m_1n_2 - m_2n_1)^2}}, \quad m = \frac{n_1l_2 - n_2l_1}{\sqrt{\Sigma(m_1n_2 - m_2n_1)^2}}$$

$$\text{and } n = \frac{l_1m_2 - l_2m_1}{\sqrt{\Sigma(m_1n_2 - m_2n_1)^2}}.$$

The line of the shortest distance is perpendicular to both the lines. Therefore, the magnitude of the shortest distance is the projection on the line of the shortest distance of the line joining any two points, one on each of the given lines (1) and (2).

Thus S. D = Projection of the join of (x_1, y_1, z_1) and (x_2, y_2, z_2) on the shortest distance whose direction cosines are l, m, n.

$$= (x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n$$

$$= \frac{(x_2 - x_1)(m_1n_2 - m_2n_1) + (y_2 - y_1)(n_1l_2 - n_2l_1) + (z_2 - z_1)(l_1m_2 - l_2m_1)}{\sqrt{\Sigma(m_1n_2 - m_2n_1)^2}}$$

$$= \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \div \sqrt{(m_1n_2 - m_2n_1)^2}$$

Equation of the shortest distance

The equation of the plane containing the line (1) and the line of the shortest distance whose direction cosines are l, m, n is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 \quad \ldots\ldots\ldots (3)$$

Similarly, the equation of the plane containing the line (2) and the shortest distance is

$$\begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix} = 0 \quad \ldots\ldots\ldots (4)$$

These two planes (3) and (4) taken together represent the equation of the line of the shortest distance.

Note. Two lines will intersect, i.e., these will be coplanar if the shortest distance between them is zero. Thus the condition for two lines to be coplanar is

$$\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

Second Method (Method of intersection of S. D)

Let the equations of the two given lines be

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1} \quad \ldots\ldots\ldots (1)$$

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and $\dfrac{x - x_2}{l_2} = \dfrac{y - y_2}{m_2} = \dfrac{z - z_2}{n_2}$ ... ... (2)

Then any point on (1) is $(l_1r_1 + x_1, m_1r_1 + y_1, n_1r_1 + z_1)$

and any point on (2) is $(l_2r_2 + x_2, m_2r_2 + y_2, n_2r_2 + z_2)$.

Let us suppose that the line joining the above two points is the shortest distance.

Then the line joining these points is perpendicular to both (1) and (2). Direction cosines of this line are proportional to

$(l_1r_1 + x_1 - l_2r_2 - x_2, m_1r_1 + y_1 - m_2r_2 - y_2, n_1r_1 + z_1 - n_2r_2 - z_2)$

Since this is perpendicular to both (1) and (2)

$\therefore\ l_1 (l_1r_1 + x_1 - l_2r_2 - x_2) + m_1 (m_1r_1 + y_1 - m_2r_2 - y_2) + n_1 (n_1r_1 + z_1 - n_2r_2 - z_2) = 0$

and $l_2 (l_1r_1 + x_1 - l_2r_2 - x_2) + m_2 (m_1r_1 + y_1 - m_2r_2 - y_2) + n_2(n_1r_1 + z_1 - n_2r_2 - z_2) = 0.$

From the above two equations, we determine the values of r_1 and r_2 and putting these values in points, we get the two points at which S. D meets the two lines.

Thus S. D = distance between these two points and equations of S. D. are the equations of the line joining these points.

Third Method

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Example 26. Find the S. D and the equations of the S. D line between the two lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$.

[D. U. S. 1989, '90; N. U. S. 1995, '96; R. U. S. 1995; C. U. S. 1977, '82]

Solution : (By the method of projection)

Let $[l, m, n]$ be the direction cosines of the S. D, then since S. D is perpendicular to both the lines, we have

$$\left.\begin{aligned} 2l + 3m + 4n &= 0 \\ \text{and } 3l + 4m + 5n &= 0 \end{aligned}\right\}$$

$$\therefore \frac{l}{15-16}=\frac{m}{12-10}=\frac{n}{8-9}$$

$$\text{or, } \frac{l}{1}=\frac{m}{-2}=\frac{n}{1}=\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{1^2+(-2)^2+1^2}}=\frac{1}{\sqrt{6}}$$

Hence the direction cosines of S. D are $\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}$.

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A point on the first line is say P(1, 2, 3) and a point on the second line is say Q(2, 4, 5).

Now length of S. D = Projection of PQ on the line S. D

$$\therefore \text{S. D} = (2-1)\frac{1}{\sqrt{6}} + (4-2).\frac{-2}{\sqrt{6}} + (5-3)\frac{1}{\sqrt{6}}$$

$$= \frac{1}{\sqrt{6}} - \frac{4}{\sqrt{6}} + \frac{2}{\sqrt{6}} = -\frac{1}{\sqrt{6}} = \frac{1}{\sqrt{6}} \text{ (magnitude)}$$

Equations of the shortest distance

The line of S. D. is botained by the intersection of

(i) a plane containing first line and parallel to S. D

(ii) a plane containing second line and parallel to S. D

The equation of the plane containg first line and S. D. is

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 1 & -2 & 1 \end{vmatrix} = 0, \text{ i.e } 11x + 2y - 7z + 6 = 0 \ldots\ldots\ldots (1)$$

and the equation of the plane containing second line and S. D. is

$$\begin{vmatrix} x-2 & y-4 & z-5 \\ 2 & 4 & 5 \\ 1 & -2 & 1 \end{vmatrix} = 0, \text{ i.e, } 7x + y - 5z + 7 = 0 \ldots\ldots\ldots (2)$$

Now (1) and (2) together give equations of S. D.

Example 27. Find the length of the S. D. between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$.

Find also its equations and points where it intersects the lines.

[D. U. H. 1979, '91; C. U. H. 1987]

Solution : (By the method of intersection of S. D)

Let the S. D. meet the given lines in points P and Q where P is $(3r_1 + 3, -r_1 + 8, r_1 + 3)$ and Q is $(-3r_2 - 3, 2r_2 - 7, 4r_2 + 6)$.

Then direction cosines of PQ are proportional to

$[3r_1 + 3 - (-3r_2 - 3), -r_1 + 8 - (2r_2 - 7), r_1 + 3 - (4r_2 + 6)]$

i.e $[3r_1 + 3r_2 + 6, -r_1 - 2r_2 + 15, r_1 - 4r_2 - 3] \ldots\ldots\ldots (1)$

Since PQ is perpendicular to both the lines, we have

$3(3r_1 + 3r_2 + 6) - 1(-r_1 - 2r_2 + 15) + 1(r_1 - 4r_2 - 3) = 0$

or, $9r_1 + 9r_2 + 18 + r_1 + 2r_2 - 15 + r_1 - 4r_2 - 3 = 0$

or, $11r_1 + 7r_2 = 0 \ldots\ldots\ldots (2)$

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and $-3(3r_1 + 3r_2 + 6) + 2(-r_1 - 2r_2 + 15) + 4(r_1 - 4r_2 - 3) = 0$

or, $-9r_1 - 9r_2 - 18 - 2r_1 - 4r_2 + 30 + 4r_1 - 16r_2 - 12 = 0$

or, $7r_1 + 29r_2 = 0$ ... ... (3)

Solving (2) and (3), we get $r_1 = r_2 = 0$.

Therefore, points P and Q are (3, 8, 3) and (–3, – 7, 6) respectively.

Also S. D. $= \sqrt{[(3+3)^2 + (8+7)^2 + (3-6)^2]}$

$= \sqrt{6^2 + (15)^2 + (-3)^2} = \sqrt{270} = 3\sqrt{30}$.

From (1) direction cosines of S. D are proportional to [6, 15, – 3] i.e [2, 5,–1].

Hence the equations of the line S. D (i.e the line PQ)

are $\dfrac{x-3}{2} = \dfrac{y-8}{5} = \dfrac{z-3}{-1}$.

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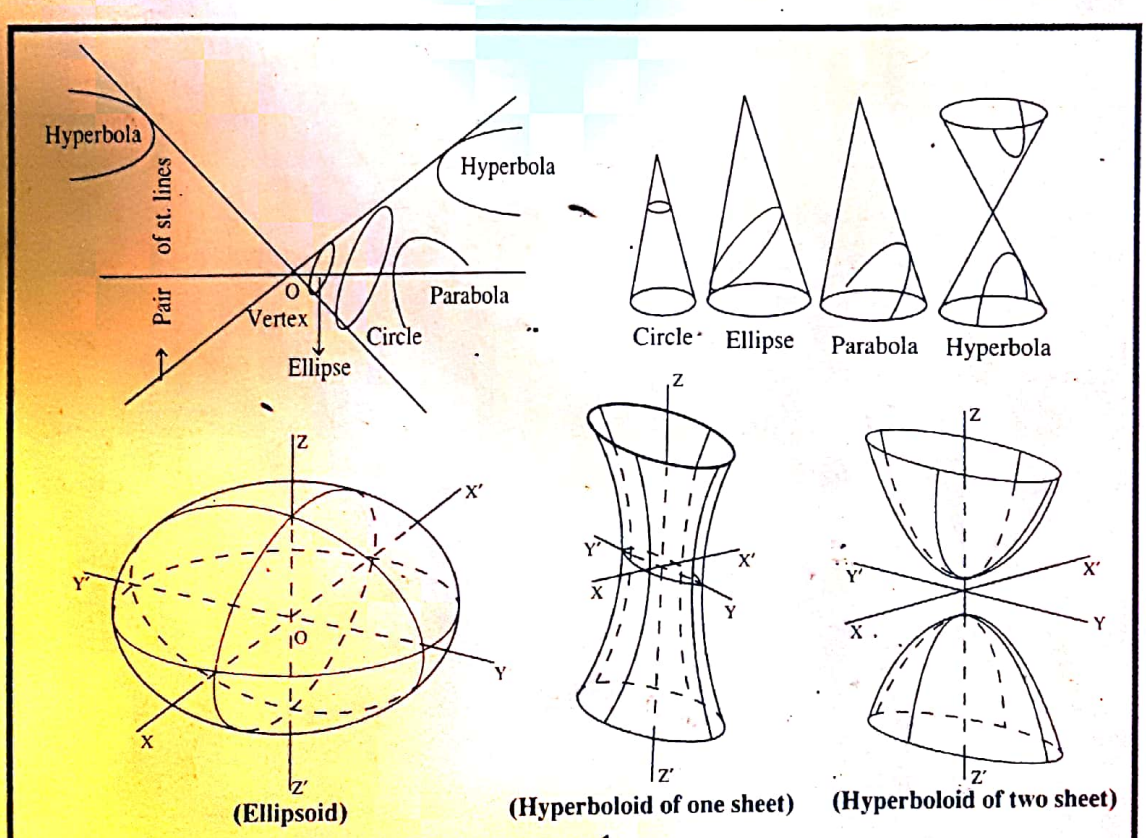
ANALYTIC AND VECTOR GEOMETRY

(TWO AND THREE DIMENSIONS)

COORDINATES & VECTORS

PROFESSOR MD. ABDUR RAHMAN



Vector area of $\Delta ABC = \frac{1}{2}(\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a})$

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