Assignment (MAT-105W) Reg: 2019831055

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Ans to the ques NO-3

3. Show that the function $f(x) = |x| + |x-1|$ is continuous at $x=1$ but not differentiable.

Ans: Given, $f(x) = |x| + |x-1|$

Here,

$$f(x) = \begin{cases} -x-(x-1) & \text{for } x<0 \\ x-(x-1) & \text{for } 0 \le x < 1 \\ x+(x-1) & \text{for } x \ge 1 \end{cases}$$

$$\therefore f(x) = \begin{cases} -2x+1 & \text{for } x<0 \\ 1 & \text{for } 0 \le x < 1 \\ 2x-1 & \text{for } x \ge 1 \end{cases}$$

for continuity at $x=1$,

Right limit, $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (2x-1) = 1$

Left limit, $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (1) = 1$

and $f(1) = 2 \cdot 1 - 1 = 1$

$\therefore \lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^-} f(x) = f(1)$

Therefore $f(x)$ is continuous at $x=1$

Reg: 2019831055

1\. for differentiability at $x=1$

Right hand derivative,

$$Rf'(1) = \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0^+} \frac{\{2(1+h)-1\} - (2\cdot1-1)}{h}$$

$$= \lim_{h\to 0^+} \frac{2+2h-1-1}{h}$$

$$= \lim_{h\to 0^+} \frac{2h}{h} = 2$$

left hand derivative,

$$Lf'(1) = \lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0^-} \frac{1-1}{h}$$

$$= \lim_{h\to 0^-} \frac{0}{h}$$

$$=0$$

Here, $Rf'(1) \neq Lf'(1)$

Therefore, $f(x)$ is not differentiable at $x=1$

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Ans to the que: NO-8

8. If $y = \cot^{-1}x$ then show that,

$$(1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$$

Ans:

Given,

$y = \cot^{-1}x$

$\Rightarrow y_1 = -\frac{1}{1+x^2}$ [Differentiating both side by x]

$\Rightarrow (1+x^2)y_1 = -1$

$\Rightarrow (1+x^2)y_2 + y_1(0+2x) = 0$ [Differentiating both sides by x again]

$\therefore (1+x^2)y_2 + 2xy_1 = 0 \quad \cdots (i)$

By using Leibnitz theorem on (i),

$D^n\{(1+x^2)y_2\} + D^n(2xy_1) = 0$

$\Rightarrow \{(1+x^2)y_{n+2} + {}^nc_1(0+2x)y_{n+1} + {}^nc_2(2)y_n\} + \{2x \cdot y_{n+1} + {}^nc_1(2)y_n\} = 0$

$\Rightarrow \{(1+x^2)y_{n+2} + 2nxy_{n+1} + \frac{n(n-1)}{2!} \cdot 2y_n\} + \{2xy_{n+1} + 2ny_n\} = 0$

$\Rightarrow (1+x^2)y_{n+2} + 2nxy_{n+1} + 2xy_{n+1} + n(n-1)y_n + 2ny_n = 0$

$\Rightarrow (1+x^2)y_{n+2} + 2(x+1)xy_{n+1} + n(n-1+2)y_n = 0$

$\therefore (1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$

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Ans to the que: NO - 13

13. Evaluate $\lim_{x \to \infty} \left[\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \dots + \frac{n}{n^2+n^2} \right]$

Ans:

The general term of the series = $\frac{n}{n^2+r^2}$

now let,

$$I = \lim_{n \to \infty} \left[\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \dots + \frac{n}{n^2+n^2} \right]$$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{n}{n^2+r^2} \quad [\because \text{By using the general term}]$$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} \left\{ \frac{1}{1+(r/n)^2} \right\}$$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{1+(\frac{r}{n})^2}$$

Substituting $\frac{1}{n} \to dx$, $\frac{r}{n} \to x$

$\therefore$ upper limit, $\lim_{n \to \infty} (\frac{n}{n}) = 1$

$\therefore$ lower limit, $\lim_{n \to \infty} (\frac{1}{n}) = 0$

Given limit become, $I = \int_0^1 \frac{1}{1+x^2} dx$

$$= \left[\tan^{-1}x \right]_0^1$$

$$= \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \frac{\pi}{4}$$

$I = \pi/4$

Reg: 2019831055

Ans:- to the que: NO - 18

Evaluate $\int e^{2x} \sin^3 x\,dx$

Ans:
let,

$$I = \int e^{2x} \sin^3 x\,dx$$

$$= \int e^{2x} \left(\frac{3\sin x - \sin 3x}{4}\right) dx$$

$$= \int \left(\frac{3}{4} e^{2x}\sin x - \frac{1}{4} e^{2x}\sin 3x\right) dx$$

$$= \frac{3}{4}\int e^{2x}\sin x\,dx - \frac{1}{4}\int e^{2x}\sin 3x\,dx \quad \ldots (i)$$

We know

$\sin 3x = 3\sin x - 4\sin^3 x$

$\sin^3 x = \frac{3\sin x - \sin 3x}{4}$

let,

$$I_1 = \int e^{2x}\sin x\,dx \quad \ldots (ii)$$

$$= \sin x\int e^{2x}dx - \int\left(\frac{d}{dx}\sin x\int e^{2x}dx\right)dx$$

$$= \sin x \cdot \frac{1}{2}e^{2x} - \int \cos x \cdot \frac{1}{2}\cdot e^{2x}dx$$

$$= \frac{1}{2}\sin x e^{2x} - \frac{1}{2}\int e^{2x}\cos x\,dx$$

$$= \frac{1}{2}\sin x e^{2x} - \frac{1}{2}\left[\cos x\int e^{2x} - \int\left(\frac{d}{dx}\cos x\int e^{2x}dx\right)dx\right]$$

$$= \frac{1}{2}\sin x\, e^{2x} - \frac{1}{2}\left[\cos x\cdot\frac{1}{2}e^{2x} - \int(-\sin x)\cdot\frac{1}{2}\cdot e^{2x}\cdot dx\right]$$

$$= \frac{1}{2}\sin x e^{2x} - \frac{1}{2}\left[\frac{1}{2}\cos x e^{2x} + \frac{1}{2}\int e^{2x}\sin x\,dx\right.$$

$$= \frac{1}{2}\sin x e^{2x} - \frac{1}{4}\cos x e^{2x} - \frac{1}{4}\int e^{2x}\sin x\,dx$$

Reg: 2019831055

Now,

$I_1 = \frac{1}{2}\sin x\, e^{2x} - \frac{1}{4}\cos x\, e^{2x} - \frac{1}{4} I_1$ $\left[I_1 = \int e^{2x}\sin x\,dx\right]$

$\Rightarrow \left(1+\frac{1}{4}\right) I_1 = \frac{1}{2}\sin x\, e^{2x} - \frac{1}{4}\cos x\, e^{2x}$

$\Rightarrow \frac{5}{4} I_1 = \frac{1}{2}\sin x\, e^{2x} - \frac{1}{4}\cos x\, e^{2x}$

$\therefore I_1 = \frac{2}{5}\sin x\, e^{2x} - \frac{1}{5}\cos x\, e^{2x} + c_1 \ldots$ (iii)

let,

$I_2 = \int e^{2x}\sin 3x\,dx \ldots$ (iv)

$= \sin 3x \int e^{2x}dx - \int\left(\frac{d}{dx}\sin 3x \int e^{2x}dx\right)dx$

$= \sin 3x \cdot \frac{1}{2}\cdot e^{2x} - \int 3\cdot\cos 3x \cdot \frac{1}{2}e^{2x}\,dx$

$= \frac{1}{2}\sin 3x\, e^{2x} - \frac{3}{2}\int e^{2x}\cos 3x\,dx$

$= \frac{1}{2}\sin 3x\, e^{2x} - \frac{3}{2}\left[\cos 3x\int e^{2x} - \int\left(\frac{d}{dx}\cos 3x\int e^{2x}dx\right)dx\right]$

$= \frac{1}{2}\sin 3x\, e^{2x} - \frac{3}{2}\left[\cos 3x \cdot \frac{1}{2}\cdot e^{2x} - \int(-3\sin x)\cdot\frac{1}{2}e^{2x}dx\right]$

$= \frac{1}{2}\sin 3x\, e^{2x} - \frac{3}{2}\left[\frac{1}{2}\cos 3x \cdot e^{2x} + \frac{3}{2}\int e^{2x}\sin x\,dx\right]$

$= \frac{1}{2}\sin 3x\, e^{2x} - \frac{3}{4}\cos 3x\, e^{2x} - \frac{9}{4}\int e^{2x}\sin x\,dx$

Now,

$I_2 = \frac{1}{2}\sin 3x\, e^{2x} - \frac{3}{4}\cos 3x\, e^{2x} - \frac{9}{4} I_2$ $\left[I_2 = \int e^{2x}\sin 3x\right]$

$\Rightarrow \left(1+\frac{9}{4}\right) I_2 = \frac{1}{2}\sin 3x\, e^{2x} - \frac{3}{4}\cos 3x\, e^{2x}$

$$\Rightarrow \frac{13}{4} I_2 = \frac{1}{2} \sin 3x\, e^{2x} - \frac{3}{4} \cos 3x\, e^{2x}$$

$$\therefore I_2 = \frac{2}{13} \sin 3x\, e^{2x} - \frac{3}{13} \cos 3x\, e^{2x} + C_2 \quad (v)$$

Now using (iii) and (v) in (i)

$$I = \frac{3}{4} \int e^{2x} \sin x\, dx - \frac{1}{4} \int e^{2x} \sin 3x\, dx$$

$$= \frac{3}{4} I_1 - \frac{1}{4} I_2$$

$$= \frac{3}{4} \left[\frac{2}{5} \sin x\, e^{2x} - \frac{1}{5} \cos x\, e^{2x} \right] - \frac{1}{4} \left[\frac{2}{13} \sin 3x\, e^{2x} - \frac{3}{13} \cos 3x\, e^{2x} \right]$$

$$= \frac{3}{10} \sin x\, e^{2x} - \frac{3}{20} \cos x\, e^{2x} - \frac{1}{26} \sin 3x\, e^{2x} - \frac{3}{52} \cos 3x\, e^{2x}$$

$$= \frac{3}{10} \left(\sin x\, e^{2x} - \frac{1}{2} \cos x\, e^{2x} \right) - \frac{1}{26} \left(\sin 3x\, e^{2x} + \frac{3}{2} \cos 3x\, e^{2x} \right)$$

$$= \frac{3}{10} e^{2x} \left(\sin x - \frac{1}{2} \cos x \right) - \frac{1}{26} e^{2x} \left(\sin 3x + \frac{3}{2} \cos 3x \right)$$

Reg: 2019831055

Ans to the que: NO- 21

ii. State Rolle's theorem verify it for,

$$f(x)=2x^3+x^2-4x-2$$

Ans:

Rolle's theorem:

Let $f(x)$ be a real valued function in interval $[a,b]$ such that,

1) $f(x)$ is continuous in closed interval $[a,b]$

2) $f(x)$ is differentiable in open interval (a,b)

3) $f(a)=f(b)$

Then there exist at least one point $x=c\in(a,b)$ such that $f'(c)=0$

2nd part:

If we solve the given equation,

$$f(x)=0$$

$$\Rightarrow 2x^3+x^2-4x-2=0$$

$$\Rightarrow x^2(2x+1)-2(2x+1)=0$$

$$\Rightarrow (x^2-2)(2x+1)=0$$

$$\therefore x=-\sqrt{2},\sqrt{2},-\frac{1}{2}$$

Now, $f'(x)=6x^2+2x-4$

considering the given function in the interval $[-\sqrt{2},\sqrt{2}]$

Reg: 2019831055

we got,

i) since $f(x)$ is polygonial so it is continuous in $[-\sqrt{2},\sqrt{2}]$

ii) $f'(x)$ is also a polygonial function so it exists for all values $x \in (\sqrt{2},\sqrt{2})$. So $f(x)$ is differentiable in $(-\sqrt{2},\sqrt{2})$.

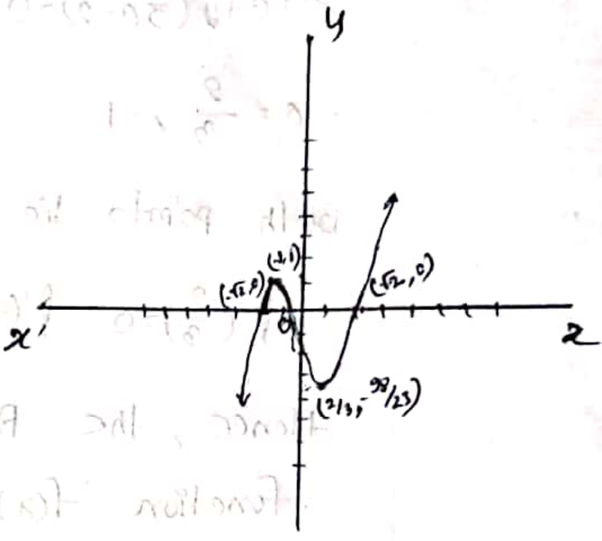


Fig: continuous f(x) function

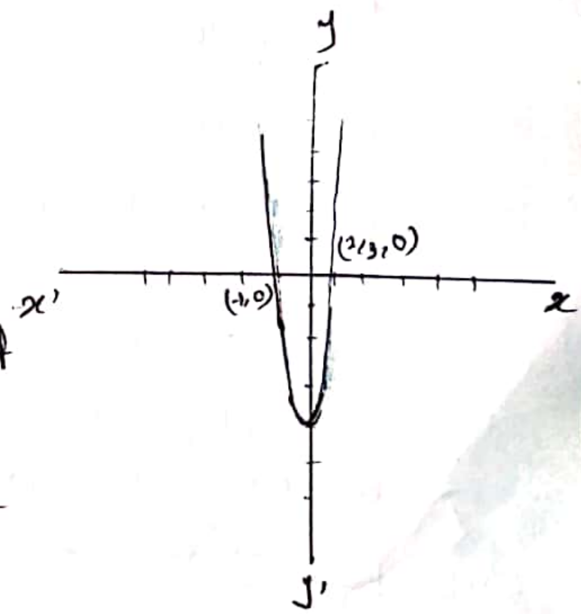
iii) $f(-\sqrt{2}) = 2(-\sqrt{2})^3 + (-\sqrt{2})^2 - 4(\sqrt{2}) - 2$

$= 0$

$f(\sqrt{2}) = 2(\sqrt{2})^3 + (\sqrt{2})^2 - 4(\sqrt{2}) - 2$

$= 0$

$\therefore f(-\sqrt{2}) = f(\sqrt{2})$



Here, $f(x)$ satisfy all these condition of Rolle's theorem,

So, there must exist a value $x = c$ such that,

$f'(c) = 0$

$\Rightarrow 6c^2 + 2c - 4 = 0$

$\Rightarrow 3c^2 + c - 2 = 0$ [dividing by 2]

$\Rightarrow 3c^2 + 3c - 2c - 2 = 0$

$\Rightarrow 3c(c+1) - 2(c+1) = 0$

Reg: 2019831055

$\Rightarrow (c+1)(3c-2)=0$

$\therefore c = \frac{2}{3}, -1$

Both points lie in $(-\sqrt{2}, \sqrt{2})$, so $c \in (-\sqrt{2}, \sqrt{2})$ and

$f'(\frac{2}{3})=0, \; f'(-1)=0$

Hence, the Rolle's theorem is verified for the function $f(x)$.

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