Fourier Theorem or Fourier Analysis

Baron Jean Baptise Joseph Fourier (1768-1830) introduced the idea that any periodic function can be represented by a series of sines and cosines which are harmonically related.

Basic Definitions

A function $f(x)$ is said to have period P if $f(x+P) = f(x)$ for all x. Let the function $f(x)$ has period 2π. In this case, it is enough to consider behavior of the function on the interval $[-\pi, \pi]$. The *Fourier series* of the function $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\} \tag{1}$$

where the *Fourier series* a_0, a_n, and b_n are defined by the integrals

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \tag{2}$$

Fourier Series of Even and Odd Functions

The Fourier series expansion of an *even* function $f(x)$ with the period of 2π does not involve the terms with sines and has the form:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{3}$$

where the Fourier coefficients are given by the formulas

$$a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx \tag{4}$$

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx \tag{5}$$

Accordingly, the Fourier series expansion of an *odd* 2π-period function $f(x)$ consists of sine terms only and has the form:

$$f(x) = \sum_{n=1}^{\infty}\{b_n \sin nx\} \tag{6}$$

where the coefficient b_n are

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx \tag{7}$$

Below we are consider expansions of 2π-periodic functions into their Fourier series.

Example 1:

Let the function $f(x)$ be 2π-periodic and suppose that it is by the Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\}$$

Calculate the coefficients a_0, a_n, and b_n

Solution :

To define a_0, we integrate the Fourier series on the interval $[-\pi, \pi]$

$$\int_{-\pi}^{\pi} f(x)\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi} \cos nx\,dx + b_n\int_{-\pi}^{\pi} \sin nx\,dx\right] \tag{8}$$

$$\frac{a_0}{2}\int_{-\pi}^{\pi} dx = \pi a_0 \tag{9}$$

For all $n > 0$,

$$\int_{-\pi}^{\pi} \cos nx \, dx = \left(\frac{\sin nx}{n}\right)\Big|_{-\pi}^{\pi} = 0 \tag{10}$$

and

$$\int_{-\pi}^{\pi} \sin nx \, dx = \left(-\frac{\cos nx}{n}\right)\Big|_{-\pi}^{\pi} = 0 \tag{11}$$

Therefore, all the terms on the right of the summation sign are zero, so we obtain

$$\int_{-\pi}^{\pi} f(x) \, dx = \pi a_0 \qquad or \qquad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \, dx \tag{12}$$

In order to find the coefficients a_n, we multiply both sides of the Fourier series by $\cos mx$ and integrate term by term:

$$\int_{-\pi}^{\pi} f(x) \cos mx \, dx = \frac{a_0}{2}\int_{-\pi}^{\pi} \cos mx \, dx + \sum_{n=1}^{\infty}\left[a_n \int_{-\pi}^{\pi} \cos nx \cos mx \, dx + b_n \int_{-\pi}^{\pi} \sin nx \cos mx \, dx\right] \tag{13}$$

The first term on the right side is zero. Then using the well-known *trigonometric identities*, we have

$$\int_{-\pi}^{\pi} \sin nx cosmx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} [\sin(n+m)x + \sin(n-m)x] \, dx = 0, \tag{14}$$

$$\int_{-\pi}^{\pi} \cos nx cosmx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} [\cos(n+m)x + \cos(n-m)x] \, dx = 0, \tag{15}$$

if $m \neq n$.
In case when $m = n$, we can write:

$$\int_{-\pi}^{\pi} \sin nx cosmx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} [\sin 2mx + \sin 0] \, dx$$

$$\Rightarrow \int_{-\pi}^{\pi} \sin 2mx \, dx = \frac{1}{2}\left[\left(-\frac{\cos 2mx}{2m}\right)\Big|_{-\pi}^{\pi}\right]$$

$$= \frac{1}{4m}[-\cos(2m\pi) + \cos(2m(-\pi))] = 0$$

Now

$$\int_{-\pi}^{\pi} \cos nx cosmx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} 2\cos^2 mx \, dx \tag{16}$$

$$\Rightarrow \frac{1}{2}\int_{-\pi}^{\pi} (\cos 2mx + 1) \, dx = \frac{1}{2}\left[\left(\frac{\sin 2mx}{2m}\right)\Big|_{-\pi}^{\pi} + 2\pi\right] = \frac{1}{4m}[\sin(2m\pi) - \cos(2m(-\pi))] + \pi = \pi. \tag{17}$$

Thus,

$$\int_{-\pi}^{\pi} f(x) \cos mx \, dx = a_m \pi, \qquad \Rightarrow a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos mx \, dx, \qquad m = 1, 2, 3, \ldots \tag{18}$$

Similarly, multiplying the Fourier series by $\sin mx$ and integrating term by term, we obtain the expression for b_m:

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin mx \, dx, \qquad m = 1, 2, 3, \ldots \tag{19}$$

Rewriting the formulas for a_n, b_n, we can write the final expressions for the Fourier coefficients:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

Example 2:

Find the Fourier series for the square 2π-periodic wave function defined on the interval $[-\pi, \pi]$:

$$f(x) = 0, \quad if \quad -\pi \le x \le 0,$$

$$f(x) = 1, \quad if \quad 0 < x \le \pi,$$

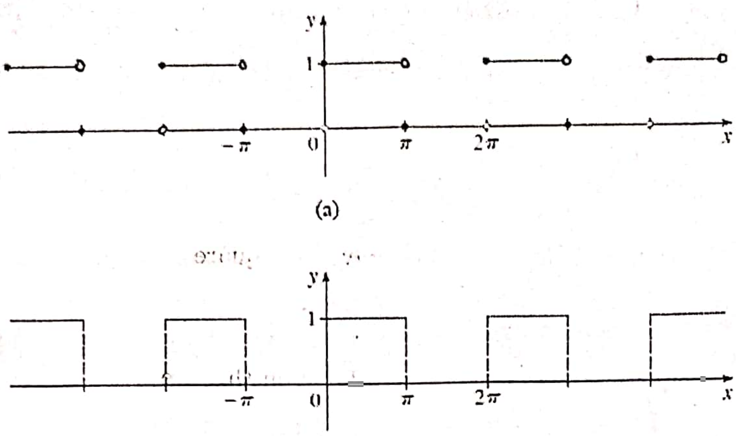

Figure 1: Square wave

Solution :

First we calculate the constant a_0 :

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{0}^{\pi} 1\,dx = \frac{1}{\pi}.\pi = 1 \tag{20}$$

Find now the Fourier coefficients for $n \neq 0$:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{0}^{\pi} 1.\cos nx\,dx = \frac{1}{\pi}\left[\left(\frac{\sin nx}{n}\right)\Big|_0^{\pi}\right] = \frac{1}{\pi n}.0 = 0 \tag{21}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{0}^{\pi} 1.\sin nx\,dx = \frac{1}{\pi}\left[\left(-\frac{\cos nx}{n}\right)\Big|_0^{\pi}\right] = -\frac{1}{\pi n}(\cos n\pi - \cos 0) = \frac{1-\cos n\pi}{n\pi} \tag{22}$$

As $\cos n\pi = (-1)^n$, we can write

$$b_n = \frac{1-(-1)^n}{n\pi} \tag{23}$$

Thus, the Fourier series for the square wave is

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\pi}\sin nx \tag{24}$$

We can easily find the first few terms of the series S_n. By setting, for example, $n+5$, we get

$$f(x) = \frac{1}{2} + \frac{1-(-1)}{\pi}\sin x + \frac{1-(-1)^2}{2\pi}\sin 2x + \frac{1-(-1)^3}{3\pi}\sin 3x + \frac{1-(-1)^4}{4\pi}\sin 4x + \frac{1-(-1)^5}{5\pi}\sin 5x + \ldots \tag{25}$$

$$f(x) = S_n = \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin 5x + \ldots \tag{26}$$

The graph of the function and the Fourier series expansion for $n = 15$ is shown below in Figure 2.

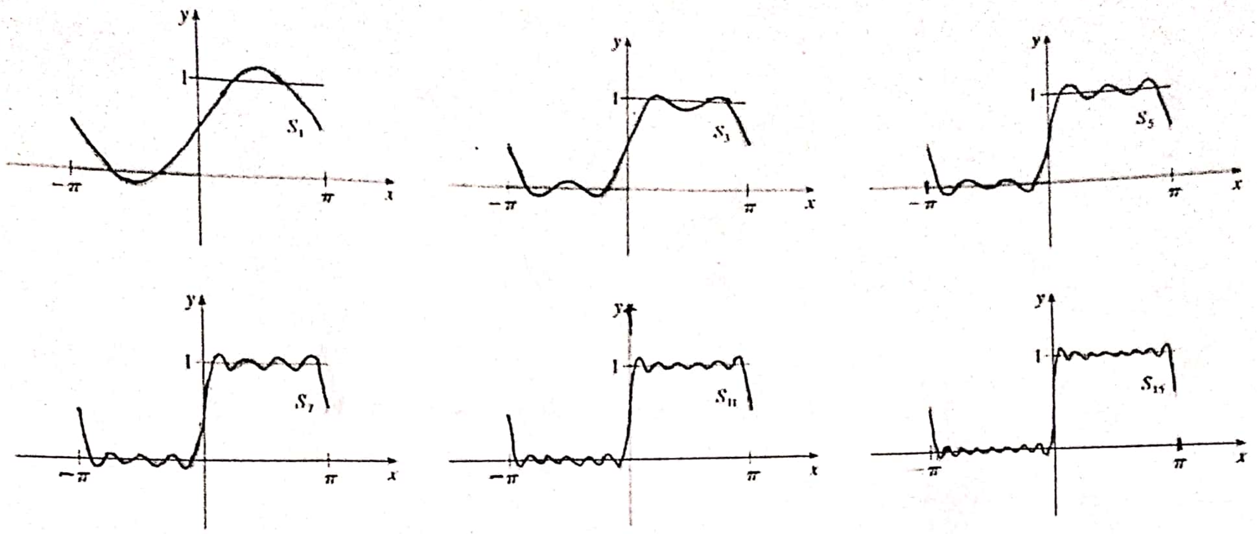

Figure 2: Fourier analysis of square wave for different n

Example 3:
Find the Fourier series for the saw-tooth wave defined on the interval $[-\pi, \pi]$ and having period 2π.

Solution:
Calculate the Fourier coefficients for the saw-tooth wave. Since this function is odd (Figure 3), then $a_0 = a_n = 0$. Find the coefficients b_n:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx \tag{27}$$

To calculate the latter integral we use *integration by parts*:

$$\int_{-\pi}^{\pi} udv = (uv)|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} vdu$$

Let $u = x, dv = \sin nx\,dx$. Then $du = dx, v = \int \sin x\,dx = -\frac{\cos nx}{n}$, so the integral becomes

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx = \frac{1}{\pi}\left[\left(-\frac{x\cos nx}{n}\right)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\left(-\frac{\cos nx}{n}\right)dx\right] = \frac{1}{n\pi}\left[-2\pi\cos n\pi + \left(\frac{\sin nx}{n}\right)\Big|_{-\pi}^{\pi}\right] \tag{28}$$

$$b_n = \frac{1}{n\pi}\left[-2\pi\cos n\pi + \frac{1}{n}(\sin n\pi - \sin(-n\pi))\right] = \frac{1}{n\pi}\left[-2\pi\cos n\pi + \frac{2\sin n\pi}{n}\right] = \frac{2}{n\pi}\left[\frac{\sin n\pi}{n} - \pi\cos n\pi\right] \tag{29}$$

Substituting $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$ for all integer values of n, we obtain

$$b_n = \frac{2}{n\pi}(-\pi(-1)^n) = -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1} \tag{30}$$

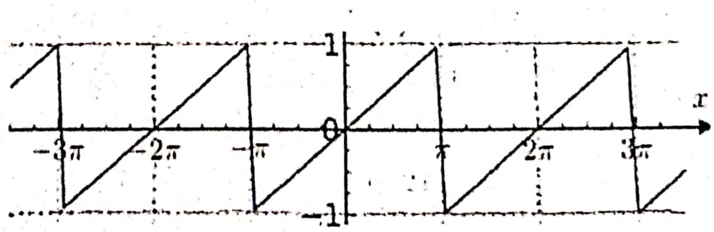

Figure 3: Sawtooth wave

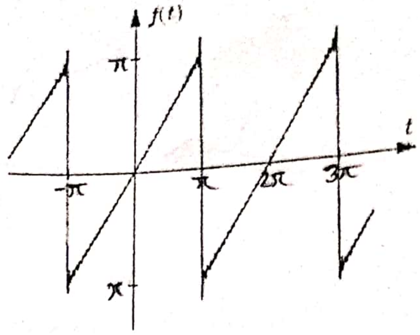

Figure 4: Fourier analysis of sawtooth wave

Thus, the Fourier series expansion of the sawtooth wave (Figure 3) is

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx \tag{3}$$

You may check the following link:
`https://www.youtube.com/watch?v=dsOcmAV-Yek`
`http://bilimneguzellan.net/en/follow-up-to-fourier-series-2/`