Homework 2: DFAs and NFAs

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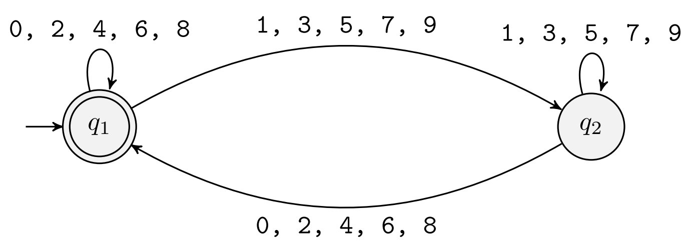
1. Divisibility Tests

Define, for all $k > 0$,

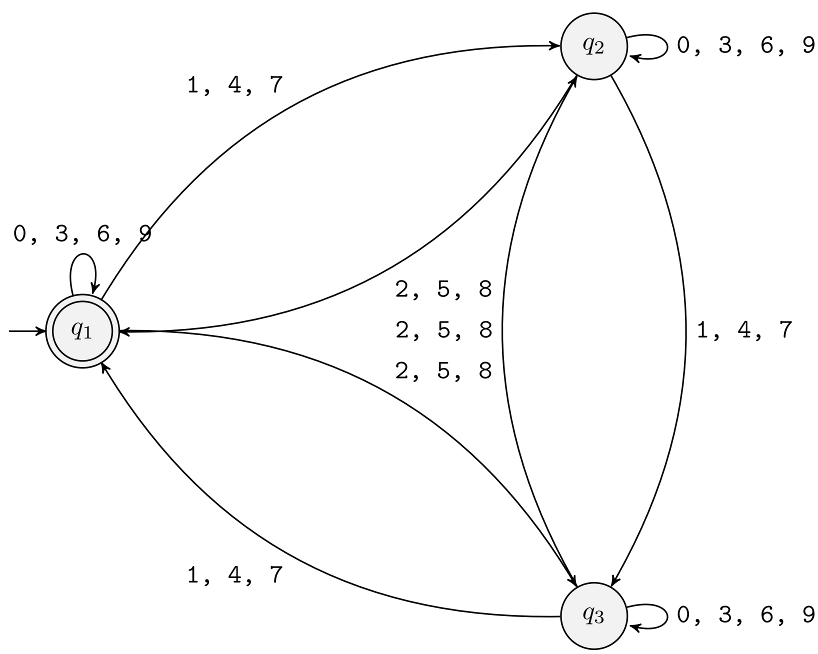
$$D_k = \{w \in \{0, \ldots, 9\}^* \mid w \text{ is the decimal representation of } k\}$$

where ε is considered to represent the number 0. For example, the strings ε, `0`, `1234`, and `01234` all belong to D_2, but `99` and `099` do not.

(∗) The DFA D_2 can be written as such:



(∗) The DFA D_3 can be written as such:



(∗) For any $k > 0$, the language D_k is regular to the DFA $M = (Q, \Sigma, \delta, q_0, q_0)$ where

- $Q = \{q_0, \dots, q_{k-1}\}$
- $\Sigma = \{0, \dots, 9\}$
- $\delta(q_n, w \in \Sigma^*) = q_{(n \times 10 + w \pmod k)}$

For any $k > 0$, any whole number divided by k has a remainder between 0 and $k-1$. Therefore, the number of possible remainders for dividing a number by k is equal to k, and thus the number of states for the DFA representing dividing a number by k is equal to k. Q is the set of all states of this DFA, where each state is numbered by the corresponding remainder ($Q = \{q_0, \dots, q_{k-1}\}$).

By having each state correspond to a remainder, the DFA then stores the remainder of the number in the string $w \in \Sigma^*$ as each digit is processed.

Mathematically, the transition function δ operates as such on string $w \in \Sigma^*$:

(a) Base Case (w_1):

The start state is q_0, so $r_{prev} = 0$. Therefore, $\delta(q_0, w_1) = q_{(0 \times 10 + w_1 \pmod k)} = q_{(w_1 \bmod k)}$

(b) Recursive Case (w_i, $0 < i \leq |w|$):

The remainder of a number can be found through calculating the remainder of its digits one by one. If N is the string of already read digits, and d is the next digit, $N_{new} = N \times 10 + d$. Taking the modulus with respect to k, we obtain the remainder of N_{new} calculated from the remainder of N:

$$\begin{aligned}
N_{new} \bmod k &= (N \times 10 + d) \bmod k \\
r_{new} &= (((N \bmod k)(10 \bmod k) \bmod k) + (d \bmod k)) \bmod k^1 \\
r_{new} &= ((R \times (10 \bmod k) \bmod k) + (d \bmod k)) \bmod k \\
r_{new} &= (R \times 10 + d) \bmod k
\end{aligned}$$

We can apply this to the transition function δ, such that for a DFA at current state q_j, $j \in [0, k-1]$, processing symbol $w_i \in w$, $\delta(q_j, w_i) = q_{((j \times 10 + w_i) \bmod k)}$.

2. References

(1) "Modulo/Properties." *Wikipedia*, Wikimedia Foundation, 21 Jan. 2026, `https://en.wikipedia.org/wiki/Modulo#Properties_(identities)`.