Computational Bionics: Project Exercise 1

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Task 1: Modeling the Leg

1.1 Description

The following section considers an ODE model of the leg.

A test person with a body mass of $m_b = 100$ kg is chosen.

Following state variables are involved:

- q_1: Extension angle of the **hip** around then transversal axis (xy-plane)
- ω_1: Angular velocity of the tight
- q_2: Rotation angle of the **knee** around the transversal axis (xy-plane)
- ω_2 Angular velocity of the shank

The segement masses are considered as point masses in the middle of each segment.

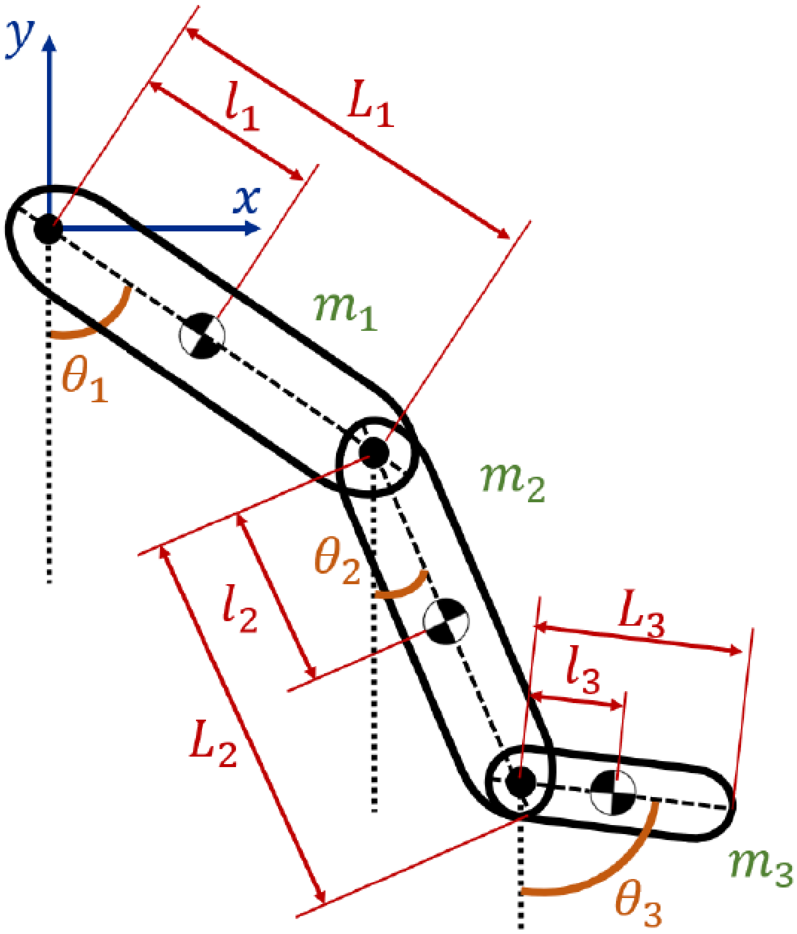
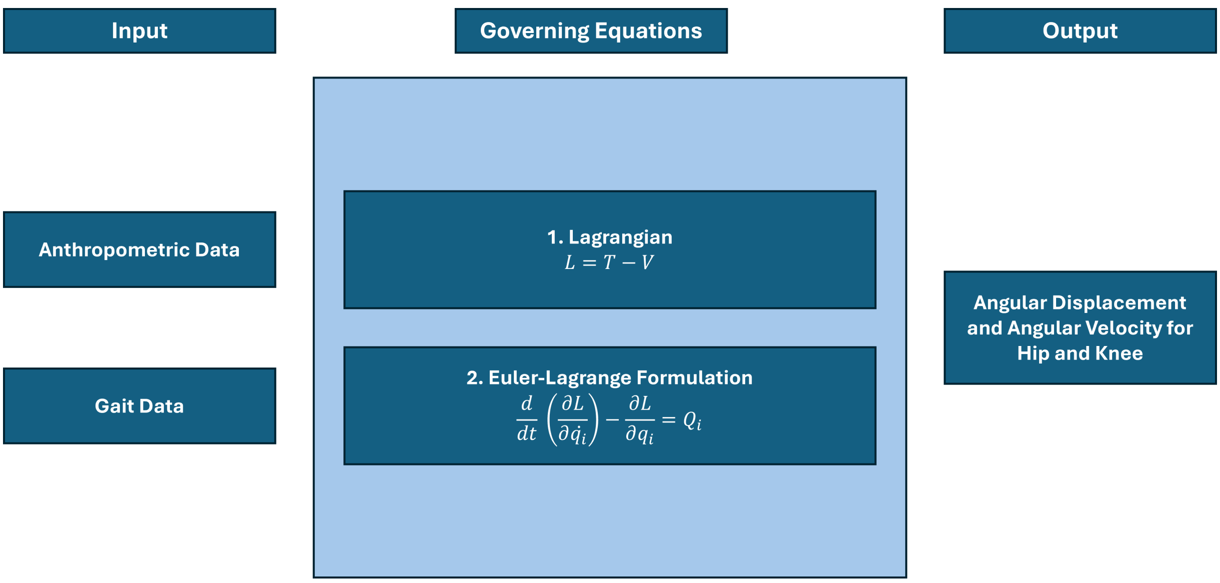


Figure 1. Representation of human lower limb for dynamic modeling

In this example the foot is treated as point mass located at the end of the shank. Hence, we only consider two sections for the tigh and the shank with the corresponding flexion and extension angles q_1 and q_2

1.2 Simplified Box Model



1.3. Anthropometric Data

Segment	Parameter	Symbol	Value	Unit
Tigh	Mass	m_1	9.73	kg
Tigh	Length	L_1	0.410	m
Tigh	Proximal COM	l_1	0.205	m
Tigh	Radius of Gyration	r_{G1}	0.132	m
Tigh	Moment of Inertia	J_1	0.1695	kgm^2
Tigh	Viscous Damping Coefficient	b_2	0.1	Nms
Shank	Mass	m_2	5.07	kg
Shank	Length	L_2	0.415	m
Shank	Proximal COM	l_2	0.2075	m
Shank	Radius of Gyration	r_{G1}	0.125	m
Shank	Moment of Inertia	J_2	0.0792	kgm^2
Shank	Viscous Damping Coefficient	b_2	0.1	Nms
Foot	Mass	m_3	0.44	kg

Sources:

https://revistas.udistrital.edu.co/index.php/reving/article/view/20333/19807

https://personal.cityu.edu.hk/meachan/Online%20Anthropometry/Chapter2/Ch2-5.htm

https://pmc.ncbi.nlm.nih.gov/articles/PMC5305206/table/pone.0172112.t001/

1.4 Derivation

In [305…

```
import sympy as sp
from sympy import sin, cos, Matrix, pi
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp
from scipy.interpolate import interp1d
```

Note: q_1, q_2 correspond to θ_1, θ_2

In [306…

```
# Define the symbolic variables
q1, q2, omega1, omega2 = sp.symbols(r'q_1 q_2 \omega_1 \omega_2')
dot_omega1, dot_omega2 = sp.symbols(r'\dot{\omega}_1 \dot{\omega}_2')
l1, l2, L1, L2 = sp.symbols(r'l_1 l_2 L_1 L_2')
m1, m2, m3 = sp.symbols(r'm_1 m_2 m_3')
J1, J2 = sp.symbols(r'J_1 J_2')
T1_sym, T2_sym = sp.symbols(r'T_1 T_2')
```

```python
b1, b2 = sp.symbols(r'b_1 b_2')
t, g = sp.symbols('t g')

q1 = sp.Function('q_1')(t)
q2 = sp.Function('q_2')(t)

w1 = q1.diff(t)
w2 = q2.diff(t)

dot_w1 = w1.diff(t)
dot_w2 = w2.diff(t)
```

Position vectors to the centre of mass of tight r_1, shank r_2, and foot r_3 with respect to the hip H

$$\mathbf{r}_1 = l_1 \cdot \begin{bmatrix} \sin q_1 \\ -\cos q_1 \\ 0 \end{bmatrix}$$

$$\mathbf{r}_2 = \begin{bmatrix} L_1 \sin q_1 + l_2 \sin q_2 \\ -L_1 \cos q_1 - l_2 \cos q_2 \\ 0 \end{bmatrix}$$

$$\mathbf{r}_3 = \begin{bmatrix} L_1 \sin q_1 + L_2 \sin q_2 \\ -L_1 \cos q_1 - L_2 \cos q_2 \\ 0 \end{bmatrix}$$

In [307…

```python
# Position of the center of mass for tigh, shank and foot
r1 = Matrix([l1 * sin(q1),
             -l1 * cos(q1)])

r2 = Matrix([L1 * sin(q1) + l2 * sin(q2),
             -L1 * cos(q1) - l2 * cos(q2)])

r3 = Matrix([L1 * sin(q1) + L2 * sin(q2),
            -L1 * cos(q1) - L2 * cos(q2)])
```

Velocity vectors to the centre of mass of tight SO, shank SU, and foot S with $\dot{q}_i = \omega_i$

$$\mathbf{v}_1 = l_1 \cdot \omega_1 \begin{bmatrix} \cos q_1 \\ \sin q_1 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_2 = L_1 \cdot \omega_1 \begin{bmatrix} \cos q_1 \\ \sin q_1 \\ 0 \end{bmatrix} + l_2 \cdot \omega_2 \begin{bmatrix} \cos q_2 \\ \sin q_2 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_3 = L_1 \cdot \omega_1 \begin{bmatrix} \cos q_1 \\ \sin q_1 \\ 0 \end{bmatrix} + L_2 \cdot \omega_2 \begin{bmatrix} \cos q_2 \\ \sin q_2 \\ 0 \end{bmatrix}$$

In [308…

```python
# Velocity of the center of mass for tigh, shank and foot
v1 = r1.diff(t)
v2 = r2.diff(t)
v3 = r3.diff(t)
```

```
# sp.print_latex(v3.subs({q1.diff(t): omega1, q2.diff(t): omega2}))
```

Kinetic Energy T

$$T_1 = \frac{1}{2} m_1 \mathbf{v}_1^T \mathbf{v}_1 + \frac{1}{2} \cdot J_1 \omega_1^2$$

$$T_2 = \frac{1}{2} m_2 \mathbf{v}_2^T \mathbf{v}_2 + \frac{1}{2} \cdot J_2 \omega_2^2$$

$$T_3 = \frac{1}{2} m_3 \mathbf{v}_3^T \mathbf{v}_3$$

Note: The foot is only considered as a point mass located at the end of the shank. It only has a translational energy component.

In [309…

```
# Kinetic energy of the system
T1 = 0.5 * m1 * v1.dot(v1) + 0.5 * J1 * q1.diff(t)**2
T2 = 0.5 * m2 * v2.dot(v2) + 0.5 * J2 * q2.diff(t)**2
T3 = 0.5 * m3 * v3.dot(v3)

T_total = T1 + T2 + T3
```

Potential Energy V

$$V_1 = m_1 \cdot g \cdot 0.5 \cdot L_1 \cdot \sin q_1$$

$$V_2 = m_2 \cdot g \cdot (L_1 \sin q_1 + 0.5 \cdot L_2 \sin q_2)$$

$$V_3 = m_3 \cdot g \cdot (L_1 \sin q_1 + \cdot L_2 \sin q_2)$$

In [310…

```
# Potential energy of the system
h1 = -l1 * cos(q1)
h2 = -L1 * cos(q1) - l2 * cos(q2)
h3 = -L1 * cos(q1) - L2 * cos(q2)

V1 = m1 * g * h1
V2 = m2 * g * h2
V3 = m3 * g * h3

V_total = V1 + V2 + V3
```

Lagrangian

$$L = T - V$$

$$L = T_1 + T_2 + T_3 - V_1 - V_2 - V_3$$

In [311…

```
L = T_total - V_total
```

Euler-Lagrange Formulation:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = T_1 - b_1\dot{q}_1 - b_2(\dot{q}_1 - \dot{q}_2)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} = T_2 - b_2(\dot{q}_2 - \dot{q}_1)$$

- q_i: Set of the generalized coordinates
- Q_i: Set of external (non-conservative) forces applied to the system (2 DOF are associated with the hip and knee joint)
- T_n: Torque acting at each joint
- b_n: viscous damping coefficient

In [312…

```
dL_d_dot_q1 = L.diff(q1.diff(t))
dL_d_dot_q2 = L.diff(q2.diff(t))

dL_d_dot_q1_dt = dL_d_dot_q1.diff(t)
dL_d_dot_q2_dt = dL_d_dot_q2.diff(t)

dL_dq1 = L.diff(q1)
dL_dq2 = L.diff(q2)

# Substitute the values of the parameters
subsDict = {q1.diff(t): omega1,
            q2.diff(t): omega2,
            q1.diff(t, 2): dot_omega1,
            q2.diff(t, 2): dot_omega2,}

dL_d_dot_q1_dt.subs(subsDict).simplify()
dL_d_dot_q2_dt.subs(subsDict).simplify()

dL_dq1.subs(subsDict).simplify()
dL_dq2.subs(subsDict).simplify();
```

Approach 1:

In [313…

```
Q1 = T1_sym - b1 * q1.diff(t) - b2 * (q1.diff(t) - q2.diff(t))
Q2 = T2_sym - b2 * (q2.diff(t) - q1.diff(t))

Q1.subs(subsDict).simplify()
Q2.subs(subsDict).simplify();
```

Approach 2:

In [314…

```
# Generalized for Q1 and Q2 - Variant 2
Q1_sym, Q2_sym = sp.symbols(r'Q_1 Q_2')

eq1_v2 = dL_d_dot_q1_dt - dL_dq1 - Q1_sym
eq2_v2 = dL_d_dot_q2_dt - dL_dq2 - Q2_sym

eq1_v2 = eq1_v2.subs(subsDict).simplify()
eq2_v2 = eq2_v2.subs(subsDict).simplify()

sol_v2 = sp.solve([eq1_v2, eq2_v2], [dot_omega1, dot_omega2])

# approx 35 seconds
```

```python
dot_omega1_sol_v2 = sol_v2[dot_omega1].simplify()
dot_omega2_sol_v2 = sol_v2[dot_omega2].simplify()
```

Setting up the system of equations:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = T_1 - b_1\dot{q}_1 - b_2(\dot{q}_1 - \dot{q}_2) \qquad (I)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} = T_2 - b_2(\dot{q}_2 - \dot{q}_1) \qquad (II)$$

In [315…

```python
# Runtime: ~ 35 seconds

eq1 = dL_d_dot_q1_dt - dL_dq1 - Q1
eq2 = dL_d_dot_q2_dt - dL_dq2 - Q2

eq1 = eq1.subs(subsDict).simplify()
eq2 = eq2.subs(subsDict).simplify()

# Solve for omega1_dot and omega2_dot
sol = sp.solve([eq1, eq2], (dot_omega1, dot_omega2))

# Extracting and simplifying the solutions
dot_omega1_sol = sol[dot_omega1].simplify()
dot_omega2_sol = sol[dot_omega2].simplify()
```

Defining the Parameters:

In [316…

```python
m_body = 100 # Body mass [kg]
m1_val = 9.73; m2_val = 5.07; m3_val = 0.44 # Masses [kg]

L1_val = 0.41; l1_val = L1_val / 2 # Lengths thigh [m]
L2_val = 0.415; l2_val = L2_val / 2 # Lengths shank [m]

r_G1 = 0.132; r_G2 = 0.125 # Radius of gyration [m]
J1_val = m1_val * r_G1**2; J2_val = m2_val * r_G2**2 # Moments of inertia [kg m^2]

b1_val = 0.1; b2_val = 0.1 # Damping coefficients [N m s/rad]

g_val = 9.81 # Gravity [m/s^2]

# Subustitute dictionary
parameterDict = {m1: m1_val, m2: m2_val, m3: m3_val,
                 L1: L1_val, l1: l1_val, L2: L2_val, l2: l2_val,
                 J1: J1_val, J2: J2_val,
                 b1: b1_val, b2: b2_val,
                 g: g_val}
```

Transform the obtained solution for $\dot{\omega}_i$ into a numerical function:

Approach 1:

In [317…

```python
# Substitute the values of the parameters
dot_omega1_sol = dot_omega1_sol.subs(parameterDict)
dot_omega2_sol = dot_omega2_sol.subs(parameterDict)

# Lambdify the equations
```

```
dot_omega1_func = sp.lambdify((q1, q2, omega1, omega2, T1_sym, T2_sym), dot_omega1_sol,
dot_omega2_func = sp.lambdify((q1, q2, omega1, omega2, T1_sym, T2_sym), dot_omega2_sol,
```

Approach 2:

In [318…

```
# Substitute the values of the parameters
dot_omega1_sol_v2 = dot_omega1_sol_v2.subs(parameterDict)
dot_omega2_sol_v2 = dot_omega2_sol_v2.subs(parameterDict)

# Lambdify the equations
dot_omega1_func_v2 = sp.lambdify((q1, q2, omega1, omega2, Q1_sym, Q2_sym), dot_omega1_sc
dot_omega2_func_v2 = sp.lambdify((q1, q2, omega1, omega2, Q1_sym, Q2_sym), dot_omega2_sc
```

Read provided Gait Data File:

In [319…

```
# Read gait data
filename = 'gait_data.xls'
gait_data = pd.read_excel(filename, engine='xlrd')

# Extract gait data
gait_step = np.array(gait_data["gait_%"]) / 100

GRFz = np.array(gait_data["GRFz[%BW]"]) * m_body
GRFx = gait_data["GRFx[%BW]"] * m_body

MX_H = np.array(gait_data["MX_H[Nm/kg]"]) * m_body
MX_K = np.array(gait_data["MX_K[Nm/kg]"]) * m_body
MX_F = np.array(gait_data["MX_F[Nm/kg]"]) * m_body

q1_gait = np.deg2rad(np.array(gait_data["Flex_Ext_H[deg]"]))
q2_gait = np.deg2rad(np.array(gait_data["Flex_Ext_K[deg]"]))
```

Extra- and Interpolation of Gait Data

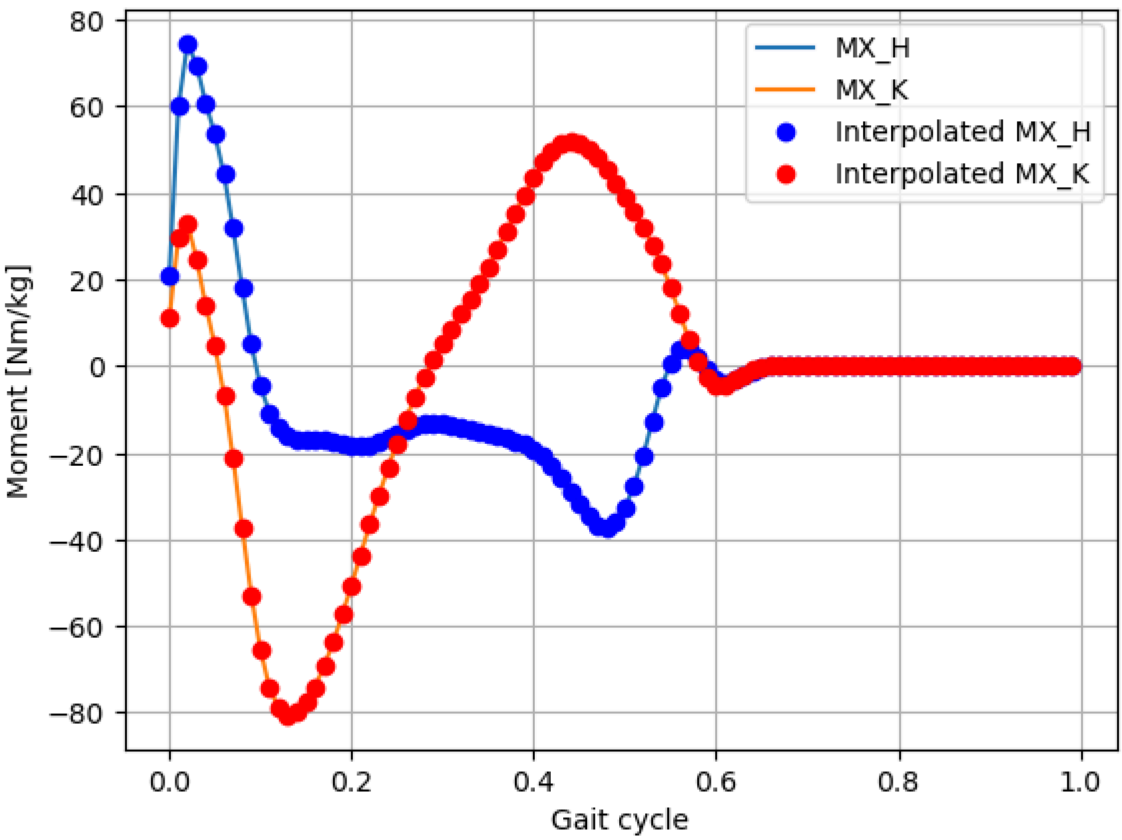
In [320…

```
# Precompute interpolating functions for M1 and M2
MX_H_interp = interp1d(gait_step, MX_H, kind='cubic', fill_value='extrapolate')
MX_K_interp = interp1d(gait_step, MX_K, kind='cubic', fill_value='extrapolate')

# Derivatives of MX_H and MX_K with respect to time using the interpolating functions
dMX_H_dt = np.gradient(MX_H_interp(gait_step), gait_step)
dMX_K_dt = np.gradient(MX_K_interp(gait_step), gait_step)

# Interpolating functions for derivatives of MX_H and MX_K
dMX_H_dt_interp = interp1d(gait_step, dMX_H_dt, kind='cubic', fill_value='extrapolate')
dMX_K_dt_interp = interp1d(gait_step, dMX_K_dt, kind='cubic', fill_value='extrapolate')

# Evaluate the interpolating functions at the gait time
dMX_H_dt_gait = dMX_H_dt_interp(gait_step)
dMX_K_dt_gait = dMX_K_dt_interp(gait_step)

# Interpolating functions for q1 and q2
q1_gait_interp = interp1d(gait_step, q1_gait, kind='cubic', fill_value='extrapolate')
q2_gait_interp = interp1d(gait_step, q2_gait, kind='cubic', fill_value='extrapolate')

# Derivatives of q1 and q2 with respect to time using the interpolated functions
dq1_dt = np.gradient(q1_gait_interp(gait_step), gait_step)
dq2_dt = np.gradient(q2_gait_interp(gait_step), gait_step)
```

```
# Interpolating functions for angular velocities
gait_omega1_interp = interp1d(gait_step, dq1_dt, kind='cubic', fill_value='extrapolate')
gait_omega2_interp = interp1d(gait_step, dq2_dt, kind='cubic', fill_value='extrapolate')

# Evaluate the interpolating functions at gait_step
gait_omega1 = gait_omega1_interp(gait_step)
gait_omega2 = gait_omega2_interp(gait_step)
```

In [321…

```
# plot interpolated moments
plt.figure()
plt.plot(gait_step, MX_H, label='MX_H')
plt.plot(gait_step, MX_K, label='MX_K')
plt.plot(gait_step, MX_H_interp(gait_step), 'bo',  label='Interpolated MX_H')
plt.plot(gait_step, MX_K_interp(gait_step), 'ro', label='Interpolated MX_K')
plt.xlabel('Gait cycle')
plt.ylabel('Moment [Nm/kg]')
plt.legend()
plt.grid()
```



Defining Time Steps

In [322…

```
# Time step
dt = 0.01
t_start = 0
t_end = 1
t_eval = np.arange(t_start, t_end, dt)
```

Initial Conditions

Approach 1:

In [323…
```
# initial conditions
q1_0 = q1_gait[0] # initial angle of the thigh
q2_0 = q2_gait[0] # initial angle of the shank

omega1_0 = gait_omega1[0]
omega2_0 = gait_omega2[0]

T1_0 = MX_H_interp(t_start)
T2_0 = MX_K_interp(t_start)

y0 = [omega1_0, omega2_0, q1_0, q2_0]
```

Approach 2:

In [324…
```
Q1_0 = T1_0 - b1_val * omega1_0 - b2_val * (omega1_0 - omega2_0)
Q2_0 = T2_0 - b2_val * (omega2_0 - omega1_0)

y0_v2 = [omega1_0, omega2_0, q1_0, q2_0, Q1_0, Q2_0]
```

ODE System of Equations

Approach 1:

In [325…
```
# Define the function to solve
def leg_model(t, y):
    omega1, omega2, q1, q2 = y

    T1 = MX_H_interp(t)
    T2 = MX_K_interp(t)

    domega1 = dot_omega1_func(q1, q2, omega1, omega2, T1, T2)
    domega2 = dot_omega2_func(q1, q2, omega1, omega2, T1, T2)

    dq1 = omega1
    dq2 = omega2

    return [domega1, domega2, dq1, dq2]
```

Approach 2:

In [326…
```
# Define the function to solve
def leg_model_2(t, y):
    omega1, omega2, q1, q2, Q1, Q2 = y

    # Get the current external torques at knee and hip
    T1 = MX_H_interp(t)
    T2 = MX_K_interp(t)

    # Calculate the generalized torques Q1 and Q2
    Q1 = T1 - b1_val * omega1 - b2_val * (omega1 - omega2)
    Q2 = T2 - b2_val * (omega2 - omega1)

    # Calculate derivatives of angular velocities
    domega1 = dot_omega1_func_v2(q1, q2, omega1, omega2, Q1, Q2)
    domega2 = dot_omega2_func_v2(q1, q2, omega1, omega2, Q1, Q2)

    # Define the relationship between the generalized torques and the external torques
```

```
    dq1 = omega1
    dq2 = omega2

    dT1 = dMX_H_dt_interp(t)
    dT2 = dMX_K_dt_interp(t)

    dQ1 = dT1 - b1_val * domega1 - b2_val * (domega1 - domega2)
    dQ2 = dT2 - b2_val * (domega2 - domega1)

    return [domega1, domega2, dq1, dq2, dQ1, dQ2]
```

Solving the System:

Approach 1:

In [327…

```
# Solve the ODE
sol = solve_ivp(leg_model, [t_start, t_end], y0, t_eval=t_eval, method='RK45')

# Extract the solution
t_sol = sol.t
omega1_sol = sol.y[0]
omega2_sol = sol.y[1]
q1_sol = sol.y[2]
q2_sol = sol.y[3]
```

Approach 2:

In [328…

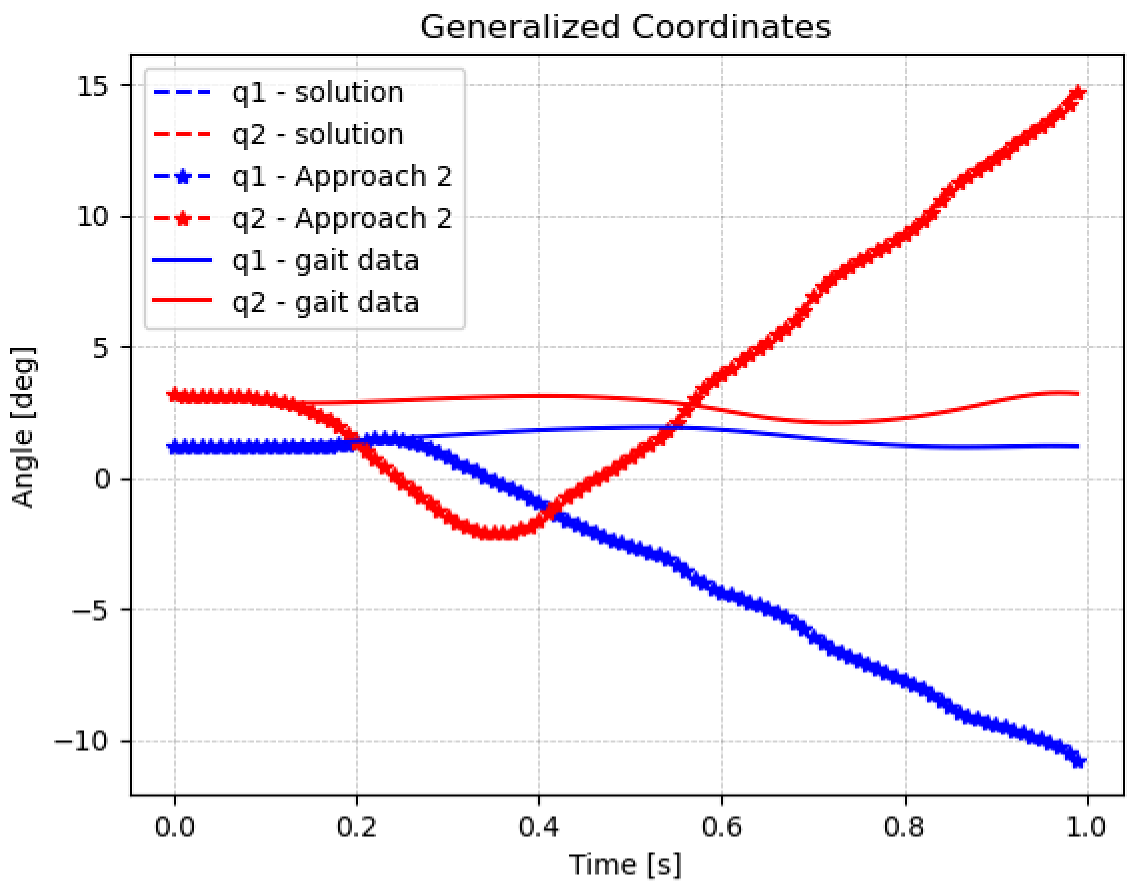
```
# Solve ODE with variant 2
sol_v2 = solve_ivp(leg_model_2, [t_start, t_end], y0_v2, t_eval=t_eval, method='RK45')

# Extract the solution v2
t_sol_v2 = sol_v2.t
omega1_sol_v2 = sol_v2.y[0]
omega2_sol_v2 = sol_v2.y[1]
q1_sol_v2 = sol_v2.y[2]
q2_sol_v2 = sol_v2.y[3]
Q1_sol_v2 = sol_v2.y[4]
Q2_sol_v2 = sol_v2.y[5]
```

Plot the Results:

Approach 1:

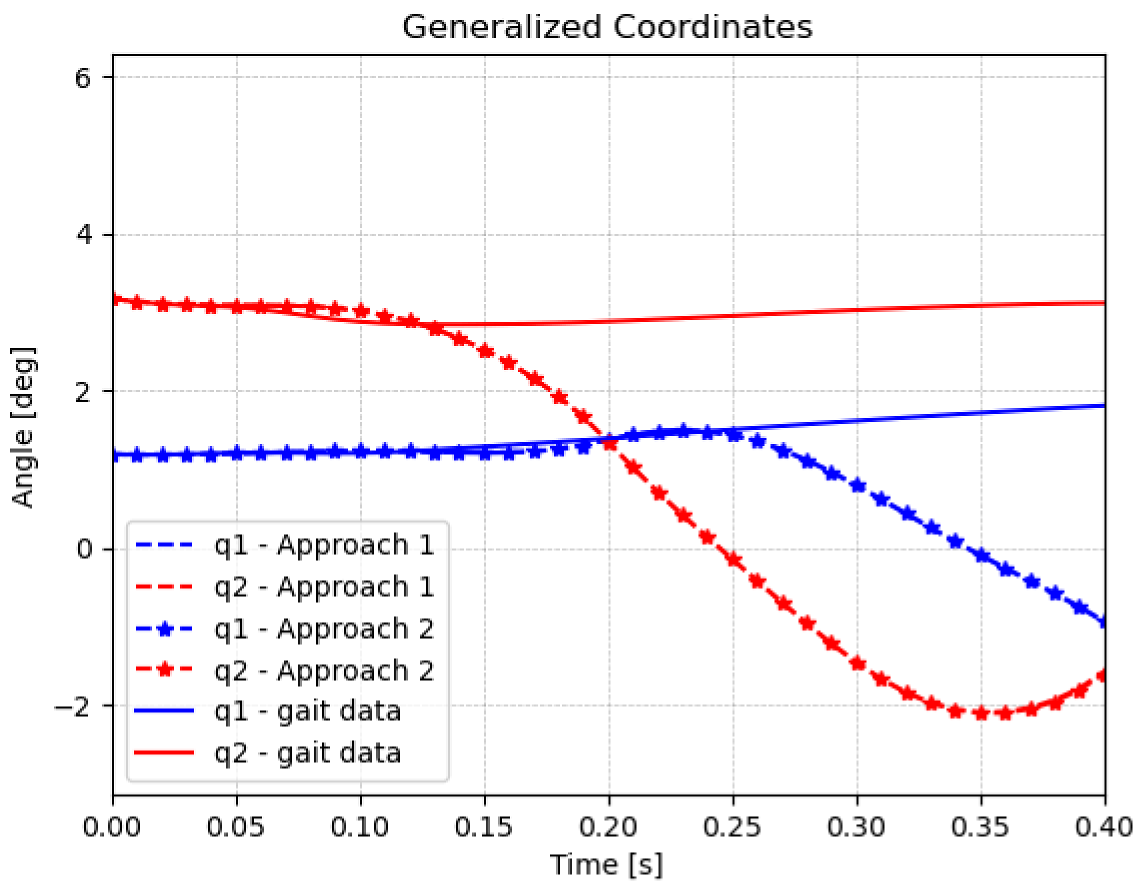
In [329…

```
# plot the angles
plt.figure()
plt.title('Generalized Coordinates')
plt.plot(t_sol, q1_sol, 'b--', label='q1 - solution')
plt.plot(t_sol, q2_sol, 'r--', label='q2 - solution')
plt.plot(t_sol_v2, q1_sol_v2, 'b--*', label='q1 - Approach 2')
plt.plot(t_sol_v2, q2_sol_v2, 'r--*',label='q2 - Approach 2')
plt.plot(gait_step, q1_gait, 'b-', label='q1 - gait data')
plt.plot(gait_step, q2_gait, 'r-',label='q2 - gait data')
plt.xlabel('Time [s]')
plt.ylabel('Angle [deg]')
plt.legend()
plt.grid(which='both', linestyle='--', linewidth=0.5, color='gray', alpha=0.5)
plt.show()
```



In [330…

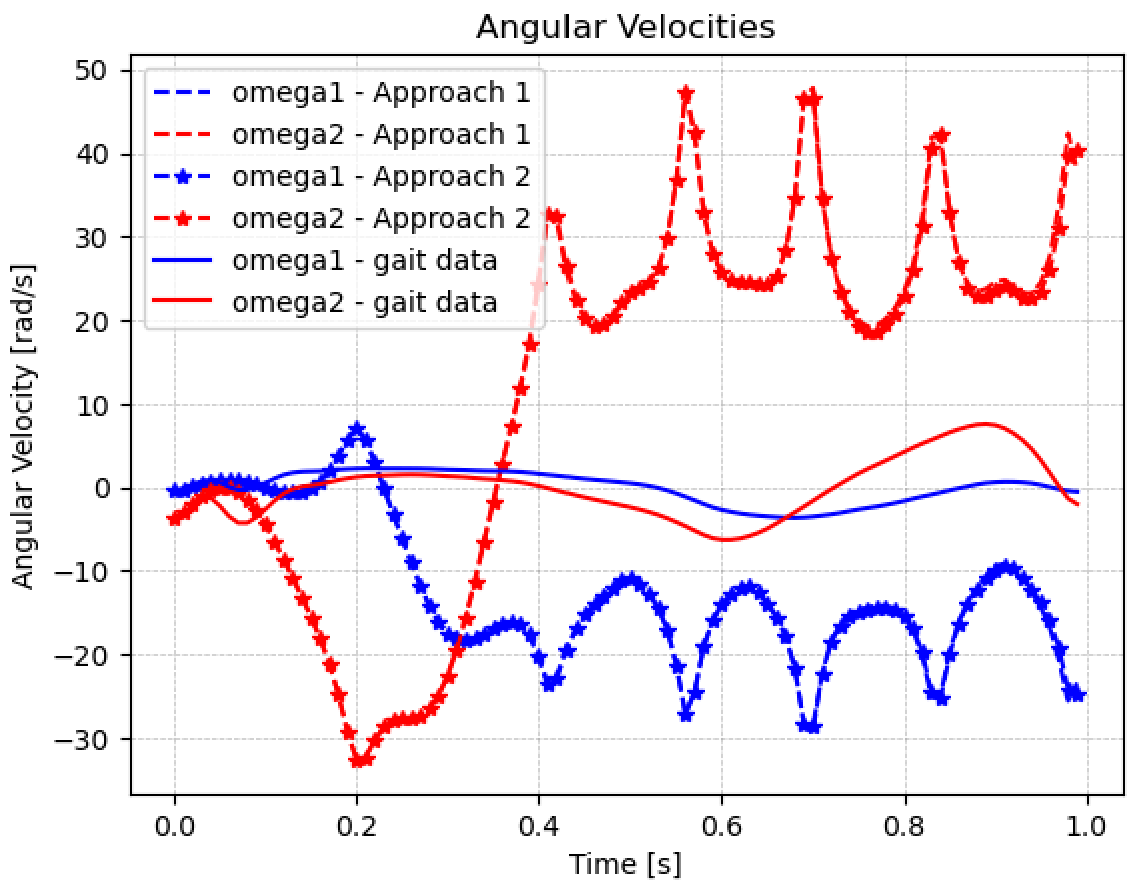
```
# plot the angles
plt.figure()
plt.title('Generalized Coordinates')
plt.plot(t_sol, q1_sol, 'b--', label='q1 - Approach 1')
plt.plot(t_sol, q2_sol, 'r--', label='q2 - Approach 1')
plt.plot(t_sol_v2, q1_sol_v2, 'b--*', label='q1 - Approach 2')
plt.plot(t_sol_v2, q2_sol_v2, 'r--*',label='q2 - Approach 2')
plt.plot(gait_step, q1_gait, 'b-', label='q1 - gait data')
plt.plot(gait_step, q2_gait, 'r-',label='q2 - gait data')
plt.xlabel('Time [s]')
plt.ylabel('Angle [deg]')
plt.ylim(-np.pi, 2*np.pi)
plt.xlim(0, 0.4)
plt.legend()
plt.grid(which='both', linestyle='--', linewidth=0.5, color='gray', alpha=0.5)
plt.show()
```



Plotting Angular Velocities

In [331…

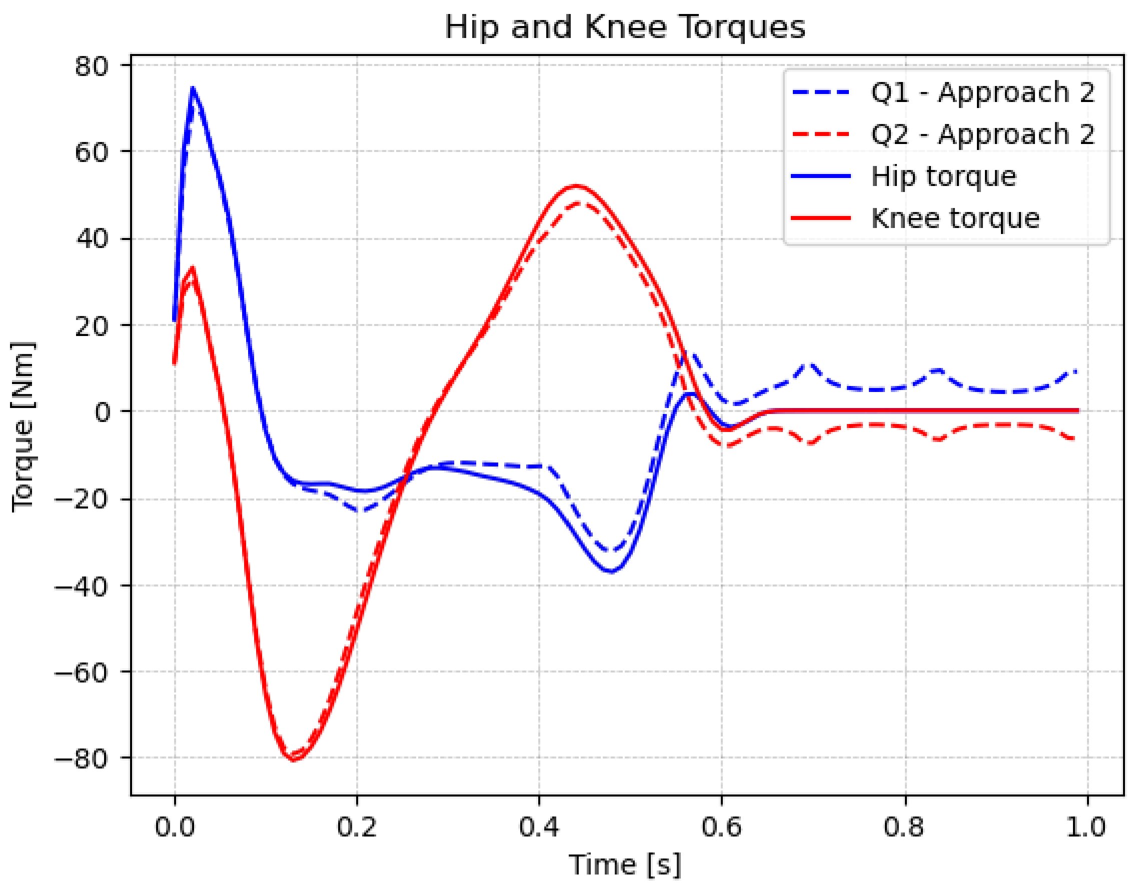
```
plt.figure()
plt.title('Angular Velocities')
plt.plot(t_sol, omega1_sol, 'b--', label='omega1 - Approach 1')
plt.plot(t_sol, omega2_sol, 'r--', label='omega2 - Approach 1')
plt.plot(t_sol_v2, omega1_sol_v2, 'b--*', label='omega1 - Approach 2')
plt.plot(t_sol_v2, omega2_sol_v2, 'r--*',label='omega2 - Approach 2')
plt.plot(gait_step, gait_omega1, 'b-', label='omega1 - gait data')
plt.plot(gait_step, gait_omega2, 'r-',label='omega2 - gait data')
plt.xlabel('Time [s]')
plt.ylabel('Angular Velocity [rad/s]')
plt.legend()
plt.grid(which='both', linestyle='--', linewidth=0.5, color='gray', alpha=0.5)
plt.show()
```



Plotting Moments around Hip and Knee

In [332…

```
plt.figure()
plt.title('Hip and Knee Torques')
plt.plot(gait_step, Q1_sol_v2, 'b--', label='Q1 - Approach 2')
plt.plot(gait_step, Q2_sol_v2, 'r--', label='Q2 - Approach 2')
plt.plot(gait_step, MX_H, 'b-', label='Hip torque')
plt.plot(gait_step, MX_K, 'r-', label='Knee torque')
plt.xlabel('Time [s]')
plt.ylabel('Torque [Nm]')
plt.legend()
plt.grid(which='both', linestyle='--', linewidth=0.5, color='gray', alpha=0.5)
plt.show()
```



1.5 Results

- The simulation data is only in a small range close to the provided gait data.
- The angle around the hip q_1 shows already after 0.15 s significant deviations from the gait data.
- The angle around the knee q_2 shows the significant deviations after 0.25 s.
- Booth approaches do not yield appropriate results since the obtained values for the considered angles around hip and knee do not match.
- The angular velocities are not matching the derived angular velocities from the gait data.
- Q_1 and Q_2 contain contributions from viscous damping which depends on the angular velocities ω_1 and ω_2. Hence, they contain artefacts of angular velocity result.
- Two approaches to solve the problem were examined. None of them yielded satisfactory results.

Task 2: Modeling the drive

2.1 Description

The following section considers a dynamic model of the drive as an ODE model with the states motor current I and motor torque M.

The motor **ILM-E50x08** from TQ systems is chosen as motor.

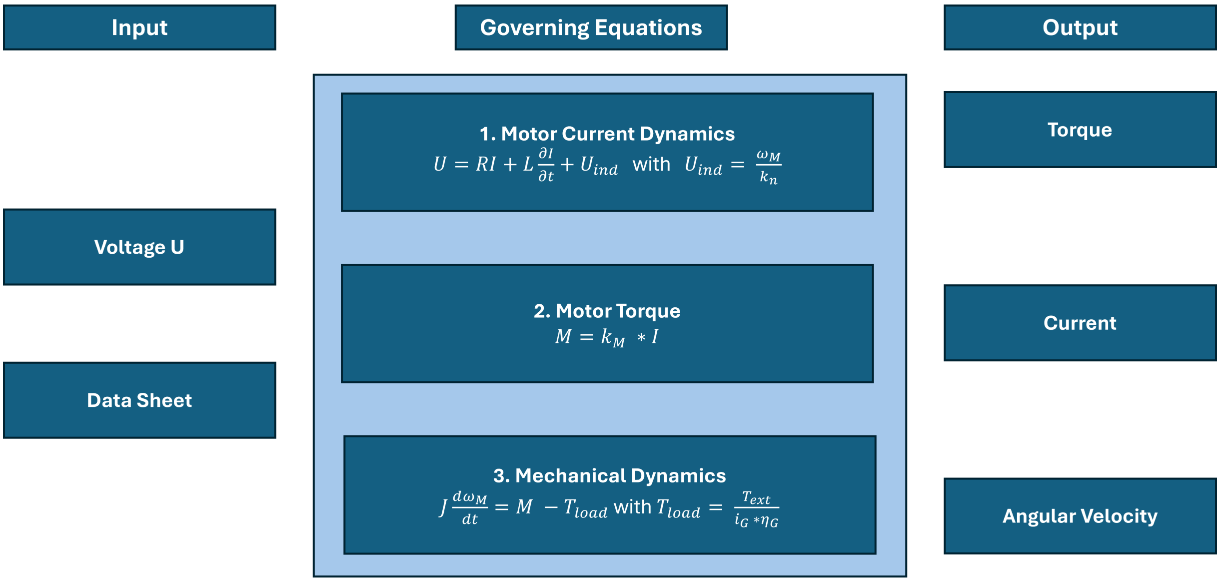
It is assumed, that this brushless drive operates in **star parallel configuration** and **operates like a brushed DC motor**. Hence, the we **approximate the *speed constant* k_n by the *no load speed* over the rated voltage**.

A **motor voltage of 16 V** is chosen.

A gearbox with a gear **ratio i_G of 60** and **efficiency of η_G 0.85** is chosen.

Parameter	Symbol	Value
Operation Mode	-	Star Parallel
Motor Voltage	U	$16\ V$
Torque Constant	k_M	$30\ mNm$ $/A^2$
Terminal Resistance	R_{TT}	$151\ m\Omega$
Terminal Inductance	L_{TT}	$121\ \mu H$
Gear Ratio	i_G	60
Gear Efficiency	η_G	0.85
Rotor Inertia	J	$0.056\ kgcm^2$

2.2 Simplified Box Model



2.3 Equations

2.3.1 Motor Current Dynamics

The mesh equation for the motor is given by:

$$U = RI + L\frac{\partial I}{\partial t} + U_{ind}$$

with

$$U_{ind} = \frac{\omega_M}{k_n}$$

where ω_M uis the motor angular frequency in rad/s and k_n is the speed constant.

Since the brushless drive is assumed in star parallel configuration and it operates like a brushed DC motor, the speed constant can be approximated by the no load speed over the rated voltage.

$$k_n = \frac{n_{no-load}}{U} = \frac{12,916\ rpm}{16\ V} = \frac{12,916\frac{rot}{min} \cdot 2\pi}{60\frac{s}{min} \cdot 16\ V} = 84.53\frac{rad}{Vs}$$

Substituting U_{ind} yields:

$$U = RI + L\frac{\partial I}{\partial t} + \frac{\omega_M}{k_n}$$

Rearranging for $\frac{\partial I}{\partial t}$ we obtain:

$$\frac{\partial I}{\partial t} = \frac{1}{L} \cdot \left(U - RI - \frac{\omega_M}{k_n}\right)$$

2.3.2 Motor Torque

The motor torque M is related to the current I via the torque constant k_M:

$$M = k_M \cdot I$$

where $k_M = 0.03 \, \frac{Nm}{A}$.

2.3.3 Mechanical Dynamics

The mechanical dynamics of the motor (Newton's Law for rotation) are:

$$J \frac{d\omega_M}{dt} = M - T_{load}$$

where:

- $J = 5.6 \cdot 10^{-4} \, kgm^2$ is the rotor inertia
- T_{load} is the load torque applied to the motor shaft

The load torque is related to the external load T_{ext} through the gearbox:

$$T_{load} = \frac{T_{ext}}{i_G \cdot \eta_G}$$

where:

- $i_G = 60$ is the gear ratio
- $\eta_G = 0.85$ is the gear efficiency

Substituting T_{load} into the mechanical dynamics equation yields:

$$J \frac{d\omega_M}{dt} = M - \frac{T_{ext}}{i_G \cdot \eta_G}$$

Using $M = k_M \cdot I$ we obtain:

$$J \frac{d\omega_M}{dt} = k_M \cdot I - \frac{T_{ext}}{i_G \cdot \eta_G}$$

2.3.4 Dynamic Model

We obtain the following ODEs for the system:

1. Current Dynamics

$$\frac{\partial I}{\partial t} = \frac{1}{L} \cdot \left(U - RI - \frac{\omega_M}{k_n} \right)$$

2. Motor Torque

$$M = k_M \cdot I$$

3. Mechanical Dynamics

$$J\frac{d\omega_M}{dt} = k_M \cdot I - \frac{T_{ext}}{i_G \cdot \eta_G}$$

2.4. Simulation

In [333…

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp
import pandas as pd
```

Defining the Parameters:

In [334…

```
# Parameters
U = 16 # V
R = 0.151 # Ohm
L = 121e-6 # H
n_0 = 12916 # rpm
w_0 = 2*np.pi * n_0 / 60 # rad/s
k_n = n_0 * 2*np.pi / (60 * U) # rad/(Vs)
k_M = 0.03 # Nm/A

J = 5.6e-6 # kgm^2

i_G = 60 # gear ratio
eta_G = 0.85 # gear efficiency

filename = 'gait_data.xls'
gait_data = pd.read_excel(filename, engine='xlrd')
m_body = 100 # kg
MX_K = np.array(gait_data["MX_K[Nm/kg]"]) * m_body

T_ext = [0, 10, 30, 50] # Nm

print("Maximal Torque at the Knee: ", np.max(MX_K))
```

```
Maximal Torque at the Knee:  51.89301739334057
```

ODE System for the Drive:

In [335…

```
# dynamic model equations
def motor_model(t, y):
    I, omega_m, M, T_ext = y

    # Current dynamics
    dI_dt = (1/L) * (U - R*I - omega_m / k_n)

    # Mechanical dynamics
    M = k_M * I
    T_load = T_ext / (i_G * eta_G)

    domega_m_dt = (1/J) * (M - T_load)

    dM = k_M * dI_dt

    return [dI_dt, domega_m_dt, dM, T_ext]
```

Initial Conditions, Time Steps and Solving the System:

In [336…

```
I0 = 0
M0 = 0
omega_M0 = 0

y0 = []
for T in T_ext:
    y0.append([I0, omega_M0, M0, T])

# Time span
t_span = [0, 0.01]
t_eval = np.linspace(t_span[0], t_span[1], 1000)

# Solve ODE
sol = []
for y0_i in y0:
    sol.append(solve_ivp(motor_model, t_span, y0_i, t_eval=t_eval, method='RK45'))
```

Extracting Results

In [337…

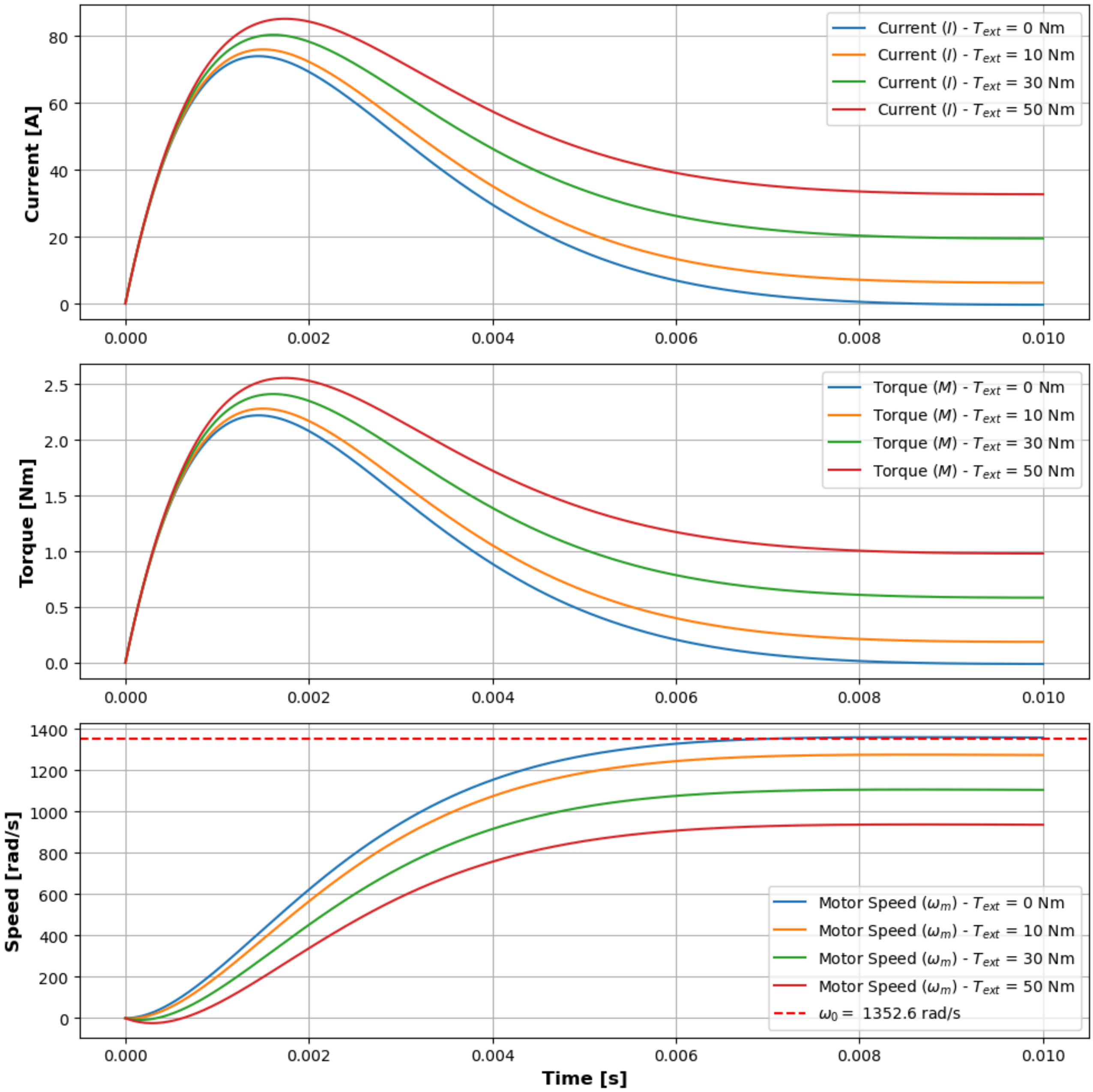
```
# Extract results
t = sol[0].t

I = [sol[i].y[0] for i in range(len(sol))]
omega_M = [sol[i].y[1] for i in range(len(sol))]
M = [sol[i].y[2] for i in range(len(sol))]
```

Plotting the Results:

In [338…

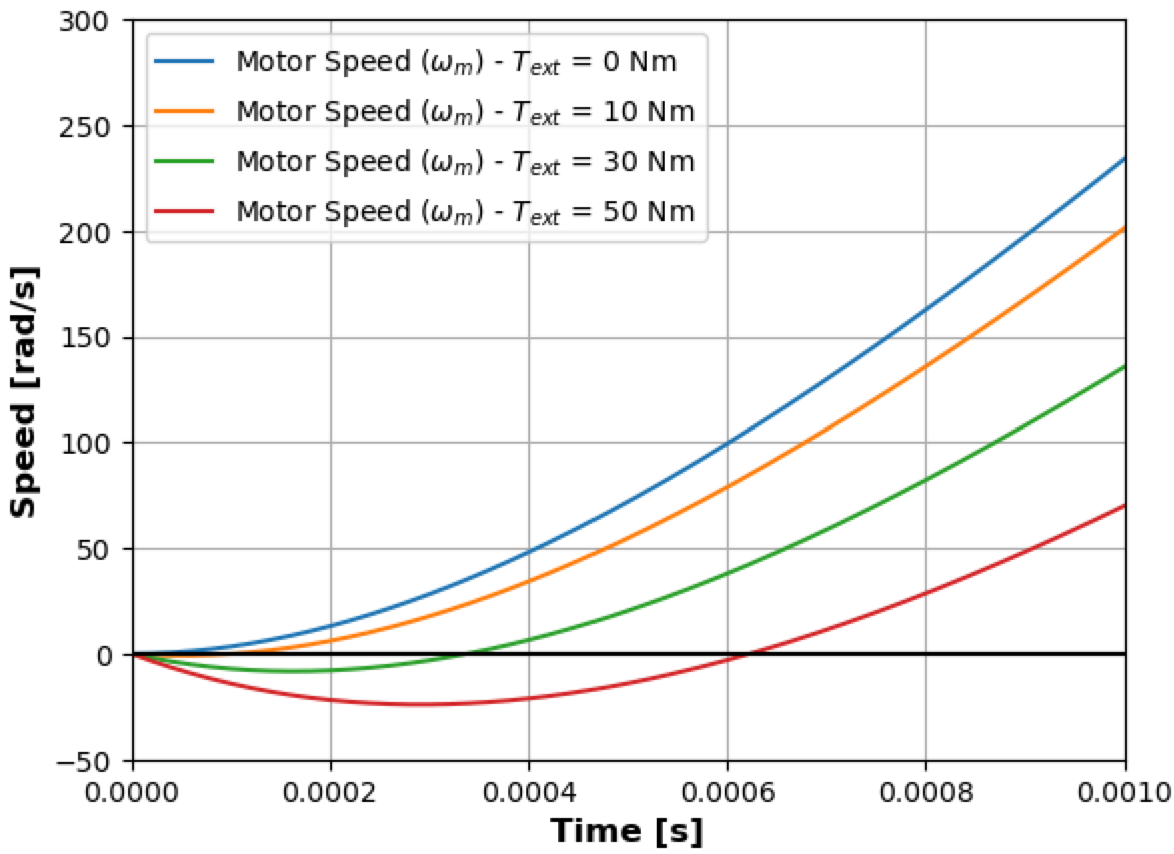
```
plt.figure(figsize=(10, 10))

# Currents
plt.subplot(3, 1, 1)
for i in range(len(T_ext)):
    plt.plot(t, I[i], label=r"Current ($I$) - $T_{ext}$ = " + str(T_ext[i]) + " Nm")
plt.ylabel("Current [A]", fontsize=12, fontweight='bold')
plt.grid()
plt.legend()

# Speed plot
plt.subplot(3, 1, 3)
for i in range(len(T_ext)):
    plt.plot(t, omega_M[i], label=r"Motor Speed ($\omega_m$) - $T_{ext}$ = " + str(T_ext
# plot w0
plt.axhline(y=w_0, color='r', linestyle='--', label=r"$\omega_0 = $ " + str(round(w_0, 1
plt.ylabel("Speed [rad/s]", fontsize=12, fontweight='bold')
plt.xlabel("Time [s]", fontsize=12, fontweight='bold')
plt.grid()
plt.legend()

# Torque plot
plt.subplot(3, 1, 2)
for i in range(len(T_ext)):
    plt.plot(t, M[i], label=r"Torque ($M$) - $T_{ext}$ = " + str(T_ext[i]) + " Nm")
plt.ylabel("Torque [Nm]", fontsize=12, fontweight='bold')
plt.grid()
plt.legend()
```

```
plt.tight_layout()
plt.show()
```



In [339…

```
# plot speed in the range up t0 0.001s
plt.figure()
for i in range(len(T_ext)):
    plt.plot(t, omega_M[i], label=r"Motor Speed ($\omega_m$) - $T_{ext}$ = " + str(T_ext
plt.ylabel("Speed [rad/s]", fontsize=12, fontweight='bold')
plt.xlabel("Time [s]", fontsize=12, fontweight='bold')
plt.axhline(y=0, color='k', linestyle='-', linewidth=1.5)
plt.grid()
plt.legend()
plt.ylim(-50, 300)
plt.xlim(0, 0.001)
plt.show()
```



2.5 Results

- The steady-state behavior varies depending on the applied external torque.
- Higher external torques T_{ext} result in increased motor current I.
- As the external torque increases, the motor speed w_M decreases.
- In the absence of external torque, the motor reaches its no-load speed of $\omega_0 = 1352.6\ rad/s$.
- When an external torque is applied, the motor initially rotates in the negative direction because the motor moment M starts at zero and builds up over time. Larger external torques require more time for the motor to generate the necessary moment to overcome them.

Task 3: Modeling the Angle Sensor

3.1 Description

This section condiseders the modeling of an angle sensor for the knee as an algebraic model.

A potentiometer is chosen as sensor.

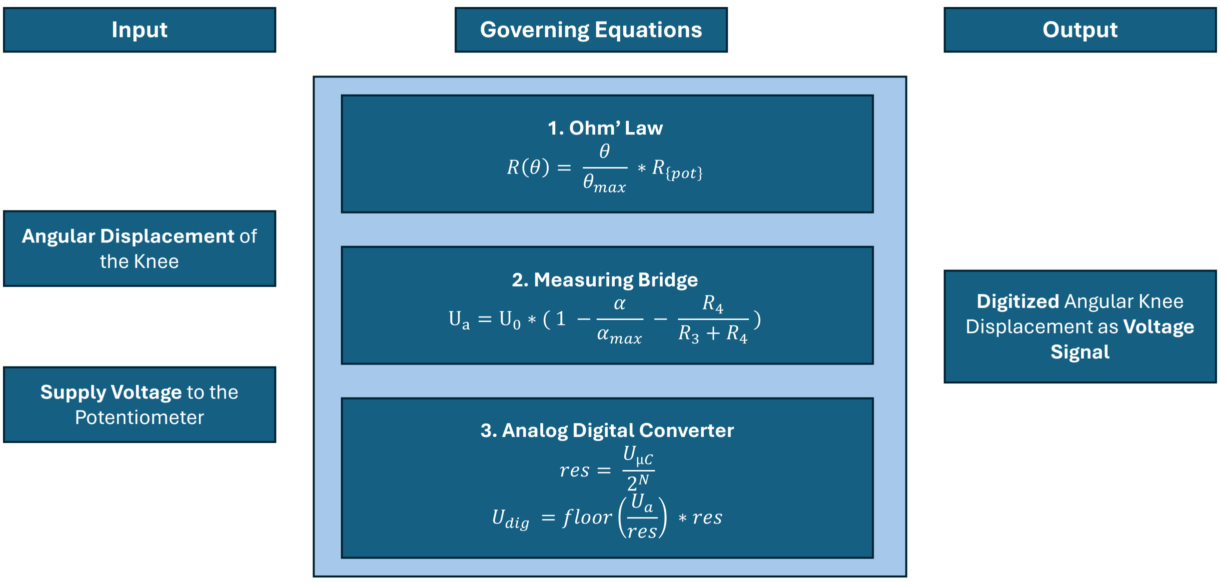
The measurement range for the angle sensor is chosen according to the range of motion of the knee, which is provided from the gait data.

The poteniometer requires a supply voltage V_{in} of 5 V and has an integrated µ-controller which has a maximum voltage input range of 3 V.

The µ-controller uses an 8-bit analog digital converter for digitization of the input voltage.

Variable	Discription	Value
V_{out}	Output Voltage	...
V_{in}	Input Voltage	$5V$
V_{micro}	Input Voltage microcontroller	$3V$
R_{pot}	Variable resistance of Potentiometer	...
\$\alpha_{max}	Angular range of Potentiometer	270°
R_{fixed}	Fixed resistance	$60k\Omega$
K_{deg}	Knee range of motion	119.7° to 184.9°
M	Microcontroller 8 bit ADC	0 to 255

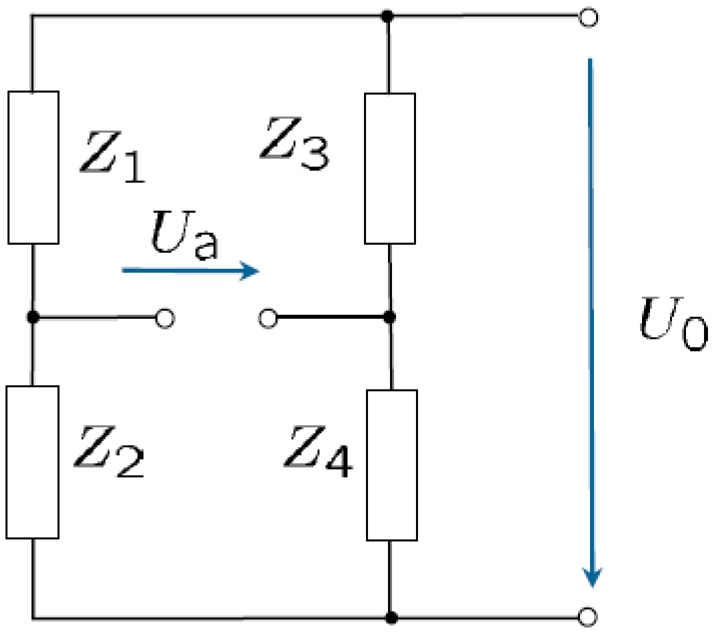
3.2 Simplified Box Model



3.3 Equations: Potentiometer in a Measuring Bridge (Wheatstone)

The potentimeter can be used with a measuring bridge, where $Z_i = R_i$ since there is only equivalent current and R_1 and R_2 correspond to the potentimeter such that

$$R_{pot} = R_1 + R_2$$



The amplifier voltage, which is used as input voltag for the µ-controller is given as:

$$U_a = U_0 \cdot \left(\frac{R_2}{R_1 + R_2} - \frac{R_4}{R_3 + R_4} \right)$$

Substituting $R_{pot} = R_1 + R_2$ yields:

$$U_a = U_0 \cdot \left(\frac{R_2}{R_{pot}} - \frac{R_4}{R_3 + R_4} \right)$$

The resistance of the potentimeter depends linearly on the rotation angle θ. R_1 and R_2 can be rewritten as:

$$R_1 = \frac{\theta}{\theta_{max}} \cdot R_{pot} \qquad R_2 = R_{pot} - R_1$$

Inserting these expressions for R_1 and R_2 yields

$$U_a = U_0 \cdot \left(\frac{R_{pot} - \frac{\theta}{\theta_{max}} \cdot R_{pot}}{R_{pot}} - \frac{R_4}{R_3 + R_4} \right)$$

$$U_a = U_0 \cdot \left(1 - \frac{\theta}{\theta_{max}} - \frac{R_4}{R_3 + R_4} \right)$$

Since the the measuring input voltage range must not exceed 3 V, the resistors R_3 and R_4 have to be selected such that the maximum and minimum angles to be measured θ_{max} and θ_{min} lie within this voltage range.

$$\theta_{min} = 119.7^\circ \approx 0.443 \cdot \alpha_{max} \qquad \theta_{max} = 184.9^\circ \approx 0.685 \cdot \alpha_{max}$$

$$\frac{R_4}{R_3 + R_4} \approx 0.443$$

We set $R_4 = 20k\Omega$ and obtain

$$R_3 = \frac{1 - 0.443}{0.443} \cdot R_4 = 25.15\ k\Omega$$

Hence, we obtain for U_a:

$$U_a = U_0 \cdot \left(1 - 0.443 - \frac{\theta}{\alpha_{max}}\right) = U_0 \cdot \left(0.557 - \frac{\theta}{\alpha_{max}}\right)$$

3.4 Simulation

In [340…

```
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
```

In [341…

```
# Load the data from the Excel file
df = pd.read_excel('gait_data.xls', header=0, engine='xlrd')
Flex_Ext_K = np.array(df['Flex_Ext_K[deg]'])
Flex_Ext_K.sort()

# Check maximum and minimum angles of knee
theta_max = round(Flex_Ext_K.max(), 2)
theta_min = round(Flex_Ext_K.min(), 2)

print(f"Maximum angle of knee (extension): {theta_max}°")
print(f"Minimum angle of knee (flexion):   {theta_min}°")
```

```
Maximum angle of knee (extension): 184.93°
Minimum angle of knee (flexion):   119.77°
```

In [342…

```
alpha_pot_max = 270 # angular range of the potentiometer

frac_min = theta_min / alpha_pot_max
frac_max = theta_max / alpha_pot_max

print(f"Fraction of the potentiometer range: {frac_max:.4f}")

R4 = 20e3 # 10 kOhm
R3 = (1 - frac_min) / frac_min * R4

print(f"Value of R3: {R3/1e3:.2f} kOhm")
```

```
Fraction of the potentiometer range: 0.6849
Value of R3: 25.09 kOhm
```

In [343…

```
U0 = 5 # Supply voltage

theta_vals = Flex_Ext_K
# theta_vals = np.linspace(theta_min, theta_max, 1000)

# Lambda function for the voltage Ua as a function of the angle alpha
Ua = lambda alpha: U0 * ( frac_max - alpha / (alpha_pot_max )) # V
```

```
Ua_vals = Ua(theta_vals)

print("Maximum voltage: ", Ua(theta_max))
print("Minimum voltage: ", Ua(theta_min))
```

```
Maximum voltage:  0.0
Minimum voltage:  1.2066666666666668
```

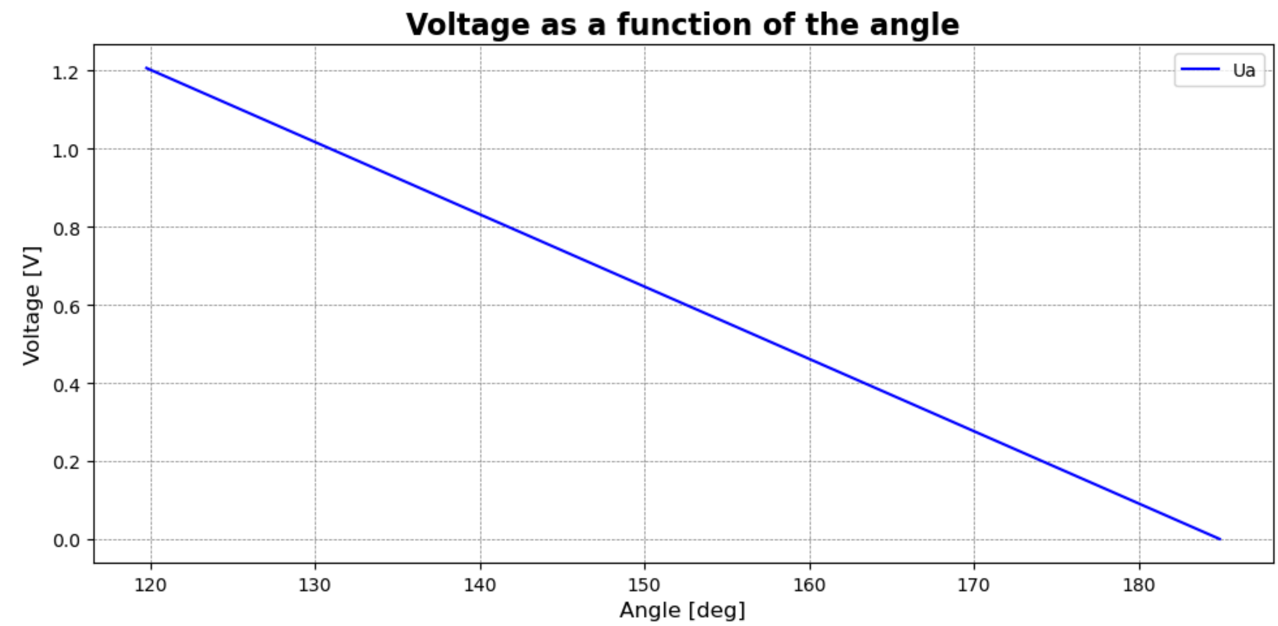
In [344…

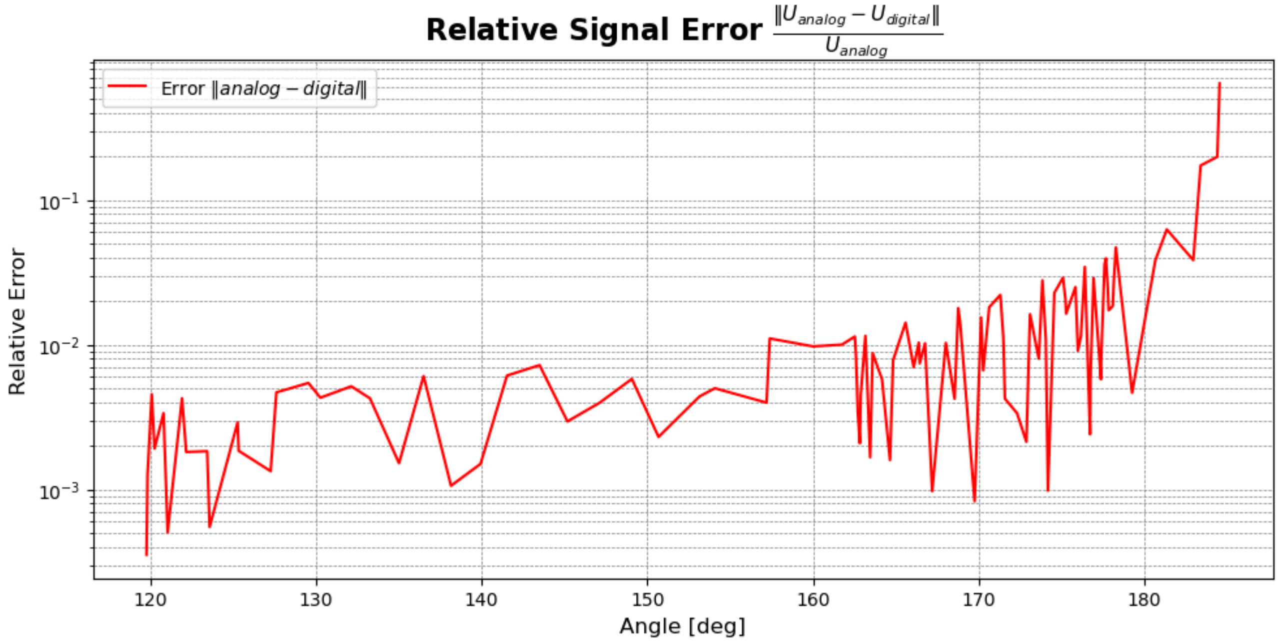
```
# 8 bit ADC Converter
levels = 2**8 # 8 bit
U_mc = 3 # V

Voltage_resolution = U_mc / levels # V / level
digitized_levels = np.round(Ua_vals / Voltage_resolution)
Ua_vals_digital = digitized_levels * Voltage_resolution

error = np.abs(Ua_vals - Ua_vals_digital) / Ua_vals

print("Resolution: ", Voltage_resolution, " V / level")
```

```
Resolution:  0.01171875  V / level
```

In [345…

```
# Plot the results
fig, ax = plt.subplots(2, 1, figsize=(10, 10))

ax[0].plot(theta_vals, Ua_vals, label='Ua', color='blue')
ax[0].set_xlabel('Angle [deg]', fontsize=12)
ax[0].set_ylabel('Voltage [V]', fontsize=12)
ax[0].set_title('Voltage as a function of the angle', fontsize=16, fontweight='bold')
ax[0].grid(which='both', axis='both', linestyle='--', color='gray', linewidth=0.5)
ax[0].legend()

ax[1].semilogy(theta_vals[:-1], error[:-1], label=r'Error $\|analog - digital\|$', color
ax[1].set_xlabel('Angle [deg]', fontsize=12)
ax[1].set_ylabel('Relative Error', fontsize=12)
ax[1].set_title(r'Relative Signal Error $\frac{\|U_{analog} - U_{digital}\|}{U_{analog}}
ax[1].grid(which='both', axis='both', linestyle='--', color='gray', linewidth=0.5)
ax[1].legend()

plt.tight_layout()
plt.show()
```





3.5 Results

- The output voltage of the potentiometer scales linearly with the flexion / extension angle of the knee.
- The maximum extension angle of 184.93° produces almost zero output voltage.
- The minimum flexion angle of 119.7 produces an output voltage at the potentiometer of 1.21 V.
- Larger knee angles result in larger output voltages.
- The resulting output voltage of the potentiometer remains within the input range of the micro controller (0 to 3 V). Hence, it scales linearly.
- The 8-bit ADC in the potentimeter converts the anlog outout voltage of the potentiometer into a digital value between 0 to 255.
- The digitized output voltage of the ADC is a multiple of the resolution ($= \frac{3V}{2^8} \approx 0.01172$ V).