In [1]:

```
Autosave disabled
```

Chapter 23 Convolutions and the Unit Step Function.

23.1 Find $f(x) * g(x)$ when $f(x) = e^{3x}$ and $g(x) = e^{2x}$.

The title of the chapter is the clue that convolution and unit step functions are related. In Wolfram Alpha, the tasked convolution can be accomplished with the entry:

!| Convolve[e^(3 t) UnitStep[t], e^(2t) UnitStep[t], t, x] |!

Exact result

$e^{2x}\,(e^{x} - 1)\,\theta(x)$

This leaves the step function in the answer. It is not in the answer in the text, therefore it is necessary to employ the rationalization that inserting the step function for proper functioning of W|A and retracting it when desired are equivalent.

text answer = $e^{3x} - e^{2x}$

23.3 Find $f(x) * g(x)$ when $f(x) = x$ and $g(x) = x^2$.

Continuing the off-label use of the function Convolve. The desired output can be obtained by using:

!| Convolve[t UnitStep[t], t^2 UnitStep[t], t, x] |!

Exact result

$\frac{x^4}{12}\,\theta(x)$

The unit step function appears as θ again.

text answer = $\frac{1}{12}\,x^4$

23.4 Find $\mathscr{L}^{-1}\left\{\frac{1}{s^2 - 5s + 6}\right\}$ by convolutions.

Once into the landscape of Laplace, the treatment of convolutions shifts. (Is that a pun?) Using Wolfram Alpha it is very straightforward.

!| simplify[inverselaplacetransform[1/(s^2 - 5 s + 6)]] |!

Result

$e^{2t}\,(e^{t} - 1)$

text answer = $e^{3x} - e^{2x}$

23.5 Find $\mathscr{L}^{-1}\left\{\frac{6}{s^2 - 1}\right\}$ by convolutions.

And similarly,

!| simplify[inverselaplacetransform[6/(s^2 - 1)]] |!

Result

$$6 \sinh(t)$$

For some reason the text did not simplify the solution as shown above. W|A can verify,

!| (exp(x)-exp(-x))/2 == sinh(x) |!

with the answer of 'True'.

23.6 Find $\mathscr{L}^{-1}\left\{\dfrac{1}{s\,(s^2+4)}\right\}$ by convolutions.

And another

!| simplify[inverselaplacetransform[1/(s (s^2 + 4))]] |!

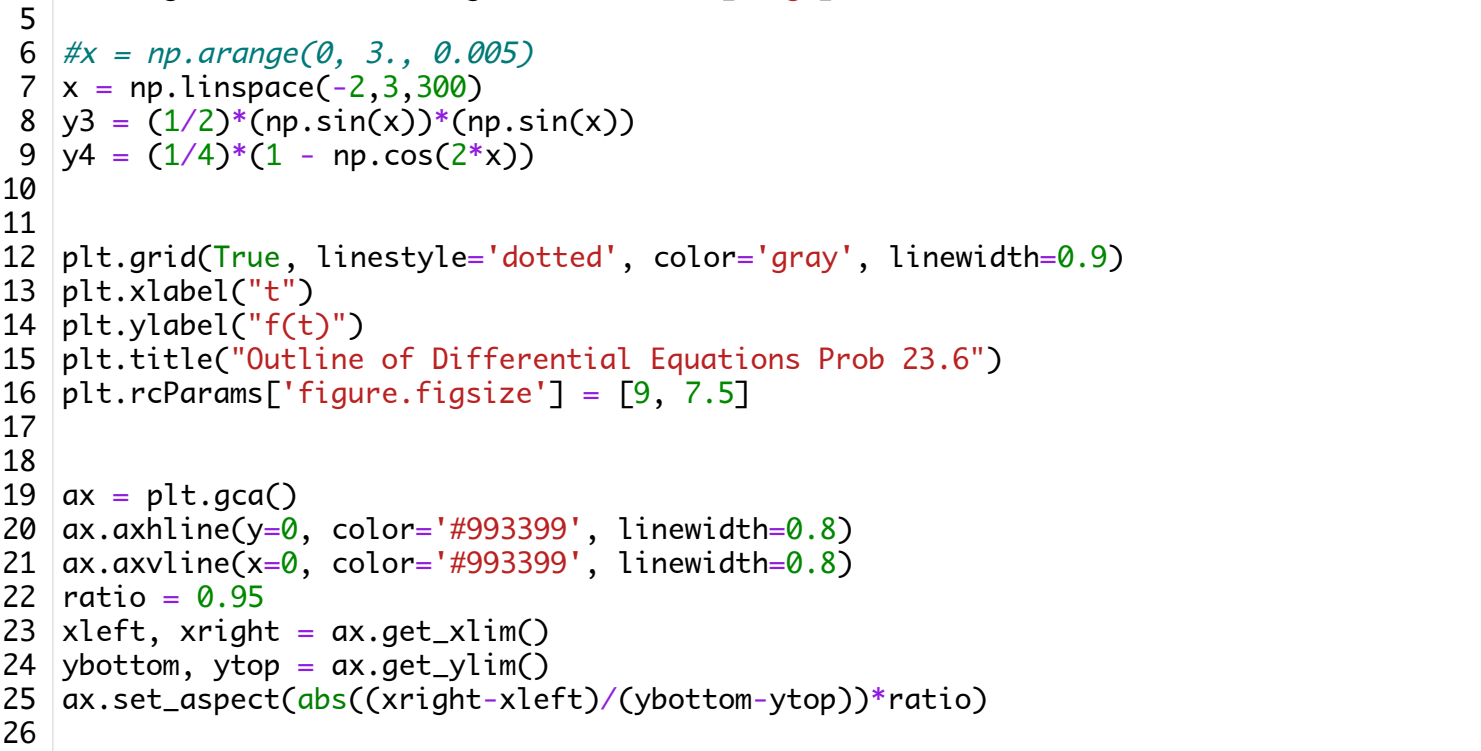
Result

$$\frac{\sin^2(t)}{2}$$

Not in the same form as text answer, so a plot will be made to consider the two variations.

In [41]:

```
import numpy as np
import matplotlib.pyplot as plt

%config InlineBackend.figure_formats = ['svg']

#x = np.arange(0, 3., 0.005)
x = np.linspace(-2,3,300)
y3 = (1/2)*(np.sin(x))*(np.sin(x))
y4 = (1/4)*(1 - np.cos(2*x))


plt.grid(True, linestyle='dotted', color='gray', linewidth=0.9)
plt.xlabel("t")
plt.ylabel("f(t)")
plt.title("Outline of Differential Equations Prob 23.6")
plt.rcParams['figure.figsize'] = [9, 7.5]


ax = plt.gca()
ax.axhline(y=0, color='#993399', linewidth=0.8)
ax.axvline(x=0, color='#993399', linewidth=0.8)
ratio = 0.95
xleft, xright = ax.get_xlim()
ybottom, ytop = ax.get_ylim()
ax.set_aspect(abs((xright-xleft)/(ybottom-ytop))*ratio)


#plt.text(965, -2.5e43, "-np.log(abs((np.cos(x/2))**2 - (np.sin(x/2))**2)))
#plt.text(965, 2.25e43, "SOLN: y = 3*np.exp(x**2) + 1/2", size=10,\
        # bbox=dict(boxstyle="square", ec=('#8C564B'),fc=(1., 1., 1),))
plt.ylim(-1,1.5)
plt.plot(x, y3, linewidth = 3)
plt.plot(x, y4, linewidth = 0.9, color = 'w')
plt.show()

```



23.7 Find $\mathscr{L}^{-1}\left\{\dfrac{1}{(s-1)^2}\right\}$ by convolutions.

And yet another

!| simplify[inverselaplacetransform[1/((s - 1)^2)]] |!

Result
$e^t\ t$

text answer = xe^x

23.9 Graph the function $f(x) = u(x-2) - u(x-3)$.

This calls for a plot of unit step functions. Seeing that

$$u(x-2) = \begin{cases} 0 & x < 2 \\ 1 & x \geq 2 \end{cases} \quad \text{and} \quad u(x-3) = \begin{cases} 0 & x < 3 \\ 1 & x \geq 3 \end{cases}$$

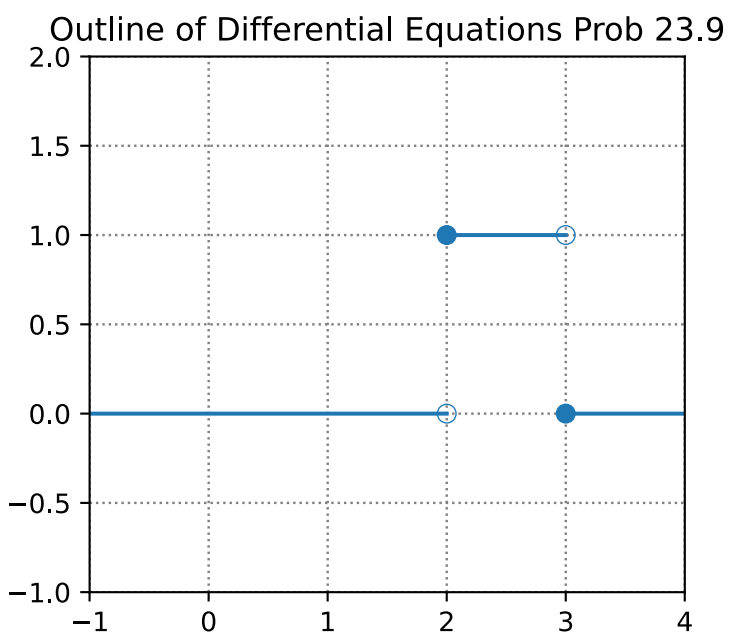
it follows that

$$f(x) = u(x-2) - u(x-3) = \begin{cases} 0-0=0 & x < 2 \\ 1-0=1 & 2 \leq x < 3 \\ 1-1=0 & x \geq 3 \end{cases}$$

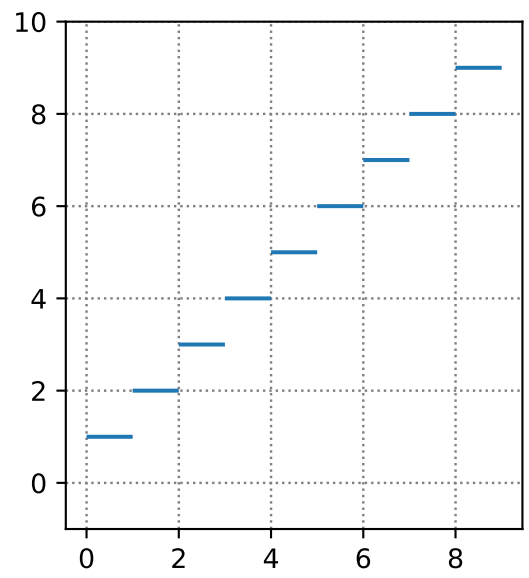
The following plot is a primitive attempt to express the above function graphically. But it works. Following the first plot is a second, which does not apply here but is merely a pointless demo.

In [13]:

```
import numpy as np
import matplotlib.pyplot as plt

%config InlineBackend.figure_formats = ['svg']

x = [2, 3]
y = [1, 1]
a = [-10, 2]
b = [0, 0]
z = [3, 5]

ax = plt.gca()
plt.grid(True, linestyle='dotted', color='gray', linewidth=0.9)
plt.rcParams['figure.figsize'] = [3, 3]
ax.set_title('Outline of Differential Equations Prob 23.9')

plt.step(x, y)
plt.step(a, b, color='#1F77B4')
plt.step(z, b, color='#1F77B4')

ratio = 0.1
xleft, xright = ax.get_xlim()
ybottom, ytop = ax.get_ylim()
ax.set_aspect(abs((xright-xleft)/(ybottom-ytop))*ratio)

xpts = np.array([2, 3])
ypts = np.array([0, 1])
wpts = np.array([2, 3])
zpts = np.array([1, 0])
plt.plot(xpts, ypts, markersize=7, color='#1F77B4', marker='o', \
         linestyle='none', mfc='none', markeredgewidth=0.5)
plt.plot(wpts, zpts, markersize=7, color='#1F77B4', linestyle ='none', \
         marker='o', mfc='#1F77B4', markeredgewidth=0.5)

plt.ylim(-1,2)
plt.xlim(-1,4)
plt.show()

```



In [63]:
```
import matplotlib.pyplot as plt

y = range(1, 10)
xmin = range(9)
xmax = range(1, 10)

ax = plt.gca()
plt.grid(True, linestyle='dotted', color='gray', linewidth=0.9)
plt.rcParams['figure.figsize'] = [5, 3.5]

ratio = 1.0
xleft, xright = ax.get_xlim()
ybottom, ytop = ax.get_ylim()
ax.set_aspect(abs((xright-xleft)/(ybottom-ytop))*ratio)

plt.ylim(-1,10)
ax.hlines(y, xmin, xmax)
plt.show()

```



23.10 Graph the function $f(x) = 5 - 5u(x - 8)$ for $x \geq 0$.

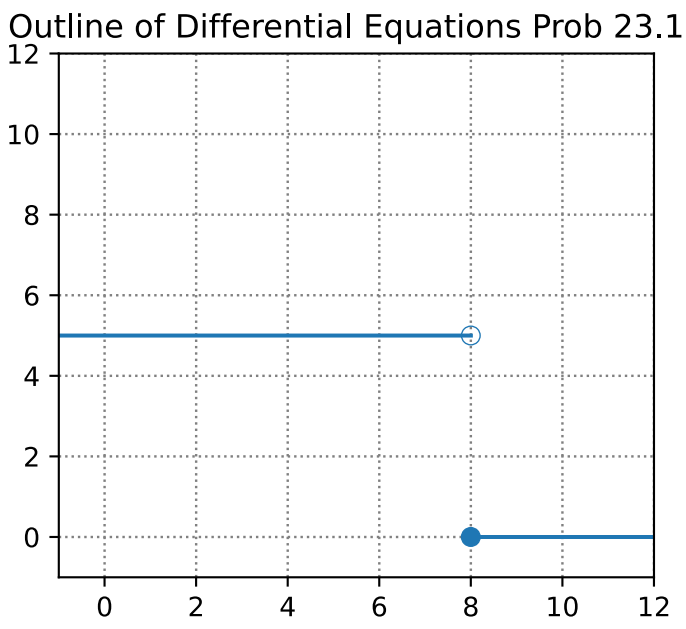
This calls for a plot of unit step functions. The function $f(x)$ is

$$f(x) = 5 - 5u(x - 8) = \begin{cases} 5 & x < 8 \\ 0 & x \geq 8 \end{cases}$$

A rough imitation of a stepwise plot is shown.

In [16]:

```
import numpy as np
import matplotlib.pyplot as plt

%config InlineBackend.figure_formats = ['svg']

x = [-10, 8]
y = [5, 5]
a = [8, 12]
b = [0, 0]
#z = [3, 5]

ax = plt.gca()
plt.grid(True, linestyle='dotted', color='gray', linewidth=0.9)
plt.rcParams['figure.figsize'] = [5, 5]
ax.set_title('Outline of Differential Equations Prob 23.10')

plt.step(x, y)
plt.step(a, b, color='#1F77B4')
#plt.step(z, b, color='#1F77B4')

ratio = 0.2
xleft, xright = ax.get_xlim()
ybottom, ytop = ax.get_ylim()
ax.set_aspect(abs((xright-xleft)/(ybottom-ytop))*ratio)

xpts = np.array([-5, 8])
ypts = np.array([5, 5])
wpts = np.array([8, 13])
zpts = np.array([0, 0])
plt.plot(xpts, ypts, markersize=7, color='#1F77B4', marker='o', \
         linestyle='none', mfc='none', markeredgewidth=0.5)
plt.plot(wpts, zpts, markersize=7, color='#1F77B4', linestyle ='none', \
         marker='o', mfc='#1F77B4', markeredgewidth=0.5)

plt.ylim(-1,12)
plt.xlim(-1,12)
plt.show()

```



23.11 Use the unit step function to give an analytic representation of the function $f(x) = \begin{cases} 0 & x < 4 \\ x - 4 & x \geq 4 \end{cases}$

By some measure, to provide the required input to Wolfram Alpha is to formulate the analytical description which the problem is requesting. Anyway, the input of

!| Piecewise[{{0, x < 4}, {(x-4) Unitstep, x >= 4}}] |!

does the job.

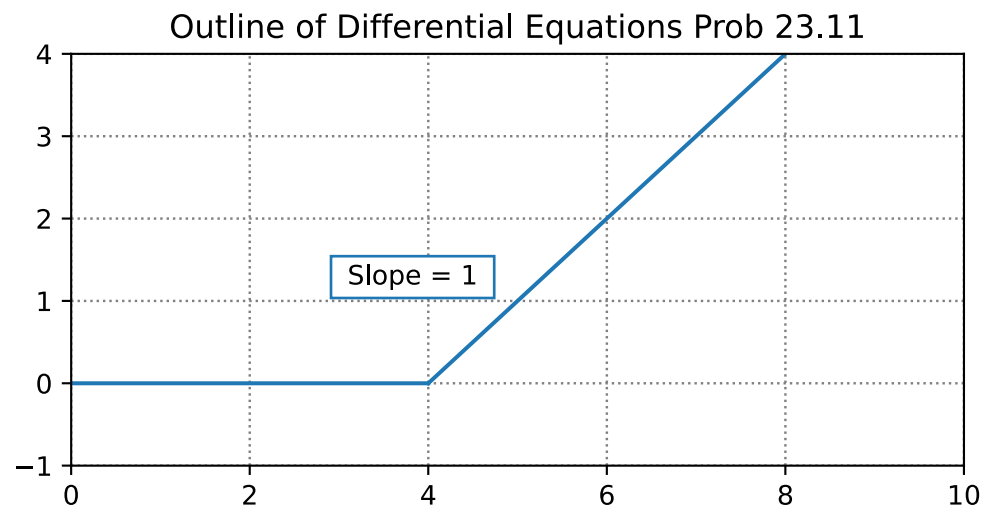
Wolfram Alpha gives a couple of alternate expressions, all in piecewise notation. W|A includes a plot showing the function, matching the text plot. The plot shown below also matches both problem description and the text plot.

The text answer is $(x - 4)u(x - 4)$.

In [77]:

```
import numpy as np
import matplotlib.pyplot as plt

%config InlineBackend.figure_formats = ['svg']

x = [-1, 4]
y = [0, 0]
xl = np.linspace(4, 8, 200)

def f(xl):
    return (xl-4)
h = f(xl)
```

```
13
14 ax = plt.gca()
15 plt.grid(True, linestyle='dotted', color='gray', linewidth=0.9)
16 plt.rcParams['figure.figsize'] = [6, 6]
17 ax.set_title('Outline of Differential Equations Prob 23.11')
18
19 plt.step(x, y)
20 plt.plot(xl, h, color = '#1F77B4')
21
22 ratio = 0.41
23 xleft, xright = ax.get_xlim()
24 ybottom, ytop = ax.get_ylim()
25 ax.set_aspect(abs((xright-xleft)/(ybottom-ytop))*ratio)
26
27 plt.text(3, 1.2, " Slope = 1 ", size=10,\
28          bbox=dict(boxstyle="square", ec=('#1F77B4'),fc=(1., 1., 1),))
29
30 plt.ylim(-1,4)
31 plt.xlim(0,10)
32 plt.show()
33
```



23.14 Find the Laplace transform of $f(x) = \begin{cases} 0 & x < 4 \\ x^2 & x \geq 4 \end{cases}$

This problem is the only one going from Ordinaryland to Laplaceland. It can be entered into Wolfram Alpha as:

!| laplace transform of Piecewise[{{0, x < 4}, {x^2, x >= 4}}] |!

In Wolfram Alpha the result is expressed as:

Result

$$\frac{2e^{-4p}(8p^2+4p+1)}{p^3}$$

which is equivalent to the text answer.

In [ ]:
```
1
```

In [ ]:
```
1
```