Limit and Continuity

i) Show that the function $f(x) = x^2 + 2$ is continuous and differentiable at $x = 1$.

→ For continuity:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2 + 2 = 1^2 + 2 = 3$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 + 2 = 1^2 + 2 = 3$$

$$f(1) = 1^2 + 2 = 3$$

$$\therefore \lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = f(1)$$

Therefore, $f(x)$ is continuous at $x = 1$ (showed)

For differentiability:

$$Rf'(1) = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0^+} \frac{(1+h)^2 + \cancel{2} - 1^2 - \cancel{2}}{h}$$

$$= \lim_{h \to 0^+} \frac{h^2 + 2h}{h} = \lim_{h \to 0^+} \frac{\cancel{h}(h+2)}{\cancel{h}}$$

$$= \lim_{h \to 0^+} (h+2) = 0 + 2 = 2$$

and $Lf'(1) = \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h}$

$$= \lim_{h \to 0^-} \frac{(1+h)^2 + 2 - 1^2 - 2}{h} = \lim_{h \to 0^-} \frac{h^2 + 2h}{h}$$

$$= \lim_{h \to 0^-} (h+2) = 0 + 2 = 2$$

Here, $Rf'(1) = Lf'(1)$.

Therefore $f(x)$ is differentiable at $x = 1$

(showed)

2) If $f(x) = \begin{cases} x ; & 0 \leq x \leq \frac{1}{2} \\ 1-x ; & \frac{1}{2} \leq x \leq 1 \end{cases}$

Show that $f(x)$ is continuous at $x = \frac{1}{2}$ but $f(x)$ is not differentiable at that point

→ for continuity:

$$\lim_{x \to \frac{1}{2}^+} f(x) = \lim_{x \to \frac{1}{2}^+} (1-x) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\lim_{x \to \frac{1}{2}^-} f(x) = \lim_{x \to \frac{1}{2}^-} x = \frac{1}{2}$$

Again, $f\left(\frac{1}{2}\right) = \cancel{\lim_{x \to 1}}\ 1 - \frac{1}{2} = \frac{1}{2}$

$$\therefore \lim_{x \to \frac{1}{2}^+} f(x) = \lim_{x \to \frac{1}{2}^-} f(x) = f\left(\frac{1}{2}\right)$$

Therefore, $f(x)$ is continuous at $x = \frac{1}{2}$. (showed)

For differentiability:

$$Rf'\left(\tfrac{1}{2}\right) = \lim_{h \to 0^+} \frac{f\left(\frac{1}{2}+h\right) - f\left(\frac{1}{2}\right)}{h}$$

$$= \lim_{h \to 0^+} \frac{1 - \left(\frac{1}{2}+h\right) - \frac{1}{2}}{h}$$

$$= \lim_{h \to 0^+} \frac{-h}{h} = \lim_{h \to 0^+} (-1) = -1$$

$$Lf'\left(\tfrac{1}{2}\right) = \lim_{h \to 0^-} \frac{f\left(\frac{1}{2}+h\right) - f\left(\frac{1}{2}\right)}{h}$$

$$= \lim_{h \to 0^-} \frac{\left(\frac{1}{2}+h\right) - \frac{1}{2}}{h}$$

$$= \lim_{h \to 0^-} \frac{h}{h} = \lim_{h \to 0^-} 1 = 1$$

Here, $Rf'\left(\frac{1}{2}\right) \neq Lf'\left(\frac{1}{2}\right)$

therefore, $f(x)$ is not differentiable at $x = \frac{1}{2}$. (showed)

3) If $f(x) = \begin{cases} 1 & \text{when } x \leq 1 \\ x & \text{when } x > 1 \end{cases}$

test the continuity and differentiability at $x = 1$.

→ For continuity:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x) = 1$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (1) = 1$$

Again, $f(1) = 1$.

$$\therefore \lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = f(1)$$

Therefore, $f(x)$ is continuous at $x = 1$.

For differentiability:

$$Rf'(1) = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0^+} \frac{1+h-1}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$$

$$Lf'(1) = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0^+} \frac{1-1}{h} = \lim_{h \to 0^+} \frac{0}{h} = 0$$

Here, $Rf'(1) \neq Lf'(1)$

Therefore, $f(x)$ is not differentiable at $x = 1$.

4) Discuss the continuity and differentiability at $x = 0$ of the function

$$f(x) = \begin{cases} 1 & \text{when } x < 0 \\ 1 + \sin x & \text{when } 0 \leq x \leq \frac{\pi}{2} \end{cases}$$

→ For continuity:

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (1 + \sin x) = 1 + \sin 0 = 1$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (1) = 1$$

Again, $f(0) = 1 + \sin 0 = 1 + 0 = 1$

$\therefore \lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = f(0)$

$\therefore f(x)$ is continuous at $x = 0$.

For differentiability:

$$Rf'(0) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0^+} \frac{1 + \sin h - 1}{h} = \lim_{h \to 0^+} \frac{\sin h}{h} = 1$$

$$Lf'(0) = \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0^-} \frac{1-1}{h} = \lim_{h\to 0^-} \frac{0}{h} = 0$$

$\therefore Rf'(0) \neq Lf'(0)$

$\therefore f(x)$ is not differentiable at $x=0$.

5) Discuss the continuity and differentiability at $x=0$ and $x=\frac{\pi}{2}$ of the function.

$$f(x) = \begin{cases} 1 & ; x<0 \\ 1+\sin x & ; 0 \leq x < \frac{\pi}{2} \\ 2+(x-\frac{\pi}{2})^2 & ; x \geq \frac{\pi}{2} \end{cases}$$

$\rightarrow$ At $x=\frac{\pi}{2}$:

For continuity:

$$\lim_{x\to \frac{\pi}{2}^+} f(x) = \lim_{x\to \frac{\pi}{2}^+} \left\{2+(x-\frac{\pi}{2})^2\right\} = 2+(\frac{\pi}{2}-\frac{\pi}{2})^2 = 2$$

$$\lim_{x\to \frac{\pi}{2}^-} f(x) = \lim_{x\to \frac{\pi}{2}^-} (1+\sin x) = 1+\sin\frac{\pi}{2} = 2$$

Again, $f(\frac{\pi}{2}) = 2+(\frac{\pi}{2}-\frac{\pi}{2})^2 = 2$

$$\therefore \lim_{x\to \frac{\pi}{2}^+} f(x) = \lim_{x\to \frac{\pi}{2}^-} f(x) = f(\frac{\pi}{2})$$

Therefore $f(x)$ is continuous at $x = \frac{\pi}{2}$

for differentiability:

~~for~~ $Rf'\left(\frac{\pi}{2}\right) = \underset{h \to 0+}{lt} \dfrac{f\left(\frac{\pi}{2}+h\right) - f\left(\frac{\pi}{2}\right)}{h}$

$$= \underset{h \to 0+}{lt} \frac{2 + \left(\frac{\pi}{2} + h - \frac{\pi}{2}\right)^2 - 2}{h}$$

$$= \underset{h \to 0+}{lt} \frac{h^2}{h} = \underset{h \to 0+}{lt} h = 0$$

$$Lf'\left(\frac{\pi}{2}\right) = \underset{h \to 0-}{lt} \frac{f\left(\frac{\pi}{2}+h\right) - f\left(\frac{\pi}{2}\right)}{h}$$

$$= \underset{h \to 0-}{lt} \frac{1 + \sin\left(\frac{\pi}{2}+h\right) - 2}{h}$$

$$= \underset{h \to 0-}{lt} \frac{\cos h - 1}{h} = \underset{h \to 0-}{lt} \frac{-2\sin^2 \frac{h}{2}}{h}$$

$$= -2 \left(\underset{h \to 0-}{lt} \frac{\sin \frac{h}{2}}{\frac{h}{2}} \right)^2 \cdot \frac{h}{4}$$

$$= -2 \cdot 0 \cdot \frac{h}{4} = 0$$

$\therefore$ $Rf'\left(\frac{\pi}{2}\right) = Lf'\left(\frac{\pi}{2}\right)$

so, $f(x)$ is differentiable at $x = \frac{\pi}{2}$.

At $x=0$:

For continuity:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (1+\sin x) = 1+\sin 0 = 1$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 1 = 1$$

Again, $f(0) = 1+\sin 0 = 1$

$$\therefore \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$$

Therefore, $f(x)$ is continuous at $x=0$.

For differentiability:

$$Rf'(0) = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0^+} \frac{1+\sin h - 1 - \sin 0}{h}$$

$$= \lim_{h\to 0^+} \frac{\sin h}{h} = 1$$

$$Lf'(0) = \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0^-} \frac{1-1}{h} = \lim_{h\to 0^-} \frac{0}{h} = 0$$

$\therefore Rf'(0) \neq Lf'(0) \quad \therefore f(x)$ is not differentiable at $x=0$.

6) Discuss the continuity and differentiability at $x=0$ and $x=1$ of the function:

$$f(x) = \begin{cases} x^2+1 ; & x \leq 0 \\ x ; & 0 < x < 1 \\ \frac{1}{x} ; & x \geq 1 \end{cases}$$

→ At $x = 0$:

For continuity: $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x^2+1) = 0^2+1 = 1$

At $x = 0$; $f(x) = f(0) = 0^2+1 = 1$

$\therefore Rf'(0) \neq Lf'(0)$

$\therefore$ $f(x)$ is not continuous at $x=0$

$\therefore$ $f(x)$ is not differentiable at $x=0$.

Therefore $f(x)$ is not differentiable

At $x = 1$:

For continuity: $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{1}{x} = \frac{1}{1} = 1$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x) = 1$

Again, $f(1) = \frac{1}{1} = 1$

$\therefore \lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = f(1)$

$\therefore$ f(x) is continuous at $x = 1$.

<u>for differentiability:</u>

$$Rf'(1) = \lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0^+} \frac{\frac{1}{1+h} - 1}{h}$$

$$= \lim_{h\to 0^+} \frac{\cancel{1} - \cancel{1} - h}{h(1+h)} = \lim_{h\to 0^+} \frac{-h}{h(1+h)}$$

$$= \lim_{h\to 0^+} \frac{-1}{1+h} = -\frac{1}{1+0} = -1$$

$$Lf'(1) = \lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0^-} \frac{1+h-1}{h} = \lim_{h\to 0^-} \frac{h}{h} = 1$$

$\therefore Rf'(1) \neq Lf'(1)$

$\therefore$ f(x) is not differentiable at $x = 1$.

7) A function f is defined as follows:

$$f(x) = \begin{cases} x & \text{when } x < 1 \\ 2-x & \text{when } 1 \le x \le 2 \\ -2+3x-x^2 & \text{when } x > 2 \end{cases}$$

Show that f is continuous at $x=1$ and $x=2$ both; & it is derivable at $x=2$ but not at $x=1$.

$\rightarrow$ At $x=2$:

For continuity:

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (-2+3x-x^2) = -2+3.2-2^2 = 0$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2-x) = 2-2 = 0$$

$$f(2) = 2-2 = 0$$

$$\therefore \lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = f(2)$$

$\therefore$ $f(x)$ is continuous at $x=2$.

For differentiability:

$$Rf'(2) = \lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0^+} \frac{\{-2+3(2+h)-(2+h)^2\}-0}{h}$$

$$= \lim_{h\to 0^+} \frac{-h^2-h}{h} = \lim_{h\to 0^+} (-h-1) = -1$$

$$Lf'(2) = \lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^-} \frac{\{2-(2+h)\}-0}{h}$$

$$= \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} (-1) = -1$$

$\therefore Rf'(2) = Lf'(2)$

$\therefore$ The function is derivable at $x=2$.

<u>At $x=1$:</u>

<u>For continuity:</u>

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2-x) = 2-1 = 1$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x) = 1$$

$$f(1) = 2-1 = 1$$

$$\therefore \lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = f(1)$$

$\therefore f(x)$ is continuous at $x=1$.

<u>For differentiability:</u>

$$Rf'(1) = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0^+} \frac{(1+h) - (2-1)}{h}$$

$$= \lim_{h\to 0^-} \frac{h}{h} = \lim_{h\to 0^-} 1 = 1$$

$\therefore$ $Rf'(1) \neq Lf'(1)$

$\therefore$ $f(x)$ is not derivable at $x=1$.

8) $f(x) = \begin{cases} 3+2x & \text{when } -\frac{3}{2} < x \le 0 \\ 3-2x & \text{when } 0 < x < \frac{3}{2} \\ -3-2x & \text{when } x \geq \frac{3}{2} \end{cases}$

Discuss the continuity and differentiability of the function at $x=0$ and $x=\frac{3}{2}$.

$\rightarrow$ At $x=0$:

For continuity:

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (3-2x) = 3-2.0 = 3$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (3+2x) = 3+2.0 = 3$

Again, $f(0) = 3+2.0 = 3$

So, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$

$\therefore$ $f(x)$ is continuous at $x=0$.

For differentiability:

$$Rf'(0) = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{3-2h-3+2\cdot 0}{h}$$

$$= \lim_{h\to 0^+} \frac{-2h}{h} = -2$$

$$Lf'(0) = \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{3+2h-3-2\cdot 0}{h}$$

$$= \lim_{h\to 0^-} \frac{2h}{h} = \lim_{h\to 0^-} 2 = 2$$

$\therefore Rf'(0) \neq Lf'(0)$

$\therefore f(x)$ is not differentiable at $x=0$.

At $x=\frac{3}{2}$:

For continuity:

$$\lim_{x\to \frac{3}{2}^+} f(x) = \lim_{x\to \frac{3}{2}^+} (-3-2x) = -3-2\cdot\frac{3}{2} = -6$$

$$\lim_{x\to \frac{3}{2}^-} f(x) = \lim_{x\to \frac{3}{2}^-} (3-2x) = 3-2\cdot\frac{3}{2} = 0$$

So, $\lim_{x\to \frac{3}{2}^+} f(x) \neq \lim_{x\to \frac{3}{2}^-} f(x)$

So, $f(x)$ is not continuous at $x=\frac{3}{2}$.

Therefore, clearly $f(x)$ is not derivable at $x=\frac{3}{2}$.

9) Show that function $f(x) = |x| + |x-1|$ is continuous at $x=1$ but not differentiable.

→ At $x=1$:

For continuity:

$$\lim_{x\to 1^+} f(x) = x + x - 1 = 2x - 1 = 2\cdot 1 - 1 = 1$$

$$\lim_{x\to 1^-} f(x) = x - (x-1) = \cancel{x} - \cancel{x} - 1 = -1$$

$$f(1) = x + x - 1 = 2x - 1 = 2\cdot 1 - 1 = 1$$

So, $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = f(1)$

$\therefore$ $f(x)$ is continuous at $x=1$.

For differentiability:

$$Rf'(1) = \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{2(1+h)-1-(2-1)}{h}$$

$$= \lim_{h\to 0^+} \frac{2+2h-1-1}{h} = \lim_{h\to 0^+} \frac{2h}{h} = 2$$

$$Lf'(1) = \lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{1-(2-1)}{h} = 0$$

$\therefore$ $Rf'(1) \neq Lf'(1)$

therefore, $f(x)$ is not differentiable at $x=1$.

10) $$f(x) = \begin{cases} 1+x & \text{when } x<0 \\ x & \text{when } 0<x<1 \\ 2-x & \text{when } 1\le x\le 2 \\ 2x-x^2 & \text{when } x>2 \end{cases}$$

Show that the function $f(x)$ is continuous at the points $x=1$ and $x=2$ but $f'(x)$ ~~is~~ doesn't at that point.

→ For continuity:

At $x=1$:

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2-x) = 2-1 = 1$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1$$

Again, $f(1) = 2-1 = 1$

$$\therefore \lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = f(1)$$

$\therefore$ $f(x)$ is continuous at $x=1$.

At $x=2$:

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 2x-x^2 = 4-4 = 0$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 2-x = 2-2 = 0$$

Again, $f(2) = 2-x = 2-2 = 0$

$\therefore \lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = f(2)$

$\therefore f(x)$ is continuous at $x = 2$.

For differentiability:

At $x = 1$: $Rf'(1) = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{2-1-h-1}{h} = -1$

$Lf'(1) = \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{1+h-1}{h} = 1$

$\therefore Rf'(1) \neq Lf'(1)$.

$\therefore f'(x)$ doesn't exist at $x = 1$.

At $x = 2$:

$Rf'(2) = \lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^+} \frac{4+2h-4-4h-h^2-0}{h}$

$= \lim_{h \to 0^+} \frac{-2h - h^2}{h} = \lim_{h \to 0^+} (-2-h) = -2$

$Lf'(2) = \lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^-} \frac{2-2-h-0}{h} = -1$

$\therefore Rf'(2) \neq Lf'(2)$

$\therefore f'(x)$ doesn't exist at $x = 2$.

Derivative

1) $\sin y = x \sin(a+y)$; $\frac{dy}{dx} = ?$

$\rightarrow$ Given, $\sin y = x \sin(a+y)$

or, $\cos y \frac{dy}{dx} = \sin(a+y) + x\cos(a+y)\frac{dy}{dx}$

or ; $\frac{dy}{dx}\{\cos y - x\cos(a+y)\} = \sin(a+y)$

$\therefore \frac{dy}{dx} = \frac{\sin(a+y)}{\cos y - x\cos(a+y)}$

2) $y = \sqrt{x}\, e^x \sec x$; $\frac{dy}{dx} = ?$

$\rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}} e^x \sec x + \sqrt{x} e^x \sec x + \sqrt{x}\, e^x \sec x \tan x$

$= e^x \sec x \left(\frac{1}{2\sqrt{x}} + \sqrt{x} + \sqrt{x}\ \cancel{\sec}\ \tan x\right)$

3) $y = (\sin x)^{\cos x} + (\cos x)^{\sin x}$; $\frac{dy}{dx} = ?$

$\rightarrow \frac{dy}{dx} = (\sin x)^{\cos x}\frac{d}{dx}(\cos x \ln \sin x) + (\cos x)^{\sin x} \frac{d}{dx}(\sin x \ln \cos x)$

$= (\sin x)^{\cos x}\{\cos x \cdot \frac{1}{\sin x} \cdot \frac{d}{dx}(\cos x) + \cos x \ln \cos x\}$

$= (\sin x)^{\cos x}(\cos x \cot x - \sin x \ln \sin x) +$

$(\cos x)^{\sin x}(\cos \ln \cos x - \sin x \tan x)$

$\rightarrow (1+x^2) y_2 + y_1 \cdot 2x = 0$

By applying Leibnitz's theorem, we get,

$(1+x^2) y_{n+2} + n \cdot 2x \cdot y_{n+1} + \frac{n(n-1)}{2} 2 y_n$

$+ y_{n+1} \cdot 2x + n \cdot 2 \cdot y_n = 0$

$\rightarrow (1+x^2) y_{n+2} + 2x(n+1) y_{n+1} + n(n+1) y_n = 0$

(showed)

3) If $y = a\cos(\ln x) + b\sin(\ln x)$

show that $x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2+1) y_n = 0$

$\rightarrow y_1 = -a\sin \ln x \cdot \frac{1}{x} + b\cos \ln x \cdot \frac{1}{x}$

$\rightarrow x y_1 = -a\sin \ln x + b\cos \ln x$

$\rightarrow y_1 + x y_2 = -a\cos \ln x \cdot \frac{1}{x} - b\sin \ln x \cdot \frac{1}{x}$

$\rightarrow x y_1 + x^2 y_2 = -a\cos \ln x - b\sin \ln x$

$\rightarrow x y_1 + x^2 y_2 + y = 0$

By applying Leibnitz's theorem, we get

$x^2 y_{n+2} + n y_{n+1} \cdot 2x + \frac{n(n-1)}{2} y_n \cdot 2 + x y_{n+1}$

$+ n y_n + y_n = 0$

$\rightarrow x^2 y_{n+2} + 2n x y_{n+1} + x y_{n+1} + (n^2 - n) y_n$

$+ n y_n + y_n = 0$

$\rightarrow x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2+1) y_n = 0$

4) $y = e^{\sin x}\sin(a^x)$; $\frac{dy}{dx} = ?$

$\rightarrow \frac{dy}{dx} = e^{\sin x}\cos x\sin(a^x) + e^{\sin x}\cos(a^x)\,a^x\ln a$

$= e^{\sin x}\{\cos x\sin(a^x) + a^x\ln a\cos(a^x)\}$

5) $y = \tan(\ln x^2)$; $\frac{dy}{dx} = ?$

$\rightarrow \frac{dy}{dx} = \sec^2(\ln x^2)\cdot\frac{1}{x^2}\cdot 2x$

$= \frac{2}{x}\sec^2(\ln x^2)$

6) $y = x^{\sin^{-1}x}$

$\rightarrow \frac{dy}{dx} = x^{\sin^{-1}x}\cdot\frac{d}{dx}\{\sin^{-1}x\ \ln(x)\}$

$= x^{\sin^{-1}x}\cdot\left\{\frac{1}{\sqrt{1-x^2}}\ln(x) + \sin^{-1}x\cdot\frac{1}{x}\right\}$

$= x^{\sin^{-1}x}\left(\frac{\ln x}{\sqrt{1-x^2}} + \frac{\sin^{-1}x}{x}\right)$

7) $y = x^{\ln x} + x\cos^{-1}x$; $\frac{dy}{dx} = ?$

$\rightarrow \frac{dy}{dx} = x^{\ln x}\frac{d}{dx}\{\ln x\cdot\ln x\} + x^{\cos^{-1}x}\frac{d}{dx}(\cos^{-1}x\cdot\ln x)$

$= x^{\ln x}\frac{d}{dx}(\ln x)^2 + x\cos^{-1}x\left(-\frac{1}{\sqrt{1-x^2}}\cdot\ln x + \cos^{-1}x\cdot\frac{1}{x}\right)$

$= x^{\ln x}\cdot 2\ln x\cdot\frac{1}{x} - \frac{x\cos^{-1}x\ln x}{\sqrt{1-x^2}} + (\cos^{-1}x)^2$

$= x^{\ln x-1}\cdot 2\ln x - \frac{x\cos^{-1}x\ln x}{\sqrt{1-x^2}} + (\cos^{-1}x)^2$

8) $y = \sin x^{\sin x}$; $\frac{dy}{dx} = ?$

$\rightarrow \frac{dy}{dx} = \sin x^{\sin x} \cdot \frac{d}{dx}\{\sin x \cdot \ln(\sin x)\}$

$= \sin x^{\sin x}\{\cos x \cdot \ln(\sin x) + \sin x \cdot \frac{1}{\sin x}\}$

$= \sin x^{\sin x} \cos x \{\ln(\sin x) + \cancel{\cos} 1\}$

9) $\ln(x+y) = xy$; $\frac{dy}{dx} = ?$

$\rightarrow \frac{1}{x+y}\left(1+\frac{dy}{dx}\right) = x\frac{dy}{dx} + \cancel{x}\, 1 \cdot y$

$\rightarrow \frac{dy}{dx}\left(\frac{1}{x+y} - x\right) = y - \frac{1}{x+y}$

$\rightarrow \frac{dy}{dx} = \frac{y - \frac{1}{x+y}}{\frac{1}{x+y} - x} = \frac{xy + y^2 - 1}{1 - x^2 - xy}$

10). $y = x^{\cos^{-1}x}$; $\frac{dy}{dx} = ?$

$\rightarrow \frac{dy}{dx} = x^{\cos^{-1}x} \cdot \frac{d}{dx}\{\cos^{-1}x \cdot \ln x\}$

$= x^{\cos^{-1}x}\left(-\frac{1}{\sqrt{1-x^2}} \cdot \ln x + \cos^{-1}x \cdot \frac{1}{x}\right)$

$= x^{\cos^{-1}x}\left(\frac{\cos^{-1}x}{x} - \frac{\ln x}{\sqrt{1-x^2}}\right)$

11) $y = x^{\sin x}$; $\frac{dy}{dx} = ?$

$\rightarrow \frac{dy}{dx} = x^{\sin x} \cdot \frac{d}{dx}\{\sin x \cdot \ln(x)\}$

$= x^{\sin x}\left(\cos x \ln x + \sin x \cdot \frac{1}{x}\right)$

$= x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right)$

12) $y = (\tan x)^{\cot x} + (\cot x)^{\tan x}$

$\rightarrow \frac{dy}{dx} = (\tan x)^{\cot x} \frac{d}{dx}(\cot x \ln \tan x) + (\cot x)^{\tan x}$
$\frac{d}{dx}(\tan x \ln \cot x)$

$= (\tan x)^{\cot x}\left\{\cot x \cdot \frac{1}{\tan x} \sec^2 x - \ln \tan x \operatorname{cosec}^2 x\right\}$
$+ (\cot x)^{\tan x}\left\{\tan x \cdot \frac{1}{\cot x}(-\operatorname{cosec}^2 x) + \ln \cot x \sec^2 x\right\}$

$= (\tan x)^{\cot x}\left\{\cot^2 x \sec^2 x - \ln \tan x \operatorname{cosec}^2 x\right\}$
$+ (\cot x)^{\tan x}\left\{\ln \cot x \sec^2 x - \sec^2 x\right\}$

$= (\tan x)^{\cot x} \operatorname{cosec}^2 x\left\{1 - \ln \tan x\right\} + (\cot x)^{\tan x}$
$\sec^2 x\left\{\ln \cot x - 1\right\}$

13) $e^x + e^y = 2xy$; $\frac{dx}{dy} = ?$

$\rightarrow e^x + e^y \frac{dy}{dx} = 2y + 2x\frac{dy}{dx}$

$\rightarrow \frac{dy}{dx}(e^y - 2x) = 2y - e^x)$

$\therefore \frac{dy}{dx} = \frac{2y - e^x}{e^y - 2x}$

14) If $y = e^{-x}\sin x$, show that $y_4 + 4y = 0$

$\rightarrow y_1 = e^{-x}\cos x - e^{-x}\sin x = e^{-x}\cos x - y$

$\rightarrow y_2 = -e^{-x}\sin x - e^{-x}\cos x - y_1$

$\rightarrow y_2 = -y - e^{-x}\cos x - y_1$

$\rightarrow y_2 = -e^{-x}\sin x - e^{-x}\cos x - e^{-x}\cos x + e^{-x}\sin x$

$\rightarrow y_2 = -2e^{-x}\cos x$

$\rightarrow y_3 = -2(-e^{-x}\sin x - e^{-x}\cos x)$

$\rightarrow y_3 = 2(e^{-x}\sin x + e^{-x}\cos x)$

$\rightarrow y_4 = 2(e^{-x}\cos x - e^{-x}\sin x - e^{-x}\sin x - e^{-x}\cos x)$

$\rightarrow y_4 = -4e^{-x}\sin x$

$\rightarrow y_4 = -4y \qquad \therefore y_4 + 4y = 0$ (showed)

15) $y = ae^{mx} + be^{-mx}$, show that, $y_2 = m^2 y$

$\rightarrow y = ae^{mx} + be^{-mx}$

or, $y_1 = ame^{mx} - bme^{-mx}$

or, $y_2 = am^2e^{mx} + bm^2e^{-mx}$

$= m^2(ae^{mx} + be^{-mx})$

or, $y_2 = m^2 y$. (showed)

Successive Differentiation

1) If $y = x^n$, then show that $y_n = n!$

→ Given, $y = x^n$

or, $y_1 = n x^{n-1}$

or, $y_2 = n(n-1) x^{n-2}$

or, $y_3 = n(n-1)(n-2) x^{n-3}$

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or, $y_n = n(n-1)(n-2) \cdots 3.2.1\, x^{n-n}$

$= n(n-1)(n-2) \cdots 3.2.1$

$= n!$ (showed)

2) If $y = (ax+b)^n$ and $n \in N$, then show that $y_n = n!\, a^n$

→ Given, $y = (ax+b)^n$

→ $y_1 = n(ax+b)^{n-1} a$

→ $y_2 = n(n-1)(ax+b)^{n-2} . a^2$

→ $y_3 = n(n-1)(n-2)(ax+b)^{n-3} a^3$

- - - - - - -

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$y_n = n(n-1)(n-2) \cdots 3.2.1.(ax+b)^{n-n} a^n$

$= n(n-1)(n-2) \cdots 3.2.1. a^n$

$= n!\, a^n$ (showed)

3) If $y = a^{bx+c}$ then show that,

$y_n = a^{bx+c} (\ln a)^n b^n$

$\rightarrow$ Given, $y = a^{bx+c}$

$\rightarrow y_1 = a^{bx+c} \ln a \cdot b$

$\rightarrow y_2 = a^{bx+c} (\ln a)^2 \cdot b^2$

$\rightarrow y_3 = a^{bx+c} (\ln a)^3 b^3$

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$y_n = a^{bx+c} (\ln a)^n b^n$. (showed)

4) If $y = \ln(ax+b)$; show that $y_n = (-1)^{n-1} (n-1)! (ax+b)^{-n} \cdot a^n$

$\rightarrow y = \ln(ax+b)$

$\rightarrow y_1 = (ax+b)^{-1} \cdot a$

$\rightarrow y_2 = (-1)(ax+b)^{-2} a^2$

$\rightarrow y_3 = (-1)(-2)(ax+b)^{-3} \cdot a^3$

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$y_n = (-1)(-2) \cdots \{-(n-1)\} (ax+b)^{-n} a^n$

$\therefore y_n = (-1)^{n-1} (ax+b)^{-n} a^n \cdot (n-1)!$ {Showed}

Leibnitz's Theorem

1) If $y = e^{a\sin^{-1}x}$ then show that

$(1-x^2)\, y_{n+2} - (2n+1)\, x\, y_{n+1} - (n^2+a^2)\, y_n = 0$

→ Given, $y = e^{a\sin^{-1}x}$

or, $y_1 = a e^{a\sin^{-1}x} \cdot \frac{1}{\sqrt{1-x^2}}$

or, $\sqrt{1-x^2} \cdot y_1 = ay$

Differentiating both sides again we get,

$\sqrt{1-x^2} \cdot y_2 - \frac{x}{\sqrt{1-x^2}} y_1 = ay_1 = a \cdot \frac{ay}{\sqrt{1-x^2}}$

Multiplying both sides by $\sqrt{1-x^2}$ gives,

$(1-x^2)\, y_2 - x y_1 - a^2 y = 0$ —(1)

By applying Leibnitz's theorem, we get;

$\{(1-x^2)\, y_{n+2} + n(-2x) \cdot y_{n+1} + \frac{n(n-1)}{2}(-2)\, y_n\}$

$- \{x\, y_{n+1} + n\, y_n\} - a^2 y_n = 0$

→ $(1-x^2)\, y_{n+2} - (2n+1)\, x\, y_{n+1} - (n^2+a^2)\, y_n = 0$

{Proved}

2) If $y = \cot^{-1}x$ then show that

$(1+x^2)\, y_{n+2} + 2(n+1)\, x\, y_{n+1} + n(n+1)\, y_n = 0$

→ Given, $y = \cot^{-1}x$ → $y_1 = -\frac{1}{1+x^2}$

→ $(1+x^2)\, y_1 = -1$

$\rightarrow (1+x^2)y_2 + y_1 \cdot 2x = 0$

By applying Leibnitz's theorem, we get,

$(1+x^2)\,y_{n+2} + n\cdot 2x \cdot y_{n+1} + \frac{n(n-1)}{2}\,2y_n$

$+ y_{n+1}\cdot 2x + n\cdot 2\cdot y_n = 0$

$\rightarrow (1+x^2)\,y_{n+2} + 2x(n+1)\,y_{n+1} + n(n+1)\,y_n = 0$

(showed)

3) If $y = a\cos(\ln x) + b\sin(\ln x)$

show that $x^2 y_{n+2} + (2n+1)x\,y_{n+1} + (n^2+1)\,y_n = 0$

$\rightarrow y_1 = -a\sin\ln x \cdot \frac{1}{x} + b\cos\ln x \cdot \frac{1}{x}$

$\rightarrow xy_1 = -a\sin\ln x + b\cos\ln x$

$\rightarrow y_1 + xy_2 = -a\cos\ln x \cdot \frac{1}{x} - b\sin\ln x \cdot \frac{1}{x}$

$\rightarrow xy_1 + x^2 y_2 = -a\cos\ln x - b\sin\ln x$

$\rightarrow xy_1 + x^2 y_2 + y = 0$

By applying Leibnitz's theorem, we get

$x^2 y_{n+2} + n\,y_{n+1}\cdot 2x + \frac{n(n-1)}{2}\,y_n \cdot 2 + x\,y_{n+1}$

$+ n y_n + y_n = 0$

$\rightarrow x^2 y_{n+2} + 2nx\,y_{n+1} + x\,y_{n+1} + (n^2-n)\,y_n$

$+ n y_n + y_n = 0$

$\rightarrow x^2 y_{n+2} + (2n+1)x\,y_{n+1} + (n^2+1)\,y_n = 0$

4) If $y = \cos\{\ln(1+x)\}$ show that

$(1+x^2) y_{n+2} + (2n+1)(1+x) y_{n+1} + (n^2+1) y_n = 0$

$\rightarrow y = \cos\{\ln(1+x)\}$

$\rightarrow y_1 = -\sin\ln(1+x) \cdot \frac{1}{1+x}$

$\rightarrow y_1(1+x) = -\sin\ln(1+x)$

$\rightarrow y_2(1+x) + y_1 = -\cos\ln(1+x) \cdot \frac{1}{1+x}$

$\rightarrow y_2(1+x)^2 + y_1(1+x) + y = 0$

$\rightarrow y_{n+2}(1+x)^2 + n\, y_{n+1}(1+x) \cdot 2 + \frac{n(n-1)}{2} y_n \cdot 2$

$+ y_{n+1}(1+x) + n y_n + y_n = 0$

$\rightarrow y_{n+2}(1+x)^2 + y_{n+1}(2n+1)(1+x) + (n^2+1) y_n$

$= 0$ (showed)

5) If $y = \tan^{-1}x$, show that

$(1+x^2) y_{n+1} + 2nx y_n + n(n-1) y_{n-1} = 0$

$\rightarrow y = \tan^{-1}x \rightarrow y_1 = \frac{1}{1+x^2} \rightarrow y_1(1+x^2) = 1$

By applying Leibnitz's theorem, we get

$y_{n+1}(1+x^2) + n \cdot y_n \cdot 2x + y_{n-1}\, 2 \cdot \frac{n(n-1)}{2!} = 0$

$\rightarrow (1+x^2) y_{n+1} + 2nx y_n + n(n-1) y_{n-1} = 0$

(showed)

6) If $\ln y = \cot^{-1}x$, show that
$(1+x^2)y_{n+2} + (2nx+2x+1)y_{n+1} + n(n+1)y_n = 0$

$\rightarrow \ln y = \cot^{-1}x \rightarrow \frac{1}{y}\cdot y_1 = -\frac{1}{1+x^2}$

$\rightarrow y_1(1+x^2) = -y$

$\rightarrow (1+x^2)y_2 + y_1\cdot 2x = -y_1$

$\rightarrow (1+x^2)y_2 + (2x+1)y_1 = 0$

By applying Leibnitz's theorem, we get,

$(1+x^2)y_{n+2} + n\cdot 2x\cdot y_{n+1} + \frac{n(n-1)}{2!}2\cdot y_n$

$+ (2x+1)\cdot y_{n+1} + n\cdot 2y_n = 0$

$\rightarrow (1+x^2)y_{n+2} + y_{n+1}(2nx+2x+1) + n(n+1)y_n = 0$

{Showed}

7) If $y = \sin(m\sin^{-1}x)$, prove that
$(1-x^2)y_{n+2} - (2n+1)x\cdot y_{n+1} + (m^2-n^2)y_n = 0$

$\rightarrow y = \sin(m\sin^{-1}x)$

$\rightarrow y_1 = \cos(m\sin^{-1}x)\cdot m\cdot\frac{1}{\sqrt{1-x^2}}$

$\rightarrow y_1^2 = \{1-\sin^2(m\sin^{-1}x)\}\cdot\frac{m^2}{1-x^2}$

$\rightarrow y_1^2(1-x^2) = (1-y^2)m^2$

$\rightarrow 2y_1y_2(1-x^2) - y_1^2\cdot 2x = m^2(-2y\cdot y_1)$

$\rightarrow y_2(1-x^2) - y_1 x = -m^2 y$ (Dividing both sides by $2y_1$)

$\rightarrow y_2(1-x^2) - xy_1 + m^2 y = 0$

By applying Leibnitz's theorem we get,

$y_{n+2}(1-x^2) + n\, y_{n+1}(-2x) + \frac{n(n-1)}{2!} y_n(-2)$

$-[y_{n+1} x + n\, y_n \cdot 1] + m^2 y_n = 0$

$\rightarrow y_{n+2}(1-x^2) + y_{n+1}(-2nx - x) + y_n\{-n(n-1) - n + m^2\}$

$= 0$

$\rightarrow y_{n+2}(1-x^2) - y_{n+1}(2n+1)x + y_n(-n^2 + n - n + m^2) = 0$

$\rightarrow (1-x^2)\, y_{n+2} - (2n+1)\, x\, y_{n+1} + (m^2 - n^2)\, y_n = 0$

[Showed]

Maxima- Minima

1) Investigate for what values of x, $f(x) = 5x^6 - 18x^5 + 15x^4 - 10$ is minimum of maximum.

→ Here, $f(x) = 5x^6 - 18x^5 + 15x^4 - 10$

$f'(x) = 30x^5 - 90x^4 + 60x^3$
$= 30(x^5 - 3x^4 + 2x^3)$

Putting, $f'(x) = 0$

we have, $30(x^5 - 3x^4 + 2x^3) = 0$

or, $x^3(x^2 - 3x + 2) = 0$

$\therefore x = 0, 1, 2$

Again, $f''(x) = 30(5x^4 - 12x^3 + 6x^2)$

when, $x = 1$, $f''(x)$ is negative and hence $f(x)$ is a maximum for $x = 1$.

when $x = 2$, $f''(x)$ is positive and so $f(x)$ is a minimum for $x = 2$.

when $x = 0$, $f''(x)$ is zero, so the test fails and we have to examine higher order derivatives.

$f'''(x) = 30(20x^3 - 36x^2 + 12x)$
$= 120(5x^3 - 9x^2 + 3x)$ $\therefore f'''(0) = 0$

$f''''(x) = 120(15x^2 - 18x + 3)$

$= 360(5x^2 - 6x + 1)$

$f''''(0) = 360$ [positive]

since even order derivative is positive for $x = 0$,

$\therefore$ for $x = 0$, $f(x)$ is a minimum.

2) Find for what values of x, the following expression is maximum or minimum respectively

$2x^3 - 21x^2 + 36x - 20$

Find also the maximum and minimum values of the expression.

$\rightarrow$ Here, $f(x) = 2x^3 + 21x^2 + 36x - 20$

$f'(x) = 6x^2 + 42x + 36$

Putting, $f'(x) = 0$

or, $6x^2 + 42x + 36 = 0$

or, $x^2 + 7x + 6 = 0$ $\quad \therefore x = -6, -1$

Again, $f''(x) = 12x + 42$

when $x = -6$; $f''(x)$ is negative and hence $f(x)$ is maximum for $x = -6$

when $x = -1$; $f''(x)$ is positive and hence $f(x)$ is minimum for $x = -1$

Minimum value, $f(-1) = 2(-1)^3 + 21(-1)^2 + 36(-1) - 20$

$= -2 + 21 - 36 - 20$

$= -37$

Maximum value, $f(-6) = 2(-6)^3 + 21(-6)^2 + 36(-6) - 20$

$= 88$

Rolle's Theorem

1) Verify Rolle's theorem for $f(x) = x^3 - 12x$ in the interval $0 \le x \le 2\sqrt{3}$.

→ Since a polynomial function is everywhere continuous and differentiable, the given function is continuous as well as differentiable on every interval. To identify the interval, we solve the equation $f(x) = 0$

Now, $f(x) = 0$

or, $x^3 - 12x = 0$

or, $x(x^2 - 12) = 0$ $\therefore x = 0, \pm 2\sqrt{3}$

So, clearly the permissible values of x are $x=0$, $x=2\sqrt{3}$ in the given interval.

Clearly, $f(0) = f(2\sqrt{3}) = 0$

Thus, all the conditions of Rolle's theorems are satisfied. So, there must exist $c \in [0, 2\sqrt{3}]$ such that $f'(c) = 0$.

But $f'(x) = 3x^2 - 12 = 3(x^2-4)$

So, $f'(c) = 0$

$\rightarrow 3(c^2-4) = 0 \rightarrow c^2 - 4 = 0 \quad \therefore c = \pm 2$

For $c = 2$, it lie in $[0, 2\sqrt{3}]$.

Hence, Rolle's theorem is verified.

2) Does Rolle's theorem apply to the function $f(x) = 1-(x-3)^{2/3}$

$\rightarrow f(x) = 1-(x-3)^{2/3}$

To identify the interval, we solve the equation $f(x) = 0$.

So, $f(x) = 0$

$\rightarrow 1 = (x-3)^{2/3}$

$\Rightarrow (x-3) = 1^{3/2} = 1\sqrt{1} = \pm 1$

$\Rightarrow x-3 = 1$ or $x-3 = -1$

$\therefore x = 4, x = 2$

So, we consider the given function in $[2, 4]$

clearly, $f(2) = f(4) = 0$

And the given function is ~~everywhere a~~ continuous as well as differentiable on every interval. Thus, all the conditions of Rolle's theorem are satisfied. So, there must exist $c \in [2, 4]$ such that $f'(c) = 0$

But $f'(x) = 0 - \frac{2}{3}(x-3)^{-1/3} = \frac{2}{3(x-3)^{1/3}}$

~~So, $f'(c) = 0$~~

$\Rightarrow \cancel{\frac{2}{3(c-3)^{1/3}}} = 0 \Rightarrow$

Evidently $f'(x)$ doest exist at $x = 3 \in (2, 4)$

Consequently, $f(x)$ is not derivable in the interval $(2, 4)$

$\therefore$ Rolle's theorem is not applicable ~~for~~ to the function $f(x)$.

3) State Rolle's theorem. Verify it for $f(x) = 2x^3 + x^2 - 4x - 2$.

→ Rolle's theorem: Let $y = f(x)$ be continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b) = 0$, then there is at least one number c in (a, b) at which $f'(c) = 0$

Verification of Rolle's theorem: Since a polynomial function is everywhere continuous and differentiable, the given function is continuous as well as differentiable on every interval.

To identify the interval, we solve the equation $f(x) = 0$.

Now, $f(x) = 0$

$\rightarrow 2x^3 + x^2 - 4x - 2 = 0$

$\rightarrow (x^2 - 2)(2x + 1) = 0$

$\rightarrow x^2 = 2$ or $x = -\frac{1}{2}$

$\Rightarrow x = \sqrt{2}$ or $x = -\sqrt{2}$ or $x = -\frac{1}{2}$

So, we consider the given function in $[-\sqrt{2}, \sqrt{2}]$

Clearly, $f(-\sqrt{2}) = f(\sqrt{2}) = 0$

Thus, all the conditions of Rolle's theorem are satisfied. So, there must exist $c \in [-\sqrt{2}, \sqrt{2}]$ such that $f'(c) = 0$

But $f'(x) = 6x^2 + 2x - 4$

So, $f'(c) = 0$

$\rightarrow 6c^2 + 2c - 4 = 0$

$\rightarrow 3c^2 + c - 2 = 0$

$\rightarrow 3c^2 + 3c - 2c - 2 = 0$

$\rightarrow 3c(c+1) - 2(c+1) = 0$

$\therefore c = \frac{2}{3}$ or $c = -1$

clearly both these points lie in $[-\sqrt{2}, \sqrt{2}]$

Hence, Rolle's theorem is verified.

Euler's theorem.

1) If $u = \tan^{-1} \dfrac{x^3+y^3}{x+y}$ show that

$$x \cdot \frac{\delta u}{\delta x} + y \cdot \frac{\delta u}{\delta y} = \sin 2u$$

→ From the given relation, we get

$$u = \tan^{-1} \frac{x^3+y^3}{x+y}$$

$$\rightarrow \tan u = \frac{x^3+y^3}{x+y}$$

$$\rightarrow \tan u = \frac{x^3 \left\{ 1 + \left(\frac{y}{x}\right)^3 \right\}}{x\left(1+\frac{y}{x}\right)} = x^2 \phi\left(\frac{y}{x}\right)$$

∴ tan u is a homogeneous function of degree 2.

Let $v = \tan u$ by Euler's theorem.

$$x = \frac{\delta v}{\delta x} + y \frac{\delta v}{\delta y} = 2v$$

$$\rightarrow x \frac{\delta \tan u}{\delta x} + y \frac{\delta \tan u}{\delta y} = 2\tan u$$

$$\rightarrow x \sec^2 u \frac{\delta u}{\delta x} + y \sec^2 x \frac{\delta u}{\delta y} = 2\tan u$$

$$\rightarrow x \frac{\delta u}{\delta x} + y \frac{\delta u}{\delta y} = \frac{2\tan u}{\sec^2 u} = 2\sin u \cos u$$

$\rightarrow x \frac{\delta u}{\delta x} + y \frac{\delta u}{\delta y} = \sin 2u$ (showed)

2) If $u = \cos^{-1} \frac{x+y}{\sqrt{x}+\sqrt{y}}$, show that $x . \frac{\delta u}{\delta x}$ $+ y . \frac{\delta u}{\delta y} + \frac{1}{2} \cot u = 0$

$\rightarrow$ from the given relation, we get,

$$u = \cos^{-1} \left\{ \frac{x+y}{\sqrt{x}+\sqrt{y}} \right\}$$

$$\rightarrow \cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x\left(1+\frac{y}{x}\right)}{\sqrt{x}\left(1+\frac{\sqrt{y}}{\sqrt{x}}\right)}$$

$$\rightarrow \cos u = \sqrt{x}\, \phi \frac{y}{x}$$

$\therefore \cos u$ is a homogeneous function of degree $\frac{1}{2}$.

Let $v = \cos u$ by Euler's theorem,

$$x \frac{\delta v}{\delta x} + y \frac{\delta v}{\delta y} = \frac{1}{2} v$$

$$\rightarrow x \frac{\delta \cos u}{\delta x} + y \frac{\delta \cos u}{\delta y} = \frac{1}{2} \cos u$$

$$\rightarrow x (-\sin u) \frac{\delta u}{\delta x} + y (-\sin u) \frac{\delta v}{\delta y} = \frac{1}{2} \cos u$$

$$\rightarrow x \frac{\delta u}{\delta x} + y \frac{\delta u}{\delta y} = \frac{1}{2} \frac{\cos u}{-\sin u} = -\frac{1}{2} \cot u$$

$$\rightarrow x \frac{\delta u}{\delta x} + y \frac{\delta u}{\delta y} + \frac{1}{2} \cot u = 0$$ [showed]

3) If $u = \tan^{-1} \frac{x+y}{\sqrt{x}+\sqrt{y}}$ then show that

$$x \frac{\delta u}{\delta x} + y \frac{\delta u}{\delta y} = \frac{1}{4} \sin 2u$$

→ from the given relation we get,

$$u = \tan^{-1} \frac{x+y}{\sqrt{x}+\sqrt{y}} \Rightarrow \tan u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$\Rightarrow \tan u = \frac{x\left(1+\frac{y}{x}\right)}{\sqrt{x}\left(1+\frac{\sqrt{y}}{\sqrt{x}}\right)} = \sqrt{x}\, \phi \frac{y}{x}$$

∴ tan u is a homogeneous function of degree $\frac{1}{2}$.

Let $v = \tan u$ by Euler's theorem

$$x \frac{\delta v}{\delta x} + y \frac{\delta v}{\delta y} = \frac{1}{2} v$$

$$\rightarrow x \frac{\delta \tan u}{\delta x} + y \frac{\delta \tan u}{\delta y} = \frac{1}{2} \tan u$$

$$\rightarrow x \sec^2 u \cdot \frac{\delta u}{\delta x} + y \sec^2 u \frac{\delta u}{\delta y} = \frac{1}{2} \tan u$$

$$\rightarrow x \frac{\delta u}{\delta x} + y \frac{\delta u}{\delta y} = \frac{1}{2} \cdot \frac{\tan u}{\sec^2 u} = \frac{1}{2} \sin u \cos u$$

$$\therefore x \frac{\delta u}{\delta x} + y \frac{\delta u}{\delta y} = \frac{1}{4} \sin 2u.$$

(showed)

Improper Integral

i) Is the area under the curve $y = \frac{1}{\sqrt{x}}$ from $x = 0$ to $x = 1$ finite? If so, what is it?

→ Although the graph of $\frac{1}{\sqrt{x}}$ goes to infinity when $x \to 0$, the area between the x axis and this graph is perfectly finite..

Here, $y = \frac{1}{\sqrt{x}}$

Let $x = \sin^2\theta$

$\Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$

Again, $\theta = \sin^{-1}\sqrt{x}$

$x \to 0,\ \theta \to 0$

$x \to 1,\ \theta \to \frac{\pi}{2}$

$$\text{Area} = \int_0^1 y\, dx$$

$$= \int_0^1 \frac{1}{\sqrt{x}}\, dx$$

$$= \int_0^{\pi/2} \frac{2\sin\theta\cos\theta\, d\theta}{\sin\theta}$$

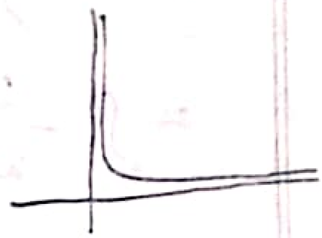
$$= \int_0^{\pi/2} 2\cos\theta\, d\theta$$

$$= 2\left[\sin\theta\right]_0^{\pi/2}$$

$$= 2\left[\sin\frac{\pi}{2} - \sin 0\right]$$

$$= 2(1 - 0)$$

$$= 2 \text{ sq unit}$$



(Ans)

∴ the area under the curve $y = \frac{1}{\sqrt{x}}$ from $x = 0$ to $x = 1$ is finit and it is 2 square unit.

2) Evaluate $\int_2^\alpha \frac{x+3}{(x-1)(x^2+1)} dx$

$\rightarrow \int \frac{x+3}{(x-1)(x^2+1)} dx$

$= \int \left(\frac{-2x-1}{x^2+1} + \frac{2}{x-1} \right) . dx$ —— (I)

Now, $\int \frac{1}{x-1} dx$

$= \int \frac{1}{u} du = \ln u$

$= \ln (x-1)$ —— (II)

Let,
$u = x-1$
$\frac{du}{dx} = 1$
$\therefore du = dx$

Again, $\int \frac{2x+1}{x^2+1} dx$

$= \int \frac{2x\, dx}{x^2+1} + \int \frac{1}{x^2+1} dx$ —— (III)

Hene, $\int \frac{x}{x^2+1} dx$

$= \frac{1}{2} \int \frac{1}{u} du$

$= \frac{1}{2} \ln|u|$

$= \frac{1}{2} \ln|x^2+1|$ —— (IV)

$u = x^2+1$
$\frac{du}{dx} = 2x$
$dx = \frac{1}{2x} du$

Again, $\int \frac{1}{x^2+1} dx = \tan^{-1}(x)$ —— (v)

From (iv) and (v) to (ii):

$$2\int \frac{x}{x^2+1} dx + \int \frac{1}{x^2+1} dx$$

$$= \frac{2\ln(x^2+1)}{2} + \tan^{-1} x$$

$$= \ln(x^2+1) + \tan^{-1} x$$

From (iii) and (ii) to (i):

$$2\int \frac{1}{x-1} dx - \int \frac{2x+1}{x^2+1} dx$$

$$= 2\ln(x-1) - \ln(x^2+1) - \tan^{-1}x$$

$$\therefore \int_2^{\alpha} \frac{x+3}{(x-1)(x^2+1)} dx = \left[2\ln(x-1) - \ln(x^2+1) - \tan^{-1}x\right]_2^{\alpha}$$

$$= 2\ln(\alpha-1) - \ln(\alpha^2+1) - \tan^{-1}\alpha - 2\ln 1 + \ln 3 + \tan^{-1}2$$

$$= \ln(\alpha-1)^2 - \ln(\alpha^2+1) + \ln 3 + \tan^{-1}2 - \tan^{-1}\alpha$$

$$= \ln\frac{3(\alpha-1)^2}{\alpha^2+1} + \tan^{-1}\frac{2-\alpha}{1+2\alpha}$$

Maclaurin's Polynomial

Maclaurin's series:

If $f(x)$, $f'(x)$, $f''(x)$ ---- $f^n(x)$ exists at $x=0$ then $f(x)$ can be expanded in a series like —

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^n}{n!}f^n(0) + \dots$$

① Find Maclaurin's series for e^x.

Sol^n: Let $f(x) = e^x$ $\quad \therefore f(0) = 1$

$f'(x) = e^x$ $\quad \therefore f'(0) = 1$.

$f''(x) = e^x$ $\quad \therefore f''(0) = 1$

Now, Maclaurin's series is —

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots$$

$$\therefore e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$ Ans

② Find Maclaurin's series for $\sin x$.

Sol^n: Let $f(x) = \sin x$ $\quad \therefore f(0) = 0$

$f'(x) = \cos x$ $\quad \therefore f'(0) = 1$

$f''(x) = -\sin x$ $\quad \therefore f''(0) = 0$

$f'''(x) = -\cos x$ $\quad \therefore f'''(0) = -1$.

$f^{iv}(x) = \sin x$ $\quad \therefore f^{iv}(0) = 0$

$f^{v}(x) = \cos x$ $\quad \therefore f^{v}(0) = 1$

Now, Maclaurin's series is —

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \frac{x^5}{5!} f^{v}(0) + \cdots$$

$$\therefore \sin x = 0 + x \cdot 1 + 0 - \frac{x^3}{3!} \cdot 1 + 0 + \frac{x^5}{5!} \cdot 1 + \cdots$$

$$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Ans

③ Find Maclaurin's series for e^{mx}

Solⁿ: Let $f(x) = e^{mx}$ $\qquad \therefore f(0) = 1$

$f'(x) = m e^{mx}$ $\qquad \therefore f'(0) = m$

$f''(x) = m^2 e^{mx}$ $\qquad \therefore f''(0) = m^2$

$f'''(x) = m^3 e^{mx}$ $\qquad \therefore f'''(0) = m^3$

$f^{iv}(x) = m^4 e^{mx}$ $\qquad \therefore f^{iv}(0) = m^4$

Now, Maclaurin's series is —

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$\therefore e^{mx} = 1 + mx + \frac{m^2 x^2}{2!} + \frac{m^3 x^3}{3!} + \cdots$$

$$\therefore e^{mx} = 1 + mx + \frac{(mx)^2}{2!} + \frac{(mx)^3}{3!} + \cdots$$

Ans

(4) Find Maclaurin's series for $\cos x$.

Sol^n: Let $f(x) = \cos x$ $\quad \therefore f(0) = 1$

$f'(x) = -\sin x$ $\quad \therefore f'(0) = 0$

$f''(x) = -\cos x$ $\quad \therefore f''(0) = -1$

$f'''(x) = \sin x$ $\quad \therefore f'''(0) = 0$

$f^{iv}(x) = \cos x$ $\quad \therefore f^{iv}(0) = 1$

Now, Maclaurin's series is –

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \cdots$$

$$\therefore \cos x = 1 + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!} + \cdots$$

$$\therefore \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Ans

(5) Find Maclaurin's series for $\ln(1+x)$

Sol^n: Let $f(x) = \ln(1+x)$ $\quad \therefore f(0) = 0$

$f'(x) = (1+x)^{-1}$ $\quad \therefore f'(0) = 1$

$f''(x) = -(1+x)^{-2}$ $\quad \therefore f''(0) = -1$

$f'''(x) = 2!(1+x)^{-3}$ $\quad \therefore f'''(0) = 2!$

$f^{iv}(x) = -3!(1+x)^{-4}$ $\quad \therefore f^{iv}(0) = -3!$

Now, Maclaurin's series is –

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \cdots$$

$$\therefore \ln(1+x) = 0 + x - 1 \cdot \frac{x^2}{2!} + \frac{x^3}{3!} \cdot 2! - \frac{x^4}{4!} \cdot 3! + \cdots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots$$

Ans

6) Find Maclaurin's series for $\sin 2x$.

Sol^n:

Let $f(x) = \sin 2x$ $\quad\therefore f(0) = 0$

$f'(x) = 2\cos 2x$ $\quad\therefore f'(0) = 2$

$f''(x) = -4\sin 2x$ $\quad\therefore f''(0) = 0$

$f'''(x) = -8\cos 2x$ $\quad\therefore f'''(0) = -8$

$f^{iv}(x) = 16\sin 2x$ $\quad\therefore f^{iv}(0) = 0$

$f^{v}(x) = 32\cos 2x$ $\quad\therefore f^{v}(0) = 32$

$f^{vi}(x) = -64\sin 2x$ $\quad\therefore f^{vi}(0) = 0$

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Maclaurin's series is –

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \frac{x^5}{5!} f^{v}(0) + \frac{x^6}{6!} f^{vi}(0) + \cdots$$

$$= 0 + 2x + 0 - \frac{8.x^3}{3!} + 0 + \frac{32x^5}{5!} + 0 - \frac{128x^7}{7!} + \cdots$$

$$= 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \cdots$$

$$\therefore \sin 2x = 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \cdots$$

Ans.

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(7) Find Maclaurin's Series for $\cos 2x$.

Soln:

Let $f(x) = \cos 2x$ $\quad \therefore f(0) = 1$

$f'(x) = -2\sin 2x$ $\quad \therefore f'(0) = 0$

$f''(x) = -4\cos 2x$ $\quad \therefore f''(0) = -4$

$f'''(x) = 8\sin 2x$ $\quad \therefore f'''(0) = 0$

$f^{iv}(x) = 16\cos 2x$ $\quad \therefore f^{iv}(0) = 16$

$f^{v}(x) = -32\sin 2x$. $\quad \therefore f^{v}(0) = 0$

$f^{vi}(x) = -64\cos 2x$. $\quad \therefore f^{vi}(0) = -64$.

Maclaurin's series is –

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0)$$

$$\frac{x^5}{5!}f^{v}(x) + \frac{x^6}{6!}f^{vi}(x) + \cdots$$

$$= 1 + 0 - \frac{4x^2}{2!} + 0 + \frac{16x^4}{4!} + 0 - \frac{64x^4}{4!} + \cdots$$

$$\therefore \cos 2x = 1 - \frac{4x^2}{2!} + \frac{16x^4}{4!} - \frac{64x^4}{4!} + \cdots$$ Ans

(8) Find Maclaurin's series for a^x.

Soln:

Let, $f(x) = a^x$. $\quad \therefore f(0) = 1$

$f'(x) = a^x \ln a$. $\quad \therefore f'(0) = \ln a$.

$f''(x) = a^x(\ln a)^2$ $\quad \therefore f''(0) = (\ln a)^2$

$f'''(x) = a^x(\ln a)^3$ $\quad \therefore f'''(0) = (\ln a)^3$

$f^{iv}(x) = a^x(\ln a)^4$ $\quad \therefore f^{iv}(0) = (\ln a)^4$

Maclaurin's series is —

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0)$$

$$\therefore a^x = 1 + x \ln a + \frac{x^2}{2!}(\ln a)^2 + \frac{x^3}{3!}(\ln a)^3 + \frac{x^4}{4!}(\ln a)^4$$

Ans

Taylor Polynomial

Taylor Series:

a Taylor series is a representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point.

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \frac{f^{iv}(x_0)}{4!}(x-x_0)^4 + \cdots\cdots$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$

(1) Find Taylor series for $f(x) = \ln x$ at $x = 2$.

Sol^n:

Here, $x_0 = 2$.

Let, $f(x) = \ln x$. $\quad \therefore f(2) = \ln 2$.

$f'(x) = \frac{1}{x}$ $\quad \therefore f'(2) = \frac{1}{2}$.

$f''(x) = -\frac{1}{x^2}$ $\quad \therefore f''(2) = -\frac{1}{4}$.

$f'''(x) = 2\frac{1}{x^3}$ $\quad \therefore f'''(2) = \frac{2}{8} = \frac{1}{4}$.

$f^{iv}(x) = -6 \cdot \frac{1}{x^4}$ $\quad \therefore f^{iv}(2) = \frac{-6}{16} = -\frac{3}{8}$

— . — — — .

the Taylor series at point $x_0 = 2$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \frac{f^{iv}(x_0)}{4!}(x-x_0)^4 \ldots$$

$$\cancel{f(\ln x)} = f(2) + f'(2)(x-2) + f''(2)(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \frac{f^{iv}(2)}{4!}(x-2)^4 \ldots$$

Therefore, we can write the answer as -

$$f(\ln 2) = \ln(2) + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3 - \frac{1}{64}(x-2)^4 + \ldots$$

<u>Ans</u>

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Some important formulas

* $\int x^n dx = \frac{x^{n+1}}{n+1} + c$
* $\int dx = x + c$
* $\int \frac{1}{x} dx = \ln x + c$
* $\int e^x dx = e^x$
* $\int a^x dx = \frac{a^x}{\ln a} + c$
* $\int \sin x \, dx = -\cos x + c$
* $\int \cos x \, dx = \sin x + c$
* $\int \sec^2 x \, dx = \tan x + c$
* $\int \operatorname{cosec}^2 x \, dx = -\cot x + c$
* $\int \sec x \tan x \, dx = \sec x + c$
* $\int \operatorname{cosec} x \cdot \cot x \, dx = -\operatorname{cosec} x + c$
* $\int \operatorname{cosec} x \, dx = \ln\left|\tan \frac{x}{2}\right| + c$
* $\int \sec x \, dx = \ln|\sec x + \tan x| + c$
* $\int \tan x \, dx = \ln|\sec x| + c$
* $\int \cot x \, dx = \ln|\sin x| + c$

* $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$
* $\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + c$
* $\int \frac{dx}{1+x^2} = \tan^{-1} x + c$
* $\int \frac{dx}{1+x^2} = -\cot^{-1} x + c$
* $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + c$
* $\int \frac{dx}{x\sqrt{x^2-1}} = -\operatorname{cosec}^{-1} x + c$

$$* \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

$$* \int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$$

$$* \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$$

$$* \int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left|\frac{\sqrt{x^2+a^2}+x}{a}\right| + c$$

$$* \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$$

$$* \int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|\frac{\sqrt{x^2-a^2}+x}{a}\right| + c$$

$$* \int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$$

$$* \int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c$$

$$* \int \sqrt{a^2+x^2}\,dx = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + c$$

① $\int \sin 2x + e^{5x}\, dx$.

Soln: Given that,

$$\int (\sin 2x + e^{5x})\, dx$$

$$= \int \sin 2x\, dx + \int e^{5x}\, dx.$$

$$= \frac{-\cos 2x}{2} + \frac{1}{5} e^{5x} + c.$$

$$= -\frac{1}{2} \cos 2x + \frac{1}{5} e^{5x} + c.$$

① $\int \frac{8^{1+x} + 4^{1-x}}{2^x} dx$

Soln: Given that, $\int \frac{8^{1+x} + 4^{1-x}}{2^x} dx$

$$= \int \frac{(2^3)^{1+x} + (2^2)^{1-x}}{2^x} dx.$$

$$= \int \frac{2^{3+3x} + 2^{2-2x}}{2^x} dx$$

$$= \int \frac{2^{3+3x}}{2^x} dx + \int \frac{2^{2-2x}}{2^x} dx$$

$$= \int 2^{3+2x} dx + \int 2^{2-3x} dx.$$

$$= \int 8 \cdot 2^{2x} . dx + \int 4 \cdot 2^{-3x}$$

$$= 8 \frac{2^{2x}}{2 \ln 2} + 4 \cdot \frac{2^{-3x}}{-3 \ln 2} + c$$

$$= \frac{4}{\ln 2} \left[2^{2x} - \frac{1}{3} 2^{-3x}\right] + c$$

Ans

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(3) $\int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} dx.$

Soln: Given that, $\int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} dx$

$$= \int \frac{(\sin^4 x + \cos^4 x)(\sin^4 x - \cos^4 x)}{1 - 2\sin^2 x \cos^2 x} dx$$

$$= \int \frac{\{(\sin^2 x)^2 + (\cos^2 x)^2 + 2\sin^2 x \cos^2 x - 2\sin^2 x \cos^2 x)(\sin^4 x - \cos^4 x)}{1 - 2\sin^2 x \cos^2 x}$$

$$= \int \frac{\{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x\}\{\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x)}{1 - 2\sin^2 x . \cos^2 x.}$$

$$= \int \frac{(1 - 2\sin^2 x . \cos^2 x)(\sin^2 x - \cos^2 x)}{1 - 2\sin^2 x . \cos^2 x} dx$$

$$= \int -\cos 2x \ dx$$

$$= -\frac{1}{2}\sin 2x + c$$ Ans.

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③ $\int \frac{dx}{x\{10+7\ln x+(\ln x)^2\}}$

Soln: Given that, $\int \frac{dx}{x\{10+7\ln x+(\ln x)^2\}}$

$= \int \frac{dz}{10+7z+z^2}$

Let, $\ln x = z$

$\therefore \frac{1}{x}dx = dz$

$= \int \frac{dz}{z^2+2\cdot z\cdot \frac{7}{2}+\left(\frac{7}{2}\right)^2+10-\frac{49}{4}}$

$= \int \frac{dz}{\left(z+\frac{7}{2}\right)^2-\frac{9}{4}} = \int \frac{dz}{(z+7/2)^2-\left(\frac{3}{2}\right)^2}$

$= \frac{1}{2\cdot\frac{3}{2}} \ln\left|\frac{z+7/2-3/2}{z+7/2+3/2}\right| + c$

$= \frac{1}{3}\ln\left|\frac{z+2}{z+5}\right| + c$

$= \frac{1}{3}\ln\left|\frac{\ln x+2}{\ln x+5}\right| + c$ Ans

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(4) $\int \frac{e^x}{e^{2x}+2e^x+5}\,dx$

Sol^n:

Given that, $\int \frac{e^x}{e^{2x}+2e^x+5}\,dx$

$$= \int \frac{e^x . dx}{(e^x)^2 + 2.e^x. + 5}$$

Let, $e^x = z$

$e^x . dx = dz$

$$= \int \frac{dz}{z^2+2z+5}$$

$$= \int \frac{dz}{z^2+2.z.1+1^2+4}$$

$$= \int \frac{dz}{(z+1)^2+2^2}$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{z+1}{2}\right)+c$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{e^x+1}{2}\right)+c$$ Ans

(5) $\int \frac{dx}{(x^2-16)\sqrt{x+1}}$

Sol^n: Given that, $\int \frac{dx}{(x^2-16)\sqrt{x+1}}$

$$= \int \frac{2z\,dz}{\{(z^2-1)^2-16\}z}$$

Let, $x+1 = z^2$

$\Rightarrow dx = 2z\,dz$

$$= \int \frac{2\,dz}{(z^2-1+4)(z^2-1-4)}$$

$$= 2\int \frac{dz}{(z^2+3)(z^2-5)}$$

$$= 2\int \left\{ \frac{1}{8(z^2-5)} - \frac{1}{8(z^2+3)} \right\} dz$$

$$= \frac{1}{4}\int \frac{dz}{z^2-(\sqrt{5})^2} - \frac{1}{4}\int \frac{dz}{z^2+(\sqrt{3})^2}$$

$$= \frac{1}{4}\cdot\frac{1}{2\sqrt{5}} \ln\left|\frac{z-\sqrt{5}}{z+\sqrt{5}}\right| - \frac{1}{4}\cdot\frac{1}{\sqrt{3}} \tan^{-1}\frac{z}{\sqrt{3}} + c$$

$$= \frac{1}{8\sqrt{5}} \ln\left|\frac{\sqrt{x+1}-\sqrt{5}}{\sqrt{x+1}+\sqrt{5}}\right| - \frac{1}{4\sqrt{3}} \tan^{-1}\frac{\sqrt{x+1}}{\sqrt{3}} + c$$

Ans

6) $\int \frac{dx}{3+2\cos x}$

Solⁿ: Given that, $\int \frac{dx}{3+2\cos x}$

$$= \int \frac{dx}{3+2\left(\frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right)}$$

$$= \int \frac{dx}{\frac{3+3\tan^2 x/2 + 2 - 2\tan^2 x/2}{1+\tan^2 x/2}}$$

$$= \int \frac{(1+\tan^2 x/2)\, dx}{5+\tan^2 x/2}$$

$$= \int \frac{\sec^2 x/2 \, dx}{5+\tan^2 x/2}$$

Let, $\tan\frac{x}{2} = z$

$\Rightarrow \frac{1}{2}\sec^2 x/2 \, dx = dz$

$$= 2\int \frac{\frac{1}{2}\sec^2 x/2 \, dx}{5+\tan^2 x/2}$$

$$= 2\int \frac{dz}{5+z^2}$$

$$= 2\cdot\frac{1}{\sqrt5}\tan^{-1}\frac{z}{\sqrt5} + c$$

$$= \frac{2}{\sqrt5}\tan^{-1}\left(\frac{\tan x/2}{\sqrt5}\right) + c \quad \text{Ans}$$

(7) $\int \frac{\cos x}{2\cos x + 3}\,dx$.

Soln: Given that, $\int \frac{\cos x}{2\cos x+3}\,dx$

$$= \int \frac{\frac{1}{2}(2\cos x+3) - \frac{3}{2}}{2\cos x + 3}\,dx.$$

$$= \int \frac{1}{2}dx - \int \frac{3}{2}\cdot\frac{dx}{2\cos x+3}$$

$$= \frac{1}{2}\cdot x - \frac{3}{2}\int \frac{dx}{2\left(\frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right)+3}$$

$$= \frac{x}{2} - \frac{3}{2}\int \frac{(1+\tan^2 x/2)\,dx}{2-2\tan^2 x/2 + 3 + 3\tan^2 x/2}$$

$$= \frac{x}{2} - \frac{3}{2}\int \frac{\sec^2\frac{x}{2}\,dx}{\tan^2 x/2 + 5}$$

Let, $\tan\frac{x}{2} = z$

$\Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx = dz$

$$= \frac{x}{2} - 3\int \frac{dz}{z^2+(\sqrt5)^2}$$

$$= \frac{x}{2} - 3\cdot\frac{1}{\sqrt5}\tan^{-1}\frac{z}{\sqrt5} + c$$

$$= \frac{x}{2} - \frac{3}{\sqrt5}\tan^{-1}\left(\frac{\tan x/2}{\sqrt5}\right) + c \quad \text{Ans}$$

(8) $\displaystyle\int \frac{dx}{4\cos x + 5}$

Soln: Let, $I = \displaystyle\int \frac{dx}{4\cos x + 5}$

$$\Rightarrow I = \int \frac{dx}{4\left(\frac{1-\tan^2(\frac{x}{2})}{1+\tan^2(\frac{x}{2})}\right)+5}$$

$$\left[\because \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right]$$

$$\Rightarrow I = \int \frac{1+\tan^2(\frac{x}{2})}{4-4\tan^2(\frac{x}{2})+5+5\tan^2(\frac{x}{2})}\,dx$$

$$= \int \frac{\sec^2(\frac{x}{2})}{\tan^2(\frac{x}{2})+9}\,dx$$

Let, $u = \tan\left(\frac{x}{2}\right)$

$\Rightarrow \frac{du}{dx} = \sec^2\left(\frac{x}{2}\right) \times \frac{1}{2}$

$\therefore \sec^2\left(\frac{x}{2}\right) \cdot dx = 2du$

$$\therefore \quad I = \int \frac{2du}{u^2+9}$$

$$= 2\int \frac{du}{u^2+3^2}$$

$$= 2 \cdot \frac{1}{3} \tan^{-1}\left(\frac{u}{3}\right) + c$$

$$\left[\because \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{3}\right) + c$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\left(\tan\frac{x}{2}\right)\right) + c$$

Ans

⑨ $\int e^{2x} \sin^3(x)\, dx$

Solⁿ: Given that,

$$I = \int e^{2x} \cdot \sin^3 x \cdot dx$$

We know that, $\sin 3x = 3\sin x - 4\sin^3 x$

$$\Rightarrow 4\sin^3 x = 3\sin x - \sin 3x$$

$$\therefore \sin^3 x = \frac{3\sin x - \sin 3x}{4}$$

$$I = \int \frac{e^{2x}(3\sin x - \sin 3x)}{4}\, dx$$

$$= \frac{3}{4}\int e^{2x} \sin x\, dx - \frac{1}{4}\int e^{2x} \sin 3x\, dx$$

$$= \frac{3}{4} I' + \frac{1}{4} I''$$

Let, $I' = \int e^{2x} \sin x\, dx$

$$\Rightarrow I' = e^{2x}\int \sin x\, dx - \int 2e^{2x}\left(\int \sin x\, dx\right) dx$$

$$\Rightarrow I' = e^{2x} \cdot (-\cos x) - \int 2e^{2x}(-\cos x)\, dx$$

$$\Rightarrow I' = -e^{2x}\cos x + 2\int e^{3x} \cos x\, dx$$

$$\Rightarrow I' = -e^{2x}\cos x + 2\left\{e^{2x}\int \cos x\,dx - \int 2e^{2x}\left(\int \cos x\,dx\right)dx\right\}$$

$$\Rightarrow I' = -e^{2x}\cos x + 2\left\{e^{2x}\sin x - 2\int e^{2x}\sin x\,dx\right\}$$

$$\Rightarrow I' = -e^{2x}\cos x + 2e^{2x}\sin x - 4I'$$

$$\Rightarrow 5I' = 2e^{2x}\sin x - e^{2x}\cos x$$

$$\therefore I' = \frac{2e^{2x}\sin x - e^{2x}\cos x}{5}.$$

Now, $I'' = \int e^{2x}\sin 3x\,dx$.

$$\Rightarrow I'' = e^{2x}\int \sin 3x\,dx - \int 2e^{2x}\left(\int \sin 3x\,dx\right)dx$$

$$= e^{2x}\cdot\left(-\frac{\cos 3x}{3}\right) - \int 2e^{2x}\cdot\left(-\frac{\cos 3x}{3}\right)dx$$

$$= -\frac{1}{3}e^{2x}\cdot\cos 3x + \frac{2}{3}\int e^{2x}\cdot\cos 3x\,dx$$

$$= -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\left\{e^{2x}\int \cos 3x\,dx - \int 2e^{2x}\left(\int \cos 3x\,dx\right)dx\right\}$$

$$= -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\left\{e^{2x}\cdot\frac{\sin 3x}{3} - 2\int e^{2x}\cdot\frac{\sin 3x}{3}\,dx\right\}$$

$$= -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{9}\left\{e^{2x}\cdot\sin 3x - 2\int e^{2x}\cdot\sin 3x\,dx\right\}$$

$$\Rightarrow I'' = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{9}e^{2x}\sin 3x - \frac{4}{9}I''$$

$$\Rightarrow \frac{13}{9}I'' = \frac{2}{9}e^{2x}\sin 3x - \frac{1}{3}e^{2x}\cos 3x$$

$$\therefore I'' = \frac{2}{13}e^{2x}\sin 3x - \frac{3}{13}e^{2x}\cos 3x$$

So, $I = \frac{3}{4} I' + \frac{1}{4} I''$

$$= \frac{3}{4}\left(\frac{2e^{2x}\sin x - e^{2x}\cos x}{5}\right) + \frac{1}{4}\left(\frac{2}{13}e^{2x}\sin 3x - \frac{3}{13}e^{2x}\cos 3x\right) + c$$

$$= \frac{3}{10}e^{2x}\sin x - \frac{3}{20}e^{2x}\cos x + \frac{1}{26}e^{2x}\sin 3x - \frac{3}{52}e^{2x}\cos 3x + c.$$

Ans

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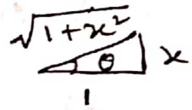
(10) $\displaystyle\int \frac{x(\tan^{-1} x)^2}{(1+x^2)^{\frac{1}{2}}\,(1+x^2)}\,dx$.

Soln: Given, $\displaystyle\int \frac{x(\tan^{-1}x)^2}{(1+x^2)^{\frac{1}{2}}(1+x^2)}\,dx$.

$$= \int \frac{x(\tan^{-1}x)^2}{(1+x^2)^{\frac{3}{2}}}\,dx.$$

Let, $x = \tan\theta$.

$\theta = \tan^{-1}x$



Again, $x = \tan\theta$.

$dx = \sec^2\theta\, d\theta$.

Now, $\displaystyle\int \frac{\tan\theta \cdot \theta^2 . \sec^2\theta\, d\theta}{(1+\tan^2\theta)^{3/2}}$.

$$= \int \frac{\theta^2 . \tan\theta . \sec^2\theta}{(\sec^2\theta)^{3/2}}\,d\theta.$$

$$= \int \frac{\theta^2 \tan\theta .\ d\theta}{\sec\theta}.$$

$$= \int \theta^2 \sin\theta\, d\theta.$$

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$$= \theta^2 \int \sin\theta \, d\theta - \int 2\theta \left(\int \sin\theta \, d\theta\right) d\theta$$

$$= \theta^2(-\cos\theta) - \int 2\theta(-\cos\theta) \, d\theta$$

$$= -\theta^2 \cos\theta + 2\int \theta \cos\theta \, d\theta$$

$$= -\theta^2 \cos\theta + 2\left\{\theta \cdot \int \cos\theta \, d\theta - \int 1\left(\int \cos\theta \, d\theta\right) d\theta\right\}$$

$$= -\theta^2 \cos\theta + 2\left\{\theta \sin\theta - \int \sin\theta \, d\theta\right\}$$

$$= -\theta^2 \cos\theta + 2\left(\theta \sin\theta - \cos\theta\right) + c$$

$$= 2\theta \sin\theta - \theta^2 \cos\theta - 2\cos\theta + c$$

$$= 2(\tan^{-1}x) \cdot \frac{x}{\sqrt{1+x^2}} - (\tan^{-1}x)^2 \cdot \frac{1}{\sqrt{1+x^2}} - 2 \cdot \frac{1}{\sqrt{1+x^2}} +$$

$$= \frac{1}{\sqrt{1+x^2}} \left\{2x(\tan^{-1}x) - (\tan^{-1}x)^2 - 2\right\} + c$$

<u>Ans</u>

(11) $\displaystyle\int \frac{x\,dx}{(x+1)\sqrt{x^2+1}}$

Sol^n: Let, $I = \displaystyle\int \frac{x\,dx}{(x+1)\sqrt{x^2+1}}$

$$I = \int \frac{x+1-1}{(x+1)\sqrt{x^2+1}}\,dx$$

$$I = \int \frac{1}{\sqrt{1+x^2}}\,dx - \int \frac{1}{(x+1)\sqrt{x^2+1}}\,dx$$

Let, $J' = \displaystyle\int \frac{1}{\sqrt{1+x^2}}\,dx$

$$= \ln\left|\sqrt{1+x^2}+x\right| + c \quad \text{—— (i)}$$

$$\left[\because \int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left|\frac{\sqrt{a^2+x^2}+x}{a}\right| + c\right]$$

Let, $I'' = \displaystyle\int \frac{1}{(x+1)\sqrt{x^2+1}}\,dx$

Let, $x+1 = \frac{1}{z} = z^{-1}$

$\Rightarrow 1 = -\frac{1}{z^2} \cdot \frac{dz}{dx}$

$\therefore dx = -\frac{dz}{z^2}$

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Now, $$I'' = \int \frac{-\frac{dz}{z^2}}{\frac{1}{z}\sqrt{\left(\frac{1}{z}-1\right)^2+1}}$$

$$= -\int \frac{dz}{z\sqrt{\left(\frac{1-z}{z}\right)^2+1}}$$

$$= -\int \frac{dz}{\sqrt{\frac{z^2(1-z)^2}{z^2}+z^2}}$$

$$= -\int \frac{dz}{\sqrt{1-2z+z^2+z^2}}$$

$$= -\int \frac{dz}{\sqrt{2z^2-2z+1}}$$

$$= -\int \frac{dz}{\sqrt{(\sqrt{2}z)^2-2\cdot\sqrt{2}z\cdot\frac{1}{\sqrt{2}}+\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}}$$

$$= -\int \frac{dz}{\sqrt{\left(\sqrt{2}z-\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}}$$

$$\left[\because \int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|\frac{\sqrt{x^2+a^2}+x}{a}\right| + c\right]$$

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$$= -\frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{2z^2-2z+1} + \sqrt{2}z - \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} \right| + c$$

$$= -\frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{2}\sqrt{z^2-z+\frac{1}{2}} + \sqrt{2}z - \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} \right| + c$$

$$= -\frac{1}{\sqrt{2}} \ln \left| 2\sqrt{z^2-z+\frac{1}{2}} + 2z - 1 \right| + c$$

$$= -\frac{1}{\sqrt{2}} \ln \left| 2\sqrt{\frac{1}{(1+x)^2} - \frac{1}{1+x} + \frac{1}{2}} + 2\cdot\frac{1}{1+x} - 1 \right| + c$$

$$= -\frac{1}{\sqrt{2}} \ln \left| 2\sqrt{\frac{2-2-2x+1+2x+x^2}{2(1+x)^2}} + \frac{2}{1+x} - 1 \right|$$

$$= -\frac{1}{\sqrt{2}} \ln \left| 2\sqrt{\frac{1+x^2}{2(1+x)^2}} + \frac{2}{1+x} - 1 \right| + c$$

$$= -\frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{2}\sqrt{1+x^2}}{(1+x)} + \frac{2}{1+x} - 1 \right| + c$$

$$= -\frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{2+2x^2} + 2 - 1 - x}{1+x} \right| + c$$

$$= -\frac{1}{\sqrt{2}} \ln \left| \frac{1-x+\sqrt{2x^2+2}}{1+x} \right| + c$$

So, $I = I' - I''$

$$= \ln\left|\sqrt{1+x^2} + x\right| + \frac{1}{\sqrt{2}} \ln\left|\frac{1-x+\sqrt{2x^2+2}}{1+x}\right| + c$$

Ans

Definite Integral as the limit of a sum

① Given,

$$\lim_{n\to\infty}\left[\frac{n}{n^2}+\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots+\frac{n}{n^2+(n-1)^2}\right]$$

General term of series $= \dfrac{n}{n^2+r^2}$.

Now Let, $I = \displaystyle\lim_{n\to\infty}\left[\frac{n}{n^2+0^2}+\frac{n}{n^2+1^2}+\cdots+\frac{n}{n^2+(n-1)^2}\right]$

$$= \lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{n}{n^2+r^2}$$

$$= \lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{n}{n^2\left(1+\frac{r^2}{n^2}\right)}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\frac{1}{1+\left(\frac{r}{n}\right)^2}$$

Substitute $\frac{1}{n}\to dx$, $\frac{r}{n}\to x$, $\displaystyle\lim_{n\to\infty}\to\int$

Lower Limit $= \displaystyle\lim_{n\to\infty}\frac{1}{n} = 0$

Upper limit $= \displaystyle\lim_{n\to\infty}\frac{n-1}{n} = 1$

Given limit become $I = \int_0^1 \frac{1}{1+x^2}\, dx$

$$= \left[\tan^{-1} x\right]_0^1$$

$$= \tan^{-1} 1 - \tan^{-1} 0$$

$$= \frac{\pi}{4} \quad \text{Ans}$$

(3) Given,

$$\lim_{n\to\infty}\left[\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \cdots \frac{n}{n^2+n^2}\right]$$

General term of series $= \frac{n}{n^2+r^2}$

Now Let,

$$I = \lim_{n\to\infty}\left[\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \cdots + \frac{n}{n^2+n^2}\right]$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{n}{n^2+r^2}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{1+\left(\frac{r}{n}\right)^2}$$

Substitute $\frac{1}{n} \to dx$, $\frac{r}{n} \to x$, $\lim_{n\to\infty} \to \int$

Lower limit $= \lim_{n\to\infty} \frac{1}{n} = 0$

Upper limit $= \lim_{n\to\infty} \frac{n}{n} = 1$.

Given Limit become,

$$I = \int_0^1 \frac{1}{1+x^2} \, dx$$

$$= \left[\tan^{-1} x\right]_0^1$$

$$= \tan^{-1} 1 - \tan^{-1} 0$$

$$= \frac{\pi}{4}$$ Ans

(3)

Given,

$$\lim_{n\to\infty}\left[\frac{\sqrt{n}}{n^{3/2}}+\frac{\sqrt{n}}{(n+3)^{3/2}}+\frac{\sqrt{n}}{(n+6)^{3/2}}+\cdots+\frac{\sqrt{n}}{\{n+3(n-1)\}^{3/2}}\right]$$

General term of series $=\dfrac{\sqrt{n}}{(n+3r)^{3/2}}$

Now, Let, $I=\lim_{n\to\infty}\left[\frac{\sqrt{n}}{n^{3/2}}+\frac{\sqrt{n}}{(n+3)^{3/2}}+\frac{\sqrt{n}}{(n+6)^{3/2}}+\cdots+\frac{\sqrt{n}}{\{n+3(n-1)\}^{3/2}}\right]$

$$=\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{\sqrt{n}}{(n+3r)^{3/2}}$$

$$=\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{\sqrt{n}}{\{n(1+3.\frac{r}{n})\}^{3/2}}$$

$$=\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{\cancel{\sqrt{n}}}{n\cancel{\sqrt{n}}\{1+3(\frac{r}{n})\}^{3/2}}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\frac{1}{\{1+3(\frac{r}{n})\}^{3/2}}$$

substitute $\frac{1}{n}\to dx$, $\frac{r}{n}\to x$ $\quad \lim_{n\to\infty}\to\int$

Lower limit $=\lim_{n\to\infty}\frac{0}{n}=0$

Upper limit $=\lim_{n\to\infty}\frac{n-1}{n}=1$

Given limit become $\quad I=\displaystyle\int_0^1\frac{dx}{(1+3x)^{3/2}}$

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$$I = \int_0^1 \frac{1}{(3x+1)^{3/2}}\,dx$$

Let, $z = 3x+1$

$$\frac{dz}{dx} = 3$$

$$dx = \frac{dz}{3}$$

if, $x = 0$

$z = 1$

and $x = 1$

$z = 4$

$$= \int_1^4 \frac{dz}{3 \cdot z^{3/2}}$$

$$= \frac{1}{3}\int_1^4 z^{-3/2}\,dz$$

$$= \frac{1}{3}\left[\frac{z^{-3/2+1}}{-3/2+1}\right]_1^4$$

$$= \frac{1}{3}\left[\frac{z^{-\frac{1}{2}}}{-\frac{1}{2}}\right]_1^4$$

$$= \frac{1}{3}\left[\frac{-2}{\sqrt{z}}\right]_1^4 =$$

$$= \frac{1}{3}\left(\frac{-2}{2} + \frac{2}{1}\right) = \frac{1}{3}(-1+2)$$

$$= \frac{1}{3}$$

(4) Given,

$$\lim_{n\to\infty}\left[\frac{1^2}{1^3+n^3}+\frac{2^2}{2^3+n^3}+\frac{3^2}{3^3+n^3}+\cdots+\frac{n^2}{n^3+n^3}\right]$$

$\therefore$ General term of series $= \dfrac{r^2}{r^3+n^3}$

Now let,

$$I=\lim_{n\to\infty}\left[\frac{1^2}{1^3+n^3}+\frac{2^2}{2^3+n^3}+\cdots+\frac{n^2}{n^3+n^3}\right]$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^2}{r^3+n^3}$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^2}{n^3\left(1+\frac{r^3}{n^3}\right)}$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\frac{\left(\frac{r}{n}\right)^2}{1+\left(\frac{r}{n}\right)^3}$$

Substitute $\frac{1}{n}\to dx$, $\frac{r}{n}\to x$, $\lim_{n\to\infty}\to\int$

Lower limit $=\lim_{n\to\infty}\frac{1}{n}=0$

Upper limit $=\lim_{n\to\infty}\frac{n}{n}=1$

Given Limit become,

$$I = \int_0^1 \frac{x^2}{1+x3}\, dx$$

$$= \frac{1}{3}\int_1^2 \frac{dz}{z}$$

$$= \frac{1}{3}\left[\ln|z|\right]_1^2$$

$$= \frac{1}{3}\left(\ln 2 - \ln 1\right)$$

$$= \frac{1}{3}\ln 2 .$$ Ans

Let, $1+x^3 = z$

$3x^2\,dx = dz$

when $x = 0$;

$z = 1$

when $x = 1$

$z = 2$

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(5) Given,

$$\lim_{n\to\infty}\left[\frac{1}{n}+\frac{\sqrt{n^2-1^2}}{n^2}+\cdots+\frac{\sqrt{n^2-(n-1)^2}}{n^2}\right]$$

$\therefore$ General term of series $= \dfrac{\sqrt{n^2-r^2}}{n^2}$

Now, Let,

$$I=\lim_{n\to\infty}\left[\frac{\sqrt{n^2-0^2}}{n^2}+\frac{\sqrt{n^2-1^2}}{n^2}+\cdots+\frac{\sqrt{n^2-(n-1)^2}}{n^2}\right]$$

$$=\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{\sqrt{n^2-r^2}}{n^2}$$

$$=\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{\sqrt{n^2\left(1-\frac{r^2}{n^2}\right)}}{n^2}$$

$$=\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{n}\sqrt{1-\left(\frac{r}{n}\right)^2}$$

Substitute $\frac{1}{n}\to dx$, $\frac{r}{n}\Rightarrow x$, $\lim\limits_{n\to\infty}\to\int$

Lower limit $=\lim\limits_{n\to\infty}=\dfrac{0}{n}=0$

Upper limit $=\lim\limits_{n\to\infty}=\dfrac{n-1}{n}=1$

Given limit become.

$$I = \int_0^1 \sqrt{1-x^2}$$

$$= \left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x\right]_0^1$$

$$= \frac{1}{2}\sin^{-1}1 - \frac{1}{2}\sin^{-1}0$$

$$= \frac{1}{2} \times \frac{\pi}{2} - 0$$

$$= \frac{\pi}{4}$$ Ans

6

Given,

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{2n-1^2}}+\frac{1}{\sqrt{4n-2^2}}+\cdots+\frac{1}{n}\right]$$

$$=\lim_{n\to\infty}\left[\frac{1}{\sqrt{2.1n-1^2}}+\frac{1}{\sqrt{2.2n-2^2}}+\cdots+\frac{1}{\sqrt{2.n.n-n^2}}\right]$$

$\therefore$ General term of series $= \dfrac{1}{\sqrt{2rn-r^2}}$

Now. Let,

$$I=\lim_{n\to\infty}\left[\frac{1}{\sqrt{2n-1^2}}+\frac{1}{\sqrt{4n-2^2}}+\cdots+\frac{1}{n}\right]$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{2rn-r^2}}$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{n^2\left\{2\left(\frac{r}{n}\right)-\left(\frac{r}{n}\right)^2\right\}}}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{\sqrt{2\left(\frac{r}{n}\right)-\left(\frac{r}{n}\right)^2}}$$

Substitute $\frac{1}{n}\to dx$, $\frac{r}{n}\to x$, $\lim\limits_{n\to\infty}\to\int$

Lower limit. $=\lim\limits_{n\to\infty}\dfrac{1}{n}=0$

Upper limit $=\lim\limits_{n\to\infty}\dfrac{n}{n}=1$

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Given limit become

$$I = \int_0^1 \frac{1}{\sqrt{2x - x^2}}\, dx$$

$$= \int_0^1 \frac{dx}{\sqrt{1-(1-2x+x^2)}}$$

$$= \int_0^1 \frac{dx}{\sqrt{1-(1-x)^2}} \qquad \left[\because \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + c\right]$$

$$= -\left[\sin^{-1}(1-x)\right]_0^1$$

$$= -\left(\sin^{-1}0 - \sin^{-1}1\right)$$

$$= -\left(0 - \frac{\pi}{2}\right)$$

$$= \frac{\pi}{2}$$

Ans

3, 4, 6

(8) Apply the reduction formula for $\int \cos^n x\,dx$ and then evaluate $\int \cos^8 x.dx$

Soln: Let, $I_n = \int \cos^n x\,dx$

$$\Rightarrow I_n = \int \cos^{n-1} x \, . \cos x . dx$$

$$= \left\{ \cos^{n-1} x \int \cos x \, dx - \int \left\{ \frac{d}{dx}(\cos^{n-1} x) \int \cos x . dx \right\} dx \right.$$

$$= \cos^{n-1} x . \sin x - \int (n-1)\cos^{n-2} x \, (-\sin x) . \sin x \; dx$$

$$= \cos^{n-1} x . \sin x + (n-1) \int \cos^{n-2} x \, (1-\cos^2 x)\, dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx - (n-1)\int \cos^n x \, dx .$$

$$= \cos^{n-1} x . \sin x + (n-1) I_{n-2} - (n-1) I_n$$

$$\Rightarrow I_n + (n-1) I_n = \cos^{n-1} x . \sin x + (n-1) I_{n-2} .$$

$$\Rightarrow I_n (1+n-1) = \sin x . \cos^{n-1} x + (n-1) I_{n-2} .$$

$$\Rightarrow I_n = \frac{\sin x . \cos^{n-1} x}{n} + \frac{n-1}{n} \int \cos^{n-2} x . dx .$$

Now, Let $I = \int \cos^8 x . dx$ ------ (i)

From the reduction formula,

$$I = \frac{\sin x . \cos^7 x}{8} + \frac{7}{8} \int \cos^6 x \, dx$$

$$= \frac{\sin x . \cos^7 x}{8} + \frac{7}{8} \left[\frac{\sin x . \cos^5 x}{6} + \frac{5}{6} \int \cos^4 x \, dx \right]$$

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$$= \frac{\sin x . \cos^7 x}{8} + \frac{7\sin x . \cos^5 x}{48} + \frac{35}{48}\int \cos^4 x . dx$$

$$= \frac{\sin x . \cos^7 x}{8} + \frac{7\sin x . \cos^5 x}{48} + \frac{35}{48}\left[\frac{\sin x . \cos^3 x}{4} + \frac{3}{4}\int \cos^2 x . dx\right]$$

$$= \frac{\sin x . \cos^7 x}{8} + \frac{7\sin x . \cos^5 x}{48} + \frac{35\sin x . \cos^3 x}{192} + \frac{105}{192}\int \cos^2 x . dx$$

$$= \frac{\sin x . \cos^7 x}{8} + \frac{7\sin x . \cos^5 x}{48} + \frac{35\sin x . \cos^3 x}{192} + \frac{105}{192}\left[\frac{\sin x . \cos x}{2} + \frac{1}{2}\int dx\right]$$

$$= \frac{\sin x . \cos^7 x}{8} + \frac{7\sin x . \cos^5 x}{48} + \frac{35\sin x . \cos^3 x}{192} + \frac{35\sin x . \cos x}{128} + \frac{35x}{128} + c .$$

Ans

⑨ Apply the reduction formula for $\int \sin^n x \, dx$ and the evaluate $\int \sin^8 x \, dx$.

Sol^n:

Let $I_n = \int \sin^n x \, dx$

$= \int \sin^{n-1} x \cdot \sin x \, dx$

$= \sin^{n-1} x \int \sin x - \int \left\{ \frac{d}{dx}(\sin^{n-1} x) \int \sin x \, dx \right\} dx$

$= -\sin^{n-1} x \cdot \cos x - \int (n-1) \sin^{n-2} x \cdot \cos x \cdot (-\cos x) \, dx$

$= -\cos x \cdot \sin^{n-1} x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x) \, dx$

$= -\cos x \cdot \sin^{n-1} x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx$

$I_n = -\cos x \cdot \sin^{n-1} x + (n-1) I_{n-2} - (n-1) I_n$

$I_n (1 + n - 1) = -\cos x \cdot \sin^{n-1} x + (n-1) I_{n-2}$

$$I_n = \frac{-\cos x \cdot \sin^{n-1} x}{n} + \frac{(n-1)}{n} \int \sin^{n-2} x \, dx$$

Now, Let $I = \int \sin^8 x \, dx$ ---- (i)

From the reduction formula,

$$I = \frac{-\cos x \cdot \sin^7 x}{8} + \frac{7}{8} \int \sin^6 x \, dx$$

$$= \frac{-\cos x \cdot \sin^7 x}{8} + \frac{7}{8} \left[\frac{-\cos x \cdot \sin^5 x}{6} + \frac{5}{6} \int \sin^4 x \, dx \right]$$

$$= \frac{-\cos x \cdot \sin^7 x}{8} - \frac{7 \cos x \cdot \sin^5 x}{48} + \frac{35}{48} \int \sin^4 x \, dx$$

$$= \frac{-\cos x \cdot \sin^7 x}{8} - \frac{7\cos x \cdot \sin^5 x}{48} + \frac{35}{48}\left[\frac{-\cos x \cdot \sin^3 x}{4} + \frac{3}{4}\int \sin^2 x \cdot dx\right]$$

$$= -\frac{\cos x \cdot \sin^7 x}{8} - \frac{7\cos x \cdot \sin^5 x}{48} - \frac{35\cos x \cdot \sin^3 x}{192} + \frac{105}{192}\left[\frac{-\cos x \cdot \sin x}{2} + \frac{1}{2}\int dx\right]$$

$$= -\frac{\cos x \cdot \sin^7 x}{8} - \frac{7\cos x \cdot \sin^5 x}{48} - \frac{35\cos x \cdot \sin^3 x}{192} - \frac{35\cos x \cdot \sin x}{128} + \frac{35x}{128} + c$$

<u>Ans</u>

3) $\int \tan^n(x)\, dx$

$= \int \tan^{n-2} x \cdot \tan^2 x\, dx$

$= \int \tan^{n-2} x \,(\sec^2 x - 1)\, dx$

$= \int \tan^{n-2} x \sec^2 x\, dx - \int \tan^{n-2} x\, dx$

Here, $\int \tan^{n-2} x \sec^2 x\, dx$ $\quad\left|\; u = \tan x,\ \frac{du}{dx} = \sec^2 x \right.$

$= \int u^{n-2}\, du = \frac{u^{n-1}}{n-1} + c$

$\therefore \int \tan^n x\, dx = \frac{u^{n-1}}{n-1} + \int \tan^{n-2} x\, dx + c$

$= \frac{\tan^{n-1} x}{n-1} + \int \tan^{n-1} x\, dx + c$

$\therefore \int \tan^8 x\, dx = \frac{\tan^7 x}{7} + \int \tan^7 x\, dx + c$

$= \frac{\tan^7 x}{7} + \frac{\tan^6 x}{6} + \int \tan^6 x\, dx + c$

$= \frac{\tan^7 x}{7} + \frac{\tan^6 x}{6} + \frac{\tan^5 x}{5} + \int \tan^5 x\, dx + c$

$= \frac{\tan^7 x}{7} + \frac{\tan^6 x}{6} + \frac{\tan^5 x}{5} + \frac{\tan^4 x}{4} + \int \tan^4 x + c$

$= \frac{\tan^7 x}{7} + \frac{\tan^6 x}{6} + \frac{\tan^5 x}{5} + \frac{\tan^4 x}{4} + \frac{\tan^3 x}{3} + \int \tan^3 x + c$

$= \frac{\tan^7 x}{7} + \frac{\tan^6 x}{6} + \frac{\tan^5 x}{5} + \frac{\tan^4 x}{4} + \frac{\tan^3 x}{3} + \frac{\tan^2 x}{2}$

$+ \int \tan^2 x + C$

$= \frac{\tan^7 x}{7} + \frac{\tan^6 x}{6} + \frac{\tan^5 x}{5} + \frac{\tan^4 x}{4} + \frac{\tan^3 x}{3}$

$+ \frac{\tan^2 x}{2} + \tan x + \int \tan x + C$

$= \frac{\tan^7 x}{7} + \frac{\tan^6 x}{6} + \frac{\tan^5 x}{5} + \frac{\tan^4 x}{4} + \frac{\tan^3 x}{3}$

$+ \frac{\tan^2 x}{2} + \tan x + \ln|\sec x| + C.$

$\int \tan x \, dx$

$= \int \frac{\sin x}{d \cos x} dx$

$= -\ln|\cos x| + C$

$= \ln|\sec x| + C$