(Custom CSS files are not reliable for controlling Jupyter font style. To establish the same appearance as the original notebook, depend on the browser to control the font, by setting the desired font faces in the browser settings. For example, Chrome 135 or Firefox 134 can do this. In this notebook series, Bookerly font is for markdown and Monaco is for code.)

Chapter 14 Applications of 2nd Order Linear Differential Equations Application such as mass and spring systems, electrical circuits, and buoyancy cycles are explored.

Cutting and pasting and Wolfram Alpha. Wolfram Alpha is amenable to accepting pasted entries. In this chapter pastable expressions are given a distinctive boundary fence, exemplified by the sample: !| abcdef |!
In the above pseudo-entry, only the alpha characters would be copied for transfer to Wolfram Alpha.

14.1 A steel ball weighing 128 lb is suspended from a spring, whereupon the spring is stretched 2 ft from its natural length. The ball is started in motion with no initial velocity by displacing it 6 in above the equilibrium position. Assuming no air resistance, find (a) an expression for the position of the ball at any time t, and (b) the position of the ball at $t = \frac{\pi}{12}\ sec.$

Verbal free body diagram A mass is suspended below a rigid surface by a spring. The experiment consists of two states, both of which are depicted. On the left, is the *Equilibrium position.* In this position a horizontal line with the label "$x = 0$" runs through the center of the mass m. Gravity is not allowed to affect the mass in the Equilibrium position. A vertical arrowed line is shown with the arrow pointing down, and the arrowhead is labeled "Positive x-direction." On the right of the diagram is the *Initial position at* $t = 0$. In this position the same $x = 0$ line extends through the view, but here it does not intersect the center of mass of the test mass, because now gravity is allowed to act on the mass, the spring is stretched, and the mass appears below the $x = 0$ line with the spring extended an amount x_0. The x_0 vertical distance is shown with a two-headed arrowed line, representing the vertical distance separating the $x = 0$ line and the center of mass of the test mass, m.

The mass of the spring is assumed to be negligible. The air resistance factor a is assumed to be proportional to velocity, which may be unrealistic. The forces acting on the system are 3: (1) $F(t)$, measured in the positive direction (down); (2) a restoring force, given by Hooke's law as $F_s = -kx,\ k > 0$; and (3) an air resistance force a, given by $F(a) = -a\dot{x},\ a > 0$ where a is the constant of proportionality. The mass m is 128/32 = 4 slugs.
The equation of motion is equivalent to Newton's second law, that is

$$\frac{F(t)}{m} = \ddot{x} + \frac{a}{m}\dot{x} + \frac{k}{m}x$$

The system starts at $t = 0$ with an initial velocity v_0 and from an initial position x_0, so that the initial conditions include: $x(0) = x_0$ and $\dot{x} = v_0 = 0$. Also given is the implied spring force, k. If 128 lb causes a 2-ft extension of the spring, Hooke's law gives

$$-128 = -k\,(2)\,, \text{or } k = 64\,\text{lb/ft.}$$

There is no externally applied force, so $F(t) = 0$, and no resistance from the surrounding medium, so $a = 0$. The equation of motion shown above becomes:

$$\ddot{x} + 16x = 0.$$

The problem is entered into Wolfram Alpha:

!| d^2x/dt^2 + 16x = 0 |!

and the answer is received:

$$x(t) = c_2 \sin(4t) + c_1 \cos(4t)$$

At $t = 0$, the position of the ball is $x_0 = -\frac{1}{2}$ ft.

transforming the position equation to:

$$-\frac{1}{2} = x(0) = c_1 \cos 0 + c_2 \sin 0 = c_1$$

thereby changing the position equation to:

$$x(t) = -\frac{1}{2} \cos 4t + c_2 \sin 4t$$

The initial velocity is given as v_0 = 0 ft/sec. To express the velocity equation, it is necessary to differentiate the position equation. Although the position equation is easy to differentiate, W|A may as well be given the job:

!| D[-1/2 * cos(4 * t) + c_2 * sin(4 * t), t] |!

resulting in:

$$2(2\,c_2 \cos(4t) + \sin(4t)) \qquad \text{or} \qquad 4\,c_2 \cos(4t) + 2\sin(4t))$$

Since

$$0 = v(0) = 4\,c_2 \cos(4(0)) + 2\sin(4(0))$$

$$\Longrightarrow c_2 = 0$$

$$\Longrightarrow x(t) = -\frac{1}{2} \cos 4t$$

which was the request of part (a) for the position of the ball at any time.

Part (b) wants to know the position of the ball at time $t = \frac{\pi}{12}$ which will be given by:

$$x\left(\frac{\pi}{12}\right) = -\frac{1}{2} \cos \frac{4\pi}{12} = -\frac{1}{4}\,\text{ft}$$

(Wolfram Alpha having been used to supply the value of $\cos \frac{\pi}{3} = \frac{1}{2}$. Note that the value $-\frac{1}{4}$ ft is taken as being vertically above the problem's x_0 position.)

14.2 A mass of 2 kg is suspended from a spring with a known spring constant of 10 N/m and allowed to come to rest. It is then set in motion by giving it an initial velocity of 150 cm/sec. Find an expression for the motion of the mass, assuming no air resistance.

The equation of motion is equivalent to Newton's second law, that is

$$\frac{F(t)}{m} = \ddot{x} + \frac{a}{m}\dot{x} + \frac{k}{m}x$$

For this problem, it is known that $a = 0, \quad m = 2, \; k = 10$, and $v_0 = 150 \text{ cm/sec.}$

$$\implies \ddot{x} + 5x = 0$$

as the equation, and this is passed to Wolfram Alpha:

!| d^2x/dt^2 + 5 x = 0 |!

which returns

$$x(t) = c_2 \sin(\sqrt{5}\,t) + c_1 \cos(\sqrt{5}\,t)$$

At time $t = 0$, the ball's position is at the equilibrium position $x_0 = 0$ m. Applying this information to the previous equation produces:

$$0 = x(0) = c_1 \cos 0 + c_2 \sin 0 = c_1$$

And putting this new found information to work,

$$x(t) = c_2 \sin(\sqrt{5}\,t)$$

Next, the velocity equation can be found by differentiating the position equation. This is too easy to resort to Wolfram Alpha, it is just

$$v(t) = \dot{x}\,t = \sqrt{5}\,c_2 \cos\sqrt{5}\,t$$

At time $t = 0$ the velocity is 150 cm/sec = 1.5 meter/sec. Updating,

$$1.5 = v(0) = \sqrt{5}\,c_2\,(cos) = \sqrt{5}\,c_2$$

$$\implies c_2 = \frac{1.5}{\sqrt{5}} \approx 0.6708$$

resulting in a position equation of

$$x(t) = 0.6708 \sin\sqrt{5}\,t$$

as the description of the ball's position at any time t.

14.3 Determine the circular frequency, natural frequency, and period for the simple harmonic motion described in Problem 14.2.

Vibrating springs, simple electrical circuits, and floating bodies are all governed by second-order linear differential equations with constant coefficients of the form

$$\ddot{x} + a_1 \dot{x} + a_0 x = f(t)$$

For vibrating spring problems, such as 14.2, $a_1 = a/m$, $a_0 = k/m$ and $f(t) = F(t)/m$.

Free undamped motion defined by the above equation with $a_1 = 0$ and $f(t) \equiv 0$ always has solutions of the form

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

which defines *simple harmonic motion* or SHM. Here, $c_1, c_2,$ and ω are constants with ω often referred to as *circular frequency*. The *natural frequency f* is

$$f = \frac{\omega}{2\pi}$$

representing the number of complete oscillations per unit time completed by the described motion. The *period* of the system, or the time required to complete one oscillation, is

$$T = \frac{1}{f}$$

As to the specific questions posed by the problem.

Looking at the solution to the general motion ODE

$$x(t) = c_2 \sin(\sqrt{5}\, t) + c_1 \cos(\sqrt{5}\, t)$$

and comparing this equation with the SHM equation, it is obvious that the circular frequency is equal to $\sqrt{5}$. That is, the motion repeats with frequency of approx 2.236 cycles per sec. The natural frequency is the circular frequency divided by 2π or $f = 0.3559$ Hz. The period is the inverse of the natural frequency, here $\frac{1}{0.3559} = 2.81$ sec.

14.4 Determine the circular frequency, natural frequency, and period for the simple harmonic motion described in Problem 14.1.

The developed position equation is:

$$x(t) = c_2 \sin(4t) + c_1 \cos(4t)$$

The circular frequency is normally the coefficient associated with the time variable, t. Here it is 4, which is taken to be the circular frequency, 4 cyles/sec. The natural frequency is the circular frequency divided by 2π, or 0.6366 Hz. The period is the inverse of the natural frequency, here $\frac{1}{.6336} \approx 1.57$ sec.

14.5 A 10-kg mass is attached to a spring, stretching it 0.7 m from its natural length. The mass is started in motion from the equilibrium position with an initial velocity of 1 m/sec in the upward direction. Find the subsequent motion, if the force due to air resistance is $-90\,\dot{x}$ N.

The general equation of motion is:

$$\frac{F(t)}{m} = \ddot{x} + \frac{a}{m}\dot{x} + \frac{k}{m} x$$

Using $g = 9.8$ m/sec^2, the weight in N is calculated as $mg = 10(9.8) = 98$ N. To obtain the k factor, the weight is divided by the excursion length, or 98/0.7 = 140. Accepting that $a = 90$ and $F(t) = 0$, the equation above can take its new shape as

$$\ddot{x} + 9\dot{x} + 14x = 0$$

This equation can be entered in Wolfram Alpha:

!| d^2x/dt^2 + 9 * x' + 14 * x = 0 |!

and the answer returned is:

$$x(t) = c_1\, e^{-7t} + c_2\, e^{-2t}$$

It may happen that the differentiated form of the above equation is needed. In that case it will be well that Wolfram Alpha should make it available:

!| d(c_1 * e^(-7 * t))/dt + d(c_2 * e^(-2 * t))/dt |!

The answer on this last one is:

$$\dot{x}\,(t) = -7c_1 e^{-7t} + -2c_2 e^{-2t}$$

One initial condition is that $x(0) = 0$, indicating that the mass starts at the equilibrium position. This condition implies that $0 = c_1 + c_2$.

A second initial condition is that $\dot{x}\,(0) = -1$ .This condition will result in the equation:

$$-1 = -7\,c_1 + -2\,c_2$$

What is needed is for Wolfram Alpha to solve these two equations simultaneously.

!| [0 = a + b, -1 = -7 * a - 2 * b] |! (Note that it is necessary to alias the arguments temporarily.)

Wolfram Alpha returns: $c_1 = \frac{1}{5},\ \ c_2 = -\frac{1}{5}$

So the final general version of the position equation is:

$$x(t) = \frac{1}{5}\,(e^{-7t} - e^{-2t})$$

14.6 A mass of 1/4 slug is attached to a spring, whereupon the spring is stretched 1.28 ft from its natural length. The mass is started in motion from the equilibrium position with an initial velocity of 4 ft/sec in the downward direction. Find the subsequent motion of the mass if the force due to air resistance is $-2\dot{x}$ lb.

The general equation of motion is:

$$\frac{F(t)}{m} = \ddot{x} + \frac{a}{m}\,\dot{x} + \frac{k}{m}\,x$$

In this case $m = 1/4$, $a = 2$, and, because there is no external force, $F(t) \equiv 0$. From Hooke's law, $k = \frac{m\,g}{l} = \frac{(1/4)(32)}{1.28} = 6.25$.

$$\ddot{x} + 8\dot{x} + 25x = 0$$

This equation can be entered in Wolfram Alpha:

!| d^2x/dt^2 + 8 * x' + 25 * x = 0 |!

and the answer returned is:

$$x(t) = c_1\, e^{-4t}\, \sin\,(3t) + c_2\, e^{-4t}\, \cos\,(3t)$$

Notice how the c_1 constant is associated with the $\sin$ factor above. This is the way that W|A serves it up, but it does not agree with the text. So for compatibility purposes, the equation will be changed to:

$$x(t) = c_1 e^{-4t} \cos(3t) + c_2 e^{-4t} \sin(3t)$$

It may happen that the differentiated form of the above equation is needed. In that case it will be well that Wolfram Alpha should make it available:

!| d(c_1 * e^(-4 * t) * cos(3t) + c_2 * e^(-4 * t) * sin(3 * t))/dt |!

With the right constant associations now in place, the velocity function is received as:

$$\dot{x}(t) = e^{-4t} (c_2 (3\cos(3t) - 4\sin(3t)) - c_1 (3\sin(3t) + 4\cos(3t)))$$

One initial condition is that $x(0) = 0$ indicating that the mass starts at the equilibrium position. This condition implies that $c_1 = 0$.

A second initial condition is that $\dot{x}(0) = 4$.This condition will result in the equation:

$$4 = 3c_2$$

Implying that $c_2 = \frac{4}{3}$.

So the final general version of the position equation is:

$$x(t) = \frac{4}{3} (e^{-4t} \sin(3t))$$

14.7 A mass of 1/4 slug is attached to a spring having a spring constant of 1 lb/ft. The mass is started in motion by initially displacing it 2 ft in the downward direction and giving it an initial velocity of 2 ft/sec in the upward direction. Find the subsequent motion of the mass, if the force due to air resistance is -1$\dot{x}$ lb.

The general equation of motion is:

$$\frac{F(t)}{m} = \ddot{x} + \frac{a}{m}\dot{x} + \frac{k}{m}x$$

In this case $m = 1/4$, $a = 1$, and $k = 1$. $F(t) \equiv 0$. The force equation becomes:

$$\ddot{x} + 4\dot{x} + 4x = 0$$

This equation can be entered in Wolfram Alpha:

!| d^2x/dt^2 + 4 * x' + 4 * x = 0 |!

and the answer returned is:

$$x(t) = c_1 e^{-2t} + c_2 e^{-2t} t$$

It may happen that the differentiated form of the above equation is needed. In that case it will be well that Wolfram Alpha should make it available:

!| d[c_1 * e^(-2 * t) + c_2 * e^(-2 * t) * t]/dt |!

The velocity function is given as:

$$\dot{x}(t) = e^{-2t} (c_2 (1 - 2t) - 2c_1)$$

One initial condition is that $x(0) = 2$ indicating that the mass starts 2 ft below the equilibrium position. This condition implies that $c_1 = 2$.

A second initial condition is that $\dot{x}(0) = -2$. This condition will result in the equation:

$$-2 = c_2 - 2c_1$$

Implying that $c_2 = 2$.

So the final general version of the position equation is:

$$x(t) = 2\,e^{-2t} + 2\,e^{-2t}\,t$$

14.9 A 10-kg mass is attached to a spring having a spring constant of 140 N/m. The mass is started in motion from the equilibrium position with an initial velocity of 1 m/sec in the upward direction and with an applied external force $F(t) = 5\sin t$. Find the subsequent motion of the mass if the force due to air resistance is $-90\,\dot{x}\,N$.

The general equation of motion is:

$$\frac{F(t)}{m} = \ddot{x} + \frac{a}{m}\dot{x} + \frac{k}{m}x$$

In this case $m = 10$, $a = 90$, and $k = 140$ $F(t) = 5\sin t$. The force equation becomes:

$$\ddot{x} + 9\dot{x} + 14x = \frac{1}{2}\sin t$$

This equation can be entered in Wolfram Alpha:

!| d^2x/dt^2 + 9 * x' + 14 * x = (1/2) * sin(t) |!

and the answer returned is:

$$x(t) = c_1\,e^{-7t} + c_2\,e^{-2t} + \frac{13\sin(t)}{500} - \frac{9\cos(t)}{500}$$

(In this case the constant term subscripts happen to match the text.)

It may happen that the differentiated form of the above equation is needed. In that case it will be well that Wolfram Alpha should make it available:

!| d[c_1 * e^(-7 * t) + c_2 * e^(-2 * t) + (13 * sin(t))/500 - (9 * cos(t))/500]/dt |!

The velocity function is given as:

$$\dot{x}(t) = -7\,c_1\,e^{-7t} - 2\,c_2\,e^{-2t} + \frac{9\sin(t)}{500} + \frac{13\cos(t)}{500}$$

One initial condition is that $x(0) = 0$ indicating that the mass starts at the equilibrium position. This condition results in the following equation:

$$0 = c_1 + c_2 - \frac{9}{500}$$

A second initial condition is that $\dot{x}(0) = -1$. This condition will result in the equation:

$$-1 = -7\,c_1 - 2\,c_2 + \frac{13}{500}$$

Wolfram Alpha will need to solve the two previous equations simultaneously in order to assign the constant values. As in a previous case, it will be necessary to give temporary aliases to c_1 and c_2 .

!| [0 = a + b - (9/500), -1 = -7 * a - 2 * b + (13/500)] |!

According to W|A's investigation,

$$c_1 = \frac{99}{500}, \qquad c_2 = -\frac{9}{500}$$

So the final general version of the position equation is:

$$x(t) = \frac{99}{500}\, e^{-7t} - \frac{9}{500}\, e^{-2t} + \frac{13 \sin(t)}{500} - \frac{9 \cos(t)}{500}$$

14.10 A 128-lb weight is attached to a spring having a spring constant of 64 lb/ft. The weight is started in motion with no initial velocity by displacing it 6 in above the equilibrium position and by simultaneously applying to the weight an external force $F(t) = 8 \sin 4t$. Assuming no air resistance, find the subsequent motion of the weight.

The general equation of motion is:

$$\frac{F(t)}{m} = \ddot{x} + \frac{a}{m}\dot{x} + \frac{k}{m}x$$

In this case $m = 4$, $a = 0$, and $k = 64$. $F(t) = 8 \sin 4\,t$. The force equation becomes:

$$\ddot{x} + 16x = 2 \sin 4\,t$$

This equation can be entered in Wolfram Alpha:

!| d^2x/dt^2 + 16 * x = 2 * sin(4 * t) |!

and the answer returned is:

$$x(t) = c_2 \sin(4\,t) + c_1 \cos(4t) - \frac{1}{4}\, t \cos(4t)$$

(In this case the constant term subscripts happen to match the text.)

It may happen that the differentiated form of the above equation is needed. In that case it will be well that Wolfram Alpha should make it available:

!| d[c_2 * sin(4 * t) + c_1 * cos(4 * t) - (1/4) * t * cos(4 * t)]/dt |!

The velocity function is given as:

$$\dot{x}(t) = -4\, c_1 \sin(4t) + 4\, c_2 \cos(4t) + t \sin(4t) - \frac{1}{4}\cos(4t)$$

One initial condition is that $x(0) = -6$ inches $= \frac{1}{2}$ foot, indicating that the mass starts 6 in above the equilibrium position. This condition results in the following equation:

$$-\frac{1}{2} = c_1$$

A second initial condition is that $\dot{x}(0) = 0$. This condition will result in the equation:

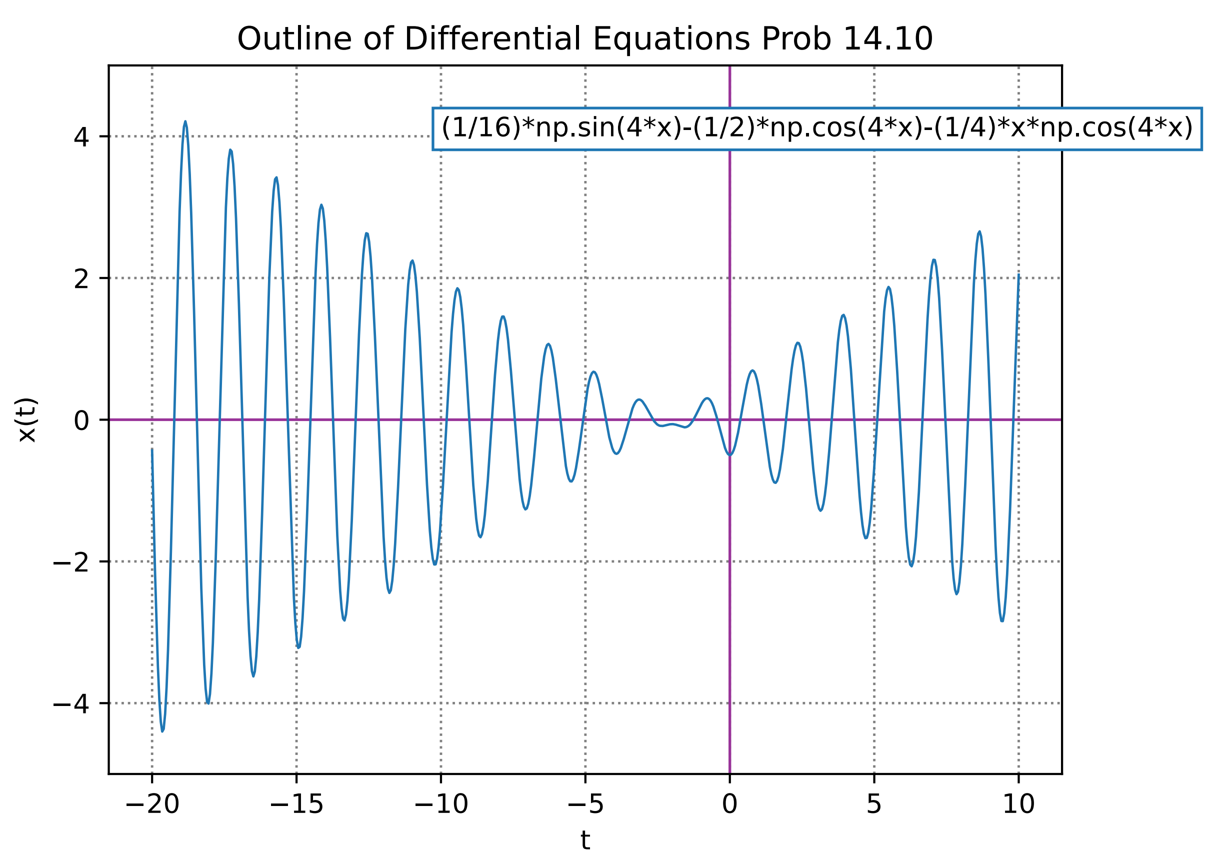
$$0 = 4\, c_2 - \frac{1}{4}$$

So the final general version of the position equation is:

$$x(t) = \frac{1}{16}\sin(4\,t) - \frac{1}{2}\cos(4t) - \frac{1}{4}\, t \cos(4t)$$

The plot in Wolfram Alpha looked interesting, so it might be worth hauling it out here.

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt

%config InlineBackend.figure_formats = ['svg']

x = np.linspace(-20,10,600)
y3 = (1/16)*np.sin(4*x) - (1/2)*np.cos(4*x) - (1/4)*x*np.cos(4*x)

plt.grid(True, linestyle='dotted', color='gray', linewidth=0.9)
plt.xlabel("t")
plt.ylabel("x(t)")
plt.title("Outline of Differential Equations Prob 14.10")
plt.rcParams['figure.figsize'] = [8, 7.5]

ax = plt.gca()
ax.axhline(y=0, color='#993399', linewidth=1)
ax.axvline(x=0, color='#993399', linewidth=1)
#ratio = 0.0002
#xleft, xright = ax.get_xlim()
#ybottom, ytop = ax.get_ylim()
#ax.set_aspect(abs((xright-xleft)/(ybottom-ytop))*ratio)

plt.text(-10, 4, "(1/16)*np.sin(4*x)-(1/2)*np.cos(4*x)-(1/4)*x*np.cos(4*x)",\
         size=10,bbox=dict(boxstyle="square", ec=('#1F77B4'),fc=(1., 1., 1),))
plt.ylim(-5,5)
plt.plot(x, y3, linewidth = 0.9)
plt.show()
```



14.12 An RCL circuit connected in series has $R = 180$ ohms, $C = \frac{1}{280}$ farad, $L = 20$ henries, and an applied voltage $E = 10 \sin t$. Assuming no initial charge on the capacitor, but in initial current of 1 ampere at $t = 0$ when the voltage is first applied, find the subsequent charge on the capacitor.

A useful equation for solving RCL problems is based on Kirchoff's loop law, and looks like:

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = \frac{1}{L}E(t)$$

Substituting the given values into the above equation brings it to:

$$\ddot{q} + \frac{180}{20}\dot{q} + 14\,q = \frac{1}{20}\,10\sin t$$

$$\Longrightarrow \ddot{q} + 9\,\dot{q} + 14\,q = \frac{1}{2}\sin t$$

This equation can be entered in Wolfram Alpha:

!| d^2q/dt^2 + 9 * dq/dt + 14 * q = (1/2) * sin(t) |!

and the answer returned is:

$$q(t) \;=\; c_1\, e^{-7t} \;+\; c_2\, e^{-2t} \;+\; \frac{13 \sin{(t)}}{500} \;-\; \frac{9 \cos{(t)}}{500}$$

It may happen that the differentiated form of the above equation is needed. In that case it will be well that Wolfram Alpha should make it available:

!| d[c_1 * e^(-7 * t) + c_2 * e^(-2 * t) + (13/500) * sin(t) - (9/500) * t]/dt |!

And the expression for the charge equation is returned:

$$\dot{q}(t) \;=\; -7\, c_1\, e^{-7t} \;-\; 2\, c_2\, e^{-2t} \;+\; \frac{9}{500} \sin{(t)} \;+\; \frac{13}{500} \cos{(t)}$$

The first initial condition is the charge on the capacitor at the beginning of the problem:

$$0 \;=\; c_1 \;+\; c_2 \;-\; \frac{9}{500}$$

The second initial condition is the initial current at the beginning of the problem:

$$1 \;=\; -7\, c_1 \;-\; 2\, c_2 \;+\; \frac{13}{500}$$

Wolfram Alpha will be tasked with finding the simultaneous solutions of these last two equations (using the usual temporary aliases).

!| [0 = a + b - (9/500), 1 = -7 * a -2 * b + (13/500)] |!

In the calculations Wolfram Alpha has found that:

$$c_1 \;=\; -\frac{101}{500} \quad \text{and} \quad c_2 \;=\; \frac{11}{50}$$

Making the final equation:

$$q(t) \;=\; \frac{101}{500}\, e^{-7t} \;+\; \frac{11}{50}\, e^{-2t} \;+\; \frac{13}{500} \sin{(t)} \;-\; \frac{9}{500} \cos{(t)}$$

14.13 An RCL circuit connected in series has $R \;=\; 10$ ohms, $C \;=\; 10^{-2}$ farad, $L \;=\; \frac{1}{2}$ henry, and an applied voltage $E \;=\; 12$ volts. Assuming no initial current and no initial charge at $t \;=\; 0$ when the voltage is first applied, find the subsequent current in the system.

The equation for current looks somewhat different than the one used for the last problem. The intention is to get a differential equation for current. In this case, since the applied voltage is a constant, it disappears on differentiation.

$$\frac{d^{\,2} I}{dt^2} \;+\; \frac{R}{L}\,\frac{dI}{dt} \;+\; \frac{1}{LC}\, I \;=\; \frac{1}{L}\,\frac{dE(t)}{dt}$$

Substituting the given values into the above equation brings it to:

$$\frac{d^2 I}{dt^2} \;+\; 20\,\frac{dI}{dt} \;+\; 200\, I \;=\; 0$$

This equation can be entered in Wolfram Alpha:

!| d^2C/dt^2 + 20 * dC/dt + 200 * C = 0 |!

The problem W|A has with the symbol 'i' or 'I' was mentioned above; therefore an alias is used until the output can be claimed.

$$I(t) \;=\; c_2\, e^{-10t}\, \sin(10t) \;+\; c_1\, e^{-10t}\, \cos(10t)$$

The equation above is almost how it comes from Wolfram Alpha. However, to adopt the convention of the problem, the subscripts of the constants have been swapped.

First dealing with initial condition expressed by $I(0) \;=\; 0$:

$$0 \;=\; c_1$$

For dealing with the second initial condition, it is necessary to have an expression for $\frac{dI}{dt}$, so it can be evaluated at $t \;=\; 0$. The text comes up with:

$$\left.\frac{dI}{dt}\right|_{t=0} \;=\; \frac{1}{L}\, E(0) \;-\; \frac{R}{L}\, I_0 \;-\; \frac{1}{LC}\, q_0$$

and plugging in the initial condition numbers from the problem description:

$$\left.\frac{dI}{dt}\right|_{t=0} \;=\; \frac{1}{1/2}\,(12) \;-\; \frac{10}{1/2}\,(0) \;-\; \frac{1}{(1/2)(10^{-2})}\,(0) \;=\; 24$$

So now the numerical value of $\frac{dI}{dt}$ is known to be 24. However, to be of use, the rhs of the derivative needs to be available for operations.

Wolfram Alpha can come up with the long derivative form:

!| d[c_2 * e^(-10t) * sin(10 * t) + c_1 * e^(-10 * t) * cos(10 * t)]/dt |!

and the output looks like:

$$\frac{dI}{dt} \;=\; -10\, e^{-10t}\, (c_1\, (\sin(10t) \;+\; \cos(10t)) \;+\; c_2\, (\sin(10t) \;-\; \cos(10t)))$$

Knowing that the lhs = 24,

$$24 \;=\; -10\, e^{-10t}\, (c_1\, (\sin(10t) \;+\; \cos(10t)) \;+\; c_2\, (\sin(10t) \;-\; \cos(10t)))$$

$$\Longrightarrow 24 \;=\; -10\,(c_1 \;-\; c_2)$$

But c_1 is already known to equal zero.

$$\Longrightarrow \frac{12}{5} \;=\; c_2$$

Making the final equation:

$$I(t) \;=\; \frac{12}{5}\, e^{-10t}\, \sin(10t)$$

14.15 An RCL circuit connected in series has a resistance of 5 ohms, an inductance of 0.05 henry, a capacitor of 4×10^4 farad, and an applied alternating emf of $200 \cos 100\, t$ volts. Find an expression for the current flowing through this circuit if the initial current and the initial charge on the capacitor are both zero.

The problem could be begun by taking an inventory of initial values. Here $\frac{R}{L} \;=\; \frac{5}{0.05} \;=\; 100;\, \frac{1}{LC} \;=\; \frac{1}{(0.05)(4{\times}10^{-4})} \;=\; 50000$, and L by itself is 0.05 That should be enough to put together the 2nd derivative expression that got the last problem started.

$$\frac{d^2 I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC} I = \frac{1}{L}\frac{dE(t)}{dt}$$

Substituting the given values into the above equation brings it to:

$$\frac{d^2 I}{dt^2} + 100\frac{dI}{dt} + 50000\, I = -400000 \sin 100t$$

In this problem, unlike the last, the rhs is not a constant, and so will not disappear on differentiation. What does happen to it is that on first differentiation it becomes -20000 sin 100t, but it is also multiplied by $\frac{1}{L}$, which puffs it up to -400000 sin 100t volts.

This equation can be entered in Wolfram Alpha:

!| d^2C/dt^2 + 100 * dC/dt + 50000 * C = -400000 * sin(100 * t) |!

When the entered expression is executed in Wolfram Alpha, it gets a "standard computation time exceeded" notification. It will have to be revisited with a different CAS in the future.

The future is here. The alternative CAS used to finish the problem is Maxima (AppImage version 20.12.2). The following cells show the output which was generated. The cell below contains alternating lines of input and output including as the 4th line a solution to the subject differential equation.

For this situation, $I = y$, $t = x$, $c_i = k_i$. Special symbols in Maxima are prefaced with a '%' sign.

```
eq: 'diff(y, x, 2) + 100·'diff(y, x) + 50000·y = −400000·sin(100·x)
```

$$\frac{d^2}{d\,x^2}\,y+100\left(\frac{d}{d\,x}\,y\right)+50000\,y=-400000\,\sin(100\,x)$$

```
sol3: ode2(eq, y, x)
```

$$y=\%e^{-50\,x}\,(\%k1\,\sin(50\sqrt{19}\,x)+\%k2\,\cos(50\sqrt{19}\,x))-\frac{160\,\sin(100\,x)-40\,\cos(100\,x)}{17}$$

So the Maxima output corresponds to the following version of the solution:

$$I(t) = e^{-50t}\,(c_2 \sin(50\sqrt{19}\,x) + c_1 \cos(50\sqrt{19}\,t)) - \frac{160 \sin(10\,x) - 40 \cos(100\,x)}{17}$$

(The subscript numbers of the constants above have been swapped to agree with the text.)

And the derivative form will also be used, and was found by Maxima to be:

$$\frac{dI}{dt} + c_1\, e^{-50t}\,(\cos(50\sqrt{19}\,t\,) - \frac{4000 \sin(100\,x) + 16000 \cos(100\,x)}{17}$$

The text has an evaluation expression, used in a problem above, for directly calculating the numerical value of the derivative of the equation:

$$\left.\frac{dI}{dt}\right|_{t=0} = \frac{1}{L}\,E(0) - \frac{R}{L}\,I_0 - \frac{1}{LC}\,q_0$$

and plugging in the initial condition numbers from the current problem description:

$$\left.\frac{dI}{dt}\right|_{t=0} = 20\,(200) - 100\,(0) - 50000\,(0) = 4000$$

Going to the solution equation for $I(t)$ and inserting a value of zero for t:

$$0 = c_2\,(0) + c_1\,(1) - \frac{40}{17}$$

$$\Longrightarrow c_1 = \frac{40}{17} \approx 2.35$$

This value can be substituted into the solution equation to produce:

$$I(t) = e^{-50t}\,(c_2 \sin(50\sqrt{19}\,x) + 2.35\cos(50\sqrt{19}\,t)) - \frac{160\sin(10\,x) - 40\cos(100\,x)}{17}$$

And now Maxima can be called on to differentiate a form which has one constant already in place out of two total. The first line below is the input line; the rest is the output line. Only the highlighted values in the following cell will be left standing after the value of $t = 0$ has been substituted.

```
diff (%e^(−50·x)·((%k2·sin(50·sqrt(19)·x))+(2.35·cos(50·sqrt(19)·x))) − (160·sin(100·x) −40·cos(100·x))/17)
```

$$(-50\,\%e^{-50\,x}\,(\%k2\,\sin(50\sqrt{19}\,x)+2.35\cos(50\sqrt{19}\,x))+\%e^{-50\,x}\,(50\sqrt{19}\,\%k2\cos(50\sqrt{19}\,x)-117.5\sqrt{19}\sin(50\sqrt{19}\,x))-\frac{4000\sin(100\,x)+16000\cos(100\,x)}{17})\,\mathrm{del}(x)+\%e^{-50\,x}\sin(50\sqrt{19}\,x)\,\mathrm{del}(\%k2)$$

Pulling survivors from the wreckage of the above.

$$-50\,(2.35) + 50\sqrt{19}\,c_2 - \frac{16000}{17}$$

And in the flagpole evaluation line above, the derivative's numerical value was to be set to 4000. Wolfram Alpha can work out the result for c_2:

!| -50 * 2.35 + 50 * sqrt(19) * c_2 - 16000/17 = 4000 |!

Delivering an answer of $c_2 = 23.2$

So that the final equation is:

$$I(t) = -2.35\,e^{-50\,t}\cos(50\sqrt{19}\,t) + 22.13\,e^{-50\,t}\sin(50\sqrt{19}\,t) + \frac{40}{17}\cos(100\,t) - \frac{160}{17}\sin(100\,t)$$

14.17 Determine the circular frequency, the natural frequency, and the period of the steady-state current found in Problem 14.16.

Looking at the final equation in Problem 14.16, the terms on the left of the rhs which involve the exponential function will all tend to go to zero as t gets large. This leaves only the two right-most terms to describe the current.

Circular frequency: $\omega = 100$ Hertz
Natural frequency: $f = \omega / 2\pi = 15.92$ Hz
Period: $T = 1/f = 2\pi/100 = 0.063$ sec

14.24 A solid cylinder partially submerged in water having weight density 62.5 $\mathrm{lb/ft^3}$, with its axis vertical, oscillates up and down within a period of 0.6 sec. Determine the diameter of the cylinder if it weighs 2 lb.

First, the density of water is given as 62.5 lb/ft^3 , but the text only refers to the density of water as 1000, which would be kg/m^3 . (Actually equal to 1001.153.) A version of Newton's 2nd Law for buoyant cylinders is:

$$\pi r^2 h\rho, = mg$$

And it can be further refined into:

$$\ddot{x} + \pi r^2 \rho x = 0$$

The text jumps immediately to this relation, giving

$$\ddot{x} + 1000\,\pi r^2 x = 0$$

If this latter line in put into Wolfram Alpha,

!| d^2x/dt^2 + 1000 * pi * r^2 * x = 0 |!

The answer is returned:

$$x(t) = c_2 \sin(10\sqrt{10\,\pi}\, r\, t) + c_1 \cos(10\sqrt{10\,\pi}\, r\, t)$$

Taking a look at the implications of the above equation. The system's circular frequency is $\omega = 10\sqrt{10\,\pi}\, r$, and its natural frequency is $f = \omega/2\pi = 8.92\,r$. The period $T = 1/f = 1/(8.92\,r)$. We are given $0.6 = T = 1/(8.92\,r)$ and thus the radius of the cylinder equals $r = 0.187\text{ ft} = 2.24\text{ in}$, giving it a diameter of 4.48 in.

14.25 A prism whose cross section is an equilateral triangle with sides of length l floats in a pool of liquid of weight density ρ with its height parallel to the vertical axis. The prism is set in motion by displacing it from its equilibrium position and giving it an initial velocity. Determine the differential equation governing the subsequent motion of this prism.

Verbal Free Body Diagram In a diagram with 2 parts, both show the prism partially submerged in liquid. The views are separated by a vertical arrowed line pointing up, with the label "Positive x direction" at the top. The prisms shown in each view are identical, except at different heights, the left lower, the right somewhat raised. On the right prism is an exposed horizontal line across the body of the prism, indicating the "Equilibrium position" and this line is at liquid surface level in the left view. An arrowed line in the right view measures the distance from the liquid surface to the "Equilibrium position" line, and this line length is given the label "x(t)" in the right view, whereas the surface level in the left view is labeled "x = 0". The prism length below the "Equilibrium position" is also adorned with an arrowed marker giving the name "h" to this length: partly submerged in right view, completely submerged in left. The exposed top end of the prism is bevel cropped; the prism possessing five faces. The end faces consist of equilateral triangles, and labels announce the equal edge lengths.

Introducing one apparent contradiction in the figure. In the description the l length is described as the triangle edge length in "cross section". In the figure however, the l length is pictured as a set of equal edge lengths in the end face, which is not possible for what appears to be an end face which is non-perpendicular to the prism body.

After the prism on the right is raised above its equilibrium position, it may be simply dropped, or imparted with a downward force. Either way it will travel past the equilibrium position, and a subsequent oscillation will ensue, until the vertical motion completely dampens out. Through a few geometrical arguments the text deploys a convincing explanation that the equations governing the motion of the prism, Newton's second law with specific details, amounts to

$$\ddot{x} + \frac{\sqrt{3}\, l^2 \rho}{4\, m} = 0$$

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