Newton's Law of Gravitation:

Every particle in the universe attracts every other particle with a force proportional to the product of their masses and inversely proportional to the square of the distance between them. The direction of the force is along the line joining the particles.

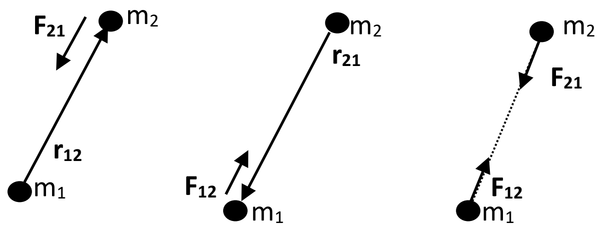
Thus the magnitude of the gravitational force F that two particles of masses m_1 and m_2 separated by a distance r exert on each other is

$$F = G\frac{m_1 m_2}{r^2}$$

G is called the gravitational constant

G= 6.673 x 10^{-11} Nm^2/kg^2

The gravitational forces between two particles are an action reaction pair. The first particle exerts a force on the second particle that is directed toward the first particle along the line joining them. The second particle exerts a force on the first particle that is directed toward the second particle along the line joining them.



The force $\mathbf{F_{21}}$ exerted on m_2 by m1 is directed opposite to the displacement $\mathbf{r_{12}}$ of m_2 from m_1. Similarly, $\mathbf{F_{12}}$ is directed opposite to $\mathbf{r_{21}}$. $\mathbf{F_{12}}$=-$\mathbf{F_{21}}$ action reaction pair. The law of gravitation is a vector law. The gravitational force $\mathbf{F_{21}}$ exerted on m_2 by m_1 is given in direction and magnitude by the vector relation

$$\vec{F}_{21} = -G\frac{m_1 m_2}{r_{12}^3}\vec{r}_{12}$$

Similarly,

$$\vec{F}_{12} = -G\frac{m_1 m_2}{r_{21}^3}\vec{r}_{21}$$

The minus sign of the equation tells that $\mathbf{F_{21}}$ points in a direction opposite to $\mathbf{r_{12}}$

Gravitational Potential Energy

From work-energy theorem we know that,

W=ΔK

Which states that the work done, W is equal to the change in kinetic energy. Conservation of energy tells us

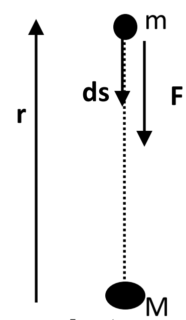
ΔK+ΔU=0

In which ΔU is the change in potential energy.

Conservation of energy holds for conservative force which is defined as the work done by this force does not depend on the path but only on the initial and final position.

$$\Delta U = -W \rightarrow U_a - U_b = -W_{ab} = -\int_a^b \vec{F}.\vec{ds}$$

Where a denotes the initial point and b denotes the final point. In the figure a particle M exerts a gravitational force F on a particle of mass m located at **r**. The particle of mass m is displaced a small distance **d**, which is in opposite direction of r. so **ds**= –**dr.**



The work done by F when the particle moves from a to b is

$$W_{ab} = \int_a^b \vec{F}.\vec{ds} = -\int_a^b F\, dr = \int_{r_a}^{r_b} \frac{GMm}{r^2}$$

We choose our reference configuration to be at an infinite separation of particle ($r_a \rightarrow \infty$) and we define U (∞) to be zero. At an arbitrary separation r, the potential energy is

$$U(r) = -W_{ab} + 0 = -G\frac{Mm}{r}$$

The minus indicates that the potential energy is negative at any finite distance, that is the potential energy is zero at infinity and decreases as the separation distance decreases. This corresponds to the fact that the gravitational force exerted on m by M is attractive.
We can reverse the previous calculation and derive the gravitational force from the potential energy. For spherically symmetric potential energy functions , the relation $F = -\frac{dU}{dr}$, gives the radial component of the force with the potential energy, $U = -G\frac{Mm}{r}$

The gravitational field and the gravitational potential

A basic fact of gravitation is that two particles exert forces on each other even though they are not contact, this point of view is called action-at-a-distance. Another point of view is called concept which regards a particle as modifying the space around it in some way and setting up a gravitational field. This field, the strength of which depends on the mass of the particle, then acts on any other particle, exerting the force of gravitational attraction on it.

We define the gravitational field strength at a point as the gravitational force per unit mass at that point, or, in terms of our mass

$$\vec{g} = \frac{\vec{F}}{m_o}$$

By moving the test mass to various positions, we can make a map showing the gravitational field at any point in space. The gravitational field is an example of vector field, each point in this field having a vector associated with it. The gravitational field arising from a fixed distribution of matter is an example of a static field, because the value of the field at a given point does not change with time.

We can also describe the gravitational field of a body by a scaler function called potential. Let us move a test particle from infinity (where the field is zero) toward a body of mass M at a distance r, where potential energy is U(r). We define the gravitational potential V at that point as

$V(r)= U(r)/m_o$

The potential is the same as the potential energy per unit test mass. Again $U(r)= -GMm_o/r$

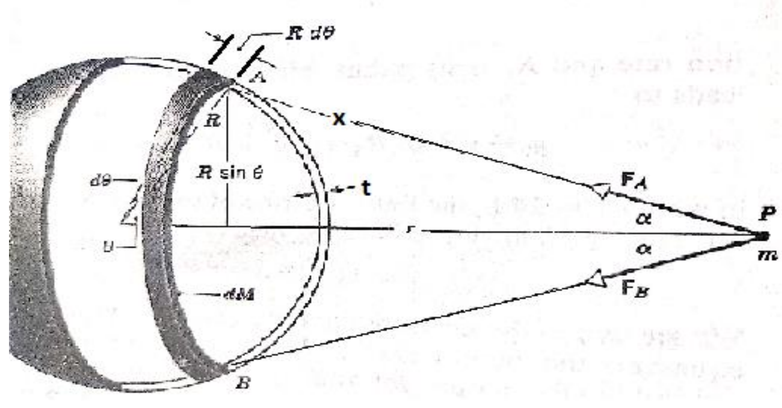
The gravitational potential can be written as $V(r)= -GM/r$. The potential V(r) is independent of the value of the test mass m_o. Similarly, the gravitational field g is also independent of m_o.

Gravitational effect of a spherical distribution of mass

Let us consider a uniformly dense spherical shell of mass "M" and whose thickness "t" is small compared to its radius "R". we will find the gravitational force it exerts on an external particle P of mass "m". A small part of the shell at A attracts m with a force $\mathbf{F_A}$ and a small part of the shell at b, equally far from m but diametrically opposite A, attracts m with a force $\mathbf{F_B}$.

The resultant of these two forces on m is $\mathbf{F_A} + \mathbf{F_B}$. Each of these forces has a component $F\cos\alpha$ along the symmetry axis and a component $F\sin\alpha$ perpendicular to the axis. The perpendicular component of the Forces cancels each other. Let us take as our element of mass of the shell a circular strip dM. Its radius is $R\sin\theta$, its length is $2\pi(R\sin\theta)$, its width is $Rd\theta$., its thichness is t. Hence its volume is:

$$dV=2\pi tR^2\sin\theta d\theta \text{.........}(1)$$



If the density of the strip is ρ then the mass within the strip is **:**

$$dM= \rho dV=2\pi t\rho R^2 \sin\theta d\theta \ldots\ldots\ldots(2)$$

Every particle of the ring, such as one of the mass dmA at A, attracts P with a force that has an axial component

$$dF_A = G\frac{m dm_A}{x^2}\cos\alpha \ldots\ldots(3)$$

Adding the contributions for all the particles in the ring gives

$$dF_A + dF_B + \cdots\cdots = G\frac{m}{x^2}(\cos\alpha)(dmA + dmB + \cdots\cdots)$$

or,

$$dF = G\frac{m}{x^2}dM(\cos\alpha) \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Where, dM is the total mass of the ring and dF is the total force on m exerted by the ring.
putting the value of dM in eq^n(4) we get:

$$dF = 2\pi t\rho m R2G\frac{\sin\theta d\theta}{x^2}(\cos\alpha) \ldots\ldots(5)$$

The variables x,α and θ are related as

$r=x\cos\alpha + R\cos\theta$

$$\cos\alpha = \frac{r-R\cos\theta}{x}\ldots\ldots\ldots(6)$$

using the law of cosine, $x^2=r^2+R^2-2rR\cos\theta$

now, $$R\cos\theta = \frac{r^2+R^2-x^2}{2r}\ldots\ldots\ldots\ldots(7)$$

putting eq^n(7) & (8) in eq^n (5) we get-

$$dF = 2\pi t\rho m R^2 G\frac{xdx}{Rrx^2}\frac{r^2-R^2+x^2}{2rx}$$

or, $$dF = \frac{\pi t\rho m RG}{r^2}\left(\frac{r^2-R^2}{x^2}+1\right)dx$$

This is the force exerted by the circular strip dM on the particle P of mass m.
Now we consider every element of mass in the shell by summing over all the circular strips in the entire shell.
The needed integral is

$$\int_{r-R}^{r+R}\left(\frac{r^2-R^2}{x^2}+1\right)dx = \left[\frac{-(r^2-R^2)}{x}+x\right]_{r-R}^{r+R} = 4R$$

so, $$F = \int_{r-R}^{r+R} dF = \frac{\pi t\rho m RG}{r^2}4R$$

$$F = G\frac{Mm}{r^2}, \qquad \text{where } M= 4\pi R^2 t\rho$$

A uniformly dense spherical shell attracts an external point mass as if all the mass of the shell were concentrated at its center .

The force exerted by a spherical shell on a particle inside it is zero. In that case the limit of needed integration over x is R-r to r+R, gives

$$\int_{R-r}^{r+R}\left(\frac{r^2-R^2}{x^2}+1\right)dx=\left[\frac{-(r^2-R^2)}{x}+x\right]_{R-r}^{r+R}=0,$$

So, F=0

A uniform spherical shell of matter exerts no gravitational force on a particle located inside it.

Gravitation near the earth surface

Let us assume the earth is spherical and that its density depends only on the radial distance from its center, the magnitudes of the gravitational force acting on aparticle of mass m located at an external point distance r from the earth's center, can be written as

$$F = G\frac{M_E m}{r^2}$$

M_E is the mass of the Earth. This gravitational force can also be written using Newton's second law of motion as

$$F=mg$$

Here, g is the free-fall acceleration due to the gravitational pull of the Earth. Combining the two equations above gives

$$g = G\frac{M_E}{r^2},$$

In the case of earth surface, this can be written as

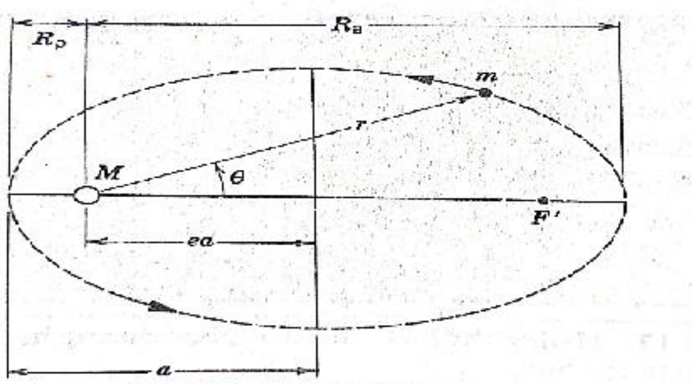
$$g_s = G\frac{M_E}{R_E^2},$$

In which g_s and R_E are the acceleration due to gravity at the earth surface and the radius of the earth respectively.

Kepler's laws of motions of motions of planets and satellites

The empirical basis for understanding the motions of the planets is Kepler's laws, these are:

1. The laws of orbits: All plants move in elliptical orbits having the Sun at one focus.



Newton was the first to realize that there is a direct mathematical relationship between inverse-square (1/r2) forces and elliptical orbit. The origin of coordinates is at the central body, the orbiting body is located at polar co-ordinates r and θ. The orbit is described by two parameters: semi major axis **a** and the eccentricity e. the distance from the center of the ellipse to the either focus is **ea**. A circular orbit is special case of an elliptical orbit with **e=0**, in which case the two foci merge to a single point at the center of the circle. For all planets in solar system, the eccentricities are small and the orbits are nearly circular.

2. The law of Areas: A line joining any planet to the sun Sweeps out equal areas in equal times.

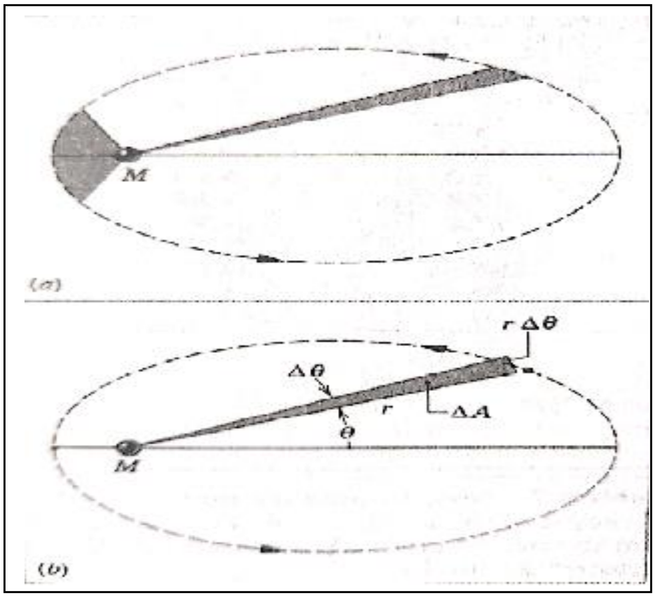


Figure 'a' illustrates this law; in effect it says that the orbiting body moves more rapidly when it is closed to the central body than it does when it is far away.

Consider a small area increment ΔA covered in a time interval Δt, as shown in figure 'b'. The area of this approximately triangular wedge is one-half its base, r Δθ, times its height r. The rate at which this area is swept out is ΔA/Δt= ½ (r Δθ) (r)/Δt. In the instantaneous limit this becomes

$$\frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{2} r^2 \left(\frac{\Delta \theta}{\Delta t}\right) = \frac{1}{2} r^2 \omega$$

The instantaneous angular momentum of the orbiting body is $L=mr^2\omega$ and so

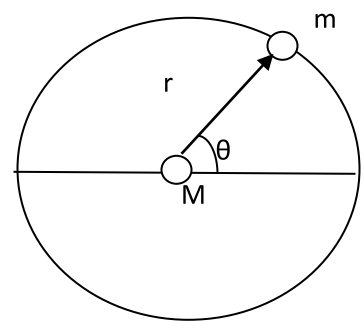
$$\frac{dA}{dt} = \frac{L}{2m}$$

To the extent that we can regard the two bodies as an isolated system, L is constant and therefore dA/dt is a constant. The speeding up of a comet as it passes close to the sun is therefore just a demonstration of the conservation of the angular momentum.

3. The Law of Periods: The square of the period of any planet about the Sun is proportional to the cube of the planet's mean distance from the Sun.

Let us prove this result for circular orbits. The gravitational force provides the necessary centripetal force for circular motion:

$$\frac{GMm}{r^2} = m\omega^2 r$$



Replacing ω by 2π/T, we obtain

$$T^2 = \left(\frac{4\pi^2}{GM}\right) r^3$$

A similar result is obtained for elliptical orbits, with the radius r replaced by the semi major axis a.

Problems

1. Two objects of masses 50kg and 500gm are situated at a distance 50cm away from each other. Find (i) the magnitude of gravitational force acting between them, (ii) Acceleration of lighter one due to the massive one (iii) Acceleration of massive one due to the lighter one, (iv) which objects actually will move to the other one?
2. Suppose you are a sphere of radius 30cm and of mass 70kg. Calculate the acceleration experienced by your friend standing 2m away from your center. (Assume that there is no other force acting on both of you)
3. Determine the mass of the Earth from the period and radius of the moon's orbit about the earth. Period is 27.3 days and radius is 3.82 x 10^5km
4. A satellite orbits at a height of h=230km above the Earth's surface. What is the period of the satellite?
5. Calculate the minimum initial speed must a projectile have at the Earth's surface to escape from the earth?