In [1]:
```
%autosave 0
```

```
Autosave disabled
```

Chapter 24: Solutions of Linear Differential Equations with Constant Coefficients by Laplace

Introduction to a far reaching system for simplification of the ODE solving process. The transform is from the time domain to the frequency domain, where algebraic operations are performed before inverse transportation back to the time domain. The main variable governing the frequency domain is s, and the time domain t.

When a Laplace transform is performed, it should be accomplished cleanly, with no leftover x or y factors. The presence of any such is an ill omen, because it can involve complexities. Like hunting through the transform table to make things come out right. In this notebook the first attempt will be without Laplace. If for some reason Laplace is needed on a specific problem, such as Problem 24.10, it will be brought in with deliberation.

24.1 Solve $y' - 5y = 0; \quad y(0) = 2.$

This problem can be entered into Wolfram Alpha:

① !| y' - 5 y = 0, y(0) = 2 |!

In Wolfram Alpha the expression is referred to as the "Differential equation solution":

$2e^{5x}$

24.2 Solve $y' - 5y = e^{5x}; \quad y(0) = 0.$

This problem can be entered into Wolfram Alpha:

!| y' - 5 y = e^(5 * x), y(0) = 0 |!

In Wolfram Alpha the expression is referred to as the "Differential equation solution":

$y(x) = e^{5x}\, x$

24.3 Solve $y' + y = \sin x; \quad y(0) = 1.$

This problem can be entered into Wolfram Alpha:

!| y' + y = sin(x), y(0) = 1 |!

In Wolfram Alpha the expression is referred to as the "Differential equation solution":

$y(x) = \frac{1}{2}(3\, e^{-x} + \sin(x) - \cos(x))$

24.4 Solve $y'' + 4y = 0; \quad y(0) = 2,\ y'(0) = 2.$

This problem can be entered into Wolfram Alpha:

!| y'' + 4 y = 0, y(0) = 2, y'(0) = 2 |!

In Wolfram Alpha the expression is referred to as the "Differential equation solution":

$y(x) = \sin(2x) + 2\cos(2x))$

24.5 Solve $y'' - 3y' + 4y = 0; \quad y(0) = 1,\ y'(0) = 5.$

This problem can be entered into Wolfram Alpha:

!| y'' - 3 y' + 4 y = 0, y(0) = 1, y'(0) = 5 |!

In Wolfram Alpha the expression is referred to as the "Differential equation solution":

$y(x) = e^{3x/2}\left(\sqrt{7}\sin\left(\frac{\sqrt{7}x}{2}\right) + \cos\left(\frac{\sqrt{7}x}{2}\right)\right)$

24.6 Solve $y'' - y' - 2y = 4x^2; \quad y(0) = 1,\ y'(0) = 4.$

This problem can be entered into Wolfram Alpha:

!| y'' - y' - 2 y = 4 x^2, y(0) = 1, y'(0) = 4 |!

In Wolfram Alpha the expression is referred to as the "Differential equation solution":

$y(x) = -2x^2 + 2x + 2e^{-x} + 2e^{2x} - 3$

24.7 Solve $y'' + 4y' + 8y = \sin x; \quad y(0) = 1,\ y'(0) = 0.$

This problem can be entered into Wolfram Alpha:

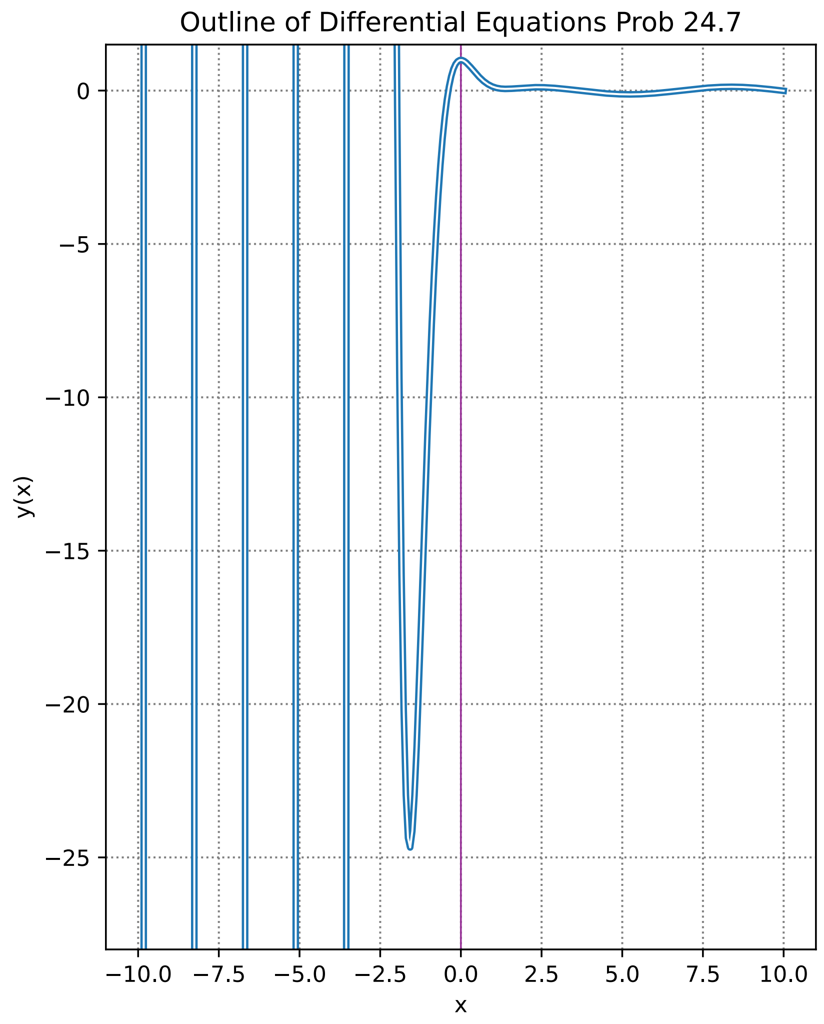
!| y'' + 4 y' + 8 y = sin(x), y(0) = 1, y'(0) = 0 |!

In Wolfram Alpha the expression is referred to as the "Differential equation solution":

$$y(x) = \frac{1}{65} e^{-2x} \left(7e^{2x} \sin(x) + 69 \cos(2x) + (131 \sin(x) - 4e^{2x}) \cos(x)\right)$$

The form assumed by the text answer versus the above W|A answer were somewhat tricky to reconcile, so a plot appears below, demonstrating their equality.

In [6]:
```
import numpy as np
import matplotlib.pyplot as plt

%config InlineBackend.figure_formats = ['svg']

#x = np.arange(0, 3., 0.005)
x = np.linspace(-10,10,300)
y3 = (1/65)*np.exp(-2*x)*(7*np.exp(2*x)*np.sin(x) + \
    69*np.cos(2*x) + (131*np.sin(x) - 4*np.exp(2*x))*np.cos(x))
y4 = (np.exp(-2*x))*((69/65)*np.cos(2*x) + (131/130)*np.sin(2*x)) + \
    (7/65)*np.sin(x) - (4/65)*np.cos(x)


plt.grid(True, linestyle='dotted', color='gray', linewidth=0.9)
plt.xlabel("x")
plt.ylabel("y(x)")
plt.title("Outline of Differential Equations Prob 24.7")
plt.rcParams['figure.figsize'] = [9, 7.5]


ax = plt.gca()
#ax.axhline(y=0, color='#993399', linewidth=0.8)
ax.axvline(x=0, color='#993399', linewidth=0.8)
ratio = 0.95
xleft, xright = ax.get_xlim()
ybottom, ytop = ax.get_ylim()
ax.set_aspect(abs((xright-xleft)/(ybottom-ytop))*ratio)


#plt.text(965, -2.5e43, "-np.log(abs((np.cos(x/2))**2 - (np.sin(x/2))**2))
#plt.text(965, 2.25e43, "SOLN: y = 3*np.exp(x**2) + 1/2", size=10,\
        # bbox=dict(boxstyle="square", ec=('#8C564B'),fc=(1., 1., 1),))
plt.ylim(-28,1.5)
plt.plot(x, y3, linewidth = 3)
plt.plot(x, y4, linewidth = 0.9, color = 'w')
plt.show()
```



24.8 Solve $y'' - 2y' + y = f(x); \quad y(0) = 0,\ y'(0) = 0.$

This problem cannot be entered into Wolfram Alpha. W|A rejects it as either a differential equation for solving, or as an input to a Laplace transform conversion.

The text marshalls resources and procedures to produce a plausible output for the problem.

24.9 Solve $y'' + y = f(x); \quad y(0) = 0,\ y'(0) = 0$ if $\quad f(x) = \begin{cases} 0 & x < 1 \\ 2 & x \geq 1 \end{cases}.$

This problem cannot be entered into Wolfram Alpha. W|A rejects it. One attempted entry is:

!| y = Piecewise[{{0, x < 1}, {1, x >=1}}], && y'' + y =0, {y(0) = 0, y'(0) = 0} |!

which brings some recognition from Wolfram Alpha, though not the desired series of calculations.

The text marshalls resources and procedures to produce a plausible output for the problem.

24.10 Solve $y''' + y' = e^x; \quad y(0) = y'(0) = y''(0) = 0.$

Here a problem has been reached which requires Laplace intervention. However, a little help is needed from Maxima, because Wolfram Alpha will neither solve the equation directly nor create a Laplace transform version of it. So accessing Maxima below:

(%i6) ode: 'diff(y(t), t, 3) + 'diff(y(t), t) = %e^t;

(%o6) $\frac{d^3}{d\,t^3}\,y(t)+\frac{d}{d\,t}\,y(t)=\%e^t$

(%i8) atvalue(y(t), t=0, 0);

(%o8) 0

(%i9) atvalue('diff(y(t), t), t=0, 0);

(%o9) 0

(%i10) atvalue('diff(y(t), t, 2), t=0, 0);

(%o10) 0

(%i11) lap_ode:laplace(ode, t, s);

(%o11) $s^3\ \mathrm{laplace}(y(t),t,s)+s\ \mathrm{laplace}(y(t),t,s)=\frac{1}{s-1}$

Note that in the first input line above, all x factors were converted to t factors to make the process work. With the transform available, Wolfram Alpha can take it from here.

!| solve[s^3 y + s y = 1/(s -1)], y |!

gives the solution to $Y(s)$ in s-space, which is

$$y = \frac{1}{s\,(s^3 - s^2 + s - 1)}$$

Then

!| inverselaplacetransform[1/(s (s^3 - s^2 + s - 1)) |!

takes the solution back to t-space, resulting in

$$\frac{1}{2}(e^t - \sin(t) + \cos(t) - 2)$$

or, un-substituting,

$$\frac{1}{2}(e^x - \sin(x) + \cos(x) - 2)$$

24.11 Solve $y' - 5y = 0$

This problem can be entered into Wolfram Alpha:

!| y' - 5 y = 0 |!

In Wolfram Alpha the expression is referred to as the "Differential equation solution":

$y(x) = c_1\,e^{5x}$

Without an initial condition, an arbitrary constant attaches.

24.12 Solve $y'' - 3y' + 2y = e^{-x}$

This problem can be entered into Wolfram Alpha:

!| y'' - 3 y' + 2 y = e^(-x) |!

In Wolfram Alpha the expression is referred to as the "Differential equation solution":

$y(x) = c_1\,e^x + c_2\,e^{2x} + \frac{e^{-x}}{6}$

Without an initial conditions, arbitrary constants for each level of order attach.

24.13 Solve $y'' - 3y' + 2y = e^{-x};\quad y(1) = 0,\ y'(1) = 0.$

This problem can be entered into Wolfram Alpha:

!| {y'' - 3 y' + 2 y = e^(-x), {y(1) = 0, y'(1) = 0 }} |!

In Wolfram Alpha the expression is referred to as the "Differential equation solution":

$y(x) = \frac{1}{6}\,e^{-x-3}\,(e - e^x)^2\,(2e^x + e)$

24.14 Solve $\frac{dN}{dt} = 0.05N;\quad N(0) = 20000.$

Wolfram Alpha seems inclined to refuse this problem, but with the right technique it can be run. The form of the entry, even the specific symbols, matter.

!| {y' = 0.05 y}, { y(0) = 20000}} |!

In Wolfram Alpha the expression is referred to as the "Differential equation solution":

$y(x) = 20000\,e^{0.05x}$

that is,

$N(t) = 20000\,e^{0.05t}$

24.15 Solve $\frac{dI}{dt} + 50I = 5;\quad I(0) = 0.$

Wolfram Alpha might be inclined to refuse this problem, but with the technique shown in Prob 24.14 it can be run. The form of the entry, even the specific symbols, matter.

!| {y' + 50 y = 5}, {y(0) = 0}} |!

In Wolfram Alpha the expression is referred to as the "Differential equation solution":

$y(x) = \frac{1}{10}\,(1 - e^{-50\,x})$

that is,

$I(t) = \frac{1}{10}\,(1 - e^{-50\,t})$

24.16 Solve $\ddot{x} + 16x = 2\sin 4t; \quad x(0) = -\frac{1}{2},\ \dot{x}(0) = 0.$

This problem is set out in a straightforward way and needs no tricks.

!| {x" + 16 x = 2 sin(4 t)}, { x(0) = -1/2, x'(0) = 0}} |!

In Wolfram Alpha the expression is referred to as the "Differential equation solution":

$x(t) = \frac{1}{16}\,(\sin(4t) - 4\,(t + 2)\cos(4t))$

In [ ]:

In [ ]:

In [ ]:

In [ ]:

In [ ]:

In [ ]:

In [ ]:

In [ ]:

In [ ]:

In [ ]:

In [ ]:

In [ ]:

In [ ]: