Chapter 3: Classifications of First-Order Differential Equations A number of exercises gathered around the concept of ODE classification. Not all common classifications are considered.

3.1. Write the differential equation $xy' - y^2 = 0$ in standard form.

Standard form conforms to the template $y' = f(x, y)$. The minimum agitation to produce the standard form would result in the equation

$$y' = \frac{y^2}{x}$$

3.2. Write the differential equation $e^x y' + e^{2x} y = \sin x$ in standard form.

Though sympy has factoring and collecting methods, they do not seem well suited to the present problem. A couple of obvious manual factoring maneuvers leads to

$$y' = -e^x y + e^{-x} \sin x$$

which seems to qualify for the standard form.

3.3. Write the differential equation $(y' + y)^5 = \sin(y'/x)$ in standard form.

To quote the answer in the text, "This equation cannot be solved algebraically for $y'$, and cannot be written in standard form." It seems that either one of the locked-in ways felt by $y'$ in this equation would be well sufficient to make it intractable of being set in standard form.

3.4. Write the differential equation $y(yy' - 1) = x$ in differential form.

In [27]:
```
import sympy as sp
from sympy import *

x,y = symbols('x, y')
yp = Symbol('yp')

eq1 = Eq(y*(y*yp - 1) -x,0)

result = solve([eq1],(yp))
result
```

Out[27]:
```
{yp: (x + y)/y**2}
```

The text points out that the last output above,

$$y' = \frac{(x + y)}{y^2}$$

is in standard form. That being the case, the simplest way to get to differential form would seem to be

$$dy = \frac{(x + y)}{y^2}\, dx$$

3.5. Write the differential equation

$$\frac{dy}{dx} = \frac{y}{x}$$

in differential form.

The simplest, and perhaps most useful would be,

$$\frac{x}{dx} = \frac{y}{dy}$$

or, maybe better

$$\frac{dx}{x} = \frac{dy}{y}$$

3.6. Write the differential equation

$$(xy + 3)\, dx + (2x - y^2 + 1)\, dy = 0$$

in standard form.

A quick manual rearrangement gives

$$\frac{dy}{dx} = \frac{-(xy + 3)}{(2x - y^2 + 1)}$$

3.7. Determine if the following differential equations are linear:

$$(a)\ \ y' = (\sin x)\, y + e^x$$
$$(b)\ \ y' = x \sin y + e^x$$
$$(c)\ \ y' = 5$$
$$(d)\ \ y' = y^2 + x$$
$$(e)\ \ y' + xy^5 = 0$$
$$(f)\ \ xy' + y = \sqrt{y}$$
$$(g)\ \ y' + xy = e^x y$$
$$(h)\ \ y' + \frac{x}{y} = 0$$

In order to be linear, all tokens representing the dependent variable, and all derivatives of it, must be to the first power. Accordingly,

(a) linear
(b) not linear
(c) linear
(d) not linear
(e) not linear
(f) not linear
(g) linear
(h) not linear

3.8. Determine whether any of the differential equations in Problem 3.7 are Bernoulli equations.

In order to be a Bernoulli equation, an ode must show the form

$$y' + p(x)y = q(x)y^n$$

Accordingly,
(a) Bernoulli
(b) not Bernoulli
(c) Bernoulli
(d) not Bernoulli
(e) Bernoulli
(f) Bernoulli
(g) Bernoulli
(h) Bernoulli

The last in the list is recognizable as Bernoulli when realizing that the second term needs to go on rhs.

3.10. Determine if the following differential equations are separable:

$$(a)\quad \sin x\, dx + y^2\, dy = 0$$
$$(b)\quad xy^2\, dx - x^2 y^2\, dy = 0$$
$$(c)\quad (1 + xy)\, dx + y\, dy = 0$$

In order to be separable, an ode must be able to sequester all x-related components on one side of the equal sign, and all y-related on the other.

Accordingly,
(a) separable
(b) separable, after division by $x^2 y^2$
(c) not separable

3.11. Determine if the following differential equations are exact:

$$(a)\quad 3x^2 y\, dx + (y + x^3)\, dy = 0 \qquad\qquad (b)\quad xy\, dx + y^2\, dy = 0$$

Test for exactness: If M(x,y) and N(x,y) are continuous functions and have continuous first partial derivatives on some rectangle of the xy-plane, then (5.1) is exact if and only if

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

Though in the test equation the M blob is associated with $dy$, it is identified in the wild equation by an association with $dx$.

So in equation (a), the partial derivative of the first blob with respect to y is: $3x^2$

And also in equation (a), the partial derivative of the second blob with respect to x is: $3x^2$

Note that the indigenous differentials in both blobs are ignored. It is assumed for the present that their only role is in identifying whether an x-blob or a y-blob is being examined.

The decisive test partials being found equal, it is concluded that equation (a) is exact.

How about for equation (b)? Here the partial derivative of the first blob with respect to y is: $x$

And also for equation (b), the partial derivative of the second blob with respect to x is: $0$

The decisive test partials being found to be unequal, it is concluded that equation (b) is not exact.

3.12. Determine whether the differential equation $y' = y/x$ is exact.

$$y' = y/x$$

The text answer on this one is interesting. Pointing out that only equations in differential form can be judged for exactness, the text examines two possible differential forms:

$$\frac{y}{x} + (-1)dy = 0$$

with the test equation equaling:

$$\frac{\partial M}{\partial y} = \frac{1}{x} \neq 0 = \frac{\partial N}{\partial x}$$

for which case, the decisive test failing, the equation must be judged not exact. But the second differential form:

$$\frac{1}{x}\,dx + \frac{-1}{y}\,dy = 0$$

leads to the decisive test

$$\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x}$$

for which case, the decisive test succeeding, the same equation, in a different differential form, must be judged exact.