

Chapter 15-16: PDEs Solvers Using Neural Network Methods

PDE study considered from the viewpoint of neural networks. Many PDEs describe the evolution of a spatially distributed system over time. The state of such a system is defined by a value  $v(x, t)$  that depends on a spatial variable  $x$  that is usually a vector and on time  $t$ . The value itself can be a vector as well.

"Learning samples" can consist of discrete values of the initial condition  $v_{i,0}(x_{i,0})$  and values  $v_{ij}(x_i, t_j)$  at later times. The initial values are presented to the network as inputs. The outputs of the (recurrent) network at later times  $t_j$  are then compared with the sample values to calculate the error.

Since PDEs are often spatially homogeneous, it makes sense to use recurrent convolutional networks here. This is the same type that is often used in image processing.

The size of the convolutional kernel depends on the degree of spatial derivatives in the PDE:

The network needs to have at least one layer for each component of the value  $v$ . Additional hidden layers might be required, especially if the PDE has higher temporal derivatives.

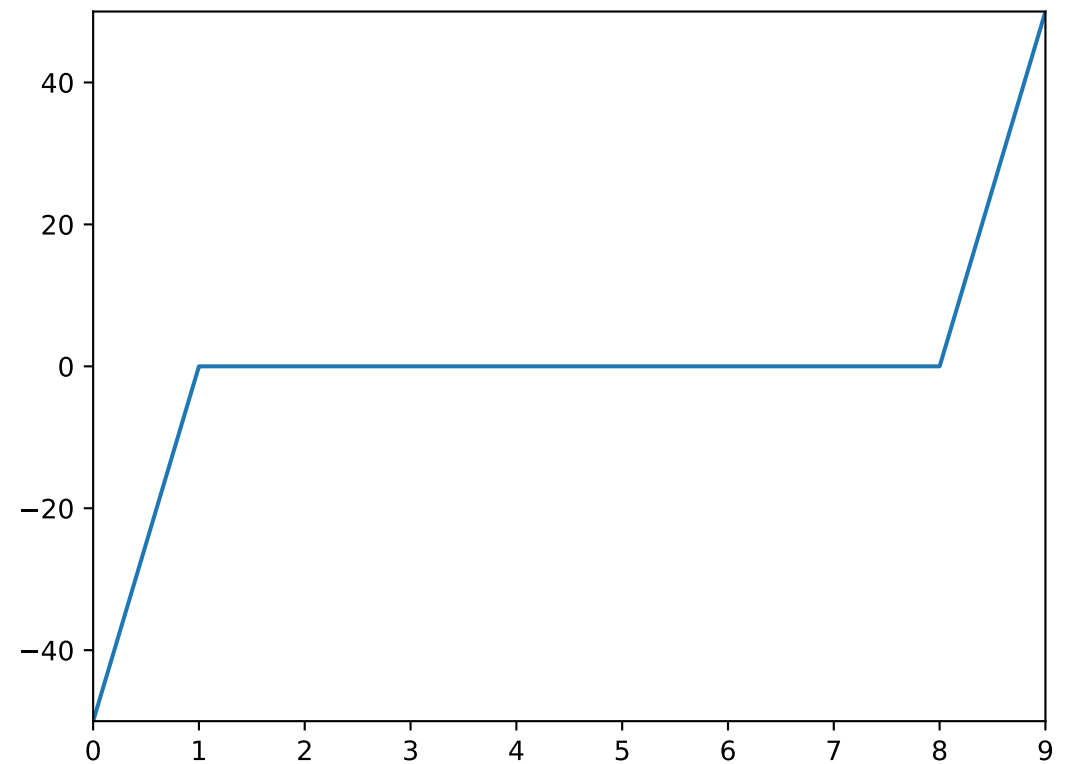
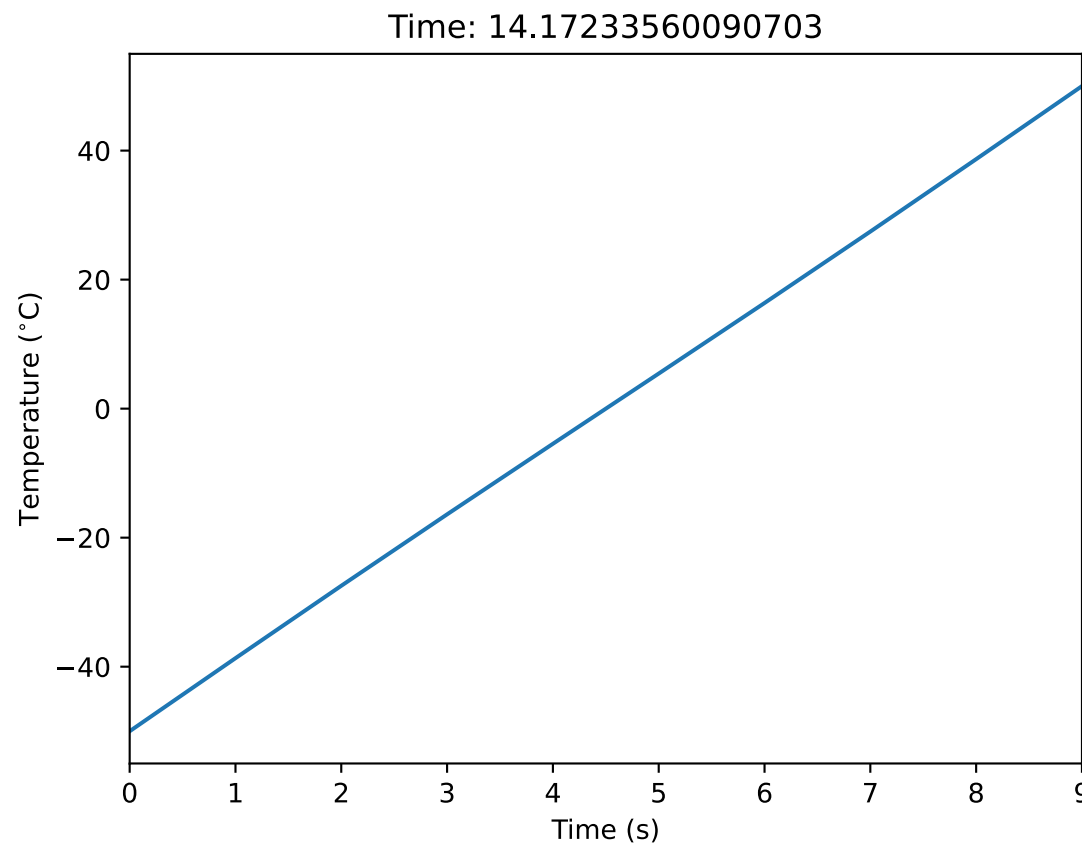
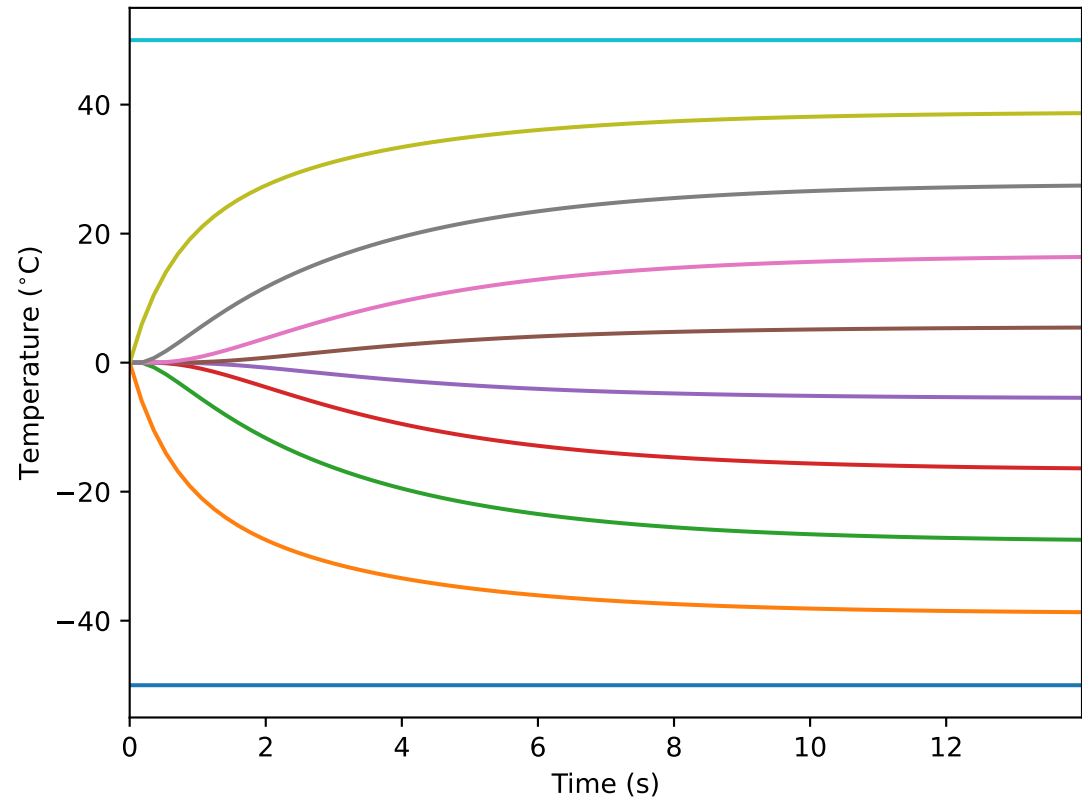
The example in the cell below was taken from the Github repository, <MatthewFilipovich / neural-network-pde-solver> . Besides this 1D heat equation example, examples for 2D heat and 1D wave are also contained there. Neural network PDE solving is a very active area, with lots of participants, and new solvers are coming out at a fast pace.

In this particular example, Mr. Filipovich has elected to show plots from new and older methods for comparison. (The 2nd plot shows a straight line, but if you look closely at the 5th plot, which is its companion, you will see that its line is not quite straight.)

The nengo\_pde module is not hosted on Pypi, and cannot be installed by pip. But due to the presence of the `_init_.py` file in the module folder, placing the folder in the Jupyter home directory allows the nengo\_pde module to be imported without difficulty into a Jupyter cell. (It is, however, necessary to install the nengo module from Pypi.)

```
In [14]: 1 """Module solves heat equation in 1D using finite difference method and nengo_pde."""
2 from nengo_pde import Solver1D
3 import matplotlib.pyplot as plt
4 %config InlineBackend.figure_formats = ['svg']
5
6
7 def feedback_connection(u):
8     return - (K/dx**2) * 2*u
9
10
11 def lateral_connection(u):
12     return K/dx**2 * u
13
14
15 # Nengo simulation
16 t_steps = 80 # Number of time steps
17 x_steps = 8 # Number of x steps
18 neurons = 500 # Number of neurons
19 radius = 100 # Radius of neurons
20 boundaries = [-50, 50] # Constant boundary conditions
21 solver = Solver1D(feedback_connection, lateral_connection)
22
23 # Grid properties
24 K = 4.2
25 x_len = 20 # mm
26 dx = x_len/x_steps
27 dt = dx**2/(2*K**2) # dt chosen for stability
28
29 # Run finite difference method simulation
30 solver.run_FDM_order1(dt, t_steps, x_steps, boundaries)
31 fig, ax = solver.plot_population(dt, False)
32 ax.set_xlabel('Time (s)')
33 ax.set_ylabel('Temperature ($^{\circ}$C)')
34 plt.show()
35 fig, ax = solver.plot_grid(t_steps, False)
36 ax.set_xlabel('Time (s)')
37 ax.set_ylabel('Temperature ($^{\circ}$C)')
38 plt.show()
39 solver.animate(nframes=t_steps)
40
41 # Run nengo_pde simulation
```

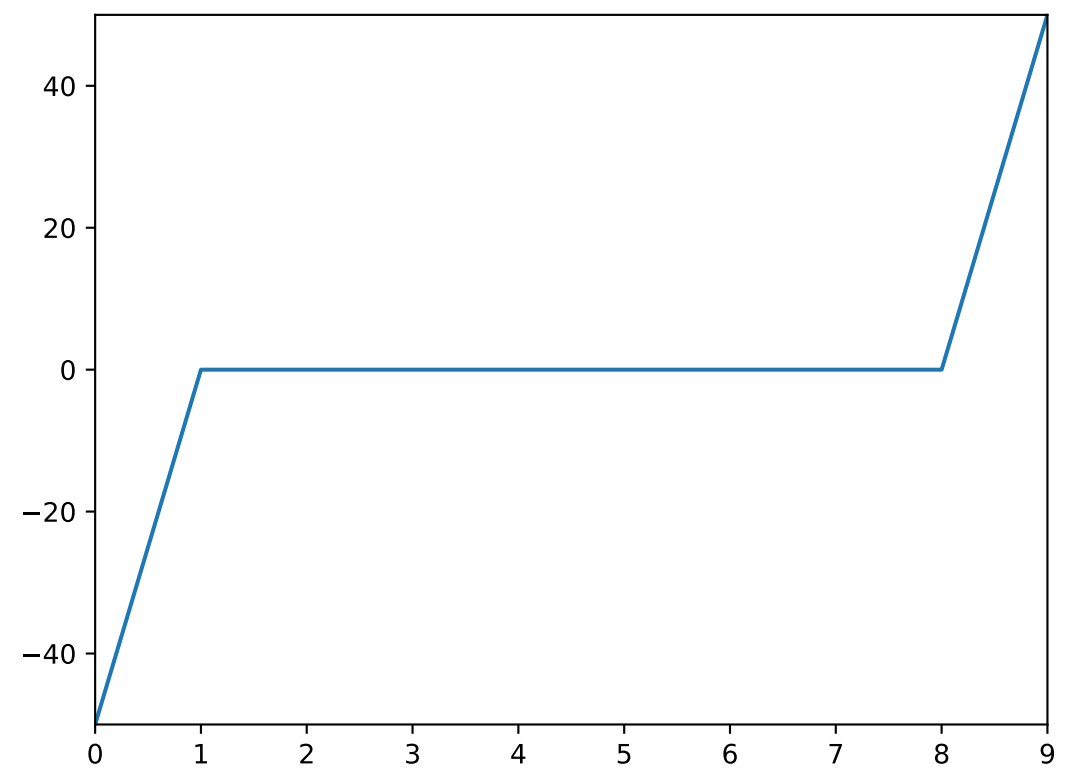
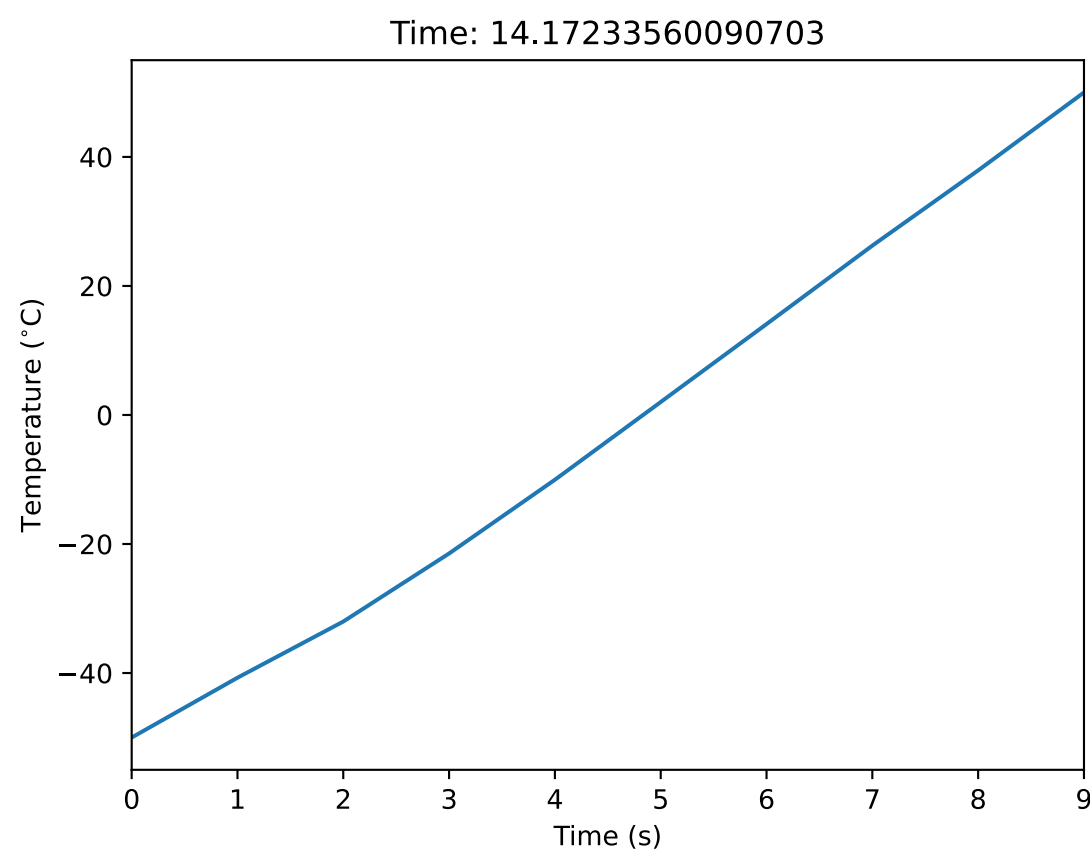
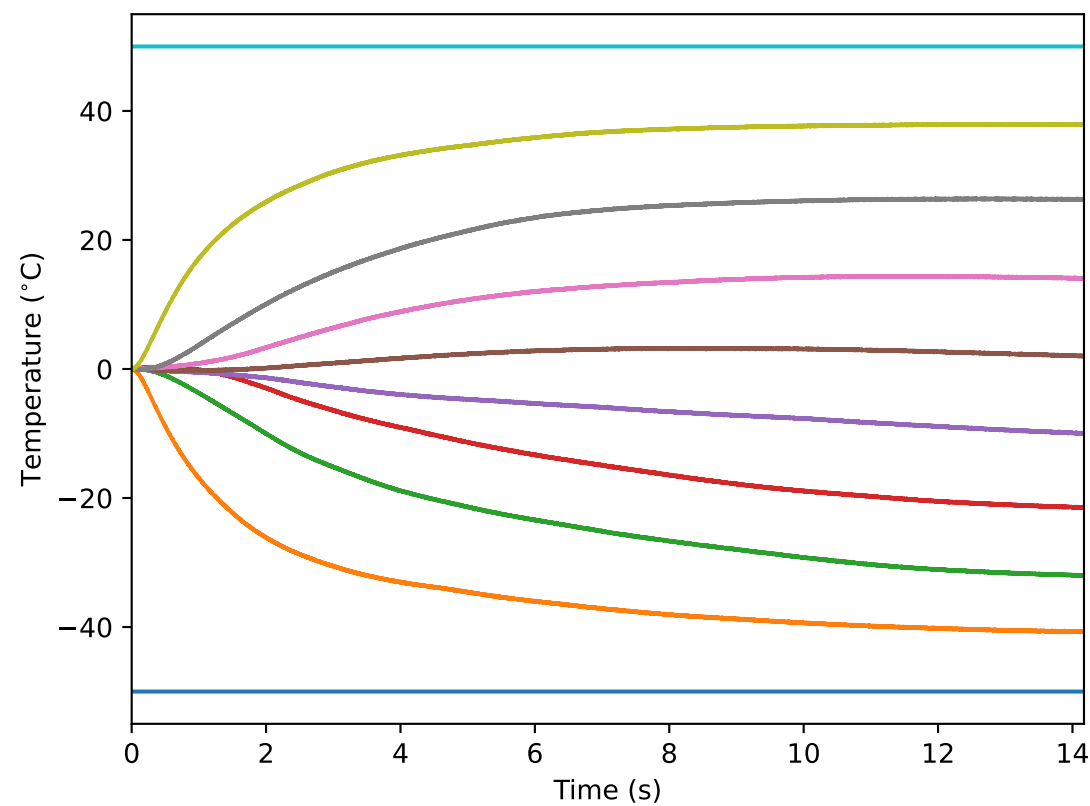
```
42 solver.run_nengo_order1(dt, t_steps, x_steps, boundaries, neurons, radius)
43 fig, ax = solver.plot_population(0.001, False)
44 ax.set_xlabel('Time (s)')
45 ax.set_ylabel('Temperature ( $^{\circ}\text{C}$ )')
46 plt.show()
47 fig, ax = solver.plot_grid(t_steps, False)
48 ax.set_xlabel('Time (s)')
49 ax.set_ylabel('Temperature ( $^{\circ}\text{C}$ )')
50 plt.show()
51 solver.animate(nframes=t_steps)
52
'''
```



HMTL progress bar requires Jupyter Notebook >= 5.0 or Jupyter Lab. Alternatively, you can use TerminalProgressBar().

Build finished in 0:00:02.

Simulation finished in 0:00:06.



Out[14]: <matplotlib.animation.FuncAnimation at 0x25558739d60>

There is an interesting paper with the title: *Variational Monte Carlo Approach to Partial Differential Equations with Neural Networks*, available on the arXiv eprint site under the name arXiv:2206.01927v2. (Because it deals with both Monte Carlo as well as Neural Networks, the work could be referred to either of two of these pde notebooks.) The Github repository for the project is RehMoritz/vmc\_pde.

(Although the arXiv site copy of the paper has good integrity when viewed online, two of the figures are mangled on PDF download. To access an undamaged version for online viewing and for printing, go to [

<https://iopscience.iop.org/article/10.1088/2632-2153/aca317/pdf> (<https://iopscience.iop.org/article/10.1088/2632-2153/aca317/pdf>) ]

Following are instructions for getting a Jupyter notebook working to show the example study from the repository, if the platform is Windows.

1. From the site: [(<https://whls.blob.core.windows.net/unstable/index.html>)] ([%5D](https://whls.blob.core.windows.net/unstable/index.html)) download one of the following versions of Jaxlib,  
cpu/jaxlib-0.1.75-cp37-none-win\_amd64.whl  
cpu/jaxlib-0.1.75-cp38-none-win\_amd64.whl  
cpu/jaxlib-0.1.75-cp39-none-win\_amd64.whl  
the choice depending on whether Python 3.7, 3.8, or 3.9 is available for use. Place the downloaded file in the Jupyter working directory.
2. Install the jaxlib file, from within Jupyter, with the command `!pip3 install jaxlib-0.1.75-cp39-none-win_amd64.whl`.
3. Install the module flax, from within Jupyter, with the command `!pip3 install flax==0.3.6`.
4. If a version of jax is installed, uninstall it or delete the site-package.
5. If action was taken in 4. above, a kernel restart is probably called for.
6. Install the module jax, from within Jupyter, with the command `!pip3 install jax==0.2.18`. Pip should do the install in spite of a slight disagreement concerning dependencies.
7. Download the zipped version of the repository and extract it, resulting in vmc\_pde-main. From the subdirectory 'vmc\_fluids', find 13 local modules (excluding 'main.py'). Copy these modules to the working directory of Jupyter.
8. Open the file `main.py` in Idle or wherever, then copy it to a Jupyter cell. The file should run, producing extensive printed data and seven time-sequence plots. The main program is shown below, but not run here. Instead, three of the plots are shown below it.

In [ ]:

```
1 import jax
2 jax.config.update("jax_enable_x64", True)
3 import jax.numpy as jnp
4 import flax.linen as nn
5 import numpy as np
6 import matplotlib.pyplot as plt
7 %config InlineBackend.figure_formats = ['svg']
8 import os
9 import time
10
11 import global_defs
12 import var_state
13 import sampler
14 import net
15 import grid
16 import train
17 import evolutionEq
18 import tdvp
19 import stepper
20 import visualization
21 import mpi_wrapper
22 import util
23
24
25 def norm_fun(v, S):
26     # norm_fun for the timesteps
27     return v @ S @ v
28
29
30 # Initializing the net
31 initKey = 1
32 sampleKey = 1
33
34 mode_dict = {"fluidpaper": {"offset": jnp.ones(2) * 0.25, "dim": 2, "latent_space_name": "cos_dis"},
35              "harmonicOsc": {"offset": jnp.ones(2) * 1, "dim": 2, "latent_space_name": "Gauss", "m"},
36              "harmonicOsc_diff": {"offset": jnp.array([1, 0, 0, 1, 0, 0]) * 1, "dim": 6, "latent_"},
37              "diffusion": {"offset": jnp.zeros(8), "dim": 8, "latent_space_name": "Student_t", "m"},
38              "diffusion_anisotropic": {"offset": jnp.zeros(12), "dim": 12, "latent_space_name": "m"},
39              "mwe": {"offset": jnp.zeros(2), "dim": 2, "latent_space_name": "Gauss", "mcmcbound": 1}
40 mode = "harmonicOsc_diff"
41 mode = "diffusion"
42 mode = "mwe"
43
44 """
45 List of things that have to be set manually before starting a run:
46 - parameter nu of the student - t in BOTH (!) sampler.py and net.py - starts with nu=2 atm.
47 - network specifications, whether to use both s and t, etc.
48     - Diffusion: noAdd
49     - harmonicOsc: DifferentAdd
50 - timestep:
51     - Diffusion: dt = 1e-7, fixed, with increasing step size, factor: 1.3, maxStep: 1e-2
52     - harmonicOsc: dt=1e-4, fixed, with increasing step size, factor: 1.3, maxStep: 1e-2
53 - blocks:
54     - Diffusion:: 4, intermediate (dim//2)
55     - harmonicOsc: 4, intermediate (dim // 2)
56 - latent space covariance matrix:
```

```

57     - Diffusion: np.eye(..) + A @ A.T
58     - harmonicOsc: L @ L.T
59     """
60
61     dim = mode_dict[mode]["dim"]
62     offset = mode_dict[mode]["offset"]
63     mcmcbound = mode_dict[mode]["mcmcbound"]
64     gridbound = mode_dict[mode]["gridbound"]
65     symgrid = mode_dict[mode]["symgrid"]
66     latent_space_name = mode_dict[mode]["latent_space_name"]
67     evolution_type = mode_dict[mode]["evolution_type"]
68
69     # set up sampler
70     sampler = sampler.Sampler(dim=dim, numChains=30, name=latent_space_name, mcmc_info={"offset": off
71
72     # set up variational state
73     print("Identifier -3")
74     vState = var_state.VarState(sampler, dim, initKey, 4, network_args={"intmediate": (dim // 2,) * 1
75     print(f"Number of Model parameters: {vState.numParameters}")
76
77
78     # Some (old) sanity checks - can be removed
79     mynet = {"net": vState.net, "params": vState.params}
80     x_real = jnp.ones(dim)
81     print(mynet["params"])
82     z_latent, _ = mynet["net"].apply(mynet["params"], x_real, evaluate=False, inv=False)
83     x_real, _ = mynet["net"].apply(mynet["params"], z_latent, evaluate=False, inv=True)
84     print(z_latent)
85     print(x_real)
86
87     x_real = - jnp.ones(dim)
88     z_latent, jac = mynet["net"].apply(mynet["params"], x_real, evaluate=False, inv=False)
89     x_real, jac_inv = mynet["net"].apply(mynet["params"], z_latent, evaluate=False, inv=True)
90     print(z_latent)
91     print(x_real)
92
93     x_real = jnp.zeros(dim)
94     z_latent, jac = mynet["net"].apply(mynet["params"], x_real, evaluate=False, inv=False)
95     x_real, jac_inv = mynet["net"].apply(mynet["params"], z_latent, evaluate=False, inv=True)
96     print(z_latent)
97     print(x_real)
98
99
100    # Initializing the grid
101    if dim == 2:
102        bounds = np.ones((dim,)) * gridbound
103        n_gridpoints = 200
104        grid = grid.Grid(bounds, n_gridpoints, sym=symgrid)
105        integral = vState.integrate(grid)
106        print("Integral value:", integral)
107
108    # time evolution
109    dt = 1e-7
110    tol = 1e-2
111    maxStep = 1e-2
112    comp_integrals = False
113    # myStepper = stepper.AdaptiveHeun(timeStep=dt, tol=tol, maxStep=maxStep)
114    myStepper = stepper.FixedStepper(timeStep=dt, mode='Heun', maxStep=maxStep, increase_fac=1.3)
115    tdvpEq = tdvp.TDVP()
116    timings = util.Timings()
117    evolutionEq = evolutionEq.EvolutionEquation(dim=dim, name=evolution_type)
118    nSamplesTDVP = 10000
119    nSamplesObs = 10000
120
121    # data to learn a specific state
122    # std_dev = 1
123    # size = (1, 1000, dim)
124    # mode = "standard_normal"
125    # data, target_fun = train.gen_data(size, mode=mode, std=std_dev)
126    # net = train.train(vState, data, grid, lr=1e-3, batchsize=100, target_fun=target_fun, epoches=20
127
128    wdir = "output/" + mode + f"/NsamplesTDVP{nSamplesTDVP}_NsamplesObs{nSamplesObs}_T10/"
129    wdir = "output/" + mode + f"/NsamplesTDVP{nSamplesTDVP}_NsamplesObs{nSamplesObs}/"
130    wdir = "output/" + mode + f"/NsamplesTDVP{nSamplesTDVP}_NsamplesObs{nSamplesObs}_Tdifferent/"
131    wdir = "output/" + mode + f"/test_NsamplesTDVP{nSamplesTDVP}_NsamplesObs{nSamplesObs}_maxStep{max
132    if mpi_wrapper.rank == 0:
133        try:
134            os.makedirs(wdir)
135        except OSError:
136            print("Creation of the directory %s failed" % wdir)
137        else:
138            print("Successfully created the directory %s " % wdir)
139
140    t = 0
141    t_end = 5
142    plot_every = 1e0
143
144    if dim == 2:
145        # visualization.plot_vectorfield(grid, evolutionEq)
146        # plt.savefig(wdir + 'vectorfield.pdf')

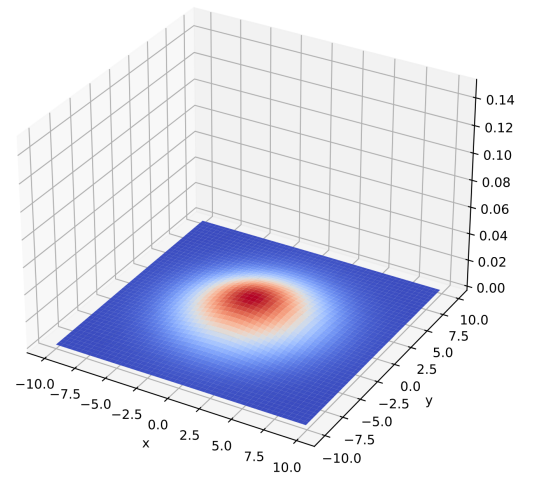
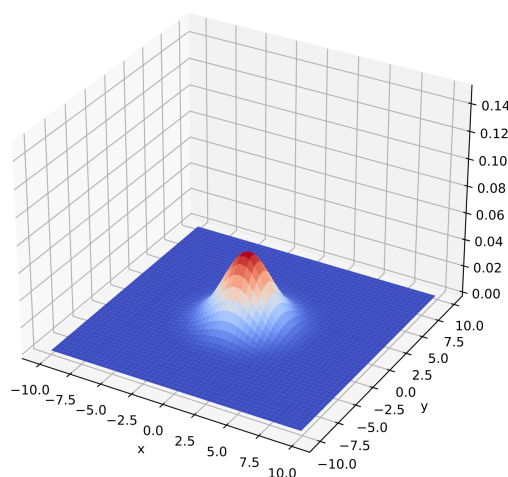
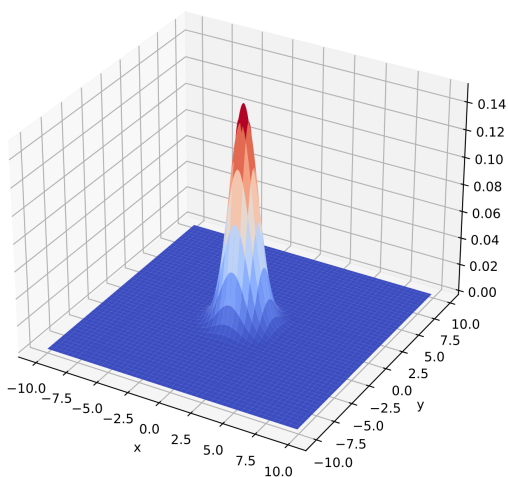
```



```

147     # plt.show()
148
149     visualization.plot(vState, grid, proj=False)
150     plt.savefig(wdir + f't_{t:.3f}.pdf')
151     plt.show()
152
153     # states = vState.sample(2000000)
154     # visualization.plot_data(states, grid, title='Samples')
155     # plt.show()
156
157
158     infos = {"times": [], "ev": [], "snr": [], "solver_res": [], "tdvp_error": [], "dist_params": []}
159     n_list = []
160     while t < t_end + dt:
161         t1 = time.perf_counter()
162         dp, dt, info = myStepper.step(0, tdvpEq, vState.get_parameters(), evolutionEq=evolutionEq, ps
163         vState.set_parameters(dp)
164         infos["times"].append(t)
165
166         print(f"t = {t:.3f}, dt = {dt:e}")
167         print("\t Timings:")
168         timings.print_timings()
169         print(f"\t Total (in main.py): {time.perf_counter() - t1}")
170
171         print("\t Data:")
172         print(f"\t > Solver Residual = {tdvpEq.solverResidual}")
173         print(f"\t > TDVP Error = {tdvpEq.tdvp_error}")
174         if comp_integrals:
175             print(f"\t > Integral 1sigma = {info['integral_1sigma']}")
176             print(f"\t > Integral 0.5sigma = {info['integral_0.5sigma']}")
177             print(f"\t > Integral 0.1sigma = {info['integral_0.1sigma']}")
178         print(f"\t > Entropy = {info['entropy']}")
179         print(f"\t > dist params = {vState.params['params']['dist_params']}")
180         print(f"\t > Means = {info['x1']}")
181         print(f"\t > Covar = {info['covar']}")
182
183         for key in info.keys():
184             if key not in infos.keys():
185                 infos[key] = []
186                 infos[key].append(info[key])
187         infos["ev"].append(tdvpEq.ev)
188         infos["snr"].append(tdvpEq.snr)
189         infos["solver_res"].append(tdvpEq.solverResidual)
190         infos["tdvp_error"].append(tdvpEq.tdvp_error)
191         infos["dist_params"].append(vState.params['params']['dist_params'])
192
193         n = round(t / plot_every)
194         if np.abs(t - n * plot_every) < dt and dim == 2 and n not in n_list:
195             n_list.append(n)
196             integral = vState.integrate(grid)
197             print("Integral value:", integral)
198
199             visualization.plot(vState, grid, proj=False)
200             plt.savefig(wdir + f't_{t:.3f}.pdf')
201             plt.show()
202
203             print(vState.net.apply(vState.params, jnp.zeros(dim,), evaluate=False, inv=True)[0])
204
205             # visualization.plot_line(vState, scale=10, fit=True, offset=offset)
206             # plt.show()
207
208             t = t + dt
209
210     util.store_infos(wdir, infos)
211     visualization.make_final_plots(wdir, infos)
212     plt.show()
213
214

```



The paper describes the method as applicable to diffusion, i.e. parabolic, problem environments. Because elliptic equations are also amenable to solution by Monte Carlo techniques, it might be possible to extend the treatment to them also.