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MAT - 107W

part: B

Ans: to the que: NO - 3 (a)

Given that,

$$Q = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

let, λ be an eigenvalue of Q.

Here,

$$Q - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{bmatrix}$$

$$|Q - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{vmatrix}$$

$$= (1-\lambda)\{(2-\lambda)(3-\lambda)-1\} + (3-\lambda) - 2 = (1-\lambda)(2-\lambda)(3-\lambda) + 0$$

Here,

$$|Q - \lambda I| = 0 \Rightarrow (1-\lambda)(2-\lambda)(3-\lambda) = 0$$

Thus, $\lambda = 1$ or $\lambda = 2$ or $\lambda = 3$

Therefore,

Spectrum of Q is $\{1, 2, 3\}$

Let $v_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigenvector associated to the eigenvalue $\lambda_1 = 1$.

Then, $(Q - \lambda_1 I) v_1 = 0$

$$\Rightarrow \begin{bmatrix} 1-1 & 1 & 2 \\ -1 & 2-1 & 1 \\ 0 & 1 & 3-1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 2 \\ -1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \left. \begin{aligned} y + 2z &= 0 \\ -x + y + z &= 0 \\ y + 2z &= 0 \end{aligned} \right\}$$

$\Rightarrow -x+y+z=0$

$y+2z=0$

Let, z is a free variable and assign $z=a$. then $y=-2a$ and $x=-a$.

Therefore, $V_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -a \\ -2a \\ a \end{bmatrix}$. where a is a nonzero real number.

Let, $V_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigenvector associated to the eigen value $\lambda_2 = 2$

Then,

$(G-\lambda_2 I)V_2 = 0$

$$\Rightarrow \begin{bmatrix} 1-2 & 1 & 2 \\ -1 & 2-2 & 1 \\ 0 & 1 & 3-2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \left.\begin{aligned} -x+y+2z &= 0 \\ -x+z &= 0 \\ y+z &= 0 \end{aligned}\right\}$$

$$\Rightarrow \left.\begin{array}{l} -x+y+2z=0 \\ -y+z=0 \\ y+z=0 \end{array}\right\}$$

$$\Rightarrow \left.\begin{array}{l} -x+y+2z=0 \\ -y-z=0 \end{array}\right\}$$

Let, z be the free variable and assign $z=b$, then

$y=-b$ and $x=b$.

Therefore, $V_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b \\ -b \\ b \end{bmatrix}$, where b is a nonzero real

number.

Let $V_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigenvector associated to the

eigenvalue $\lambda_3 = 3$.

Then $(Q-\lambda_3 I)V_3 = 0$

$$\Rightarrow \begin{bmatrix} 1-3 & 1 & 2 \\ -1 & 2-3 & 1 \\ 0 & 1 & 3-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -2 & 1 & 2 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\Rightarrow$ $\left.\begin{aligned} -2x+y+2z&=0 \\ -x-y+z&=0 \\ y&=0 \end{aligned}\right\}$

$\Rightarrow$ $\left.\begin{aligned} -2x+y+2z&=0 \\ -3y&=0 \\ y&=0 \end{aligned}\right\}$

$\Rightarrow$ $\left.\begin{aligned} -2x+y+2z&=0 \\ y&=0 \end{aligned}\right\}$

let, z is the free variable and assign $z=c$, Then $x=c$.

Therefore, $V_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c \\ 0 \\ c \end{bmatrix}$, where c is a nonzero real number.

for $a=b=c=1$, we have $V_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$, $V_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $V_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Ans: to the que: NO-3 (b)

Let,

$S = \{S_1, S_2, S_3\}$

and $T = \{T_1, T_2, T_3, T_4\}$

Given, $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \\ z \\ x \end{bmatrix}$

$\therefore L(S_1) = L\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

$L(S_2) = L\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

$L(S_3) = L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

Now, $L(S_1)$ is some linear combination of T.

$\therefore L(S_1) = aT_1 + bT_2 + cT_3 + dT_4$

$$= \begin{bmatrix} a+b+c+d \\ a+c \\ a+b+d \\ a+b+c \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ $a+b+c+d = 2 \ldots\ldots$ (i)

$a+c = 0 \ldots\ldots\ldots\ldots$ (ii)

$a+b+d = 1 \ldots\ldots\ldots$ (iii)

$a+b+c = 1 \ldots\ldots\ldots$ (iv)

(iv) − (iii) $\Rightarrow b = 1$

(i) − (iii) $\Rightarrow c = 1$

(iv) $\Rightarrow a+1+1 = 1 \Rightarrow a = -1$

(iii) $d = 1-a-b = 1-1+1 = 1$

$\therefore (a,b,c,d) = (-1,1,1,1)$

$\therefore L(S_1) = -T_1 + T_2 + T_3 + T_4$

again, $L(S_2)$ is some linear combination of T.

$L(S_2) = aT_1 + bT_2 + cT_3 + dT_4$

$$= \begin{bmatrix} a+b+c+d \\ a+c \\ a+b+d \\ a+b+c \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore a+b+c+d = 1 \ldots\ldots\ldots$ (i)

$a+c = -1 \ldots\ldots\ldots$ (ii)

$a+b+d = 1 \ldots\ldots\ldots$ (iii)

$a+b+c = 0 \ldots\ldots\ldots$ (iv)

(iv) $-$ (ii) $\Rightarrow b = 0-(-1) = 1$

(i) $-$ (iii) $\Rightarrow c = 0$

(iv) $\Rightarrow a+1+0 = 0 \Rightarrow a = -1$

(iii) $\Rightarrow -1+1+d = 1 \Rightarrow d = 1$

$\therefore (a,b,c,d) = (-1,1,0,1)$

$\therefore L(S_2) = -T_1 + T_2 + T_4$

again, $L(S_3)$ is some linear combination of T

$\therefore L(S_3) = aT_1 + bT_2 + cT_3 + dT_4$

So, $\begin{bmatrix} a+b+c+d \\ a+c \\ a+b+d \\ a+b+c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$\therefore a+b+c+d=0 \ldots\ldots$ (i)

$a+c=0 \ldots\ldots\ldots\ldots$ (ii)

$a+b+d=1 \ldots\ldots\ldots\ldots$ (iii)

$a+b+c=0 \ldots\ldots\ldots$ (iv)

(iv) - (ii) $\Rightarrow b=0$

(i) - (iii) $\Rightarrow c=-1$

(iv) $\Rightarrow a+0-1=0 \quad \Rightarrow a=1$

(iii) $\Rightarrow 1+0+d=1 \quad \Rightarrow d=0$

$\therefore (a,b,c,d)=(1,0,-1,0)$

$\therefore L(S_3)=T_1-T_3$

$\therefore \left[L(S_1)\right]_T = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$\left[L(S_2)\right]_T = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

$\left[L(S_3)\right]_T = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$

$$\therefore [L]_{S.T} = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

And, this is the transformation matrix.

Ans: to the que: NO - 4 (a)

Order of an element:

For $x \in G$, order of x is the least integer n such that $x^n = e$

Second part:

Let, G be a group and $a \in G$.

Let, $a^n = e$

Then,

$$e = (aa^{-1})^n = a^n (a^{-1})^n$$

$$= e(a^{-1})^n$$

$= (a^{-1})^n$

Again,

let, $(a^{-1})^n = e$

Then,

$e = (aa^{-1})^n = a^n (a^{-1})^n = a^n e$

$= a^n$

So,

$a^n = e \Longleftrightarrow (a^{-1})^n = e$

$\therefore$ The order of an element and that of its inverse are same in a group.

Ans: to the que: NO - 4(b)

Here,

$Z_6 = \{0,1,2,3,4,5\}$

Generators of Z_6 under addition are given below in the table -

(+)	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4