

```
In [3]: 1 %autosave 0  
2  
Autosave disabled
```

```
In [1]: 1 !pip3 install scikit-fem  
2  
Collecting scikit-fem  
  Downloading scikit_fem-8.0.0-py3-none-any.whl (157 kB)  
    ----- 157.8/157.8 kB 3.1 MB/s eta 0:00:00  
Requirement already satisfied: scipy in c:\users\gary\appdata\local\programs\python\python39\lib\site-packages (from scikit-fem) (1.9.3)  
Requirement already satisfied: numpy in c:\users\gary\appdata\local\programs\python\python39\lib\site-packages (from scikit-fem) (1.23.5)  
Installing collected packages: scikit-fem  
Successfully installed scikit-fem-8.0.0
```

Chapter 31-15 Solving PDEs with the Finite Element Method.

The finite element method (FEM) is a popular method for numerically solving differential equations arising in engineering and mathematical modeling. Typical problem areas of interest include the traditional fields of structural analysis, heat transfer, fluid flow, mass transport, and electromagnetic potential.

The FEM is a general numerical method for solving partial differential equations in two or three space variables (i.e., some boundary value problems). To solve a problem, the FEM subdivides a large system into smaller, simpler parts that are called finite elements. This is achieved by a particular space discretization in the space dimensions, which is implemented by the construction of a mesh of the object: the numerical domain for the solution, which has a finite number of points. The finite element method formulation of a boundary value problem finally results in a system of algebraic equations. The method approximates the unknown function over the domain. The simple equations that model these finite elements are then assembled into a larger system of equations that models the entire problem. The FEM then approximates a solution by minimizing an associated error function via the calculus of variations.

Note1: from Wikipedia

Note2: This notebook contains work in both Octave and Python formats

1. This problem depends on material from Chapters 6 and 8 of the book *Numerical Solution of Differential Equations -- Introduction to Finite Difference and Finite Element Methods* by Li, Qiao, and Tang. At the address <https://zhilin.math.ncsu.edu/> (<https://zhilin.math.ncsu.edu/>) is Li's page featuring details about the book. There is a button on the page, just underneath the book title, with the label "Matlab Codes". The download will be entitled "DE_FEM_Matlab_Code.zip" All of the files in the folder with the title "Chapter 6" should be loaded into Octave. Only one will execute in Octave, the one entitled drive.m. This file is shown below.

```
In [ ]: 1 clear all; close all; % Clear every thing so it won't mess up with other  
2 % existing variables.  
3  
4 %%%%%% Generate a triangulation  
5  
6 x(1)=0; x(2)=0.1; x(3)=0.3; x(4)=0.333; x(5)=0.5; x(6)=0.75;x(7)=1;  
7  
8 %x=0:0.05:1;  
9  
10 %%%%%% Call fem1d function to get the FEM solution at nodal points.  
11  
12 U = fem1d(x);  
13  
14 %%%%%% Compare errors:  
15  
16 x2 = 0:0.01:1; k2 = length(x2);  
17 for i=1:k2,  
18   u_exact(i) = soln(x2(i));  
19   u_fem(i) = fem_soln(x,U,x2(i)); % Compute FEM solution at x2(i)  
20 end  
21  
22 error = norm(u_fem-u_exact,inf) % Compute the infinity error  
23  
24 plot(x2,u_fem,'.-', x2,u_exact) % Solid: the exact, %dotted: FEM solution  
25 hold; plot(x,U,'o') % Mark the solution at nodal points.  
26 % set(gca,'FontSize',18);  
27 xlabel('x'); ylabel('u(x) & u_{fem}(x)');  
28 title('Solid line: Exact solution, Dotted line: FEM solution')  
29  
30 figure(2); % set(gca,'FontSize',18);  
31 plot(x2,u_fem-u_exact); title('Error plot')
```

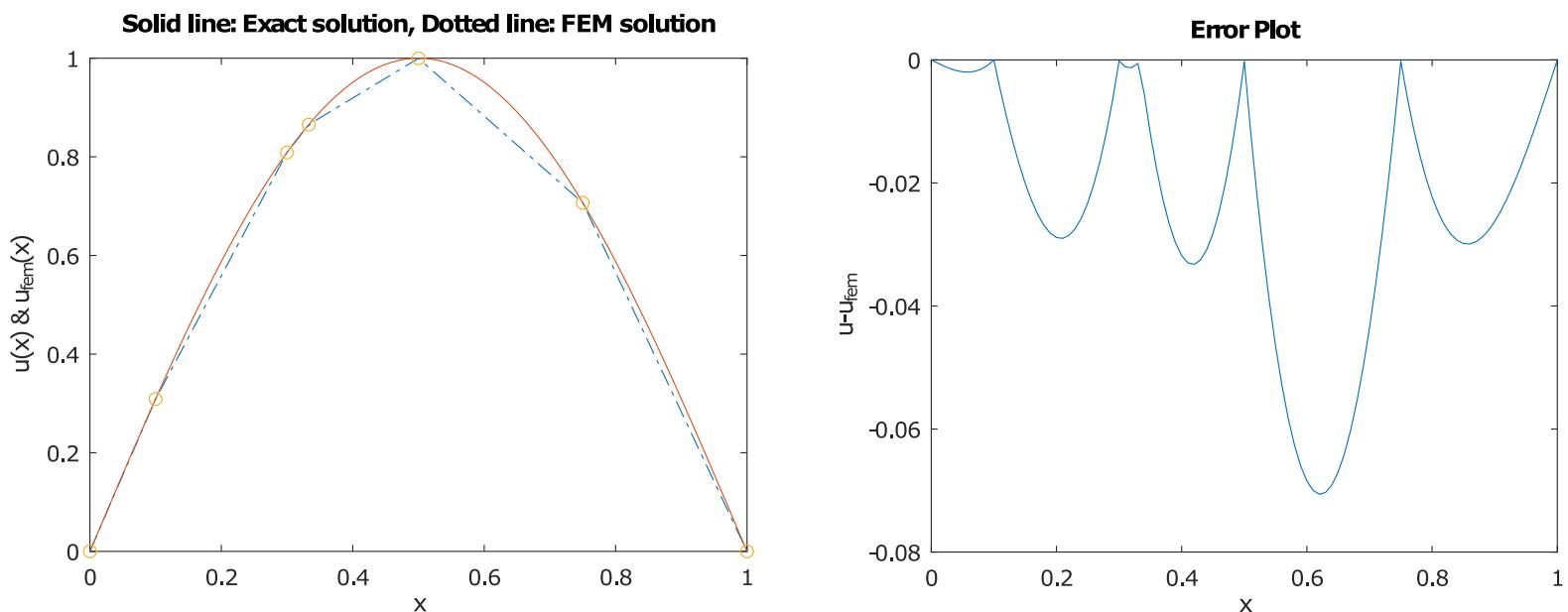
```

32 xlabel('x'); ylabel('u-u_{fem}'); title('Error Plot')
33

```

If access to the book mentioned above is possible, some good info regarding the relation between PDEs and finite element modeling can be learned.

The plots which are produced by running the code in the above cell are shown below.



3. For the initial conditions shown in the comment banner in the cell below, construct a 1-dimensional finite element solution and plot.

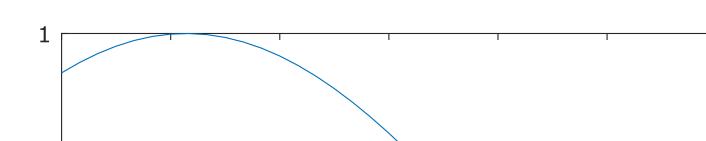
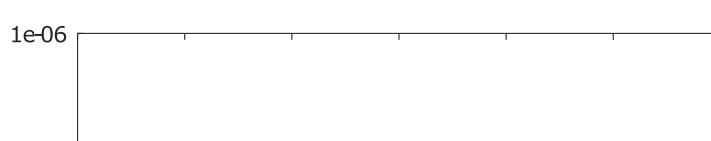
The plot below on the left shows the error; the plot on the right shows the analytical solution.

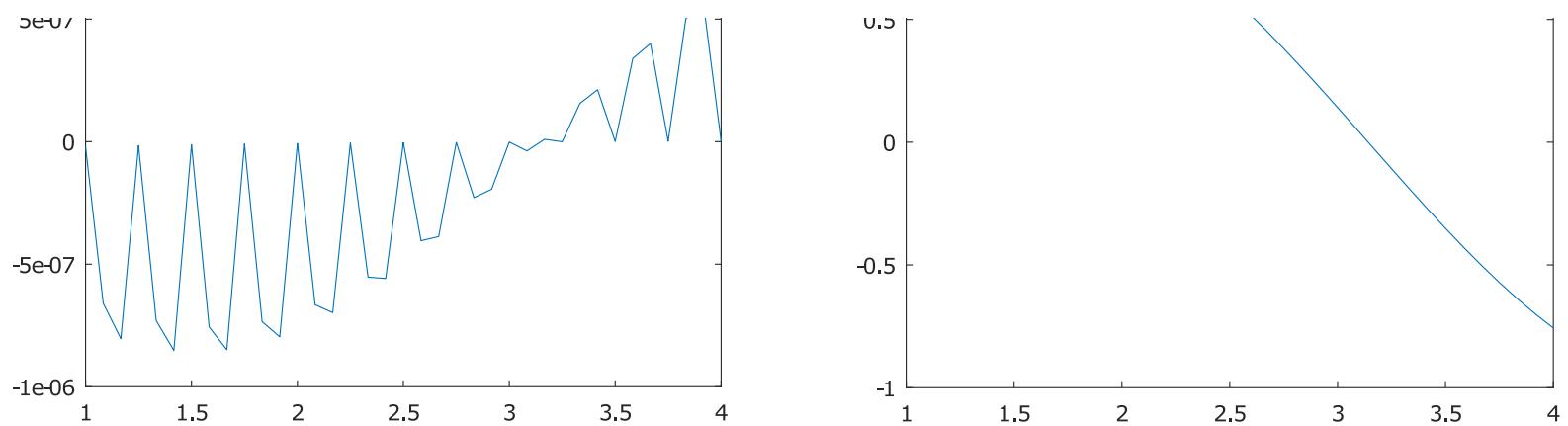
In []:

```

1 %%%%%%
2 %
3 %
4 % Matlab program for one-dimensinal finite element method for
5 %
6 % -(k(x)u_x)_x + c(x) u_x + b(x) u(x) = f(x)
7 %
8 % with different boundary condition at x =0, and x = l
9 %
10 %
11
12 function [x,u]= fem1d
13
14 clear
15 % Preprocessor:
16
17 global nnod nelem
18 global gk gf
19 global xi w
20
21 %%%%%% Start the program %%%%%%
22
23 % clear
24
25 [xi,w] = setint; % Get Gaussian points and weights.
26
27 % Preprocessor:
28 [x,kbc,vbc,kind,nint,nnodes] = propset; % Input data
29
30 formkf(kind,nint,nnodes,x,xi,w); % Form the discrete system
31
32 applybc(kbc,vbc);
33
34 u = gk\gf; % Solve the linear system.
35
36 for i=1:nnod,
37 e(i) = u(i) - ueexact(x(i));
38 end
39 figure(1); plot(x,e)
40 figure(2); plot(x,u)
41

```





In []:

The FEnics package requires Dolfin as a component, and Dolfin can be hard to install. NGSolve requires the MKL library, which is a big download if the CPU is not Intel. Some other finite element packages require Petsc, which can be difficult to get running, even on Linux.

In Contrast, the Python module scikit-fem is a lightweight but sturdy implement for finite element calculations in Python. It does not need compiling, and also entails no dependencies for the average Python installation. The next few cells below contain the first exercise in the Getting Started section of the scikit-fem documentation.

4. Solve the Poisson problem

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

where $\Omega = (0, 1)^2$ is a square domain and $f(x, y) = \sin \pi x \sin \pi y$

In [1]:

```
1 import skfem as fem
2 >>> from skfem.helpers import dot, grad # helpers make forms look nice
3 >>> @fem.BilinearForm
4 ... def a(u, v, _):
5 ...     return dot(grad(u), grad(v))
6
```

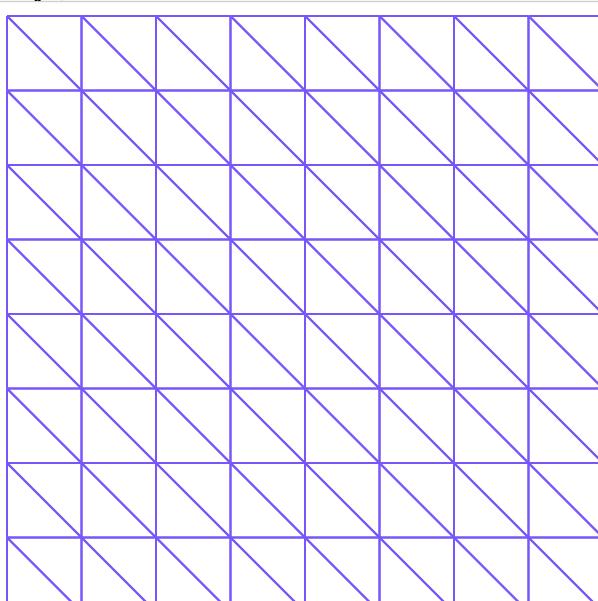
In [2]:

```
1 import numpy as np
2 >>> @fem.LinearForm
3 ... def L(v, w):
4 ...     x, y = w.x # global coordinates
5 ...     f = np.sin(np.pi * x) * np.sin(np.pi * y)
6 ...     return f * v
7
```

In [4]:

```
1 mesh = fem.MeshTri().refined(3) # refine thrice
2 mesh
3
```

Out[4]:



In [6]:

```
1 Vh = fem.Basis(mesh, fem.ElementTriP1())
2 Vh
3
```

Out[6]:

```
<skfem CellBasis(MeshTri1, ElementTriP1) object>
Number of elements: 128
Number of DOFs: 81
Size: 27648 B
```

```
In [9]: 1 A = a.assemble(Vh)
2 l = L.assemble(Vh)
3 A.shape
4
```

```
Out[9]: (81, 81)
```

```
In [10]: 1 l.shape
2
```

```
Out[10]: (81,)
```

```
In [11]: 1 D = Vh.get_dofs()
2 D
3
```

```
Out[11]: <skfem.DofsView(MeshTri1, ElementTriP1) object>
Number of nodal DOFs: 32 ['u']
```

```
In [13]: 1 x = fem.solve(*fem.condense(A, l, D=D))
2 x.shape
3
```

```
Out[13]: (81,)
```

```
In [15]: 1 @fem.Functional
2 def error(w):
3     x, y = w.x
4     uh = w['uh']
5     u = np.sin(np.pi * x) * np.sin(np.pi * y) / (2. * np.pi ** 2)
6     return (uh - u) ** 2
7 round(error.assemble(Vh, uh=Vh.interpolate(x)), 9)
8
```

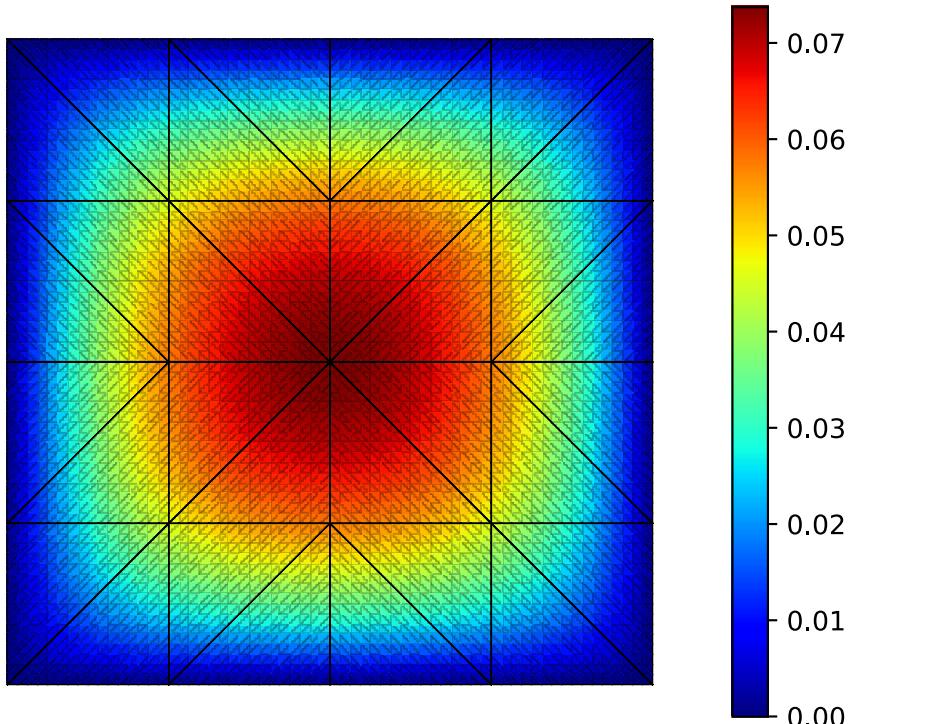
```
Out[15]: 1.069e-06
```

In [17]:

```

1 """Discontinuous Galerkin method."""
2
3 from skfem import *
4 from skfem.helpers import grad, dot, jump
5 from skfem.models.poisson import laplace, unit_load
6
7 m = MeshTri.init_sqsymmetric().refined()
8 e = ElementTriDG(ElementTriP4())
9 alpha = 1e-3
10
11 ib = Basis(m, e)
12 bb = FacetBasis(m, e)
13 fb = [InteriorFacetBasis(m, e, side=i) for i in [0, 1]]
14
15 @BilinearForm
16 def dgform(u, v, p):
17     ju, jv = jump(p, u, v)
18     h = p.h
19     n = p.n
20     return ju * jv / (alpha * h) - dot(grad(u), n) * jv - dot(grad(v), n) * ju
21
22 @BilinearForm
23 def nitscheform(u, v, p):
24     h = p.h
25     n = p.n
26     return u * v / (alpha * h) - dot(grad(u), n) * v - dot(grad(v), n) * u
27
28 A = asm(laplace, ib)
29 B = asm(dgform, fb, fb)
30 C = asm(nitscheform, bb)
31 b = asm(unit_load, ib)
32
33 x = solve(A + B + C, b)
34
35 M, X = ib.refinterp(x, 4)
36
37 def visualize():
38     from skfem.visuals.matplotlib import plot, draw
39     %config InlineBackend.figure_formats = ['svg']
40
41     ax = draw(M, boundaries_only=True)
42     return plot(M, X, shading="gouraud", ax=ax, colorbar=True)
43
44 if __name__ == "__main__":
45     visualize().show()
46

```



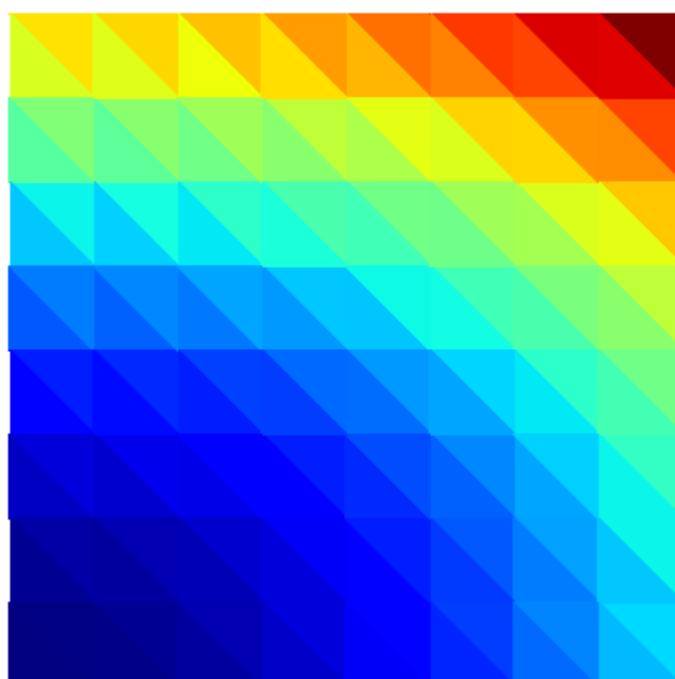
The Github repository of [gdmcbain/fenics-tuto-in-skfem](https://github.com/gdmcbain/fenics-tuto-in-skfem) contains eight exercises from the FEnics tutorial book. FEnics's great popularity no doubt stems largely from the book by Langtangen and Logg, entitled *Solving PDEs in Python: The FEnics Tutorial I*, a full-length book about using FEnics for finite element solutions of PDEs. This book can be accessed online or downloaded for free. Since FEnics is such a popular package, it is natural for someone to wonder if the new, little-known and somewhat brash program scikit-fem could reproduce the results of the FEnics tutorials. (Spoiler: it can, with considerably less trouble.) Below are six of the eight problems of the collection. (Note: problems 2 and 6 could not be attempted because they require a commercial program for execution.)

FEnics exercise #1 below, executed by scikit-fem.

```
In [1]: 1 import numpy as np
2
3 from skfem import *
4 from skfem.models.poisson import laplace, unit_load, mass
5
6 mesh = MeshTri().refined(3)
7 mesh
8
9 V = InteriorBasis(mesh, ElementTriP1())
10
11 u_D = 1 + [1, 2] @ mesh.p ** 2
12
13 boundary = mesh.boundary_nodes()
14
15 u = np.zeros_like(u_D)
16 u[boundary] = u_D[boundary]
17
18 a = asm(laplace, V)
19 L = -6.0 * asm(unit_load, V)
20
21 u = solve(*condense(a, L, u, D=boundary))
22
23 ax = mesh.plot(u)
24 ax.get_figure().savefig("poisson.svg")
25
26 mesh.save("fenics01.ply")
27
28 error = u - u_D
29 print("error_L2 =", np.sqrt(error.T @ asm(mass, V) @ error))
30 print("error_max =", np.linalg.norm(error, np.inf))
31
```

Warning: PLY doesn't support 64-bit integers. Casting down to 32-bit.

```
error_L2 = 3.090730095650652e-16
error_max = 1.1102230246251565e-15
```



Type *Markdown* and *LaTeX*: α^2

FEnics exercise #3 below, executed by scikit-fem.

```
In [2]: 1 from matplotlib.pyplot import subplots, pause
2 #%matplotlib qt
3 import numpy as np
4 %config InlineBackend.figure_formats = ['svg']
5
6 from skfem import *
7 from skfem.models.poisson import laplace, mass, unit_load
8
9 alpha = 3.0
10 beta = 1.2
11
12 nx = ny = 2 ** 3
13
14 time_end = 2.0
```

```

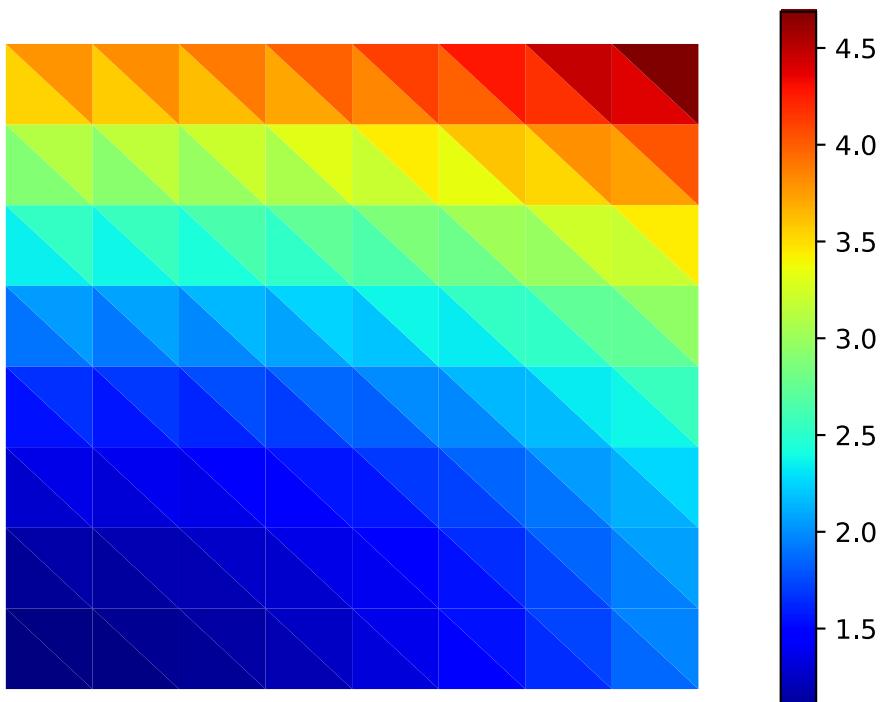
15 num_steps = 10
16 dt = time_end / num_steps
17
18 mesh = (
19     MeshLine(np.linspace(0, 1, nx + 1)) * MeshLine(np.linspace(0, 1, ny + 1))
20 ).to_meshtri()
21 basis = InteriorBasis(mesh, ElementTriP1())
22
23 boundary = basis.get_dofs().all()
24 interior = basis.complement_dofs(boundary)
25
26 M = asm(mass, basis)
27 A = M + dt * asm(laplace, basis)
28 f = (beta - 2 - 2 * alpha) * asm(unit_load, basis)
29
30 fig, ax = subplots()
31
32
33 def dirichlet(t: float) -> np.ndarray:
34     return 1.0 + [1.0, alpha] @ mesh.p ** 2 + beta * t
35
36
37 t = 0.0
38 u = dirichlet(t)
39
40 zlim = (0, np.ceil(1 + alpha + beta * time_end))
41
42 for i in range(num_steps + 1):
43
44     ax.cla()
45     ax.axis("off")
46     fig.suptitle("t = {:.4f}".format(t))
47     mesh.plot(u, ax=ax, zlim=zlim)
48     if t == 0.0:
49         fig.colorbar(ax.get_children()[0])
50     fig.show()
51     pause(1.0)
52
53     t += dt
54     b = dt * f + M @ u
55
56     u_D = dirichlet(t)
57     u = solve(*condense(A, b, u_D, D=boundary))
58     error = np.linalg.norm(u - u_D)
59     print("t = %.2f: error = %.3g" % (t, error))
60

```

C:\Users\gary\AppData\Local\Temp\ipykernel_7460/3832573378.py:50: UserWarning: Matplotlib is currently using module://matplotlib_inline.backend_inline, which is a non-GUI backend, so cannot show the figure.

```
fig.show()
```

t = 0.0000



```

t = 0.20: error = 9.44e-15
t = 0.40: error = 1.25e-14
t = 0.60: error = 1.45e-14
t = 0.80: error = 1.7e-14
t = 1.00: error = 1.68e-14
t = 1.20: error = 1.89e-14
t = 1.40: error = 2.01e-14
t = 1.60: error = 2.01e-14
t = 1.80: error = 2.16e-14
t = 2.00: error = 2.34e-14
t = 2.20: error = 2.46e-14

```

FEnics exercise #4 below, executed by scikit-fem.

In [3]:

```

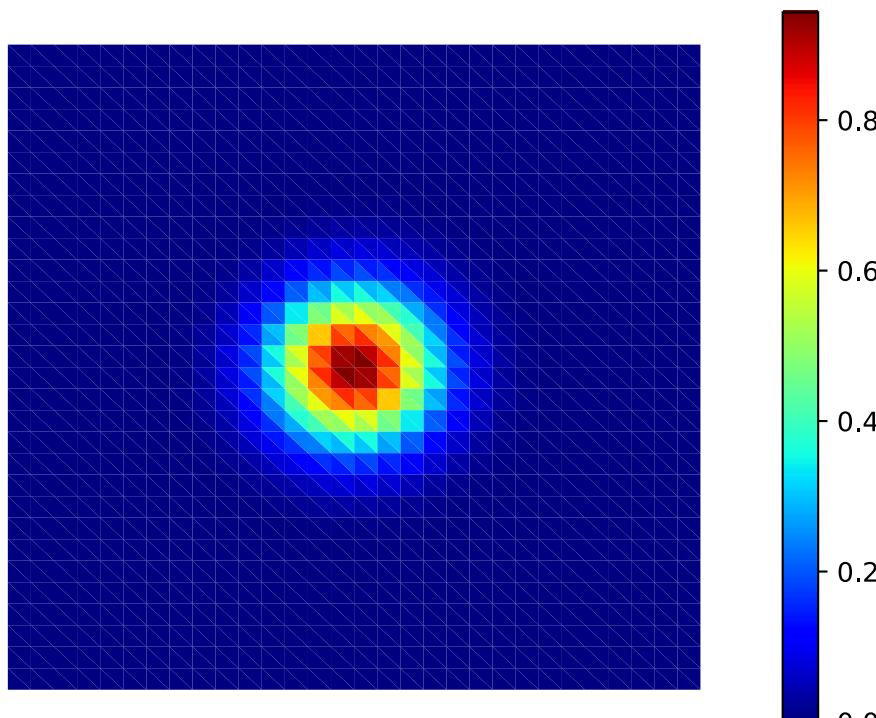
1 from pathlib import Path
2
3 from matplotlib.pyplot import subplots, pause
4 import numpy as np
5
6 from skfem import *
7 from skfem.models.poisson import laplace, mass
8
9 a = 5.0
10
11 nx = ny = 30
12
13 time_end = 2.0
14 num_steps = 50
15 dt = time_end / num_steps
16
17 mesh = (
18     MeshLine(np.linspace(-2, 2, nx + 1)) * MeshLine(np.linspace(-2, 2, ny + 1))
19 ).to_meshtri()
20 basis = InteriorBasis(mesh, ElementTriP1())
21
22 boundary = basis.get_dofs().all()
23 interior = basis.complement_dofs(boundary)
24
25 M = asm(mass, basis)
26 A = M + dt * asm(laplace, basis)
27
28 fig, ax = subplots()
29
30 t = 0.0
31 u = np.exp(-a * (np.sum(mesh.p ** 2, axis=0))) # initial condition, P1 only
32
33 output_dir = Path("heat_gaussian")
34 try:
35     output_dir.mkdir()
36 except FileExistsError:
37     pass
38
39 for i in range(num_steps + 1):
40
41     ax.cla()
42     ax.axis("off")
43     fig.suptitle("t = {:.4f}".format(t))
44     mesh.plot(u, ax=ax, zlim=(0, 1))
45     if t == 0.0:
46         fig.colorbar(ax.get_children()[0])
47         fig.savefig("initial.png")
48     fig.show()
49     pause(0.01)
50
51     t += dt
52     b = M @ u
53
54     u = solve(*condense(A, b, D=boundary))
55
56     mesh.save(str(output_dir.joinpath(f"solution{i:06d}.msh")), {"temperature": u})
57
58

```

C:\Users\gary\AppData\Local\Temp\ipykernel_7460/3973729471.py:48: UserWarning: Matplotlib is currently using module://matplotlib_inline.backend_inline, which is a non-GUI backend, so cannot show the figure.

fig.show()

t = 0.0000



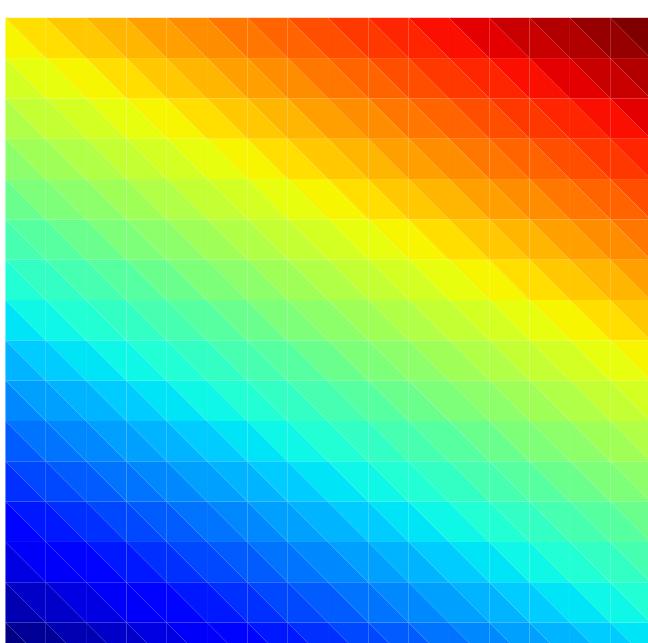
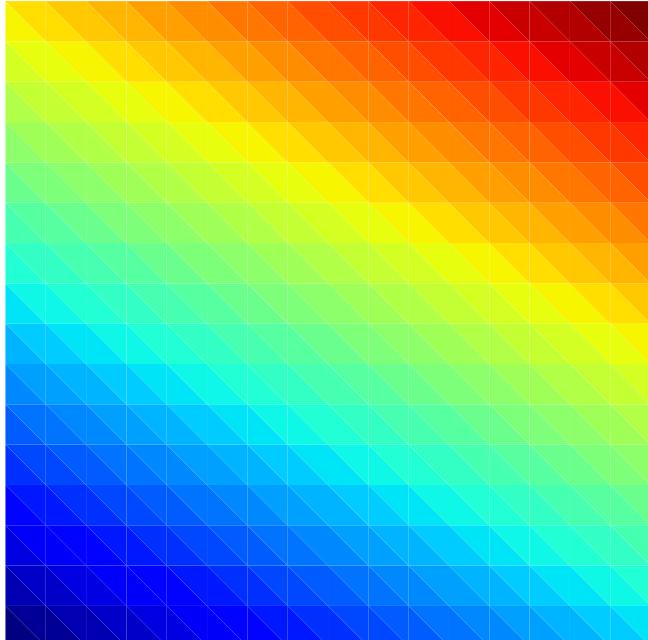
FEnics exercise #5 below, executed by scikit-fem.

```
In [18]: 1 from pathlib import Path
2
3 import numpy as np
4 from scipy.optimize import root
5
6 from sympy import symbols
7 from sympy.vector import CoordSys3D, gradient, divergence
8 from sympy.utilities.lambdify import lambdify
9
10 from skfem import (
11     MeshTri,
12     InteriorBasis,
13     ElementTriP1,
14     BilinearForm,
15     LinearForm,
16     asm,
17     solve,
18     condense,
19 )
20 from skfem.models.poisson import laplace
21 from skfem.visuals.matplotlib import plot
22 %config InlineBackend.figure_formats = ['svg']
23
24
25 output_dir = Path("poisson_nonlinear")
26
27 try:
28     output_dir.mkdir()
29 except FileExistsError:
30     pass
31
32
33 def q(u):
34     """Return nonlinear coefficient"""
35     return 1 + u * u
36
37
38 R = CoordSys3D("R")
39
40
41 def apply(f, coords):
42     x, y = symbols("x y")
43     return lambdify((x, y), f.subs({R.x: x, R.y: y}))(*coords)
44
45
46 u_exact = 1 + R.x + 2 * R.y # exact solution
47 f = -divergence(q(u_exact)) * gradient(u_exact) # manufactured RHS
48
49 mesh = MeshTri().refined(3) # refine thrice
50
51 V = InteriorBasis(mesh, ElementTriP1())
52
53 boundary = V.get_dofs().all()
54 interior = V.complement_dofs(boundary)
55
56
57 @LinearForm
58 def load(v, w):
59     return v * apply(f, w.x)
60
61
62 b = asm(load, V)
63
64
65 @BilinearForm
66 def diffusion_form(u, v, w):
67     return sum(v.grad * (q(w["w"])) * u.grad)
68
69
70 def diffusion_matrix(u):
71     return asm(diffusion_form, V, w=V.interpolate(u))
72
73
74 dirichlet = apply(u_exact, mesh.p) # P1 nodal interpolation
75 plot(V, dirichlet).get_figure().savefig(str(output_dir.joinpath("exact.png")))
76
77
78 def residual(u):
79     r = b - diffusion_matrix(u) @ u
80     r[boundary] = 0.0
81     return r
82
83
```

```

84 u = np.zeros(V.N)
85 u[boundary] = dirichlet[boundary]
86 result = root(residual, u, method="krylov")
87
88 if result.success:
89     u = result.x
90     print("Success. Residual =", np.linalg.norm(residual(u), np.inf))
91     print("Nodal Linf error =", np.linalg.norm(u - dirichlet, np.inf))
92     plot(V, u).get_figure().savefig(str(output_dir.joinpath("solution.png")))
93 else:
94     print(result)
95
Success. Residual = 1.4058151617812875e-07
Nodal Linf error = 1.2228777324096995e-08

```



FEnics exercise #7 below, executed by scikit-fem.

```

In [3]: 1 from pathlib import Path
2
3 import numpy as np
4
5 import skfem
6 from skfem.models.poisson import vector_laplace, laplace
7 from skfem.models.general import divergence
8
9 from meshio.xdmf import TimeSeriesWriter
10
11
12 @skfem.BilinearForm
13 def vector_mass(u, v, w):
14     return sum(v * u)
15
16
17 @skfem.BilinearForm
18 def port_pressure(u, v, w):
19     return sum(v * (u * w.n))
20
21
22 p_inlet = 8.0
23
```

```

24 #mesh = skfem.MeshTri()
25 #mesh.refine(3)
26 mesh = MeshTri().refined(3)
27
28 boundary = {
29     "inlet": mesh.facets_satisfying(lambda x: x[0] == 0),
30     "outlet": mesh.facets_satisfying(lambda x: x[0] == 1),
31     "wall": mesh.facets_satisfying(lambda x: np.logical_or(x[1] == 0, x[1] == 1)),
32 }
33 boundary["ports"] = np.concatenate([boundary["inlet"], boundary["outlet"]])
34
35 element = {"u": skfem.ElementVectorH1(skfem.ElementTriP2()), "p": skfem.ElementTriP1()}
36 basis = {
37     **{v: skfem.InteriorBasis(mesh, e, intorder=4) for v, e in element.items()},
38     **{
39         "label": skfem.FacetBasis(mesh, element["u"], facets=boundary[label])
40         for label in ["inlet", "outlet"]
41     },
42 }
43
44
45 M = skfem.asm(vector_mass, basis["u"])
46 L = {"u": skfem.asm(vector_laplace, basis["u"]), "p": skfem.asm(laplace, basis["p"])}
47 B = -skfem.asm(divergence, basis["u"], basis["p"])
48 P = B.T + skfem.asm(
49     port_pressure,
50     *(
51         skfem.FacetBasis(mesh, element[v], facets=boundary["ports"], intorder=3)
52         for v in ["p", "u"]
53     )
54 )
55
56 t_final = 1.0
57 dt = 0.05
58
59 dirichlet = {
60     "u": basis["u"].get_dofs(boundary["wall"]).all(),
61     "p": np.concatenate([basis["p"].get_dofs(boundary["ports"]).all()]),
62 }
63 inlet_pressure_dofs = basis["p"].get_dofs(boundary["inlet"]).all()
64
65 uv_, p_ = (np.zeros(basis[v].N) for v in element.keys()) # penultimate
66 p__ = np.zeros_like(p_) # antepenultimate
67
68 K = M / dt + L["u"]
69
70 t = 0
71
72 with TimeSeriesWriter("channel.xdmf") as writer:
73     writer.write_points_cells(mesh.p.T, {"triangle": mesh.t.T})
74
75     while t < t_final:
76
77         t += dt
78
79         # Step 1: Momentum prediction (Ern & Guermond 2002, eq. 7.40, p. 274)
80
81         uv = skfem.solve(
82             *skfem.condense(
83                 K,
84                 (M / dt) @ uv_ - P @ (2 * p_ - p__),
85                 np.zeros_like(uv_),
86                 D=dirichlet["u"],
87             )
88         )
89
90         # Step 2: Projection (Ern & Guermond 2002, eq. 7.41, p. 274)
91
92         dp = np.zeros(basis["p"].N)
93         dp[inlet_pressure_dofs] = p_inlet - p_[inlet_pressure_dofs]
94
95         dp = skfem.solve(*skfem.condense(L["p"], B @ uv, dp, D=dirichlet["p"]))
96
97         # Step 3: Recover pressure and velocity (E. & G. 2002, p. 274)
98
99         p = p_ + dp
100        print(min(p), "<= p <= ", max(p))
101
102        du = skfem.solve(*skfem.condense(M / dt, -P @ dp, D=dirichlet["u"]))
103        u = uv + du
104
105        uv_ = uv
106        p_, p__ = p, p_
107
108        # postprocessing
109        writer.write_data(
110            t,
111            point_data={
112                "pressure": p,

```

```

114         "velocity": np.pad(
115             u[basis["u"].nodal_dofs].T, ((0, 0), (0, 1)), "constant"
116         ),
117     },
118 )
119
120     print(min(u[::2]), "<= u <= ", max(u[::2]), "||v|| = ", np.linalg.norm(u[1::2]))
121
122
123 # References
124
125 # Ern, A., Guermond, J.-L. (2002). _Éléments finis : théorie,
126 # applications, mise en œuvre_ (Vol. 36). Paris: Springer. ISBN:
127 # 3540426159
128
0.0 <= p <= 8.0
0.0 <= u <= 0.5322508243494332 ||v|| = 4.026720129295742e-14
0.0 <= p <= 8.0
0.0 <= u <= 0.6309113809656185 ||v|| = 1.436588597740157e-05
0.0 <= p <= 8.0
0.0 <= u <= 0.7616343911111542 ||v|| = 1.90715233764314e-05
0.0 <= p <= 8.0
0.0 <= u <= 0.8422995060746478 ||v|| = 2.4372603252675528e-05
0.0 <= p <= 8.0
0.0 <= u <= 0.8947839812242627 ||v|| = 2.810672594345035e-05
0.0 <= p <= 8.0
0.0 <= u <= 0.9296216090981441 ||v|| = 3.052360984815118e-05
0.0 <= p <= 8.0
0.0 <= u <= 0.9528899755044427 ||v|| = 3.190213298235965e-05
0.0 <= p <= 8.0
0.0 <= u <= 0.9684589961632456 ||v|| = 3.2489532651000665e-05
0.0 <= p <= 8.0
0.0 <= u <= 0.9788815550706454 ||v|| = 3.2490550626528445e-05
0.0 <= p <= 8.0
0.0 <= u <= 0.9858598213872569 ||v|| = 3.206844857583792e-05
0.0 <= p <= 8.0
0.0 <= u <= 0.9905321915694375 ||v|| = 3.135004182450136e-05
0.0 <= p <= 8.0
0.0 <= u <= 0.9936606578490389 ||v|| = 3.0431839802365636e-05
0.0 <= p <= 8.0
0.0 <= u <= 0.9957553825354923 ||v|| = 2.9386031114155746e-05
0.0 <= p <= 8.0
0.0 <= u <= 0.9971579467570855 ||v|| = 2.826577517080062e-05
0.0 <= p <= 8.0
0.0 <= u <= 0.9980970614624239 ||v|| = 2.7109626371259528e-05
0.0 <= p <= 8.0
0.0 <= u <= 0.9987258643521538 ||v|| = 2.594510013397419e-05
0.0 <= p <= 8.0
0.0 <= u <= 0.9991468917776665 ||v|| = 2.479147375473926e-05
0.0 <= p <= 8.0
0.0 <= u <= 0.9994287989893178 ||v|| = 2.3661944160698367e-05
0.0 <= p <= 8.0
0.0 <= u <= 0.9996175555793892 ||v|| = 2.256526542369644e-05
0.0 <= p <= 8.0
0.0 <= u <= 0.9997439454833684 ||v|| = 2.150697652766544e-05

```

FEnics exercise #8 below, executed by scikit-fem. Note: In this case it is necessary to set the inline backend to "retina", because the .svg backend scrambles the graphic. (Whatever it is, "retina" seems to match vector image quality, so it is considered a player of equal skill.)

In [5]:

```

1 r"""
2 This example demonstrates the solution of a slightly more complicated problem
3 with multiple boundary conditions and a fourth-order differential operator. We
4 consider the `Kirchhoff plate bending problem
5 <https://en.wikipedia.org/wiki/Kirchhoff%20Love\_plate\_theory>`_ which
6 finds its applications in solid mechanics. For a stationary plate of constant
7 thickness :math:`d`, the governing equation reads: find the deflection :math:`u`
8 :math:\Omega \rightarrow \mathbb{R}` that satisfies
9 .. math::
10   \frac{Ed^3}{12(1-\nu^2)} \Delta^2 u = f \quad \text{in } \Omega,
11 \text{where } \Omega = (0,1)^2, \text{ :math:`f` is a perpendicular force,}
12 \text{ :math:`E` and :math:`\nu` are material parameters.}
13 \text{In this example, we analyse a :math:`1 \times 1` plate of steel with thickness :math:`d=0.1` ,}
14 \text{The Young's modulus of steel is :math:`E = 200 \times 10^9` Pa and Poisson}
15 \text{ratio :math:`\nu = 0.3` .}
16 \text{In reality, the operator}
17 .. math::
18   \frac{Ed^3}{12(1-\nu^2)} \Delta^2
19 \text{is a combination of multiple first-order operators:}
20 .. math::
21   \boldsymbol{K}(u) = - \boldsymbol{\nabla} \cdot \boldsymbol{\epsilon}(u), \quad \boldsymbol{\epsilon} = \boldsymbol{\nabla} u
22 .. math::
23   \boldsymbol{M}(u) = \frac{d^3}{12} \mathbb{C} \boldsymbol{K}(u), \quad \mathbb{C} \text{ is the identity matrix. In particular,}
24 .. math::

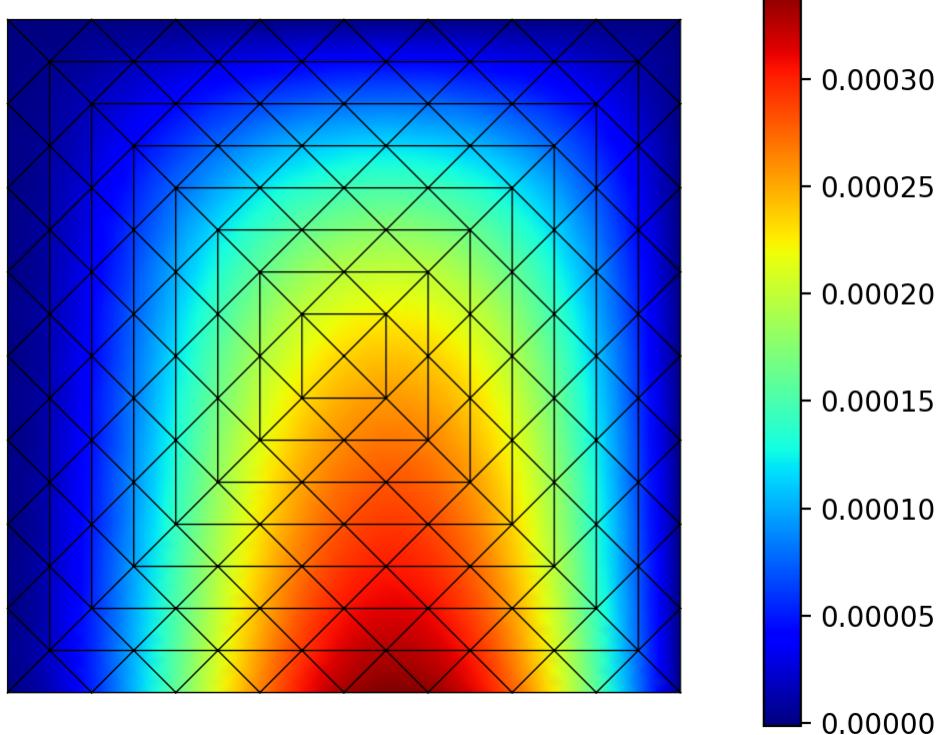
```

```

26 \frac{Ed^3}{12(1-\nu^2)} \Delta u = - \text{div}, \text{div}, \boldsymbol{M}(u).
27 There are several boundary conditions that the problem can take.
28 The *fully clamped* boundary condition reads
29 .. math::
30 u = \frac{\partial u}{\partial \boldsymbol{n}} = 0,
31 where :math:`\boldsymbol{n}` is the outward normal.
32 Moreover, the *simply supported* boundary condition reads
33 .. math::
34 u = 0, \quad M_{nn}(u)=0,
35 where :math:`M_{nn} = \boldsymbol{n} \cdot (\boldsymbol{M} \boldsymbol{n})` .
36 Finally, the *free* boundary condition reads
37 .. math::
38 M_{nn}(u)=0, \quad V_n(u)=0,
39 where :math:`V_n` is the Kirchhoff shear force https://arxiv.org/pdf/1707.08396.pdf_. The exa
40 definition is not needed here as this boundary condition is a
41 natural one.
42 The correct weak formulation for the problem is: find :math:`u \in V` such that
43 .. math::
44 \int_\Omega \boldsymbol{M}(u) : \boldsymbol{K}(v) \, dx = \int_\Omega f v \, , \quad \forall v \in V
45 where :math:`V` is now a subspace of :math:`H^2` with the essential boundary
46 conditions for :math:`u` and :math:`\frac{\partial u}{\partial \boldsymbol{n}}` .
47 Instead of constructing a subspace for :math:`H^2` , we discretise the problem
48 using the non-conforming Morley finite element
49 https://users.aalto.fi/~jakke74/WebFiles/Slides-Niiranen-ADMOS-09.pdf_ which
50 is a piecewise quadratic :math:`C^0`-continuous element for biharmonic problems.
51 The full source code of the example reads as follows:
52 .. literalinclude:: examples/ex02.py
53 :start-after: EOF"""
54 from skfem import *
55 from skfem.models.poisson import unit_load
56 import numpy as np
57 %config InlineBackend.figure_formats = ['retina']
58
59 m = (
60     MeshTri.init_symmetric()
61     .refined(3)
62     .with_boundaries(
63         {
64             "left": lambda x: x[0] == 0,
65             "right": lambda x: x[0] == 1,
66             "top": lambda x: x[1] == 1,
67         }
68     )
69 )
70
71 e = ElementTriMorley()
72 ib = Basis(m, e)
73
74
75 @BilinearForm
76 def bilinf(u, v, w):
77     from skfem.helpers import dd, ddot, trace, eye
78     d = 0.1
79     E = 200e9
80     nu = 0.3
81
82     def C(T):
83         return E / (1 + nu) * (T + nu / (1 - nu) * eye(trace(T), 2))
84
85     return d**3 / 12.0 * ddot(C(dd(u)), dd(v))
86
87
88 K = asm(bilinf, ib)
89 f = 1e6 * asm(unit_load, ib)
90
91 D = np.hstack([ib.get_dofs("left"), ib.get_dofs({"right", "top"}).all("u")])
92
93 x = solve(*condense(K, f, D=D))
94
95 def visualize():
96     from skfem.visuals.matplotlib import draw, plot
97     ax = draw(m)
98     return plot(ib,
99                x,
100               ax=ax,
101               shading='gouraud',
102               colorbar=True,
103               nrefs=2)
104
105 if __name__ == "__main__":
106     visualize().show()
107
108
109
110
111
112
113
114
115

```

116
117



The following cell shows an example from the scikit-fem documentation, [Example 11: Three-dimensional linear elasticity](#).

In [12]:

```
1 r"""Linear elasticity.  
2 This example solves the linear elasticity problem using trilinear elements. The  
3 weak form of the linear elasticity problem is defined in  
4 :func:`skfem.models.elasticity.linear_elasticity`.  
5 """  
6  
7 import numpy as np  
8 from skfem import *  
9 from skfem.models.elasticity import linear_elasticity, lame_parameters  
10  
11 m = MeshHex().refined(3)  
12 e1 = ElementHex1()  
13 e = ElementVector(e1)  
14 ib = Basis(m, e, MappingIsoparametric(m, e1), 3)  
15  
16 K = asm(linear_elasticity(*lame_parameters(1e3, 0.3)), ib)  
17  
18 dofs = {  
19     'left' : ib.get_dofs(lambda x: x[0] == 0.0),  
20     'right': ib.get_dofs(lambda x: x[0] == 1.0),  
21 }  
22  
23 u = ib.zeros()  
24 u[dofs['right'].nodal['u^1']] = 0.3  
25  
26 I = ib.complement_dofs(dofs)  
27  
28 u = solve(*condense(K, x=u, I=I))  
29  
30 sf = 1.0  
31 m = m.translated(sf * u[ib.nodal_dofs])  
32  
33 if __name__ == "__main__":  
34     from os.path import splitext  
35     from sys import argv  
36  
37 #m.save(splitext(argv[0])[0] + '.vtk')  
38 m.save('testimg' + '.vtk')  
39  
40  
41  
42  
43  
44  
45  
46  
47  
48  
49  
50  
51  
52  
53  
54  
55  
56
```

