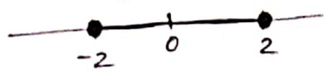
MSR

Find domain and range of the following functions.

(i) $f(x) = \sqrt{4-x^2}$

Solution: Here the value of $f(x)$ will be real if

$$4-x^2 \geq 0$$



$$\Rightarrow -x^2 \geq -4$$

$$\Rightarrow x^2 \leq 4$$

$$\Rightarrow -2 \leq x \leq 2$$

$$\therefore D_f = \{x ; -2 \leq x \leq 2\} = [-2, 2]$$

Again, $y = f(x) = \sqrt{4-x^2} \longrightarrow (1)$

In (1) the value of y can not be negative, that is, the value of y will be Positive or zero.

$$(1) \Rightarrow y^2 = 4-x^2 \text{ when } y \not< 0$$

$$\Rightarrow x^2 = 4-y^2, \quad y \not< 0$$

$$\Rightarrow x = \pm\sqrt{4-y^2}, \quad y \not< 0$$

Here the value of x will be exist if $4-y^2 \geq 0$ and $y \not< 0$

$$\Rightarrow y^2-4 \leq 0 \text{ \& } y \not< 0$$

$$\Rightarrow y^2 \leq 4 \text{ \& } y \not< 0$$

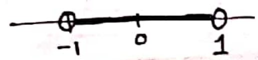
$$\Rightarrow -2 \leq y \leq 2 \text{ \& } y \not< 0$$

$$\Rightarrow 0 \leq y \leq 2$$

$$\therefore R_f = \{y : 0 \leq y \leq 2\} = [0, 2]$$

(ii) $f(x) = \ln\left(\frac{1+x}{1-x}\right)$

Solution: Here the value of $f(x)$ will be real if



$$\frac{1+x}{1-x} > 0$$

$$\Rightarrow \frac{x+1}{x-1} < 0$$

$$\Rightarrow -1 < x < 1$$

$$\therefore D_f = \{x : -1 < x < 1\}$$

$$= (-1, 1)$$

Again, $y=f(x)=\ln\left(\frac{1+x}{1-x}\right)$

$$\Rightarrow \ln\left(\frac{1+x}{1-x}\right)=y$$

$$\Rightarrow \frac{1+x}{1-x}=e^y$$

$$\Rightarrow 1+x=e^y-xe^y$$

$$\Rightarrow x=\frac{e^y-1}{e^y+1}$$

Here the value of y is real for all real values of x.

$\therefore R_f=\mathbb{R}$

③ $f(x)=\frac{x-3}{2x+1}$

Solution: Given that, $f(x)=\frac{x-3}{2x+1}$

$f(x)$ gives real values for all real values of x except $2x+1=0$ or $x=-\frac{1}{2}$

$\therefore D_f=\mathbb{R}-\{-\frac{1}{2}\}$

Again, $y=f(x)=\frac{x-3}{2x+1}$

$$\Rightarrow 2xy+y=x-3$$

$$\Rightarrow x(2y-1)=-(y+3)$$

$$\therefore x=-\frac{y+3}{2y-1}$$

x gives real values for all real values of y except $y=\frac{1}{2}$

$\therefore R_f=\mathbb{R}-\{\frac{1}{2}\}$

④ $f(x)=\frac{x-3}{x^2-9}$

Solution: Given that, $f(x)=\frac{x-3}{x^2-9}$

Here $f(x)$ gives real values for all real values of x except $x^2-9=0$ or $x=\pm 3$

$\therefore D_f=\mathbb{R}-\{-3,3\}$

Again, $y=f(x)=\frac{x-3}{x^2-9}$

$\Rightarrow y=\frac{1}{x+3}$ when $x\neq 3$

$\Rightarrow x+3 = \frac{1}{y}$ when $x \neq 3$ or $y \neq \frac{1}{6}$

$\therefore x = \frac{1}{y} - 3, \; y \neq \frac{1}{6}$

Here x is defined for all real values of y except $y = \frac{1}{6}$ and 0

$\therefore R_f = \mathbb{R} - \{0, \frac{1}{6}\}$

(5) $f(x) = \frac{|x|}{x}$

Solution: Here, $f(x) = \frac{|x|}{x}$

obviously, $f(x)$ is defined for all real values of x, except $x=0$.

Hence, the domain of $f(x)$ is $-\infty < x < \infty$, except $x=0$

Again, $\because |x| = x$, when $x > 0$

$= -x$, when $x < 0$,

$\frac{|x|}{x} = 1$, when $x > 0$ and

$\frac{|x|}{x} = -1$ when $x < 0$

So that range of $f(x)$ is $[-1, 1]$.

Improper Integral. (Definition)

* Show that $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}$

Solution: From definition.

$$\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \lim_{\epsilon \to \infty} \int_0^\epsilon \frac{dx}{(x^2+a^2)(x^2+b^2)}$$

$$= \frac{1}{a^2-b^2} \lim_{\epsilon \to \infty} \int_0^\epsilon \left(\frac{1}{x^2+b^2} - \frac{1}{x^2+a^2} \right) dx$$

$$= \frac{1}{a^2-b^2} \lim_{\epsilon \to \infty} \left[\frac{1}{b} \tan^{-1}\frac{x}{b} - \frac{1}{a} \tan^{-1}\frac{x}{a} \right]_0^\epsilon$$

$$= \frac{1}{a^2-b^2} \lim_{\epsilon \to \infty} \left(\frac{1}{b} \tan^{-1}\frac{\epsilon}{b} - \tan^{-1}\frac{\epsilon}{a} - 0 \right)$$

$$= \frac{1}{a^2-b^2} \left(\frac{1}{b} \tan^{-1}\infty - \frac{1}{a} \tan^{-1}\infty \right)$$

$$= \frac{1}{a^2-b^2} \left(\frac{\pi}{2b} - \frac{\pi}{2a} \right) = \frac{1}{a^2-b^2} \frac{\pi(a-b)}{2ab}$$

$$= \frac{\pi}{2ab(a+b)} \quad \text{(showed)}$$

H.W * Show that $\displaystyle\int_0^\infty \frac{x\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{a^2-b^2} \ln\frac{b}{a}$ when $a, b > 0$

Soln: ~~For~~ By definition, $\displaystyle\int_0^\infty \frac{x\,dx}{(x^2+a^2)(x^2+b^2)} = \lim_{\epsilon\to\infty} \int_0^\epsilon \frac{x\,dx}{(x^2+a^2)(x^2+b^2)}$

Putting, $x^2 = z$

$\Rightarrow 2x\,dx = dz$

$\Rightarrow x\,dx = \frac{1}{2}\,dz$

when $x=\epsilon$, $z=\epsilon^2$

" $x=0$, $z=0$

$$= \lim_{\epsilon\to\infty} \frac{1}{2} \int_0^{\epsilon^2} \frac{dz}{(z+a^2)(z+b^2)}$$

$$= \frac{1}{2(a^2-b^2)} \lim_{\epsilon\to\infty} \left\{ \int_0^{\epsilon^2} \left(\frac{1}{z+b^2} - \frac{1}{z+a^2} \right) dz \right\}$$

$$= \frac{1}{2(a^2-b^2)} \lim_{\epsilon\to\infty} \left[\ln(z+b^2) - \ln(z+a^2) \right]_0^{\epsilon^2}$$

$$= \frac{1}{2(a^2-b^2)} \lim_{\epsilon\to\infty} \left[\ln\frac{z+b^2}{z+a^2} \right]_0^{\epsilon^2}$$

$$= \frac{1}{2(a^2-b^2)} \lim_{\epsilon\to\infty} \left\{ \left(\ln\frac{\epsilon^2+b^2}{\epsilon^2+a^2} - \ln\frac{b^2}{a^2} \right) \right\}$$

$$= \frac{1}{2(a^2-b^2)} \left[\lim_{\epsilon\to\infty} \ln\frac{1+(b^2/\epsilon^2)}{1+(a^2/\epsilon^2)} + \ln\left(\frac{a}{b}\right)^2 \right]$$

$$= \frac{1}{2(a^2-b^2)} \left[\ln 1 + 2\ln\frac{a}{b} \right]$$

$$= \frac{1}{2(a^2-b^2)} \left(0 + 2\ln\frac{a}{b} \right)$$

$$= \frac{1}{a^2-b^2} \ln\frac{a}{b}$$

H.W: Show that the function $f(x) = \frac{1}{4}x^3 + 1$ satisfies the hypothesis of the Mean-value Theorem over the interval $[0,2]$ and find all values of c in the interval $(0,2)$ at which the tangent line to the graph of f is parallel to the secant line joining the points $(0, f(0))$ and $(2, f(2))$.