Non-uniform dependences partitioned by recurrence chains

Yijun Yu
CS Department, University of Toronto, Canada

Erik H. D'Hollander
EE Department, Ghent University, Belgium

Abstract

Non-uniform distance loop dependences are a known obstacle to find parallel iterations. To find the outermost loop parallelism in these "irregular" loops, a novel method is presented based on recurrence chains. The scheme organizes non-uniformly dependent iterations into lexicographically ordered monotonic chains. While the initial and final iterations of monotonic chains form two parallel sets, the remaining iterations form an intermediate set that can be partitioned further. When there is only one pair of coupled array references, the non-uniform dependences are represented by a single recurrence equation. In that case, the chains in the intermediate set do not bifurcate and each can be executed as a WHILE loop. The independent and the initial iterations of monotonic dependence chains constitute the outermost parallelism. The proposed approach compares favorably with other treatments of non-uniform dependences in the literature. When there are multiple recurrence equations, a dataflow parallel execution can be scheduled using the technique to find maximum loop parallelism.

Keywords *non-uniform dependences, loop parallelization, recurrence chains, iteration space partitioning, imperfectly nested loop*

1. Introduction

Data dependences of a program lead to maximal parallelism by a data flow execution. A more restricted parallelism is implemented in common parallel programming languages because the loop control limits the traversal of the iteration space to regular patterns. Common language support for parallelism is parallel DO loops (DOALL). It is one of the most important goals for parallelizing compilers to reveal loop parallelism [1, 16, 30, 8, 27]. The loop can run in parallel when no data dependences exist between any two iterations with different index values. This is judged by various dependence tests [29, 14, 18, 22]. Moreover, some loop nests being tested as sequential can still be parallelized after suitable loop transformations.

Existing DOALL loop transformation methods focus on loops with uniform distance dependences [2, 9]. Many loops, however, are with non-uniform distance dependences where these transformations can not be applied. For instance, we found that more than 46% of the nested loops in the `SPECfp95` benchmark contain non-uniform data dependences. Furthermore, coupled array subscripts, i.e. indices that appear in both dimensions, often cause non-uniform distances. Another study of 12 benchmarks [21] found out 45% of two dimensional array reference pairs with coupled linear subscripts. Including one-dimensional arrays, about 12.8% of the coupled subscripts in the `SPECfp95` benchmarks generate non-uniform dependences. The percentage of loops with non-uniform dependences is higher because a single pair of non-uniform coupled subscripts will cause non-uniform dependences.

A way to catch all dependences by sequential linear steps through the iteration space is to find a set of vectors whose linear combinations compose all distance vectors [23, 26, 6]. To enable DOALL partitioning, a previous scheme in [27] replaces the non-uniform distance vectors with a set of pseudo distance vectors. They allow DOALL loop transformations to work as if they are uniform distance vectors because an integer linear combination of these lexicographically positive (LP) vectors covers all the non-uniform dependences. It finds maximal parallelism when the dependences are uniform, but may generate artificial dependences in the non-uniform case.

To avoid introducing artificial dependences into the loop, another way of treating the non-uniform dependences is based on exact solution of the dependence equations [11]. Although it is generally impossible to enforce data flow scheduling at compile time, recurrence chain partitioning makes it possible when there is only one pair of coupled subscripts or there is no compile-time unknown variable in the loop bounds. The recurrence chain partitioning separates the iteration space into three sequential partitions. The first and the last sets are fully parallel. When there is only one pair of coupled subscripts, the intermediate set is partitioned as disjoint monotonic recurrence chains that can be executed as WHILE loops. When there are multiple coupled subscripts and loop bounds are known at compile time, the intermediate set is successively partitioned into subsets following the dataflow until it is empty.

The remainder of the paper is organized as follows. Section 2 presents the program model for non-uniform distance dependences. Section 3 illustrates the recurrence chain partitioning scheme by showing what kinds of dependences form the recurrence chains and how to generate the parallel partitions. Section 4 presents the results of applying different schemes on some program examples and section 5 compares it with related work.

2. Program model

Consider the coupled array subscripts occur in m nested loops: $L_1, \ldots, L_m$. For $l = 1, \ldots, m$, every loop L_l is normalized to have a unit stride and each loop index variable I_l is constraint by a lower bound p_l and an upper bound q_l. The loop bounds are affine functions of the index variables in the outer loops. The *iteration space* Φ of the loop is a vector set of all values taken by the integer loop indices

$$\{\mathbf{i} = (i_1 \ldots i_m) \mid p_l \leq i_l \leq q_l, 1 \leq l \leq m, \mathbf{i} \in \mathbb{Z}^m\}. \quad (1)$$

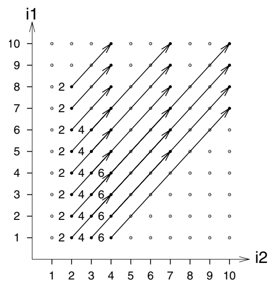
Loop carried dependences occur when the same array-element is addressed in two different iterations, where at least one of them is a write. Assume an array X is referred with affine index expressions: $X[\mathbf{IA} + \mathbf{a}]$ and $X[\mathbf{IB} + \mathbf{b}]$ where $\mathbf{I}$ denotes the vector of loop indexing variables, integer matrices $\mathbf{A}, \mathbf{B}$ and vectors $\mathbf{a}, \mathbf{b}$ are constants. A *direct dependence* occurs between iterations $\mathbf{i}$ and $\mathbf{j}$ if 1) the diophantine equation has a solution

$$\mathbf{i}\,\mathbf{A} + \mathbf{a} = \mathbf{j}\,\mathbf{B} + \mathbf{b} \quad (2)$$

2) and the solution is within the loop bounds: $\mathbf{i}, \mathbf{j} \in \Phi$.
A solution of the diophantine equations results in a pair of directly dependent iterations $(\mathbf{i}, \mathbf{j})$, also noted as $(\mathbf{i} \rightarrow \mathbf{j})$. The union of these pairs is called the *direct dependences set* of the loop, Δ. An *indirect dependence* between $\mathbf{i}$ and $\mathbf{j}$ occurs when there exists a chain of $M > 1$ direct dependences: $(\mathbf{i}_k, \mathbf{i}_{k+1}) \in \Delta$ for $1 \leq k \leq M$, $\mathbf{i}_1 = \mathbf{i}$ and $\mathbf{i}_M = \mathbf{j}$. Including both direct and indirect dependences, the *dependence set* is $\Delta^+ = \{(\mathbf{i}, \mathbf{j}) \mid (\mathbf{i}, \mathbf{j}) \in \Delta \vee \exists\, \mathbf{k} : (\mathbf{i}, \mathbf{k}), (\mathbf{k}, \mathbf{j}) \in \Delta\}$. The *dependence distance* between two dependent iterations $\mathbf{i}$ and $\mathbf{j}$ is $\mathbf{d} = \mathbf{j} - \mathbf{i}$. The union of all distances in a dependence set is called *the dependence distance set*. Therefore the direct dependence set Δ gives rise to the *direct distance set* $\mathcal{D}$. Likewise the dependence distances in a loop are represented by the *distance set* $\mathcal{D}^+$. For any two direct dependent iterations $(\mathbf{i}, \mathbf{j}) \in \Delta$ and any non-zero vector $\mathbf{c} \in \mathbb{Z}^m$, if $(\mathbf{i} + \mathbf{c}, \mathbf{j} + \mathbf{c}) \in \Delta$ as long as $\mathbf{i} + \mathbf{c}, \mathbf{j} + \mathbf{c} \in \Phi$, then the loop has *uniform* dependences. In all other cases, the dependences of the loop are *non-uniform*.
Consider an example from [27], as listed in figure 1. The iteration space of this loop is: $\Phi = \{(i_1, i_2) \mid 1 \leq i_1 \leq N_1, 1 \leq i_2 \leq N_2, i_1, i_2 \in \mathbb{Z}\}$. A dependence equation is established as a system of diophantine equations:

$$\left\{ \begin{array}{llllll} 3i_1 & & +1 & = & j_1 & +3 \\ 2i_1 & +i_2 & -1 & = & j_2 & +1 \end{array} \right. \quad (3)$$

The solutions of (eq.3) are a set of direct dependences. When $N_1 = N_2 = 10$, the dependences are non-uniform because e.g. $(1,2) \rightarrow (3,4)$ does not imply $(1,1) \rightarrow (3,3)$. The detection of parallelism in uniform dependence loops has received widespread attention, e.g. by unimodular



```
DO I1=1,N1
 DO I2=1,N2
  a(3*I1+1,2*I1+I2-1)
    =a(I1+3,I2+1)
 ENDDO
ENDDO
```

Figure 1. An example loop and its iteration space. The direct dependences are shown as arrows with direct distance (d, d) where $d = 2, 4, 6$ are marked to the left of the arrow tails.

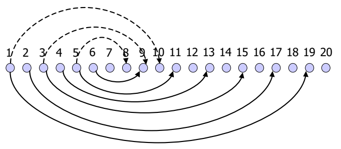
transformations[4, 24], partitioning[9] and others [15]. Unfortunately, many loops contain non-uniform dependences, for which no general mathematical approach is available to detect the parallel iterations. A number of alternatives have been proposed for the case of affine index expressions, e.g. uniformization oriented techniques [23, 26, 6, 19, 27] and dataflow oriented techniques [20, 11].
In this paper, the loops with non-uniform dependences are parallelized using WHILE loops with irregular strides. Dataflow oriented code in the cases where the uniformization method such as PDM [27] allows for extra parallelism, a recurrence chain partitioning in section 3 constructs WHILE loops with irregular strides to follow the non-uniform dependences.

3. Recurrence chain partitioning

To avoid artificial dependences introduced by uniformization methods [27], our recurrence partitioning scheme discovers dataflow parallelism by solving exact dependences. Using exact dependences, each step of *dataflow* partitioning puts the iterations without lexicographically predecessors into an initial fully parallel set and partitions the remaining iterations successively until no more iteration is left. Besides the initial set, a three-sets dataflow partitioning also separates the iterations without lexicographically successors as a fully parallel final set. When the dependences can be solved as dependence convex hulls, however, the dataflow partitioning may not terminate at compile-time for unknown loop bounds. Therefore a special treatment is proposed here to allow partitioning for loops with unknown bounds and a single pair of coupled subscripts. In one step, it separates the intermediate set of the three-set dataflow partitioning into disjoint monotonic dependence chains.

Definition 1. Monotonic dependence chain. *A monotonic dependence chain is a sequence of lexicographically ordered iterations in which each iteration directly depends on a unique immediate predecessor iteration.* □

For example, the loop in figure 2 has non-uniform dependences. The dependences are separated into monotonic



```
DO I=1,20
  a(2*I)=a(21-I)
ENDDO
```

Figure 2. The loop dependences are solved as $\{i \rightarrow j \mid 2i = 21 - j\}$ where $i < j$ or $i > j$ are respectively solid or dashed arrows.

chains. A solution chain $6 \rightarrow 9 \rightarrow 3 \rightarrow 15$ is separated into three monotonic chains: $6 \rightarrow 9$, $3 \rightarrow 9$ and $3 \rightarrow 15$. Each monotonic chain has only two iterations, thus the iteration space is partitioned into two sets. The first set is the union of the initial iterations $\{1, 2, 3, 4, 5, 6\}$ and the independent iterations $\{7, 12, 14, 16, 18, 20\}$.

The dependences can be specified as a relation in the iteration space. Consider a dependence equation $\mathbf{iA} + \mathbf{a} = \mathbf{jB} + \mathbf{b}$ established from two references in two iterations with index vectors $\mathbf{i}$ and $\mathbf{j}$ respectively. If $\mathbf{i} \prec \mathbf{j}$, iteration $\mathbf{i}$ is called a *predecessor* of iteration $\mathbf{j}$ and $\mathbf{j}$ a *successor* of $\mathbf{i}$. The exact dependences are the union of the predecessor and successor relations:

$$\begin{aligned} R_d = R_{pred} \cup R_{succ} = \{\mathbf{j} \rightarrow \mathbf{i} \mid \mathbf{i}\,\mathbf{A} + \mathbf{a} = \mathbf{j}\,\mathbf{B} + \mathbf{b}, \mathbf{j} \prec \mathbf{i}\} \\ \cup \{\mathbf{i} \rightarrow \mathbf{j} \mid \mathbf{i}\,\mathbf{A} + \mathbf{a} = \mathbf{j}\,\mathbf{B} + \mathbf{b}, \mathbf{i} \prec \mathbf{j}\} \end{aligned} \tag{4}$$

For multiple coupled subscripts, the combined dependence relation unions all the dependence relations of each dependence equation. An accurate solution to a union of integer convex sets can be found by the algorithm [18] implemented in an integer programming tool, the Omega library.

3.1. Partitioning the iteration space into three sets

Intuitively, the iteration space can be partitioned into separate monotonic chains. Starting from an initial iteration, i.e. an iteration without predecessors, a WHILE loop can be formed for each monotonic chain by updating the iteration index iteratively until it exceeds the border of iteration space. However, even when the dependent iterations are on separate recurrence chains, the lexicographical order is not always followed by a WHILE loop. In that case, several monotonic dependence chains may intersect at the same iteration, e.g. figure 2 shows that a WHILE loop updating indices successively by $i' = 21 - 2i$, forms chain $6 \rightarrow 9 \rightarrow 3 \rightarrow 15$ which violates the lexicographical ordering, whereas monotonic chains $6 \rightarrow 9$, $3 \rightarrow 9$, $3 \rightarrow 15$ intersect, such that iterations $3, 9$ will be executed twice.

The recurrence chain partitioning only executes the initial and final iterations once. The iteration space is separated into two fully parallel sets and one intermediate set so that the monotonic chains in the intermediate set are separate, or as in the above example, the monotonic chains in the intermediate set are empty. For the monotonic chains in the intermediate set to be disjoint, it requires a single pair of coupled subscripts. According to the dependence relation in (eq.4), an *independent* iteration that has neither predecessor nor successor; otherwise it is a *dependent* iteration with predecessors or successors. A dependent iteration that has no predecessor is an *initial* iteration, a dependent iteration that has no successor is a *final* iteration, otherwise a dependent iteration that has both predecessors and successors is an *intermediate* iteration. Therefore the whole iteration space is composed of independent, initial, intermediate and final iterations. Using *dom* (R) and *ran* (R) respectively to denote the relation R's domain dom $R \equiv \{\mathbf{x} \mid (\mathbf{x} \rightarrow \mathbf{y}) \in R\}$ and range ran $R \equiv \{\mathbf{y} \mid (\mathbf{x} \rightarrow \mathbf{y}) \in R\}$, the sets are calculated from the dependence relation R_d and the iteration space Φ as: dep $=$ (dom $R_d \cup$ ran R_d), initial $=$ dep $\setminus$ ran R_d, intermediate $=$ dom $R_d \cap$ ran R_d and final $=$ dep $\setminus$ dom R_d. The independent and initial iterations are in an *initial set* P_1 of the iteration space, the intermediate iterations are in an *intermediate set* P_2 and the final iterations are in an *final set* P_3. They are calculated as

$$\begin{aligned} &P_1 = \Phi \setminus \text{ran } R_d, \qquad P_2 = \text{ran } R_d \cap \text{dom } R_d \\ &P_3 = \text{ran } R_d \setminus \text{dom } R_d \end{aligned} \tag{5}$$

A dependence occurs only from an initial iteration to an intermediate one, from an intermediate iteration to another, or from an intermediate iteration to a final one. Thus the three sets can be executed in the order of $P_1 \rightarrow P_2 \rightarrow P_3$. The intermediate set P_2 needs to be further partitioned for dependences that occur inside $\{\mathbf{i} \rightarrow \mathbf{j} \mid (\mathbf{i} \rightarrow \mathbf{j}) \in R_d, \mathbf{i}, \mathbf{j} \in P_2\}$.

3.2. Partitioning the intermediate set

Starting from dependence equation (eq.2): $\mathbf{iA}+\mathbf{a} = \mathbf{jB}+\mathbf{b}$, when both $\mathbf{A}$ and $\mathbf{B}$ are full rank square matrices, there is an one-to-one recurrence relation between the dependent iterations.

Lemma 1. *When there is only one pair of coupled references with full rank coefficient matrices* $\mathbf{A}$ *and* $\mathbf{B}$*, the monotonic dependence chains in the intermediate set* P_2 *are disjoint, i.e., there is only one predecessor and one successor for each iteration in* P_2.

Proof. Each iteration in P_2 has at least one predecessor and one successor because P_2 is the intersection of the domain and range of the dependence relation. Since $\mathbf{A}$ and $\mathbf{B}$ are full rank, let $\mathbf{T} = \mathbf{BA}^{-1}, \mathbf{u} = (\mathbf{b} - \mathbf{a})\mathbf{A}^{-1}$.The dependence (eq.2) is rewritten as: $\mathbf{i} = \mathbf{j}\;\mathbf{T} + \mathbf{u}$. Suppose $\exists \mathbf{j} \in P_2$ such that there are two different predecessors $\mathbf{i}_1$ and $\mathbf{i}_2$, thus $\mathbf{i}_1 = \mathbf{jT} + \mathbf{u}$ and $\mathbf{i}_2 = \mathbf{jT} + \mathbf{u}$. However, $\mathbf{i}_1 = \mathbf{i}_2$. Thus there is only one predecessor for all iteration $\mathbf{j} \in P_2$. Similarly only one successor follows each iterations $\mathbf{i} \in P_2$ by replacing $\mathbf{T}$ with $\mathbf{AB}^{-1}$ and $\mathbf{u}$ with $(\mathbf{a} - \mathbf{b})\mathbf{B}^{-1}$. $\square$

Since the monotonic chains are disjoint in the intermediate set, a compile-time recurrence chain partitioning is applicable to the intermediate set instead of doing unlimited steps of dataflow partitioning when loop bounds contain unknown variables.

The dependence relation is $R_d = \{\mathbf{j} \to \mathbf{i} \mid \mathbf{i} = (\mathbf{j} - \mathbf{u})\mathbf{T}^{-1}, \mathbf{j} \prec \mathbf{i}\} \cup \{\mathbf{i} \to \mathbf{j} \mid \mathbf{j} = \mathbf{i}\mathbf{T} + \mathbf{u}, \mathbf{i} \prec \mathbf{j}\}$. The initial iterations $\mathbf{i}_0$ are those without preceding solution in the iteration space Φ (either is not integer or is outside the bounds): $(\mathbf{i}_0 - \mathbf{u})\mathbf{T}^{-1} \notin \Phi$. The sequence of the recurrent dependent iterations is on a *dependence recurrence chain*. The general solution of an iteration on the recurrence chain beginning with $\mathbf{i}_0$ is $\mathbf{i}_k = \mathbf{i}_0\mathbf{T}^k + \mathbf{u}(\mathbf{T}^{k-1} + \cdots + \mathbf{T}^0)$. The distance vector between the dependent iterations $\mathbf{i}_{k+1}$ and $\mathbf{i}_k$ is

$$\begin{aligned} \mathbf{d}_k &= \mathbf{i}_{k+1} - \mathbf{i}_k = (\mathbf{i}_0(\mathbf{T} - \mathbf{I}) + \mathbf{u})\mathbf{T}^k = \mathbf{d}_0\mathbf{T}^k \\ \mathbf{d}_0 &= \mathbf{i}_0(\mathbf{T} - \mathbf{I}) + \mathbf{u}. \end{aligned} \tag{6}$$

Removing the initial and final iterations, a recurrence chain will be separated into disjoint monotonic chains. WHILE loops are constructed to sequentially execute these monotonic chains. If initially $\mathbf{i}_0 \in R_{\text{pred}}$, the WHILE loop changes index by R_{pred}: i.e. $\mathbf{I} = (\mathbf{I} - \mathbf{u})\mathbf{T}^{-1}$, otherwise the WHILE loop changes index by R_{succ} i.e. $\mathbf{I} = \mathbf{I}\mathbf{T} + \mathbf{u}$. Each WHILE loop starts at an iteration that depends on an initial iteration in P_1. The starting iterations are in the following set: $W = \{\mathbf{j} \mid (\mathbf{i} \to \mathbf{j}) \in R_d, \mathbf{i} \in P_1, \mathbf{j} \in P_2\}$ and the WHILE loop stops when the successor becomes a final iteration. Thus the condition for the WHILE loop to continue is $\mathbf{i} \in (\Phi \setminus final) = \mathbf{i} \in (\Phi \cap \text{dom } R_d)$.

Only $\cap, \cup, \setminus, dom, ran$ operations are applied to the union of convex sets Φ and R_d to obtain the fully parallel sets P_1, W, P_3. Therefore each of them can be specified by a union of convex sets which is the logical conjunctive normal form where each logical operand is a linear inequality. Although Fourier-Motzkin elimination can be used to generate a DO loop nest for each convex set, it is first necessary to make the convex sets disjoint. An algorithm exists [13] to generate loops from a lexicographically ordered union of convex sets. The lexicographical order of the convex sets is not necessary here because they are fully independent.

3.3. Extending the iteration space to statement-level

To reveal statement-level parallelism in case of imperfectly nested loops or multiple statements in a loop body, each instance of a statement $S(\mathbf{I})$ with loop index vector $\mathbf{I} = \mathbf{i}$ needs to be associated with a unique index vector $\mathbf{s_i}$ such that 1) the lexicographical ordering of $\mathbf{s_i}$ reflects the statement instances execution order; 2) the set of statement index vectors forms a union of convex sets.

An example of such extension has been proposed by the affine mapping framework in [12]. Assume there are l surrounding loops for a statement $S(\mathbf{I})$. For any instance $S(\mathbf{i})$, a statement index s_k is inserted after each loop index i_k for $k = 1, \cdots, l$ and s_0 is given before the outermost loop index i_1 to indicate the position of the whole loop nest in the program. A unique index vector $\mathbf{s_i} = (s_0, i_1, s_1, \cdots, i_l, s_l)$ is thus formed. In order to apply lexicographical ordering on the statement index vectors, dummy zeroes are appended to the unique index vectors for statements outsides the innermost loop. To make sure the set of statement index vectors forms a union of convex sets, the statements in the same loop are associated with a sequence of numbers with unit increment. The first statement in the loop L_k is associated with $s_k = 1$ for convenience. Both the unified iteration space with statement instances and the iteration space with loop body instances are a union of convex sets. Thus the partitioning method for them are inherently the same, the only difference is that we calculate statement-level dependences from the following relations for any two instances of statements $S_1(\mathbf{I}; \mathbf{I}^1)$ and $S_2(\mathbf{I}; \mathbf{I}^2)$ with unique index vectors $\mathbf{s_i}$ and $\mathbf{t_j}$:

$$\begin{aligned} R_d = &\{\mathbf{t_j} \to \mathbf{s_i} \mid \mathbf{iA} + \mathbf{a} = \mathbf{j\,B} + \mathbf{b}, \mathbf{t_j} \prec \mathbf{s_i}\} \\ \cup &\{\mathbf{s_i} \to \mathbf{t_j} \mid \mathbf{iA} + \mathbf{a} = \mathbf{j\,B} + \mathbf{b}, \mathbf{s_i} \prec \mathbf{t_j}\} \end{aligned} \tag{7}$$

3.4. The recurrence partitioning algorithm

Algorithm 1 summarizes the recurrence chains partitioning scheme. Initially the unified iteration space Φ and the dependence relation R_d are calculated. If there is a single pair of coupled subscripts with full rank coefficient matrices $\mathbf{A}$ and $\mathbf{B}$, the recurrence chain partitioning is applied to the intermediate set after a three-set dataflow partitioning according to Φ and R_d. WHILE loops are generated for each monotonic chain in the intermediate set. Otherwise, if the loop bounds are known at compile-time, the dataflow partitioning is successively done to the iteration space Φ and the dependence sub-relation R_d until Φ is empty.

Algorithm 1. The recurrence partitioning scheme

Input: *A sequential loop nest with a single pair of coupled linear array subscripts or with compile-time known loop bounds. The loop body is denoted as* $S(\mathbf{I})$.

Output: *A sequence of DOALL loop nests.*

```
let $\Phi$ be the unified iteration space, calculate dependences as:
$R_d = \bigcup \{\mathbf{s_i} \to \mathbf{t_j} \mid (\mathbf{iA} + \mathbf{a} = \mathbf{jB} + \mathbf{b} \vee \mathbf{iA} + \mathbf{a} = \mathbf{jB} + \mathbf{b}) \wedge \mathbf{s_i} \prec \mathbf{t_j} \wedge \mathbf{s_i}, \mathbf{t_j} \in \Phi\}$
if $X(\mathbf{IA} + \mathbf{a})$, $X(\mathbf{IB} + \mathbf{b})$ are the single reference pair in $S(\mathbf{I})$
    and $\mathbf{A}, \mathbf{B}$ are full rank then
  $P_1 = \Phi \setminus (ran\ R_d)$; $P_2 = (ran\ R_d) \cap (dom\ R_d)$;
  $P_3 = (ran\ R_d) \setminus (dom\ R_d)$;
  $W = \{\mathbf{j} \mid (\mathbf{i} \to \mathbf{j}) \in R_d \wedge \mathbf{i} \in P_1 \wedge \mathbf{j} \in P_2\}$;
  call DOALLCodeGeneration($P_1$, $S(\mathbf{I})$);
  call DOALLCodeGeneration($W$, $S'(\mathbf{I})$)
                          if ($\mathbf{I} \in R_{pred}$) then
                            $\mathbf{T} = \mathbf{AB}^{-1}$; $\mathbf{u} = (\mathbf{a} - \mathbf{b})\mathbf{B}^{-1}$
                          else
                            $\mathbf{T} = \mathbf{BA}^{-1}$; $\mathbf{u} = (\mathbf{b} - \mathbf{a})\mathbf{A}^{-1}$
    where $S'(\mathbf{I}) \equiv$  end if
                          do while($\mathbf{I} \in (\Phi \cap dom\ R_d)$)
                            $S(\mathbf{I})$; $\mathbf{I} = \mathbf{IT} + \mathbf{u}$;
                          end do while
  call DOALLCodeGeneration($P_3$, $S(\mathbf{I})$);
else if the loop bounds are constant then
```

```
  do while ($\Phi$ is not empty)
    $P_1 = \Phi \setminus (ran\ R_d)$; $\Phi = \Phi \setminus P_1$;
    $R_d = \{\mathbf{i} \to \mathbf{j} \mid (\mathbf{i} \to \mathbf{j}) \in R_d \wedge \mathbf{i}, \mathbf{j} \in \Phi\}$;
    call DOALLCodeGeneration($P_1$, $S(\mathbf{I})$)
  enddo while
endif
subroutine DOALLCodeGeneration(Set, Body)
  separate Set into $N$ disjoint convex sets $CH_1, \cdots, CH_N$ [13];
  do i=1, N
    generate a DOALL loop nest with the body Body
      bounded by $CH_i$ [3];
  enddo
□return
```

If there are multiple coupled subscripts and the loop bounds are unknown at compile-time, the recurrence partitioning can not apply. In that case, the pseudo distance partitioning in [27] is used instead.

The theoretical speedup of the partitioning is determined by the execution time of the critical path in proportion to the number of iterations on the critical path. The following theorem states the lower bound of the speedup when the monotonic chains do not bifurcate. Consequently the theoretical parallel speedup is at least $\frac{|\Phi|}{l}$ on $O(|\Phi|)$ parallel processors, where $|\Phi|$ denotes the number of iterations in iteration space Φ.

Theorem 1. *Given a recurrence equation* $\mathbf{i}_{k+1} = \mathbf{i}_k\mathbf{T} + \mathbf{u}$ *with non-singular matrix* $\mathbf{T}$, *let* $a = \max(|det(\mathbf{T})|, |det(\mathbf{T}^{-1})|)$. *In the iteration space* Φ, *the critical path found by algorithm 1 contains at most* $l = \lfloor \log_a(L) + 1 \rfloor$ *iterations, where L is the maximum Euclid distance between any two iterations:* $L = \max_{\mathbf{i}_1, \mathbf{i}_2 \in \Phi} ||\mathbf{i}_2 - \mathbf{i}_1||$.

Proof. For each distance vector $\mathbf{d} = \mathbf{i}_2 - \mathbf{i}_1$, the Euclid distance is $||\mathbf{d}|| = \sqrt{d_1^2 + \cdots + d_m^2}$. Suppose n is the length of a recurrence chain by (eq.6), $||\mathbf{d}_n|| = ||\mathbf{d}_0||a^n$ or $a^n = \frac{||\mathbf{d}_n||}{||\mathbf{d}_0||} \leq ||\mathbf{d}_n|| \leq L$. Since $a > 1$, the length of the critical path is $n + 1 \leq \lfloor \log_a(L) + 1 \rfloor$. □

4. Results

This section applies the recurrence partitioning on several examples and compares their speedups with other schemes.

Example 1 The example in figure 1 after our recurrence chain partitioning is:

```
C initial partition
 DOALL i1=1,min(N1,3)
  DOALL i2=1,N2
   s(i1,i2)
  ENDDOALL
 ENDDOALL
 DOALL i1=4,N1
  DOALL i2=1,min((2*i1)/3,N2)
   s(i1,i2)
  ENDDOALL
  DOALL i2=(2*i1+3)/3,N2
   IF (i1-3.le.3*((i1-2)/3)) THEN
    s(i1,i2)
    ENDIF
  ENDDOALL
 ENDDOALL
C intermediate partition and while start
DOALL i1=4,min((3*N2+5)/8,min((N1+2)/3,7)),3
 DOALL i2=(2*i1+3)/3,N2-2*i1+2
  chain(i1,i2)
 ENDDOALL
ENDDOALL
DOALL i1=10,min((3*N2+5)/8,(N1+2)/3),3
 DOALL i2=(2*i1+3)/3,min(N2-2*i1+2,(8*i1-2)/9)
  chain(i1,i2)
 ENDDOALL
 DOALL i2=(8*i1+9)/9,N2-2*i1+2
  IF (i1-7.le.9*((i1-4)/9)) THEN
   chain(i1,i2)
  ENDIF
 ENDDOALL
ENDDOALL
C final partition
 DOALL i1=4,min((N1+2)/3,(3*N2-1)/2),3
  DOALL i2=max(N2-2*i1+3,(2*i1+3)/3),N2
   s(i1,i2)
  ENDDOALL
 ENDDOALL
 DOALL i1=3*(((N1+5)/3+1)/3)+1,
*          min(N1,(3*N2-1)/2),3
  DOALL i2=(2*i1+3)/3,N2
   s(i1,i2)
  ENDDOALL
 ENDDOALL
 ...
 SUBROUTINE chain(i,j)
 DO WHILE (2.le.i.and.3*i.le.2+N1
*     .and.1.le.j.and.2*i+j.le.2+N2)
  s(i,j);
  IF (i.mod.3.ne.1) RETURN;
  ip = 3*i-2
  jp = 2*i+j-2
  i = ip
  j= jp
 ENDDO
 END
```

The original loop body is represented as an inlined function $s(i,j)$. The first partition index set splits as a union of convex sets without dependences. Similarly no dependence is within the intermediate set and the final set. The monotonic recurrence chains in the intermediate set are executed by a WHILE loop in the subroutine "chain" that can be inlined. Since $det(\mathbf{T}) = 3$, the largest partition has at most $\lfloor 1 + log_3(\sqrt{N_1^2 + N_2^2}) \rfloor$ iterations by theorem 1.

Example 2 Consider another non-uniform dependence example used by Ju et al [11].

```
DO I=1,N
 DO J=1,N
  a(2*I+3,J+1) = a(I+2*J+1,I+J+3)
 ENDDO
ENDDO
```

The PDM partitioning can only find a parallelism of two in the innermost loop, thus the recurrence chain partitioning is applied using algorithm 1:

```
DOALL i=1,12
 IF(mod(i,2).eq.1)THEN
  DOALL j=1,min(-i+10,(i-1)/2)
 4    a(2*i+3,j+1)=a(i+2*j+1,i+j+3)
 5   ENDDOALL
 6  ENDIF
 7  DOALL j=(i+2)/2,min(i+3,-i+10)
 8   a(2*i+3,j+1)=a(i+2*j+1,i+j+3)
 9  ENDDOALL
10  DOALL j=max(-i+11,1),min(i+3,12)
11   a(2*i+3,j+1)=a(i+2*j+1,i+j+3)
12  ENDDOALL
13  DOALL j=(3*i+8)/2,12
14   a(2*i+3,j+1)=a(i+2*j+1,i+j+3)
15  ENDDOALL
16 ENDDOALL
17 i=2
18 j=6
19 a(2*i+3,j+1)=a(i+2*j+1,i+j+3)
20 DOALL i=2,8
21  IF(mod(i,2).eq.0)THEN
22   DOALL j=1,min(-i+10,i/2)
23    a(2*i+3,j+1)=a(i+2*j+1,i+j+3)
24   ENDDOALL
25  ENDIF
26  IF(i.eq.3)a(2*i+j,i+1)=a(i+2*j+1,i+j+3)
27  IF(i.ge.4)THEN
28   DOALL j=i+4,min((3*i+6)/2,12)
29    a(2*i+3,j+1)=a(i+2*j+1,i+j+3)
30   ENDDOALL
31  ENDIF
32 ENDDOALL
```

For this N=12 case, there is only a single iteration in the intermediate set, particularly iteration $(2, 6)$. Therefore the WHILE loop is simplified away. For general N the WHILE loop can not be removed. When $n > 1$, the maximum distance between any two iterations in the iteration space is $L = \sqrt{2}n$. Let $a = |det(\mathbf{T})| = 2$, thus the longest critical path has at most $\lfloor \log_a(L)+1 \rfloor = \lfloor \log_2(n)+0.5 \rfloor$ iterations by theorem 1.

Example 3 Consider the previous imperfect nested loop example from Chen et al [6]:

```
DO I=1,N
 DO J=1,I
  DO K=J,I
   ... = a(I+2*K+5,4*K-J)
  ENDDO
  a(I-J,I+J)= ...
 ENDDO
ENDDO
```

Our recurrence chain partitioning is applied to find an empty intermediate set P_2, the result code is generated as follows (a visualization can be seen in [28]).

```
 1 DOALL I=1,N
 2  DOALL J=1,I
 3   DOALL K=J,I
 4    ... = a(I+2*K+5,4*K-J)
 5   ENDDOALL
 6   IF (I-J-7.LE.3*((I+J)/4)) THEN
 7    a(I-J,I+J)=...
 8   ENDIF
 9  ENDDOALL
10 ENDDOALL
11 DOALL I=30,N
12  DOALL J=1,(I-23)/7
13   IF (I+J+1.LE.4*((I-J-5)/3)) THEN
14    a(I-J,I+J)=...
15   ENDIF
16  ENDDOALL
17 ENDDOALL
```

Lines 1-10 are P_1 and 11-17 are P_3. Compare with the DOACROSS loop generated in [6], this code has only DOALL loops and theoretically can finish in two iteration time.

Example 4 Cholesky is a kernel in the NASA benchmarks, in which two imperfectly nested loops contain non-uniform dependences.

```
  DO 1 J=0, N
   I0=MAX( -M, -J)
   DO 2 I=I0, -1
    DO 3 JJ=I0-I, -1
     DO 3 L=0, NMAT
 3  a(L,I,J)=a(L,I,J)-a(L,JJ,I+J)*a(L,I+JJ,J)
    DO 2 L=0, NMAT
 2   a(L,I,J)=a(L,I,J)*a(L,0,I+J)
   DO 4 L=0, NMAT
 4  epss(L)=EPS*a(L,0,J)
   DO 5 JJ=I0, -1
    DO 5 L=0, NMAT
 5  a(L,0,J)=a(L,0,J)-a(L,JJ,J)**2
   DO 1 L=0, NMAT
 1  a(L,0,J)=1./SQRT(ABS(epss(L)+a(L,0,J)))
  DO 6 I=0, NRHS
   DO 7 K=0, N
    DO 8 L=0, NMAT
 8   b(I,L,K)=b(I,L,K)*a(L,0,K)
    DO 7 JJ=1, MIN(M, N-K)
     DO 7 L=0, NMAT
 7    b(I,L,K+JJ)=b(I,L,K+JJ)
  *      -a(L,-JJ,K+JJ)*b(I,L,K)
   DO 6 K=N, 0, -1
    DO 9 L=0, NMAT
 9   b(I,L,K)=b(I,L,K)*a(L,0,K)
    DO 6 JJ=1, MIN(M, K)
     DO 6 L=0, NMAT
 6    b(I,L,K-JJ)=b(I,L,K-JJ)
  *       -a(L,-JJ,K)*b(I,L,K)
```

When parameters `NMAT=250, M=4, N=40, NHRS=3`, it takes 238 partitioning steps for the compiler to finish the recurrence dataflow partitioning (the result code is not shown here to save space). Because there are multiple coupled subscripts and generally compile-time unknown parameters `NMAT, M, N, NHRS`, the PDM partitioning is applied:

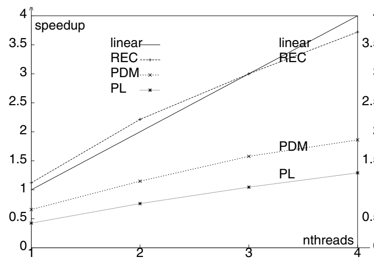
```
 1   DOALL 6 L=0, NMAT
 2    DO 1 J=0, N
 3     I0=MAX(-M,-J)
 4     DO 2 I=I0, -1
 5      DO 3 JJ=I0-I, -1
 6 3     A(L,I,J)=A(L,I,J)-A(L,JJ,I+J)*A(L,I+JJ,J)
 7 2    A(L,I,J)=A(L,I,J)*A(L,0,I+J)
 8 4   EPSS(L)=EPS*A(L,0,J)
 9     DO 5 JJ=I0, -1
10 5    A(L,0,J)=A(L,0,J)-A(L,JJ,J)**2
11 1   A(L,0,J)=1./SQRT(ABS(EPSS(L)+A(L,0,J)))
12     DOALL 6 I=0, NRHS
13      DO 7 K=0, N
14 8     B(I,L,K)=B(I,L,K)*A(L,0,K)
15       DOALL 7 JJ=1, MIN(M, N-K)
16 7      B(I,L,K+JJ)=B(I,L,K+JJ)
17   *         -A(L,-JJ,K+JJ)*B(I,L,K)
```

```
      DO 6 K=N, 0, -1
9       B(I,L,K)=B(I,L,K)*A(L,0,K)
        DOALL 6 JJ=1, MIN(M, K)
6         B(I,L,K-JJ)=B(I,L,K-JJ)
  *          -A(L,-JJ,K)*B(I,L,K)
```

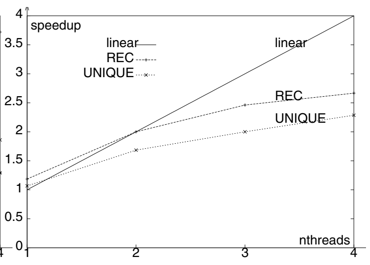
Experiments To observe the performance results, one has to take the parallel loop overhead and loop granularity into considerations. The experiments have been performed on a SMP Linux system with 4 identical Itanium CPU's. The back-end Intel compiler accepts OpenMP directives [7] to generate light-weighted threads. A code region is indicated as parallel by a directive pair: `c$omp parallel` and `c$omp end parallel`. Nested outermost DOALL loops are coalesced into a single parallel loop. Barrier synchronization is only necessary at the borders of the partition sets P1/P2 and P2/P3, directive `c$omp end do nowait` is used between the DOALL nests that are generated from a fully parallel set. The speedup is given as the ratio between the original sequential execution time and the multi-threads execution time where environment variable `OMP_NUM_THREADS` specifies the number of CPU used. The four examples subjected to the partitioning methods are shown in figure 3. For Example 1 with parameters `N1`=300, `N2`=1000, the PL [9], PDM [27] and REC speedups are compared. The REC speedup is better than linear when the number of threads is smaller than 3 because array subscripts calculations are simplified in the recurrence WHILE loop. However, it drops below linear when number of threads is larger than 3 because the loop bounds calculation gets more overhead. For Example 2 with parameter `N`=300, the UNIQUE [11] and REC speedups are compared. They both outperforms linear speed when executed on single CPU because the convex loop index calculations are optimized by Omega calculator. REC outperforms UNIQUE because it generates shorter sequence of fully parallel regions. For Example 3 with parameter `N`=300, speedups of the REC partitioning, inner loop J, K parallelization [25], and the DOACROSS parallelization [6] are compared. REC performs the best because it has least synchronizations. For Example 4 with parameters `NMAT`=250, `M`=4, `N`=40 and `NRHS`=3, PDM [27] and REC dataflow partitioning speedup results are shown. REC partitioning outperforms PDM and even linear program when nthread is smaller than three because of the loop bounds optimization by Omega calculator. When the number of threads is larger than 3, however, the simpler PDM partitioning performs better because it has better load balance.

5. Related work

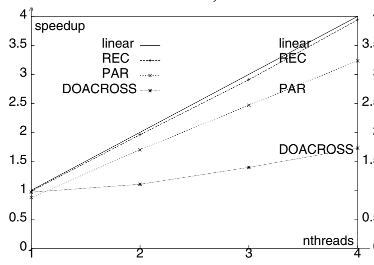
To test loop parallelism for non-uniform dependences, the *range test* [5] is based on intersection of the value range of non-linear expressions to mark a loop parallel for an empty range. Since the dependence range is less exact, our recurrence chain partitioning uses the *Omega* test [17] to solve the dependence relation based on exact integer programming. The zero columns of pseudo distance matrix (PDM) are first used to test for parallel loops, then the Omega test is used for those loops with non-zero columns in the PDM.



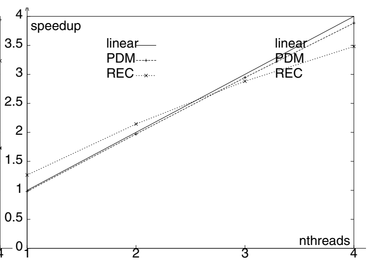
Ex. 1: N1=300, N2=1000



Ex. 2: N=300



Ex. 3: N=300



Ex. 4: NMAT=250, M=4, N=40, NRHS=3

Figure 3. Speedup on Itanium multiprocessor

Wolf et al [24] extend the uniform distance vectors to *dependence vectors*, i.e., each element of the dependence vector is either a constant or a direction sign. Both distance and direction vectors are treated in the same framework of dependence vectors. This leads to outermost loop parallelization as well as innermost loop parallelization by unimodular transformations. For non-uniform dependences, however, the direction vector representation introduces more artificial distance vectors than dependence uniformization: it is equivalent to use the basis of the vector space as pseudo distance vectors which may have a higher rank and a smaller determinant than the PDM derived from distance vectors. An algorithm in [24] can find a legal unimodular transformation that reduces the outermost columns of a distance matrix to zero. However, this algorithm can not be used for the pseudo distance matrix because the unimodular transformation found are not always legal when there are non-uniform dependences.

Shang et al [26] represent the non-uniform distances as an affine (*non-negative* linear) combination of the basic dependence vectors (BDV), which are not lexicographically positive. The Basic Ideas I and III generate a set of full-rank BDV which inhibits parallelizing the outermost loops by a unimodular transformation, while the Basic Idea II searches for a set of *cone-optimal* BDV, i.e., the BDV are minimal in rank. Because the lexicographical positiveness is not carried by the BDV, an additional *linear scheduling* [10] is needed to maintain the lexicographical order.

Tzen et al [23] and Chen et al [6] implement the BDV linear scheduling by DO-ACROSS loops synchronization. DOACROSS loops allow the iterations to be asynchronously executed within a delay, which is enforced by P/V synchronization on the loop index. DOACROSS synchronization is more complex than the barrier synchronization of DOALL loops. Though no parallelism is obtained using PDM partitioning for their example shown in example 3, two perfectly nested DOALL loops can be obtained using recurrence chain partitioning.

Punyamurtula et al [19] propose the minimum distance tiling that runs the adjacent iterations in parallel as long as their distance is smaller than the minimum dependence distances. After making the minimum distances tiling of the iteration space, Tzen or Chen's method is used for the inter-tile dependences. This method creates innermost parallelism whereas PDM partitioning creates outermost parallelism. Theoretically, it speedups Example 2 by 4 times.

Ju et al [11] propose unique-set oriented partitioning to exploit exact non-uniform dependences: The dependence convex hulls are separated into head or tail sets by lexicographical order. The first recurrence equation is called "flow" and the second is called "anti", which split the head or tail sets. The intersections among the (head, tail) $\times$ (flow,anti) sets yield 5 individual cases. The method also applies only to one pair of subscripts with non-singular A, B matrices, otherwise their coefficients calculation will divide by zero. Using their approach on Example 2, 5 perfectly nested DOALL loops were obtained in sequence [11]p.334. The number of iterations is not 144 due to apparent errors in the loop bounds of the 3rd and 4th loop nests. We recalculated the example with their method and found that two of the 5 unique sets can not be written as perfectly nested loops because they are not convex sets. Among the five unique sets, the third one is sequential. Whereas applying the recurrence chain partitioning, only 3 fully parallel partitions are obtained, resulting in more parallelism.

6. Conclusion

This paper presents two partitioning schemes, based on recurrence chains, to find outermost parallelism for loops with non-uniform dependences. Comparing to the previously discovered pseudo distance matrix (PDM) method [27], recurrence chains partitioning is an enhancement when the loops has a single pair of coupled subscripts or with symbolic affine bounds. When the loop has non-linear bounds and multiple pairs of coupled subscripts, PDM can still be applied. The advantage of REC lies in the dataflow partitioning for non-uniform dependences.

References

[1] R. Allen and K. Kennedy. Automatic translation of Fortran programs to vector form. *TOPLAS*, 9(4):491–542, Oct 1987.

[2] U. Banerjee. Unimodular transformations of double loops. In *Advances in Languages and Compilers for Parallel Computing, 1990 Workshop*, pages 192–219, Aug. 1990.

[3] U. Banerjee. *Loop transformations for restructuring compilers: the foundations*. Kluwer Academic, 1993. 305 p.

[4] U. Banerjee, R. Eigenmann, A. Nicolau, and D. A. Padua. Automatic program parallelization. *Proc. of the IEEE*, 81(2):211–243, Feb 1993.

[5] W. Blume and R. Eigenmann. The Range test: a dependence test for symbolic, non-linear expressions. In *Proceedings, Supercomputing '94*, pages 528–537. IEEE, 1994.

[6] D. Chen and P. Yew. On the effective execution of nonuniform DOACROSS loops. *TPDS*, 7(5):463–476, May 1996.

[7] D. Clark. OpenMP: A parallel standard for the masses. *IEEE Concurrency*, 6(1):10–12, JAN-MAR 1998.

[8] E. D'Hollander, F. Zhang, and Q. Wang. The Fortran parallel transformer and its programming environment. *Journal of Information Sciences*, 106:293–317, 1998.

[9] E. H. D'Hollander. Partitioning and labeling of loops by unimodular transformations. *TPDS*, 3(4):465–476, Jul 1992.

[10] P. Feautrier. Some efficient solutions to the affine scheduling problem. I. One-dimensional time. *International Journal of Parallel Programming*, 21(5):313–347, Oct 1992.

[11] J. Ju and V. Chaudhary. Unique sets oriented parallelization of loops with non-uniform dependences. *The Computer Journal*, 40(6):322–339, 1997.

[12] W. Kelly and W. Pugh. Minimizing communication while preserving parallelism. In *Supercomputing'96*, pages 52–60. ACM, 1996.

[13] W. Kelly, W. Pugh, and E. Rosser. Code generation for multiple mappings. In *The 5th Symposium on the Frontiers of Massively Parallel Computation*, 1995.

[14] X. Kong, D. Klappholz, and K. Psarris. The I-test - an improved dependence test for automatic parallelization and vectorization. *TPDS*, 2(3):342–349, jul 1991.

[15] A. W. Lim and M. S. Lam. Maximizing parallelism and minimizing synchronization with affine partitions. *Parallel Computing*, 24(3-4):445–475, May 1998.

[16] D. A. Padua and M. J. Wolfe. Advanced compilers optimizations for supercomputers. *CACM*, 29(12):1184–1201, Dec 1986.

[17] P. M. Petersen and D. A. Padua. Static and dynamic evaluation of data dependence analysis techniques:. *TPDS*, 7(11):1121–1132, Nov 1996.

[18] W. Pugh. A practical algorithm for exact array dependence analysis. *CACM*, 35(8):102–114, Aug 1992.

[19] S. Punyamurtula, V. Chaudhary, J. Ju, and S. Roy. Compile time partitioning of nested loop iteration spaces with non-uniform dependences. *Journal of Parallel Algorithms and Applications*, 13(1):113–141, Jan. 1999.

[20] L. Rauchwerger, N. M. Amato, and D. A. Padua. A scalable method for run-time loop parallelization. *International Journal of Parallel Programming*, 23(6):537–576, 1995.

[21] Z. Shen, Z. Li, and P.-C. Yew. An empirical study of Fortran programs for parallelizing compilers. *TPDS*, 1(3):356–364, July 1990.

[22] J. Subhlok and K. Kennedy. Integer programming for array subscript analysis. *TPDS*, 6(6):662–668, June 1995.

[23] T. Tzen and L. Ni. Dependence uniformization: A loop parallelization technique. *TPDS*, 4:547–558, May 1993.

[24] M. E. Wolf and M. S. Lam. A loop transformation theory and an algorithm to maximize parallelism. *TPDS*, 2(4):452–471, Oct 1991.

[25] M. Wolfe and C. Tseng. The POWER test for data dependence. *TPDS*, 3(5):591–601, sep 1992.

[26] W.Shang, E.Hodzic, and Z.Chen. On uniformization of affine dependence algorithms. *IEEE Trans. Computers*, 45(7):827–40, 1996.

[27] Y. Yu and E. D'Hollander. Partitioning loops with variable dependence distances. In *ICPP'00*, pages 209–218. IEEE, Aug 2000.

[28] Y. Yu and E. D'Hollander. Loop parallelization using the 3D iteration space visualizer. *Journal of Visual Languages and Computing*, 12(2):163–181, April 2001.

[29] C.-Q. Zhu and P.-C. Yew. A scheme to enforce data dependence on large multiprocessor systems. *TSE*, 13(6):726–739, Jun 1987.

[30] H. Zima, H. Bast, and M. Gerndt. SUPERB - a tool for semi-automatic MIMD SIMD parallelization. *Parallel Computing*, 6(1):1–18, Jan 1988.