Chapter 8: Linear Differential Equations: Theory of Solutions

Discussing linearly independent solutions, the Wronskian, and Homogeneous and Nonhomogeneous linear equations.

Cutting and pasting and Wolfram Alpha. Wolfram Alpha is amenable to accepting pasted entries. In this chapter pastable expressions are given a distinctive boundary fence, exemplified by the sample: !| abcdef |!

In the above pseudo-entry, only the alpha characters would be copied for transfer to Wolfram Alpha.

8.1 State the order of each of the following differential equations and determine whether any are linear:

$(a)\;\; 2xy'' + x^2y' - (\sin x)\,y = 2$
$(b)\;\; yy''' + xy' + y = x^2$
$(c)\;\; y'' - y = 0$
$(d)\;\; 3y' + xy = e^{-x^2}$
$(e)\;\; 2e^xy''' + e^xy'' = 1$
$(f)\;\; \frac{d^4y}{dx} + y^4 = 0$
$(g)\;\; y'' + \sqrt{y'} + y = x^2$
$(h)\;\; y' + 2y + 3 = 0$

8 expressions

$(a)\;\; order\ 2-$
$(b)\;\; order\ 3-$
$(c)\;\; order\ 2-$
$(d)\;\; order\ 1-linear$
$(e)\;\; order\ 3-$
$(f)\;\; order\ 4-$
$(g)\;\; order\ 2-$
$(h)\;\; order\ 1-linear$

8.2 Which of the linear differential equations given in Problem 8.1 are homogeneous?

Only part (c) is homogeneous, comprising elements only concerned with y. Under the strict definition of homogeneous, even part (h), which has no visible x element, does not quality as homogeneous because of the constant term, which could be interpreted as an x function to the zero degree.

8.3 Which of the linear differential equations given in Problem 8.1 have constant coefficients?

The equations in parts (c) and (h) have constant coefficients.

8.4 Find the general form of a linear differential equation of (a) order two and (b) order one.

The answer is:

$$(a) \qquad b_2(x)y'' + b_1(x)y' + b_0(x)y = g(x)$$

$$(b) \qquad b_1(x)y' + b_0(x)y = g(x)$$

8.5 Find the Wronskian of the set $\{e^x, e^{-x}\}$.

There is no need to attempt to use curly braces in W|A; it just converts them to parentheses. The entry:

!| Wronskian (e^x, e^(-x)) |!

does the trick, producing the answer: -2

8.6 Find the Wronskian of the set $\{\sin 3x, \cos 3x\}$.

The entry:

!| Wronskian (sin(3 * x), cos(3 * x)) |!

produces the answer: -3

8.7 Find the Wronskian of the set $\{x,\ x^2,\ x^3\}$.

The entry:

!| Wronskian (x, x^2, x^3) |!

produces the answer: $2x^3$

8.8 Find the Wronskian of the set $\{1 - x,\ 1 + x,\ 1 - 3x\}$.

The entry:

!| Wronskian (1 - x, 1 + x, 1 - 3 * x) |!

produces the answer: 0

8.9 Determine whether the set $\{e^x,\ e^{-x}\}$ is linearly dependent on $(-\infty,\ \infty)$

The entry:

!| linearly independent (e^x, e^(-x)) |!

produces the answer:

" $(e^x,\ e^{-x})$ is linearly independent"

W|A does not give information about the environment for which the statement is true, though in other cases it does. Applying the definition of linear independence will show easily that the W|A statement is true on the Real line. (See next problem.)

8.10 Redo Problem 8.9 by testing directly how the definition of linear independence is satisfied by the set under consideration.

The applicable equation (which may be referred to as the "verif") is:

$$c_1 e^x + c_2 e^{-x} \equiv 0$$

The task is to find out whether there are values of c_1 and c_2, not both zero, which will satisfy the above equation. It is possible to isolate one of the constants by rearranging the above equation as:

$$c_2 \equiv -c_1 e^{2x}$$

For any nonzero value of c_1, the lhs is constant whereas the rhs varies depending on x. Therefore the only solution to the above equation, and therefore to the "verif", is for $c_1 = c_2 = 0$. Thus the set under consideration in not linearly dependent and must therefore be linearly independent.

8.15 Find the general solution of $y'' + 9y = 0$ if it is known that two solutions are:

$$y_1(x) = \sin 3x \qquad \text{and} \qquad y_2(x) = \cos 3x$$

The entry is made in Wolfram Alpha:

!| y'' + 9 * y = 0 |!

and the answer received is:

$$y(x) = c_2 \sin(3x) + c_1 \cos(3x)$$

without addressing the calculation procedure foreseen by the text.

8.16 Find the general solution of $y'' - y = 0$ if it is known that two solutions are:

$$y_1(x) = e^x \qquad \text{and} \qquad y_2(x) = e^{-x}$$

The entry is made in Wolfram Alpha:

!| y'' - y = 0 |!

and the answer received is:

$$y(x) = c_1 e^x + c_2 e^{-x}$$

without addressing the calculation procedure foreseen by the text.

8.24 Determine whether the set $\{x^3,\ |x^3|\}$ is linearly dependent on [-1, 1]

The entry is made in Wolfram Alpha:

!| linear independent (x^3, |x^3|) |!

and the answer received is:

$(x^3, \ |x|^3)$ is linearly independent

without addressing the calculation procedure foreseen by the text. Also without describing the environment in which the offered answer is valid. As the W|A's instinct for detecting restrictions is rather sensitive, the answer is accepted.

8.25 Find $W(x^3, |x^3|)$ on [-1, 1].

The entry is made in Wolfram Alpha:

!| Wronskian (x^3, |x^3|) |!

and the answer received is:

$$\begin{cases} \tilde{\infty} & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

The tilde infinity symbol apparently represents complex infinity, and as long as the discussion remains in the Real realm, the answer is plain zero.

8.29 Determine all solutions of the initial-value problem
$y'' + e^x y' + (x + 1)y = 0; \ y(1) = 0; \ y'(1) = 0$.

The entry is made in Wolfram Alpha:

!| y'' + e^x * y' + (x + 1) * y = 0, y(1) = 0, y'(1) = 0|!

and no answer received. The equation is identified as a Sturm-Liouville equation, and there are some alternate forms shown. Then the entry is modified as:

!| solve [y'' + e^x * y' + (x + 1) * y = 0, y(1) = 0, y'(1) = 0] using r k f algorithm |!

and a plot is developed, although no solution expression is given. The plot, however, is a smooth flatline on $y = 0$, which may be taken for the zero answer that the text comes up with.

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