Chain Rule Assignment

1. Given,

$$f(z) = \log_e(1+z) \text{ where } z = x^T x,\ x \in \mathbb{R}^d$$

If, $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix}$, then $x^T = \begin{bmatrix} x_1 & x_2 & \dots & x_d \end{bmatrix}$

$$x^T x = \left[x_1^2 + x_2^2 + \dots + x_d^2\right]$$

Applying chain rule,

$$\frac{d}{dx}(f) = \frac{d}{dz}(f) \cdot \frac{d}{dx}(z)$$

$$= \frac{d}{dz}\left(\log_e(1+z)\right) \cdot \frac{d}{dx}(x^T x)$$

$$= \frac{1}{1+z} \cdot \frac{d}{dz}(z) \cdot \frac{d}{dx}(x_1^2 + x_2^2 + \dots + x_d^2)$$

$$= \frac{1}{1+z}(2x_1 + 2x_2 + \dots + 2x_d)$$

$$= \frac{1}{1+z} \cdot 2 \cdot (x_1 + x_2 + \dots + x_d)$$

$$= \frac{2}{1+z} \sum_{i=1}^{d} x_i$$

Ans:

2. $f(z) = e^{-z/2}$; where $z = g(y)$, $g(y) = y^T S^{-1} y$, $y = h(x)$, $h(x) = x - \mu$

$\Rightarrow$ Using chain rule,

$$\frac{d}{dz}(f) = \frac{d}{dz}(f) \cdot \frac{d}{dy}(z) \cdot \frac{d}{dx}(y)$$

here,

$$\frac{d}{dz}(f) = \frac{d}{dz}\left(e^{-z/2}\right) = -\frac{e^{-z/2}}{2}$$

$$\frac{d}{dy}(z) = \frac{d}{dy}\left(y^T S^{-1} y\right)$$

$$= \lim_{h \to 0} \frac{g(y+h) - y(y)}{h}$$

$$= \lim_{h \to 0} \frac{(y^T + h) S^{-1} (y+h) - y^T S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{(y^T S^{-1} + h S^{-1})(y+h) - y^T S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{y^T S^{-1} y} + y^T S^{-1} h + h S^{-1} y + h^2 S^{-1} - \cancel{y^T S^{-1} y}}{h}$$

$$= \lim_{h \to 0} \left(\frac{\cancel{h}(y^T S^{-1} + S^{-1} y + h S^{-1})}{\cancel{h}} \right)$$

$$= \lim_{h \to 0} \left(y^T S^{-1} + S^{-1} y + h S^{-1}\right)$$

$$= y^T S^{-1} + S^{-1} y$$

$$\frac{dy}{dx} = \frac{d(x-M)}{dx}$$

$$= 1$$

$$\therefore \frac{d}{dx}(f) = \frac{d}{dz}(f) \cdot \frac{d}{dy}(z) \cdot \frac{d}{dx}(y)$$

$$= -\frac{e^{-z/2}}{2} \cdot \left(y^T S^{-1} + S^{-1} y\right) \cdot 1$$

$$= -\frac{e^{-z/2}}{2} \cdot \frac{1}{S}\left(y^T + y\right)$$

A.