

7-4 Work

If you accelerate an object to a greater speed by applying a force to the object, you increase the kinetic energy $K (= \frac{1}{2}mv^2)$ of the object. Similarly, if you decelerate the object to a lesser speed by applying a force, you decrease the kinetic energy of the object. We account for these changes in kinetic energy by saying that your force has transferred energy *to* the object from yourself or *from* the object to yourself. In such a transfer of energy via a force, **work W** is said to be *done on the object by the force*. More formally, we define work as follows:

Work W is energy transferred to or from an object by means of a force acting on the object. Energy transferred to the object is positive work, and energy transferred from the object is negative work.

“Work,” then, is transferred energy; “doing work” is the act of transferring the energy. Work has the same units as energy and is a scalar quantity.

The term *transfer* can be misleading. It does not mean that anything material flows into or out of the object; that is, the transfer is not like a flow of water. Rather, it is like the electronic transfer of money between two bank accounts: The number in one account goes up while the number in the other account goes down, with nothing material passing between the two accounts.

Note that we are not concerned here with the common meaning of the word “work,” which implies that *any* physical or mental labor is work. For example, if you push hard against a wall, you tire because of the continuously repeated muscle contractions that are required, and you are, in the common sense, working. However, such effort does not cause an energy transfer to or from the wall and thus is not work done on the wall as defined here.

To avoid confusion in this chapter, we shall use the symbol W only for work and shall represent a weight with its equivalent mg .

7-5 Work and Kinetic Energy

Finding an Expression for Work

Let us find an expression for work by considering a bead that can slide along a frictionless wire that is stretched along a horizontal x axis (Fig. 7-2). A constant force \vec{F} , directed at an angle ϕ to the wire, accelerates the bead along the wire. We can relate the force and the acceleration with Newton’s second law, written for components along the x axis:

$$F_x = ma_x, \quad (7-3)$$

where m is the bead’s mass. As the bead moves through a displacement \vec{d} , the force changes the bead’s velocity from an initial value \vec{v}_0 to some other value \vec{v} . Because the force is constant, we know that the acceleration is also constant. Thus, we can use Eq. 2-16 to write, for components along the x axis,

$$v^2 = v_0^2 + 2a_x d. \quad (7-4)$$

Solving this equation for a_x , substituting into Eq. 7-3, and rearranging then give us

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = F_x d. \quad (7-5)$$

The first term on the left side of the equation is the kinetic energy K_f of the bead at the end of the displacement d , and the second term is the kinetic energy K_i of the bead at the start of the displacement. Thus, the left side of Eq. 7-5 tells us the kinetic energy has been changed by the force, and the right side tells us the change is equal to $F_x d$. Therefore, the work W done on the bead by the force

(the energy transfer due to the force) is

$$W = F_x d. \quad (7-6)$$

If we know values for F_x and d , we can use this equation to calculate the work W done on the bead by the force.

To calculate the work a force does on an object as the object moves through some displacement, we use only the force component along the object's displacement. The force component perpendicular to the displacement does zero work.

From Fig. 7-2, we see that we can write F_x as $F \cos \phi$, where ϕ is the angle between the directions of the displacement \vec{d} and the force \vec{F} . Thus,

$$W = F d \cos \phi \quad (\text{work done by a constant force}). \quad (7-7)$$

Because the right side of this equation is equivalent to the scalar (dot) product $\vec{F} \cdot \vec{d}$, we can also write

$$W = \vec{F} \cdot \vec{d} \quad (\text{work done by a constant force}), \quad (7-8)$$

where F is the magnitude of \vec{F} . (You may wish to review the discussion of scalar products in Section 3-8.) Equation 7-8 is especially useful for calculating the work when \vec{F} and \vec{d} are given in unit-vector notation.

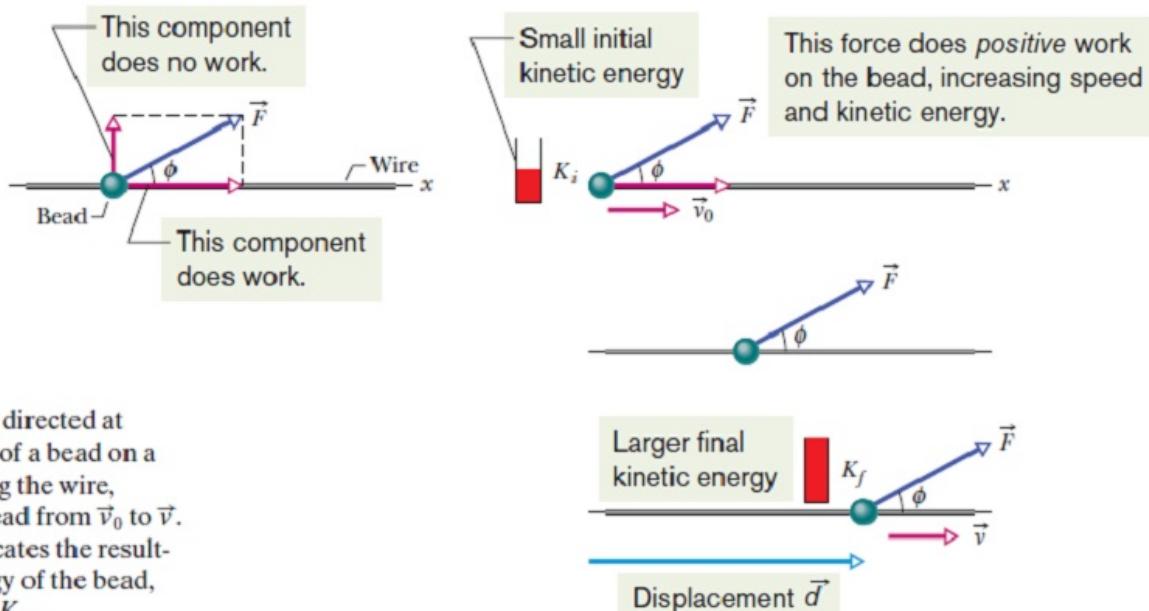


Fig. 7-2 A constant force \vec{F} directed at angle ϕ to the displacement \vec{d} of a bead on a wire accelerates the bead along the wire, changing the velocity of the bead from \vec{v}_0 to \vec{v} . A “kinetic energy gauge” indicates the resulting change in the kinetic energy of the bead, from the value K_i to the value K_f .

Units for work. Work has the SI unit of the joule, the same as kinetic energy. However, from Eqs. 7-6 and 7-7 we can see that an equivalent unit is the newton-meter ($\text{N} \cdot \text{m}$). The corresponding unit in the British system is the foot-pound ($\text{ft} \cdot \text{lb}$). Extending Eq. 7-2, we have

$$1 \text{ J} = 1 \text{ kg} \cdot \text{m}^2/\text{s}^2 = 1 \text{ N} \cdot \text{m} = 0.738 \text{ ft} \cdot \text{lb}. \quad (7-9)$$

Net work done by several forces. When two or more forces act on an object, the net work done on the object is the sum of the works done by the individual forces. We can calculate the net work in two ways. (1) We can find the work done by each force and then sum those works. (2) Alternatively, we can first find the net force \vec{F}_{net} of those forces. Then we can use Eq. 7-7, substituting the magnitude F_{net} for F and also the angle between the directions of \vec{F}_{net} and \vec{d} for ϕ . Similarly, we can use Eq. 7-8 with \vec{F}_{net} substituted for \vec{F} .

Work-Kinetic Energy Theorem

Equation 7-5 relates the change in kinetic energy of the bead (from an initial $K_i = \frac{1}{2}mv_0^2$ to a later $K_f = \frac{1}{2}mv^2$) to the work $W (= F_x d)$ done on the bead. For such particle-like objects, we can generalize that equation. Let ΔK be the change in the kinetic energy of the object, and let W be the net work done on it. Then

$$\Delta K = K_f - K_i = W, \quad (7-10)$$

which says that

$$\left(\begin{array}{l} \text{change in the kinetic} \\ \text{energy of a particle} \end{array} \right) = \left(\begin{array}{l} \text{net work done on} \\ \text{the particle} \end{array} \right).$$

We can also write

$$K_f = K_i + W, \quad (7-11)$$

which says that

$$\left(\begin{array}{l} \text{kinetic energy after} \\ \text{the net work is done} \end{array} \right) = \left(\begin{array}{l} \text{kinetic energy} \\ \text{before the net work} \end{array} \right) + \left(\begin{array}{l} \text{the net} \\ \text{work done} \end{array} \right).$$

These statements are known traditionally as the **work-kinetic energy theorem** for particles. They hold for both positive and negative work: If the net work done on a particle is positive, then the particle's kinetic energy increases by the amount of the work. If the net work done is negative, then the particle's kinetic energy decreases by the amount of the work.

For example, if the kinetic energy of a particle is initially 5 J and there is a net transfer of 2 J to the particle (positive net work), the final kinetic energy is 7 J. If, instead, there is a net transfer of 2 J from the particle (negative net work), the final kinetic energy is 3 J.

7-6 Work Done by the Gravitational Force

We next examine the work done on an object by the gravitational force acting on it. Figure 7-6 shows a particle-like tomato of mass m that is thrown upward with initial speed v_0 and thus with initial kinetic energy $K_i = \frac{1}{2}mv_0^2$. As the tomato rises, it is slowed by a gravitational force \vec{F}_g ; that is, the tomato's kinetic energy decreases because \vec{F}_g does work on the tomato as it rises. Because we can treat the tomato as a particle, we can use Eq. 7-7 ($W = Fd \cos \phi$) to express the work done during a displacement \vec{d} . For the force magnitude F , we use mg as the magnitude of \vec{F}_g . Thus, the work W_g done by the gravitational force \vec{F}_g is

$$W_g = mgd \cos \phi \quad (\text{work done by gravitational force}). \quad (7-12)$$

For a rising object, force \vec{F}_g is directed opposite the displacement \vec{d} , as indicated in Fig. 7-6. Thus, $\phi = 180^\circ$ and

$$W_g = mgd \cos 180^\circ = mgd(-1) = -mgd. \quad (7-13)$$

The minus sign tells us that during the object's rise, the gravitational force acting on the object transfers energy in the amount mgd from the kinetic energy of the object. This is consistent with the slowing of the object as it rises.

After the object has reached its maximum height and is falling back down, the angle ϕ between force \vec{F}_g and displacement \vec{d} is zero. Thus,

$$W_g = mgd \cos 0^\circ = mgd(+1) = +mgd. \quad (7-14)$$

The plus sign tells us that the gravitational force now transfers energy in the amount mgd to the kinetic energy of the object. This is consistent with the speeding up of the object as it falls.

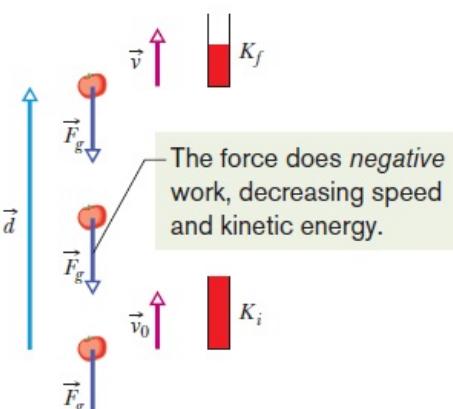


Fig. 7-6 Because the gravitational force \vec{F}_g acts on it, a particle-like tomato of mass m thrown upward slows from velocity v_0 to velocity v during displacement \vec{d} . A kinetic energy gauge indicates the resulting change in the kinetic energy of the tomato, from $K_i (= \frac{1}{2}mv_0^2)$ to $K_f (= \frac{1}{2}mv^2)$.

Work Done in Lifting and Lowering an Object

Now suppose we lift a particle-like object by applying a vertical force \vec{F} to it. During the upward displacement, our applied force does positive work W_a on the object while the gravitational force does negative work W_g on it. Our applied force tends to transfer energy to the object while the gravitational force tends to transfer energy from it. By Eq. 7-10, the change ΔK in the kinetic energy of the object due to these two energy transfers is

$$\Delta K = K_f - K_i = W_a + W_g, \quad (7-15)$$

in which K_f is the kinetic energy at the end of the displacement and K_i is that at the start of the displacement. This equation also applies if we lower the object, but then the gravitational force tends to transfer energy *to* the object while our force tends to transfer energy *from* it.

In one common situation, the object is stationary before and after the lift—for example, when you lift a book from the floor to a shelf. Then K_f and K_i are both zero, and Eq. 7-15 reduces to

$$W_a + W_g = 0$$

or $W_a = -W_g \quad (7-16)$

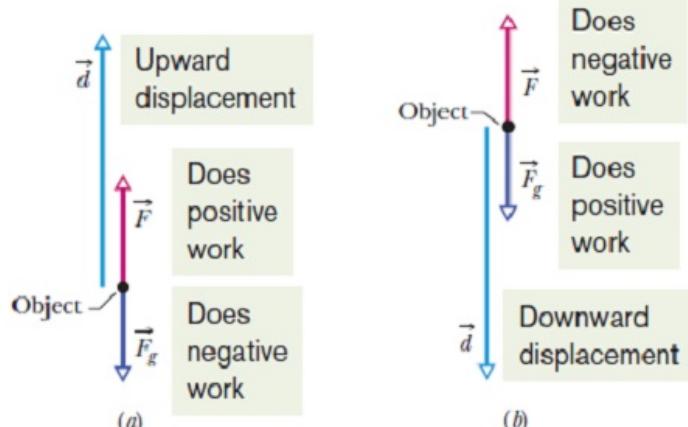
Note that we get the same result if K_f and K_i are not zero but are still equal. Either way, the result means that the work done by the applied force is the negative of the work done by the gravitational force; that is, the applied force transfers the same amount of energy to the object as the gravitational force transfers from the object. Using Eq. 7-12, we can rewrite Eq. 7-16 as

$$W_a = -mgd \cos \phi \quad (\text{work done in lifting and lowering; } K_f = K_i), \quad (7-17)$$

with ϕ being the angle between \vec{F}_g and \vec{d} . If the displacement is vertically upward (Fig. 7-7a), then $\phi = 180^\circ$ and the work done by the applied force equals mgd . If the displacement is vertically downward (Fig. 7-7b), then $\phi = 0^\circ$ and the work done by the applied force equals $-mgd$.

Equations 7-16 and 7-17 apply to any situation in which an object is lifted or lowered, with the object stationary before and after the lift. They are independent of the magnitude of the force used. For example, if you lift a mug from the floor to over your head, your force on the mug varies considerably during the lift. Still, because the mug is stationary before and after the lift, the work your force does on the mug is given by Eqs. 7-16 and 7-17, where, in Eq. 7-17, mg is the weight of the mug and d is the distance you lift it.

Fig. 7-7 (a) An applied force \vec{F} lifts an object. The object's displacement \vec{d} makes an angle $\phi = 180^\circ$ with the gravitational force \vec{F}_g on the object. The applied force does positive work on the object. (b) An applied force \vec{F} lowers an object. The displacement \vec{d} of the object makes an angle $\phi = 0^\circ$ with the gravitational force \vec{F}_g . The applied force does negative work on the object.



7-7 Work Done by a Spring Force

We next want to examine the work done on a particle-like object by a particular type of *variable force*—namely, a **spring force**, the force from a spring. Many forces in nature have the same mathematical form as the spring force. Thus, by examining this one force, you can gain an understanding of many others.

The Spring Force

Figure 7-9a shows a spring in its **relaxed state**—that is, neither compressed nor extended. One end is fixed, and a particle-like object—a block, say—is attached to the other, free end. If we stretch the spring by pulling the block to the right as in Fig. 7-9b, the spring pulls on the block toward the left. (Because a spring force acts to restore the relaxed state, it is sometimes said to be a *restoring force*.) If we compress the spring by pushing the block to the left as in Fig. 7-9c, the spring now pushes on the block toward the right.

To a good approximation for many springs, the force \vec{F}_s from a spring is proportional to the displacement \vec{d} of the free end from its position when the spring is in the relaxed state. The *spring force* is given by

$$\vec{F}_s = -k\vec{d} \quad (\text{Hooke's law}), \quad (7-20)$$

which is known as **Hooke's law** after Robert Hooke, an English scientist of the late 1600s. The minus sign in Eq. 7-20 indicates that the direction of the spring force is always opposite the direction of the displacement of the spring's free end. The constant k is called the **spring constant** (or **force constant**) and is a measure of the stiffness of the spring. The larger k is, the stiffer the spring; that is, the larger k is, the stronger the spring's pull or push for a given displacement. The SI unit for k is the newton per meter.

In Fig. 7-9 an x axis has been placed parallel to the length of the spring, with the origin ($x = 0$) at the position of the free end when the spring is in its relaxed state. For this common arrangement, we can write Eq. 7-20 as

$$F_x = -kx \quad (\text{Hooke's law}), \quad (7-21)$$

where we have changed the subscript. If x is positive (the spring is stretched toward the right on the x axis), then F_x is negative (it is a pull toward the left). If x is negative (the spring is compressed toward the left), then F_x is positive (it is a push toward the right). Note that a spring force is a *variable force* because it is a function of x , the position of the free end. Thus F_x can be symbolized as $F(x)$. Also note that Hooke's law is a *linear* relationship between F_x and x .

The Work Done by a Spring Force

To find the work done by the spring force as the block in Fig. 7-9a moves, let us make two simplifying assumptions about the spring. (1) It is *massless*; that is, its mass is negligible relative to the block's mass. (2) It is an *ideal spring*; that is, it obeys Hooke's law exactly. Let us also assume that the contact between the block and the floor is frictionless and that the block is particle-like.

We give the block a rightward jerk to get it moving and then leave it alone. As the block moves rightward, the spring force F_x does work on the block, decreasing the kinetic energy and slowing the block. However, we *cannot* find this work by using Eq. 7-7 ($W = Fd \cos \phi$) because that equation assumes a constant force. The spring force is a variable force.

To find the work done by the spring, we use calculus. Let the block's initial position be x_i and its later position x_f . Then divide the distance between those two

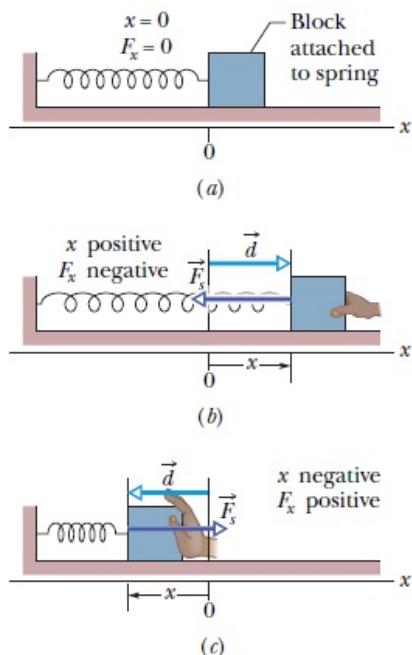


Fig. 7-9 (a) A spring in its relaxed state. The origin of an x axis has been placed at the end of the spring that is attached to a block. (b) The block is displaced by \vec{d} , and the spring is stretched by a positive amount x . Note the restoring force \vec{F}_s exerted by the spring. (c) The spring is compressed by a negative amount x . Again, note the restoring force.

positions into many segments, each of tiny length Δx . Label these segments, starting from x_i , as segments 1, 2, and so on. As the block moves through a segment, the spring force hardly varies because the segment is so short that x hardly varies. Thus, we can approximate the force magnitude as being constant within the segment. Label these magnitudes as F_{x1} in segment 1, F_{x2} in segment 2, and so on.

With the force now constant in each segment, we *can* find the work done within each segment by using Eq. 7-7. Here $\phi = 180^\circ$, and so $\cos \phi = -1$. Then the work done is $-F_{x1} \Delta x$ in segment 1, $-F_{x2} \Delta x$ in segment 2, and so on. The net work W_s done by the spring, from x_i to x_f , is the sum of all these works:

$$W_s = \sum -F_{xj} \Delta x, \quad (7-22)$$

where j labels the segments. In the limit as Δx goes to zero, Eq. 7-22 becomes

$$W_s = \int_{x_i}^{x_f} -F_x dx. \quad (7-23)$$

From Eq. 7-21, the force magnitude F_x is kx . Thus, substitution leads to

$$\begin{aligned} W_s &= \int_{x_i}^{x_f} -kx dx = -k \int_{x_i}^{x_f} x dx \\ &= (-\frac{1}{2}k)[x^2]_{x_i}^{x_f} = (-\frac{1}{2}k)(x_f^2 - x_i^2). \end{aligned} \quad (7-24)$$

Multiplied out, this yields

$$W_s = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2 \quad (\text{work by a spring force}). \quad (7-25)$$

This work W_s done by the spring force can have a positive or negative value, depending on whether the *net* transfer of energy is to or from the block as the block moves from x_i to x_f . *Caution:* The final position x_f appears in the *second* term on the right side of Eq. 7-25. Therefore, Eq. 7-25 tells us:

Work W_s is positive if the block ends up closer to the relaxed position ($x = 0$) than it was initially. It is negative if the block ends up farther away from $x = 0$. It is zero if the block ends up at the same distance from $x = 0$.

If $x_i = 0$ and if we call the final position x , then Eq. 7-25 becomes

$$W_s = -\frac{1}{2}kx^2 \quad (\text{work by a spring force}). \quad (7-26)$$

The Work Done by an Applied Force

Now suppose that we displace the block along the x axis while continuing to apply a force \vec{F}_a to it. During the displacement, our applied force does work W_a on the block while the spring force does work W_s . By Eq. 7-10, the change ΔK in the kinetic energy of the block due to these two energy transfers is

$$\Delta K = K_f - K_i = W_a + W_s, \quad (7-27)$$

in which K_f is the kinetic energy at the end of the displacement and K_i is that at the start of the displacement. If the block is stationary before and after the displacement, then K_f and K_i are both zero and Eq. 7-27 reduces to

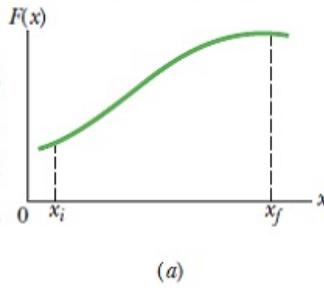
$$W_a = -W_s. \quad (7-28)$$

7-8 Work Done by a General Variable Force

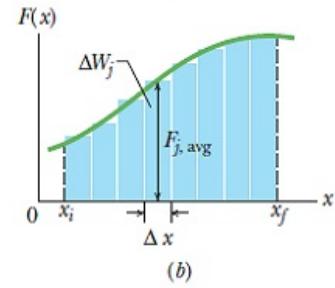
One-Dimensional Analysis

Let us return to the situation of Fig. 7-2 but now consider the force to be in the positive direction of the x axis and the force magnitude to vary with position x . Thus, as the bead (particle) moves, the magnitude $F(x)$ of the force doing work on it changes. Only the magnitude of this variable force changes, not its direction, and the magnitude at any position does not change with time.

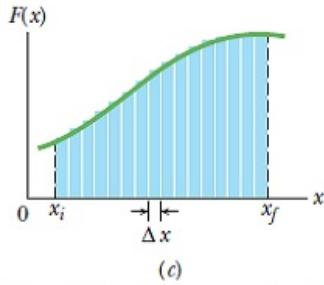
Work is equal to the area under the curve.



We can approximate that area with the area of these strips.



We can do better with more, narrower strips.



For the best, take the limit of strip widths going to zero.

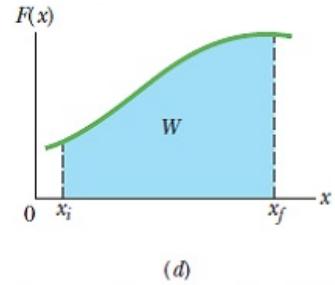


Fig. 7-11 (a) A one-dimensional force $\vec{F}(x)$ plotted against the displacement x of a particle on which it acts. The particle moves from x_i to x_f . (b) Same as (a) but with the area under the curve divided into narrow strips. (c) Same as (b) but with the area divided into narrower strips. (d) The limiting case. The work done by the force is given by Eq. 7-32 and is represented by the shaded area between the curve and the x axis and between x_i and x_f .

Figure 7-11a shows a plot of such a *one-dimensional variable force*. We want an expression for the work done on the particle by this force as the particle moves from an initial point x_i to a final point x_f . However, we *cannot* use Eq. 7-7 ($W = Fd \cos \phi$) because it applies only for a constant force \vec{F} . Here, again, we shall use calculus. We divide the area under the curve of Fig. 7-11a into a number of narrow strips of width Δx (Fig. 7-11b). We choose Δx small enough to permit us to take the force $F(x)$ as being reasonably constant over that interval. We let $F_{j,\text{avg}}$ be the average value of $F(x)$ within the j th interval. Then in Fig. 7-11b, $F_{j,\text{avg}}$ is the height of the j th strip.

With $F_{j,\text{avg}}$ considered constant, the increment (small amount) of work ΔW_j done by the force in the j th interval is now approximately given by Eq. 7-7 and is

$$\Delta W_j = F_{j,\text{avg}} \Delta x. \quad (7-29)$$

In Fig. 7-11b, ΔW_j is then equal to the area of the j th rectangular, shaded strip.

To approximate the total work W done by the force as the particle moves from x_i to x_f , we add the areas of all the strips between x_i and x_f in Fig. 7-11b:

$$W = \sum \Delta W_j = \sum F_{j,\text{avg}} \Delta x. \quad (7-30)$$

Equation 7-30 is an approximation because the broken “skyline” formed by the tops of the rectangular strips in Fig. 7-11b only approximates the actual curve of $F(x)$.

We can make the approximation better by reducing the strip width Δx and using more strips (Fig. 7-11c). In the limit, we let the strip width approach zero; the number of strips then becomes infinitely large and we have, as an exact result,

$$W = \lim_{\Delta x \rightarrow 0} \sum F_{j,\text{avg}} \Delta x. \quad (7-31)$$

This limit is exactly what we mean by the integral of the function $F(x)$ between the limits x_i and x_f . Thus, Eq. 7-31 becomes

$$W = \int_{x_i}^{x_f} F(x) dx \quad (\text{work: variable force}). \quad (7-32)$$

If we know the function $F(x)$, we can substitute it into Eq. 7-32, introduce the proper limits of integration, carry out the integration, and thus find the work.

(Appendix E contains a list of common integrals.) Geometrically, the work is equal to the area between the $F(x)$ curve and the x axis, between the limits x_i and x_f (shaded in Fig. 7-11d).

Three-Dimensional Analysis

Consider now a particle that is acted on by a three-dimensional force

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}, \quad (7-33)$$

in which the components F_x , F_y , and F_z can depend on the position of the particle; that is, they can be functions of that position. However, we make three simplifications: F_x may depend on x but not on y or z , F_y may depend on y but not on x or z , and F_z may depend on z but not on x or y . Now let the particle move through an incremental displacement

$$d\vec{r} = dx \hat{i} + dy \hat{j} + dz \hat{k}. \quad (7-34)$$

The increment of work dW done on the particle by \vec{F} during the displacement $d\vec{r}$ is, by Eq. 7-8,

$$dW = \vec{F} \cdot d\vec{r} = F_x dx + F_y dy + F_z dz. \quad (7-35)$$

The work W done by \vec{F} while the particle moves from an initial position r_i having coordinates (x_i, y_i, z_i) to a final position r_f having coordinates (x_f, y_f, z_f) is then

$$W = \int_{r_i}^{r_f} dW = \int_{x_i}^{x_f} F_x dx + \int_{y_i}^{y_f} F_y dy + \int_{z_i}^{z_f} F_z dz. \quad (7-36)$$

If \vec{F} has only an x component, then the y and z terms in Eq. 7-36 are zero and the equation reduces to Eq. 7-32.

Work–Kinetic Energy Theorem with a Variable Force

Equation 7-32 gives the work done by a variable force on a particle in a one-dimensional situation. Let us now make certain that the work is equal to the change in kinetic energy, as the work–kinetic energy theorem states.

Consider a particle of mass m , moving along an x axis and acted on by a net force $F(x)$ that is directed along that axis. The work done on the particle by this force as the particle moves from position x_i to position x_f is given by Eq. 7-32 as

$$W = \int_{x_i}^{x_f} F(x) dx = \int_{x_i}^{x_f} ma dx, \quad (7-37)$$

in which we use Newton's second law to replace $F(x)$ with ma . We can write the quantity $ma dx$ in Eq. 7-37 as

$$ma dx = m \frac{dv}{dt} dx. \quad (7-38)$$

From the chain rule of calculus, we have

$$\frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = \frac{dv}{dx} v, \quad (7-39)$$

and Eq. 7-38 becomes

$$ma dx = m \frac{dv}{dx} v dx = mv dv. \quad (7-40)$$

Substituting Eq. 7-40 into Eq. 7-37 yields

$$\begin{aligned} W &= \int_{v_i}^{v_f} mv dv = m \int_{v_i}^{v_f} v dv \\ &= \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2. \end{aligned} \quad (7-41)$$

Note that when we change the variable from x to v we are required to express the limits on the integral in terms of the new variable. Note also that because the mass m is a constant, we are able to move it outside the integral.

Recognizing the terms on the right side of Eq. 7-41 as kinetic energies allows us to write this equation as

$$W = K_f - K_i = \Delta K, \quad \text{which is the work–kinetic energy theorem.}$$

7-9 Power

The time rate at which work is done by a force is said to be the **power** due to the force. If a force does an amount of work W in an amount of time Δt , the **average power** due to the force during that time interval is

$$P_{\text{avg}} = \frac{W}{\Delta t} \quad (\text{average power}). \quad (7-42)$$

The **instantaneous power** P is the instantaneous time rate of doing work, which we can write as

$$P = \frac{dW}{dt} \quad (\text{instantaneous power}). \quad (7-43)$$

Suppose we know the work $W(t)$ done by a force as a function of time. Then to get the instantaneous power P at, say, time $t = 3.0$ s during the work, we would first take the time derivative of $W(t)$ and then evaluate the result for $t = 3.0$ s.

The SI unit of power is the joule per second. This unit is used so often that it has a special name, the **watt** (W), after James Watt, who greatly improved the rate at which steam engines could do work. In the British system, the unit of power is the foot-pound per second. Often the horsepower is used. These are related by

$$1 \text{ watt} = 1 \text{ W} = 1 \text{ J/s} = 0.738 \text{ ft} \cdot \text{lb/s} \quad (7-44)$$

and $1 \text{ horsepower} = 1 \text{ hp} = 550 \text{ ft} \cdot \text{lb/s} = 746 \text{ W}. \quad (7-45)$

Inspection of Eq. 7-42 shows that work can be expressed as power multiplied by time, as in the common unit **kilowatt-hour**. Thus,

$$\begin{aligned} 1 \text{ kilowatt-hour} &= 1 \text{ kW} \cdot \text{h} = (10^3 \text{ W})(3600 \text{ s}) \\ &= 3.60 \times 10^6 \text{ J} = 3.60 \text{ MJ}. \end{aligned} \quad (7-46)$$

Perhaps because they appear on our utility bills, the watt and the kilowatt-hour have become identified as electrical units. They can be used equally well as units for other examples of power and energy. Thus, if you pick up a book from the floor and put it on a tabletop, you are free to report the work that you have done as, say, 4×10^{-6} kW · h (or more conveniently as 4 mW · h).

We can also express the rate at which a force does work on a particle (or particle-like object) in terms of that force and the particle's velocity. For a particle that is moving along a straight line (say, an x axis) and is acted on by a constant force \vec{F} directed at some angle ϕ to that line, Eq. 7-43 becomes

$$\begin{aligned} P &= \frac{dW}{dt} = \frac{F \cos \phi \, dx}{dt} = F \cos \phi \left(\frac{dx}{dt} \right), \\ \text{or } P &= \vec{F} \cdot \vec{v} \cos \phi. \end{aligned} \quad (7-47)$$

Reorganizing the right side of Eq. 7-47 as the dot product $\vec{F} \cdot \vec{v}$, we may also write the equation as

$$P = \vec{F} \cdot \vec{v} \quad (\text{instantaneous power}). \quad (7-48)$$

There power is defined as the rate at which work is done by a force. In a more general sense, power P is the rate at which energy is transferred by a force from one type to another. If an amount of energy ΔE is transferred in an amount of time Δt , the **average power** due to the force is $P_{\text{avg}} = \frac{\Delta E}{\Delta t}$

Similarly, the **instantaneous power** due to the force is

$$P = \frac{dE}{dt}.$$

Kinetic Energy

Kinetic energy K is energy associated with the *state of motion* of an object. The faster the object moves, the greater is its kinetic energy. When the object is stationary, its kinetic energy is zero.

For an object of mass m whose speed v is well below the speed of light,

$$K = \frac{1}{2}mv^2 \quad (\text{kinetic energy}).$$

For example, a 3.0 kg duck flying past us at 2.0 m/s has a kinetic energy of $6.0 \text{ kg} \cdot \text{m}^2/\text{s}^2$; that is, we associate that number with the duck's motion.

The SI unit of kinetic energy (and every other type of energy) is the **joule** (J), named for James Prescott Joule, an English scientist of the 1800s. It is defined directly from above Eqn terms of the units for mass and velocity:

$$1 \text{ joule} = 1 \text{ J} = 1 \text{ kg} \cdot \text{m}^2/\text{s}^2.$$

Thus, the flying duck has a kinetic energy of 6.0 J.

8-2 Work and Potential Energy

In Chapter 7 we discussed the relation between work and a change in kinetic energy. Here we discuss the relation between work and a change in potential energy.

Let us throw a tomato upward (Fig. 8-2). We already know that as the tomato rises, the work W_g done on the tomato by the gravitational force is negative because the force transfers energy *from* the kinetic energy of the tomato. We can now finish the story by saying that this energy is transferred by the gravitational force *to* the gravitational potential energy of the tomato–Earth system.

The tomato slows, stops, and then begins to fall back down because of the gravitational force. During the fall, the transfer is reversed: The work W_g done on the tomato by the gravitational force is now positive—that force transfers energy *from* the gravitational potential energy of the tomato–Earth system *to* the kinetic energy of the tomato.

For either rise or fall, the change ΔU in gravitational potential energy is defined as being equal to the negative of the work done on the tomato by the gravitational force. Using the general symbol W for work, we write this as

$$\Delta U = -W. \quad (8-1)$$

This equation also applies to a block–spring system, as in Fig. 8-3. If we abruptly shove the block to send it moving rightward, the spring force acts leftward and thus does negative work on the block, transferring energy from the kinetic energy of the block to the elastic potential energy of the spring–block system. The block slows and eventually stops, and then begins to move leftward because the spring force is still leftward. The transfer of energy is then reversed—it is from potential energy of the spring–block system to kinetic energy of the block.

Conservative and Nonconservative Forces

Let us list the key elements of the two situations we just discussed:

1. The *system* consists of two or more objects.
2. A *force* acts between a particle-like object (tomato or block) in the system and the rest of the system.
3. When the system configuration changes, the force does *work* (call it W_1) on the particle-like object, transferring energy between the kinetic energy K of the object and some other type of energy of the system.

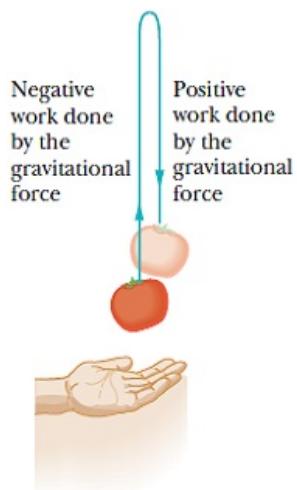


Fig. 8-2 A tomato is thrown upward. As it rises, the gravitational force does negative work on it, decreasing its kinetic energy. As the tomato descends, the gravitational force does positive work on it, increasing its kinetic energy.

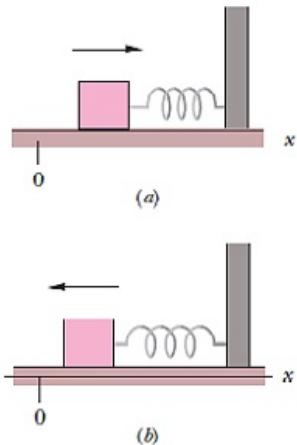


Fig. 8-3 A block, attached to a spring and initially at rest at $x = 0$, is set in motion toward the right. (a) As the block moves rightward (as indicated by the arrow), the spring force does negative work on it. (b) Then, as the block moves back toward $x = 0$, the spring force does positive work on it.

4. When the configuration change is reversed, the force reverses the energy transfer, doing work W_2 in the process.

In a situation in which $W_1 = -W_2$ is always true, the other type of energy is a potential energy and the force is said to be a **conservative force**. As you might suspect, the gravitational force and the spring force are both conservative (since otherwise we could not have spoken of gravitational potential energy and elastic potential energy, as we did previously).

A force that is not conservative is called a **nonconservative force**. The kinetic frictional force and drag force are nonconservative. For an example, let us send a block sliding across a floor that is not frictionless. During the sliding, a kinetic frictional force from the floor slows the block by transferring energy from its kinetic energy to a type of energy called *thermal energy* (which has to do with the random motions of atoms and molecules). We know from experiment that this energy transfer cannot be reversed (thermal energy cannot be transferred back to kinetic energy of the block by the kinetic frictional force). Thus, although we have a system (made up of the block and the floor), a force that acts between parts of the system, and a transfer of energy by the force, the force is not conservative. Therefore, thermal energy is not a potential energy.

When only conservative forces act on a particle-like object, we can greatly simplify otherwise difficult problems involving motion of the object. The next section, in which we develop a test for identifying conservative forces, provides one means for simplifying such problems.

8-3 Path Independence of Conservative Forces

The primary test for determining whether a force is conservative or nonconservative is this: Let the force act on a particle that moves along any *closed path*, beginning at some initial position and eventually returning to that position (so that the particle makes a *round trip* beginning and ending at the initial position). The force is conservative only if the total energy it transfers to and from the particle during the round trip along this and any other closed path is zero. In other words:

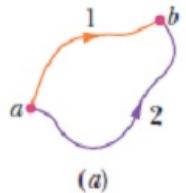
The net work done by a conservative force on a particle moving around any closed path is zero.

We know from experiment that the gravitational force passes this *closed-path test*. An example is the tossed tomato of Fig. 8-2. The tomato leaves the launch point with speed v_0 and kinetic energy $\frac{1}{2}mv_0^2$. The gravitational force acting on the tomato slows it, stops it, and then causes it to fall back down. When the tomato returns to the launch point, it again has speed v_0 and kinetic energy $\frac{1}{2}mv_0^2$. Thus, the gravitational force transfers as much energy *from* the tomato during the ascent as it transfers *to* the tomato during the descent back to the launch point. The net work done on the tomato by the gravitational force during the round trip is zero.

An important result of the closed-path test is that:

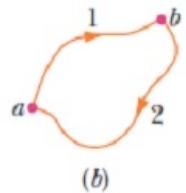
The work done by a conservative force on a particle moving between two points does not depend on the path taken by the particle.

For example, suppose that a particle moves from point *a* to point *b* in Fig. 8-4a along either path 1 or path 2. If only a conservative force acts on the particle, then the work done on the particle is the same along the two paths. In symbols, we can



The force is conservative.
Any choice of path between
the points gives the same
amount of work.

Fig. 8-4 (a) As a conservative force acts on it, a particle can move from point *a* to point *b* along either path 1 or path 2. (b) The particle moves in a round trip, from point *a* to point *b* along path 1 and then back to point *a* along path 2.



And a round trip gives
a total work of zero.

write this result as

$$W_{ab,1} = W_{ab,2}, \quad (8-2)$$

where the subscript *ab* indicates the initial and final points, respectively, and the subscripts 1 and 2 indicate the path.

This result is powerful because it allows us to simplify difficult problems when only a conservative force is involved. Suppose you need to calculate the work done by a conservative force along a given path between two points, and the calculation is difficult or even impossible without additional information. You can find the work by substituting some other path between those two points for which the calculation is easier and possible.

Proof of Equation 8-2

Figure 8-4b shows an arbitrary round trip for a particle that is acted upon by a single force. The particle moves from an initial point *a* to point *b* along path 1 and then back to point *a* along path 2. The force does work on the particle as the particle moves along each path. Without worrying about where positive work is done and where negative work is done, let us just represent the work done from *a* to *b* along path 1 as $W_{ab,1}$ and the work done from *b* back to *a* along path 2 as $W_{ba,2}$. If the force is conservative, then the net work done during the round trip must be zero:

$$W_{ab,1} + W_{ba,2} = 0,$$

and thus

$$W_{ab,1} = -W_{ba,2}. \quad (8-3)$$

In words, the work done along the outward path must be the negative of the work done along the path back.

Let us now consider the work $W_{ab,2}$ done on the particle by the force when the particle moves from *a* to *b* along path 2, as indicated in Fig. 8-4a. If the force is conservative, that work is the negative of $W_{ba,2}$:

$$W_{ab,2} = -W_{ba,2}. \quad (8-4)$$

Substituting $W_{ab,2}$ for $-W_{ba,2}$ in Eq. 8-3, we obtain

$$W_{ab,1} = W_{ab,2},$$

which is what we set out to prove.

Gravitational Potential Energy

We first consider a particle with mass m moving vertically along a y axis (the positive direction is upward). As the particle moves from point y_i to point y_f , the gravitational force \vec{F}_g does work on it. To find the corresponding change in the gravitational potential energy of the particle–Earth system, we use Eq. 8-6 with two changes: (1) We integrate along the y axis instead of the x axis, because the gravitational force acts vertically. (2) We substitute $-mg$ for the force symbol F , because \vec{F}_g has the magnitude mg and is directed down the y axis. We then have

$$\Delta U = - \int_{y_i}^{y_f} (-mg) dy = mg \int_{y_i}^{y_f} dy = mg \left[y \right]_{y_i}^{y_f},$$

which yields

$$\Delta U = mg(y_f - y_i) = mg \Delta y. \quad (8-7)$$

Only *changes* ΔU in gravitational potential energy (or any other type of potential energy) are physically meaningful. However, to simplify a calculation or a discussion, we sometimes would like to say that a certain gravitational potential value U is associated with a certain particle–Earth system when the particle is at a certain height y . To do so, we rewrite Eq. 8-7 as

$$U - U_i = mg(y - y_i). \quad (8-8)$$

Then we take U_i to be the gravitational potential energy of the system when it is in a **reference configuration** in which the particle is at a **reference point** y_i . Usually we take $U_i = 0$ and $y_i = 0$. Doing this changes Eq. 8-8 to

$$U(y) = mgy \quad (\text{gravitational potential energy}). \quad (8-9)$$

This equation tells us:

The gravitational potential energy associated with a particle–Earth system depends only on the vertical position y (or height) of the particle relative to the reference position $y = 0$, not on the horizontal position.

Elastic Potential Energy

We next consider the block–spring system shown in Fig. 8-3, with the block moving on the end of a spring of spring constant k . As the block moves from point x_i to point x_f , the spring force $F_x = -kx$ does work on the block. To find the corresponding change in the elastic potential energy of the block–spring system, we substitute $-kx$ for $F(x)$ in Eq. 8-6. We then have

$$\Delta U = - \int_{x_i}^{x_f} (-kx) dx = k \int_{x_i}^{x_f} x dx = \frac{1}{2}k \left[x^2 \right]_{x_i}^{x_f},$$

or

$$\Delta U = \frac{1}{2}kx_f^2 - \frac{1}{2}kx_i^2. \quad (8-10)$$

To associate a potential energy value U with the block at position x , we choose the reference configuration to be when the spring is at its relaxed length and the block is at $x_i = 0$. Then the elastic potential energy U_i is 0, and Eq. 8-10 becomes

$$U - 0 = \frac{1}{2}kx^2 - 0,$$

which gives us

$$U(x) = \frac{1}{2}kx^2 \quad (\text{elastic potential energy}). \quad (8-11)$$

8-5 Conservation of Mechanical Energy

The **mechanical energy** E_{mec} of a system is the sum of its potential energy U and the kinetic energy K of the objects within it:

$$E_{\text{mec}} = K + U \quad (\text{mechanical energy}). \quad (8-12)$$

In this section, we examine what happens to this mechanical energy when only conservative forces cause energy transfers within the system—that is, when frictional and drag forces do not act on the objects in the system. Also, we shall assume that the system is *isolated* from its environment; that is, no *external force* from an object outside the system causes energy changes inside the system.

When a conservative force does work W on an object within the system, that force transfers energy between kinetic energy K of the object and potential energy U of the system. From Eq. 7-10, the change ΔK in kinetic energy is

$$\Delta K = W \quad (8-13)$$

and from Eq. 8-1, the change ΔU in potential energy is

$$\Delta U = -W. \quad (8-14)$$

Combining Eqs. 8-13 and 8-14, we find that

$$\Delta K = -\Delta U. \quad (8-15)$$

In words, one of these energies increases exactly as much as the other decreases.

We can rewrite Eq. 8-15 as

$$K_2 - K_1 = -(U_2 - U_1), \quad (8-16)$$

where the subscripts refer to two different instants and thus to two different arrangements of the objects in the system. Rearranging Eq. 8-16 yields

$$K_2 + U_2 = K_1 + U_1 \quad (\text{conservation of mechanical energy}). \quad (8-17)$$

In words, this equation says:

$$\left(\begin{array}{l} \text{the sum of } K \text{ and } U \text{ for} \\ \text{any state of a system} \end{array} \right) = \left(\begin{array}{l} \text{the sum of } K \text{ and } U \text{ for} \\ \text{any other state of the system} \end{array} \right),$$

when the system is isolated and only conservative forces act on the objects in the system. In other words:

In an isolated system where only conservative forces cause energy changes, the kinetic energy and potential energy can change, but their sum, the mechanical energy E_{mec} of the system, cannot change.

This result is called the **principle of conservation of mechanical energy**. (Now you can see where *conservative* forces got their name.) With the aid of Eq. 8-15, we can write this principle in one more form, as

$$\Delta E_{\text{mec}} = \Delta K + \Delta U = 0. \quad (8-18)$$

The principle of conservation of mechanical energy allows us to solve problems that would be quite difficult to solve using only Newton's laws:

When the mechanical energy of a system is conserved, we can relate the sum of kinetic energy and potential energy at one instant to that at another instant *without considering the intermediate motion and without finding the work done by the forces involved*.

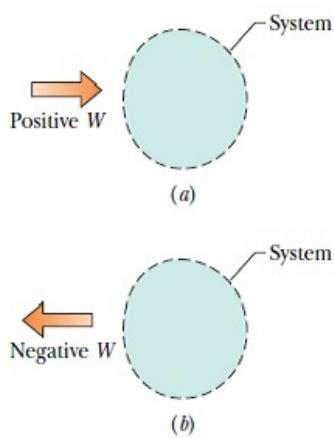


Fig. 8-11 (a) Positive work W done on an arbitrary system means a transfer of energy to the system. (b) Negative work W means a transfer of energy from the system.

8-7 Work Done on a System by an External Force

In Chapter 7, we defined work as being energy transferred to or from an object by means of a force acting on the object. We can now extend that definition to an external force acting on a system of objects.

Work is energy transferred to or from a system by means of an external force acting on that system.

Figure 8-11a represents positive work (*a* transfer of energy *to* a system), and Fig. 8-11b represents negative work (*a* transfer of energy *from* a system). When more than one force acts on a system, their *net work* is the energy transferred to or from the system.

These transfers are like transfers of money to and from a bank account. If a system consists of a single particle or particle-like object, as in Chapter 7, the work done on the system by a force can change only the kinetic energy of the system. The energy statement for such transfers is the work–kinetic energy theorem $\Delta K = W$; that is, a single particle has only one energy account, called kinetic energy. External forces can transfer energy into or out of that account. If a system is more complicated, however, an external force can change other forms of energy (such as potential energy); that is, a more complicated system can have multiple energy accounts.

Let us find energy statements for such systems by examining two basic situations, one that does not involve friction and one that does.

No Friction Involved

To compete in a bowling-ball-hurling contest, you first squat and cup your hands under the ball on the floor. Then you rapidly straighten up while also pulling your hands up sharply, launching the ball upward at about face level. During your upward motion, your applied force on the ball obviously does work; that is, it is an external force that transfers energy, but to what system?

To answer, we check to see which energies change. There is a change ΔK in the ball's kinetic energy and, because the ball and Earth become more separated, there is a change ΔU in the gravitational potential energy of the ball–Earth system. To include both changes, we need to consider the ball–Earth system. Then your force is an external force doing work on that system, and the work is

$$W = \Delta K + \Delta U, \quad (8-25)$$

$$\text{or} \quad W = \Delta E_{\text{mec}} \quad (\text{work done on system, no friction involved}), \quad (8-26)$$

where ΔE_{mec} is the change in the mechanical energy of the system. These two equations, which are represented in Fig. 8-12, are equivalent energy statements for work done on a system by an external force when friction is not involved.

Friction Involved

We next consider the example in Fig. 8-13a. A constant horizontal force \vec{F} pulls a block along an x axis and through a displacement of magnitude d , increasing the block's velocity from \vec{v}_0 to \vec{v} . During the motion, a constant kinetic frictional force \vec{f}_k from the floor acts on the block. Let us first choose the block as our

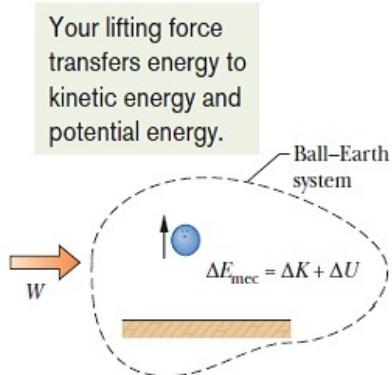
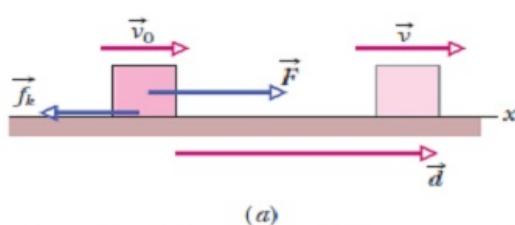


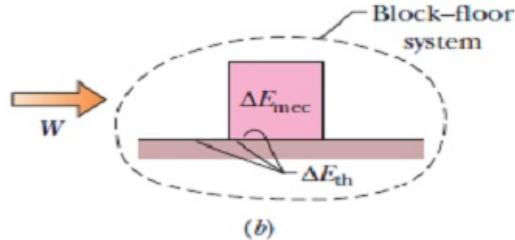
Fig. 8-12 Positive work W is done on a system of a bowling ball and Earth, causing a change ΔE_{mec} in the mechanical energy of the system, a change ΔK in the ball's kinetic energy, and a change ΔU in the system's gravitational potential energy.

The applied force supplies energy.
The frictional force transfers some of it to thermal energy.

So, the work done by the applied force goes into kinetic energy and also thermal energy.



(a)



(b)

Fig. 8-13 (a) A block is pulled across a floor by force \vec{F} while a kinetic frictional force \vec{f}_k opposes the motion. The block has velocity \vec{v}_0 at the start of a displacement \vec{d} and velocity \vec{v} at the end of the displacement. (b) Positive work W is done on the block–floor system by force \vec{F} , resulting in a change ΔE_{mec} in the block’s mechanical energy and a change ΔE_{th} in the thermal energy of the block and floor.

system and apply Newton’s second law to it. We can write that law for components along the x axis ($F_{\text{net},x} = ma_x$) as

$$F - f_k = ma. \quad (8-27)$$

Because the forces are constant, the acceleration \vec{a} is also constant. Thus, we can use Eq. of motion

$$v^2 = v_0^2 + 2ad.$$

Solving this equation for a , substituting the result into Eq. 8-27, and rearranging then give us

$$Fd = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 + f_k d \quad (8-28)$$

or, because $\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \Delta K$ for the block,

$$Fd = \Delta K + f_k d. \quad (8-29)$$

In a more general situation (say, one in which the block is moving up a ramp), there can be a change in potential energy. To include such a possible change, we generalize Eq. 8-29 by writing

$$Fd = \Delta E_{\text{mec}} + f_k d. \quad (8-30)$$

By experiment we find that the block and the portion of the floor along which it slides become warmer as the block slides.

The temperature of an object is related to the object’s thermal energy E_{th} (the energy associated with the random motion of the atoms and molecules in the object). Here, the thermal energy of the block and floor increases because (1) there is friction between them and (2) there is sliding. Recall that friction is due to the cold-welding between two surfaces. As the block slides over the floor, the sliding causes repeated tearing and re-forming of the welds between the block and the floor, which makes the block and floor warmer. Thus, the sliding increases their thermal energy E_{th} .

Through experiment, we find that the increase ΔE_{th} in thermal energy is equal to the product of the magnitudes f_k and d :

$$\Delta E_{\text{th}} = f_k d \quad (\text{increase in thermal energy by sliding}). \quad (8-31)$$

Thus, we can rewrite Eq. 8-30 as

$$Fd = \Delta E_{\text{mec}} + \Delta E_{\text{th}}. \quad (8-32)$$

Fd is the work W done by the external force \vec{F} (the energy transferred by the force). The block’s mechanical energy changes, and the thermal energies of the block and floor also change. Therefore, the work done by force \vec{F} is done on the block–floor system. That work is

$$W = \Delta E_{\text{mec}} + \Delta E_{\text{th}} \quad (\text{work done on system, friction involved}). \quad (8-33)$$