RESEARCH STATEMENT

WEI WANG

Abstract. My research interests mainly focus on characteristic classes, bordism theory and classification of manifolds. In this statement, I will give the description of my previous works and discuss some future programs.

1. Background

1.1. **Equivalence relations on manifolds.** Given two manifolds M_1 and M_2, there is a basic question: when M_1 is "equal" to M_2?

Before dealing with this question, one has to clarify what is the meaning of "M_1 is equal to M_2".

In fact, many equivalence relations have been introduced into topology, for example, homeomorphism, diffeomorphism. Then one can use topological methods to determine whether M_1 is equal to M_2 under various equivalence relations.

We list some frequently used equivalence relations in topology. Let M_1 and M_2 be two closed smooth manifolds:

- **Bordism:** M_1 is bordant to M_2 if $\exists$ W s.t. $\partial W = M_1 \sqcup M_2$.
- **Homotopy equivalence:** $\exists$ continuous maps $f : M_1 \to M_2$ and $g : M_2 \to M_1$, s.t $f \circ g$ and $g \circ f$ are homotopic to identity maps ($f \circ g$ and $g \circ f$ can deform continuously to identity maps).
- **Homeomorphism:** $\exists$ continuous maps $f : M_1 \to M_2$ and $g : M_2 \to M_1$, s.t $f \circ g = id_{M_2}$ and $g \circ f = id_{M_1}$.
- **Diffeomorphism:** $\exists$ smooth maps $f : M_1 \to M_2$ and $g : M_2 \to M_1$, s.t $f \circ g = id_{M_2}$ and $g \circ f = id_{M_1}$.

1.2. **Topological invariants.** We can use topological invariants to discuss the question when M_1 is equal to M_2. The more topological invariants we find, the more possibility we can determine the question whether M_1 is equal to M_2 under certain equivalence relation.

In algebraic topology, many topological invariants have been invented. For example:

- Homology Groups, Cohomology Rings, K-Groups, Bordism Rings, Generalized Homology groups and Cohomology groups.

- Characteristic classes and Characteristic numbers.
- Homotopy Groups, Cohomology Operations, Secondary Operations.

1.3. **Main motivations.** We are very interested in:

- Calculating topological invariants, especially invariants of differential manifolds.
- Applying topological invariants to discuss topological properties of manifolds.
- Discussing certain topological invariants and characteristic classes relating to geometric structures.

2. Review of previous works

2.1. **Calculation of certain topological invariants.** In my thesis, I concentrate on calculating homology and cohomology groups of two classes of topological spaces.

The first class are symplectic connected sums of certain symplectic 4-manifolds and 6-manifolds.

From the point of view of differential topology, symplectic connected sum is a very typical surgery on submanifolds. We study the topological construction of symplectic connected sum and find that under some topological restrictions, we can calculate $H^*(M_1\sharp_{sym}M_2)$ from $H^*(M_1)$ and $H^*(M_2)$. These results were published in [4].

The second class are loop spaces

$$\Omega M := \{f : S^1 \to M | f(s_0) = m_0, f \text{ continuous}\}$$

of closed $n-1$-connected $2n$-dimensional manifolds.

We use Leray-Serre spectral sequence and some homological techniques to calculate the Hopf-algebra structures of $H_*(\Omega M)$ and $H^*(\Omega M)$. The methods and techniques are not new and these results are not published.

2.2. **Bordism.** Bordism is a relative coarse equivalence relation on closed smooth manifolds. However, manifolds under this relation admits two algebraic operations: disjoint union and product of manifolds. More precisely, consider the set:

$$\Omega_*^O := \{[M]|M \text{ closed smooth manifold}\}/\sim_{bordism}$$

then $(\Omega_*^O, \times, \sqcup)$ becomes a ring, which is called the unoriented bordism ring.

If one continues to consider manifolds with specific structure on their stable normal bundles, more bordism rings can be defined. For example:

- the Oriented Bordism Ring: Ω_*^{SO}
- the Spin Bordism Ring: Ω_*^{Spin}
- the Unitary Bordism Ring: Ω_*^{U}
- the Special Unitary Bordism Ring: Ω_*^{SU}
- the String Bordism Ring: Ω_*^{String}

We are interested in three questions on bordism rings:

- The algebraic structure of bordism rings.
- How to represent algebraic elements in bordism rings by good manifolds.
- Determine when two manifolds M_1 and M_2 are bordant.

In 2014-2015, we are working on the geometric representation problem in Ω_*^{SU}. Buchstaber, Panov and Ray in [1] asked if all the toric SU-manifolds (c_1 of normal bundle is zero) are bordant to zero in Ω_*^{SU}. Zhi Lü found a very typical way to construct toric SU-manifolds and we use this method to construct toric SU-manifolds and determine whether these manifolds are zero in Ω_*^{SU}. After some calculations, we gave the negative answer to their question. These works were published in [2].

2.3. **Equivariant bordism.** Roughly speaking, equivariant bordism is a bordism theory of manifolds with group action. Similar to the classical bordism theory, one can define when a manifold with G-action is the boundary of a "big" manifold with G-action and can construct certain equivariant bordism rings.

Analogy to classical bordism theory, we are still interested in the algebraic structures, the geometric representation, and the determination question of equivariant bordism rings. However, unlike the classical case, the algebraic structures of equivariant bordism rings are far from known. We have to start from determination question.

In classical bordism theory, there are many good results of the determination question. For example:

- In Ω_*^{O}, a closed smooth manifold is a boundary if and only if all Stifiel-Whitney characteristic numbers vanish.
- In Ω_*^{SO}, a closed smooth manifold is a boundary if and only if all Stifiel-Whitney characteristic numbers and Pontrjagin numbers vanish.
- In Ω_*^{Spin}, a closed smooth manifold is a boundary if and only if all Stifiel-Whitney characteristic numbers and KO-characteristic numbers vanish.
- In Ω_*^{U}, a closed smooth manifold is a boundary if and only if all Chern numbers vanish.

Now in equivariant case, we are interested in the case when G is the torus group. In 2016-2017, joint with Zhi Lü, we are working on the problem when a unitary T^n-manifold is bordant to zero. V. Guillemin, V. Ginzburg and Y. Karshon conjectured that they are determined by all the equivariant cohomology characteristic numbers.

We give the confirmative answer to their question in [3] and discuss some further relations between various equivariant characteristic numbers in [5].

2.4. **Generalized Montgomery-Yang correspondence.** In 1966, D.Montgomery and C.T. Yang constructed a one-to-one correspondence between the group of differentiable knotted 3-spheres in 6-sphere and the set of homotopy $\mathbb{C}P^3$, i.e.:

$$\Phi : \Sigma^{6,3} := \{i : S^3 \hookrightarrow S^6\} \longrightarrow \Pi_1^6 := \{M | M \simeq \mathbb{C}P^3\}.$$

The very interesting part of this correspondence is that they found a group structure on the set of homotopy $\mathbb{C}P^3$. In general classification problem of manifolds, usually the set of manifold classes does not admit algebraic structure.

We are interested in the question what kind of manifold classes can admit certain algebraic structure. The classical example is the set of homotopy n-spheres under connected sum and it is well-studied by Kervaie and Milnor.

Now we are working on the generalized Montgomery-Yang correspondence in high dimension. The work is ongoing.

3. Future programs

3.1. **Seminar program.** In recent years, some new ideas and new theories affect a lot in algebraic topology. For example, "Using algebraic geometry to study algebraic topology" becomes dominant in Topological Modular Forms,"Equivariant Homotopy theory" becomes the crucial step in solving the Kervaire Invariant One problem.

In order to catch up these new trends, if it is possible, we would like to organize a long-term topology seminar concentrating on the following new topics:

- Topological Modular Forms
- Chromatic Homotopy Theory
- Motivic Homotopy Theory
- Equivariant Homotopy Theory
- Topics of intersections between Algebraic Topology and Algebraic Geometry
- New methods in Topology of high-dimensional manifolds and its application to Geometry

3.2. **Topological invariants and geometric structures.** In some cases, topological invariants can detect geometric structures on certain manifolds. For example, for simply-connected closed spin manifold M, Stolz's theorem confirms that the existence of positive scalar curvature is equivalent to the vanishing of the alpha-invariant $\alpha[M]$, which is equal to certain KO-characteristic numbers. From the point view of topology, the following questions are very interesting:

- For String manifolds, there are also some questions relating the Witten-genus and certain String-characteristic numbers to the existence of certain geometric structures.

- For some manifolds with special holonomy groups, for example G_2, F_4, we are interested to know the topological obstructions and their related topological properties

3.3. **String bordism and string characteristic classes.** String bordism Ω_*^{String} is a very special bordism ring. It admits many fascinating relations with other areas. One the one hand, Witen genus relates Ω_*^{String} to $\pi_* TMF$, whose group structure is isomorphic to certain modular forms. On the other hand, the arithmetic of elliptic curves and algebraic geometry methods play a very important role in the study of Ω_*^{String} (or $MString$), and TMF.

For us, we are interested in some concrete questions:

- Try to calculate more structure of Ω_*^{String}.
- Try to study more characteristic classes and numbers of Ω_*^{String} and TMF.
- Use these $String$ and TMF characteristic classes to study the topology of certain manifolds.

3.4. **Problems in equivariant bordism.** Equivariant stable homotopy theory is an "old and new" area in algebraic topology. It developed quickly in 1960's. Many people obtained fruitful results and used them to solve problems in transformation groups.

In 2010's, the equivariant homotopy theory is the crucial step in solving Kervaire invariant one problem, it becomes active again with new insights and new methods. We would like to learn these new ideas and new methods to study equivariant bordism theory. More precisely, we want to

- Calculate algebraic structures of certain equivariant bordism ring
- Study certain equivariant characteristic classes and numbers
- Work on the determination question: when two manifolds with G-action are equivalent up to equivariant bordism.

References

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