Chapter 7: Applications of First-Order Differential Equations

A number of problems in Physics, Engineering, and Epidemiology and considered,weighing and acknowledging the contributions made by differential equations.

Cutting and pasting and Wolfram Alpha. Wolfram Alpha is amenable to accepting pasted entries. In this chapter pastable expressions are given a distinctive boundary fence, exemplified by the sample: !| abcdef |!

In the above pseudo-entry, only the alpha characters would be copied for transfer to Wolfram Alpha.

7.1 A $20,000 savings account pays 5 percent interest per year, compounded continuously. Find (a) the amount in the account after three years, and (b) the time required for the account to double (if static).

N represents the current balance. At first, $N(0) = 20,000$. The balance grows by interest payments, in proportion to the amount in the account. The constant of proportionality is $k = 0.05$ and the relationship is $\frac{dN}{dt} - 0.05N = 0$. Entering this expression in Wolfram Alpha yields

$$N(t) = c_1 e^{0.05t}$$

At time $t = 0, \quad N(0) = 20000$, and

$$\implies 20000 = c_1$$

Substituting upward produces

$$\implies N(t) = 20000e^{0.05t}$$

and the last is a description of the balance at any time $t \quad$ (with t in years)

Plugging in 3 for t, and putting the rhs into Wolfram Alpha gives:

$$\$23,236.68$$

which answers (a).

When will the account equal 40000?

$$40000 = 20000e^{0.05t}$$

$$\implies 2 = e^{0.05t}$$

$$\implies \log(\text{abs}(2)) = 0.05t$$

In Wolfram Alpha, the entry line is:

!| log(abs(2)) = 0.05t, t |!

and the answer is 13.86, which answers (b)

7.2 A bank account of $\$5000$ earns interest compounded continuously. If undisturbed, what will the account be worth after 7 years if the interest rate is 8.5 percent for the first 4 years and 9.25 percent for the last three years?

Having two different interest rates in the same problem effectually turns it into two problems.

Let $N(t)$ be the balance in the account at any time t. At the beginning, $N(0) = 5000$. For the first 4 years, $k = 0.085$ and

$$\frac{dN}{dt} - 0.085\, N = 0$$

Entering this expression into Wolfram Alpha

!| dN/dt - 0.085 N = 0 |!

produces the following

$$N(t) = c_1\, e^{0.085t}$$

At time $t = 0$, $N(0) = 5000$, and both of these initial conditions can be substituted into the previous equation, changing it to

$$5000 = c_1 e^{0.085\,(0)} = c_1 \qquad (0 \le t \le 4)$$

and 5000 can take the place of c_1 in the previous equation. Wanting to know the value of the bank account after 4 years means entering the line:

!| N(t) = 5000 e^(0.085 4) |!

into Wolfram Alpha. The response is an answer of 7024.74. So that is the signpost at the 4-year mark. The second part of the problem can be started now.

The interest rate increases to 9.25 percent, which means entering the following line in Wolfram Alpha:

!| dN/dt - 0.0925 N = 0 |!

The answer given, in a line that can be referred to as "4plus" is:

$$N(t) = c_1\, e^{0.0925\, t}$$

Now the 4plus line can be modified to read:

$$7024.74 = c_1\, e^{0.0925\,(4)}$$

Consulting W|A,

!| 7024.74 = c1 e^(0.0925 4), c1 |!

it is found that $c_1 = 4852.23$, and therefore the calculation equation has become

$$N(t) = 4852.23\, e^{0.0925\, t}$$

Only one more calculation is required, and the entry in W|A is made:

!| 4852.23 e^(0.0925 * 7) |!

resulting in the answer, in dollars, for the whole 7-year period: 9271.44. Using the full period here instead of only the last 3 years is not intuitive, and suggests the following test:

!| 4852.23 * e^(0.0925 * 4) |!

which results in the answer: 7024.44, exactly the same amount which the original $\$5000$ became under the original lower interest rate after 4 years. So in a sense the value 4852.23 seems a fictitious one, invented so that the 2-step process can be feigned to occur in only 1 step.

7.3 What constant interest rate is required if an initial deposit placed into an account that accrues interest compounded continuously is to double its value in six years?

The account balance is described by

$$\frac{dN}{dt} - kN = 0$$

And entering this expression into Wolfram Alpha results in the answer:

$$N(t) = c_1\, e^{kt}$$

At time $t = 0$ the starting balance is N_0, though what that amount is remains unknown. What is known is that

$$N_0 = c_1\, e^{k\,(0)} = c_1$$

and $N_0 = c_1$ means that

$$N(t) = N_0\, e^{kt}$$

What the problem wants is the rate of interest, k, that will allow N_0 to double in six years. Ideally, Wolfram Alpha could do this in one step with the line:

!| N_0 = 2 * N_0 * e^(6 * k * t), k |!

Unfortuately, W|A has no clue to what this multi-variable snipe hunt could mean. It is necessary to pry things apart by hand first. For the sake of unconfusing nomenclature for W|A, proceed in this way:

$$2N_0 = N_0 e^{6k}$$

$$\Longrightarrow e^{6k} = 2$$

Then apply a log identity to get:

$$6 * k = log|2|$$

Then enter into W|A the following:

!| N[6*k = log|2|], k |!

To encourage W|A to respond with numerical output. The answer is k = 0.115525; that is, an interest rate of 11.5 percent would be required to get the balance doubled in six years.

7.4 A bacteria culture is known to grow at a rate proportional to the amount present. After 1 hour, 1000 strands of bacteria are observed in the culture; and after 4 hours, 3000 strands. Find (a) an expression for the approximate number of strands of the bacteria present in the culture at any time t and (b) the approximate number of strands of the bacteria originally in the culture.

Similar to the problems on interest, the bacteria ODE is:

!| dN/dt - kN = 0 |!

in the form entered in Wolfram Alpha. The result:

$$N(t) = ce^{kt}$$

(Losing the subscript on c conveniently.) This equation translates the 2 given time points given by the problem, i.e. $1000 = ce^{k}$ and $3000 = ce^{4k}$ into conditions that can be used for a general solution.

In Wolfram Alpha it is simply entered as:

!| 1000 = c * e^k, 3000 = c * e^(4 * k) |!

and an answer is provided, to the effect that $c = 693.361$ and $k = 0.366204$.

This establishes finding the equation for bacterial count at any time marker, that being

$$N(t) = 693\, e^{0.366t}$$

constituting the answer to part (a)

Part (b) of the problem is extremely easy, consisting of simply setting t to zero and checking the value of the remaining frame, which is 693. So at the start, there were approximately 693 strands of bacteria.

7.5 The population of a certain country is known to increase at a rate proportional to the number of people presently living in the country. If after two years the population has doubled, and after three years the population is 20,000, estimate the number of people initially living in the country.

The increase in population is proportional to the number living in the country. Watching the proportional change is the business of the ODE. If the information is entered in Wolfram Alpha as:

!| dN/dt - k * N = 0 |!

then what is returned is the equation for the number of people at any time, expressed as:

$$N(t) = ce^{kt}$$

Looking at the equation at $t = 0$, the e drops out and N_0 is seen to equal c. Since the parameters are equal, a substitution can be made, such that

$$N(t) = N_0\, e^{kt}$$

The problem's given data include the condition that at $t = 2,\ N(t) = 2N_0$.

Plugging these numbers in produces the equation

$$2N_0 = N_0\, e^{2k}$$

and in converting from exponential to log form the relevent equation is:

$$k = \frac{1}{2}\log 2$$

In Wolfram Alpha the entry can be made:

!| N[k = 1/2 * log 2], k |!

which on execution reveals that $k = 0.346574$

Substitution now updates the population-at-any-time formula to become general:

$$N(t) = N_0\, e^{0.347\, t}$$

When $t = 3,\ N = 20000$. Substituting this info into the 'p-a-a-t' formula gives:

$$20000 = N_0\, e^{(0.347)(3)} = N_0\,(2.2832)$$

In W|A it could be tried as:

!| N[20000 = N_0 * e^(0.347 * 3)], N_0 |!

This gives an initial population size of 7062, in agreement with the text answer.

7.6 A certain radioactive material is known to decay at a rate proportional to the amount present. If initially there are 50 milligrams of the material present and after 2 hours it is observed that the material has lost 10 percent of its original mass, find (a) an expression for the mass of the material remaining at any time t, (b) the mass of the material after 4 hours, and (c) the time at which the material has decayed to $\frac{1}{2}$ of its original mass.

The symbol N stands for the mass of material at time t. If the equation is entered into Wolfram Alpha as

!| dN/dt - k * N = 0 |!

then the answer shows as $N = c_1 e^{kt}$ (and the subscript on c can be dropped for present purposes).

We know that at time 0, $N = 50$. Referring this info into the above equation, $c = 50$ and the general equation for mass decrease with time becomes

$$N = 50\, e^{kt}$$

However, k is still not known. Another data point available is that at time $t = 2$, 10 percent of the original mass of 50 mg, or 5 mg, has decayed. That information can be used to solve for k.

$$45 = 50\, e^{2k}$$

$$\Longrightarrow k = \frac{1}{2}\log\frac{45}{50}$$

The following line is entered into Wolfram Alpha:

!| N[k = 1/2 * log(45/50)], k |!

which reveals the value of k to be -0.053. Now the general mass decay formula has been found:

$$N = 50\, e^{-0.053\, t}$$

and part (a) of the problem has been solved. Part (b) is to find the remaining mass at time $t = 4$. In Wolfram Alpha the line is run:

!| N [N = 50 * e^(-0.053 * 4)], N |!

and the answer is supplied, 40.45 mg., which takes care of (b). Part (c) wants to know how long it will take for half the material to decay. This will be when $25 = 50\, e^{-0.053\, t}$ or $-0.053\, t = \log\frac{1}{2}$. The Wolfram Alpha version would be:

!| N [-0.053 * t = log(1/2)], t |!

and the answer is 13.08 hrs.

7.7 Five mice in a stable population of 500 are intentionally infected with a contagious disease to test a theory of epidemic spread that postulates the rate of change in the infected population is proportional to the product of the number of mice who have the disease with the number that are disease free. Assuming the theory is correct, how long will it take half the population to contract the disease?

N(t) will be the number of mice with the disease at time t. At $N(0)$ the number of infected is 5, and the number of uninfected will be $500 - N(t)$. The theory predicts that

$$\frac{dN}{dt} = k\,N(500 - N)$$

The character k represents the constant of proportionality. The proportionality here relates to the number of disease free mice. The last equation can be rearranged:

$$\frac{dN}{N\,(500 - N)} - k\,dt = 0$$

Wolfram Alpha can process this equation using partial fraction decomposition:

!| [1/(N*(500 - N))] /Apart |!

which results in the expression:

$$\frac{1}{500N} - \frac{1}{500\,(N - 500)}$$

which can be rearranged in the somewhat unorthodox way:

$$\frac{\frac{1}{500}}{N} + \frac{\frac{1}{500}}{500 - N}$$

and then joined up with the part concerning the k factor to give:

$$\frac{1}{500}\left(\frac{1}{N} + \frac{1}{500 - N}\right) dN - k\,dt = 0$$

To get the solution to this equation, in Wolfram Alpha the line can be entered:

!| 1/500 * integrate[1/N + 1/(500 - N) dN] - integrate[k dt] |!

which results in:

$$\frac{1}{500}\left(\log|N| - \log|500 - N|\right) - kt = c$$

(Constant c and absolute value bars added manually to the last equation.)

$$\Longrightarrow \log\left|\frac{N}{500 - N}\right| = 500(c + kt),$$

$$\Longrightarrow \frac{N}{500 - N} = e^{500\,(c + kt)}$$

$$\Longrightarrow \frac{N}{500 - N} = c_1\,e^{500\,kt}$$

At time $t = 0$, $N = 5$, and substituting this condition shows:

$$\frac{5}{495} = c_1\,e^{500\,k\,(0)} = c_1$$

So the general equation for calculating the number of infected is modified by the substitution of the above to become:

$$\frac{N}{500 - N} = \frac{1}{99}\,e^{500\,kt}$$

The situation we are tasked to find out about is the one with 250 infected mice. This adjusts the general equation to point to:

$$1 = \frac{1}{99}\,e^{500\,kt}$$

This equation can be solved directly by Wolfram Alpha, by entering the line:

!| N[1 = 1/99 * e^(500 * k * t)], t |!

and the answer delivered is:

$$t \approx \frac{0.00919024}{k}$$

7.8 A metal bar at a temperature of 100° is placed in a room at a constant temperature of 0° F. If after 20 minutes the temperature of the bar is 50° F, find (a) the time it will take the bar to reach a temperature of 25° F and (b) the temperature of the bar after 10 minutes.

The ODE which applies to proportional temperature changes is:

$$\frac{dT}{dt} + kT = kT_m$$

where the T_m term refers to the ambient temperature of the environment. Since in the problem description it is seen that the room temperature is 0° F, the equation is now:

$$\frac{dT}{dt} + kT = 0$$

Entering this ODE into Wolfram Alpha with the line

!| dT/dt + k * T = 0 |!

elicits the response:

$$T(t) = c_1 e^{-kt}$$

Note: The subscript on c|, can be dropped if desired. Since at time $t = 0$ the temperature of the bar is 100° F, the modification of the affected parts of the above equation equate 100 with c and it is okay to write:

$$T(t) = 100\, e^{-kt}$$

as the general temperature equation for the problem environment; however, no value for k is available yet. A remedy readily appears, since the problem offers that at time $t = 20$ the value of T is 50, from which the following can be deduced:

$$50 = 100e^{-20k}$$

This gives Wolfram Alpha enough to work with to solve for k :

!| N[50 = 100 * e^(-20 * k)], k |!

(By including the N brackets in the request, W|A is induced to offer a real as well as mixed format answer.) The real answer is:

$$k \approx 0.0346574$$

Now the general version of the equation for temperature is available:

$$T = 100e^{-0.035\, t}$$

Part (a) of the problem wants to know the time when the bar temperature reaches $25^\circ\ F$. W|A can tell that:

!| N[25 = 100*e*^(-0.035t)], t |!

A matter of 39.6 minutes.

Part (b) of the problem wants to know what the bar temperature will be 10 minutes after time $t = 0$. The line for that question looks like:

!| N[T = 100*e*^(-0.03510)], t |!

And the answer is 70.4688° F.

7.9 A body at a temperature of 50° F is placed outdoors where the temperature is 100° F. If after 5 minutes the temperature of the body is 60° F, find (a) how long it will take the body to reach a temperature of 75° F and (b) the temperature of the body after 20 minutes.

The ODE which applies to proportional temperature changes is:

$$\frac{dT}{dt} + kT = kT_m$$

The problem advises that the ambient temperature is 100° F, which alters the equation above to become:

$$\frac{dT}{dt} + kT = 100k$$

First order of business is to get Wolfram Alpha to solve the ODE:

!| dT/dt + k * T = 100 * k |!

produces

$$T(t) = c_1 e^{-kt} + 100$$

(The subscript on c can be dropped.) An initial condition can help fill in one of the unknown parameters. At time $t = 0$, the temperature T equals 50° F. Substituting these numbers into the equation above constrains the constant c to equal -50. This leaves the evolving equation at

$$T(t) = -50e^{-kt} + 100$$

What other information is available to tighten things up? At time $t = 5$, it occurs that $T = 60$. This should be enough evidence to fix the equation.

!| N[60 = -50 * e^(-5 * k) + 100], k |!

When the above line is run in Wolfram Alpha, it reveals that $k \approx 0.0446287$, which makes the general equation equal to:

$$T(t) = -50e^{-0.045t} + 100$$

Part (a) of the problem wants to know at what time t the temperature will be 75° F. Referring the question to Wolfram Alpha with

!| N[75 = -50*e*^(-0.045t) + 100], t |!

provides the answer: $t = 15.4033$

Part (b) of the problem wants to know what the temperature of the body will be at time $t = 20$. Wolfram Alpha declares this time to equal 79.67 minutes.

7.10 A body at an unknown temperature is placed in a room which is held at a constant temperature of 30° F. If after 10 minutes the temperature of the body is 0° F and after 20 minutes the temperature of the body is 15° F, find the unknown initial temperature.

Editing the primary temperature proportion equation to reflect the ambient temperature of the problem:

$$\frac{dT}{dt} + kT = 30k$$

And submitting it to Wolfram Alpha for solution:

!| dT/dt + k * T = 30 * k |!

results in the solution:

$$T(t) = c_1 e^{-kt} + 30$$

(And the subscript on c can be dropped.)

At time mark $t = 10$ the problem informs that $T = 0$. Therefore, from the equation above,

$$0 = ce^{-10k} + 30$$

Likewise, at time mark $t = 20$ the problem informs that $T = 15$. Therefore, backing up two equations for the form

$$15 = ce^{-20k} + 30$$

The last two equations can be solved by Wolfram Alpha simultaneously, using the entry line:

!| [0 = c * e^(-10 * k) + 30, 15 = c * e^(-20 * k) + 30] |!

The solutions which come out of W|A match those found by the text: $c = -60$ and $k = \frac{\log(2)}{10}$

Note that it takes a second execution, of

!| N[log(2)/10] |!

to get the numerical form of the answer for k : 0.0693147.

This will make the general equation equal to:

$$T(t) = -60e^{-0.069\,t} + 30$$

The specific answer sought by the problem is not worth bothering W|A with: the initial temperature, that of time $t = 0$:

$$T(0) = -60e^{(-0.069)(0)} + 30 = -60 + 30 = -30^\circ \text{ F}$$

7.11 A body of mass 5 slugs is dropped from a height of 100 ft with zero velocity. Assuming no air resistance, find (a) an expression for the velocity of the body at any time t, (b) an expression for the position of the body at any time t, and (c) the time required to reach the ground.

Verbal free body diagram Picture a ball bearing suspended above a plane in cross section representing the ground. The center of the ball bearing has a label indicating vertical height $x = 0$. Ground level has a label indicating vertical height $x = 100$, positive direction is down.

Since there is no air resistance, the equation of motion which is applicable is:

!| dv/dt = g |!

Wolfram Alpha solves the equation with the answer:

$$v(t) = c_1 + gt$$

At time $t = 0$ there is no velocity, $v = 0$, and therefore $c_1 = 0$. Thus,

$$v = gt$$

or assuming that $g = 32\ ft/sec^2$, then problem part (a) is answered:

$$v = 32t$$

Velocity being the rate of change with time, the above can be presented as the ODE with the Wolfram Alpha input:

!| dx/dt = 32 * t |!

and the reply is:

$$x(t) = c_1 + 16t^2$$

But since $x = 0$ at time $t = 0$, $c_1 = 0$ leaving only

$$x(t) = 16t^2$$

as the answer to part (b)

Part (c) inquires as to the time corresponding to a travel distance of 100. In W|A

!| N[100 = 16t^2], t |!

returns the answer that $t = 2.5$ seconds.

7.12 A steel ball weighing 2 lb is dropped from a height of 3000 ft with no velocity. As it falls, the ball encounters air resistance equal to $v/8$ (in pounds), where v denotes the velocity of the ball (in feet per second). Find (a) the limiting velocity for the ball and (b) the time required for the ball to hit the ground.

In falling body problems involving air resistance the force of the fall is reduced by the air resistance. Instead of $F = mg$ there is now the formula $F = mg - kv$. Irrelevant note: This is to be understood as applicable to small bodies like ball bearings; for larger, real world bodies the resisting force is proportional to v^2 instead of v. For the present (idealized) problem the equation of motion is set forth as:

$$F = mg - kv$$

and the velocity is modeled as

$$\frac{dv}{dt} + \frac{k}{m} v = g$$

The coordinate system is as described in the previous problem, except that x is now located at 3000 units from the drop point. The weight of the ball is 2 lb, and its mass is 2/32 = 1/16 slug. Plugging in 1/8 for the k quantity and 1/16 for the mass, results in:

$$\frac{dv}{dt} + 2v = 32$$

Entering this equation into Wolfram Alpha produces the ODE solution,

$$v(t) = c_1 e^{-2t} + 16$$

It is given that $v = 0$ at time $t = 0$, which can bring the generalized equation into better focus. Making the substitutions,

$$0 = c_1 e^{-2\,(0)} + 16 = c_1 + 16$$

from which it is deduced that $c_1 = -16$ and the $v(t)$ equation becomes:

$$v(t) = -16e^{-2t} + 16$$

With the generalized equation in hand it is possible to address the specific questions posed by the problem:

(a) What is the terminal velocity?

The terminal velocity will be seen in the limit as $t \to \infty$, or as posed to Wolfram Alpha,

!| lim (-16 * e^(-2 * t) + 16) as t->infinity |!

which is given by W|A as equal to 16.

(b) What is the time elapsed when the ball strikes the ground?

Since $v = \frac{dx}{dt}$ the equation for $v(t)$ can be transformed into:

$$\frac{dx}{dt} = -16\,e^{-2t} + 16$$

Integrating as:

!| integrate[-16e^(-2 * t) + 16, t] |! yields:

$$8\,(2t + e^{-2t}) + c_2$$

The left side of the velocity equation was integrated at the same time though not shown, but it should be noted that it is now position rather than velocity that is being dealt with.

Putting in the starting values of zero position and zero time, it is found that

$$0 = 8 + c_2$$

or $$c_2 = -8$$

The position equation can give what is desired now:

$$x(t) = 8e^{-2t} + 16t - 8$$

by entering in Wolfram Alpha

!| N[3000 = 8 * e^(-2 * t) + 16 * t - 8], t |!

The answer is returned: 188 seconds.

7.13 A body weighing 64 lb is dropped from a height of 100 ft with an initial velocity of 10 ft/sec. Assume that the air resistance is proportional to the velocity of the body. If the limiting velocity is known to be 128 ft/sec, find (a) an expression for the velocity of the body at any time t and (b) an expression for the position of the body at any time t.

The coordinate system is the same as used before. The height above ground will not figure in this problem. The body weighs 64 lb. Using $w = mg$, it follows that the mass of the body is 2 slugs. There is an accepted observation that when $k > 0$ the limiting velocity v_l is defined by

$$v_l = \frac{mg}{k}$$

Given that v_l is 128 ft/sec and both m and g are known, k can be quickly calculated to equal 1/2.

The 'falling with resistance' formula,

$$\frac{dv}{dt} + \frac{k}{m} v = g$$

can be filled in as

$$\frac{dv}{dt} + \frac{1}{4} v = 32$$

Wolfram Alpha, applied to with the entry

!| dv/dt + 1/4 * v = 32 |!

can quickly solve this with the answer: $v(t) = c_1 e^{-t/4} + 128$

Starting info for the problem includes the fact that at time $t = 0$ the velocity equals 10. Substituting into the previous equation and with some simple rearrangement, it is revealed that $c_1 = -118$. The general equation for velocity becomes:

$$v = -118e^{-t/4} + 128$$

Part (a) of the problem is answered.

As for part (b) , since $v = \frac{dx}{dt}$ the velocity equation can be rewritten

$$\frac{dx}{dt} = -118e^{-t/4} + 128$$

Entering it into the Wolfram Alpha execution line:

!| dx/dt = -118e^(-t/4) + 128 |!

results in the solution:

$$x(t) = 128t + 472e^{-t/4} + c_2$$

Looking at the initial values from the perspective of position, $x = 0$ at time $t = 0$, and this reflected in

$$0 = 472e^0 + (128)(0) + c_2$$

$$\Longrightarrow c_2 = -472$$

and the equation for displacement at any time is given as:

$$x(t) = 128t + 472e^{-t/4} - 472$$

7.14 A body of mass m is thrown vertically into the air with an initial velocity v_0 . If the body encounters an air resistance proportional to its velocity, find (a) the equation of motion in the coordinate system of the *verbal free body diagram* below, (b) an expression for the velocity of the body at any time t , and , (c) the time at which the body reaches its maximum height.

Verbal Free Body Diagram Picture a box which has just been flung vertically into the air. The velocity vector, v, indicating positive velocity, is pointing up. However, both the mg vector and the kv vector are pointing down, towards the ground. The ground plane has a label attached which says $x = 0$.

Because of the negative direction of gravity, the normal equation of motion of a falling body may not apply. Two forces act on the body, (1) the force due to the gravity given by mg and (2) the force due to air resistance given by kv, which both slow the velocity of the body. In the resulting equation of motion

$$\frac{dv}{dt} + \frac{k}{m} v = -g$$

the factors on the lhs and rhs point in the up direction because it has had its normal sign reversed. This gives the answer to part (a).

Wolfram Alpha can be used to get the answer to the ODE above:

!| dv/dt + k/m * v = -g |!

which equals:

$$v(t) = c_1 e^{-(k\,t)/m} - \frac{gm}{k}$$

It is desirable to get the c_1 out of the last equation and replace it with known parameters. Looking at $t = 0$, this time step happens when $v = v_0$, and the last equation can be written as:

$$v_0 = c_1 e^{-k\,(0)/m} - \frac{gm}{k}$$

implying that

$$c_1 = v_0 + \frac{mg}{k}$$

and the previous velocity equation changes to

$$\Longrightarrow v = \left(v_0 + \frac{mg}{k}\right) e^{(-k/m)t} - \frac{mg}{k}$$

Since the above describes the motion of the body at any time, it is the answer to part (b). The body will reach its maximum height when it stops it vertical climb, that is, when $v = 0$. Substituting $v = 0$ into the motion equation as it stands above, and with no other changes, looks like

!| Solve [0 = (v_0 + ((m * g)/k)) * e^((-k * t)/m) - (m * g)/k] for t |!

and results in the real answer

$$t = \left(\frac{m \log\left(\frac{gm + k\,v_0}{gm}\right)}{k} \right)$$

for the maximum time of travel to the top of the trajectory, the answer to part (c)

7.15 A body of mass 2 slugs is dropped with no initial velocity and encounters an air resistance that is proportional to the square of its velocity. Find an expression for the velocity of the body at any time t.

In this problem the force due to air resistance is $-kv^2$, so that the mass-acceleration equation is:

$$m\frac{dv}{dt} = mg - kv^2$$

And the mass is known, so that

$$2\frac{dv}{dt} = 64 - kv^2$$

Switching to differential form:

$$\frac{2}{64 - kv^2}\, dv - dt = 0$$

Handing this one off to Wolfram Alpha:

!| (2/(64 - k * v^2)) * dv - dt = 0 |!

is met with the solution:

$$v(t) = \frac{8 \tanh\left(4\left(c_1\sqrt{k} + \sqrt{k}\,t\right)\right)}{\sqrt{k}}$$

At time $t = 0$ the problem specifies that v is also zero. These circumstances result in the following equation:

$$0 = \frac{8 \tanh\left(4\left(c_1\sqrt{k}\right)\right)}{\sqrt{k}}$$

If Wolfram Alpha is applied to in order to find c_1, it will reply that no answer can be found. However, looking at the forms on the two sides of the equation, it is possible to logic through. There is no physical way that $k = 0$, yet at some point the combined factors of c_1 and k must be the argument of the inverse tanh function, and the outcome of that operation must be to produce the zero sitting on the lhs. The only argument that will produce zero in the inverse tanh function is zero itself. Therefore c_1 must equal zero, and the final equation describing the velocity at any time, is:

$$v(t) = \frac{8 \tanh\left(4\sqrt{k}\,t\right)}{\sqrt{k}}$$

This answers the problem requirements. It is the answer of the text, though arrived at through a slightly different path.

7.16 A tank initially holds 100 gal of a brine solution containing 20 lb of salt. At $t = 0$, fresh water is poured into the tank at the rate of 5 gal/min, while the well-stirred mixture leaves the tank at the same rate. Find the amount of salt in the tank at any time t.

Salt solution concentration problems are a commonly seen species for differential equations. The parameters include:

Q : pounds of salt in the tank at any time t
V_0 : volume of tank
a : lbs of salt
b : external brine solution containing b lbs of salt per gallon
e : the rate at which the b solution is poured into the tank
f : the rate at which stirred solution leaves the tank
t : time

The concentration of salt in the tank at any time is:

$$\frac{Q}{V_0 + et + ft}$$

from which it follows that salt leaves the tank at the rate of

$$f\left(\frac{Q}{V_0 + et - ft}\right) \qquad \text{lb/min}$$

And the salt balance can be described by

$$\frac{dQ}{dt} = be - f\left(\frac{Q}{V_0 + et - ft}\right)$$

or

$$\frac{dQ}{dt} + \frac{f}{V_0 + (e - f)\,t}\,Q = be$$

For this specific problem, $V_0 = 100, a = 20, b = 0,$ and $e = f = 5.$ The previous equation becomes, as seen by Wolfram Alpha,

!| dQ/dt + (1/20) * Q = 0 |!

What returns is $Q(t) = c_1\,e^{-t/20}$

The initial conditions give additional insight into the problem. At time $t = 0$, the condition is given that $Q = a = 20$. The setting of $t = 0$ makes the exponential factor disappear, revealing that $c_1 = 20$, and this will alter the general equation to

$$Q = 20e^{-t/20}$$

which is what the problem wanted to find out.

7.17 A tank initially holds 100 gal of a brine solution containing 1 lb of salt. At $t = 0$, another brine solution containing 1 lb of salt per gallon is poured into the tank at the rate of 3 gal/min, while the well-stirred mixture leaves the tank at the same rate. Find (a) the amount of salt in the tank at any time t and (b) the time at which the mixture in the tank contains 2 lb of salt.

Keeping the salt solution concentration parameters visible for this problem

Q : pounds of salt in the tank at any time t
V_0 : volume of tank
a : lbs of salt
b : external brine solution containing b lbs of salt per gallon
e : the rate at which the b solution is poured into the tank
f : the rate at which stirred solution leaves the tank
t : time

The concentration of salt in the tank at any time is:

$$\frac{Q}{V_0 + et + ft}$$

from which it follows that salt leaves the tank at the rate of

$$f\left(\frac{Q}{V_0 + et - ft}\right) \qquad \text{lb/min}$$

And the salt balance can be described by

$$\frac{dQ}{dt} = be - f\left(\frac{Q}{V_0 + et - ft}\right)$$

or

$$\frac{dQ}{dt} + \frac{f}{V_0 + (e - f)\,t}\,Q = be$$

For this specific problem, $V_0 = 100, a = 1, b = 1,$ and $e = f = 3.$ The previous equation becomes, as seen by Wolfram Alpha,

!| dQ/dt + (0.03) * Q = 3 |!

What returns is $Q(t) = c_1\,e^{-0.03\,t} + 100$

The initial conditions give additional insight into the problem. At time $t = 0$, the condition is given that $Q = a = 1$. The setting of $t = 0$ makes the exponential factor disappear, revealing that $c_1 = 99$, and this will alter the general equation to

$$Q = -99e^{-0.03t} + 100$$

which is what part (a) of the problem wanted to find out.

Part (b) of the problem wants to find out how long after the start time will the tank contain 2 lb of salt. Substituting $Q = 2$ into the equation above, results in:

!| N[2 = -99 * e^(-0.03 * t) + 100] ,t |!

in W|A format, and the answer that is returned is

0.338 min. (The exact format of the entry line was tweaked a couple of times to get W|A to give a single float answer, but the exact entry above did work.)

7.18 A 50-gal tank initially contains 10 gal of fresh water. At $t = 0$, a brine solution containing 1 lb of salt per gallon is poured into the tank at the rate of 4 gal/min, while the well-stirred mixture leaves the tank at the rate of 2 gal/min. Find (a) the amount of time required for overflow to occur, and (b) the amount of salt in the tank at the moment of overflow.

Keeping the salt solution concentration parameters visible for this problem

Q : pounds of salt in the tank at any time t
V_0 : volume of tank
a : lbs of salt
b : external brine solution containing b lbs of salt per gallon
e : the rate at which the b solution is poured into the tank
f : the rate at which stirred solution leaves the tank
t : time

For this specific problem, $V_0 = 10, a = 0, b = 1, e = 4, f = 2.$

The volume of brine in the tank at any time is:

$$V_0 + et - ft = 10 + 2t$$

and the overflow will occur when the tank is fully filled, that is, when $10 + 2t = 50$, that is, when $t = 20$ minutes. That answers part (a) of the problem.

Part (b) of the problem wants to know the amount of salt in the tank at the moment of overflow. Finding this out involves the salt balance equation

$$\frac{dQ}{dt} = be - f\left(\frac{Q}{V_0 + et - ft}\right)$$

or

$$\frac{dQ}{dt} + \frac{f}{V_0 + (e - f)t}Q = be$$

In this particular case

$$\frac{dQ}{dt} + \frac{2}{10 + 2t}Q = 4$$

The previous equation becomes, as seen by Wolfram Alpha,

!| dQ/dt + (2/(10 + 2 * t)) * Q = 4 |!

What returns is $Q(t) = \frac{c_1 + 2t^2 + 20t}{t + 5}$

The initial conditions give additional insight into the problem. At time $t = 0$, the condition is given that $Q = a = 0$. Putting these values into the last equation reveals that

$$\frac{c_1}{5} = 0 \Rightarrow c_1 = 0$$

So the final equation for Wolfram Alpha is the simple one:

!|(2 * t^2 + 20 * t)/(t + 5), t=20 |!

and the returned answer is 48 lbs of salt in the tank at the time of overflow.

7.19 An RL circuit has an emf of 5 volts, a resistance of 50 ohms, an inductance of 1 henry, and no initial current. Find the current in the circuit at any time t.

The basic equation governing the amount of current I (in amperes) in a simple RL circuit consisting of a resistance R (in ohms), an inductor L (in henries), and an electromotive force (abbreviated emf) E (in volts) is:

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E}{L}$$

In this particular problem, $E = 5, R = 50,$ and $L = 1$, so the EMF steady state equation becomes:

$$\frac{dI}{dt} + 50I = 5$$

The ODE is fed to Wolfram Alpha with the line

!| dC/dt +50 * C = 5 |!

and the returned answer is:

$$I(t) = c_1\, e^{-50t} + \frac{1}{10}$$

Wolfram Alpha reveals a wart in this procedure, because introduction of the $'I'$ character, even capitalized, is taken for Euler's imaginary i. So an alternate character, as above, needs to be substituted, then retracted on exit.

Initial conditions will solidify the general equation for current. At time $t = 0$ $I = 0$. Applying this condition to the above equation reveals that the value of c_1 must be zero. Therefore, the current at any time is equal to:

$$I = -\frac{I}{10}e^{-50t} + \frac{1}{10}$$

The above equation is the answer to the question posed by the problem.

7.20 An RL circuit has an emf given (in volts) by $3 \sin 2t$ a resistance of 10 ohms, and inductance of 0.5 henry, and an initial current of 6 amperes. Find the current in the circuit at any time t.

Another RL circuit problem. In this one, $E = 3\sin 2t,\ R = 10,$ and $L = 0.5$. Therefore the steady-state current equation takes the form:

$$\frac{dI}{dt} + 20I = 6\sin 2t$$

Entering this into Wolfram Alpha in the form:

!| dC/dt + 20 * C = 6 * sin(2 * t) |!

produces the answer:

$$I(t) = c_1 e^{-20t} + \frac{30}{101}\sin(2t) - \frac{3}{101}\cos(2t)$$

At time $t = 0$, $I = 0$

$$\Longrightarrow 6 = c_1 e^{-20(0)} + \frac{30}{101}\sin 2(0)$$

$$\Longrightarrow 6 = c_1 - \frac{3}{101}$$

Implying that $c_1 = \frac{609}{101}$. The general current equation takes the form:

$$I = \frac{606}{101}e^{-20t} + \frac{30}{101}\sin 2t - \frac{3}{101}\cos 2t$$

Expressing the current "at any time" can mean observing the current immediately at switch on, when a transient condition exists, or observing later, when a steady state condition exists. As in most situations, in this problem the former condition is represented by all 3 of the terms in the final equation, whereas the latter condition consists of only the final two terms.

7.22 An RC circuit has an emf given (in volts) by $400\cos 2t$, a resistance of 100 ohms, and a capacitance of 10^{-2} farad. Initially there is no charge on the capacitor. Find the current in the circuit at any time t.

Another RL circuit problem. In this one, $E = 400\cos 2t,\ R = 100,$ and $C = 10^{-2}$. Therefore the steady-state current equation takes the form:

$$\frac{dq}{dt} + q = 4\cos 2t$$

Entering this into Wolfram Alpha in the form:

!| dq/dt + q = 4 * cos(2 * t) |!

produces the answer:

$$q(t) = c_1 e^{-t} + \frac{8}{5}\sin(2t) + \frac{4}{5}\cos(2t)$$

At time $t = 0$, $q = 0$

$$\Longrightarrow 0 = c_1 e^{(0)} + \frac{8}{5}\sin(2(0)) + \frac{4}{5}\cos(2(0))$$

$$\Longrightarrow 0 = c_1 + \frac{4}{5}$$

Implying that $c_1 = -\frac{4}{5}$. The general equation for charge takes the form:

$$q(t) = -\frac{4}{5}e^{-t} + \frac{8}{5}\sin(2t) + \frac{4}{5}\cos(2t)$$

Getting the general current equation means taking the derivative of the above equation, in W|A easily done with the entry:

!| D[-4/5 * e^(-t) + 8/5 * sin(2 * t) + 4/5 * cos(2 * t)] |!

So that the general current equation is the return from Wolfram Alpha

$$I = \frac{dq}{dt} = \frac{4}{5}\left(e^{-t} - 2\sin(2t) + 4\cos(2t)\right)$$

which answers the question posed by the problem.

7.23 Find the orthogonal trajectories of the family of curves $x^2 + y^2 = c^2$

Consider a one-parameter family of curves in the $xy-$plane defined by

$$F(x, y, c) = 0$$

where c stands for the parameter. The problem is to find another one-parameter family of curves, called the orthogonal trajectories of the family cited above, and given analytically by

$$G(x, y, k) = 0$$

such that every curve in the new family intersects at right angles with every curve in the original family.

The family, which is given by $F(x, y, c) = x^2 + y^2 - c^2$ is made up of circles whose centers are at the origin and whose radii are given by c. Wolfram Alpha will do an implicit differentiation with the entry:

!| implicit differentiation [x^2 + y^2 = c^2] |!

returning the expression

$$y'(x) = -\frac{x}{y}$$

Orthogonal trajectories are solutions of the equation

$$\frac{dy}{dx} = -\frac{1}{f(x, y}$$

It is the procedure with o.p. problems to process the above form into

$$\frac{dy}{dx} = \frac{y}{x}$$

The ODE just shown can be solved by Wolfram Alpha routinely:

!| dy/dx = y/x |!

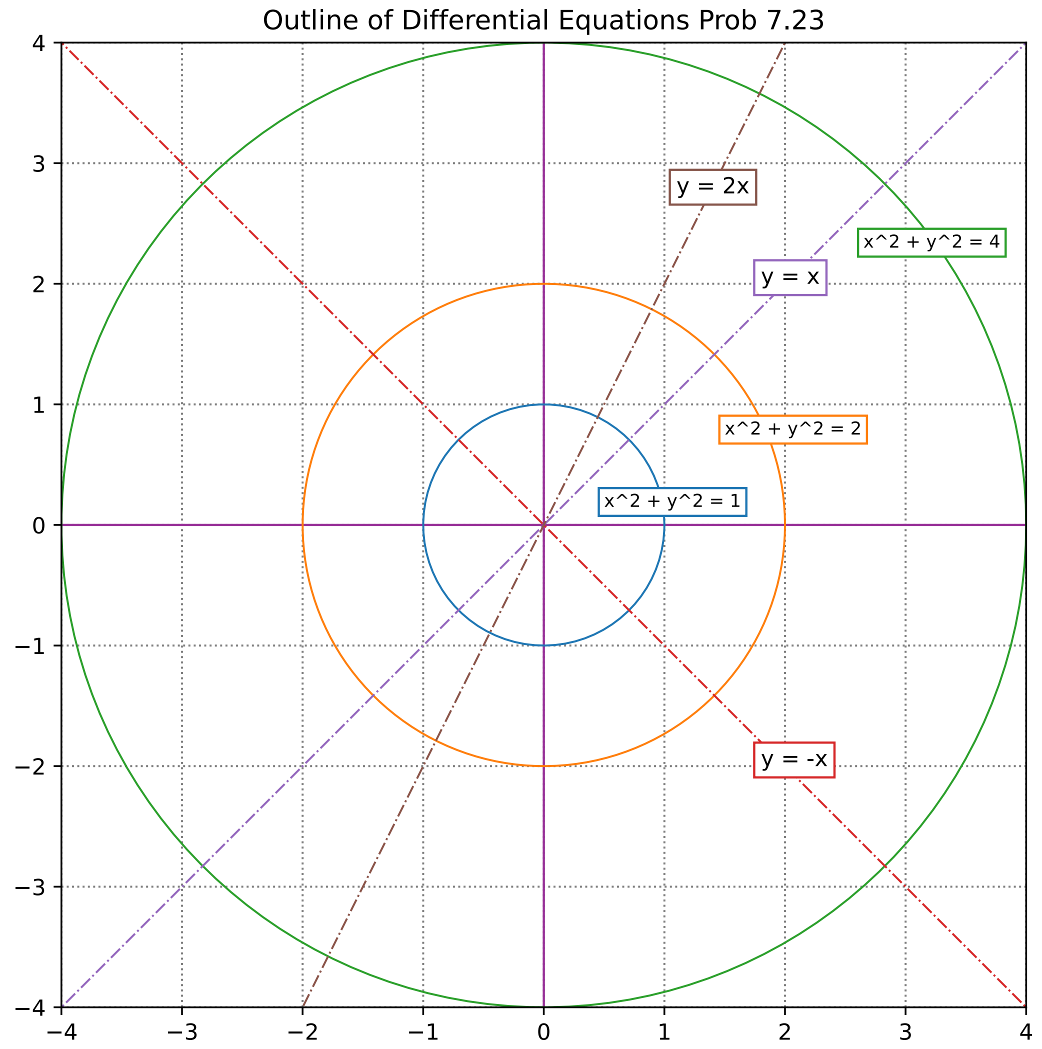
The solution is:

$$y(x) = c_1 x$$

Next comes a plot of some representative family members along with some orthogonal projections. Note that as required, intersections occur at right angles. The parametric plot style is convenient.

In [69]:

```
import numpy as np
import matplotlib.pyplot as plt

%config InlineBackend.figure_formats = ['svg']

x = np.linspace(-10,10,1000)
y1 = -x
y2 = x
y3 = 2*x

angle = np.linspace( 0 , 2 * np.pi , 150 )
fig, axes = plt.subplots( 1 )
ax = plt.gca()
ax.axhline(y=0, color='#993399', linewidth=1)
ax.axvline(x=0, color='#993399', linewidth=1)


plt.text(1.8, -2, "y = -x", size=10,bbox=dict\
         (boxstyle="square", ec=('#D62728'),fc=(1., 1., 1),))
plt.text(1.8, 2, "y = x", size=10,bbox=dict\
         (boxstyle="square", ec=('#9467BD'),fc=(1., 1., 1),))
plt.text(1.1, 2.75, "y = 2x", size=10,bbox=dict\
         (boxstyle="square", ec=('#86564B'),fc=(1., 1., 1),))
plt.text(0.5, 0.15, "x^2 + y^2 = 1", size=8,bbox=dict\
         (boxstyle="square", ec=('#1F77B4'),fc=(1., 1., 1),))
plt.text(1.5, 0.75, "x^2 + y^2 = 2", size=8,bbox=dict\
         (boxstyle="square", ec=('#FF7F0E'),fc=(1., 1., 1),))
plt.text(2.65, 2.3, "x^2 + y^2 = 4", size=8,bbox=dict\
         (boxstyle="square", ec=('#2CA02C'),fc=(1., 1., 1),))

#radius = 1.0
plt.rcParams['figure.figsize'] = [8, 8]

axes.plot(1.0*np.cos( angle ), 1.0*np.sin( angle ), linewidth = 0.9)
axes.plot(2.0*np.cos( angle ), 2.0*np.sin( angle ), linewidth = 0.9)
axes.plot(4.0*np.cos( angle ), 4.0*np.sin( angle ), linewidth = 0.9)

axes.plot(x, y1, linewidth = 0.9, linestyle='dashdot')
axes.plot(x, y2, linewidth = 0.9, linestyle='dashdot')
axes.plot(x, y3, linewidth = 0.9, linestyle='dashdot')

axes.set_aspect( 1 )
axes.grid(True, linestyle='dotted', color='gray', linewidth=0.9)
plt.xlim(-4, 4)
plt.ylim(-4,4)
plt.title( 'Outline of Differential Equations Prob 7.23' )
plt.show()
```



7.24 Find the orthogonal trajectories of the family of curves $y = cx^2$

Consider a one-parameter family of curves in the xy–plane defined by

$$F(x, y, c) = 0$$

where c stands for the parameter. The problem is to find another one-parameter family of curves, called the orthogonal trajectories of the family cited above, and given analytically by

$$G(x, y, k) = 0$$

such that every curve in the new family intersects at right angles with every curve in the original family.

The family, which is given by $F(x, y, c) = y - cx^2$ is made up of parabolas. Wolfram Alpha will do an implicit differentiation with the entry:

!| implicit differentiation [y = c * x^2] |!

returning the expression

$$y'(x) = 2cx$$

At this point it is necessary to get rid of the constant c.

Note that this is not an arbitrary constant added to an integral, but an actual element of the family inventory. So the original equation can be solved to give $c = \frac{y}{x^2}$. Then the substitution of the rhs can be made into the derivative, yielding $\frac{dy}{dx} = \frac{2y}{x}$. This is where the procedure does that weird flipping+sign change thing, giving

$$\frac{dy}{dx} = \frac{-x}{2y}$$

or

$$x\,dx + 2y\,dy = 0$$

which is another ODE for Wolfram Alpha to solve.

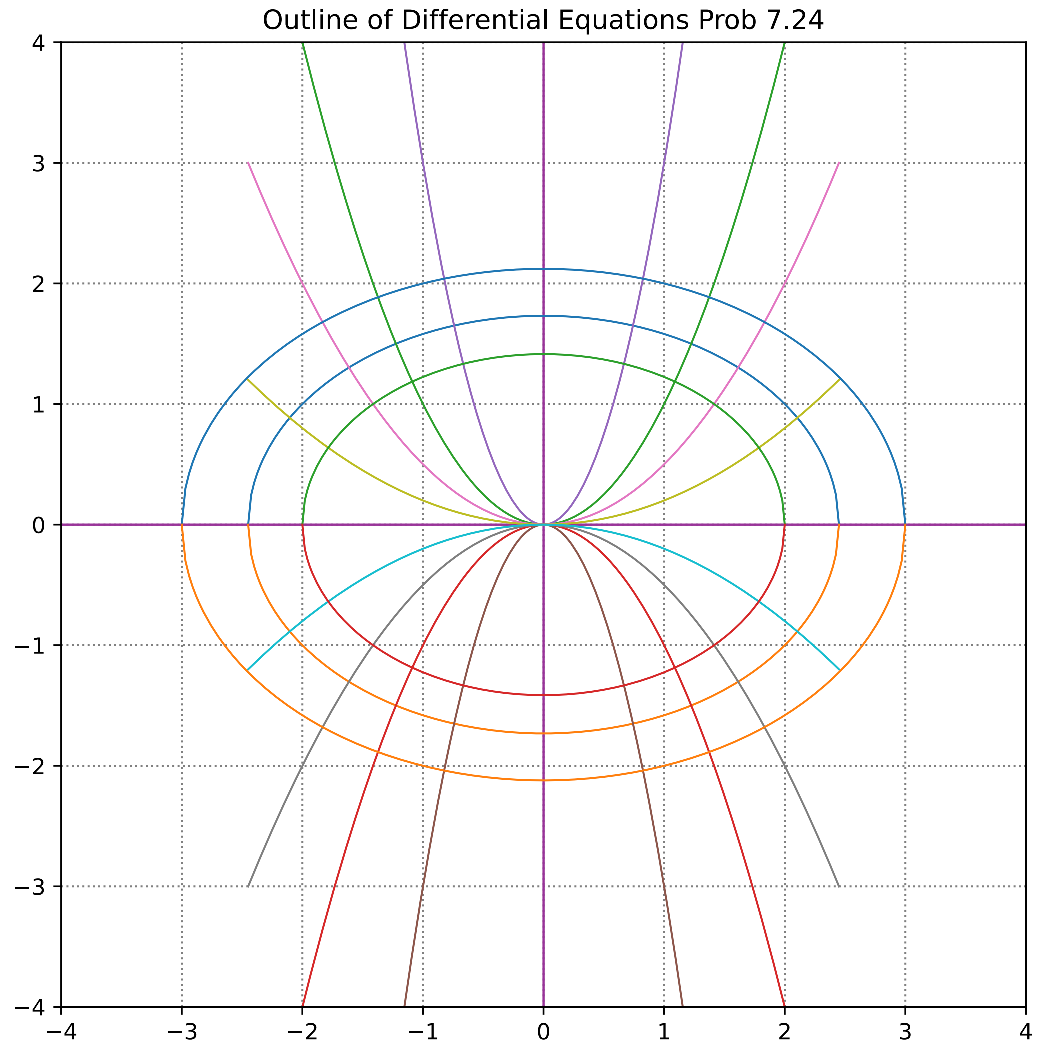
!| x * dx + 2 * y * dy = 0 |!

and Wolfram Alpha comes back with:

$$y(x) = -\frac{\sqrt{c_1 - x^2}}{\sqrt{2}} \qquad \text{and} \qquad y(x) = \frac{\sqrt{c_1 - x^2}}{\sqrt{2}}$$

Testing out the solutions devised by Wolfram Alpha is shown in the multi-curve plot below. As far as native visual judgment goes, the intersections look square. Not all the ellipses would plot cleanly, making it necessary to reject some which were tested.

In [141]:
```
import numpy as np
import matplotlib.pyplot as plt

%config InlineBackend.figure_formats = ['svg']

#x = np.linspace(-10,10,1000)
x1 = np.linspace(-np.sqrt(6),np.sqrt(6),200)
x2 = np.linspace(-np.sqrt(9),np.sqrt(9),200)
x3 = np.linspace(-np.sqrt(4),np.sqrt(4),200)
y1p = np.sqrt(6 - x1**2)/np.sqrt(2)
y1n = -np.sqrt(6 - x1**2)/np.sqrt(2)
yp1p = x1**2
yp1n = -x1**2
yp2p = 3*x1**2
yp2n = -3*x1**2
yp4p = .5*x1**2
yp4n = -.5*x1**2
yp5p = .2*x1**2
yp5n = -.2*x1**2
y2p = np.sqrt(9 - x2**2)/np.sqrt(2)
y2n = -np.sqrt(9 - x2**2)/np.sqrt(2)
y3p = np.sqrt(4 - x3**2)/np.sqrt(2)
y3n = -np.sqrt(4 - x3**2)/np.sqrt(2)

angle = np.linspace( 0 , 2 * np.pi , 150 )
fig, axes = plt.subplots( 1 )
ax = plt.gca()
ax.axhline(y=0, color='#993399', linewidth=1)
ax.axvline(x=0, color='#993399', linewidth=1)

plt.rcParams['figure.figsize'] = [8, 8]


axes.plot(x1, y1p, linewidth = 0.9)
axes.plot(x1, y1n, linewidth = 0.9)
axes.plot(x1, yp1p, linewidth = 0.9)
axes.plot(x1, yp1n, linewidth = 0.9)
axes.plot(x1, yp2p, linewidth = 0.9)
axes.plot(x1, yp2n, linewidth = 0.9)
axes.plot(x1, yp4p, linewidth = 0.9)
axes.plot(x1, yp4n, linewidth = 0.9)
axes.plot(x1, yp5p, linewidth = 0.9)
axes.plot(x1, yp5n, linewidth = 0.9)
axes.plot(x2, y2p, linewidth = 0.9)
axes.plot(x2, y2n, linewidth = 0.9)
axes.plot(x3, y3p, linewidth = 0.9)
axes.plot(x3, y3n, linewidth = 0.9)
#axes.plot(x, y3, linewidth = 0.9, linestyle='dashdot')

axes.set_aspect( 1 )
axes.grid(True, linestyle='dotted', color='gray', linewidth=0.9)
plt.xlim(-4, 4)
plt.ylim(-4,4)
plt.title( 'Outline of Differential Equations Prob 7.24' )
plt.show()
```



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