if-then

(iii) Deductive proof:

A deductive proof consists of a sequence of statements whose truth leads us from initial statement (hypothesis or given statement) to a conclusion statement.

* Each step of proof follows –
- accepted logical principle
- previous statements in the deductive proof
- given facts

Example (1.3) If $x \geq 4$, then $2^x \geq x^2$.

Solution: The hypothesis H is "$x \geq 4$" which has a parameter x, and it's neither true nor false. Rather it's truth depends on the value of x. For example, H is true for $x = 6$ and false for $x = 2$.

Steps: 1) check x value for hypothesis
2) " x " " conclusion

Likewise, the conclusion ~~is~~ C is $2^x \leq x^2$. This statement also uses parameter x not others. For example, C is false for $x=3$, since $2^3=8$ which is less than $3^2=9$.

On the other hand, C is true for $x=4$ as $2^4=16$ and $4^2=16$.

For $x=5$, the statement is also true, since $2^5=32$ is greater than $5^2=25$. ~~The intuitive~~

Thus we can say that $2^x \leq x^2$ will be true whenever $x \geq 4$.

Example 1.4) If x is the sum of the squares of four positive integers, than $2^x \geqslant x^2$

Solution: Let a, b, c and d be the four positive integers.

So, $x = a^2 + b^2 + c^2 + d^2$ - - - (1)

and $a \geqslant 1$, $b \geqslant 1$, $c \geqslant 1$, $d \geqslant 1$ - - - (2)

* From (2) and properties of arithmetic; we get

$a^2 \geqslant 1$, $b^2 \geqslant 1$, $c^2 \geqslant 1$, $d^2 \geqslant 1$ - - (3)

$\{$If $y \geqslant 1$, $y^2 \geqslant 1\}$

* From (1), (3) and properties of arithmetic,

we get $x \geqslant 4$ - - - (4)

x is at least $1+1+1+1 = 4$

* From (4) and theorem 1.3, we get

$2^x \geqslant x^2$ - - - (5)

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* Contrapositive: A statement and its contrapositive are
  are equivalent
  - either both true
  - or both false

So we can prove either to prove the other.

<u>Ex:</u> If H then C $\approx$ if not C then not H
contrapositive

* <u>Converse:</u> if H then C
  not equivalent
  $\downarrow$ (converse)
  if C then H

* Proof by contradiction: completing the proof showing that something known to be false, starting by assuming the hypothesis true and the conclusion false.

Ex: Prove that $\sqrt{5}$ is an irrational number.

$\Rightarrow$ Let $\sqrt{5}$ be a rational number.

So, $\sqrt{5} = \frac{p}{q}$ $[p, q \in \mathbb{Z}, q \neq 0]$

and p and q are coprime numbers

$q > 1$

$\Rightarrow 5 = \frac{p^2}{q^2}$ [squaring both sides]

Multiplying both sides by q, we get,

$\Rightarrow 5q = \frac{p^2}{q}$

Here, 5q clearly is an integer, but $\frac{p^2}{q}$ is not as p and q are coprimes and $q > 1$.

So, $5q \neq \frac{p^2}{q}$

$\therefore \sqrt{5} \neq \frac{p}{q}$

Therefore, $\sqrt{5}$ is an irrational number.

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↗ Disproof

Proofs by counterexamples:

shows that a given statement can't possibly be correct by showing an instance that contradicts a universal statement.

Theorem 1.13: All primes are odd.

→ The integer 2 is a prime, but is even.

Theorem 1.14: There is no pair of integers a and b such that

$$a \bmod b = b \bmod a$$

⇒ (1) If $a > b$ then

$a \bmod b = c$ is a unique integer between 0 and $b-1$

and $b \bmod a = b$

So, $b \bmod a > a \bmod b$

(2) If $a < b$, then

$b \bmod a = c'$ is a unique integer between 0 and $a-1$

and $a \bmod b = a$

$\therefore$ $b \bmod a < a \bmod b$

(3) if $a = b$ then

$a \bmod b = a \bmod a = 0$

$b \bmod a = b \bmod b = 0$

So, if $a \neq b$, then there is no pair of integers a and b such that $a \bmod b = b \bmod a$

Theorem 1.15: $a \bmod b = b \bmod a$ if and only if $a = b$

$\longrightarrow$ same as 1.14

Induction : used to prove a statement is true for all values of n.

2 steps:

Basis step: S(i) is true for i = 0 or 1

i → particular integer

Inductive step: if S(n) is true,

S(n+1) is also true

Example: for all $n \geq 0$:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

⇒ Base step: for $n = 0$,

$\sum_{i=1}^{0} i^2$ has no terms, so the sum is 0.

$$\therefore \sum_{i=1}^{0} i^2 = 0$$

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Inductive step: Let, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Again, $\sum_{i=1}^{n+1} i^2 = \frac{(n+1)(n+2)(2(n+1)+1)}{6}$

$$= \frac{(n^2+3n+2)(2n+3)}{6}$$

$$= \frac{1}{6}(2n^3+3n^2+6n^2+9n+4n+6)$$

$$= \frac{1}{6}(2n^3+9n^2+13n+6) \quad \cdots\cdots(1)$$

We can also write,

$$\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2$$

$$= \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$= \frac{1}{6}\{(n^2+n)(2n+1) + 6n^2+12n+6\}$$

$$= \frac{1}{6}\{2n^3+n^2+2n^2+n+6n^2+12n+6\}$$

$$= \frac{1}{6}(2n^3+9n^2+13n+6) \quad \cdots\cdots(2)$$

[Proved]

Example: If $x \geq 4$ then $2^x \geq x^2$

→ **Base step:** For $x = 4$, $2^x = 2^4 = 16$.

$x^2 = 4^2 = 16$

$\therefore 2^x = x^2$ —— (1)

Inductive step: For $x > 4$

Let $2^{x+1} \geq (x+1)^2$

$\rightarrow 2^x \cdot 2 \geq x^2 + 2x + 1$

$\rightarrow x^2 \cdot 2 \geq x^2 + 2x + 1$ [From (1)]

$\rightarrow 2x^2 \geq x^2 + 2x + 1$

$\rightarrow x^2 \geq 2x + 1$

$\rightarrow x \geq 2 + \frac{1}{x}$ [dividing both sides by x]

For $x \geq 4$, the maximum value of $\left(2 + \frac{1}{x}\right)$ is 2.25, so, L.H.S $\geq$ R.H.S

$\boxed{\therefore 2^x \geq x^2}$