SWE 123

Discrete Mathematics

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Chapter: 09

Graph Theory

Definition:

What is Graph?

Graph:

A graph G = (V, E) consists of V, a nonempty set of vertices (or nodes) and E, a set of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints.

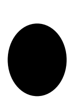
Also,

A graph G = (V, E) consists of a set V of vertices (also called nodes) and a set E of edges.

Graph is a structure defined as G (V, E),

where V is a set of vertices, V = {V1, V2, …, Vn} and

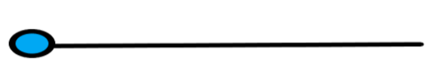
E is a set of edges, E = {E1, E2, …, En}.

 **Node / Vertex** **Edge / Line**

Incident:

If an edge connects to a vertex, we say the edge is incident to the vertex and say the vertex is an endpoint of the edge.

Also, a vertex is incident to an edge if the vertex is one of the two vertices the edge connects.



Incidence:

An incidence is a pair (**u**, **e**) where **u** is a vertex and **e** is an edge incident to **u**.

Two distinct incidences (u, e) and (v, f) are adjacent if and only if u = v, e = f or uv = e or f.

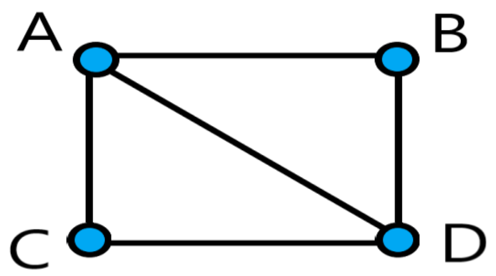
Adjacent

Adjacent vertex:

If two vertices are joined by the same edge, they are called adjacent vertex. Simply, two vertices are called **adjacent** if they share a common edge.

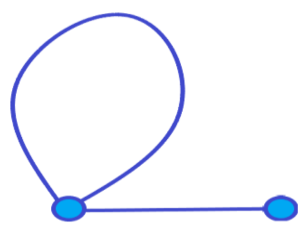
Adjacent edge:

If two edges are incident on same vertex, they are called adjacent vertex. Simply, two edges of a graph are called **adjacent** if they share a common vertex.



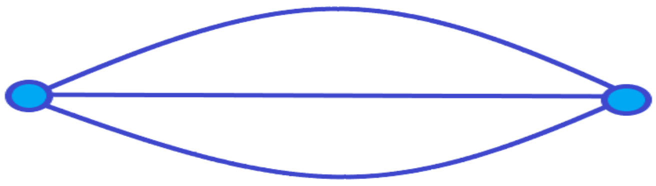
Self-loop:

If an edge has only one end-point then it is called a self-loop or loop edge.



Parallel edges:

If two or more edges have the same endpoints then they are called multiple or parallel edges.



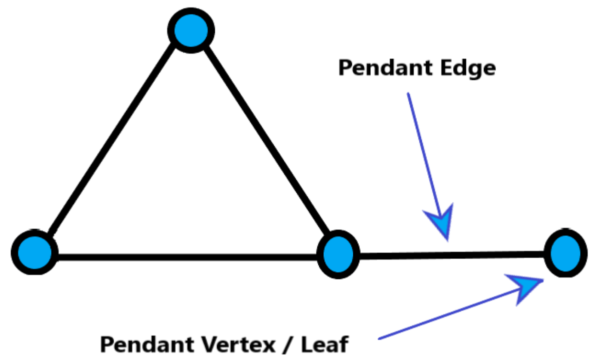
Pendant:

Pendant Vertex:

A pendant vertex is a vertex that is connected to exactly one other vertex by a single edge.

Pendant Edge:

An edge of a graph is said to be pendant if one of its vertices is a pendant vertex.

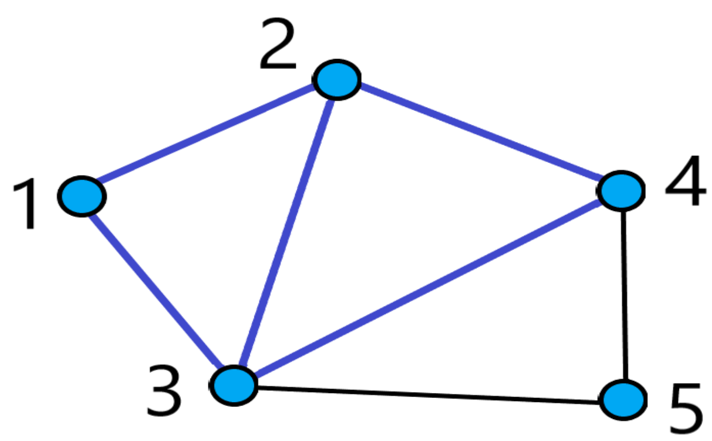


Walk:

A walk in a graph is a sequence of alternating vertices and edges v1 e1 v2 e2 …. vn en vn+1 with n ≥ 0. If we traverse a graph then we get a walk.

Vertex can be repeated

Edges can be repeated



Here 1->2->3->4->2->1->3 is a walk.

Walk can be open or closed.

Walk can repeat anything (edges or vertices).

Open walk:

A walk is said to be an open walk if the starting and ending vertices are different i.e., the origin vertex and terminal vertex are different.

In the above diagram:
1->2->3->4->5->3-> is an open walk.

Closed walk:

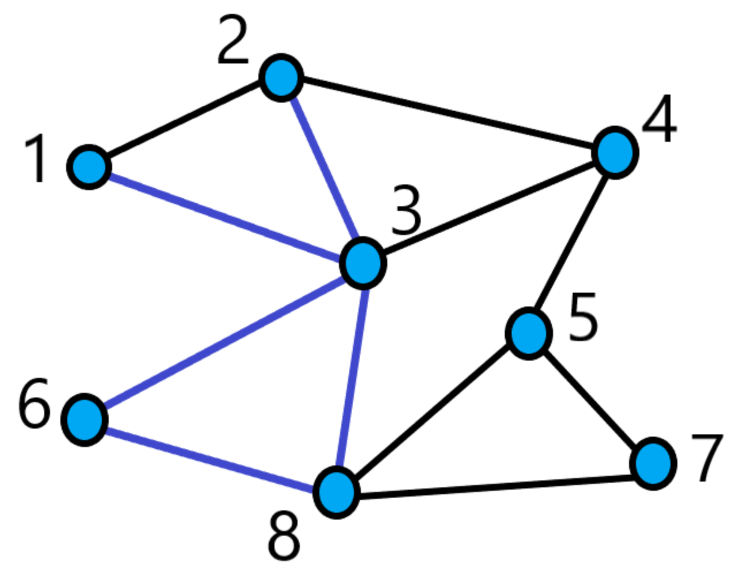
A walk is said to be a closed walk if the starting and ending vertices are identical i.e., if a walk starts and ends at the same vertex, then it is said to be a closed walk.

In the above diagram:
1->2->3->4->5->3->1-> is a closed walk.

Trail:

Trail is an open walk in which no edge is repeated.

Vertex can be repeated.



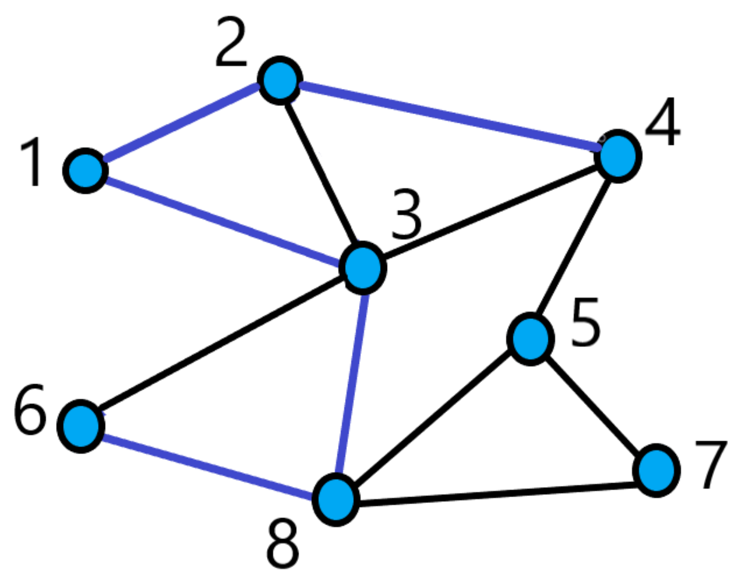
Here 1->3->8->6->3->2 is trail.
Also 1->3->8->6->3->2->1 will be a closed trail.

Path:

Path is a trail in which no vertex is repeated.

It is a trail in which neither vertices nor edges are repeated i.e., if we traverse a graph such that we do not repeat a vertex and nor we repeat an edge. As path is also a trail, thus it is also an open walk.

Vertex not repeated
Edge not repeated



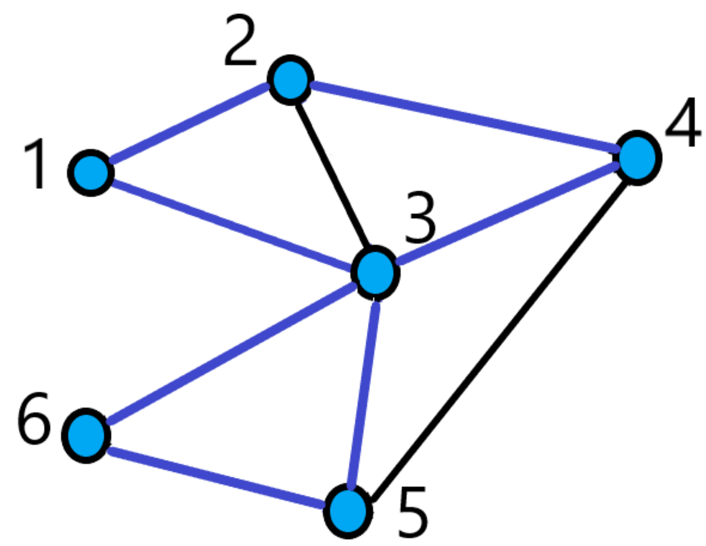
Here 6->8->3->1->2->4 is a Path.

Circuit:

Circuit is a trail in which vertex can be repeated.

Traversing a graph such that not an edge is repeated but vertex can be repeated and it is closed also i.e., it is a closed trail.

Vertex can be repeated
Edge not repeated



Here 1->2->4->3->6->8->3->1 is a circuit

Circuit is a closed trail.

These can have repeated vertices only.

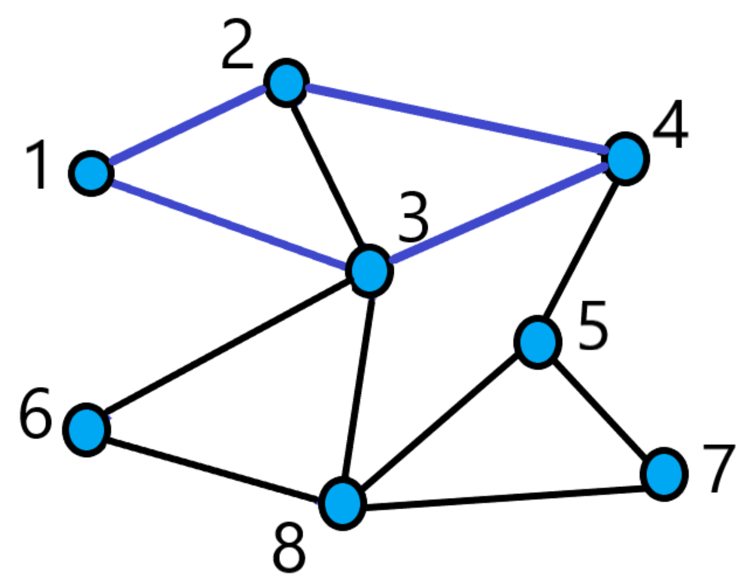
Cycle or Simple Circuit:

Cycle:

Cycle is a path in which only the starting and ending vertex are same.

Simple Circuit:

A simple circuit is a circuit that does not have any other repeated vertex except the first and last.



Here 1->2->4->3->1 is a cycle.

Cycle is a closed path.

These cannot have repeat anything (neither edges nor vertices).

Cycle or Simple Circuit:

Traversing a graph such that we do not repeat a vertex nor we repeat a edge but the starting and ending vertex must be same i.e. we can repeat starting and ending vertex only then we get a cycle.

Vertex not repeated
Edge not repeated

Walk

Trail

Path **Circuit**

Cycle **Simple circuit**

Types of GRAPHS

Though, there are a lot of different types of graphs depending upon the number of vertices, number of edges, interconnectivity, and their overall structure, some of such common types of graphs are as follows:

1)Finite Graph:

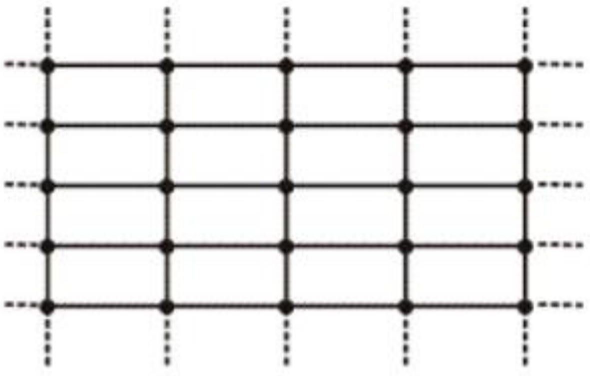
A graph with a finite vertex set and a finite edge set is called a finite graph.

Example:

2)Infinite Graph:

A graph with an infinite vertex set or an infinite number of edges is called an infinite graph.

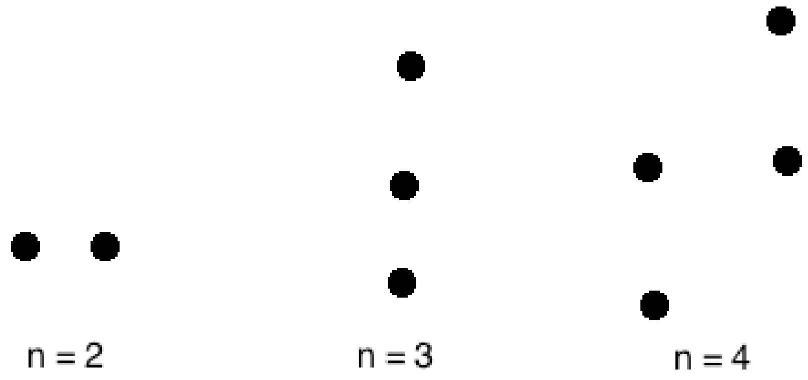
Example:



3) Null Graph:

A **null graph** is a graph in which there are no edges between its vertices. A null graph is also called empty graph.

Example:

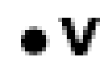


A null graph with n vertices is denoted by Nn.

4)Trivial Graph:

A trivial graph is the graph which has only one vertex.

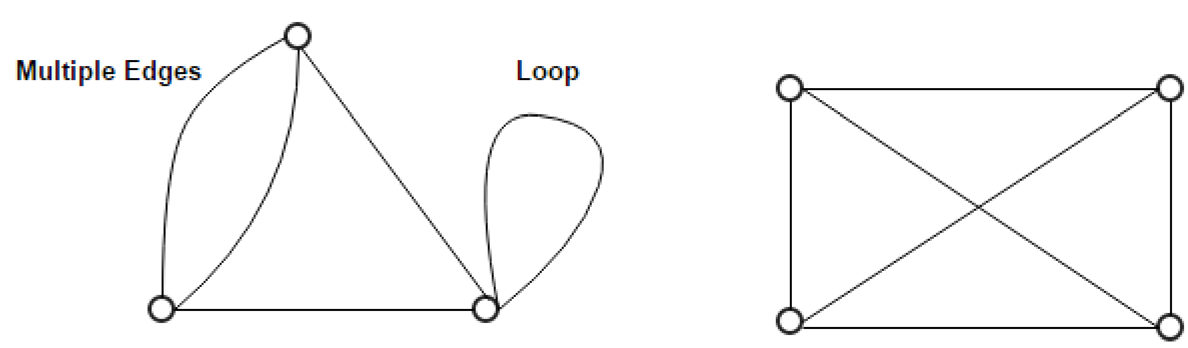
Example



In the above graph, there is only one vertex 'v' without any edge. Therefore, it is a trivial graph.

5)Simple Graph:

A **simple graph** is the undirected graph with **no parallel edges** and **no loops**. A simple graph which has n vertices, the degree of every vertex is at most n – 1.



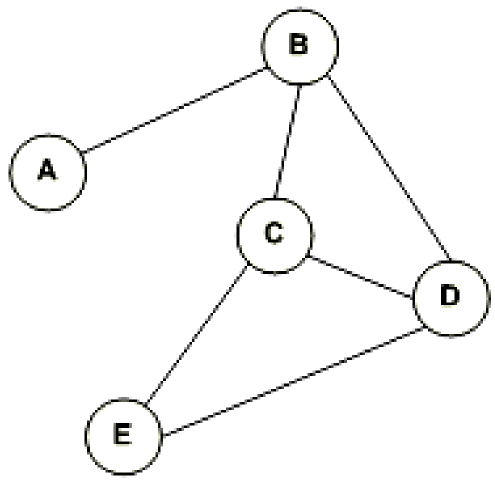
Not a Simple Graph Simple Graph

In the above example, first graph is not a simple graph because it has two edges between the vertices A and B and it also has a loop.
Second graph is a simple graph because it does not contain any loop and parallel edges.

6)Undirected Graph:

An **undirected graph** is a graph whose edges are **not directed**.

Example:

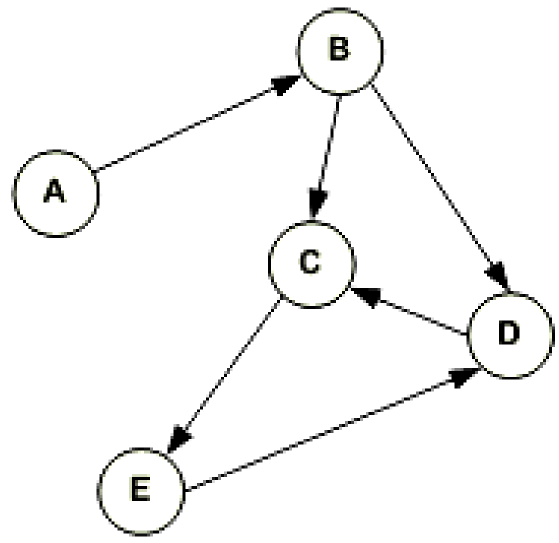


In the above graph since there is no directed edges, therefore it is an undirected graph.

7)Directed Graph:

A **directed graph** is a graph in which the **edges are directed** by arrows.
Directed graph is also known as **digraphs**.

Example:

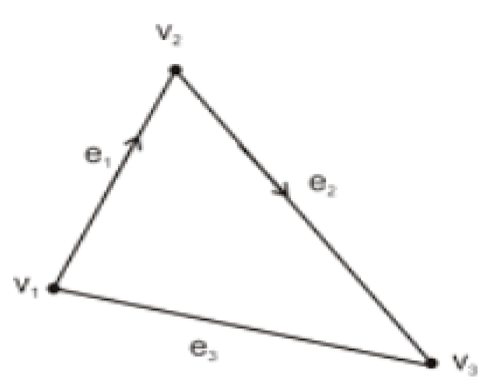


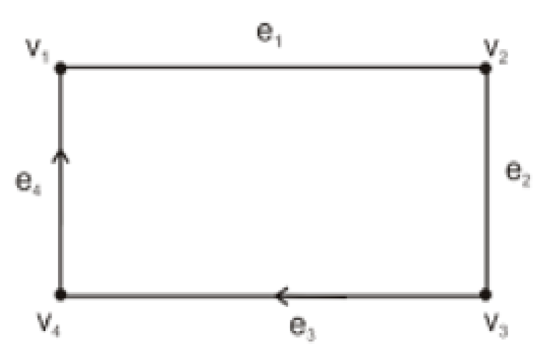
In the above graph, each edge is directed by the arrow. A directed edge has an arrow from A to B, means A is related to B, but B is not related to A.

8)Mixed Graph:

A graph with both directed and undirected edge is called a mixed graph.

Example:

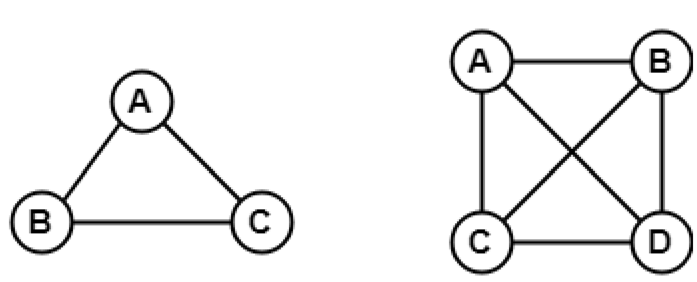




9)Complete Graph:

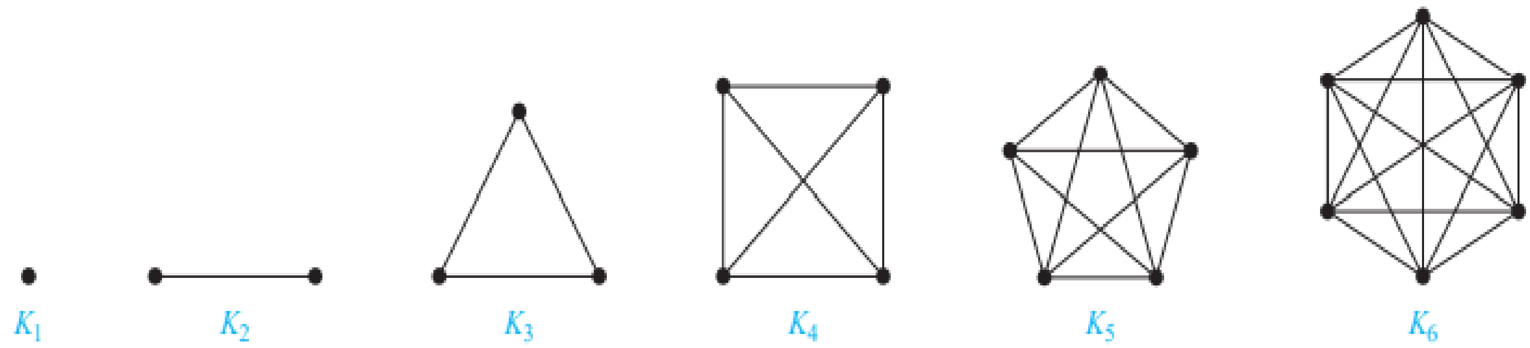
A graph in which every pair of vertices is joined by exactly one edge is called **complete graph**. It contains all possible edges.

A complete graph with n vertices contains exactly nC2 edges and is represented by Kn.
Example:



In the above example, since each vertex in the graph is connected with all the remaining vertices through exactly one edge therefore, both graphs are complete graph.

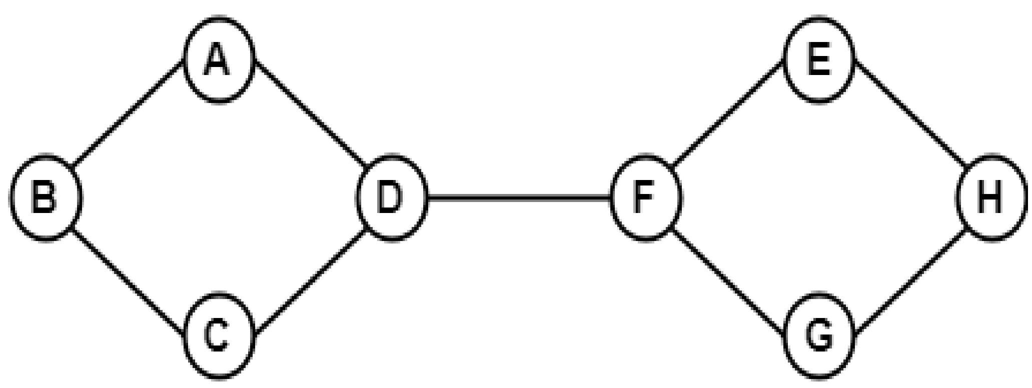
The graphs Kn, for n = 1, 2, 3, 4, 5, 6, are displayed here:



10) Connected Graph:

A **connected graph** is a graph in which we can visit from any one vertex to any other vertex. In a connected graph, at least one edge or path exists between every pair of vertices.

Example:

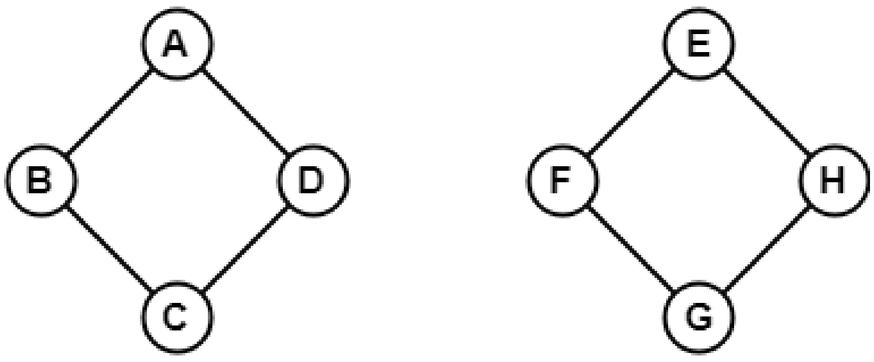


In the above example, we can traverse from any one vertex to any other vertex. It means there exists at least one path between every pair of vertices therefore, it a connected graph.

11) Disconnected Graph:

A **disconnected graph** is a graph in which any path does not exist between every pair of vertices.

Example:

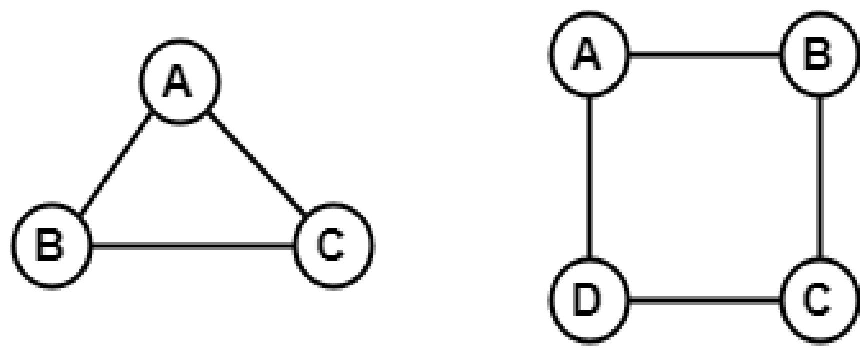


The above graph consists of two independent components which are disconnected. Since it is not possible to visit from the vertices of one component to the vertices of other components therefore, it is a disconnected graph.

12) Regular Graph:

A **Regular graph** is a graph in which degree of all the vertices is same.
If the degree of all the vertices is k, then it is called k-regular graph.

Example:



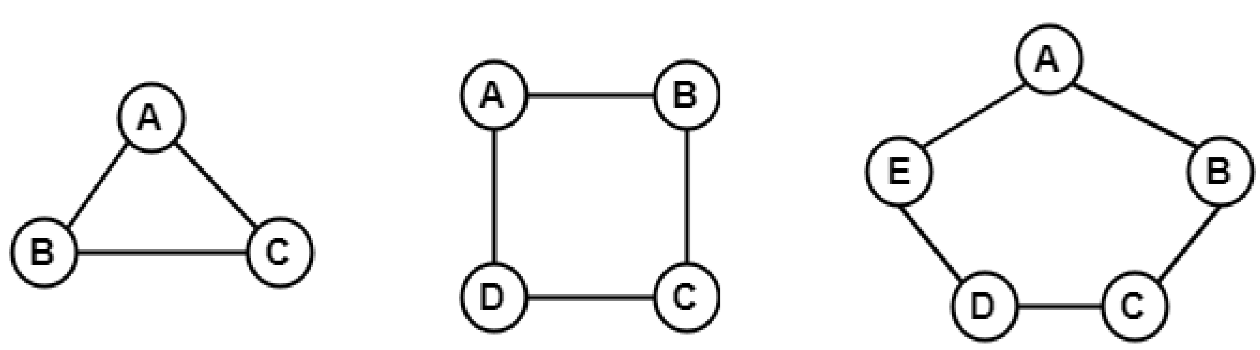
In the above example, all the vertices have degree 2. Therefore, they are called 2- **Regular graph**.

13) Cyclic Graph:

A graph with 'n' vertices (where, n>=3) and 'n' edges forming a cycle of 'n' with all its edges is known as **cycle graph**.
A graph containing at least one cycle in it is known as a **cyclic graph**.

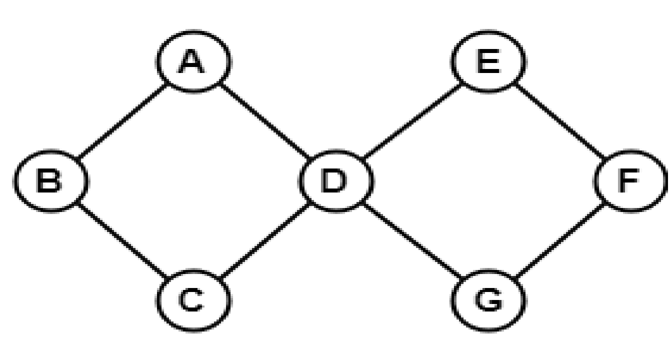
In the cycle graph, degree of each vertex is 2.
The cycle graph which has n vertices is denoted by Cn.

Example-1:



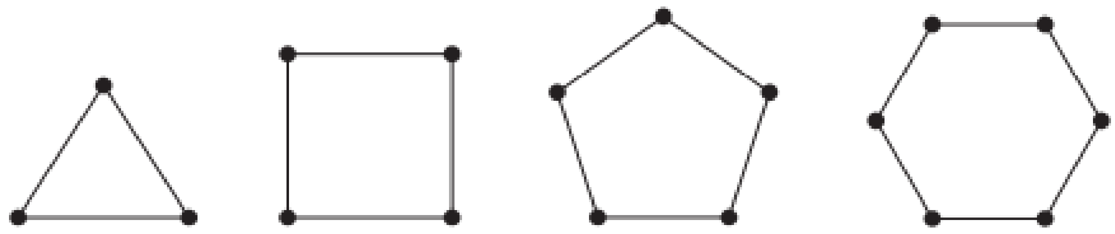
In the above example, all the vertices have degree 2. Therefore, they all are cyclic graphs.

Example-2:



Since, the above graph contains two cycles in it therefore, it is a cyclic graph.

The cycles C3, C4, C5, and C6 are displayed in here:



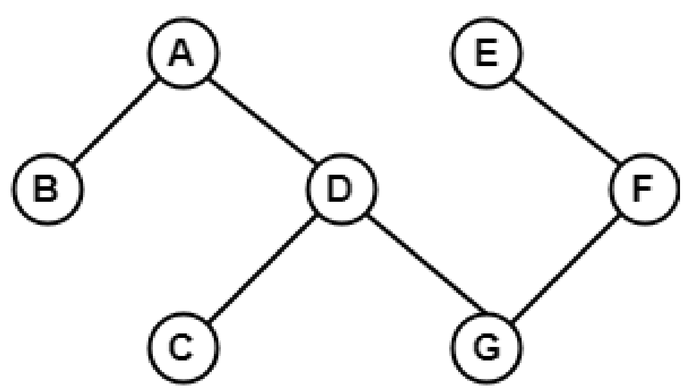
C_3 C_4 C_5 C_6

The Cycles C3, C4, C5, and C6.

14) Acyclic Graph:

A graph which does not contain any cycle in it is called as an **acyclic graph**.

Example:

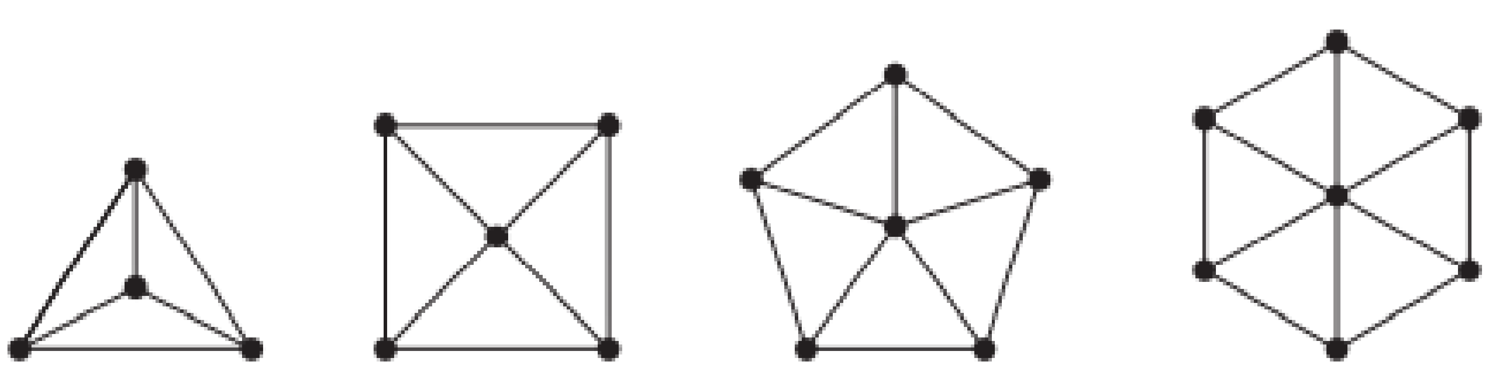


Since, the above graph does not contain any cycle in it therefore, it is an acyclic graph.

15) Wheels:

We obtain a wheel Wn when we add an additional vertex to a cycle Cn, for n ≥ 3, and connect this new vertex to each of the n vertices in Cn, by new edges.

The wheels W3, W4, W5, and W6 are displayed in here:



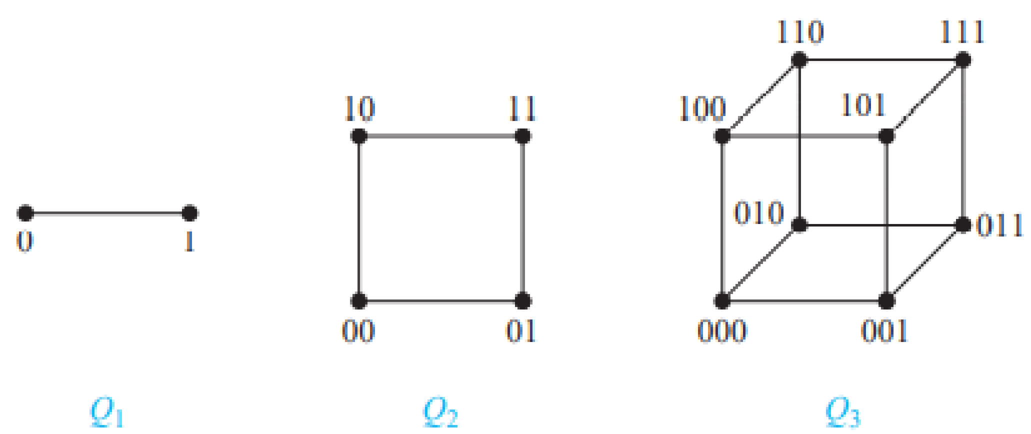
W_3 W_4 W_5 W_6

The Wheels W3, W4, W5, and W6.

16) n-Cubes

An n-dimensional hypercube, or n-cube, denoted by Qn, is a graph that has vertices representing the 2n bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position.

We display Q1, Q2, and Q3 in here:



The n-cube Qn, n = 1, 2, 3.

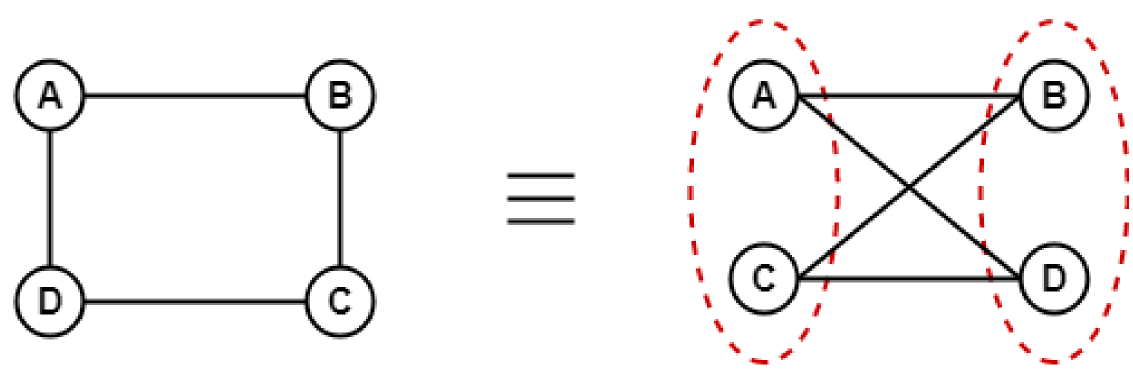
17) Bipartite Graph:

A **bipartite graph** is a graph in which the vertex set can be partitioned into two sets such that edges only go between sets, not within them.

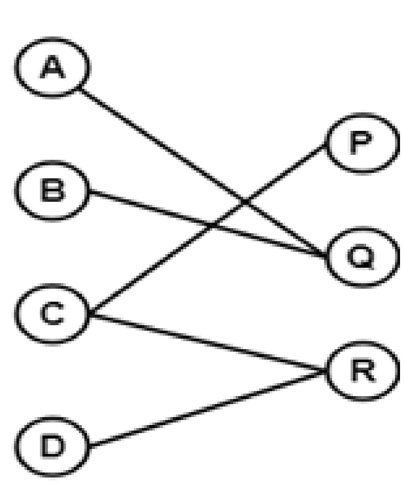
A graph G (V, E) is called bipartite graph if its vertex-set V(G) can be decomposed into two non-empty disjoint subsets V1(G) and V2(G) in such a way that each edge e ∈ E(G) has its one last joint in V1(G) and other last point in V2(G).

The partition V = V1 ∪ V2 is known as bipartition of G.

Example-1:



Example-2:



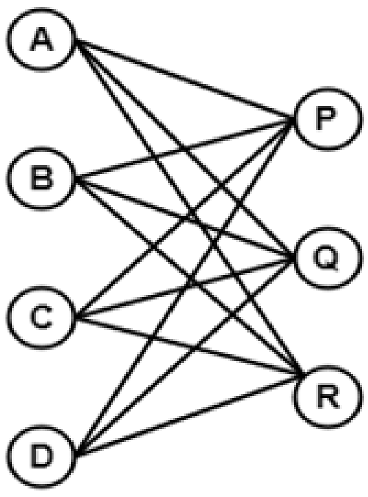
18) Complete Bipartite Graph:

A **complete bipartite graph** is a bipartite graph in which each vertex in the first set is joined to each vertex in the second set by exactly one edge.

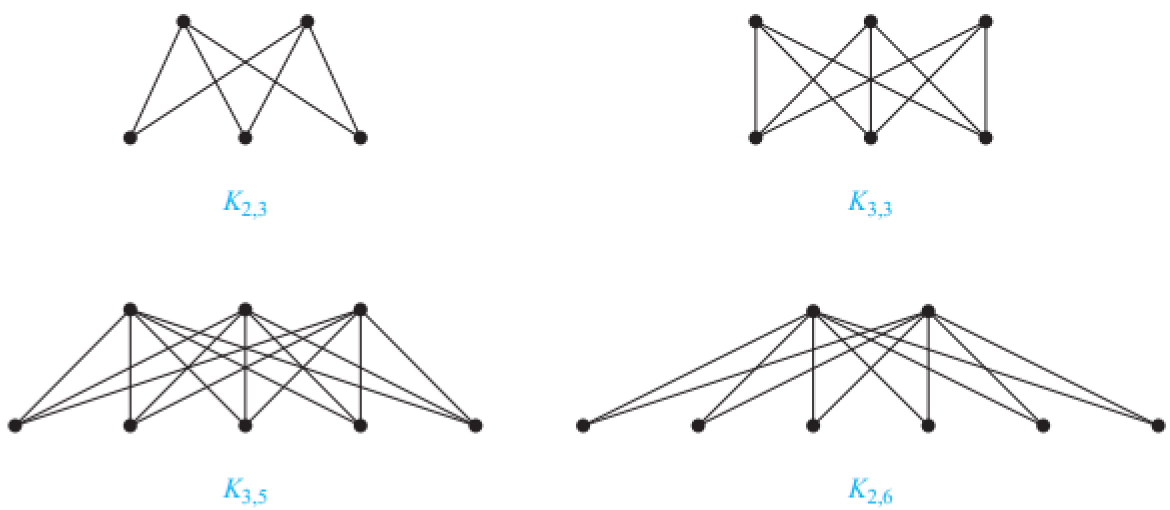
A complete bipartite graph is a bipartite graph which is complete.

1. Complete Bipartite graph = Bipartite graph + Complete graph

Example:



The above graph is known as $K_{4,3}$.

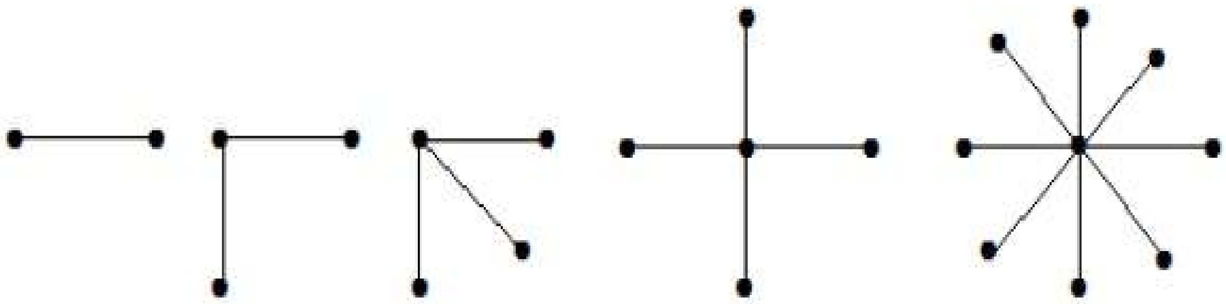


19) Star Graph:

A star graph is a complete bipartite graph in which n-1 vertices have degree 1 and a single vertex have degree (n -1). This exactly looks like a star where (n - 1) vertices are connected to a single central vertex.

A star graph with n vertices is denoted by S_n.

Example:



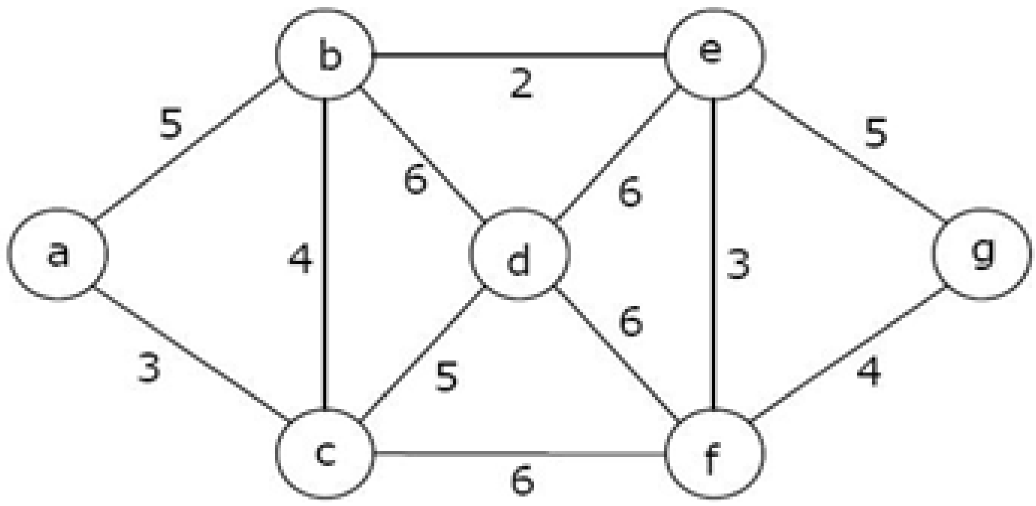
In the above example, out of n vertices, all the (n-1) vertices are connected to a single vertex. Hence, it is a star graph.

20) Weighted Graph:

A weighted graph is a graph whose edges have been labeled with some weights or numbers.

The length of a path in a weighted graph is the sum of the weights of all the edges in the path.

Example:

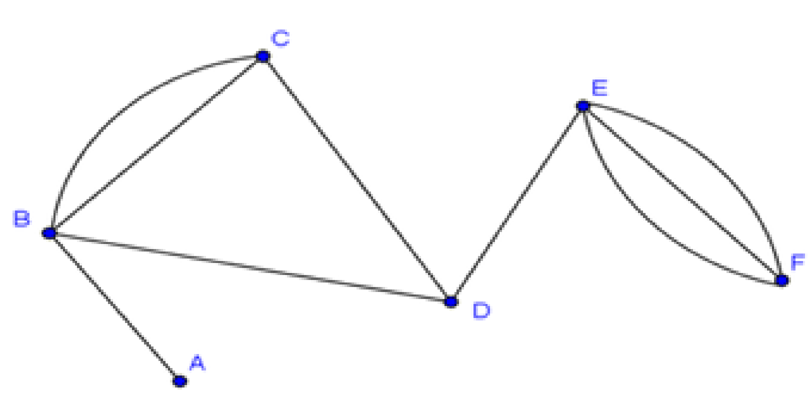


In the above graph, if path is a -> b -> c -> d -> e -> g
Then the length of the path is 5 + 4 + 5 + 6 + 5 = 25.

21) Multiple Graph:

Any graph which contains some multiple edges is called a multigraph. In a multigraph, no loops are allowed.

Example:

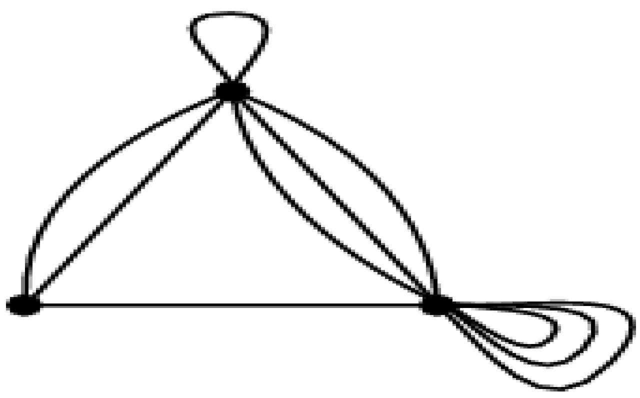


In the above graph, vertex-set B and C are connected with two edges. Similarly, vertex sets E and F are connected with 3 edges. Therefore, it is a multi-graph.

22) Pseudograph:

A graph in which **loops and multiple edges are allowed** is called **pseudograph**.

Example:

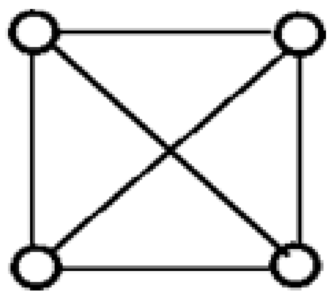


pseudograph

23) Planar Graph:

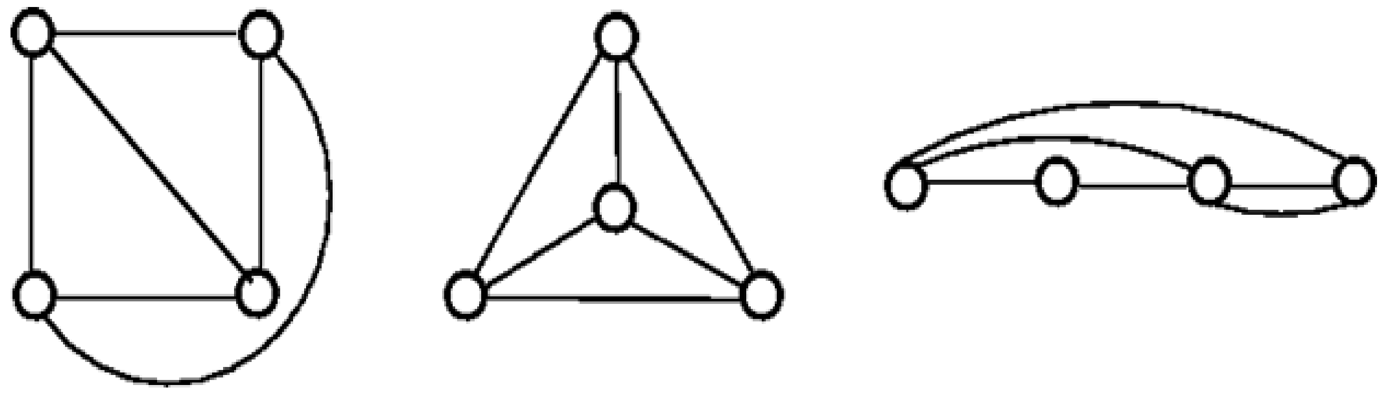
A **planar graph** is a graph that we can draw in a plane in such a way that no two edges of it cross each other except at a vertex to which they are incident.

Example:



The above graph may not seem to be planar because it has edges crossing each other. But we can redraw the above graph.

The three plane drawings of the above graph are:

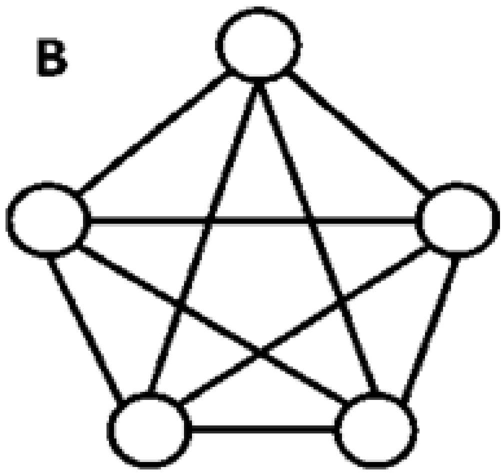


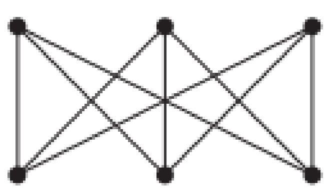
The above three graphs do not consist of two edges crossing each other and therefore, all the above graphs are planar.

24) Non – Planar Graph:

A graph that is not a planar graph is called a non-planar graph. In other words, a graph that cannot be drawn without at least on pair of its crossing edges is known as non-planar graph.

Example:





$K_{3,3}$

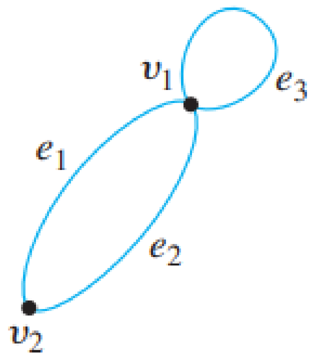
The above graph is a non - planar graph.

25) Subgraph:

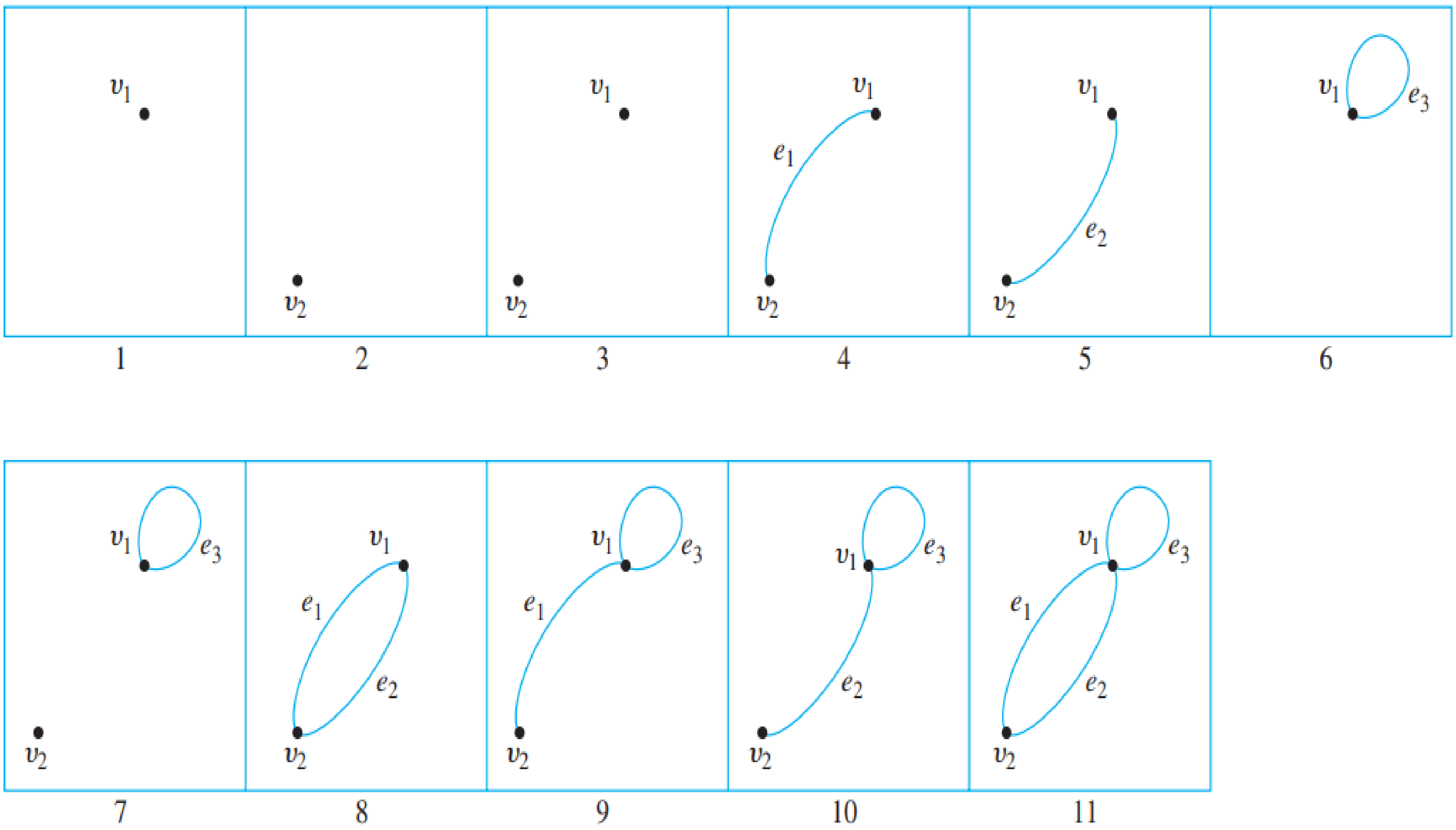
A graph H is said to be a subgraph of a graph G if, and only if, every vertex in H is also a vertex in G, every edge in H is also an edge in G, and every edge in H has the same endpoints as it has in G.

Example:

List all subgraphs of the graph G with vertex set {v1, v2} and edge set {e1, e2, e3}, where the endpoints of e1 are v1 and v2, the endpoints of e2 are v1 and v2, and e3 is a loop at v1.
G can be drawn as shown below.



There are 11 subgraphs of G, which can be grouped according to those that do not have any edges, those that have one edge, those that have two edges, and those that have three edges. The 11 subgraphs are shown here



HANDSHAKING THEOREM

Theorem:

The sum of degree of all vertices in G is twice the number of edges in G.

If G is any graph, then the sum of the degrees of all the vertices of G equals twice the number of edges of G. Specifically, if the vertices of G are $v_1, v_2, \ldots, v_n$, where n is a nonnegative integer, then

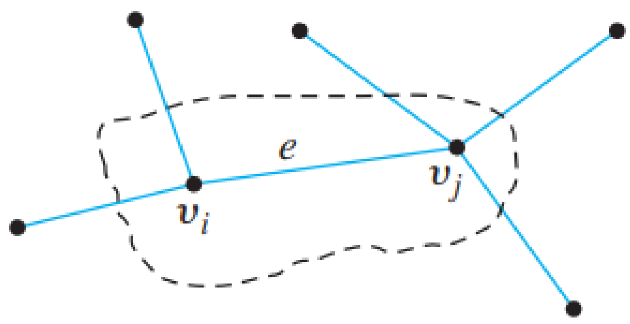
$$\boldsymbol{the\ total\ degree\ of\ G} = \mathbf{deg}(\boldsymbol{v_1}) + \mathbf{deg}(\boldsymbol{v_2}) + \cdots + \mathbf{deg}(\boldsymbol{v_n})$$
$$= \mathbf{2} \times (\boldsymbol{the\ number\ of\ edges\ of\ G})$$
$$\sum_{v \in V} \mathbf{deg}(\boldsymbol{v}) = \mathbf{2}\boldsymbol{e}$$

Means, if we sum up the degree of every vertex, it will be equal to 2 * total edges

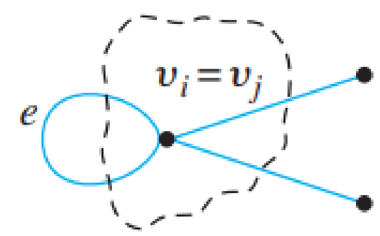
This applies even when if multiple edges and loops are present.

Proof:

Let G be a particular but arbitrarily chosen graph, and suppose that G has n vertices $v_1, v_2, \ldots, v_n$, and m edges, where n is a positive integer and m is a nonnegative integer. We claim that each edge of G contributes 2 to the total degree of G. For suppose e is an arbitrarily chosen edge with endpoints vi and vj. This edge contributes 1 to the degree of vi and 1 to the degree of vj. As shown below, this is true even if $\boldsymbol{i = j}$, because an edge that is a loop is counted twice in computing the degree of the vertex on which it is incident.



$i \neq j$



$i = j$

Therefore, e contributes 2 to the total degree of G. Since e was arbitrarily chosen, this shows that each edge of G contributes 2 to the total degree of G. Thus,

$$\boldsymbol{the\ total\ degree\ of\ G} = 2 \times (\boldsymbol{the\ number\ of\ edges\ of\ G})$$

Concept behind Handshaking Theorem:

Degree of a vertex does not depend on vertex. It depends on edges connected to it.

Corollary:

The total degree of a graph is even.

The total degree of a graph is even.

Proof:

By Handshaking Theorem, the total degree of G equals 2 times the number of edges of G, which is an integer, and so the total degree of G is even.

Proposition:

In any graph there is an even number of vertices of odd degree.

In any graph there is an even number of vertices of odd degree.

Proof:

There are two types of vertices: one with odd degree and another even degree. Suppose G is any graph, and suppose G has n vertices of odd degree and m vertices of even degree, where n is a positive integer and m is a nonnegative integer.

Consider:

Ve-> the set of vertices of even degree

Vo-> the set of vertices of odd degree

Let E be the sum of the degrees of all the vertices of even degree, O the sum of the degrees of all the vertices of odd degree, and T the total degree of G. If $v_{e_1}, v_{e_2}, \dots, v_{e_m}$ are the vertices of even degree and $v_{o_1}, v_{o_2}, \dots, v_{o_n}$ are the vertices of odd degree, then

$$\boldsymbol{E = \deg(v_{e_1}) + \deg(v_{e_2}) + \cdots + \deg(v_{e_m})}$$

$$\boldsymbol{O = \deg(v_{o_1}) + \deg(v_{o_2}) + \cdots + \deg(v_{o_n})}$$

$$\boldsymbol{T = \deg(v_{e_1}) + \deg(v_{e_2}) + \cdots + \deg(v_{e_m}) + \deg(v_{o_1}) + \deg(v_{o_2}) + \cdots + \deg(v_{o_n}) = E + O}$$

$$\boldsymbol{2e = \sum_{v \in V} \deg(V) = \sum_{v \in V_e} \deg(v_e) + \sum_{v \in V_o} \deg(v_o)}$$

$$\boldsymbol{2 \times total\ edges = sum\ of\ vertices\ having\ even\ deg + sum\ of\ vertices\ having\ odd\ deg}$$

2*total edges = sum of vertices having even deg + sum of vertices having odd deg

$$T = E + O$$

Here,

T is even because T = 2*number of edges

E is even number because the sum of even numbers is always even

$$O = T(even) - E(even)$$

Hence, O is a difference of two even integers, and so O is even.

By assumption, $\deg(v_i)$ is odd for every integer $i = 1,2,3,\dots,n$. Thus 0, an even integer, is a sum of the n odd integers $\deg(v_{o_1}), \deg(v_{o_2}), \dots, \deg(v_{o_n})$. But if a sum of n odd integers is even, then n is even. Therefore, n is even [as was to be shown].

The sum of even numbers of odd number is always even. 1+3+5+7=16

But, the sum of odd numbers of odd number is always odd. 1+3+5=9

So, the sum of odd numbers is an even number if and only if there are even number of odd numbers.

So, sum of vertices having odd degree = even.

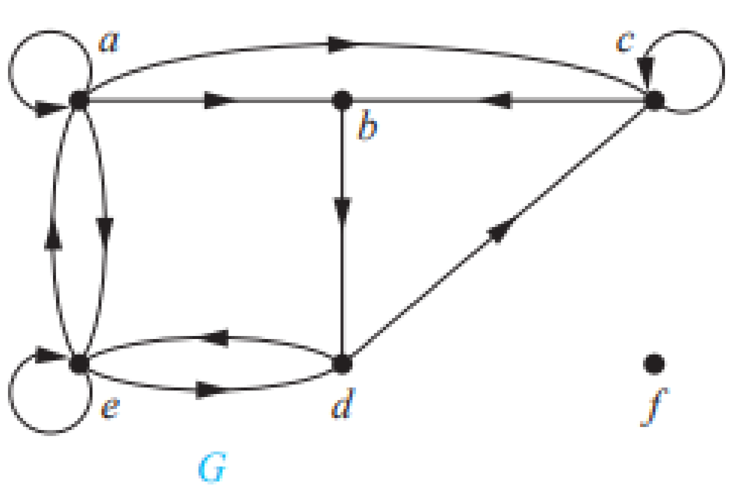
If sum is even, then number is even too.

The number of vertices of odd degree is always even.

Theorem:

Let G = (V, E) be a graph with directed edges. Then

$$\sum \deg^{-}(v) = \sum \deg^{+}(v) = |E| .$$



The in-degrees in G are deg−(a) = 2, deg−(b) = 2, deg−(c) = 3, deg−(d) = 2, deg−(e) = 3, and deg−(f) = 0. The out-degrees are deg+(a) = 4, deg+(b) = 1, deg+(c) = 2, deg+(d) = 2, deg+(e) = 3, and deg+(f) = 0.

Because each edge has an initial vertex and a terminal vertex, the sum of the in-degrees and the sum of the out-degrees of all vertices in a graph with directed edges are the same. Both of these sums are the number of edges in the graph.

Hall's Marriage Theorem

Application to marriage

Suppose there are n women and n men, all of whom want to get married to someone of the opposite sex. Suppose further that the women each have a list of the men they would be happy to marry, and that every man would be happy to marry any woman who is happy to marry him, and that each person can only have one spouse.

In this case, Hall's marriage theorem says that the men and women can all be paired off in marriage so that everyone is happy, if and only if the marriage condition holds: if in any group of women, the total number of men who are acceptable to at least one of the women in the group is greater than or equal to the size of the group.

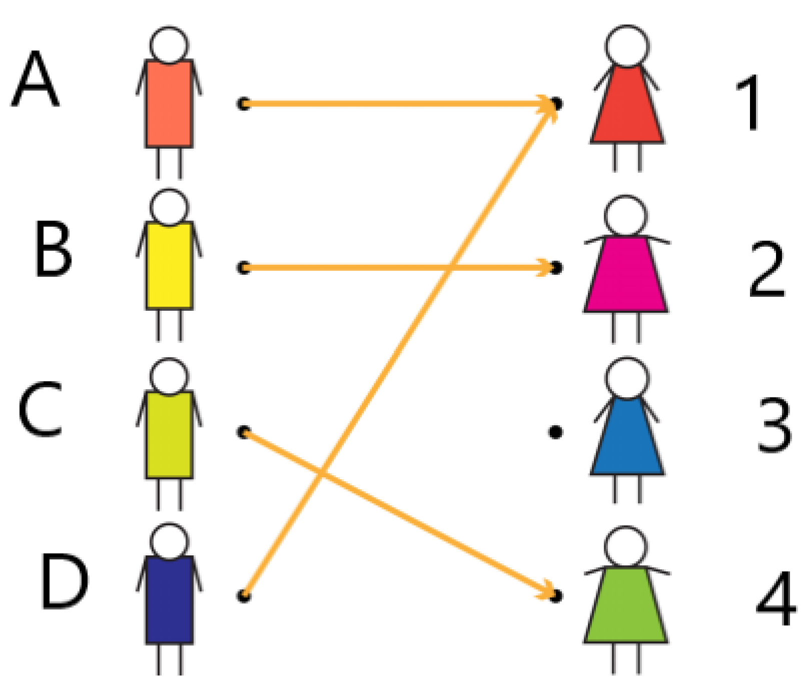
Again, it is clear that this condition is necessary. Hall's marriage theorem is that it is sufficient as well.

Hall's Marriage Theorem:

Suppose G is a bipartite graph with bipartition (A, B). There is a matching that covers A if and only if for every subset $\boldsymbol{X} \subseteq \boldsymbol{A}$, $\boldsymbol{N}(\boldsymbol{X}) \geq |\boldsymbol{X}|$ where $\boldsymbol{N}(\boldsymbol{X})$ is the number of neighbors of X.

The bipartite graph G = (V, E) with bipartition (V1, V2) has a complete matching from V1 to V2 if and only if |N(A)|≥|A| for all subsets A of V1.

Hall's Theorem tells us when we can have the perfect matching:



Let, X = {A, D} where X ⊆ Boys Group.

Now, neighbor of X, N(X) = {1}

|N(X)| = 1 and |X| = 2

So, $\boldsymbol{N}(\boldsymbol{X})\, ! \geq |\boldsymbol{X}|$

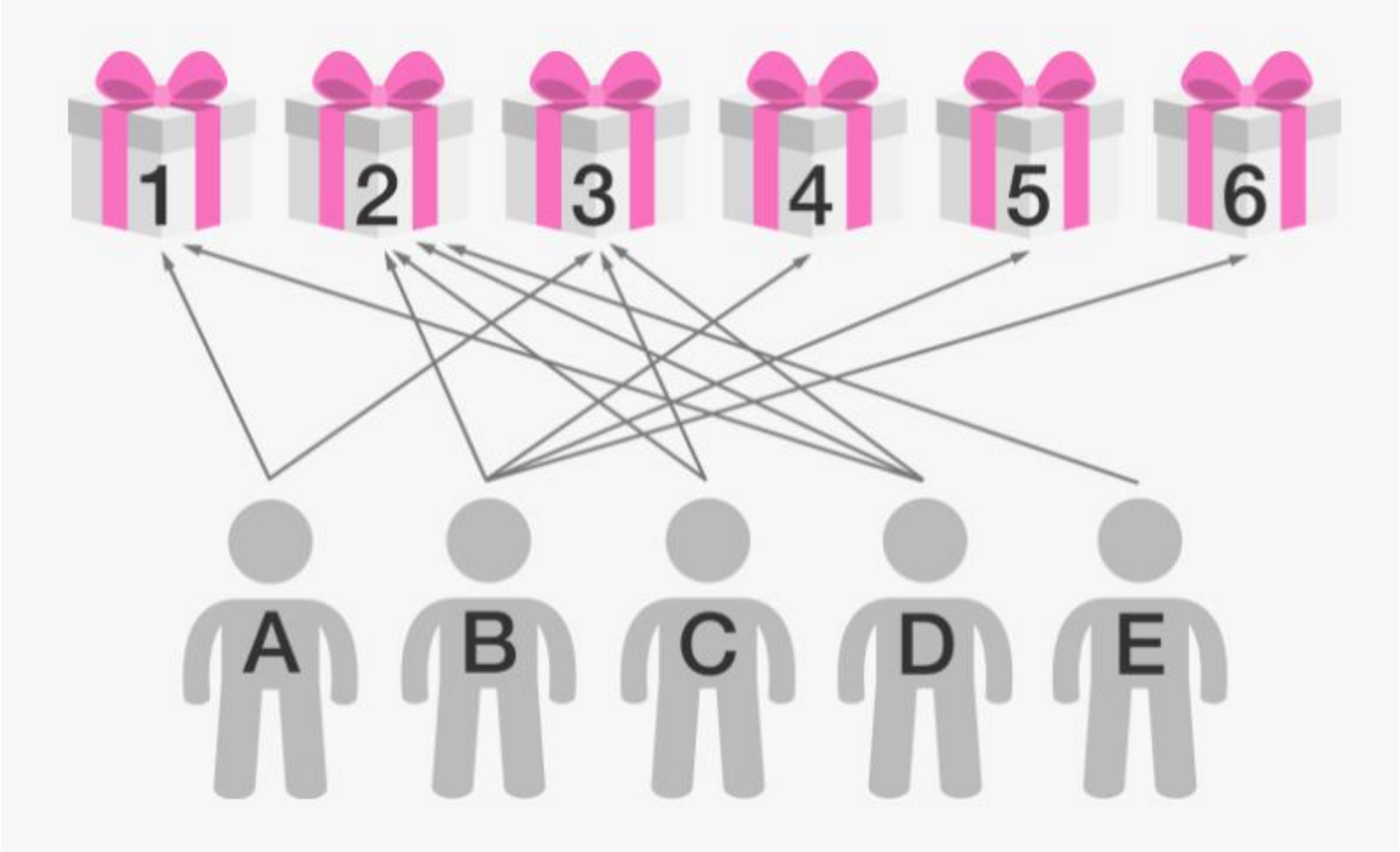
So, complete matching is not possible in this graph.

Suppose I have 6 gifts (labeled 1,2,3,4,5,6) to give at Christmas, to 5 friends (Alice, Bob, Charles, Dot, Edward).

Can I distribute one gift to each person so that everyone gets something they want?

Certainly, this depends on the preferences of my friends. If none of them like any of the gifts, then I am out of luck. But even if they all like some of the gifts, I may still not be able to give them out satisfactorily. For instance, if none of them like gifts 55 or 66, then I will have only 4 gifts to give to my 5 friends. Or suppose that

Alice wants: 1,3
Bob wants: 2,4,5,6
Charles wants: 2,3
Dot wants: 1,2,3
Edward wants: 2



There is still no way to distribute the gifts to make everyone happy. In fact, notice that I have four friends (Alice, Charles, Dot, Edward) who only want one of the first three gifts, which makes it clear that the problem is impossible.

It turns out, however, that this is the *only* way for the problem to be impossible. As long as there isn't a subset of the friends that collectively likes fewer gifts than there are friends in the subset, there will always be a way to give everyone something they want. This is the crux of Hall's marriage theorem.

EULER PATH AND CIRCUIT

An Euler path is a path that uses every edge of a graph exactly once.

An Euler circuit is a circuit that uses every edge of a graph exactly once.

I An Euler path starts and ends at different vertices.

I An Euler circuit starts and ends at the same vertex.

An Euler circuit in a graph G is a simple circuit containing every edge of G. An Euler path in G is a simple path containing every edge of G.

N.B: G must be a connected graph.

*** This theory came from a problem known as "The Seven Bridges of Königsberg". The Swiss mathematician Leonhard Euler solved this problem and he invented this theory.

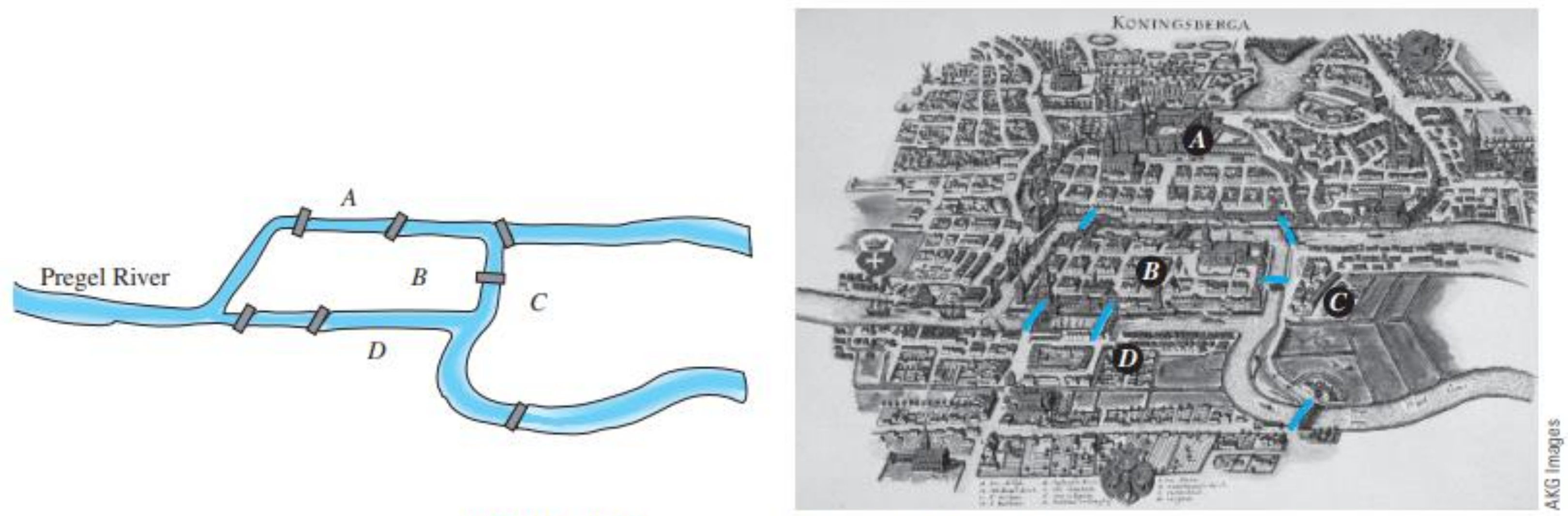
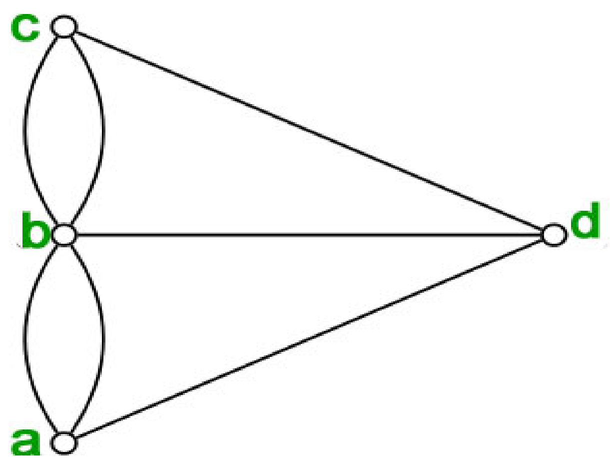


FIGURE 10.1.1 The Seven Bridges of Königsberg

The Konigsberg bridge problem's graphical representation:

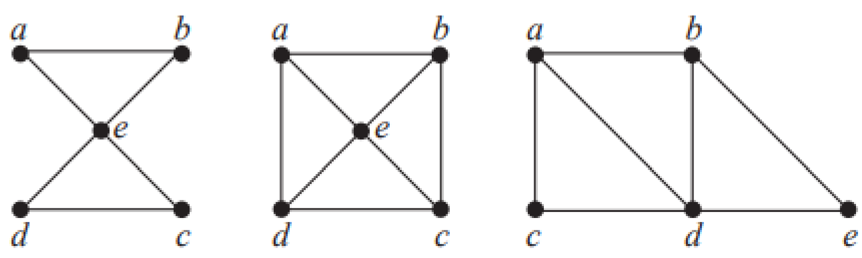


Euler Circuit:

Let G be a graph. An Euler circuit for G is a circuit that contains every vertex and every edge of G. That is, an Euler circuit for G is a sequence of adjacent vertices and edges in G that has at least one edge, starts and ends at the same vertex, uses every vertex of G at least once, and uses every edge of G exactly once.

Theorem 1: Euler Circuit's Existence for Undirected Graph

"An undirected and connected graph has an Euler Circuit if and only if each of its vertices has positive even degree."



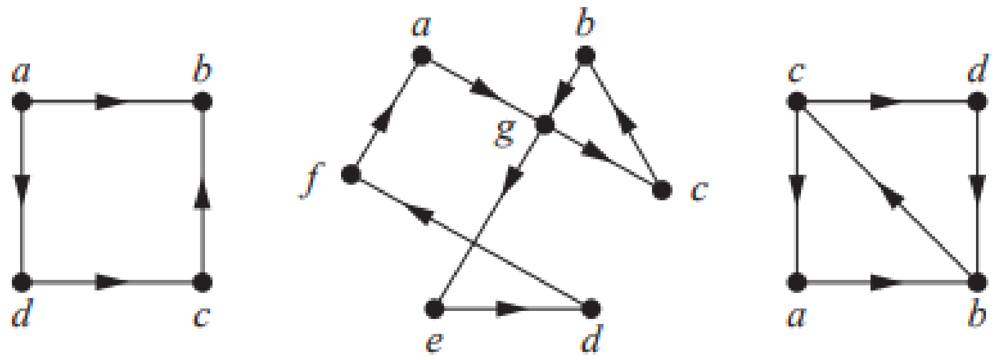
G_1 G_2 G_3

The graph G1 has an Euler circuit, for example, a, e, c, d, e, b, a. Neither of the graphs G2 or G3 has an Euler circuit (the reader should verify this). However, G3 has an Euler path, namely, a, c, d, e, b, d, a, b. G2 does not have an Euler path (as the reader should verify).

Lemma 1: Euler Circuit's Existence for Directed Graph

"A directed and connected graph has an Euler Circuit if and only if each of its vertices has degree zero."

*** Degree zero means: total_incoming = total_outgoing



H_1 H_2 H_3

The graph H2 has an Euler circuit, for example, a, g, c, b, g, e, d, f, a. Neither H1 nor H3 has an Euler circuit (as the reader should verify).

Question:

How to determine if a connected (and Directed / Undirected) graph has an Euler circuit?

Answer:

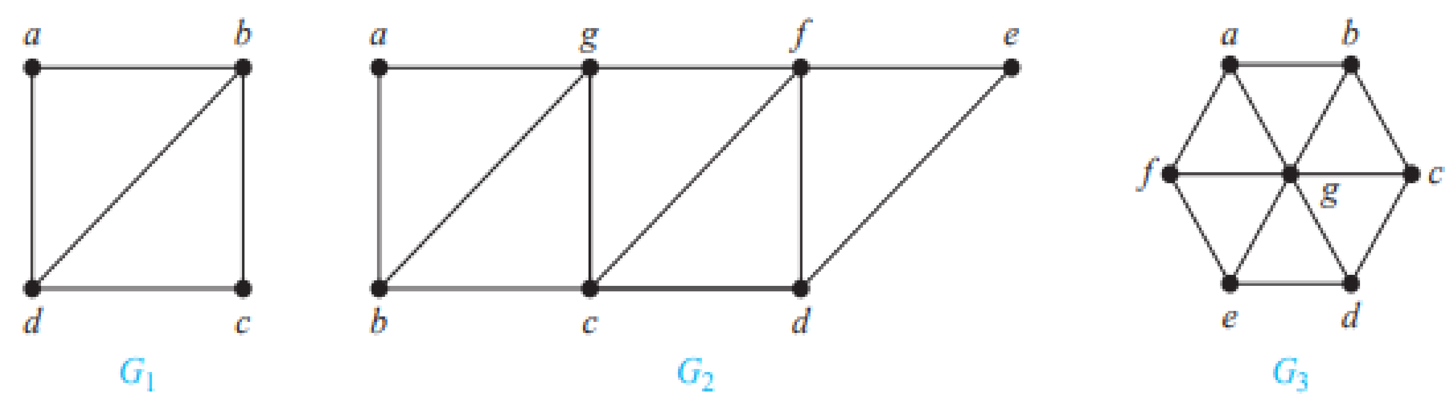
Show that the graph satisfies or doesn't satisfy the Theorem (for Undirected graph) or the Lemma (for Directed graph). If the given graph the conditions, it's an Euler Circuit, otherwise doesn't have an Euler Circuit.

Euler Path:

Let G be a graph, and let v and w be two distinct vertices of G. An Euler trail from v to w is a sequence of adjacent edges and vertices that starts at v, ends at w, passes through every vertex of G at least once, and traverses every edge of G exactly once.

Theorem 2: Euler Path's Existence for Undirected Graph

"A connected and undirected multigraph has an Euler Path but not an Euler Circuit if and only if it has two vertices of odd degree."



G1 contains exactly two vertices of odd degree, namely, b and d. Hence, it has an Euler path that must have b and d as its endpoints. One such Euler path is d, a, b, c, d, b. Similarly, G2 has exactly two vertices of odd degree, namely, b and d. So, it has an Euler path that must have b and d as endpoints. One such Euler path is b, a, g, f, e, d, c, g, b, c, f, d. G3 has no Euler path because it has six vertices of odd degree.

*** As you see there are two points of degree. One should be the beginning of your journey and the other should be the ending. You can exchange the beginning point and the ending point among them, but if you want to start and/or end from/at other vertices which have even degree, you'll not be able to complete the path, remember it.

Lemma 2: Euler Path's Existence for Directed Graph

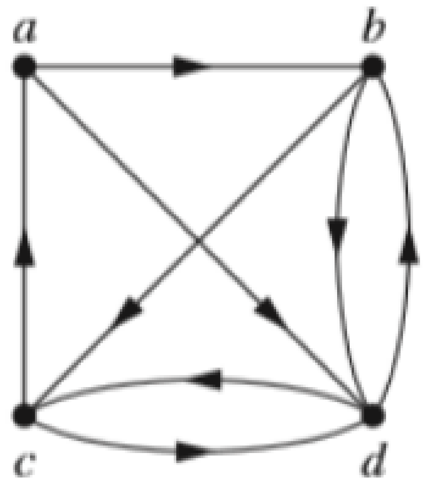
"A connected and directed multi-graph has an Euler Path but not an Euler circuit if and only if –

(i) **It has exactly one vertex of degree +1.**

(ii) **It has exactly one vertex of degree -1 and**

(iii) **The rest vertices have equal degree 0."**

*** An undirected graph has to satisfy all conditions mentioned in the lemma. For finding a path you have to start from +1 and end at -1. Otherwise, you cannot complete the path, remember it. (Assuming: ingoing = -ve & outgoing = +ve).



In this graph, point a has degree +1 and point d has degree -1

(Assuming: ingoing = -ve & outgoing = +ve).

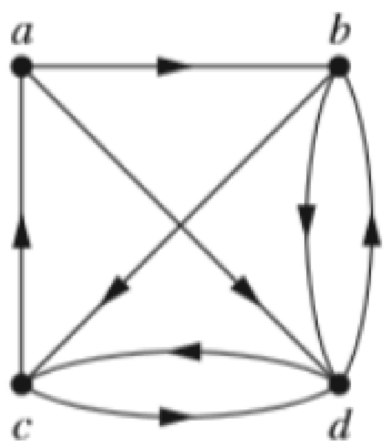
*** This is the better way of assuming, as we consider something is positive when it grows. Here, outgoing edges are considered growing from the vertex. Whatever it is, you have to check outgoing > ingoing for a starting and the reverse idea for the ending.

Question:

How to determine if a connected (and Directed / Undirected) graph has an Euler Path?

Solution:

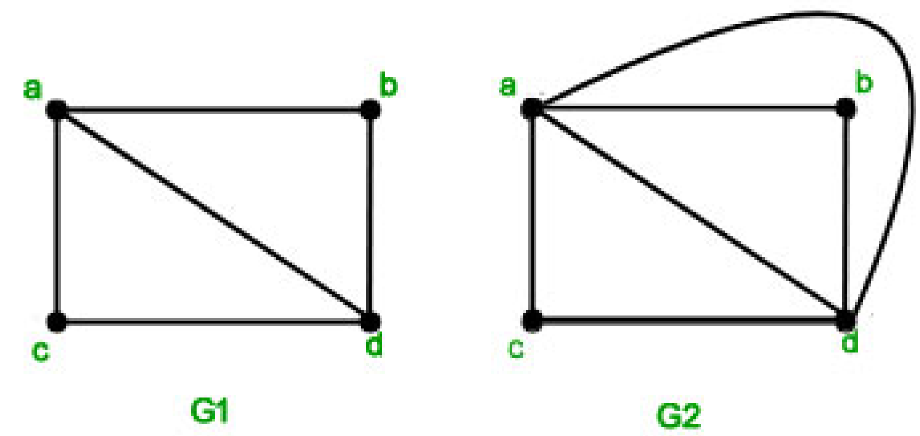
Show that the graph satisfies or doesn't satisfy the Theorem (for Undirected graph) or the Lemma (for Directed graph). If the given graph the conditions, it's an Euler Path, otherwise doesn't have an Euler Path.



The graph in the left, point a has degree +1, point b has degree -2, point c has degree 0 and point d has degree +1. So, it doesn't have an Euler Path as it doesn't satisfy the lemma.

Question:

Which graphs shown below have an Euler path or Euler circuit?



Solution:

G1 has two vertices of odd degree and and the rest of them have even degree. So, this graph has an Euler path but not an Euler circuit. The path starts and ends at the vertices of odd degree. The path is- a, c, d, a, b, d.

G2 has four vertices all of even degree, so it has a Euler circuit. The circuit is – a, d, b, a, c, d, a.

Question:

Which graphs shown in Figure 7 have an Euler path?

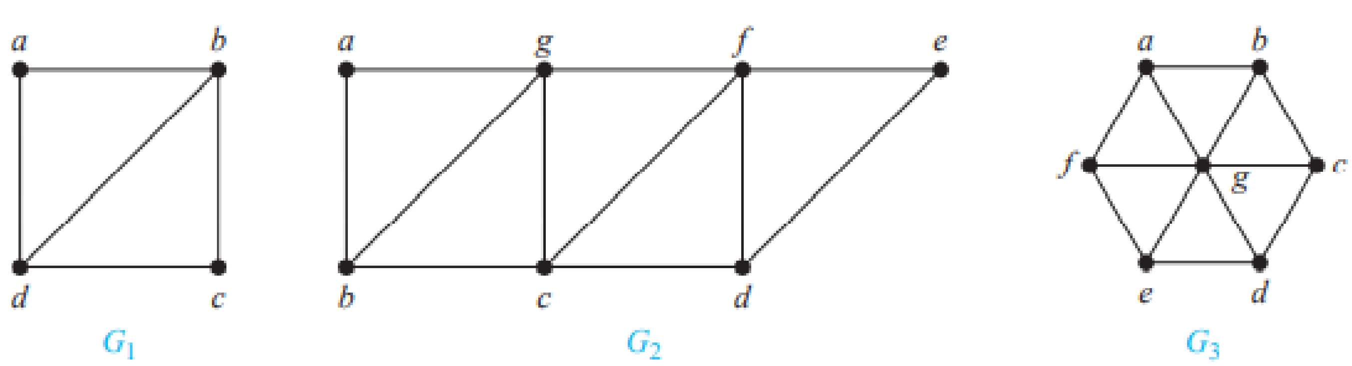


FIGURE 7 Three Undirected Graphs.

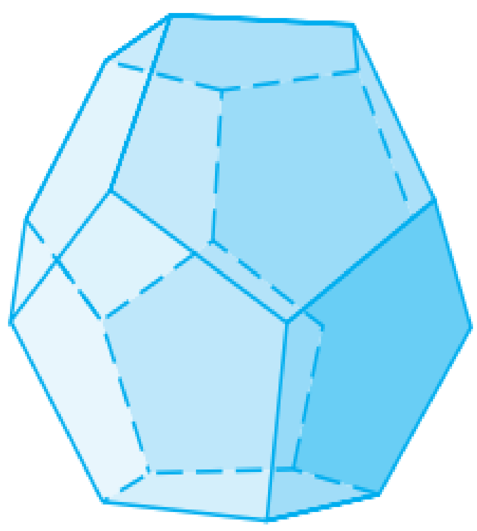
Solution:

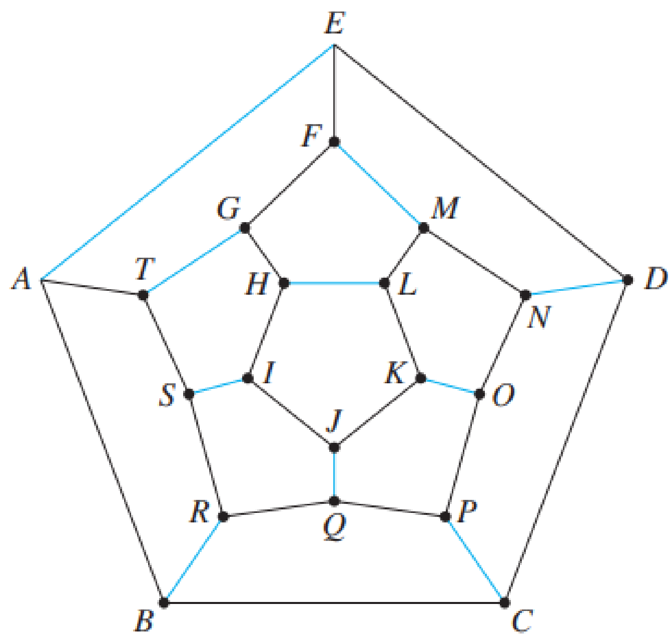
G1 contains exactly two vertices of odd degree, namely, b and d. Hence, it has an Euler path that must have b and d as its endpoints. One such Euler path is d, a, b, c, d, b. Similarly, G2 has exactly two vertices of odd degree, namely, b and d. So it has an Euler path that must have b and d as endpoints. One such Euler path is b, a, g, f, e, d, c, g, b, c, f, d. G3 has no Euler path because it has six vertices of odd degree.

HAMILTON PATH AND CIRCUIT

A **Hamiltonian circuit** is a **circuit** that visits every vertex once with no repeats. Being a **circuit**, it must start and end at the same vertex. A **Hamiltonian path** also visits every vertex once with no repeats, but does not have to start and end at the same vertex.

#The terminology comes from a game, called the ICOSIAN PUZZLE.

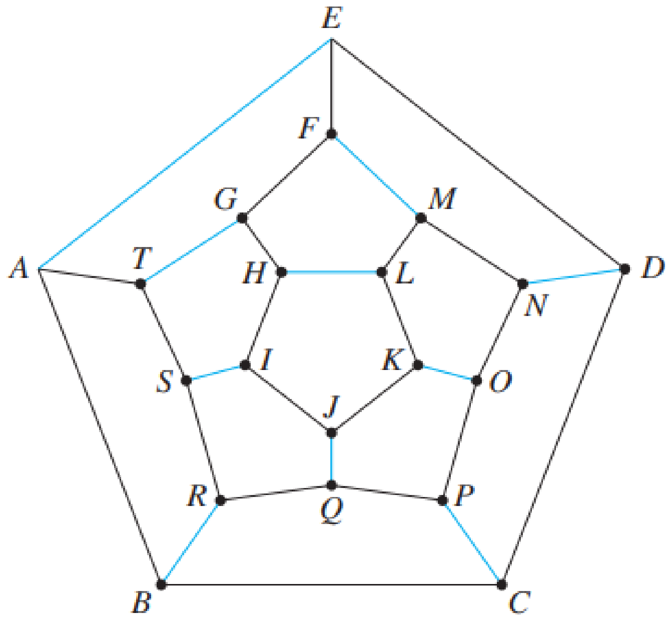




Given a graph G, is it possible to find a circuit for G in which all the vertices of G (except the first and the last) appear exactly once? In 1859 the Irish Mathematician Sir William Rowan Hamilton introduced a puzzle in the shape of a dodecahedron (DOH-dek-a-HEE-dron) named "A Voyage Round the World". For solving this puzzle, we need to learn about Hamilton Path and Circuit. However, this theory may seem easy but its application is very deep.

Hamilton Circuit:

Given a graph G, a Hamiltonian circuit for G is a simple circuit that includes every vertex of G. That is, a Hamiltonian circuit for G is a sequence of adjacent vertices and distinct edges in which every vertex of G appears exactly once, except for the first and the last, which are the same.



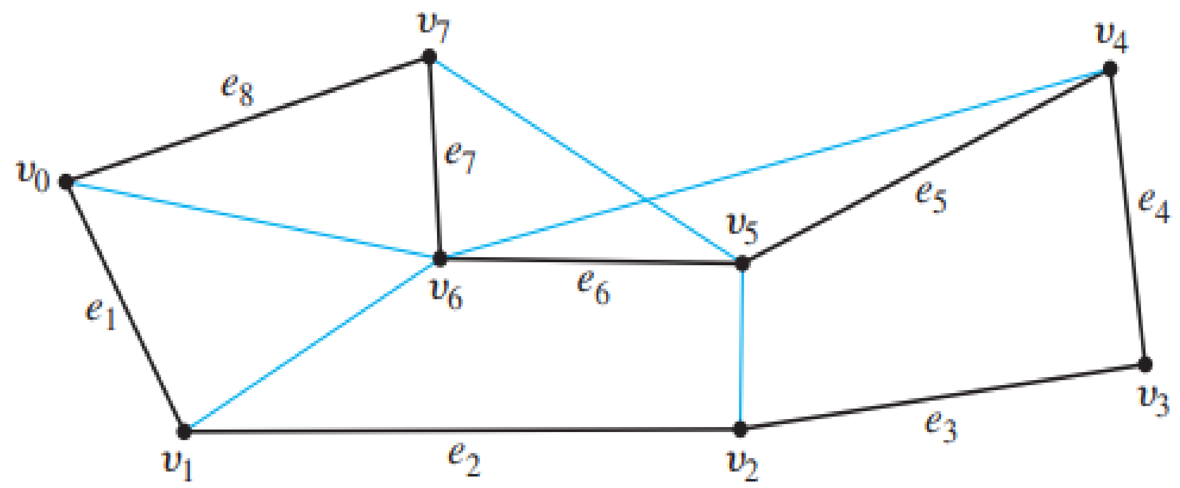
The graph G has Hamilton circuit.

One solution is the circuit A B C D E F G H I J K L M N O P Q R S T A,

Propositions:

If a graph G has a Hamilton Circuit, then G has a subgraph H with the following properties –

(i) H contains every vertex of G.
(ii) H is connected.
(iii) H has the same number of edges as vertices.
(iv) Every vertex of H has degree 2.



C: $v_0\, e_1 v_1 e_2\, v_2\, e_3 v_3\, e_4 v_4\, e_5 v_5\, e_6 v_6\, e_7 v_7 e_8\, v_0$

The edges of H are shown in black.

*** A fixed graph may have multiple subgraphs H who satisfy the properties mention above. In this case, all of them are considered to be the Hamilton Circuits of the given graph G.

Dirac's Theorem: The Existence of Hamilton Circuit

"If G is a simple graph with n vertices with n>=3 such that the degree of every vertex in G is at least n/2, then G has a Hamilton Circuit."

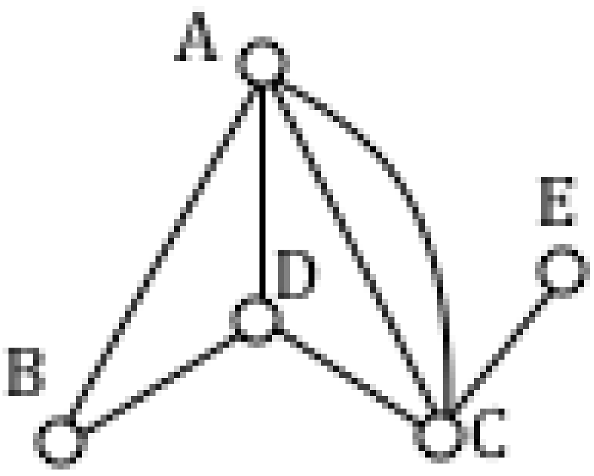
Ore's Theorem: The Existence of Hamilton Circuit

"If G is a simple graph with n vertices with n>= 3 such that deg(u) + deg(v) >= n for every pair of non-adjacent vertices u and v in G, then G has a Hamilton Circuit."

*** Ore's Theorem is much more efficient for ensuring the existence of Hamilton Circuit.

Hamilton Path:

Given a graph G, a Hamiltonian path for G is a simple path that includes every vertex of G. That is, a Hamiltonian path for G is a sequence of adjacent vertices and distinct edges in which every vertex of G appears exactly once, but does not have to start and end at the same vertex.

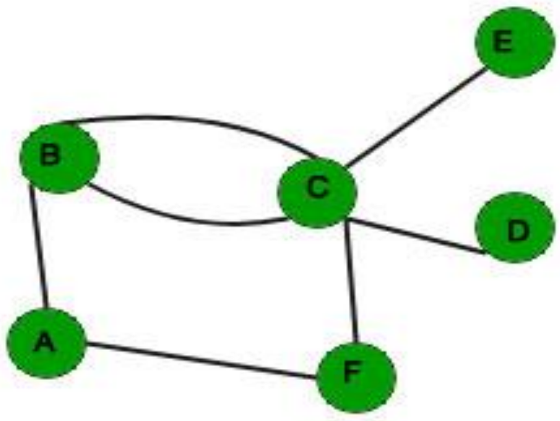


The Graph G is a Hamilton Path.

As we can see that once we travel to vertex E there is no way to leave without returning to C, so there is no possibility of a Hamiltonian circuit. If we start at vertex E we can find several Hamiltonian paths, such as ECDAB and ECABD.

Question:

Does the following graph have a Hamiltonian Circuit?



Solution

No, the above graph does not have a Hamiltonian circuit as there are two vertices with degree one in the graph.

Question:

Which of the simple graphs in Figure 10 have a Hamilton circuit or, if not, a Hamilton path?

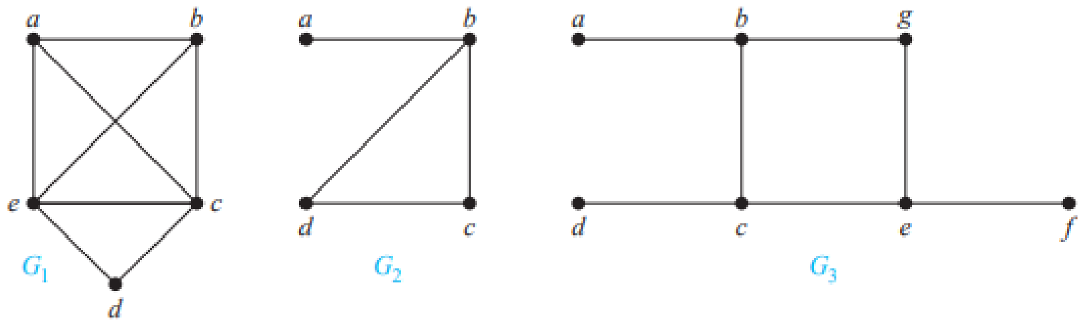


FIGURE 10 **Three Simple Graphs.**

Solution

G1 has a Hamilton circuit: a, b, c, d, e, a. There is no Hamilton circuit in G2 (this can be seen by noting that any circuit containing every vertex must contain the edge {a, b} twice), but G2 does have a Hamilton path, namely, a, b, c, d. G3 has neither a Hamilton circuit nor a Hamilton path, because any path containing all vertices must contain one of the edges {a, b}, {e, f }, and {c, d} more than once.

Question:

Show that neither graph displayed in Figure 11 has a Hamilton circuit.

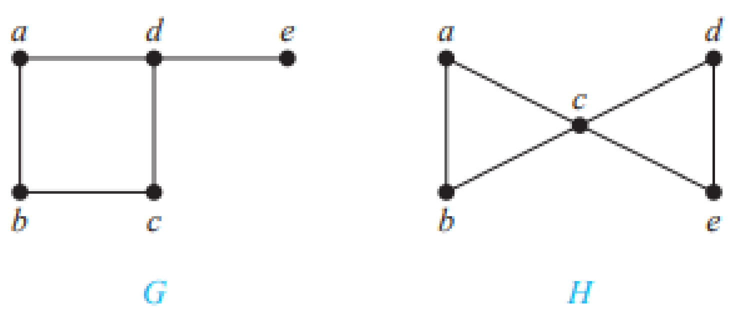


FIGURE 11 **Two Graphs That Do Not Have a Hamilton Circuit.**

Solution:

There is no Hamilton circuit in G because G has a vertex of degree one, namely, e. Now consider H. Because the degrees of the vertices a, b, d, and e are all two, every edge incident with these vertices must be part of any Hamilton circuit. It is now easy to see that no Hamilton circuit can exist in H, for any Hamilton circuit would have to contain four edges incident with c, which is impossible.

Question:

Show that Kn has a Hamilton circuit whenever n ≥ 3.

Solution:

We can form a Hamilton circuit in Kn beginning at any vertex. Such a circuit can be built by visiting vertices in any order we choose, as long as the path begins and ends at the same vertex and visits each other vertex exactly once. This is possible because there are edges in Kn between any two vertices.

MATRIX REPRESENTATIONS OF GRAPHS

Adjacency Matrix for undirected graph:

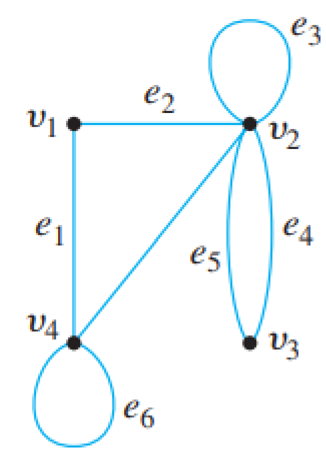
Let G be an undirected graph with ordered vertices $\boldsymbol{v_1, v_2, \dots, v_n}$. The adjacency matrix of G is the $\boldsymbol{n \times n}$ matrix $\boldsymbol{A} = (\boldsymbol{a_{ij}})$ over the set of nonnegative integers such that

$$\boldsymbol{a_{ij} = \textit{the number of edges connecting } v_i \textit{ to } v_j \qquad \textit{for every } i, j = 1, 2, \dots, n.}$$

In other words, if its adjacency matrix is $\boldsymbol{A} = (\boldsymbol{a_{ij}})$, then

$$a_{ij} = \begin{cases} 1 & if \{v_i, v_j\} \text{ is an edge of } G \\ 0 & otherwise \end{cases}$$

Example:



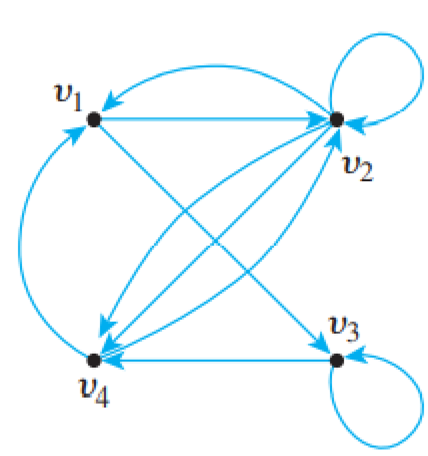
$$\boldsymbol{A} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{0} & \mathbf{2} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} \end{bmatrix} \end{array}$$

The entries of A satisfy the condition, $a_{ij} = a_{ji}$, for every $i, j = 1, 2, \dots, n$. This implies that the appearance of A remains the same if the entries of A are flipped across its main diagonal. A matrix, like A, with this property is said to be symmetric.

Adjacency Matrix for directed graph:

Let G be a directed graph with ordered vertices $\boldsymbol{v_1, v_2, \dots, v_n}$. The adjacency matrix of G is the $\boldsymbol{n \times n}$ matrix $\boldsymbol{A} = (\boldsymbol{a_{ij}})$ over the set of nonnegative integers such that

$$\boldsymbol{a_{ij} = \textit{the number of arrows from } v_i \textit{ to } v_j \qquad \textit{for all } i, j = 1, 2, \dots, n.}$$

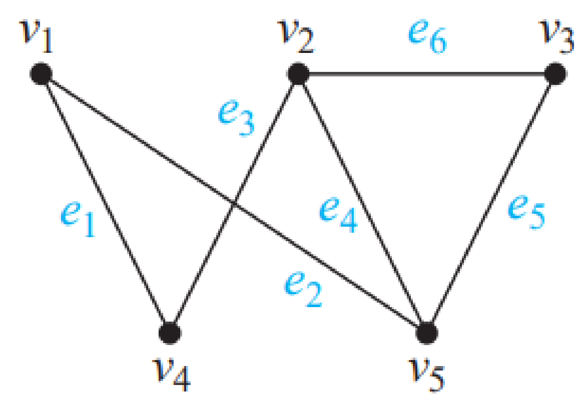


$$A = \begin{array}{c} \\ \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \boldsymbol{v_3} \\ \boldsymbol{v_4} \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{2} & \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix} \end{array}$$

Incidence Matrix for undirected graph:

Let G = (V, E) be an undirected graph. Suppose that $\boldsymbol{v_1, v_2, \dots, v_n}$. are the vertices and $\boldsymbol{e_1, e_2, \dots, e_m}$. are the edges of G. Then the incidence matrix with respect to this ordering of V and E is the n × m matrix $M = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{where } e_j \text{ is incident with } v_i, \\ 0 & \text{otherwise} \end{cases}$$

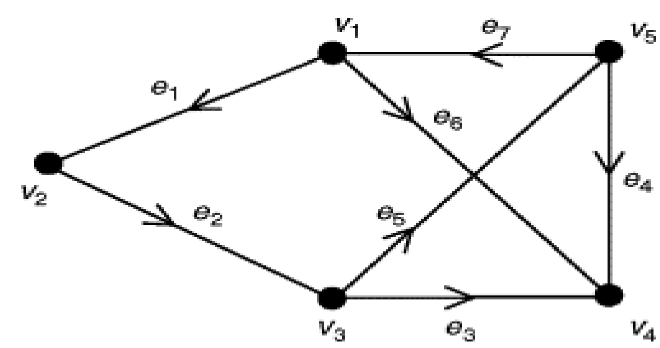


$$\boldsymbol{M} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \end{bmatrix} \end{array}$$

Incidence Matrix for directed graph:

Let G = (V, E) be a directed graph. Suppose that $\boldsymbol{v_1, v_2, \dots, v_n}$. are the vertices and $\boldsymbol{e_1, e_2, \dots, e_m}$. are the edges of G. Then the incidence matrix with respect to this ordering of V and E is the n × m matrix $M = [m_{ij}]$, where

$$m_{ij} = \begin{cases} +1 & \text{where } e_j \text{ is outgoing from } v_i \\ -1 & \text{where } e_j \text{ is ingoing to } v_i \\ 0 & \text{otherwise} \end{cases}$$



$$\begin{array}{cc} & \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ M = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} & \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{-1} & \mathbf{0} & \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{1} \end{bmatrix} \end{array}$$

Isomorphism of Graphs

Two graphs are said to be isomorphic if they are perhaps the same graphs, just drown differently with different names they have identical behavior for any graph-theoretic properties.

Formally speaking,

Let G and G′ be graphs with vertex sets V(G) and V(G′) and edge sets E(G) and E(G′), respectively. **G is isomorphic to G′** if, and only if, there exist one-to-one correspondences g: V(G) -> V(G′) and h: E(G) -> E(G′) that preserve the edge-endpoint functions of G and G′ in the sense that for each v∈ V(G) and e ∈ E(G),

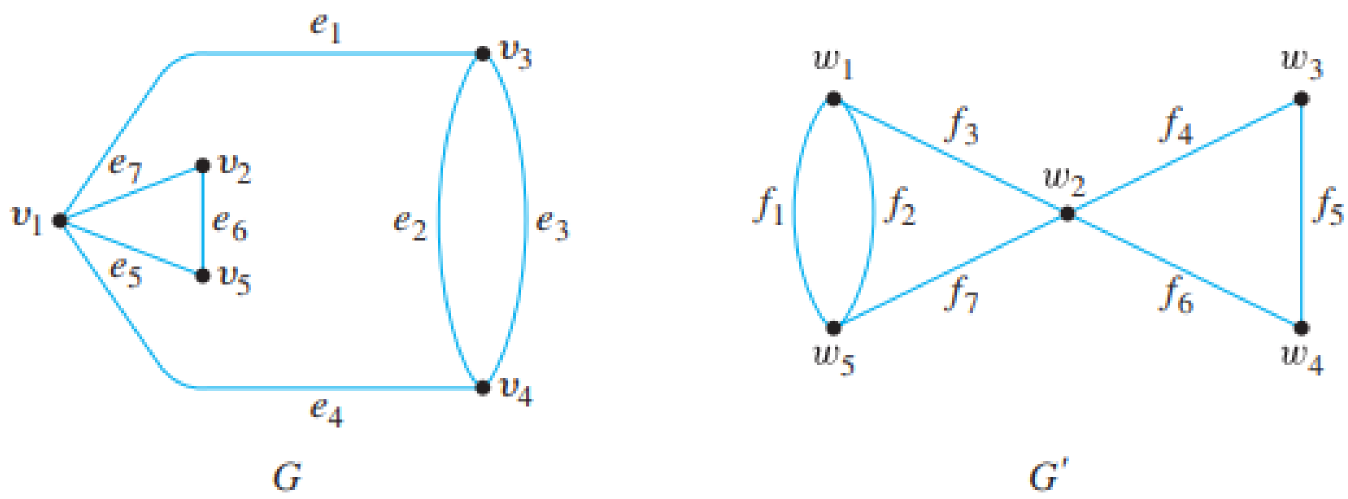
v is an endpoint of e ⇔ g(v) is an endpoint of h(e).

{u, v} is an edge in G ⇔ {g(u), g(v)} is an edge in G′.

The simple graphs G1 = (V1, E1) and G2 = (V2, E2) are isomorphic if there exists a oneto-one and onto function f from V1 to V2 with the property that a and b are adjacent in G1 if and only if f (a) and f (b) are adjacent in G2, for all a and b in V1. Such a function f is called an isomorphism. ∗ Two simple graphs that are not isomorphic are called non-isomorphic.

How to check if two graphs are isomorphic or not:

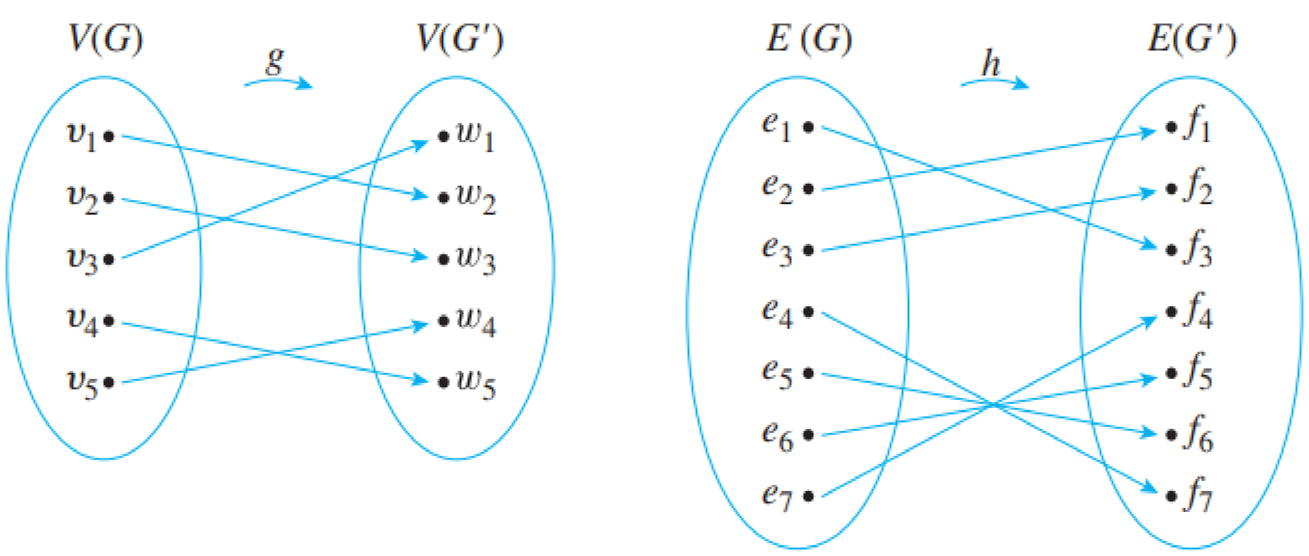
1. Check number of edges and vertices are same
2. Check number of vertices with same degree
3. Check degree of vertices along with their neighbor
4. Minimum cycle length, maximum cycle length, cycle with specific length



To solve this problem, you must find functions g: V(G) -> V(G') and h: E(G) -> E(G') such that for each v ∈ V(G) and e ∈ E(G), v is an endpoint of e if, and only if, g(v) is an endpoint of h(e). Setting up such functions is partly a matter of trial and error and partly a matter of deduction.

Here, $g(v_1) = w_2$, $g(v_2) = w_3, g(v_3) = w_1, g(v_4) = w_5, g(v_5) = w_4$

$$h(e_1) = f_3, h(e_2) = f_1, h(e_3) = f_2, h(e_4) = f_7, h(e_5) = f_6, h(e_6) = f_5, h(e_7) = f_4$$



So, Graph G and G' are isomorphic.

Question 1:

Determine whether the graphs G and H shown in Figure 6 are isomorphic.

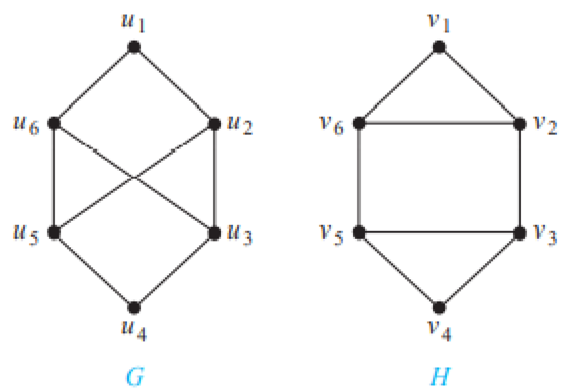


FIGURE 6 **The Graphs *G* and *H*.**

Solution:

Both G and H have six vertices and eight edges. Each has four vertices of degree three, and two vertices of degree two. So, the three invariants—number of vertices, number of edges, and degrees of vertices—all agree for the two graphs. However, H has a simple circuit of length three, namely, v1, v2, v6, v1, whereas G has no simple circuit of length three, as can be determined by inspection (all simple circuits in G have length at least four). Because the existence of a simple circuit of length three is an isomorphic invariant, G and H are not isomorphic.

Question 2:

Determine whether the graphs G and H shown in Figure 7 are isomorphic.

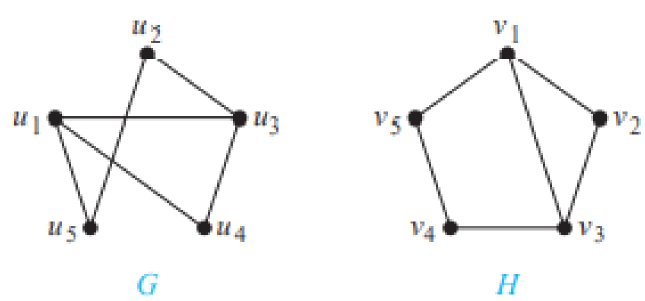


FIGURE 7 The Graphs *G* and *H*.

Solution:

Both G and H have five vertices and six edges, both have two vertices of degree three and three vertices of degree two, and both have a simple circuit of length three, a simple circuit of length four, and a simple circuit of length five. Because all these isomorphic invariants agree, G and H may be isomorphic.

To find a possible isomorphism, we can follow paths that go through all vertices so that the corresponding vertices in the two graphs have the same degree. For example, the paths u1, u4, u3, u2, u5 in G and v3, v2, v1, v5, v4 in H both go through every vertex in the graph; start at a vertex of degree three; go through vertices of degrees two, three, and two, respectively; and end at a vertex of degree two. By following these paths through the graphs, we define the mapping f with f (u1) = v3, f (u4) = v2, f (u3) = v1, f (u2) = v5, and f (u5) = v4. The reader can show that f is an isomorphism, so G and H are isomorphic, either by showing that f preserves edges or by showing that with the appropriate orderings of vertices the adjacency matrices of G and H are the same.

Question 3:

Show that the graphs G = (V, E) and H = (W, F), displayed in Figure 8, are isomorphic.

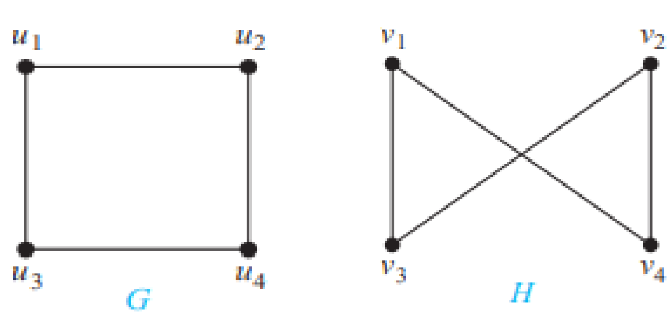


FIGURE 8 The Graphs G and H.

Solution:

The function f with f (u1) = v1, f (u2) = v4, f (u3) = v3, and f (u4) = v2 is a oneto-one correspondence between V and W. To see that this correspondence preserves adjacency, note that adjacent vertices in G are u1 and u2, u1 and u3, u2 and u4, and u3 and u4, and each of the pairs f (u1) = v1 and f (u2) = v4, f (u1) = v1 and f (u3) = v3, f (u2) = v4 and f (u4) = v2, and f (u3) = v3 and f (u4) = v2 consists of two adjacent vertices in H.

Question 4:

Show that the graphs displayed in Figure 9 are not isomorphic.

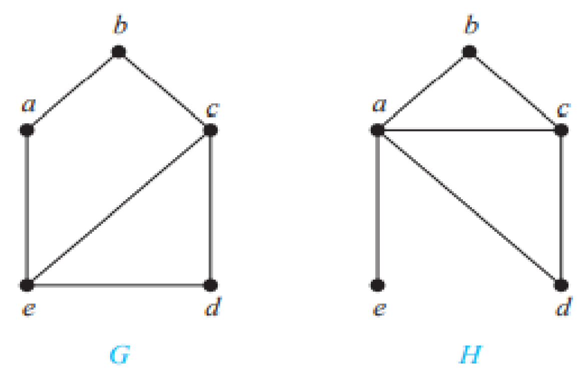


FIGURE 9 **The Graphs *G* and *H*.**

Solution:

Both G and H have five vertices and six edges. However, H has a vertex of degree one, namely, e, whereas G has no vertices of degree one. It follows that G and H are not isomorphic.

Question 5:

Determine whether the graphs shown in Figure 10 are isomorphic.

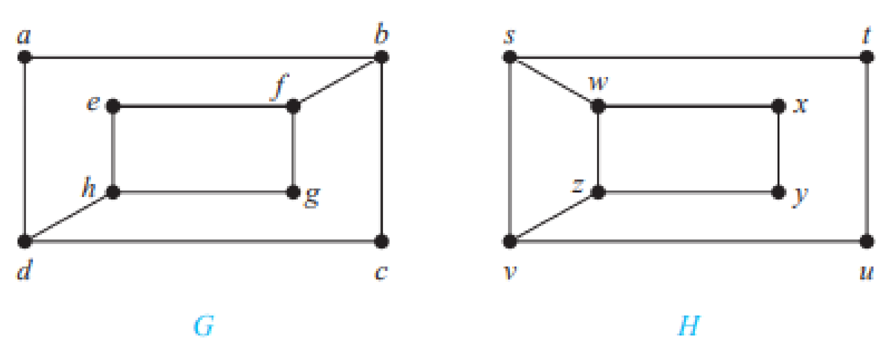


FIGURE 10 **The Graphs *G* and *H*.**

Solution:

The graphs G and H both have eight vertices and 10 edges. They also both have four vertices of degree two and four of degree three. Because these invariants all agree, it is still conceivable that these graphs are isomorphic. b f h d s w z v FIGURE 11 The Subgraphs of G and H Made Up of Vertices of Degree Three and the Edges Connecting Them. However, G and H are not isomorphic. To see this, note that because deg(a) = 2 in G, a must correspond to either t, u, x, or y in H, because these are the vertices of degree two in H. However, each of these four vertices in H is adjacent to another vertex of degree two in H, which is not true for a in G. Another way to see that G and H are not isomorphic is to note that the subgraphs of G and H made up of vertices of degree three and the edges connecting them must be isomorphic if these two graphs are isomorphic (the reader should verify this). However, these subgraphs, shown in Figure 11, are not isomorphic

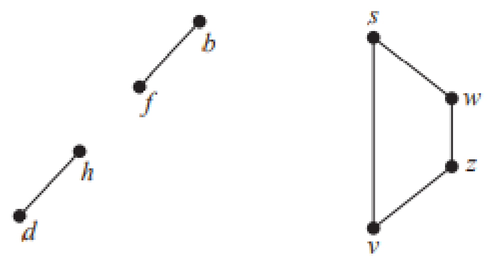


FIGURE 11 The Subgraphs of G and H Made Up of Vertices of Degree Three and the Edges Connecting Them.

Question 6:

Determine whether the graphs G and H displayed in Figure 12 are isomorphic.

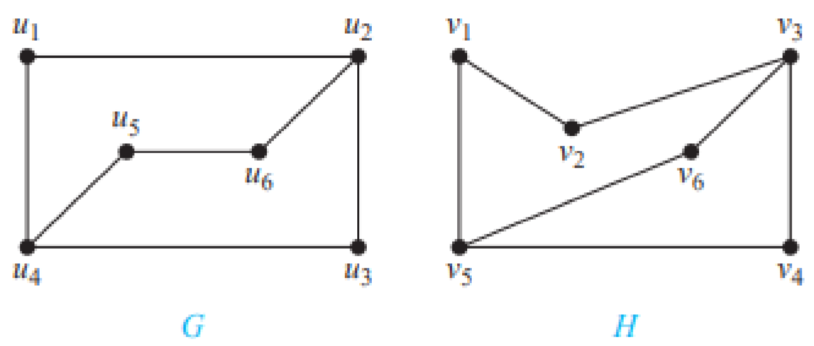


FIGURE 12 Graphs *G* and *H*.

Solution:

Both G and H have six vertices and seven edges. Both have four vertices of degree two and two vertices of degree three. It is also easy to see that the subgraphs of G and H consisting of all vertices of degree two and the edges connecting them are isomorphic (as the reader should verify). Because G and H agree with respect to these invariants, it is reasonable to try to find an isomorphism f.

We now will define a function f and then determine whether it is an isomorphism. Because deg(u1) = 2 and because u1 is not adjacent to any other vertex of degree two, the image of u1 must be either v4 or v6, the only vertices of degree two in H not adjacent to a vertex of degree two. We arbitrarily set f (u1) = v6. [If we found that this choice did not lead to isomorphism, we would then try f (u1) = v4.] Because u2 is adjacent to u1, the possible images of u2 are v3 and v5. We arbitrarily set f (u2) = v3. Continuing in this way, using adjacency of vertices and degrees as a guide, we set f (u3) = v4, f (u4) = v5, f (u5) = v1, and f (u6) = v2. We now have a one-to-one correspondence between the vertex set of G and the vertex set of H, namely, f (u1) = v6, f (u2) = v3, f (u3) = v4, f (u4) = v5, f (u5) = v1, f (u6) = v2. To see whether f preserves edges, we examine the adjacency matrix of G,

$$\mathbf{A}_G = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array},$$

and the adjacency matrix of H with the rows and columns labeled by the images of the corresponding vertices in G,

$$\mathbf{A}_H = \begin{array}{c} \\ v_6 \\ v_3 \\ v_4 \\ v_5 \\ v_1 \\ v_2 \end{array} \begin{array}{c} \begin{array}{cccccc} v_6 & v_3 & v_4 & v_5 & v_1 & v_2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}.$$

Because AG = AH, it follows that f preserves edges. We conclude that f is an isomorphism, so G and H are isomorphic. Note that if f turned out not to be an isomorphism, we would not have established that G and H are not isomorphic, because another correspondence of the vertices in G and H may be an isomorphism.

Planar Graph

A graph is called planar if it can be drawn in the plane without any crossing edges.

A graph is called planar if it can be drawn in the plane without any edges crossing (where a crossing of edges is the intersection of the lines or arcs representing them at a point other than their common endpoint). Such a drawing is called a planar representation of the graph.

A graph may be planar even if it is usually drawn with crossings, because it may be possible to draw it in a different way without crossings.

Question 1

Is K4 (shown in Figure 1 with two edges crossing) planar?

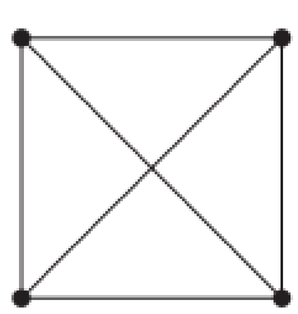
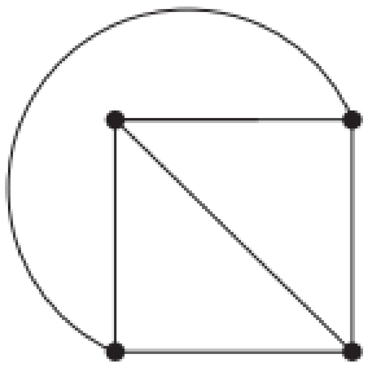


Figure: 1

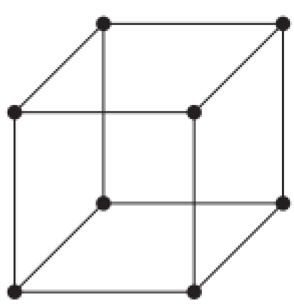
Solution:

K4 is planar because it can be drawn without crossings, as shown in Figure 2.



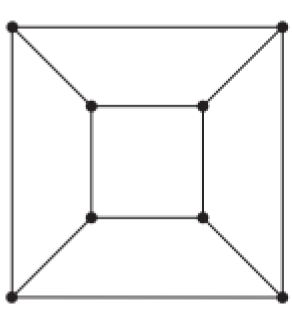
Question 2:

Is Q3, shown in Figure 3, planar?



Solution:

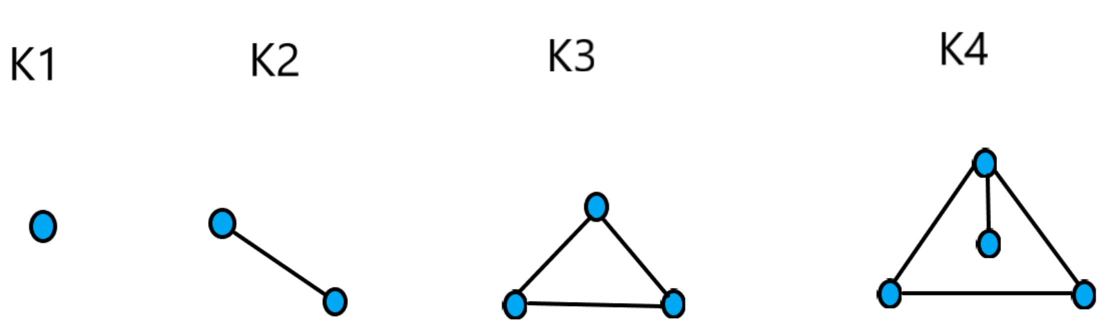
Q3 is planar, because it can be drawn without any edges crossing, as shown in Figure 4



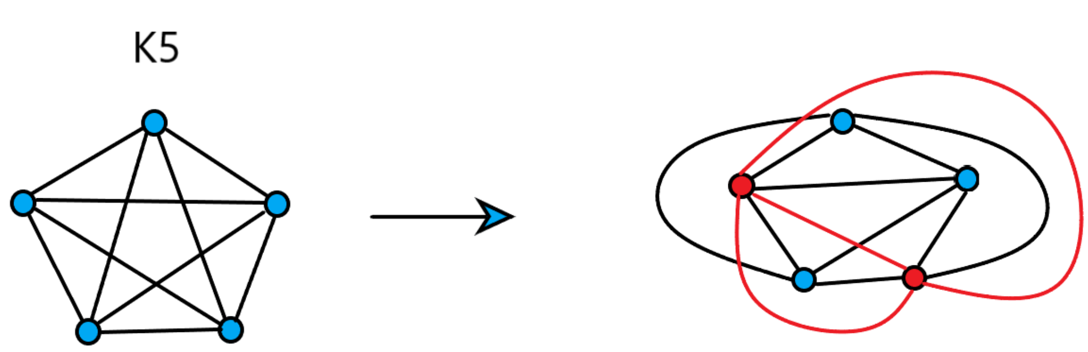
Question 3:

Which complete graphs are planar?

Solution:



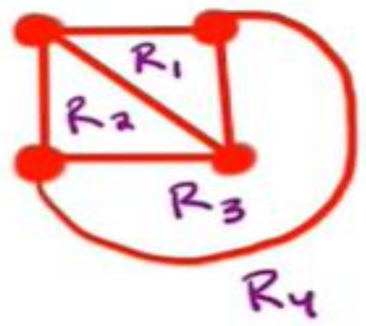
Trying to make K5 planar:



So, K5 is not a planar, because it can be drawn without any edges crossing.

A plane graph is a planar graph that has been drawn in the plane without any edge crossings

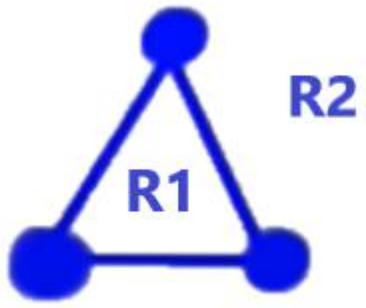
A plane graph divides the plane into **regions**



Notice that r1, r2 and r3 are bounded they're contained but r4 is an unbounded region.

In fact, every plane graph has an unbounded region called the exterior region.

So, if we draw the triangle, we can see that it just divides the plane into two regions:

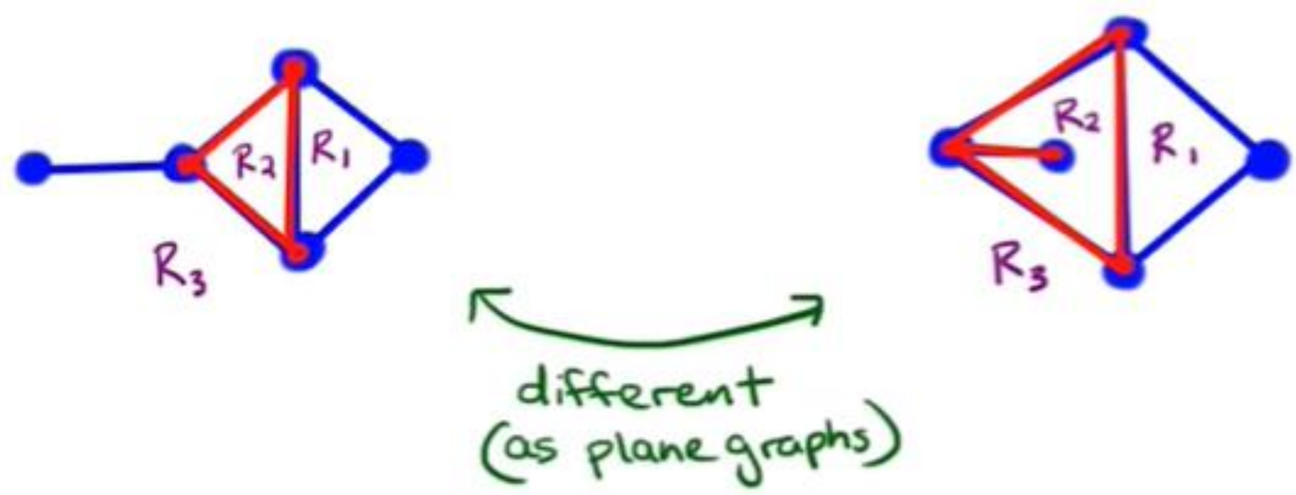


The one inside the Triangle and the one outside.

If we draw a tree on some number of vertices, we can see that it has just one region which is the unbounded region.

Every tree can be drawn as a plane graph and has only one region which is the unbounded region.

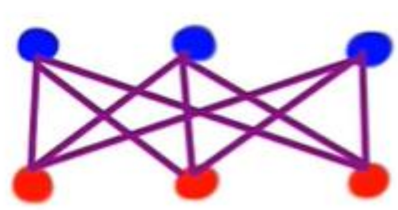
The boundary of a region in a plane graph is the set of vertices and edges that "outline" it.



Although the graphs are isomorphic, these are different plane graphs. They have different boundaries We can see, the boundary of r2 in the first picture, it contains only the triangle whereas the boundary of r2 in the other picture looks different. Even though these graphs are clearly the same in the sense of that they are isomorphic, they could be drawn in different ways in terms of a plane drawing.

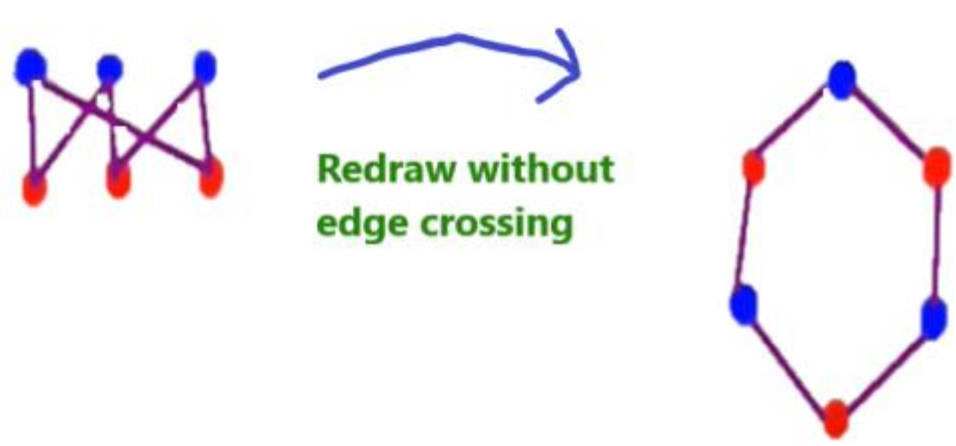
Question

Can you redraw this k33 so that no edges cross?

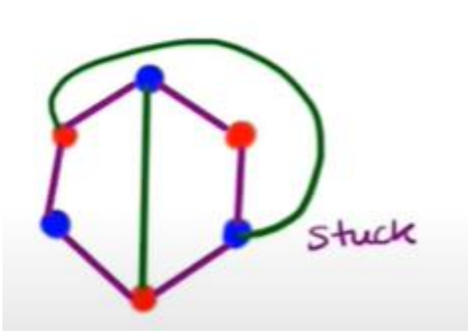


Informal proof:

Notice that k33 has a Hamilton cycle.



We now have our Hamilton cycle without any crossing. But we are kind of missing 3 edges because in the original graph every blue vertex is connected to every red vertex. 3 remaining edges need to connect vertices on opposite sides of the cycle. Let's try to draw the remaining 3 edges.



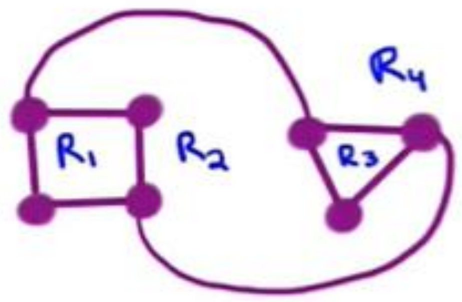
By the pigeonhole principle we have 2 edges both going across the middle of the Hamilton cycle or both going across outside of the Hamilton cycle.

So, there will be an edge crossing.

So, K33 is not planar.

Theorem Proof:

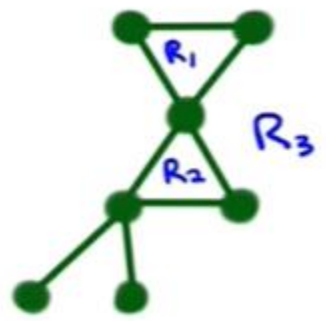
Let's draw an example:



n=7, m=9, r=4

n-m+r=2

Let's try another example:



n=7, m=8, r=3

n-m+r=2

This is not just a coincidence; it actually holds for every connected plane graph.

Theorem 2:

If G is a connected plane graph with n vertices, m edges and r regions, then n – m + r = 2

This is known as Euler formula for plane graph

Proof:

(Induction on m)

Basis: m=0, Then G=k1

So, n=1, m=0, r=1 and n-m+r=2.

Inductive Hypothesis:

Suppose the theorem is true for all connected plane graphs with <m edges (where m>=1)

Now consider a connected plane graph G on m edges, n vertices, r regions

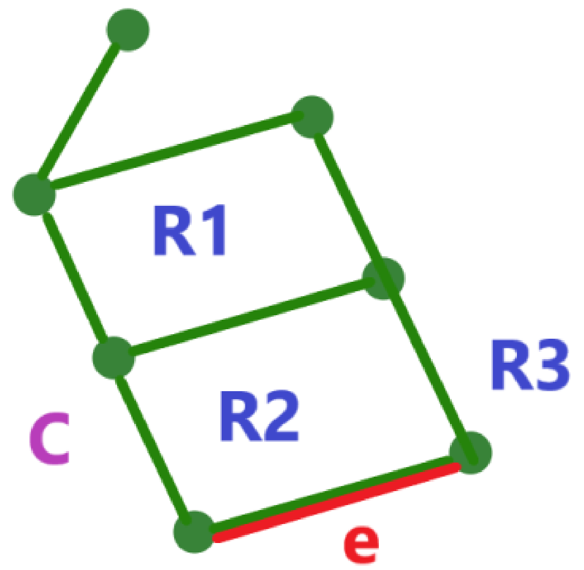
Case1: If G is a tree, then m = n-1 and r=1

So, n-m+r = n-(n-1) + 1 = 2

Case2: If G is not a tree, Then G has a cycle C.

Let e be an edge of C, then e is not a bridge.

So, G\e is connected and planar and has n vertices, m-1 edges and r-1 regions



Removing e means regions R2 and R3 join into 1 region.

By the inductive hypothesis the theorem holds for G\e

So, n-(m-1) + (r-1) = 2

⇨ n-m+r=2

A connected plane graph satisfies n-m+r=2

Corollary:

If G is a plane graph with c(G) connected components then n - m + r = 1 + c(G)

Example:

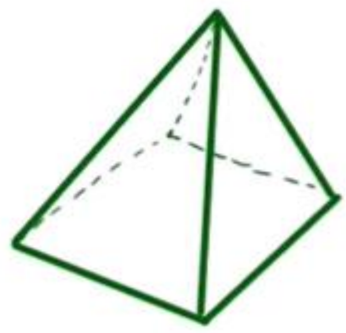
Notice that it has 3 connected components, 5 regions, 12 vertices and 13 edges.

n=12, m=13, r=5, c(G)=3

Now, n-m+r = 12-13+5 = 4 = 1+3 = 1+c(G)

So, the theorem does generalize to the disconnected case as well.

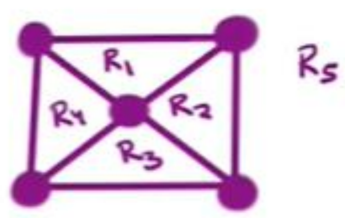
Euler formula is often talked about in the context of polyhedra. Here we will see a tetrahedron.



Polyhedron with 5 vertices, 8 edges, 5 faces

V=5, E=8, F=5

We can also redraw this graph which looks like a bird's eye view of this tetrahedron. And that's the associated graph to that polyhedron.



Notice that it has 5 vertices, 8 edges, 5 regions

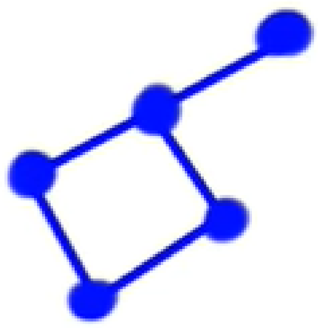
In this picture the bottom of the tetrahydron corresponds to the region that is the exterior unbounded region in our plane graph.

If V, E, F are the number of vertices, edges and faces of a polyhedron then V-E+F=2

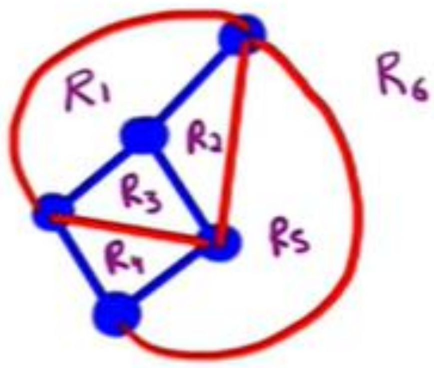
This is known as **Euler polyhedron formula**

Maximal Planar Graph

Let's try an example



Keep adding edges but keep it planar. We want to keep on adding red edges as long as they're not crossing any other edges and just keep going until we're forced to stop. So, this is what happens we end up with this graph here.



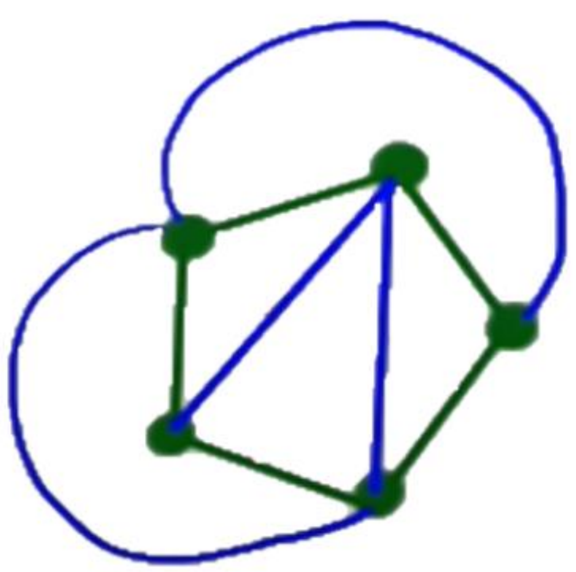
If we add any new edge, the graph would no longer be a plane graph. So, that is what's called a maximal plane graph.

Furthermore, if you look at any of the regions in this graph, you'll see that all of their boundaries are Triangles even the exterior region.

So, this is also sometimes called a **triangulation.**

Let's try another example:

We'll start with this plane graph and then we'll add edges until we get a new graph which is a Triangulation of the old graph.



This is a triangulation

Theorem:

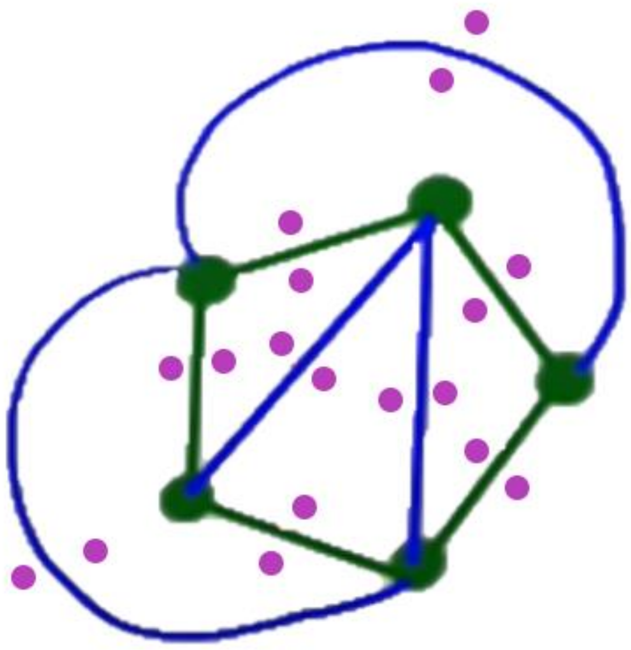
If G is a maximum planer graph with n vertices (n>=3) and m edges then m=3n-6

Proof:

Take a plane drawing of G with r regions.

The boundary of every region is a Triangle.

$$\sum_{all\ regions\ r} (number\ of\ edges\ boundary\ R) = 3r$$



Notice that each edge is on the boundary of 2 regions.

So, in that sum each gets get counted twice, one for each of those regions. That means the sum just counts all of the edges twice.

Now, 2m=3r

Euler's formula: $n - m + r = 2$

$=> n - m + 2m/3 = 2$

$=> 3n - 3m + 2m = 6$

$=> 3n - m = 6$

$=> \ m = 3n - 6$

Corollary:

If G is a planar graph with n>=3 vertices and m edges then m<=3n-6

Proof:

Add enough edges to G so that the resulting graph G' is maximal planar and it has m' edges.

Then, $m' = 3n - 6$

Obviously, $m \leq m'$

So, $m \leq 3n - 6.$

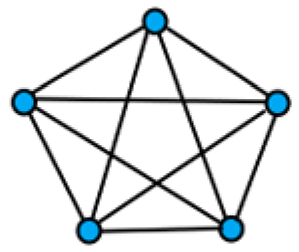
Recall:

maximum planar => $m = 3n - 6$

Planar=> $m \leq 3n - 6$

Question

Prove that K5 is not planar



Proof:

K5 has 5 vertices, 10 edges, n=5, m=10

If K5 is planar, then m <= 3n-6 = 3(5)-6 = 9

But m =10

So, K5 cannot be planar.

Lemma:

If G is a planar graph with n vertices and m edges and no triangles, then m<=2n-4

Proof:

Considering a plane drawing of G.

Add edges while keeping the resulting graph a planar graph with no triangles until we get G' which is maximal with respect to this property.

So, G' has m' edges (m'>=m) and

$$\sum_{all\ regions\ r} (number\ of\ edges\ boundary\ R) \geq \ 4r$$

this is because G' has at least 4 edges on the boundary of each region

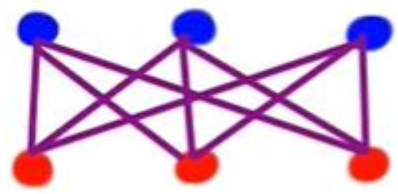
$$2m' \ >= \ 4r$$
$$Thus\ r \ <= \ m'/2$$

From Euler formula, If G' is a plane graph, then

$$2 \ = \ n - m' + r \ <= \ n - m' + m'/2$$
$$2 <= \ n - m'/2$$
$$m'/2 \ <= \ n \ - 2$$
$$m' \ <= \ 2n \ - \ 4$$
$$m \ <= \ m' \ <= \ 2n - 4$$

Question

Prove that K33 is not planar



Proof:

K33 has n=6, m=9 Also k33 is bipartite, so K33 has no odd cycles.

In particular, k33 has no triangles.

If k33 is no planar then $m \leq 2n - 4 = 8$

But $m = 9$

So, K33 cannot be planar

K5 and K33 are non-planar graphs.

If a graph G contains K5 or K33 as a subgraph, then G is not planar.

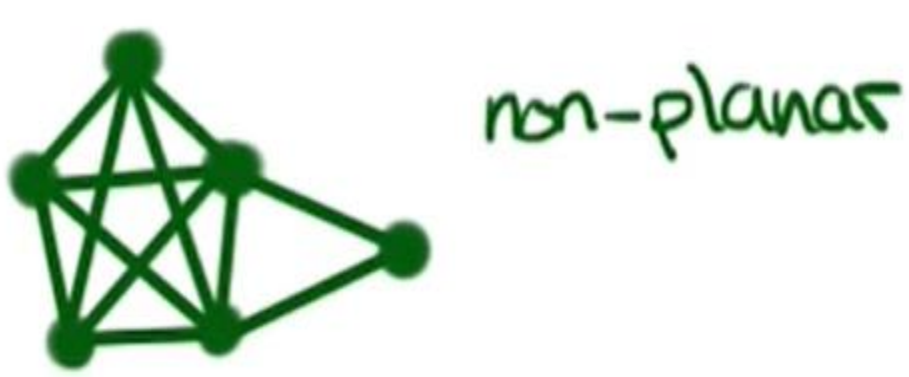
Characterizations of Planar Graph:

Recall:

K33 is not planar K5 is not planar.

If G contains a non-planar subgraph, then G is non-planar.

Example:



Observe:

In this graph K5 is not a subgraph but it's still true that this graph is non-planar. This graph is called a subdivision of K5.

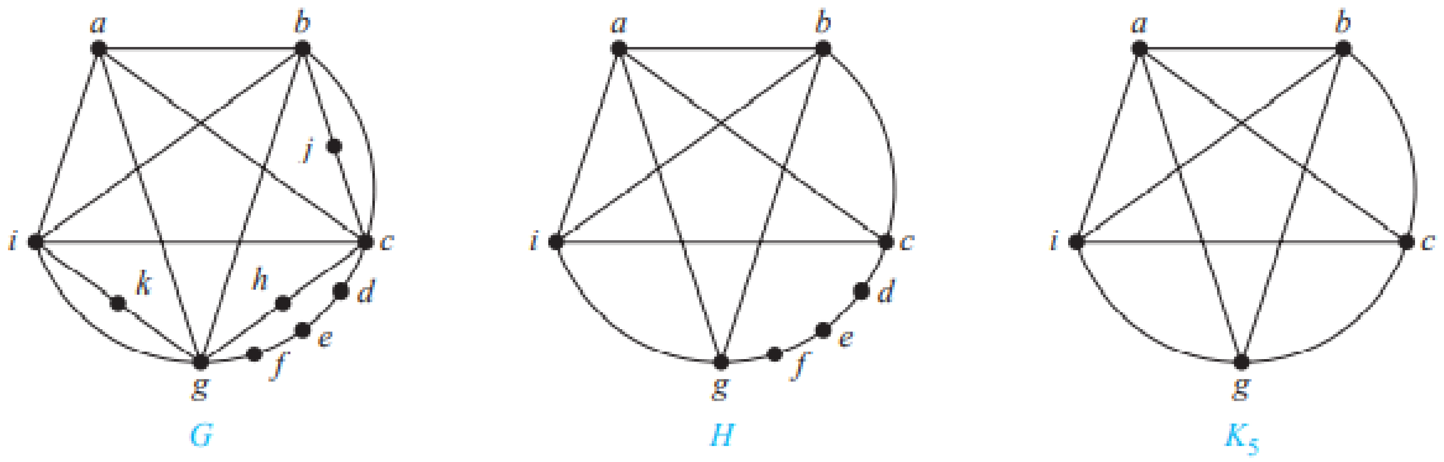
Elementary subdivision:

An elementary subdivision of a non-empty graph G is a graph obtained from G by removing an edge e = uv and adding a new vertex w and new edges uw and vw.

So, G is an elementary subdivision of k5

Homomorphic:

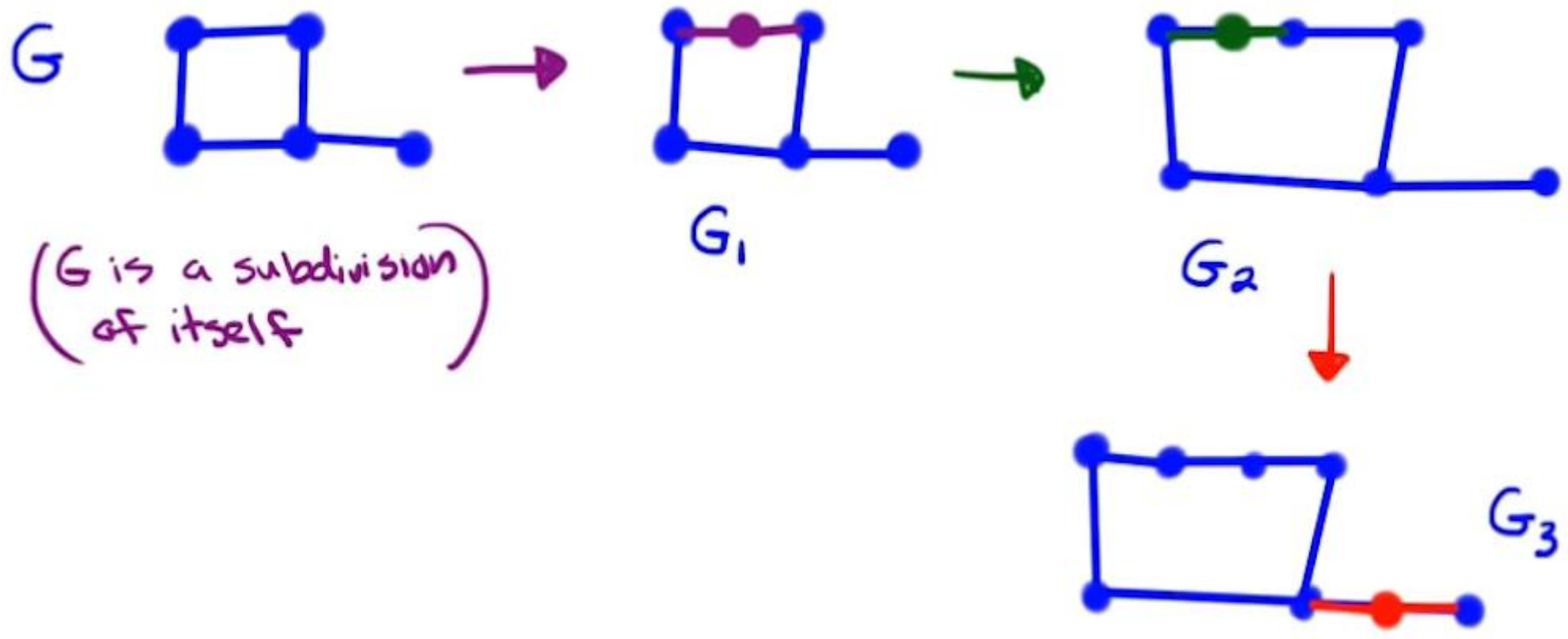
The graphs G1 = (V1, E1) and G2 = (V2, E2) are called homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivisions.



The Undirected Graph G, a Subgraph H Homeomorphic to K5, and K5.

Let try another example:

A subdivision of a graph G is a graph obtained from G by a sequence of zero or more elementary subdivisions.



Here graph G1, G2 and G3 are Homomorphic to G

Fact:

Any subdivision H of a graph G is planar if and only if G is planar

Because subdividing an edge does not change the planarity of the graph.

Fact:

If a graph G is a subdivision of K5 or K33 then G is nonplanar.

Fact:

If a graph G contains a subgraph that is a subdivision of K5 or K33 then G is non planar.

Kuratowski's Theorem:

A graph G is planar if and only if G contains no subgraph which is a subdivision of K5 or K33.

Petersen graph:

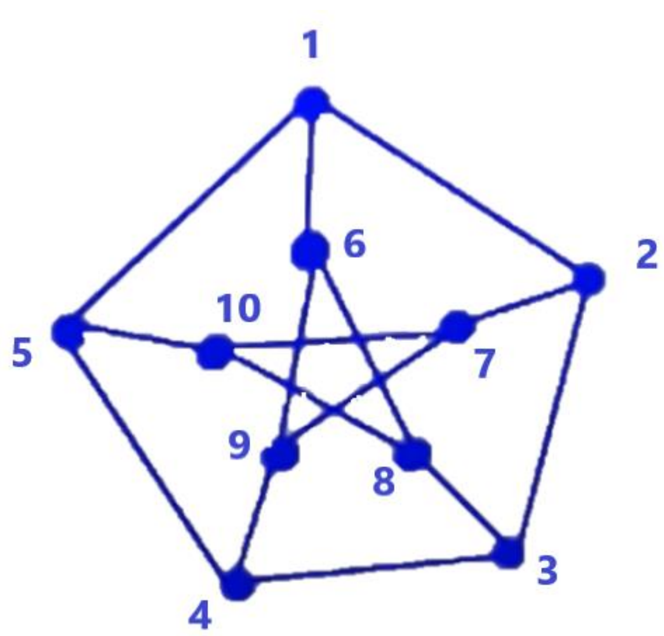
Fact:

The Petersen graph contains a subgraph that is a subdivision of K33

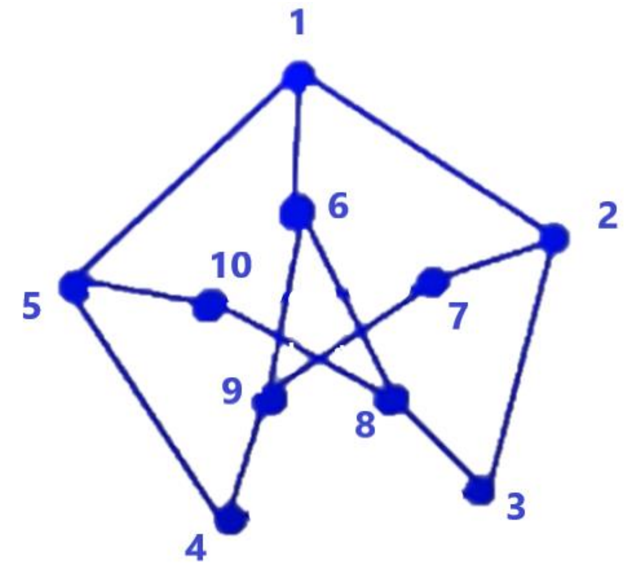
Why am I choosing K33! Doesn't it have a subdivision of K5?

Even though the petersen graph looks remarkably similar to K5, it does not contain a subdivision of K5. Because K5 is the complete graph on five vertices. Necessarily has every vertex of degree 4. But that doesn't happen in petersen graph. Because the petersen graph is three regular.

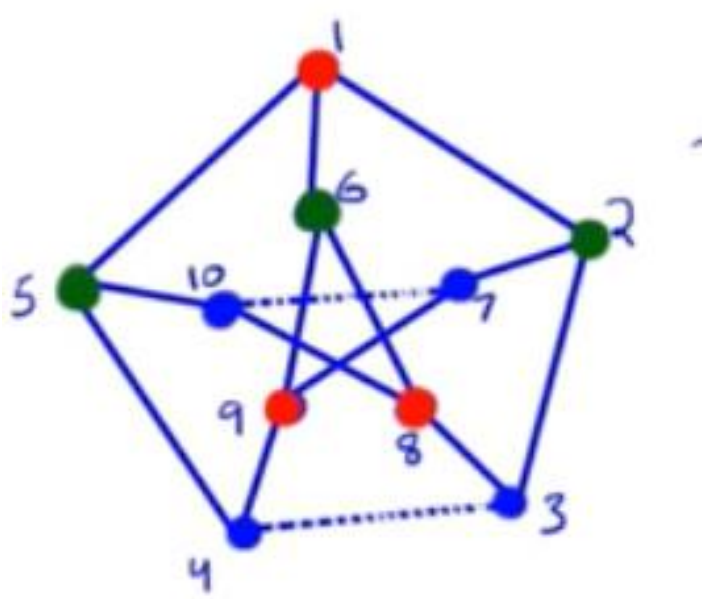
So, even though the Petersen graph does not have a sub graph which is a subdivision of K5. Let's redraw the Petersen graph and then label the vertices 1 through 10.



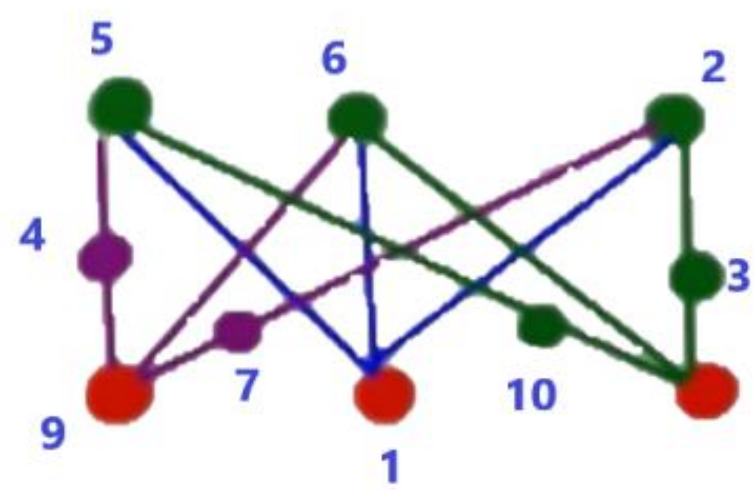
Next, I want to look at a particular subgraph. So, I'm going to erase two of the edges. The edges 3-4 and 7-10 are removed.



So, I'm going to highlight 3 vertices in red and 3 vertices in green.



Let's redraw this subgraph with this new picture in mind.



Now it's really clear that the subgraph we were looking at is indeed just a subdivision of K33.

Therefore, by kuratowski's theorem, The Petersen graph is non-planar.

Graph Coloring:

A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

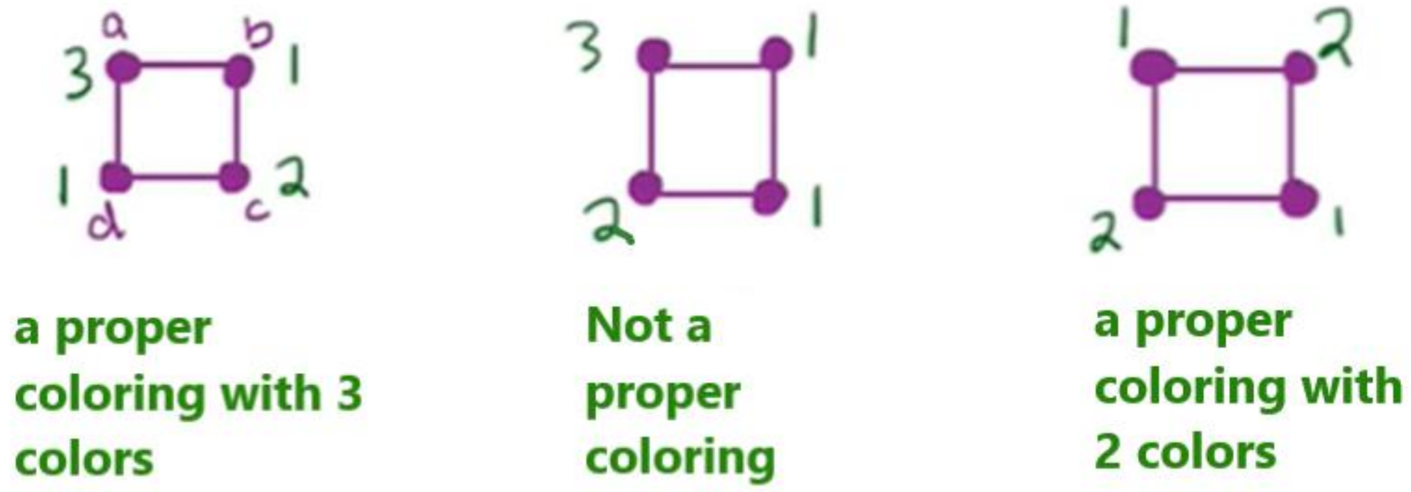
Let G be a graph with vertex set V(G).

A proper vertex coloring of G is a labelling of the vertex set

F: V(G) -> {1,2, 3, …, k}

where the labels are called colors and no two adjacent vertices get the same color.

Example:



In most cases, "coloring" means "proper vertex coloring"

A k-coloring of a graph G is a coloring of G using k colors.

If G has a k coloring, then it is k-colorable.

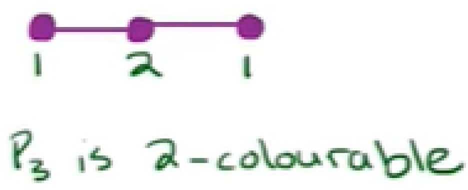
Example:

Chromatic Number:

The chromatic number of a graph is the least number of colors needed for a coloring of this graph. The chromatic number of a graph G is denoted by χ (G). (Here χ is the Greek letter chi.)

The chromatic number of a graph G, denoted x(G), is the smallest k such that G is k-colorable.

Example:



$\chi(P_3) = 2$

If $\chi(G) = k$, we say that G is k-chromatic.

Facts:

- **If G has n vertices, then $\chi(G) \leq n$**
- **$\chi(G) = 1$ if and only if G has no edges.**
- **$\chi(C_{2n}) = 2$ and $\chi(C_{2n+1}) = 3$**
- **$\chi(K_n) = n$ (Because every pair of vertices is adjacent in complete graph. So, we have to use up n colors in order to make a proper vertex coloring)**
- **If H is a subgraph of G, then $\chi(G) \geq \chi(H)$**

Let G = (V, E)

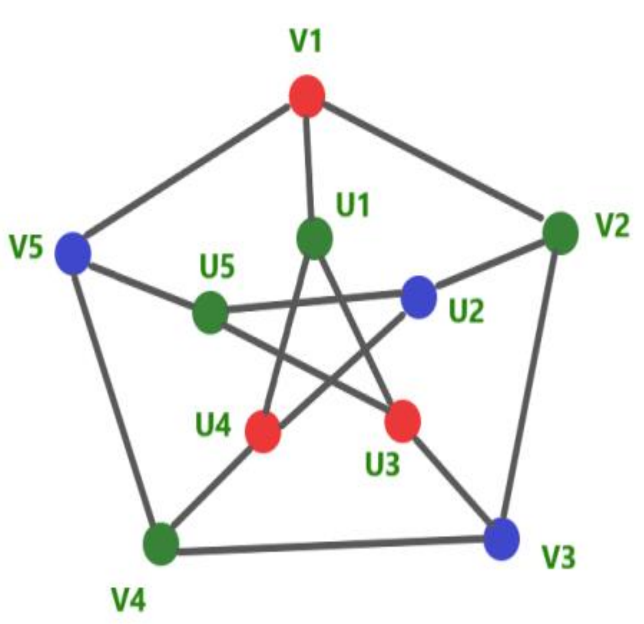
A k-coloring of G partitions the vertex set V into k sets $V_1, V_2, \ldots, V_k$, where each set Vi is an independent set (no two vertices in the set are adjacent)

This means $V = V_1 \cup V_2 \cup \ldots \cup V_k$

And $V_i \cap V_j = \emptyset \quad For\ all\ i \neq j$

The independent sets $V_1, V_2, \ldots, V_k$ are called color classees.

Example:



Color Classes:

$V_1 = \{V_1, U_3, U_4\}$

$V_2 = \{V_2, V_4, U_1, U_5\}$

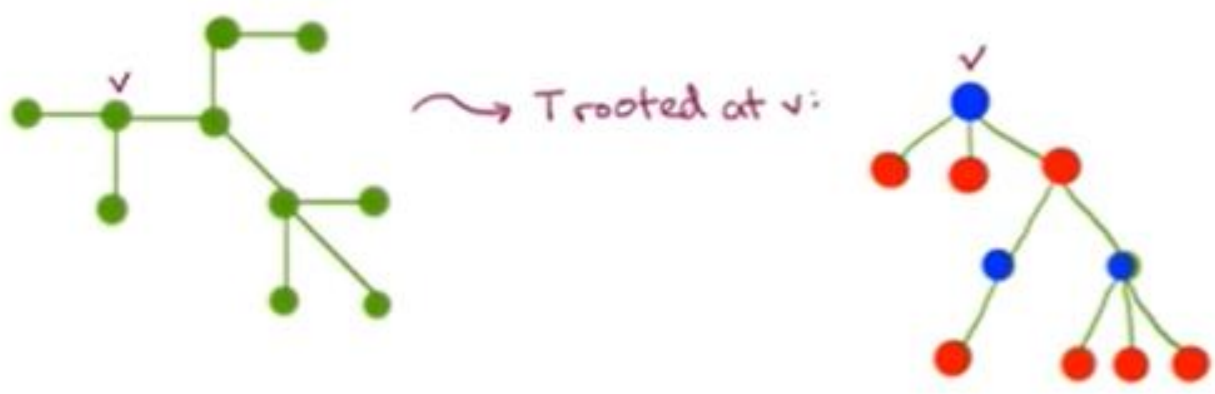
$V_3 = \{V_3, V_5, U_2\}$

Fact: Every tree with at least 2 vertices is 2-chromatic.

Proof:

Let T be a tree with $V(T) \geq 2$.

Let $v \in V(T)$ and consider T to be rooted at v.



We can sort of layer our tree in levels.

Let v be colored with color 1

Let the neighbors of v be colored with color 2

Let the neighbors of those vertices be color 1

Continue …

What I have done here: I've colored all of the vertices **blue** that belonged to the set X where X is the set of vertices whose distance from V is even. And we've colored all of the vertices **red** that belong to the set Y where Y is the set of vertices of the tree whose distance from our special vertex V is odd.

$$X = \{ u \in V(T) \mid d(u,v) \text{ is even} \}$$

$$Y = \{ u \in V(T) \mid d(u,v) \text{ is odd} \}$$

In a tree there is a unique path between any 2 vertices. Now along any path in T, the vertices alternate colors. So, no pair of adjacent vertices receive the same color.

This is a 2-coloring of T and $\chi(T) \geq 2$ since $V(T) \geq 2$

We have

G is a tree $=> \chi(T) = 2$

What about the converse?

$\chi(T) = 2 =>$ G is a tree

Is this true?

The answer is NO.

For example, we could take the 4 cycle which we know can be colored in 2 colors.

In fact, any cycle of even length is colorable in 2 colors. So, it's not true that if we have chromatic number 2, that must be a tree. But we have noticed in the proof that we just saw that a tree is in fact bipartite.

A Graph G is bipartite if and only if it contains no odd cycle.

Theorem:

$\chi(G) = 2$ *if and only if* G **is bipartite (for non-trivial graphs)**

Recall: (G is bipartite <=> G has no odd cycle)

Proof:

Aim to show:

$\chi(\boldsymbol{G}) = \mathbf{2}$ => $\boldsymbol{G}$ is bipartite

Contrapositive: G is not bipartite => $\chi(\boldsymbol{G}) > \mathbf{2}$

Suppose G is not bipartite.

Then G has an odd cycle $\boldsymbol{C_{2n+1}}$ and hence $\chi(\boldsymbol{G}) \geq \chi(\boldsymbol{C_{2n+1}}) = \mathbf{3}$

Aim to show:

G is bipartite => $\chi(\boldsymbol{G}) = \mathbf{2}$

Suppose G is a bipartite. Then $\boldsymbol{V}(\boldsymbol{G}) = \boldsymbol{X} \cup \boldsymbol{Y}$ such that every edge of G is of the form $\boldsymbol{e} = \boldsymbol{xy}$ where $\boldsymbol{x} \in \boldsymbol{X}$ and $\boldsymbol{y} \in \boldsymbol{Y}$.

Then color all vertices of X color 1 and all vertices of Y color 2.

So, $\chi(\boldsymbol{G}) = \mathbf{2}$.

Question 1:

What are the chromatic numbers of the graphs G and H shown in Figure 3?

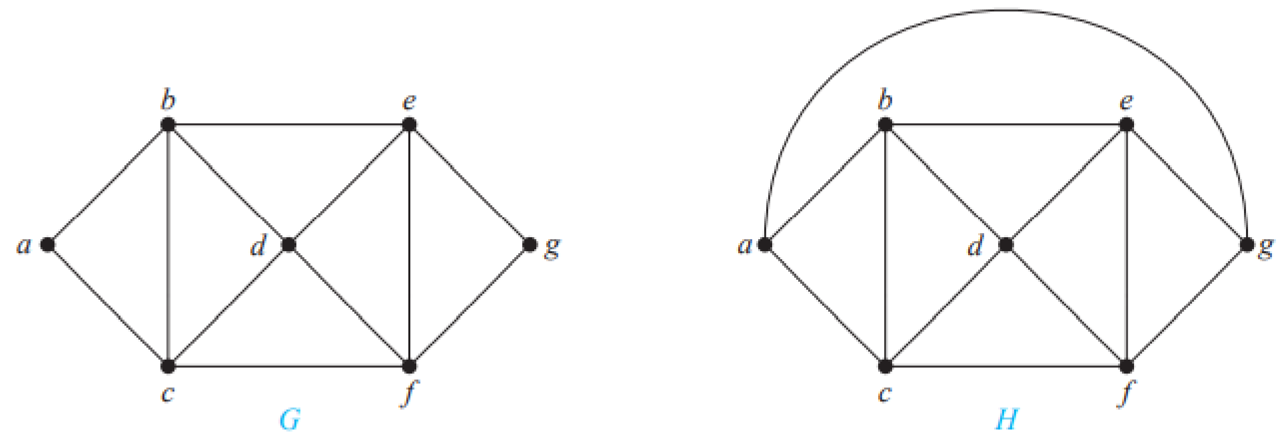


FIGURE 3 **The Simple Graphs G and H.**

Solution:

The chromatic number of G is at least three, because the vertices a, b, and c must be assigned different colors. To see if G can be colored with three colors, assign red to a, blue to b, and green to c. Then, d can (and must) be colored red because it is adjacent to b and c. Furthermore, e can (and must) be colored green because it is adjacent only to vertices colored red and blue, and f can (and must) be colored blue because it is adjacent only to vertices colored red and green. Finally, g can (and must) be colored red because it is adjacent only to vertices colored blue and green. This produces a coloring of G using exactly three colors. Figure 4 displays such a coloring.

The graph H is made up of the graph G with an edge connecting a and g. Any attempt to color H using three colors must follow the same reasoning as that used to color G, except at the last stage, when all vertices other than g have been colored. Then, because g is adjacent (in H) to vertices colored red, blue, and green, a fourth color, say brown, needs to be used. Hence, H has a chromatic number equal to 4. A coloring of H is shown in Figure 4.

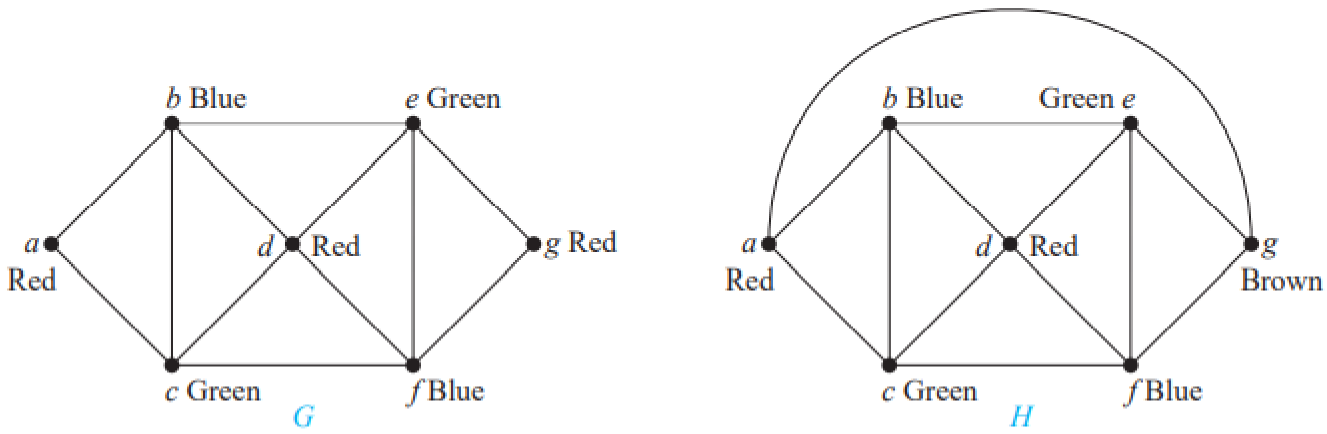


FIGURE 4 Colorings of the Graphs *G* and *H*.

Question 2:

What is the chromatic number of Kn?

Solution:

A coloring of Kn can be constructed using n colors by assigning a different color to each vertex. Is there a coloring using fewer colors? The answer is no. No two vertices can be assigned the same color, because every two vertices of this graph are adjacent. Hence, the chromatic number of Kn is n. That is, χ (Kn) = n. (Recall that Kn is not planar when n ≥ 5, so this result does not contradict the four-color theorem.) A coloring of K5 using five colors is shown in Figure 5.

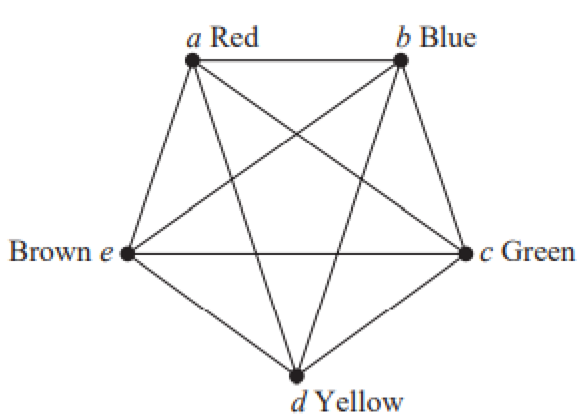


FIGURE 5 A Coloring of K_5.

Question 3:

What is the chromatic number of the complete bipartite graph Km,n, where m and n are positive integers?

Solution:

The number of colors needed may seem to depend on m and n. However, as Theorem tells us, only two colors are needed, because Km, n is a bipartite graph. Hence, χ (Km, n) = 2. This means that we can color the set of m vertices with one color and the set of n vertices with a second color. Because edges connect only a vertex from the set of m vertices and a vertex from the set of n vertices, no two adjacent vertices have the same color. A coloring of K3,4 with two colors is displayed in Figure 6.

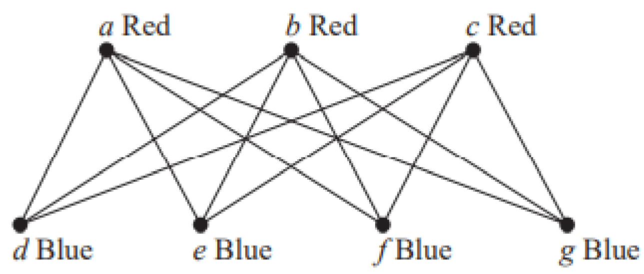


FIGURE 6 **A Coloring of $K_{3,4}$.**

Question 4:

What is the chromatic number of the graph Cn, where n ≥ 3? (Recall that Cn is the cycle with n vertices.)

Solution:

We will first consider some individual cases. To begin, let n = 6. Pick a vertex and color it red. Proceed clockwise in the planar depiction of C6 shown in Figure 7. It is necessary to assign a second color, say blue, to the next vertex reached. Continue in the clockwise direction; the third vertex can be colored red, the fourth vertex blue, and the fifth vertex red. Finally, the sixth vertex, which is adjacent to the first, can be colored blue. Hence, the chromatic number of C6 is 2. Figure 7 displays the coloring constructed here.

Next, let n = 5 and consider C5. Pick a vertex and color it red. Proceeding clockwise, it is necessary to assign a second color, say blue, to the next vertex reached. Continuing in the clockwise direction, the third vertex can be colored red, and the fourth vertex can be colored blue. The fifth vertex cannot be colored either red or blue, because it is adjacent to the fourth vertex and the first vertex. Consequently, a third color is required for this vertex. Note that we would have also needed three colors if we had colored vertices in the counterclockwise direction. Thus, the chromatic number of C5 is 3. A coloring of C5 using three colors is displayed in Figure 7.

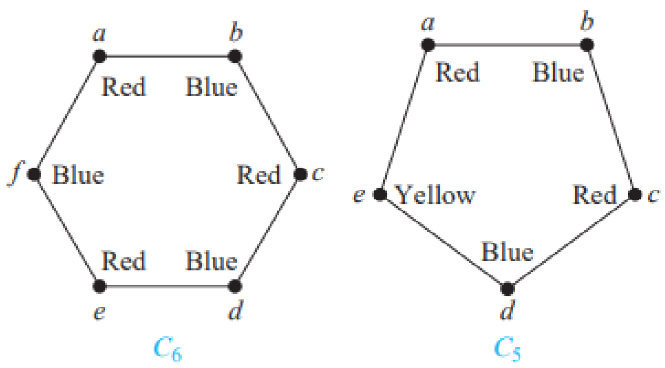


FIGURE 7 **Colorings of C_5 and C_6.**

In general, two colors are needed to color Cn when n is even.

When n is odd and n > 1, the chromatic number of Cn is 3.