Lecture # 01 **Date:** Aug 11, 2020

$$M = \begin{bmatrix} 2 & 4 & -7 \\ 6 & 8 & 0 \\ -3 & 5 & 8 \end{bmatrix}$$

In general, $G = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots \quad \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

A matrix is a rectangular arrangement of numbers.

In short, $M = [a_{ij}], 1 \leq i \leq m, 1 \leq j \leq n$.

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Zero matrix: A matrix $M = [a_{ij}], 1 \leq i \leq m, 1 \leq j \leq n$ is called a zero matrix if $a_{ij} = 0$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

Row matrix: A matrix $M = [a_{ij}], 1 \leq i \leq m, 1 \leq j \leq n$ is called a row matrix if $m = 1$.

Column matrix: A matrix $M = [a_{ij}], 1 \leq i \leq m, 1 \leq j \leq n$ is called a column matrix if $n = 1$.

$$A = [5 \quad 0 \; -1 \quad 10]$$

$$B = \begin{bmatrix} 7 \\ -8 \\ 4 \\ 6 \\ 7 \end{bmatrix}$$

Square matrix: A matrix $M = [a_{ij}], 1 \leq i \leq m, 1 \leq j \leq n$ is called a square matrix if $m = n$. Then we write $M = [a_{ij}]$, $1 \leq i, j \leq m$. Here m is called the **order** of the matrix M.

Identity matrix: A square matrix $M = [a_{ij}], 1 \leq i, j \leq m$ is called an identity matrix if $a_{ij} = 1$ when $i = j$ and $a_{ij} = 0$ otherwise.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Upper triangular matrix: A square matrix $M = [a_{ij}], 1 \leq i, j \leq m$ is called an upper triangular matrix if $a_{ij} = 0$ when $i > j$.

Lower triangular matrix: A square matrix $M = [a_{ij}], 1 \leq i, j \leq m$ is called a lower triangular matrix if $a_{ij} = 0$ when $i < j$.

$$L = \begin{bmatrix} 5 & 0 & 0 \\ -9 & 4 & 0 \\ 6 & 0 & 6 \end{bmatrix}$$

Diagonal matrix: A square matrix $M = [a_{ij}], 1 \leq i, j \leq m$ is called a diagonal matrix if $a_{ij} = 0$ when $i \neq j$.

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -14 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Scalar matrix: $S = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

A square matrix $M = [a_{ij}]$, $1 \le i, j \le m$ is called a scalar matrix if $a_{ij} = \lambda$ when $i = j$ and $a_{ij} = 0$ otherwise. Here λ is a constant.

Next Class: Basic Operations in Matrix …

Addition:

Let $A = [a_{ij}], 1 \leq i \leq m, 1 \leq j \leq n$ and $B = [b_{ij}], 1 \leq i \leq m, 1 \leq j \leq n$ be two matrices. Then $C = [c_{ij}], 1 \leq i \leq m, 1 \leq j \leq n$ is the sum of A and B, that is, $C = A + B$ if $c_{ij} = a_{ij} + b_{ij}$ for all $1 \leq i \leq m, 1 \leq j \leq n$.

Example: $\begin{bmatrix} 2 & 3 \\ -4 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 4 \\ 0 & -3 & 3 \end{bmatrix}$=Undefined

Subtraction:

Let $A = [a_{ij}], 1 \leq i \leq m, 1 \leq j \leq n$ and $B = [b_{ij}], 1 \leq i \leq m, 1 \leq j \leq n$ be two matrices. Then $C = [c_{ij}], 1 \leq i \leq m, 1 \leq j \leq n$ is the difference of A and B, that is, $C = A - B$ if $c_{ij} = a_{ij} - b_{ij}$ for all $1 \leq i \leq m, 1 \leq j \leq n$.

Scalar Multiplication:

Let $A = [a_{ij}], 1 \leq i \leq m, 1 \leq j \leq n$ be any matrix. Then $C = [c_{ij}], 1 \leq i \leq m, 1 \leq j \leq n$ is a scalar multiple of A, that is, $C = \alpha A$ if $c_{ij} = \alpha a_{ij}$ for all $1 \leq i \leq m, 1 \leq j \leq n$.

Example: $\frac{1}{2}\begin{bmatrix} 2 & -4 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$

Matrix Multiplication:

Let $A = [a_{ij}], 1 \le i \le m, 1 \le j \le n$ and $B = [b_{ij}], 1 \le i \le n, 1 \le j \le l$ be two matrices. Then $C = [c_{ij}], 1 \le i \le m, 1 \le j \le l$ is the matrix product of A and B, that is, $C = A \times B$ if $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$.

$c_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + \cdots + a_{1n}b_{n1} = \sum_{k=1}^{n} a_{1k} b_{k1}$

Example: $[1 \quad 2 \quad 0] \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \text{Undefined}$

$[1 \quad 2 \quad 3] \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = [14], \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [1 \quad 2 \quad 3] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

Here $AB \neq BA$. Thus matrix multiplication is noncommutative.

Matrix Transpose:

Let $A = [a_{ij}], 1 \le i \le m, 1 \le j \le n$ be any matrix. Then $C = [c_{ij}], 1 \le i \le n, 1 \le j \le m$ is the transpose of A, denoted by $C = A^t$, if $c_{ij} = a_{ji}$ for all $1 \le i \le n, 1 \le j \le m$.

Example: $A = \begin{bmatrix} 1 & 5 & 7 & 0 \\ 5 & 0 & 4 & -1 \\ 7 & 4 & 9 & 2 \\ 0 & -1 & 2 & 3 \end{bmatrix}, \ A^t = A.$

Symmetric matrix:

A square matrix $M = [a_{ij}], 1 \le i, j \le m$ is called a symmetric matrix if $a_{ij} = a_{ji}$ for all $1 \le i, j \le m$.

Remark: $M = M^t$ if M is symmetric.

Skew-symmetric matrix:

A square matrix $M = [a_{ij}], 1 \le i, j \le m$ is called a skew-symmetric matrix if $a_{ij} = -a_{ji}$ for all $1 \le i, j \le m$.

Remark: $M = -M^t$ if M is skew-symmetric.

Properties of matrix transpose:

1. $(A + B)^t = A^t + B^t$
2. $(A - B)^t = A^t - B^t$
3. $(\alpha A)^t = \alpha A^t$
4. $(AB)^t = B^t A^t$, Check it from a text book or any sources.
5. $(A^t)^t = A$

Class work:

1. $A + A^t$, Symmetric
2. $A - A^t$, Skew-symmetric
3. AA^t, Symmetric
4. **Theorem:** Show that for any square matrix A, $A + A^t$ is symmetric, $A - A^t$ is skew-symmetric, and AA^t is symmetric.

   Proof: (i) Let $X = A + A^t$.

   We have

   $$X^t = (A + A^t)^t = A^t + (A^t)^t = A^t + A = A + A^t = X$$

Thus X is symmetric, that is, $A + A^t$ is symmetric.

Problem: Let A be any square matrix. Find B and C such that $A = B + C,$ where B is symmetric and C is skew-symmetric.

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & -2 & 5 \\ -1 & 7 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ 1 & -3 & 6 \\ 0 & \frac{7}{2} & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 3 \\ -1 & 1 & -1 \\ -1 & \frac{7}{2} & 3 \end{bmatrix}$$

Lecture # 03 **Date:** Sep 29, 2020

Determinant:

Exercise 1. C, C++, html or Php code for evaluating determinant of a square matrix.

Example: Let $M = \begin{bmatrix} 2 & 5 \\ -3 & 4 \end{bmatrix}$. Then $|M| = \begin{vmatrix} 2 & 5 \\ -3 & 4 \end{vmatrix} = 2 \times 4 - 5 \times (-3) = 23.$

In general, for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

For $B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $|B| = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} + (-1)^{1+2} b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

For $B = [b_{ij}], 1 \leq i, j \leq n,$ $|B| = \sum_{j=1}^{n} b_{1j}B_{1j} = \sum_{i=1}^{n} b_{i2}B_{i2},$ B_{1j} is the cofactor of the element b_{1j}.

$B_{ij} = (-1)^{i+j}$ $\times$(determinant of the submatrix obtained from B by omitting the ith row and jth column of the matrix B)

Lecture # 04 **Date:** Oct 01, 2020

Inverse of a real number:

1. **Multiplicative inverse:** $ab = 1,\ 3 \times \frac{1}{3} = 1$
2. **Additive inverse:** $a + b = 0, 3 + (-3) = 0$

Inverse of a square matrix:

$$AB = I = BA$$

We write, $B = A^{-1}$ is the inverse of A.

Finding inverse of a square matrix A:

1. **Cofactor expansion method:**

Step 1: Check if A^{-1} exists.

Step 2: Find $cofA$, that is, cofactor matrix of A.

Step 3: Find $adjA$, that is, adjoint matrix of A.

Step 4: Find A^{-1} by using $A^{-1} = \frac{1}{|A|} \times adjA$.

Step 5: Verify your answer.

Example: $A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

Here $|A| = -7 \neq 0$. Thus A is nonsingular and hence A^{-1} exists.

We have, $cofA = \begin{bmatrix} \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} -2 & 1 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} -2 & 2 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 0 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} \end{bmatrix} =$

$\begin{bmatrix} -3 & -2 & -2 \\ 2 & -1 & -1 \\ 2 & -1 & 6 \end{bmatrix}$

Then $adjA = cofA^T = \begin{bmatrix} -3 & 2 & 2 \\ -2 & -1 & -1 \\ -2 & -1 & 6 \end{bmatrix}$

Then $A^{-1} = \frac{1}{-7}\begin{bmatrix} -3 & 2 & 2 \\ -2 & -1 & -1 \\ -2 & -1 & 6 \end{bmatrix}$

Exercise: Let $B = \begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix}$. Find B^{-1}.

Ans: $B^{-1} = \frac{1}{8}\begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix}$

2. Row reduction method:

Singular and nonsingular matrices: M is singular if and only if $|M| = 0$. Otherwise, M is nonsingular.

Echelon form of a matrix: $E = \begin{bmatrix} 0 & -1 & 2 & 4 & 2 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

The number of starting zeros in a row increases row by row until zero rows remain.

Lecture # 05 **Date:** Oct 05, 2020

Gaussian elimination method: $M = \begin{bmatrix} 1 & 0 & 2 & 0 \\ -1 & 0 & 1 & -1 \\ 2 & -1 & 0 & -1 \\ -3 & -1 & 0 & 1 \end{bmatrix}$

$$\begin{matrix} R_2 \leftarrow R_2 + R_1 \\ R_3 \leftarrow R_3 - 2R_1 \\ R_4 \leftarrow R_4 + 3R_1 \end{matrix} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & -1 & -2 & -1 \\ 0 & -1 & 6 & 1 \end{bmatrix}$$

$$R_3 \leftrightarrow R_2 \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 3 & -1 \\ 0 & -1 & 6 & 1 \end{bmatrix}$$

$$R_4 \leftarrow R_4 - R_2 \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 8 & 2 \end{bmatrix}$$

$$R_4 \leftarrow 3R_4 - 8R_3 \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 14 \end{bmatrix}$$

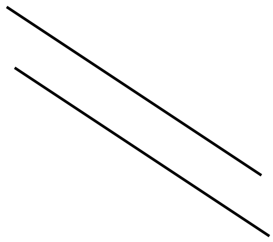
- Computer code for function det(M)

Lecture # 06 **Date:** Oct 08, 2020

System of linear Equations (SLE):

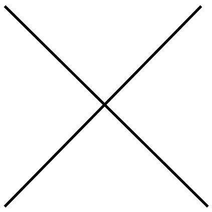
SLE with two variables:

$$\left.\begin{aligned} x + y &= 1 \\ x + y &= 2 \end{aligned}\right\}$$



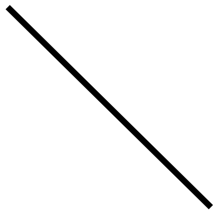
- Lines are parallel. SLE is inconsistent, that is, it has no solution.

(1) $\left.\begin{aligned} x + y &= 0 \\ x - y &= 0 \end{aligned}\right\}$



- Lines have a common point. SLE is consistent and has unique solution.

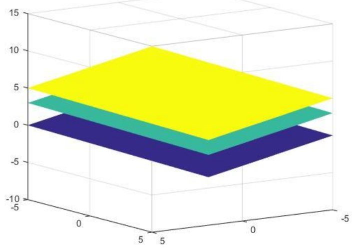
(2) $\left.\begin{aligned} x + y &= 1 \\ 2x + 2y &= 2 \end{aligned}\right\}$



- Lines coincide. SLE is consistent and has one free variable, that is, it has infinite solutions.

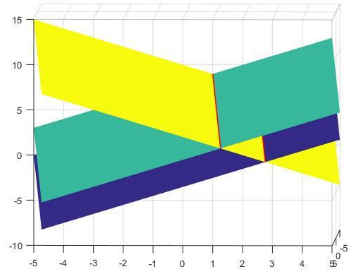
SLE with three variables:

(1) $\left.\begin{aligned} x + y - z &= 0 \\ x + y - z &= -3 \\ x + y - z &= -5 \end{aligned}\right\}$



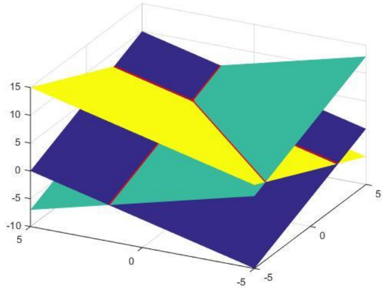
- All the planes are parallel. SLE is inconsistent, that is, it has no solution.

(2) $\left.\begin{aligned} x + y - z &= 0 \\ x + y - z &= -3 \\ -x + y - z &= -5 \end{aligned}\right\}$



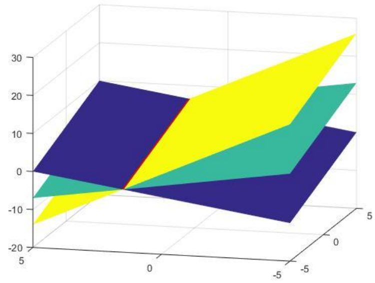
- Two planes are parallel. SLE is inconsistent, that is, SLE has no solution.

(3) $\left.\begin{aligned} x + y - z &= 0 \\ x - y - z &= -3 \\ -x + y - z &= -5 \end{aligned}\right\}$



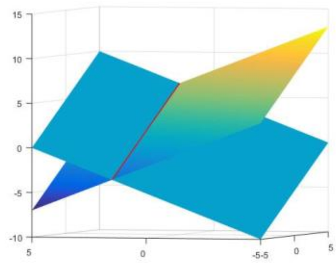
- Planes have a common point. SLE is consistent and has unique solution.

(4) $\left.\begin{aligned} x + y - z &= 0 \\ x - y - z &= -3 \\ x - 3y - z &= -6 \end{aligned}\right\}$



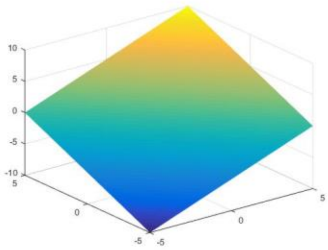
- Planes have a common line. SLE is consistent and has one free variable, that is, SLE has infinite solutions.

(5) $\left.\begin{aligned} x + y - z &= 0 \\ x - y - z &= -3 \\ 2x - 2y - 2z &= -6 \end{aligned}\right\}$



- Two planes coincide and have a common line. SLE is consistent and has one free variable, that is, SLE has infinite solutions.

$$\textbf{(6)} \quad \left.\begin{aligned} x - y - z &= -3 \\ 2x - 2y - 2z &= -6 \\ 3x - 3y - 3z &= -9 \end{aligned}\right\}$$



- All the planes coincide. SLE is consistent and has two free variables, that is, SLE has infinite solutions.

Lecture # 07 **Date:** Oct 10, 2020

Observing solutions of different SLEs from Online Algebra Solution Engine (oase.daffodilvarsity.edu.bd)

$\phi(x, y, z) = c$ Equation of a surface

$\phi(x, y, z) - c = 0$

Lecture # 08 **Date:** Oct 14, 2020

Eigenvalues and Eigenvectors:

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 15 \\ 15 \end{bmatrix} = 5 \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$Av = \lambda v \Longrightarrow Av - \lambda v = \mathbf{0} \Longrightarrow (A - \lambda I)v = \mathbf{0}$

Let A be a square matrix and λ be any scalar number. Then

- The polynomial $|A - \lambda I|$ is called the **characteristic polynomial** of the matrix A.

- The equation $|A - \lambda I| = 0$ **characteristic equation** of A.
- The values of λ satisfying the characteristic equation are called **eigenvalues** of A.
- The set of all eigenvalues of A is called the **spectrum** of A.

Example: Let $M = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ and λ be the eigenvalues of M.

Then $M - \lambda I = \begin{bmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{bmatrix}$.

Then $|M - \lambda I| = \begin{vmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda) - 3 =$ $5 - 6\lambda + \lambda^2$

Then $|M - \lambda I| = 0 \Longrightarrow 5 - 6\lambda + \lambda^2 = 0 \Longrightarrow \lambda = 1$ or $\lambda = 5$

Therefore, spectrum of M is $\{1,5\}$.

Let A be any square matrix and λ be a particular eigenvalue of A. Then

- A nonzero vector v is called an **eigenvector** of A corresponding to the eigenvalue λ if and only if $(A - \lambda I)v = \mathbf{0}$.
- The set of all eigenvectors of A together with the zero vector is called the **eigenspace** of A associated to the eigenvalue λ.

Problem: It is found that 1 is an eigenvalue of the matrix $M = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$. Find the eigenspace of M associated to the eigenvalue 1.

Solution: Let $v = \begin{bmatrix} x \\ y \end{bmatrix}$ be an eigen vector of M associated to the eigenvalue $\lambda = 1$.

Then $(A - \lambda I)v = \mathbf{0}$

$$\Rightarrow \left(\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{matrix} x + 3y = 0 \\ x + 3y = 0 \end{matrix}\right\} \Rightarrow x + 3y = 0$$

Let y be the free variable and $y = a$, where a is any nonzero real number. Then $x = -3a$ and $y = a$. Thus the eigenspace of M is $\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3a \\ a \end{bmatrix}\right\}$ where a is any nonzero real number.

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

Problem: Let $B = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$. Find the spectrum of B and the eigenspace associated to every eigenvalue of B.

Lecture # 09 **Date:** Oct 18, 2020

Solution: Let λ be an eigenvalue of B.

Here $B - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{bmatrix}$

Then $|B - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{vmatrix} = (1-\lambda)\{(2-\lambda)(3-\lambda)-1\} + (3-\lambda) - 2 = (1-\lambda)(2-\lambda)(3-\lambda) - (1-\lambda) + (3-\lambda) - 2 = (1-\lambda)(2-\lambda)(3-\lambda) + 0$

Here $|B - \lambda I| = 0 \Rightarrow (1-\lambda)(2-\lambda)(3-\lambda) = 0$

Thus $\lambda = 1$ or $\lambda = 2$ or $\lambda = 3$

Therefore, spectrum of B is $\{1, 2, 3\}$.

Let $v_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigenvector associated to the eigenvalue $\lambda_1 = 1$.

Then $(B - \lambda_1 I)v_1 = \mathbf{0}$

$$\Rightarrow \begin{bmatrix} 1-1 & 1 & 2 \\ -1 & 2-1 & 1 \\ 0 & 1 & 3-1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 2 \\ -1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$$

$$\Rightarrow \left.\begin{matrix} y + 2z = 0 \\ -x + y + z = 0 \\ y + 2z = 0 \end{matrix}\right\}$$

$$\Rightarrow \left.\begin{array}{r} -x+y+z=0 \\ y+2z=0 \end{array}\right\}$$

Let z is the free variable and assign $z = a$. Then $y = -2a$ and $x = -a$.

Therefore, $v_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -a \\ -2a \\ a \end{bmatrix}$, where a is a nonzero real number.

Let $v_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigenvector associated to the eigenvalue $\lambda_2 = 2$.

Then $(B - \lambda_2 I)v_2 = \mathbf{0}$

$$\Rightarrow \begin{bmatrix} 1-2 & 1 & 2 \\ -1 & 2-2 & 1 \\ 0 & 1 & 3-2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$$

$$\Rightarrow \left.\begin{array}{r} -x+y+2z=0 \\ -x+z=0 \\ y+z=0 \end{array}\right\}$$

$$\Rightarrow \left.\begin{array}{r} -x+y+2z=0 \\ -y-z=0 \\ y+z=0 \end{array}\right\}$$

$$\Longrightarrow \left.\begin{array}{r} -x+y+2z=0 \\ -y-z=0 \\ \end{array}\right\}$$

Let z is the free variable and assign $z=b$. Then $y=-b$ and $x=b$.

Therefore, $v_2=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}b\\-b\\b\end{bmatrix}$, where b is a nonzero real number.

Let $v_3=\begin{bmatrix}x\\y\\z\end{bmatrix}$ be the eigenvector associated to the eigenvalue $\lambda_3=3$.

Then $(B-\lambda_3 I)v_3=\mathbf{0}$

$$\Longrightarrow \begin{bmatrix}1-3 & 1 & 2\\ -1 & 2-3 & 1\\ 0 & 1 & 3-3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\mathbf{0}$$

$$\Longrightarrow \begin{bmatrix}-2 & 1 & 2\\ -1 & -1 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\mathbf{0}$$

$$\Longrightarrow \left.\begin{array}{r} -2x+y+2z=0 \\ -x-y+z=0 \\ y=0 \end{array}\right\}$$

$$\Longrightarrow \left.\begin{array}{r} -2x+y+2z=0 \\ -3y=0 \\ y=0 \end{array}\right\}$$

$$\Longrightarrow \left.\begin{array}{r} -2x + y + 2z = 0 \\ y = 0 \end{array}\right\}$$

Let z is the free variable and assign $z = c$. Then $x = c$.

Therefore, $v_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c \\ 0 \\ c \end{bmatrix}$, where c is a nonzero real number.

For $a = b = c = 1$, we have $v_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Observation: $P = \begin{bmatrix} -1 & 1 & 1 \\ -2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ (Modal matrix)

$$P^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 0 & 1 \\ 2 & -2 & -2 \\ -1 & 1 & 3 \end{bmatrix}$$

$$P^{-1}BP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Modal matrix: P is a **modal** matrix associated to the matrix B if $P^{-1}BP$ is a diagonal matrix.

Problem: Find the modal matrix for a given matrix.

Problem: Find the matrix that diagonalizes a given matrix.

Problem: Let $B = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$. Find B^{100}.

Lecture # 10 **Date:** Oct 20, 2020

$$(P^{-1}BP)^{100} = P^{-1}B^{100}P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{100}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{100} & 0 \\ 0 & 0 & 3^{100} \end{bmatrix}$$

Then $B^{100} = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{100} & 0 \\ 0 & 0 & 3^{100} \end{bmatrix} P^{-1}$

$$= \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ -2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{100} & 0 \\ 0 & 0 & 3^{100} \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 \\ 2 & -2 & -2 \\ -1 & 1 & 3 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -1 & 2^{100} & 3^{100} \\ -2 & -2^{100} & 0 \\ 1 & 2^{100} & 3^{100} \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 \\ 2 & -2 & -2 \\ -1 & 1 & 3 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 + 2^{101} - 3^{100} & -2^{101} + 3^{100} & -1 - 2^{101} + 3^{101} \\ 2 - 2^{101} & 2^{101} & -1 + 2^{101} \\ -1 + 2^{101} - 3^{100} & -2^{101} + 3^{100} & 1 - 2^{101} + 3^{101} \end{bmatrix}$$

Consider $|A - \lambda I| = 0$ be the characteristic equation of A.

Here, we have $|A - A| = 0$. Thus the matrix A satisfies its characteristic equation.

Cayley-Hamilton Theorem: Every square matrix satisfies its characteristic equation.

Problem: Apply Cayley-Hamilton theorem to find the inverse of $B = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$.

Solution: $(1-\lambda)(2-\lambda)(3-\lambda) = 0 \Longrightarrow (2 - 3\lambda + \lambda^2)(3 - \lambda) = 0 \Longrightarrow$

Lecture # 11 **Date:** Nov 02, 2020

Problem: Apply Cayley-Hamilton theorem to find the inverse of $B = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$.

Solution: Recall the characteristic equation of B:

$$(1-\lambda)(2-\lambda)(3-\lambda) = 0 \Longrightarrow (2 - 3\lambda + \lambda^2)(3-\lambda) = 0$$
$$\Longrightarrow 6 - 11\lambda + 6\lambda^2 - \lambda^3 = 0$$

Apply Cayley-Hamilton Theorem:

$$6I - 11B + 6B^2 - B^3 = \mathbf{0} \Longrightarrow 6I = B^3 - 6B^2 + 11B$$
$$= B(B^2 - 6B + 11I)$$

$\Longrightarrow B^{-1} = \frac{1}{6}(B^2 - 6B + 11I)$ (Multiplying both sides by $\frac{1}{6}B^{-1}$)

$$\Rightarrow B^{-1} = \frac{1}{6}\left(\begin{bmatrix} 0 & 5 & 9 \\ -3 & 4 & 3 \\ -1 & 5 & 10 \end{bmatrix} - 6\begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \frac{1}{6}\begin{bmatrix} 5 & -1 & -3 \\ 3 & 3 & -3 \\ -1 & -1 & 3 \end{bmatrix}$$

Problem: Let $v_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$. Find the matrix A such that $Av_1 = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$, $Av_2 = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$ and $Av_3 = \begin{bmatrix} -7 \\ 5 \end{bmatrix}$.

Hence find Av for any vector $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \Re^3$.

Take Home Exam: Part 01

Course Code: MAT107W **Course Title:** Linear and Abstract Algebra

Total Marks: 10 **Submission Deadline:** 15 November 2020

1. Write **BY HAND** on plain papers with your **REGISTRTATION NUMBER** on the top right corner of each page.
2. Prepare **ONE PDF** file containing the images of all the pages consecutively.
3. Rename the file as **REGISTRTATION NUMBER _FULL NAME** (For example, 20198310[][]_Mr. uvw xyz)
4. Send a copy of the file to *salahuddin-mat@sust.edu* by the **DEAD LINE** with **Take Home Exam: Part 01** as the **SUBJECT** of the email.
5. **Note:** Priority will be given to the earlier received files. **RECEIVED** date and time will be considered as the **SUBMISSION** date and time.

Problem 1: Let $M = \begin{bmatrix} -1 & 2 & 0 \\ a-b & 3 & 0 \\ a & -b & 1 \end{bmatrix}$ where a and b are the last two digits of your university registration number, that is, $20198310ab$. Find the matrix that diagonalizes M. Hence compute M^{199}.

Problem 2: Let $v_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$. Find the matrix A such that $Av_1 = \begin{bmatrix} -5 \\ 8 \\ -1 \\ 0 \end{bmatrix}$, $Av_2 = \begin{bmatrix} 5 \\ -2 \\ -1 \\ -1 \end{bmatrix}$ and $Av_3 = \begin{bmatrix} 2 \\ 4 \\ -2 \\ -3 \end{bmatrix}$. Hence find Av for any vector $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \Re^3$.

Take Home Exam: Part 02

Course Code: MAT107W **Course Title:** Linear and Abstract Algebra

Total Marks: 10 **Submission Deadline:** 10 December 2020

1. Write **BY HAND** on plain papers with your **REGISTRTATION NUMBER** on the top right corner of each page.
2. Prepare **ONE PDF** file containing the images of all the pages consecutively.
3. Rename the file as **REGISTRTATION NUMBER _FULL NAME** (For example, 20198310[][]_Mr. Uvw Xyz)
4. Send a copy of the file to *salahuddin-mat@sust.edu* by the **DEAD LINE** with **Take Home Exam: Part 02** as the **SUBJECT** of the email.
5. **Note:** Priority will be given to the earlier received files. **RECEIVED** date and time will be considered as the **SUBMISSION** date and time.

Problem 1: Let $\boldsymbol{M} = \begin{bmatrix} \mathbf{2} & \mathbf{0} & -\mathbf{1} \\ \mathbf{5} & \boldsymbol{x}+\boldsymbol{y} & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{y} & \mathbf{3} \end{bmatrix}$ where $\boldsymbol{x}$ and $\boldsymbol{y}$ real numbers. Find the values of $\boldsymbol{x}$ and $\boldsymbol{y}$ so that $\boldsymbol{M}$ is nonsingular. Use Cayley-Hamilton theorem to find $\boldsymbol{M}^{-\mathbf{1}}$ for $\boldsymbol{x} = \mathbf{2}$ and $\boldsymbol{y} = -\mathbf{1}$.

Problem 2: Let the linear transformation $L: \mathfrak{R}^3 \to \mathfrak{R}^4$ be defined by $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \\ z \\ x \end{bmatrix}$, $\forall x, y, z \in \mathfrak{R}$. Find the transformation matrix $[L]$ with respect to the bases $S = \left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ and $T = \left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\}$.

[Use any text book of linear algebra or the site https://yutsumura.com/linear-algebra/linear-transformation-from-rn-to-rm/ for sample solutions.**]**

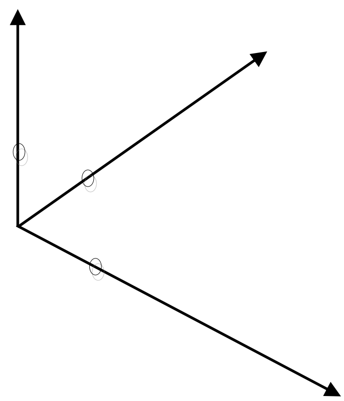
Lecture # 12 **Date:** Aug 27, 2021

Change of Basis:

Vector space: $\mathbb{R}^3$

Example:

Standard bases:



For $\mathbb{R}^3$: $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

For $\mathbb{R}^2$: $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$

For $\mathbb{R}^4$: $\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$

For a vector $u = \begin{bmatrix}x\\y\\z\\w\end{bmatrix} \in \mathbb{R}^4$, we have

$$u = \begin{bmatrix}x\\y\\z\\w\end{bmatrix} = x\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + y\begin{bmatrix}0\\1\\0\\0\end{bmatrix} + z\begin{bmatrix}0\\0\\1\\0\end{bmatrix} + w\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$$[u]_{Std} = \begin{bmatrix}x\\y\\z\\w\end{bmatrix} = u$$

Obviously, it true for any vector space, its standard basis, and any vector belonging to it.

Consider a nonstandard ordered basis: $S = \left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

For a vector $v = \begin{bmatrix}2\\5\\-3\end{bmatrix}$, we have

$$v = \begin{bmatrix}2\\5\\-3\end{bmatrix} = 2\begin{bmatrix}1\\1\\1\end{bmatrix} + 3\begin{bmatrix}0\\1\\1\end{bmatrix} - 8\begin{bmatrix}0\\0\\1\end{bmatrix}$$

Component vector of v w.r.t. S: $[v]_S = \begin{bmatrix} 2 \\ 3 \\ -8 \end{bmatrix}$

Another basis: $T = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}$

$$v = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Component vector of v w.r.t. T: $[v]_T = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$

For another vector: $v_1 = \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}$, we have

$$v_1 = \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} - 7\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Component vector of v w.r.t. T: $[v_1]_T = \begin{bmatrix} 3 \\ 5 \\ -7 \end{bmatrix}$

Question: What is $[v_1]_S$?

In general,

Two ordered bases: $S = \{s_1,\ s_2,\ s_3\}$ and $T = \{t_1,\ t_2,\ t_3\}$

Consider any vector $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$. Then v can be expressed as linear combinations of the basis vectors:

$$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = as_1 + bs_2 + cs_3$$

$$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = et_1 + ft_2 + gt_3$$

Component vectors: $[v]_S = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $[v]_T = \begin{bmatrix} e \\ f \\ g \end{bmatrix}$

We want to find a matrix:

$M_{T\leftarrow S}$ such that

$$M_{T\leftarrow S}[v]_S = M_{T\leftarrow S}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}e\\f\\g\end{bmatrix} = [v]_T$$

and $M_{S\leftarrow T}$ such that

$$M_{S\leftarrow T}[v]_T = M_{S\leftarrow T}\begin{bmatrix}e\\f\\g\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix} = [v]_S$$

Change of basis matrix from S to T**:**

$$M_{T\leftarrow S} = [[s_1]_T \quad [s_2]_T \quad [s_3]_T]$$

We have

$$s_1 = m_{11}t_1 + m_{12}t_2 + m_{13}t_3$$

$$s_2 = m_{21}t_1 + m_{22}t_2 + m_{23}t_3$$

$$s_3 = m_{31}t_1 + m_{32}t_2 + m_{33}t_3$$

Then

$$M_{T\leftarrow S} = \begin{bmatrix}m_{11} & m_{21} & m_{31}\\ m_{12} & m_{22} & m_{32}\\ m_{13} & m_{23} & m_{33}\end{bmatrix}$$

Example:

$$s_1 = \begin{bmatrix}1\\1\\1\end{bmatrix} = 1\begin{bmatrix}1\\1\\0\end{bmatrix} + 0\begin{bmatrix}0\\-1\\0\end{bmatrix} + (-1)\begin{bmatrix}0\\0\\-1\end{bmatrix} = 1t_1 + 0t_2 + (-1)t_3$$

$$s_2 = \begin{bmatrix}0\\1\\1\end{bmatrix} = 0\begin{bmatrix}1\\1\\0\end{bmatrix} + (-1)\begin{bmatrix}0\\-1\\0\end{bmatrix} + (-1)\begin{bmatrix}0\\0\\-1\end{bmatrix} = 0t_1 + (-1)t_2 + (-1)t_3$$

$$s_3 = \begin{bmatrix}0\\0\\1\end{bmatrix} = 0\begin{bmatrix}1\\1\\0\end{bmatrix} + 0\begin{bmatrix}0\\-1\\0\end{bmatrix} + (-1)\begin{bmatrix}0\\0\\-1\end{bmatrix} = 0t_1 + 0t_2 + (-1)t_3$$

Then

$$M_{T\leftarrow S} = \begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ -1 & -1 & -1\end{bmatrix}$$

Verification:

$$v = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} = 2s_1 + 3s_2 - 8s_3, \qquad [v]_s = \begin{bmatrix} 2 \\ 3 \\ -8 \end{bmatrix}$$

$$v = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} = 2t_1 - 3t_2 + 3t_3, \qquad [v]_T = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$$

$$M_{T \leftarrow S}[v]_S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -8 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} = [v]_T$$

Change of basis matrix from T to S:

$$M_{S \leftarrow T} = [[t_1]_S \quad [t_2]_S \quad [t_3]_S]$$

We have

$$t_1 = n_{11}s_1 + n_{12}s_2 + n_{13}s_3$$

$$t_2 = n_{21}s_1 + n_{22}s_2 + n_{23}s_3$$

$$t_3 = n_{31}s_1 + n_{32}s_2 + n_{33}s_3$$

Then

$$M_{S \leftarrow T} = \begin{bmatrix} n_{11} & n_{21} & n_{31} \\ n_{12} & n_{22} & n_{32} \\ n_{13} & n_{23} & n_{33} \end{bmatrix}$$

Example:

$$t_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1s_1 + 0s_2 + (-1)s_3$$

$$t_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0s_1 + (-1)s_2 + 1s_3$$

$$t_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0s_1 + 0s_2 + (-1)s_3$$

Then

$$M_{S \leftarrow T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 1 & -1 \end{bmatrix}$$

Verification:

$$v = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} = 2s_1 + 3s_2 - 8s_3, \qquad [v]_s = \begin{bmatrix} 2 \\ 3 \\ -8 \end{bmatrix}$$

$$v = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} = 2t_1 - 3t_2 + 3t_3, \qquad [v]_T = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$$

$$M_{S\leftarrow T}[v]_T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -8 \end{bmatrix} = [v]_S$$

Answer to the question:

$$v_1 = 3s_1 + 5s_2 - 7s_3 \qquad [v_1]_T = \begin{bmatrix} 3 \\ 5 \\ -7 \end{bmatrix}$$

Question: What is $[v_1]_S$?

$$M_{S\leftarrow T}[v_1]_T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \\ -7 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 9 \end{bmatrix} = [v_1]_S$$

Verification:

$$v_1 = \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 5\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 9\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 3s_1 - 5s_2 + 9s_3$$

Lecture # 13 — **Date:** Aug 27, 2021

Groups, Rings, and Fields

Common sets:

1. $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$
2. $\mathbb{R}^+ = (0, \infty)$, Set of all positive real numbers.
3. $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, Set of all positive integers.
4. $n\mathbb{Z} = \{x : x = ny \text{ for some } y \in \mathbb{Z}\}, n > 1.$
   Example: $2\mathbb{Z} = \{0, \pm 2, \pm 4, \pm 6, \ldots\}$
5. $\mathcal{M}_{m\times n}$, Set of all m-by-n matrices.
   Example: $V = \{v : v \in \mathcal{M}_{n\times 1}\}$, Set of all vectors (column matrices) with n elements.
6. $P(x)$, Set all polynomials in x with coefficients from $\mathbb{R}$.
7. $\mathbb{Z}_n \subseteq \mathbb{Z}$

8. $\mathbb{Z}_n = \{x : x = y \bmod n \text{ for some } y \in \mathbb{Z}\}$, Set of all positive integers (remainder) modulo n.
   $Example$: $\mathbb{Z}_4 = \{0,1,2,3\}$.
9. $\mathbb{Z}_p = \{x : x = y \bmod p \text{ for some } y \in \mathbb{Z}\}$, p is a prime, Set of all positive integers (remainder) modulo p.
10. $\overline{\mathbb{Z}}_p = \{1, x : x \text{ is a prime and } x = y \bmod p \text{ for some } y \in \mathbb{Z}\}$, p is a prime; Set of all primes including 1 for positive integers modulo p.
    $Example$: $\overline{\mathbb{Z}}_7 = \{1,2,3,5\}$.
11. $\{1, -1, i, -i\}$, Here i is the imaginary unit defined as $i^2 = -1$.
12. $\{0\}$ and $\{1\}$, Two singleton sets

Binary Operations:

1. $" + "$, addition of real numbers
2. $" . "$ or $" \times "$, multiplication of real numbers
3. $" . "$, dot product of vectors
4. $" . "$, element wise multiplication for matrices
5. $" \times "$, matrix multiplication

Some Properties:

Let $" * "$ be a binary operation defined on the set X.

1. **Closure property:** X is called closed under $" * "$ if $a * b \in X$ for all $a, b \in X$.
2. **Commutative Property:** $" * "$ is called commutative if $a * b = b * a$ for all $a, b \in X$.
3. **Associative property:** $" * "$ is called associative if $a * (b * c) = (a * b) * c$ for all $a, b, c \in X$.
4. **Existence of identity:** Identity w.r.t. $" * "$ exists in X if there is an element e in X s.t. $a * e = e * a = a$ for all $a \in X$.
5. **Existence of inverses:** Inverses w.r.t. $" * "$ exist in X if there is an element a^{-1} in X s.t. $a * a^{-1} = a^{-1} * a = e$ for every $a \in X$.

Some Observations:

1. $\mathbb{R}$
   a. Closed under addition and multiplication.
   b. Addition and multiplication are associative.
   c. Additive identity “0” and inverses exist.
   d. Multiplicative identity “1” exists and inverses exist for all elements except “0”.

2. $\mathbb{Q} \subseteq \mathbb{R}$
   a. Closed under addition and multiplication.
   b. Addition and multiplication are associative.

c. Additive identity “0” and inverses exist.
d. Multiplicative identity “1” exists and inverses exist for all elements except “0”.

3. $\mathbb{Z} \subseteq \mathbb{Q}$
a. Closed under addition and multiplication.
b. Addition and multiplication are associative.
e. Additive identity “0” and inverses exist.
c. Multiplicative identity “1” exists and inverses do not exist for any element except “1”.

4. $n\mathbb{Z} \subseteq \mathbb{Z}, n > 1$
a. Closed under addition and multiplication.
b. Addition and multiplication are associative.
c. Additive identity “0” and inverses exist.
d. Multiplicative identity and inverses do not exist.

5. $\mathbb{Z}^{+} \subseteq \mathbb{Z}$
a. Closed under addition and multiplication.
b. Addition and multiplication are associative.
c. Additive identity “0” and inverses do not exist.
d. Multiplicative identity “1” exists but inverses do not exist for any element except “1”.

6. $\mathcal{M}_{m\times n}$, (Multiplication is defined only in the cases when $m = n$)
a. Closed under addition and multiplication.
b. Addition and multiplication are associative.
c. Additive identity “$Z_{m\times n}$” and inverses exist. ($Z_{m\times n}$ are the zero matrices)
d. Multiplicative identity “I_n” and inverses exist for all elements except the matrices $\mathcal{M}_n$ with $|\mathcal{M}_n| = 0$.

7. $P(x)$
a. Closed under addition and multiplication.
b. Addition and multiplication are associative.
c. Additive identity “0” and inverses exist.
d. Multiplicative identity “1” exists but inverses do not exist for any element except “1”.

8. $\mathbb{Z}_n \subseteq \mathbb{Z}$
a. Closed under addition and multiplication.
b. Addition and multiplication are associative.

c. Additive identity "0" and inverses of all elements exist.
d. Multiplicative identity "1" exists and inverses exist for all elements except the "0".

9. $\mathbb{Z}_p \subseteq \mathbb{Z}$, p is a prime
a. Closed under addition and multiplication.
b. Addition and multiplication are associative.
c. Additive identity and inverses exist.
d. Multiplicative identity exists and inverses exist for all elements except the "0".

10. $\overline{\mathbb{Z}}_p \subseteq \mathbb{Z}$, p is a prime
a. Closed under multiplication.
b. Addition and multiplication are associative.
c. Multiplicative identity and inverses exist.

11. $\{1, -1, i, -i\}$ and $\{1\}$
a. Closed under multiplication only.
b. Multiplication is associative.
c. Multiplicative identity and inverses exist.

12. $\{0\}$
a. Closed under addition and multiplication.
b. Addition and multiplication are associative.
c. Only additive identity "0" and inverse exist.

Groups:
Consider a nonempty set X and a binary operation "$*$". Then X is a group under "$*$" if
1. X is closed under "$*$".
2. "$*$" is associative on X.
3. An identity "e" w.r.t. "$*$" exists in X.
4. Inverses "x^{-1}" w.r.t. "$*$" exist for every $x \in X$.

A group is called c**ommutative** (**abelian**) if the operation "$*$" is commutatative.

Examples:
1. $\mathbb{R}$ under addition
2. $\mathbb{Q}$ under addition
3. $\mathbb{Z}$ under addition
4. $n\mathbb{Z}$, $n > 1$ under addition
5. $\mathcal{M}_{m \times n}$ under addition
6. $P(x)$ under addition

7. $\mathbb{Z}_n$ under addition
8. $\mathbb{Z}_p \subseteq \mathbb{Z}$, p is a prime, under addition
9. $\{1, -1, i, -i\}$ under multiplication
10. $\{1\}$ under multiplication
11. $\{0\}$ under addition
12. $\{0\}$ under multiplication

Explain why not a group?

1. $\mathbb{R}$ under multiplication
2. $\mathbb{Q}$ under multiplication
3. $\mathbb{Z}$ under multiplication
4. $n\mathbb{Z}$, $n > 1$ under multiplication
5. $\mathbb{Z}^+$ under addition
6. $\mathbb{Z}^+$ under multiplication
7. $\mathbb{Z}_p$, p is a prime, under multiplication
8. $\overline{\mathbb{Z}}_p$, p is a prime, under addition or multiplication
9. $\mathcal{M}_{m\times n}$ under matrix multiplication (Here, $m = n$)
10. $\{1, -1, i, -i\}$ under addition
11. $\{1\}$ under addition

Rings:

Consider a nonempty set X and two binary operations "$+$" and "$\times$". Then X is a ring if

1. X is a commutative group under "$+$". Here additive identity is "0".
2. X is closed under "$\times$".
3. "$\times$" is associative on X.
4. An identity "1" w.r.t. "$\times$" exists in X.
5. The operations are *distributive*, that is,
   $a \times (b + c) = a \times b + a \times c$ and $(a + b) \times c = a \times c + b \times c$.

Examples:

1. $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$
2. $P(x)$
3. $\mathbb{Z}_n$
4. $\mathbb{Z}_p$, p is a prime
5. $\{0\}$ with multiplicative inverse

Fields:

Consider a nonempty set X and two binary operations "$+$" and "$\times$". Then X is a ring if

1. X is a commutative group under "$+$". Here additive identity is "0".

2. $X - \{0\}$ is a commutative group under "$\times$". Here multiplicative identity is "1".
3. The operations are *distributive*, that is,
   $a \times (b + c) = a \times b + a \times c$ and $(a + b) \times c = a \times c + b \times c$.

Examples:

1. $\mathbb{R}$ and $\mathbb{Q}$
2. $\mathbb{Z}_p$, p is a prime

Important Questions:

1. Is there any subset of $\mathcal{M}_{m\times n}$ that can be a group under some operation? (There are at least three groups, even more)
2. Is a vector space a group? If so, what is group operation?
3. Is a vector space a ring or a field?

We write xy for $x * y$.

Subgroup:

A subset H of a group G is called a subgroup if it is nonempty, closed under the group operation and has inverses.
We write $H \le G$ when H is a subgroup of G.

Since H, there exists $x \in H$. Then H contains inverses of its elements $x^{-1} \in H$. Since H is closed under the group operation, $e = xx^{-1} \in H$.

Example:

1. $\{e\} \le G$ for any group G
2. $n\mathbb{Z} \le \mathbb{Z} \le \mathbb{Q} \le \mathbb{R},\ n > 1$, all under addition
3. $\mathcal{M}_m \le \mathcal{M}_{m\times n},\ m \le n$, under addition

Center of a group:

The center of a group G is the set of elements which commute with every element of G and it is denoted by $C(G)$. Thus

$$C(G) = \{x \in G : xh = hx \text{ for all } h \in G\}$$

Problem: The $C(G)$ is the subgroup of G.

Solution: $C(G)$ is itself a group because

1. For $e \in G$, $ex = xe$ for all $x \in G$. Thus $e \in C(G)$ and hence $C(G)$ is nonempty.
2. Let $x, y \in C(G)$. Let $h \in G$ be any element. Then $xh = hx$ and $yh = hy$. Then $(xy)h = x(yh) = x(hy) = (xh)y = (hx)y = h(xy) = h(xy)$. Then $xy \in C(G)$.
3. Let $x \in C(G)$. Let $h \in G$ be any element. Then $xh = hx$. Then $(xh)x^{-1} = (hx)x^{-1}$. Then $x(hx^{-1}) = h$. Then $x^{-1}x(hx^{-1}) = x^{-1}h$. Then $x^{-1}h = hx^{-1}$. Thus $x^{-1} \in C(G)$.

Hence $C(G)$ is a subgroup of G.
We write x^2 for $x * x$, for example, $x^2 = x + x = x \times x$

Order of an element:
For $x \in G$, order of x is the least integer n such that $x^n = e$.

Cyclic groups:
A group is called cyclic if it can be generated by a single element. In other words, there exist an element $x \in G$ such that $G = \{e,\ x,\ x^2,\ x^3, \dots, x^n\}$.
Example: $\{1, -1, i, -i\} = \{1, i, i^2, i^3\}$

Cosets:
The **left** coset of $H \leq G$ with respect to an element $g \in G$ is the set of all elements which can be obtained by multiplying g with an element of H, $gH = \{gh : h \in H\}$.

1. $eH = He = H$
2. Let $G = \mathbb{Z}$ and $H = 2\mathbb{Z}$. Then $0H = 2H = \cdots = H$ and $1H = 3H = \cdots (2n+1)H, n \in \mathbb{Z}$.

Lagrange Theorem:
Given a group G and a subgroup H of this group, the order of H divides the order of the group G.

Conjugates:
Let two elements g and n from a group G. The conjugate of n by g is the group element gng^{-1}.
We define the conjugate of a set $N \subseteq G$ by g as $gNg^{-1} = \{gng^{-1} :\ n \in N\}$.

Normal subgroup:
A subgroup N of G is normal if for every element g in G, the conjugate of N is N itself, that is, $gNg^{-1} = N$.

Quotient group:
Given a group G and a normal subgroup N, the group of cosets formed is known as the quotient group and is denoted by $\frac{G}{N}$.
Example: $\frac{\mathbb{Z}}{2\mathbb{Z}} = \{2\mathbb{Z}, (2n+1)2\mathbb{Z}, n \in \mathbb{Z}\}$. Then $\left|\frac{\mathbb{Z}}{2\mathbb{Z}}\right| = 2$.

Theorem. Given a group G and a normal subgroup N, $|G| = |N| \left|\frac{G}{N}\right|$.