Simple Harmonic Oscillation(SHO)

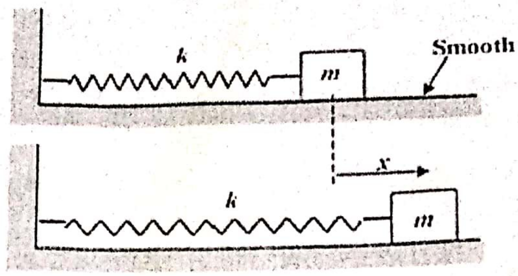

Figure 1: Spring-mass system

Consider a spring-mass system, where one-end of spring with spring constant k is fastened to a rigid support and another end is attached to the block of mass m. Figure 1 shows a spring-mass system in relaxed position or state $x = 0$(broken line) is called equilibrium position. Now, if you deform the system with an external force F_{ext} in the positive $x - direction$, an restoring force F_s, will be produced due to the elastic property of the spring. Upon releasing the block from a distance say$x$, the block will move back and forth about its equilibrium position. Restoring force, $F = -kx$, is opposite to the distance creates ɛ n acceleration to the block. Newton's of motion for the spring-mass system is

$$F_s = -kx \Rightarrow ma = -kx \Rightarrow a = -\frac{k}{m}x \tag{1}$$

Equation 1 is the defining equation of SHO, which tells us that restoring force or acceleration is proportional and opposite to the displacement.

Quantities involved in SHO

Observations reveals that the block does not cross the limit upto which it was displaced from its equilibrium position. Hence, it travels upto maximum distance either side of equilibrium position which is called *amplitude or maximum displacement and denoted by A or* x_m.
The spring-mass system maintains periodic motion, i. e., it returns a certain point after every equal time interval. This time interval is known as *time-period* of the motion and expressed as T. The reciprocal of T is called meaning how many times block reaches the certain point in unit time as $f = 1/T$.
Another quantity epoch will be discussed later.

Differential equation of SHO and its Solution:

Defining equation of SHO is

$$a = -\frac{k}{m}x \tag{2}$$

but $a = \frac{dv}{dt}$ and $v = \frac{dx}{dt}$. Substituting these in Equation 2 gives

$$\frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} = -\frac{k}{m}x \tag{3}$$

Equation 3 is the differential equation of SHO. Its solution i.e., expression of the displacement is essential to know position, velocity and acceleration at any instant. The expression of x will be sinusoidal (sine or cosine) or exponential since its second order derivative gives it back with a minus sign and a constant $\frac{k}{m}$.

The quantity related to $\frac{k}{m}$ can be found considering their units as

$$\frac{k}{m} = \frac{N/m}{Kg} = \frac{Kgms^{-2}}{(Kg)m} = s^{-2} = (Hz)^2 \tag{4}$$

So, the constant will have some form of frequency f. Since position is a function of time, t having unit sec, the sinusoidal function takes $2\pi ft = \omega t$ as its angle. ω is called angular frequency where $\omega^2 = k/m$. Generally, the expression of x is written as

$$x(t) = \begin{cases} A\cos(\omega t + \phi) \\ A\sin(\omega t + \phi) \\ Ae^{i(\omega t+\phi)} \end{cases} \tag{5}$$

where ϕ is called initial phase or epoch at $t = 0$ and $(\omega t + \phi)$ is called the phase. For any SHO A, ω, ϕ are constants. Now, we will see that $A\cos(\omega t + \phi)$ is the solution of Eq. 2. So,
Firstly,

$$\begin{aligned}
\frac{dx}{dt} &= \frac{d}{dt}A\cos(\omega t + \phi) \\
\frac{dx}{dt} &= A\frac{d}{dt}\cos(\omega t + \phi) \\
\frac{dx}{dt} &= A[-\sin(\omega t + \phi)]\frac{d}{dt}(\omega t + \phi) \\
\frac{dx}{dt} &= -A\sin(\omega t + \phi)\left[\omega\frac{dt}{dt} + \frac{d\phi}{dt}\right] \\
\frac{dx}{dt} &= v(t) = -\omega A\sin(\omega t + \phi)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{d^2x}{dt^2} &= \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}[-\omega A\sin(\omega t + \phi)] \\
\frac{d^2x}{dt^2} &= -\omega A\frac{d}{dt}[\sin(\omega t + \phi)] \\
\frac{d^2x}{dt^2} &= -\omega^2 A\cos(\omega t + \phi) \\
\frac{d^2x}{dt^2} &= a(t) = -\omega^2 x
\end{aligned} \tag{6}$$

Hence $x = A\cos(\omega t + \phi)$ is one of the solutions of differential equation of SHO or SHM.

Graphical presentation in SHM

The dynamical quantities involved in SHM are position, velocity and acceleration; all them are functions of time. Consider the followings are position, velocity, and acceleration

$$x(t) = A\cos(\omega t + \phi)) \tag{7}$$
$$v(t) = -\omega A\sin(\omega t + \phi) \tag{8}$$
$$a(t) = -\omega^2 A\cos(\omega t + \phi) \tag{9}$$

To know the graphical behavior of x, v, and a with time; at first, we have to make the following table:

t	0	$\frac{T}{4}$	$\frac{T}{2}$	$\frac{3T}{4}$	T
ωt	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos \omega t$	1	0	-1	0	1
$x = A\cos\omega t$	A	0	$-A$	0	A
$\sin\omega t$	0	1	0	-1	0
$v = -\omega A \sin\omega t$	0	$-\omega A$	0	$+\omega A$	0
$a = -\omega^2 A \cos\omega t$	$-\omega^2 A$	0	$+\omega^2 A$	0	$-\omega^2 A$

In the above table t is the independent variable and the chosen points are of equal intervals $T/4$ of time period. Since x, v, and a are sinusoidal in nature, they their nature after every time period.

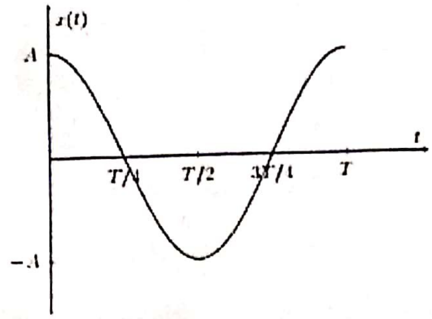

Figure 2: Displacement versus time graph of a simple harmonic oscillator

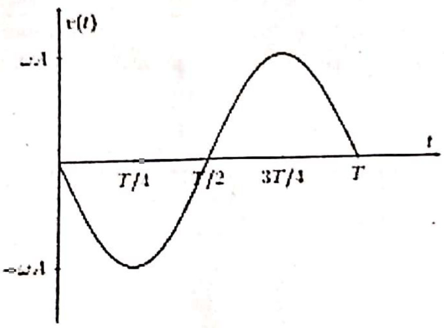

Figure 3: Velocity versus time graph of a simple harmonic oscillator

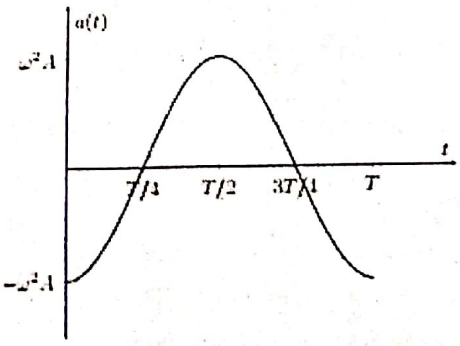

Figure 4: Acceleration versus time graph of a simple harmonic oscillator

Energy in SHM

Energy i.e., mechanical energy has two forms, namely, potential energy and kinetic energy. Now, we will derive the expressions of at any instant. Potential energy, U, at position x is

$$U = \int_0^x F(x)\, dx$$

$$U = \int_0^x (-kx)\, dx \tag{10}$$

$$U = \frac{1}{2}kx^2 \tag{11}$$

in Eq. 10 we used $F(x) = -kx$ as the spring's force. If We choose $x = a \sin \omega t$, we obtain from Eq. 11

$$U = \frac{1}{2}kA^2 \sin^2 \omega t \tag{12}$$

The kinetic energy, T at x is

$$T = \frac{1}{2}mv^2$$

$$T = \frac{1}{2}m\omega^2 \cos^2 \omega t \tag{13}$$

$$T = \frac{1}{2}kA^2 \cos^2 \omega t \tag{14}$$

where $v = \omega A \cos \omega t$ in Eq. 13 and $\omega^2 = k/m$ in Eq. 14. Hence, the total energy, E at position x is

$$E = T + U = \frac{1}{2}kA^2 \cos^2 \omega t + \frac{1}{2}kA^2 \sin^2 \omega t = \frac{1}{2}kA^2 \tag{15}$$

where we used $\sin^2 \omega t + \cos^2 \omega t = 1$, Eq. 15 shows that the energy in SHM is independent of both *position and time* involves only the constant quantities *spring constant or springiness and amplitude of the motion.* Therefore energy in SHM is constant at any position and at any time. graphically we can these as:

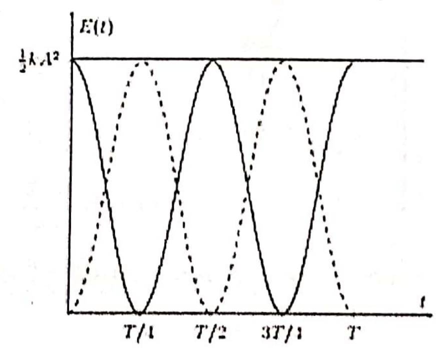

Figure 5: Energy versus time graph of a simple harmonic oscillator

In Figs. 5 and 6 dashed line and solid line represent potential energy and kinetic energy respectively.

Systems executing SHO

Spring-mass system:

We have already describe the spring-mass system and we have

$$F_s = -kx = ma \quad and \quad \omega^2 = \frac{k}{m} \tag{16}$$

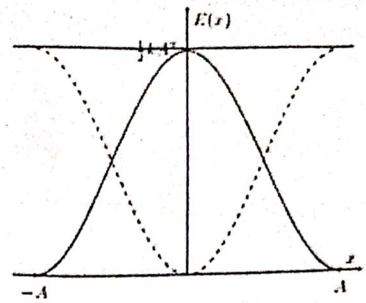

Figure 6: Energy versus position graph of a simple harmonic oscillator

where k is the spring-constant(springiness) of the spring considering the mass of the spring is very negligible. m is the mass of the block attached to the spring.
Spring mass system is an example of simple harmonic oscillator. This system maintains periodicity of time i.e., it has a time period which is related to the angular frequency, ω as $\omega = 2\pi/T$. So, from Eq. 16 we can write

$$\frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$
$$\Rightarrow T = 2\pi\sqrt{\frac{m}{k}} \tag{17}$$

For greater k we will have less time period or rapid oscillation and for massive body we will have larger time period or sluggish oscillation.

Simple Pendulum

Anything which can oscillate about an horizontal axis is called the pendulum.A simple pendulum is made attaching a small object(bob) of mass m with a string of negligible mass. Consider a simple pendulum is

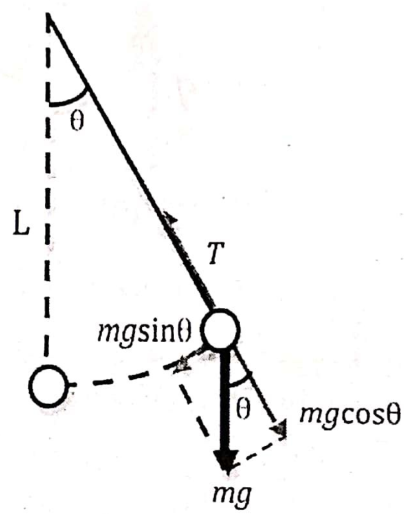

Figure 7: Simple pendulum

hanged from the ceiling. The length of the pendulum from the suspension to the midpoint of the bob is L. If we displace the bob by an angle θ from its equilibrium position, the bob will oscillate simple harmonically as shown in Fig. 7.
The equation of motion in this case is

$$\tau = (-mg\sin\theta)L = I\alpha \tag{18}$$

$-mg\sin\theta$ works as a restoring force, $mg\cos\theta$ balances with the tension of the string. I is the moment of inertia of the bob about point of suspension mL^2 and $\alpha = d^2\theta/dt^2$ is the angular acceleration of the bob. Thus Eq. 16 reduces to

$$(-mg\sin\theta)L = mL^2\frac{d^2\theta}{dt^2}$$
$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta \tag{19}$$

For small angle approximation, $\sin\theta \approx \theta$, Eq. 19 finally has the form

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta \tag{20}$$

This is the differential form of SHO in angular motion with constant $\omega^2 = g/L$. so, the time period for simple pendulum is

$$\frac{2\pi}{T} = \sqrt{\frac{g}{L}}$$
$$\Rightarrow T = 2\pi\sqrt{\frac{L}{g}} \tag{21}$$

Compound pendulum(Physical pendulum)

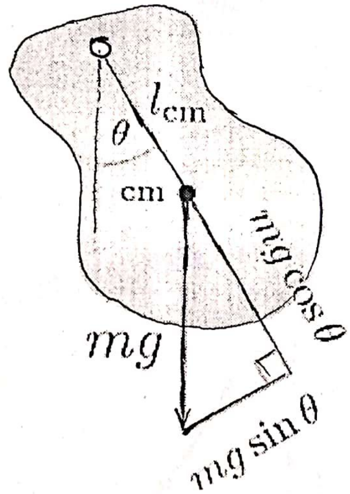

Figure 8: Physical pendulum

Simple pendulum is an ideal case but the pendulums except simple are compound pendulum. Fig. 8 shows a compound pendulum of mass m displaced from it s equilibrium position at an angle θ. The equation of motion is

$$\tau = I\alpha$$
$$I\frac{d^2\theta}{dt^2} = (-mg\sin\theta)l \tag{22}$$

where l is the distance between point of suspension and center of mass(cm), I is the moment of inertia about the point of suspension. The small angle approximation $\sin\theta = \theta$ reduces the Eq. 22 as

$$\frac{d^2\theta}{dt^2} = -mg\frac{l}{I}\theta \tag{23}$$

The expression of time period for this case can be from

$$\omega^2 = \frac{mgl}{I}$$

$$\frac{2\pi}{T} = \sqrt{\frac{mgl}{I}}$$

$$T = 2\pi\sqrt{\frac{I}{mgl}} \tag{24}$$

To calculate the time-period we have to use parallel-axes theorem for moment of inertia since the axis of rotation is the parallel to the axis passing through the center of mass, in which $I = I_G + ml^2$.

Torsional pendulum

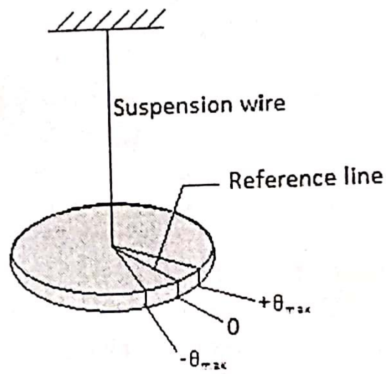

Figure 9: Torsional pendulum

Consider one end of a wire is attached to a flat disc and another end of the wire is attached to a rigid support. If we now displace the the disc from its equilibrium position by a small angle, a twist is produced in the connecting wire. The twist then helps the disc to oscillate harmonically. The equation of motion for torsional pendulum is

$$\tau = -\kappa\theta = I\alpha \tag{25}$$

κ is the torsional constant of the wire and I is the moment of inertia of the disc about the point of connection to the wire. From Eq. 25

$$\omega^2 = \frac{\kappa}{I}$$

$$\frac{2\pi}{T} = \sqrt{\frac{\kappa}{I}}$$

$$T = 2\pi\sqrt{\frac{I}{\kappa}} \tag{26}$$

Eq. 26 is the equation of time period for torsional pendulum in SHO or *angular harmonic oscillator.*

LC Circuit

The simplest example of an oscillating electrical circuit consists of an inductor L and a capacitor C connected together in series with a switch as shown in Fig. 10. We assume that the resistance in the circuit is negligible. This analogous to the assumption for mechanical systems that there are no frictional forces present. Initially the switch is open and the capacitor is charged to voltage V_C. The charge q on the capacitor is given by $q = V_C C$ where C is the capacitance. When switch is closed the charge begins to flow throw the inductor and

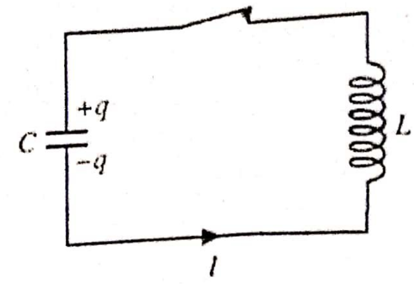

Figure 10: An electrical oscillator consisting of an inductor L and a capacitor C connected in series.

a current $I = dq/dt$ flows in the circuit. This *time-varying* produces a voltage across the inductor given by $V_L = Idq/dt$, using Kirchhoff's law, we can write

$$\begin{aligned} V_C + V_L &= 0 \\ \frac{q}{C} + L\frac{dq}{dt} &= 0 \\ \frac{q}{C} + L\frac{d^2q}{dt^2} &= 0 \\ \frac{d^2q}{dt^2} &= -\frac{1}{LC}q \end{aligned} \tag{27}$$

This describes how the charge on a plate of the capacitor varies with time, which represents SHM. the frequency of the oscillator is $\omega = \sqrt{1/LC}$. Since at time $t = 0$ the charge on the capacitor is maximum value say, q_0, then the solution of the Eq. 27 is $q = q_0 \cos \omega t$. The variation of charge q with respect to t is shown in Fig 11.

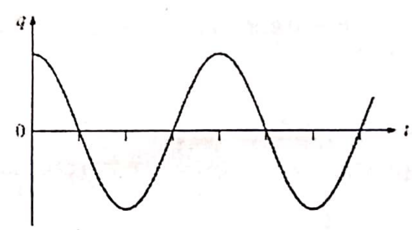

Figure 11: The variation of charge q with time on the capacitor in a series LC circuit.

Damped harmonic oscillation

A bob swinging back and forth at the end of a string is not an ideal oscillating system. After we set the bob in motion, the amplitude of oscillation steadily reduces and bob eventually comes to rest. This is because there are dissipative forces acting and system readily loses energy. For example, the bob will experience a frictional force as it moves through the air. All real oscillating systems are subject to damping forces and will cease to cease to oscillate if energy is no feed back into them.

The equation of motion for damped harmonic oscillator

An example of a damped harmonic oscillator is shown in Fig. 12. It is a similar to spring-mass system but now the mass is immersed in viscous fluid. When an object move through a viscous fluid it experiences a frictional force. This force dampens the motion: higher the velocity the greater the frictional force. The damping forceF_d acting on the mass in fig. 12 is proportional to its velocity v so long as v is not too large, i.e., $F_d = -bv$ where the minus sign indicates that the force always acts in the opposite direction to the motion. The constant b depends on the shape of the mass and the viscosity of the fluid and has unit of the force per

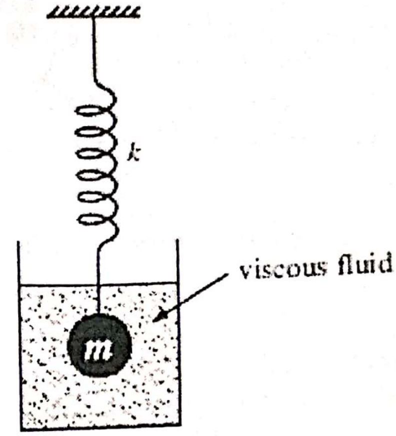

Figure 12: A damped harmonic oscillator with an oscillating mass immersed in a viscous fluid

unit velocity. When the mass is displaced from its equilibrium position there will be restoring force due to the spring and in addition the damping force $-bv$ due to the fluid. the resulting equation of the motion is

$$ma = -kx - bv$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \tag{28}$$

we introduce the parameters

$$\omega_0^2 = \frac{k}{m} \tag{29}$$

$$\gamma = \frac{b}{m} \tag{30}$$

In terms of these, eq. 28 becomes

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0 \tag{31}$$

This is the equation of damped harmonic oscillator. ω_0 is called natural frequency or oscillation of the frequency if there is no damping. This allows the possibility that the damping does change the frequency of oscillation. Eq. 31 has different solutions depending on the degree of damping involved, corresponding to the cases of (i) light damping, (ii) heavy or over damping and (iii) critical damping. Light damping is the most important case for as because it involves oscillatory motion whereas the other two cases do not.

Light damping

This corresponds to the mass in Fig. 12 being immersed in a fluid of low viscosity like thin oil or even just air. For a lightly damped oscillator we suggest an expression for the displacement that has the form $x = A_0 e^{(-\beta t)} \cos\omega t$. Then

$$\frac{dx}{dt} = -A_0 e^{-\beta t}(\omega \sin\omega t + \beta\cos\omega t) \tag{32}$$

$$\frac{d^2x}{dt^2} = A_0 e^{-\beta t}[2\beta\omega \sin\omega t + (\beta^2 - \omega^2)\cos\omega t] \tag{33}$$

Substituting these into Eq. 31 and collecting terms in $\sin\omega t$ and $\cos\omega t$ gives

$$A_0 e^{-\beta t}[(2\beta\omega - \gamma\omega) sin\omega t + (\beta^2 - \omega^2 - \gamma\beta + \omega_0^2)\cos\omega t] = 0 \tag{34}$$

This can only be true for all times if the $\sin\omega t$ and $\cos\omega t$ terms are both equal to zero. Therefore,

$$2\beta\omega - \gamma\beta = 0$$

giving $\beta = \gamma/2$ and

$$\beta^2 - \omega^2 - \gamma\beta + \omega_0^2 = 0$$

Substituting for β we obtain

$$\omega^2 = \omega_0^2 - \frac{\gamma^2}{4} \tag{35}$$

So our solution for the equation of the lightly damped oscillator is

$$x = A_0 e^{-\gamma t/2} \cos \omega t \tag{36}$$

where $\omega = (\omega_0^2 - \gamma^2/4)^{1/2}$Eq. 6 represents oscillatory motion if ω is real, i.e., $\omega_0^2 > \gamma^2/4$ is the condition for light damping. Eq. 35 shows the angular frequency of oscillation ω is approximately equal to the undamped value ω_0 when $\omega_0^2 >> \gamma^2/4$. The general solution of Eq. 31 is

$$x = A_0 e^{-\gamma t/2} \cos(\omega t + \phi) \tag{37}$$

A graph of $x = A_0 e^{-\gamma t/2} \cos \omega t$ is shown in Fig. 13 where the steady decrease in the amplitude of the

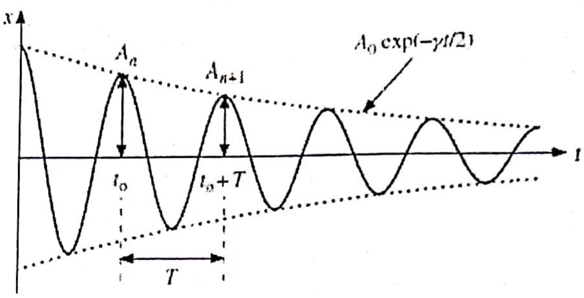

Figure 13: Decay of amplitude in damped harmonic oscillation

oscillation is apparent. The dotted line represent the $e^{-\gamma t/2}$ term which form an envelope for the oscillations.

Rate of energy loss in a damped harmonic oscillation

The mechanical energy of a damped harmonic oscillator is eventually dissipated as heat t its surroundings. We can deduce the rate at which energy is lost by considering how the total energy of the oscillation changes with time. The total energy E is given by

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \tag{38}$$

For the case of very lightly damped oscillator, i.e., $\omega_0^2 >> \gamma^2/4$, the displacement is

$$x\text{[redacted]} = A_0 e^{-\gamma t/2} \cos \omega_0 t \tag{39}$$

Hence

$$v = \frac{dx}{dt} = A_0 \omega_0 e^{-\gamma t/2} [\sin \omega_0 t + (\gamma/2\omega_0) \cos \omega_0 t] \tag{40}$$

Since $\omega_0^2 >> \gamma^2/4$, we can neglect the second term in the square brackets and write

$$v = \frac{dx}{dt} = A_0 \omega_0 e^{-\gamma t/2} \sin \omega_0 t \tag{41}$$

Then

$$E = \frac{1}{2} A_0^2 e^{-\gamma t} [m\omega_0^2 \sin^2 \omega_0 t + k \cos^2 \omega_0 t] \tag{42}$$

Substituting for $\omega_0^2 = k/m$, we obtain

$$E = \frac{1}{2}kA_0^2 e^{-\gamma t} = E_0 e^{-\gamma t} \tag{43}$$

where E_0 is the total energy of the oscillator at $t = 0$. The energy of the oscillator decays exponentially with time as shown in figure 14

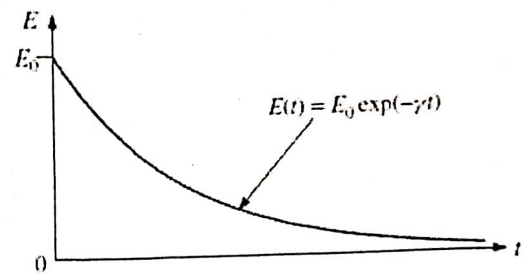

Figure 14: The exponential Decay of the energy for a very lightly damped oscillator

Forced oscillations and resonance

So far we have considered free oscillations where a system is disturbed from rest and then oscillates about its equilibrium position with steadily decreasing amplitude, as when we strike a bell. We now turn our attention to forced oscillations where we apply a periodic driving force to the system. We are surrounded by examples of such forced oscillations. We give a push to a playground swing at regular intervals to sustain its motion. We can observe the main physical characteristics of forced harmonic motion using a simple pendulum. We drive the pendulum by moving its point of suspension backwards and forwards harmonically, along a horizontal direction. At very low driving frequencies the pendulum mass closely follows the movement of the point of suspension with them both moving in the same direction as each other, i.e. they have the same amplitude and move in phase. As the driving frequency is increased the amplitude of oscillation increases dramatically and becomes much larger than the movement of the point of suspension. We might rightly suspect that the maximum amplitude occurs when the pendulum is driven close to its natural frequency of oscillation. The system is then said to be in resonance. We get the largest amplitude at resonance because this is the frequency at which the pendulum 'wants' to oscillate. As the driving frequency is increased further the amplitude of oscillation decreases but perhaps more surprisingly the mass now moves in the opposite direction to the point of suspension, although still with the same frequency. At even higher frequencies we reach the situation where the pendulum mass hardly moves at all. This is because it has inertia. The simple pendulum serves as a useful example, but all forced oscillators behave in this manner.

Problems

Consider the following expressions :

$(i) x = a \sin \omega t$ $\quad (ii) x = A\cos(\omega t - \pi/2)$ $\quad (iii) x = A\cos(\omega t - \pi)$

$(iv) x = A\sin(\omega t + \pi/2)$ $\quad (v) x = A\sin(\omega t - 3\pi/2)$ $\quad (vii) x = A\cos(\omega t + 3\pi/2)$

(a) Show that, above expressions are the displacements of SHM.

(b) Draw the graphs of x versus t, v versus t, and a versus t in $-2\pi \leq t \leq 2\pi$.

(c) Find the displacement, velocity , and acceleration at time $t = 5sec$.