1) For continuity:

At $x = 1$:

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2-x) = 2-1 = 1$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x) = 1$$

$$f(1) = 2-1 = 1$$

$\therefore \lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = f(1)$

$\therefore$ the function f is continuous at $x = 1$

At $x = 2$:

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (-2+3x+x^2) = -2+6-4 = 0$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2-x) = 2-2 = 0$$

$$f(2) = 2-2 = 0$$

$\therefore \lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = f(2)$

The function f is continuous at $x = 2$.

For differentiability:

At $x = 1$:

$$Rf'(1) = \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{\{2-(1+h)\}-(2-1)}{h}$$

$$= \lim_{h\to 0^+} \frac{-h}{h} = \lim_{h\to 0^+} -1 = -1$$

$$Lf'(1) = \lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{(1+h)-(2-1)}{h}$$

$$= \lim_{h\to 0^-} \frac{h}{h} = \lim_{h\to 0^-} 1 = 1$$

$\therefore Rf'(1) \neq Lf'(1)$

The function f is not differentiable at $x=1$.

At $x=2$:

$$Rf'(2) = \lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0^+} \frac{\{-2+3(2+h)-(2+h)^2\}-0}{h}$$

$$= \lim_{h\to 0^+} \frac{-h^2-h}{h} = \lim_{h\to 0^+} (-h-1) = -1$$

$$Lf'(2) = \lim_{h\to 0^-} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0^-} \frac{\{2-(2+h)\}-0}{h}$$

$$= \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} (-1) = -1$$

$\therefore Rf'(2) = Lf'(2)$

$\therefore$ the function is differentiable at $x=2$.

At $x = 0$:

2) For continuity:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (3-2x) = 3-2.0 = 3$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (3+2x) = 3+2.0 = 3$$

Again, $f(0) = 3+2.0 = 3$

So, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$

$\therefore$ $f(x)$ is continuous at $x = 0$.

for differentiability:

$$Rf'(0) = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{f(h)-f(0)}{h}$$

$$= \lim_{h\to 0^+} \frac{3-2h-3+2.0}{h} = \lim_{h\to 0^+} \frac{-2h}{h} = -2$$

$$Lf'(0) = \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{f(h)-f(0)}{h}$$

$$= \lim_{h\to 0^-} \frac{3+2h-3-2.0}{h} = \lim_{h\to 0^-} \frac{2h}{h} = \lim_{h\to 0^-} 2 = 2$$

$\therefore$ $Rf'(0) \neq Lf'(0)$

$\therefore$ $f(x)$ is not differentiable at $x = 0$.

At point $x = \frac{3}{2}$:

For continuity:

$$\lim_{x \to \frac{3}{2}^+} f(x) = \lim_{x \to \frac{3}{2}^+} (-3-2x) = -3 - 2 \cdot \frac{3}{2} = -6$$

$$\lim_{x \to \frac{3}{2}^-} f(x) = \lim_{x \to \frac{3}{2}^-} (3-2x) = 3 - 2 \cdot \frac{3}{2} = 0$$

So, $\lim_{x \to \frac{3}{2}^+} f(x) \neq \lim_{x \to \frac{3}{2}^-} f(x)$

So, $f(x)$ is not continuous at $x = \frac{3}{2}$

Therefore, clearly $f(x)$ is not differentiable at $x = \frac{3}{2}$.

3) At $x = 1$:

For continuity:

$$\lim_{x \to 1^+} f(x) = x + x - 1 = 2x - 1 = 2 - 1 = 1$$

$$\lim_{x \to 1^-} f(x) = x - (x-1) = 1$$

$$f(1) = x + x - 1 = 2x - 1 = 2 - 1 = 1$$

So, $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = f(1)$

$\therefore$ $f(x)$ is continous at $x = 1$.

For differentiability;

$$Rf'(1) = \lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^+} \frac{2(1+h) - 1 - (2-1)}{h} = 2$$

$$Lf'(1) = \lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^-} \frac{1-(2-1)}{h} = 0$$

$\therefore f(x)$ is not differentiable at $x = 1$. (Showed)

4) For Continuity:

$$\lim_{x\to 0^+} f(x) = 1 + \sin 0 = 1 + 0 = 1$$

$$\lim_{x\to 0^-} f(x) = 1$$

$$f(0) = 1 + \sin 0 = 1$$

$$\therefore \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$$

$\therefore f(x)$ is continuous at $x = 0$.

For differentiability:

$$Rf'(0) = \lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0^+} \frac{1+\sin h - 1 - \sin 0}{h} = \lim_{h\to 0^+} \frac{\sin h}{h} = 1$$

$$Lf'(0) = \lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{1-1}{h} = 0$$

$\therefore Rf'(0) \neq Lf'(0)$

$\therefore f(x)$ is not differentiable at $x = 0$.

5) For Continuity:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2-x) = 2-1 = 1$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1$$

$$f(1) = 2-1 = 1$$

$$\therefore \lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = f(1)$$

$\therefore f(x)$ is continuous at $x = 1$

Again, $$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 2x - x^2 = 4-4 = 0$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 2-x = 2-2 = 0$$

And, $$f(2) = 2-x = 2-2 = 0$$

$$\therefore \lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = f(2)$$

$\therefore f(x)$ is continuous at $x = 2$.

For Differentiability:

At $x = 1$:

$$Rf'(1) = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{2-1-h-1}{h} = -1$$

$$Lf'(1) = \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{1+h-1}{h} = 1$$

$\therefore Rf'(1) \neq Lf'(1)$

$\therefore f'(x)$ doesn't exist at $x=1$.

At $x=2$:

$$Rf'(2) = \lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0^+} \frac{4+2h-4-4h-h^2-0}{h}$$

$$= \lim_{h\to 0^+} \frac{-2h-h^2}{h} = \lim_{h\to 0^+} -2-h = -2$$

$$Lf'(2) = \lim_{h\to 0^-} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0^-} \frac{2-2-h-0}{h} = -1$$

$\therefore Rf'(2) \neq Lf'(2)$

$\therefore f'(x)$ doesn't exist at $x=2$. (showed)

6) Given, $y = e^{a\sin^{-1}x}$

$\rightarrow y_1 = a e^{a\sin^{-1}x} \cdot \frac{1}{\sqrt{1-x^2}}$

$\rightarrow \sqrt{1-x^2} \cdot y_1 = ay$

Differentiating both sides again we get

$\rightarrow \sqrt{1-x^2} \cdot y_2 - \frac{x}{\sqrt{1-x^2}} y_1 = ay_1 = a \cdot \frac{ay}{\sqrt{1-x^2}}$

Multiplying both sides by $\sqrt{1-x^2}$ gives

$(1-x^2)y_2 - xy_1 - a^2y = 0$ —(1)

By applying Leibnitz's theorem, we get

$\{(1-x^2)y_{n+2} + n(-2x) \cdot y_{n+1} + \frac{n(n-1)}{2}(-2)y_n\}$

$-\{x y_{n+1} + n y_n\} - a^2 . y_n = 0$

$\rightarrow (1-x^2) y_{n+2} - (2n+1) x y_{n+1} - (n^2+a^2) y_n = 0$

{Proved}

7) Given, $y = \sin(m \sin^{-1} x)$

$\rightarrow y_1 = \cos(m \sin^{-1} x) . m . \frac{1}{\sqrt{1-x^2}}$

$\rightarrow y_1^2 = \cos^2 (m \sin^{-1} x) . m^2 . \frac{1}{1-x^2}$

$\rightarrow y_1^2 = \{1 - \sin^2(m \sin^{-1} x)\} . m^2 . \frac{1}{1-x^2}$

$\rightarrow y_1^2 = (1 - y^2) \frac{m^2}{1-x^2}$

$\rightarrow y_1^2 (1-x^2) = (1-y^2) m^2$

$\rightarrow 2y_1 y_2 (1-x^2) - y_1^2 . 2x = m^2 (-2y y_1)$

[Differentiating both sides in perspective of x]

$\rightarrow y_2 (1-x^2) - y_1 x = -m^2 y$

[Dividing both sides by $2y_1$]

$\rightarrow y_2 (1-x^2) - x y_1 + m^2 y = 0$

By applying Leibnitz's theorem we get

$y_{n+2} (1-x^2) + n y_{n+1} (-2x) + \frac{n(n-1)}{2!} y_n (-2)$

$- [y_{n+1} x + n y_n . 1] + m^2 y_n = 0$

$\Rightarrow y_{n+2}(1-x^2) + y_{n+1}(-2nx - x) + y_n\{-n(n-1-n+m^2\}$
$= 0$

$\Rightarrow y_{n+2}(1-x^2) - y_{n+1}(2n+1)x + y_n\{-n^2+n-n+m^2\} = 0$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)x\,y_{n+1} + (m^2-n^2)y_n = 0$

[showed]

8) $y = \cot^{-1}x \Rightarrow y_1 = -\frac{1}{1+x^2}$

$\Rightarrow (1+x^2)y_1 = -1$

$\Rightarrow (1+x^2)y_2 + y_1 \cdot 2x = 0$

By applying Leibnitz's theorem, we get.

$(1+x^2)y_{n+2} + n.2x.y_{n+1} + \frac{n(n-1)}{2}.2y_n$
$+ y_{n+1}.2x + n.y_n.2 = 0$

$\Rightarrow (1+x^2)y_{n+2} + 2x(n+1)y_{n+1} + n(n+1)y_n = 0$

9) $y = a\cos(\ln x) + b\sin(\ln x)$

$\Rightarrow y_1 = -a\sin\ln x \cdot \frac{1}{x} + b\cos\ln x \cdot \frac{1}{x}$

$\Rightarrow xy_1 = -a\sin\ln x + b\cos\ln x$

$\Rightarrow y_1 + xy_2 = -a\,\cancel{\sin}\cos\ln x \cdot \frac{1}{x} - b\sin\ln x \cdot \frac{1}{x}$

$\Rightarrow xy_1 + x^2y_2 = -a\cos\ln x - b\sin\ln x$

$\Rightarrow xy_1 + x^2y_2 + y = 0$

By applying Leibnitz's theorem, we get.

$x^2 y_{n+2} + n\, y_{n+1} \cdot 2x + \frac{n(n-1)}{2} y_n . 2 + x\, y_{n+1}$

$+ n\, y_n + y_n = 0$

$\Rightarrow x^2 y_{n+2} + 2n\, x\, y_{n+1} + x\, y_{n+1} + (n^2 - n)\, y_n$

$+ n\, y_n + y_n = 0$

$\Rightarrow x^2 y_{n+2} + (2n+1)\, x\, y_{n+1} + (n^2+1)\, y_n = 0$

10) $y = \cos\{\ln(1+x)\}$

$\Rightarrow y_1 = -\sin \ln(1+x) . \frac{1}{1+x}$

$\Rightarrow y_1 (1+x) = -\sin \ln(1+x)$

$\Rightarrow y_2 (1+x) + y_1 = -\cos \ln(1+x) . \frac{1}{1+x}$

$\Rightarrow y_2 (1+x)^2 + y_1 (1+x) + y = 0$

$\Rightarrow y_{n+2} (1+x)^2 + n\, y_{n+1} (1+x) . 2 + \frac{n(n-1)}{2} y_n . 2$

$+ y_{n+1} (1+x) + n\, y_n + y_n = 0$

$\Rightarrow y_{n+2} (1+x)^2 + y_{n+1} (2n+1)(1+x) + (n^2+1)\, y_n = 0$

{Showed}

11) $\lim_{n \to \infty} \left[\frac{1^2}{1^3+n^3} + \frac{2^2}{2^3+n^3} + \cdots \frac{n^2}{n^3+n^3} \right]$

$= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{r^2}{r^3+n^3}$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{r^2}{n^3\left(1+\left(\frac{r}{n}\right)^3\right)}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{(r/n)^2}{1+\left(\frac{r}{n}\right)^3}$$

$$= \int_0^1 \frac{x^2}{1+x^3}\,dx$$

[Substituting $\frac{r}{n} \to x$, $\frac{1}{n} \to dx$

Upper limit $= \lim_{n\to\infty} \frac{n}{n} = 1$

Lower limit $= \lim_{n\to\infty} \frac{0}{n} = 0$]

$$= \int_1^2 \frac{\frac{1}{3}\,dz}{z}$$

$$= \frac{1}{3} \int_1^2 \frac{dz}{z}$$

$$= \frac{1}{3} \left[\ln|z|\right]_1^2$$

$$= \frac{1}{3}(\ln 2 - \ln 1) = \frac{1}{3} \ln 2$$

Let $1+x^3 = z$

$\rightarrow 3x^2\,dx = dz$

$\rightarrow x^2\,dx = \frac{1}{3}dz$

$x \to 0 : z \to 1$

$x \to 1 : z \to 2$

12) $$\lim_{n\to\infty} \left[\frac{1}{n} + \frac{\sqrt{n^2-1}}{n^2} + \cdots \frac{\sqrt{n^2-(n-1)^2}}{n^2}\right]$$

$$= \lim_{n\to\infty} \left[\frac{\sqrt{n^2-0^2}}{n^2} + \frac{\sqrt{n^2-1^2}}{n^2} + \cdots \frac{\sqrt{n^2-(n-1)^2}}{n^2}\right]$$

$$= \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{\sqrt{n^2-r^2}}{n^2} = \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{n\sqrt{1-\left(\frac{r}{n}\right)^2}}{n^2}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{n-1} \sqrt{1-\left(\frac{r}{n}\right)^2}$$

$$= \int_0^1 \sqrt{1-x^2}\,dx$$

substitute, $\frac{1}{n} \to dx$

$\frac{r}{n} \to x$

$U.L = \lim_{n\to\infty} \frac{n-1}{n} = 1$

$L.L = \lim_{n\to\infty} \frac{0}{n} = 0$

$$= \int_0^{\pi/2} \sqrt{1-\sin^2\theta}\,\cos\theta\, d\theta$$

Let $x = \sin\theta$
$\Rightarrow dx = \cos\theta\, d\theta$
$\Rightarrow \theta = \sin^{-1} x$
$x \to 1 \,;\, \theta \to \frac{\pi}{2}$
$x \to 0 \,;\, \theta \to 0$

$$= \int_0^{\pi/2} \cos\theta \cos\theta\, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} 2\cos^2\theta\, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{2} \left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{1}{2}\sin\pi - 0\right] = \frac{\pi}{4}$$

(13) $$\lim_{n\to\infty} \left[\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \cdots \frac{n}{n^2+n^2}\right]$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{n}{n^2+r^2} = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{n}{n^2\left(1+\left(\frac{r}{n}\right)^2\right)}$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} \cdot \frac{1}{1+\left(\frac{r}{n}\right)^2}$$

Substitute, $\frac{1}{n} \to dx$
$\frac{r}{n} \to x$
U.L = 1
L.L = 0

$$= \int_0^1 \frac{1}{1+x^2}\, dx$$

$$= \left[\tan^{-1} x\right]_0^1$$

$$= \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

14) $\underset{n\to\infty}{Lt}\left[\frac{1}{\sqrt{2n-1^2}}+\frac{1}{\sqrt{4n-2^2}}+\cdots\frac{1}{n}\right]$

$= \underset{n\to\infty}{Lt}\sum_{r=1}^{n}\frac{1}{\sqrt{2nr-r^2}} = \underset{n\to\infty}{Lt}\sum_{r=1}^{n}\frac{1}{n}\cdot\frac{1}{\sqrt{2\frac{r}{n}-\frac{r^2}{n^2}}}$

$= \int_0^1\frac{dx}{\sqrt{2x-x^2}} = \int_0^1\frac{dx}{\sqrt{1-1+2x-x^2}}$

$= \int_0^1\frac{dx}{\sqrt{1-(x-1)^2}} = \left[\sin^{-1}(x-1)\right]_0^1$

$= \sin^{-1}(1-1)-\sin^{-1}(0-1) = 0-\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$

15) $\underset{n\to\infty}{Lt}\left[\frac{\sqrt{n}}{n^{3/2}}+\frac{\sqrt{n}}{(n+3)^{3/2}}+\cdots\frac{\sqrt{n}}{\{n+3(n-1)\}^{3/2}}\right]$

$= \underset{n\to\infty}{Lt}\sum_{r=0}^{n-1}\frac{\sqrt{n}}{(n+3r)^{3/2}}$

$= \underset{n\to\infty}{Lt}\sum_{r=0}^{n-1}\frac{\sqrt{n}}{n^{3/2}\left(1+3\frac{r}{n}\right)^{3/2}} = \underset{n\to\infty}{Lt}\sum_{r=0}^{n-1}\frac{1}{n}\cdot\frac{1}{\left(1+3\frac{r}{n}\right)^{3/2}}$

$= \int_0^1\frac{dx}{(1+3x)^{3/2}}$

Let $1+3x=z$
$\Rightarrow 3dx=dz$
$x\to 0;\ z\to 1$
$x\to 1;\ z\to 4$

$= \frac{1}{3}\int_1^4\frac{dz}{z^{3/2}} = \frac{1}{3}\int_1^4 z^{-3/2}dz$

$= \frac{1}{3}\left[\frac{z^{-1/2}}{-1/2}\right]_1^4 = -\frac{2}{3}\left[\frac{1}{\sqrt{4}}-\frac{1}{1}\right] = -\frac{2}{3}\cdot-\frac{1}{2}$

$= \frac{1}{3}$

16) $\int \frac{dx}{3\sin x + 2\cos x + 5}$

$= \int \frac{dx}{3 \cdot \frac{2\tan \frac{x}{2}}{1+\tan^2\frac{x}{2}} + 2 \cdot \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} + 5}$

$= \int \frac{\sec^2\frac{x}{2}\, dx}{6\tan\frac{x}{2} + 2 - 2\tan^2\frac{x}{2} + 5 + 5\tan^2\frac{x}{2}}$

$= \int \frac{\sec^2\frac{x}{2}\, dx}{3\tan^2\frac{x}{2} + 6\tan\frac{x}{2} + 7}$

Let $\tan\frac{x}{2} = z$

$\Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\, dx = dz$

$= \int \frac{2\, dz}{3\left(z^2 + 2z + \frac{7}{3}\right)}$

$= \frac{2}{3} \int \frac{dz}{(z+1)^2 + \frac{4}{3}} = \frac{2}{3} \cdot \frac{\sqrt{3}}{2} \tan^{-1} \frac{\frac{z+1}{2}}{\sqrt{3}}$

$= \frac{1}{\sqrt{3}} \tan^{-1} \frac{\sqrt{3}\left(\tan\frac{x}{2} + 1\right)}{2} + c$

17) $\int \frac{dx}{5 + 4\cos x} = \int \frac{dx}{5 + 4 \cdot \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$

$= \int \frac{\sec^2\frac{x}{2}\, dx}{5 + 5\tan^2\frac{x}{2} + 4 - 4\tan^2\frac{x}{2}}$

Let $\tan\frac{x}{2} = z$

$\therefore \frac{1}{2}\sec^2\frac{x}{2}\, dx = dz$

$= \int \frac{\sec^2\frac{x}{2}\, dx}{9 + \tan^2\frac{x}{2}}$

$$= \int \frac{2\,dz}{9+z^2} = 2\int \frac{dz}{3^2+z^2} = 2\cdot\frac{1}{3}\tan^{-1}\frac{z}{3}+c$$

$$= \frac{2}{3}\tan^{-1}\frac{\tan\frac{x}{2}}{3}+c.$$

18) Let, $I = \int e^{2x}\sin^3 x\,dx$

$$= \int e^{2x}\left(\frac{3\sin x - \sin 3x}{4}\right)dx$$

$$= \int \left(\frac{3}{4}e^{2x}\sin x - \frac{1}{4}e^{2x}\sin 3x\right)dx$$

$$= \frac{3}{4}\int e^{2x}\sin x\,dx - \frac{1}{4}\int e^{2x}\sin 3x\,dx \quad \text{——(1)}$$

Let, $I_1 = \int e^{2x}\sin x\,dx$ ——(ii)

$u = \sin x$

$dv = e^{2x}dx$

$\therefore v = \frac{1}{2}e^{2x}$

$du = \cos x\,dx$

$$= \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\int e^{2x}\cos x\,dx$$

$$= \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\left[\frac{1}{2}e^{2x}\cos x + \frac{1}{2}\int e^{2x}\sin x\,dx\right]$$

Again, $u = \cos x$

$dv = e^{2x}dx$

$\therefore v = \frac{1}{2}e^{2x}$

$du = -\sin x\,dx$

$$= \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x - \frac{1}{4}I_1$$

$$\therefore I_1 + \frac{1}{4}I_1 = \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x$$

$$\Rightarrow I_1 = \frac{4}{5}\left(\frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x\right)$$

$$\Rightarrow I_1 = \frac{2}{5}e^{2x}\sin x - \frac{1}{5}e^{2x}\cos x$$

Again Let, $\int e^{2x}\sin 3x\,dx = I_2$

$$= \frac{1}{2}e^{2x}\sin 3x - \frac{1}{2}\int e^{2x}\cdot 3\cos 3x\,dx$$

$$= \frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\int e^{2x}\cos 3x\,dx$$

$= \frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\left[\frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\int e^{2x}\sin 3x\,dx\right]$

$= \frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x - \frac{9}{4}I_2$

$\therefore I_2 + \frac{9}{4}I_2 = \frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x$

$\Rightarrow I_2 = \frac{4}{13}\left(\frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x\right)$

$\Rightarrow I_2 = \frac{2}{13}e^{2x}\sin 3x - \frac{3}{13}e^{2x}\cos 3x$

$\therefore I = \frac{3}{4}I_1 - \frac{1}{4}I_2$

$= \frac{3}{4}\left(\frac{2}{5}e^{2x}\sin x - \frac{1}{5}e^{2x}\cos x\right) - \frac{1}{4}\left(\frac{2}{13}e^{2x}\sin 3x - \frac{3}{13}e^{2x}\cos 3x\right)$

$= \frac{3}{10}e^{2x}\sin x - \frac{3}{20}e^{2x}\cos x - \frac{1}{26}e^{2x}\sin 3x - \frac{3}{52}e^{2x}\cos 3x + c$

19) $\int \frac{x(\tan^{-1}x)^2}{(1+x^2)^{1/2}(1+x^2)}\,dx$

let, $\tan^{-1}x = z$

$\cancel{\rightarrow \frac{dx}{1+x^2}}$

$\rightarrow x = \tan z$

$\rightarrow dx = \sec^2 z\,dz$

$= \frac{\tan z \cdot z^2}{(1+\tan^2 z)^{3/2}}\sec^2 z\,dz$

$= \frac{z^2\tan z}{\sec^3 z}\sec^2 z\,dz$

$= \frac{z^2\tan z}{\sec z}\,dz = \int z^2\sin z\,dz$

$= -z^2\cos z + 2\int z\cos z\,dz$

Let $u = z^2$

$\rightarrow du = 2z\,dz$

$dv = \sin z\,dz$

$\therefore v = -\cos z$

$$= -z^2\cos z + 2\left[z\sin z - \int \sin z\, dz\right]$$

$$= -z^2\cos z + 2z\sin z + 2\cos z + c$$

$$= -(\tan^{-1}x)^2\cos(\tan^{-1}x) + 2\tan^{-1}x\sin(\tan^{-1}x) + 2\cos(\tan^{-1}x) + c$$

Again, $u = z$

$\therefore du = dz$

$dv = \cos z\, dz$

$\therefore v = \sin z$

$$= -(\tan^{-1}x)^2 \cdot \frac{1}{\sqrt{1+x^2}} + 2\tan^{-1}x \cdot \frac{x}{\sqrt{1+x^2}} + 2\cdot\frac{1}{\sqrt{1+x^2}} + c$$

$$= \frac{2 + 2x\tan^{-1}x - (\tan^{-1}x)^2}{\sqrt{1+x^2}} + c$$

20) $$\int \frac{x\, dx}{(1+x)\sqrt{1+x^2}} = \int \frac{(1+x)-1}{(1+x)\sqrt{1+x^2}}\, dx$$

$$= \int \frac{dx}{\sqrt{1+x^2}} - \int \frac{dx}{(1+x)\sqrt{1+x^2}}$$

$$= \ln(x+\sqrt{1+x^2}) - \int \frac{-dz}{z^2\cdot\frac{1}{z}\sqrt{1+\left(\frac{1}{z}-1\right)^2}}$$

$$= \ln(x+\sqrt{1+x^2}) + \frac{dz}{z\sqrt{1+\frac{1}{z^2}-\frac{2}{z}+1}}$$

let, $1+x = \frac{1}{z}$

$\Rightarrow dx = -\frac{dz}{z^2}$

$\therefore x = \frac{1}{z} - 1$

$$= \ln(x+\sqrt{1+x^2}) + \frac{dz}{\sqrt{z^2+1-2z+z^2}}$$

$$= \ln(x+\sqrt{1+x^2}) + \frac{1}{\sqrt{2}}\cdot\frac{dz}{\sqrt{z^2-z+\frac{1}{2}}}$$

$$= \ln(x+\sqrt{1+x^2}) + \frac{1}{\sqrt{2}}\cdot\frac{dz}{\sqrt{\left(z-\frac{1}{2}\right)^2+\frac{1}{4}}}$$

$$= \ln(x+\sqrt{1+x^2}) + \frac{1}{\sqrt{2}}\ln\left(\sqrt{\left(z-\frac{1}{2}\right)^2+\frac{1}{4}} + \left(z-\frac{1}{2}\right)\right) + c$$

$$= \ln(x+\sqrt{1+x^2}) + \frac{1}{\sqrt{2}} \ln\left| z - \frac{1}{2} + \sqrt{z^2 - z + \frac{1}{2}} \right| + c$$

$$= \ln(x+\sqrt{1+x^2}) + \frac{1}{\sqrt{2}} \ln\left| \frac{1}{x+1} - \frac{1}{2} + \sqrt{\frac{1}{(x+1)^2} - \frac{1}{x+1} + \frac{1}{2}} \right| + c$$

$$= \ln(x+\sqrt{1+x^2}) + \frac{1}{\sqrt{2}} \ln\left| \frac{2-1-x}{2(1+x)} + \frac{\sqrt{2-2x-2+x^2+2x+1}}{\sqrt{2}(x+1)} \right| + c$$

$$= \ln(x+\sqrt{1+x^2}) + \frac{1}{\sqrt{2}} \ln\left| \frac{1-x}{2(1+x)} + \frac{\sqrt{1+x^2}}{\sqrt{2}(x+1)} \right| + c$$

$$= \ln(x+\sqrt{1+x^2}) + \frac{1}{\sqrt{2}} \ln\left| \frac{1-x+\sqrt{2+2x^2}}{2(1+x)} \right| + c$$

$$= \ln(x+\sqrt{1+x^2}) + \frac{1}{\sqrt{2}} \ln\left|1-x+\sqrt{2+2x^2}\right| - \frac{1}{\sqrt{2}} \ln\left|2(1+x)\right| + c$$

21) Rolle's Theorem: Let $y = f(x)$ be continuous at every point of the closed interval $[a,b]$ and differentiable at every point of its interior (a,b). If $f(a) = f(b) = 0$, then there is at least one number c in (a,b) at which $f'(c) = 0$.

Verification of Rolle's Theorem:

Since a polynomial function is everywhere continuous and differentiable, the given function is continuous as well as differentiable on every interval. To identify the interval, we solve the equation $f(x) = 0$.

Now, $f(x) = 0$

$\Rightarrow 2x^3 + x^2 - 4x - 2 = 0$

$\Rightarrow (x^2 - 2)(2x + 1) = 0$

$\Rightarrow x^2 = 2$ or $= -\frac{1}{2}$

$\therefore x = \sqrt{2}$ or $x = -\sqrt{2}$ or $x = -\frac{1}{2}$

So, we consider the given function in $[-\sqrt{2}, \sqrt{2}]$

Clearly, $f(-\sqrt{2}) = f(\sqrt{2}) = 0$

Thus, all the conditions of Rolle's theorem are satisfied. So, there must exist $c \in [-\sqrt{2}, \sqrt{2}]$ such that $f'(c) = 0$

But $f'(x) = 6x^2 + 2x - 4$

So, $6c^2 + 2c - 4 = 0$

$\Rightarrow 3c^2 + c - 2 = 0$

$\Rightarrow 3c^2 + 3c - 2c - 2 = 0$

$\Rightarrow 3c(c+1) - 2(c+1) = 0$

So, $c = \frac{2}{3}$ or $c = -1$

Clearly both these points lie in $[-\sqrt{2}, \sqrt{2}]$

Hence, Rolle's theorem is verified.