Chapter 21 The Laplace Transform. Introduction to a far reaching system for simplification of the ODE solving process. The transform is from the time domain to the frequency domain, where algebraic operations are performed before inverse transportation back to the time domain. The main variable governing the frequency domain is s, and the time domain t.

21.1 Determine whether the improper integral

$$\int_2^\infty \frac{1}{x^2}\, dx$$

converges.

This problem can be entered into Wolfram Alpha:

!| integrate[1/x^2 dx] from 2 to infinity |!

In Wolfram Alpha the expression is merely referred to as a "Definite integral":

$$\int_x^\infty \frac{1}{x^2}\, dx = \frac{1}{2} = 0.5$$

21.2 Determine whether the improper integral

$$\int_9^\infty \frac{1}{x}\, dx$$

converges.

This problem can be entered into Wolfram Alpha:

!| integrate[1/x dx] from 9 to infinity |!

In Wolfram Alpha the result is expressed as:

Result (integral does not converge)

21.3 Determine those values of s for which the improper integral

$$\int_0^\infty e^{-sx}\, dx$$

converges.

This problem can be entered into Wolfram Alpha:

!| integrate[e^(-s * x) dx] from 0 to infinity |!

In Wolfram Alpha the result is expressed as:

Definite integral $\int_0^\infty e^{-sx}\, dx = \frac{1}{s}$ for $\mathrm{Re}(s) > 0$

So relying on Wolfram Alpha to set restrictive conditions works in this case.

21.4 Find the Laplace transform of $f(x) = 1$.

This problem can be entered into Wolfram Alpha:

!| find laplace transform of [1] |!

In Wolfram Alpha the result is expressed as:

Result $\frac{1}{s}$

21.5 Find the Laplace transform of $f(x) = x^2$.

This problem can be entered into Wolfram Alpha:

!| find laplace transform of [x^2] |!

In Wolfram Alpha the result is expressed as:

Result $\frac{2}{p^3}$

In Laplaceland the focused-on entity is usually referred to as s, but some some reason that differs here. The p comes from the standard notation of the transform operator,

$$\mathscr{L}[f(x)] = \int_0^x e^{-p\,x} f(x)\, dx$$

In comparing the answers to problems in this Chapter, it is evident that the s and p styles of representation are interchangeable.

21.6 Find $\mathscr{L}\{e^{ax}\}$.

This problem can be entered into Wolfram Alpha:

!| find laplace transform of [e^(ax)] |!

In Wolfram Alpha the result is expressed as:

Result $\frac{1}{p-a}$

Focusing on p here, as noted above.

21.7 Find $\mathscr{L}\{\sin ax\}$.

This problem can be entered into Wolfram Alpha:

!| find laplace transform of [sin(ax)] |!

In Wolfram Alpha the result is expressed as:

Result $\frac{a}{a^2+p^2}$

Focusing on p here, as noted above.

21.8 Find the Laplace transform of $f(x) = \begin{cases} e^x & x \le 2 \\ 3 & x > 2 \end{cases}$

This problem can be entered into Wolfram Alpha:

!| find laplace transform of Piecewise[{{e^x, x <= 2}, {3, x >2}}] |!

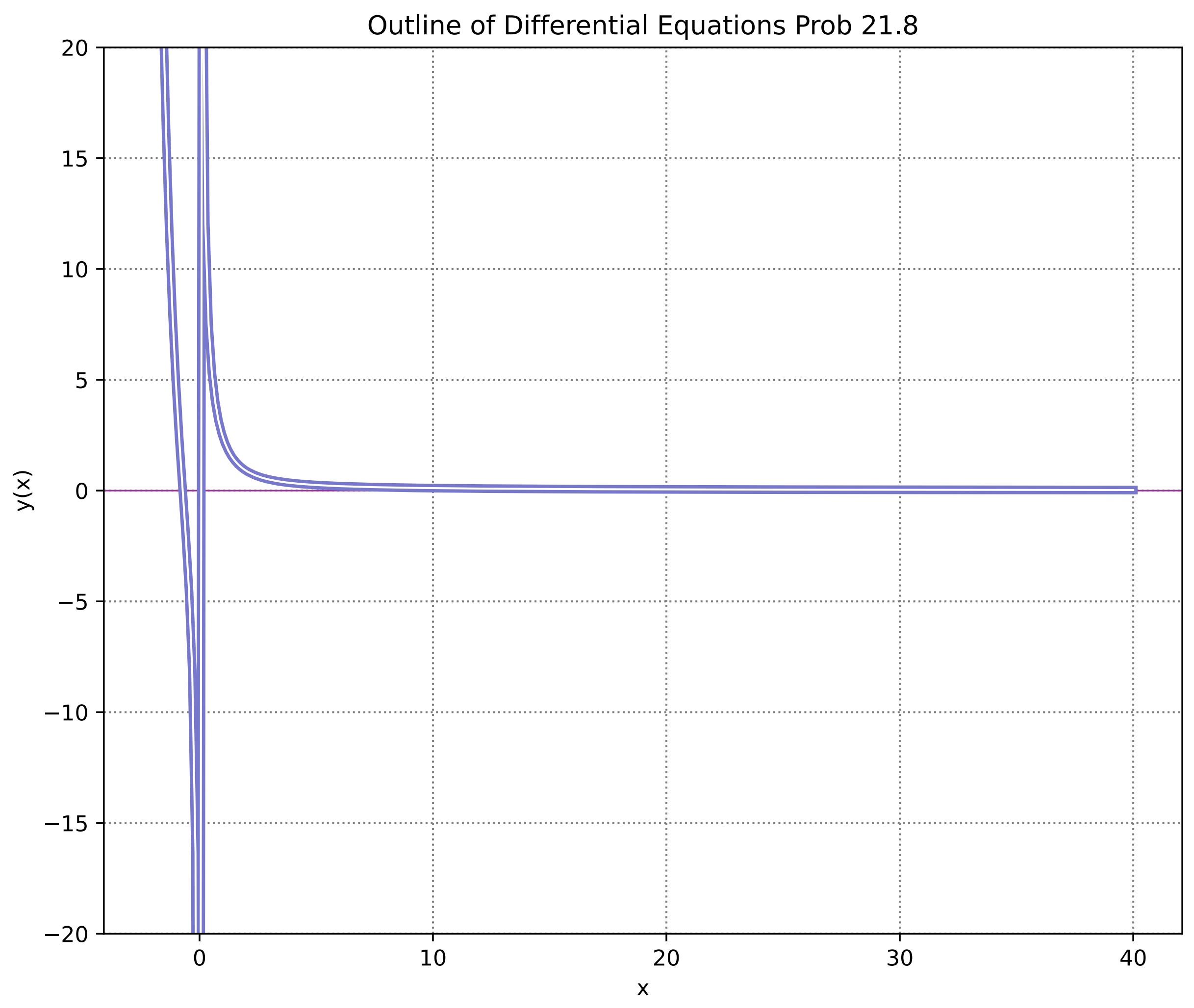
In Wolfram Alpha the result is expressed as:

Result

$$\frac{1}{p-1} - \frac{e^{-2p}\,(e^2 p + 3p + 3)}{(p-1)\,p}$$

The above result differs in appearance somewhat from the text answer, so a plot here would be prudent. Several alternate forms of the result equation in Wolfram Alpha were available, and one in particular matches the text well.

In [55]:
```
import numpy as np
import matplotlib.pyplot as plt

%config InlineBackend.figure_formats = ['svg']

x = np.linspace(-2,40,300)
y4 = (1 - np.exp(-2*(x - 1))   )/(x - 1) + (3 * np.exp(-2 * x))/x
y6 = 1/(x - 1) -(np.exp(-2 * x)*(np.exp(2)*x - 3 * x + 3)) / ((x - 1)*x)

plt.grid(True, linestyle='dotted', color='gray', linewidth=0.9)
plt.xlabel("x")
plt.ylabel("y(x)")
plt.title("Outline of Differential Equations Prob 21.8")
plt.rcParams['figure.figsize'] = [9, 7.5]


ax = plt.gca()
ax.axhline(y=0, color='#993399', linewidth=0.7)
ax.axvline(x=0, color='#993399', linewidth=0.7)
ratio = 0.95
xleft, xright = ax.get_xlim()
ybottom, ytop = ax.get_ylim()
ax.set_aspect(abs((xright-xleft)/(ybottom-ytop))*ratio)


#plt.text(965, -2.5e43, "-np.log(abs((np.cos(x/2))**2 - (np.sin(x/2))**2)))
#plt.text(965, 2.25e43, "SOLN: y = 3*np.exp(x**2) + 1/2", size=10,\
        # bbox=dict(boxstyle="square", ec=('#8C564B'),fc=(1., 1., 1),))
plt.ylim(-20,20)
plt.plot(x, y4, linewidth = 4, color = '#7777CC')
plt.plot(x, y6, linewidth = 0.9, color = 'w')
plt.show()
```



21.9 Find the Laplace transform of the function graphed in Fig.21-1: $f(x) = \begin{cases} -1 & x \le 4 \\ 1 & x > 4 \end{cases}$

This problem can be entered into Wolfram Alpha:

!| find Laplace transform of Piecewise[{{-1, x <= 4}, {1, x >4}}] |!

In Wolfram Alpha the result is expressed as:

Result

$\frac{2e^{-4p}-1}{p}$

The curvy plot of the above, contrasted with the step function appearance of the input function, reminds that things in Laplaceland look very different from the way they look in Regularland.

21.10 Find the Laplace transform of $f(x) = 3 + 2x^2$.

This problem can be entered into Wolfram Alpha:

!| find Laplace transform of f(x) = 3 + 2 * x^2 |!

In Wolfram Alpha the result is expressed as:

Result

$\frac{3p^2+4}{p^3}$

21.11 Find the Laplace transform of $f(x) = 5\sin 3x - 17e^{-2x}$.

This problem can be entered into Wolfram Alpha:

!| find the Laplace transform of f(x) = 5 * sin(3 * x) - 17 * e^(-2 * x) |!

In Wolfram Alpha the result is expressed as:

Result

$\frac{15}{p^2+9} - \frac{17}{p+2}$

21.12 Find the Laplace transform of $f(x) = 5\sin 3x - 17e^{-2x}$.

This problem can be entered into Wolfram Alpha:

!| find the Laplace transform of f(x) = 2 * sin(x) + 3 * cos(2 * x) |!

In Wolfram Alpha the result is expressed as:

Result

$\frac{3p}{p^2+4} + \frac{2}{p^2+1}$

21.13 Find the Laplace transform of $f(x) = 5\sin 3x - 17e^{-2x}$.

This problem can be entered into Wolfram Alpha:

!| find the Laplace transform of f(x) = 2 * x^2 - 3x + 4 |!

In Wolfram Alpha the result is expressed as:

Result

$\frac{4p^2-3p+4}{p^3}$

21.14 Find $\mathscr{L}\{ xe^{4x} \}$.

This problem can be entered into Wolfram Alpha:

!| find the Laplace transform of x * e^(4 * x) |!

In Wolfram Alpha the result is expressed as:

Result

$\frac{1}{(p-4)^2}$

21.15 Find $\mathscr{L}\{ xe^{4x} \}$.

This problem can be entered into Wolfram Alpha:

!| find the Laplace transform of e^(-2 * x) * sin(5 * x) |!

In Wolfram Alpha the result is expressed as:

Result

$\frac{5}{(p+2)^2+25}$

21.16 Find $\mathscr{L}\{\ x\cos\sqrt{7}x\ \}$.

The following entry did not work in Wolfram Alpha:

!| find the Laplace transform of x * cos(sqrt(7) * x) |!

A couple of small changes made it work:

!| laplace transform of x * cos(sqrt(7) * x) |!

In Wolfram Alpha the result is expressed as:

Result

$$\frac{p^2-7}{(p^2+7)^2}$$

21.17 Find $\mathscr{L}\{\ e^x\ x\cos 2x\ \}$.

This problem can be entered into Wolfram Alpha:

!| laplace transform of e^x * x * cos(2 * x) |!

In Wolfram Alpha the result is expressed as:

Result

$$\frac{p^2-2p-3}{(p^2-2p+5)^2}$$

The above answer does not match the answer in the text. However, with the answer above, performing an inverse Laplace transform recovers the problem expression, whereas, with the text answer, it does not. The text answer is:

$$\frac{(s+1)^2-4}{((s+1)^2+4)^2}$$

21.18 Find $\mathscr{L}\{\ x^{7/2}\ \}$.

This problem can be entered into Wolfram Alpha:

!| laplace transform of x^(7/2) |!

In Wolfram Alpha the result is expressed as:

Result

$$\frac{105\sqrt{\pi}}{16p^{9/2}}$$

21.19 Find $\mathscr{L}\left\{\ \frac{\sin 3x}{x}\ \right\}$.

This problem can be entered into Wolfram Alpha:

!| laplace transform of sin(3 * x)/x |!

In Wolfram Alpha the result is expressed as:

Result

$$\frac{1}{2}\left(\pi-2\tan^{-1}\left(\frac{p}{3}\right)\right)$$

21.20 Find $\mathscr{L}\{\int_0^x \sinh 2t\,dt\ \}$.

This problem can be entered into Wolfram Alpha:

!| laplace transform of integrate[sinh(2 * t) dt] from 0 to x |!

In Wolfram Alpha the result is expressed as:

Result

$$\frac{2}{p\,(p^2-4)}$$

21.22 Find the Laplace transform of $f(x)=\begin{cases}1 & 0<x\le 1\\ -1 & 1<x\le 2\end{cases}$

This problem is restated from the text to try to capture the function depicted there in a plot. However, apparently the attempt was not successful. The entry in Wolfram Alpha goes like:

!| laplace transform of Piecewise[{{1, 0 < x <= 1}, {-1, 1 < x <=2}}] |!

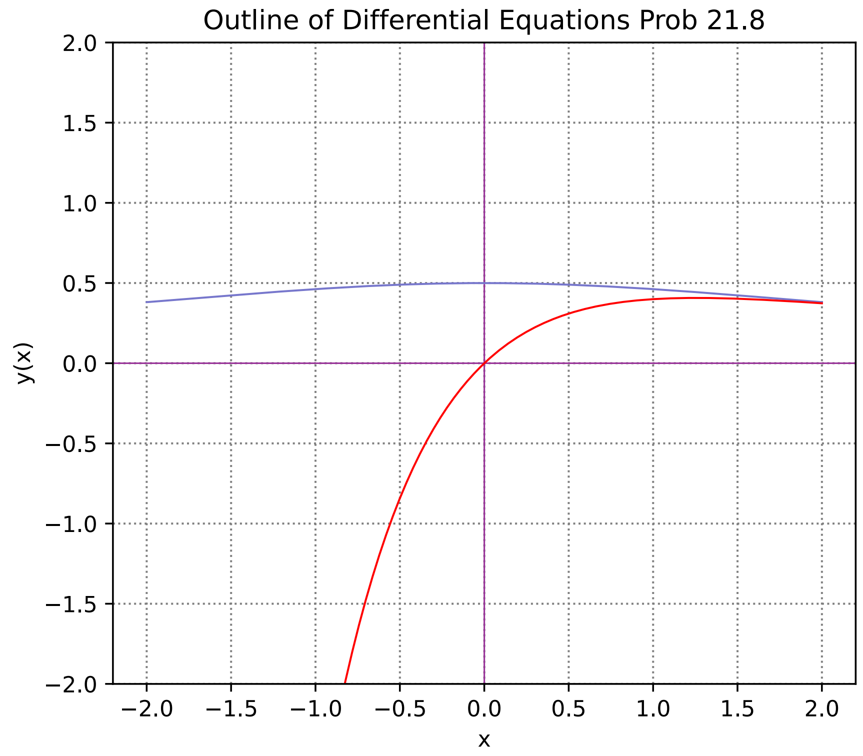
In Wolfram Alpha the result is expressed as:

Result

$$\frac{e^{-2p}\,(e^p-1)^2}{p}$$

This result does not agree with that of the text, and neither does the general scale of the proceedings. A plot shows that the two functions are not the same in the frequency domain, so cannot be expected to coincide in the time domain either.

In [8]:
```
import numpy as np
import matplotlib.pyplot as plt

%config InlineBackend.figure_formats = ['svg']

x = np.linspace(-2,2,300)
y4 = 1/x*np.tanh(x/2)
y6 = (np.exp(-2*x))*(np.exp(x) - 1)**2/x

plt.grid(True, linestyle='dotted', color='gray', linewidth=0.9)
plt.xlabel("x")
plt.ylabel("y(x)")
plt.title("Outline of Differential Equations Prob 21.8")
plt.rcParams['figure.figsize'] = [4, 4]


ax = plt.gca()
ax.axhline(y=0, color='#993399', linewidth=0.7)
ax.axvline(x=0, color='#993399', linewidth=0.7)
ratio = 0.95
xleft, xright = ax.get_xlim()
ybottom, ytop = ax.get_ylim()
ax.set_aspect(abs((xright-xleft)/(ybottom-ytop))*ratio)


#plt.text(965, -2.5e43, "-np.log(abs((np.cos(x/2))**2 - (np.sin(x/2))**2)))
#plt.text(965, 2.25e43, "SOLN: y = 3*np.exp(x**2) + 1/2", size=10,\
        # bbox=dict(boxstyle="square", ec=('#8C564B'),fc=(1., 1., 1),))
plt.ylim(-2,2)
plt.plot(x, y4, linewidth = 0.9, color = '#7777CC')
plt.plot(x, y6, linewidth = 0.9, color = 'r')
plt.show()
```



21.24 Find $\mathscr{L}\left\{e^{4x}\, x \int_0^x \frac{1}{t}\, e^{-4t} \sin 3t\, dt\right\}$

This problem can be entered into Wolfram Alpha:

!| laplace transform [e^(4 x) x int[(1/t) * e^(-4 t) * sin(3 t) dt]] |!

In Wolfram Alpha the result is expressed as:

Result

$$\frac{e^{-2p}\,(e^p-1)^2}{p}$$

This result does not agree with that of the text, or have any identifiable resemblance. The only condition under which Wolfram Alpha would solve the equation was that the integral should be indefinite. This makes it unusable for Laplace procedures. An interesting and not-understood feature of the output is that it contains an x factor, about which any meaning is unclear.

Result

$$-\frac{i\, e^{4x}\, x\left(\log\left(1+\left(\frac{4}{25}-\frac{3i}{25}\right)s\right)-\log\left(1+\left(\frac{4}{25}+\frac{3i}{25}\right)s\right)\right)}{2s}$$

In [ ]:

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