In [3]:

Autosave disabled

Chapter 25: Solutions of Linear Systems by Laplace Transforms. Extending the convenience of Laplace transform solution to systems of two or more equations. When it comes to solving systems of ODEs, the Laplace method can sometimes allow successful solutions that would not otherwise be possible. In this problem set, Wolfram Alpha and Maxima team up to grind the grist, efficiently achieving a series of satisfactory outcomes.

25.1 Solve the following system for the unknown functions $u(x)$ and $v(x)$:

$$u' + u - v = 0$$

$$v' - u + v = 2;$$

$$u(0) = 1, \qquad v(0) = 2$$

The first two equation lines above need to be taken independently into Laplaceland. Doing this with Wolfram Alpha will be deferred until Prob 25.3. For now, Maxima will do the honors.

Note: When transcribing text for entry into Wolfram Alpha the "fenceposts" !| ... |! are excluded from the text.

Consider the interactive steps in the cell below:

(%i1) ode: 'diff(u(t), t) + u(t) − v(t) = 0;

(%o1) $\frac{d}{d\,t}u(t)-v(t)+u(t)=0$

(%i2) atvalue(u(t), t=0, 1);

(%o2) 1

(%i4) atvalue(v(t), t=0, 2);

(%o4) 2

(%i5) lap_ode: laplace(ode, t, s);

(%o5) $-\mathrm{laplace}(v(t),t,s)+s\,\mathrm{laplace}(u(t),t,s)+\mathrm{laplace}(u(t),t,s)-1=0$

(%i7) ode: 'diff(v(t) ,t)− u(t) + v(t) = 2;

(%o7) $\frac{d}{d\,t}v(t)+v(t)-u(t)=2$

(%i8) atvalue(u(t), t=0, 1);

(%o8) 1

(%i9) atvalue(v(t), t=0, 2);

(%o9) 2

(%i10) lap_ode: laplace(ode, t, s);

(%o10) $s\,\mathrm{laplace}(v(t),t,s)+\mathrm{laplace}(v(t),t,s)-\mathrm{laplace}(u(t),t,s)-2=\frac{2}{s}$

Output steps 5 and 10 express the relationships of the functions $U(s)$ and $V(s)$ in Laplacelandese. These correspond to:

$$-V(s) + s\,U(s) + U(s) - 1 = 0$$

$$s\,V(s) + V(s) - U(s) - 2 = \frac{2}{s}$$

with minor grooming

$$-V + (s + 1)\,U = 1$$

$$(s + 1)\,V - U = \frac{2\,(1 + s)}{s}$$

The last two equations can be solved simultaneously by either Wolfram Alpha or sympy. W|A will be given the first try. The entry line is:

!| solve[-v + (s + 1) u = 1, (s + 1) v - u = (2 (1 + s))/s], u, v |!

and the results are:

$$U(s) = \frac{s + 1}{s^2}$$

$$V(s) = \frac{2s + 1}{s^2}$$

Taking these back to Regularland. The first entry line is:

!| inverse laplace transform[(s + 1)/s^2] |!, yielding $t + 1$

and the second entry line is:

!| inverse laplace transform[(2 s + 1)/s^2] |!, yielding $t + 2$

meaning that the base functions are $f_1(x) = x + 1$ and $f_2(x) = x + 2$

25.2 Solve the system:

$$y' + z = x$$

$$z' + 4y = 0;$$

$$y(0) = 1, \quad z(0) = -1$$

The first two equation lines in the problem statement need to be taken independently into Laplaceland. Again, Maxima gets the assignment.

Consider the interactive steps in the cell below (concatenating 8 cells from the Maxima worksheet):

(%i18) ode: 'diff(y(t), t) + z(t) = t;

(%o18) $\frac{d}{d\,t}\,y(t)+z(t)=t$

(%i19) atvalue(y(t), t=0, 1);

(%o19) 1

(%i20) atvalue(z(t), t=0, −1);

(%o20) −1

(%i21) lap_ode: laplace(ode, t, s);

(%o21) $\mathrm{laplace}(z(t),t,s)+s\,\mathrm{laplace}(y(t),t,s)-1=\frac{1}{s^2}$

(%i22) ode: 'diff(z(t) ,t)+ 4·y(t) = 0;

(%o22) $\frac{d}{d\,t}\,z(t)+4\,y(t)=0$

(%i23) atvalue(y(t), t=0, 1);

(%o23) 1

(%i24) atvalue(z(t), t=0, −1);

(%o24) −1

(%i25) lap_ode: laplace(ode, t, s);

(%o25) $s\,\mathrm{laplace}(z(t),t,s)+4\,\mathrm{laplace}(y(t),t,s)+1=0$

In the Maxima output, steps 21 and 25 express the relationships of the functions $Y(s)$ and $Z(s)$ in Laplacelandese. These correspond to:

$$Z(s) + s\,Y(s) - 1 = \frac{1}{s^2}$$

$$s\,Z(s) + 4Y(s) + 1 = 0$$

with minor grooming

$$Z(s) + s\,Y(s) = \frac{s^2 + 1}{s^2}$$

$$s\,Z(s) + 4Y(s) = -1$$

The last two equations can be solved simultaneously by either Wolfram Alpha or sympy. W|A again gets the nod, being slightly more convenient. The entry line is:

!| solve [z + s y = (s^2 + 1)/s^2, s z + 4 y = - 1], y, z |!

and the results are:

$$Y(s) = \frac{s^2 + s + 1}{s\,(s^2 - 4)}$$

$$Z(s) = \frac{s^3 + 4s^2 + 4}{4s^2 - s^4}$$

(For a moment it looks like the solution herein will differ from that of the text. The inconsistency is solved, however. Still, there is a cell from sympy checking the simultaneous equation calculation, and that is included below.)

Taking the solutions back to Regularland. The first entry line is:

!| inverse laplace transform[(s^2 + s + 1)/(s (s^2 - 4))] |!, yielding $\frac{3\,e^{-2t}}{8} + \frac{7\,e^{2t}}{8} - \frac{1}{4}$

and the second entry line is:

!| inverse laplace transform[(s^3 + 4 s^2 + 4)/(4 s^2 - s^4)] |!, yielding $t + \frac{3e^{-2t}}{4} - \frac{7\,e^{2t}}{4}$,

meaning that the base functions will follow the last templates with x substituting for t.

In [20]:

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from sympy import *

y, z, s = symbols('y, z, s')
eq1 = Eq(z + s*y, (s**2 + 1)/s**2)
eq2 = Eq(s*z + 4*y, -1)
res = solve([eq1, eq2], (y, z))
print(res)

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{y: (s**2 + s + 1)/(s**3 - 4*s), z: (-s**3 - 4*s**2 - 4)/(s**4 - 4*s**2)}
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25.3 Solve the system:

$$w' + y = \sin x$$

$$y' - z = e^x;$$

$$z' + w + y = 1;$$

$$w(0) = 0, \quad y(0) = 1 \quad z(0) = 1$$

The first three equation lines in the problem statement need to be taken independently into Laplaceland. This time Wolfram Alpha will be the engine. The format that will be tried for the first step is:

① !| laplace transform[w'(t) + y(t) = sin(t), t, s, {w(0) = 0, y(0) = 1}] |!

With only w and y involved in this step, there is no need to enter the initial value for z. The result of the above command is:

$$s\,W(s) + Y(s) = \frac{1}{s^2 + 1} + w(0)$$

The entry that will be tried for the second step is:

② !| laplace transform[y'(t) - z(t) = e^t, t, s, {y(0) = 1, z(0) = 1}] |!

With only y and z involved in this step, there is no need to enter the initial value for w. The result of the above command is:

$$s\,Y(s) - Z(s) = \frac{1}{s - 1} + y(0)$$

The entry that will be tried for the third step is:

③ !| laplace transform[z'(t) + w(t) + y(t) = 1, t, s, {w(0) = 0, y(0) = 1, z(0) = 1}] |!

Since all three variables are involved in this step, it is necessary to include all three initial conditions. The result of the above command is:

$$W(s) + Y(s) + s\,Z(s) = \frac{1}{s} + z(0)$$

There is no reformulation necessary on the three equations before simultaneous solution. Therefore, inserting the initial conditions as they make appearance, the simultaneous command in Wolfram Alpha looks like:

!| solve[s w + y = 1/(s^2 + 1), s y - z = 1/(s - 1) +1, w + y + s z =1/s + 1], w, y, z |!

and the result is:

$$W(s) = -\frac{1}{(s-1)\,s}$$

$$Y(s) = \frac{s^2 + s}{(s-1)(s^2+1)}$$

$$Z(s) = \frac{s}{s^2+1}$$

Time now to take these expressions back to Regularland. The first entry in Wolfram Alpha is:

① !| inverse laplace transform[-1/(s (s - 1)), s, t] |!

and the result is:

$$1 - e^t$$

The second entry in Wolfram Alpha is:

② !| inverse laplace transform[(s^2 + s)/((s - 1) (s^2 + 1)), s, t] |!

and the result is:

$$e^t + \sin(t)$$

The third entry in Wolfram is:

③ !| inverse laplace transform[s/(s^2 + 1), s, t] |!

and the result is:

$$\cos(t)$$

In Regularland the t symbols are replaced with x, which concludes the answer.

25.4 Solve the system:

$$y'' + z + y = 0$$

$$z' + y' = 0;$$

$$y(0) = 0, \quad y'(0) = 0 \quad z(0) = 1$$

The first two equation lines in the problem statement need to be taken independently into Laplaceland. This time Wolfram Alpha will be the engine. The format that will be tried for the first step is:

① !| laplace transform[y''(t) + z(t) +y(t) = 0, t, s, {y(0) = 0, y'(0) = 0, z(0) =1}] |!

Both y and z are involved in this step, so all three initial conditions need to be included. The result of the above command is:

$$(s^2 + 1)\,Y(s) + Z(s) = s\,y(0) + y'(0)$$

The entry that will be tried for the second step is:

② !| LaplaceTransform[z'(t) + y'(t) = 0, t, s, {y(0) = 0, y'(0) = 0, z(0) = 1}] |!

The second step fails, Wolfram Alpha will not accept it, because both $z(t)$ and $y(t)$ are expressed as differentials. There may be a special method to avoid this difficulty, but a little experimentation does not reveal it. Switching horses in midstream, the second step will be transferred to Maxima.

Picking up with Maxima, the interaction necessary to take the second problem description line to Laplaceland is shown below, culminating with output line 17.

(%i13) ode: 'diff(z(t), t) + 'diff(y(t), t) = 0;

(%o13) $\frac{d}{d\,t}\,z(t)+\frac{d}{d\,t}\,y(t)=0$

(%i14) atvalue(y(t), t=0, 0);

(%o14) 0

(%i15) atvalue('diff(y(t), t), t=0, 0);

(%o15) 0

(%i16) atvalue(z(t), t=0, 1);

(%o16) 1

(%i17) lap_ode: laplace(ode, t, s);

(%o17) $s\ \mathrm{laplace}(z(t),t,s)+s\ \mathrm{laplace}(y(t),t,s)-1=0$

Translating the Maxima output:

$$s\,Z(s)\ +\ s\,Y(s)\ -\ 1\ =\ 0$$

So the two equations to be solved simultaneously will be the first from Wolfram Alpha and the second from Maxima, and the input line in W|A will be:

!| solve[(s^2 + 1) y + z = 0, s z + s y - 1 = 0], y, z |!

and the result is:

$$Y(s)\ =\ -\frac{1}{s^3}$$

$$Z(s)\ =\ \frac{1}{s^3}\ +\ \frac{1}{s}$$

Time now to take these expressions back to Regularland. The first entry in Wolfram Alpha is:

① !| inverse laplace transform[-1/s^3, s, t] |!

and the result is:

$$-\frac{t^2}{2}$$

The second entry in Wolfram Alpha is:

② !| inverse laplace transform[1/s^3 + 1/s, s, t] |!

and the result is:

$$\frac{t^2}{2}\ +\ 1$$

In Regularland the t symbols are replaced with x, which concludes the answer.

25.5 Solve the system:

$$z''\ +\ y'\ =\ \cos x$$

$$y''\ -\ z\ =\ \sin x;$$

$$z(0)\ =\ -1,\qquad z'(0)\ =\ -1\qquad y(0)\ =\ 1\qquad y'(0)\ =\ 0$$

The role of Maxima.

Wolfram Alpha will not accept this system of equations, there seems to be a superfluity of differential signs. So Maxima will have to pick up the slack. Both the first and second lines of the problem description are generated by Maxima, as shown on output lines 41 and 47 of the following two cells.The smoothed expression of these Laplaceland equations are shown below.

$$s^2\,Z(s)\ +\ s\,Y(s)\ +\ s\ =\ \frac{s}{s^2\ +\ 1}$$

$$-Z(s) + s^2 Y(s) - s = \frac{1}{s^2 + 1}$$

The two equations above could be simplified slightly, but regardless, there will be a lot of s symbols lying around afterwards, and since they are only intermediate results, it is just as well to leave everything as it is for the simultaneous equation:

!| solve[s^2 z + s y + s = s/(s^2 + 1), -z + s^2 y - s = 1/(s^2 + 1)], y, z |!

and the result is:

$$y = \frac{s}{s^2 + 1}$$

$$z = -\frac{s + 1}{s^2 + 1}$$

The equations above are staged and ready to be transformed back to Regularland. The first is inversely transformed with the line:

!| inverse laplace transform[s/(s^2 + 1), s, t] |!

and the result is:

$$\cos(t)$$

The second is inversely transformed with the line:

!| inverse laplace transform[-(s + 1)/(s^2 + 1), s, t] |!

and the result is:

$$-\sin(t) - \cos(t)$$

In Regularland the t symbols are replaced with x, which concludes the answer.

(%i36) ode: 'diff(z(t), t, 2) + 'diff(y(t), t) = cos(t);

(%o36) $\frac{d^2}{d t^2} z(t) + \frac{d}{d t} y(t) = \cos(t)$

(%i37) atvalue(z(t), t=0, −1);

(%o37) −1

(%i38) atvalue('diff(z(t), t), t=0, −1);

(%o38) −1

(%i39) atvalue(y(t), t=0, 1);

(%o39) 1

(%i34) atvalue('diff(y(t), t), t=0, 0);

(%o34) 0

(%i41) lap_ode: laplace(ode, t, s);

(%o41) $s^2 \operatorname{laplace}(z(t), t, s) + s \operatorname{laplace}(y(t), t, s) + s = \frac{s}{s^2 + 1}$

(%i42) ode2: 'diff(y(t), t, 2) − z(t) = sin(t);

(%o42) $\frac{d^2}{d t^2} y(t) - z(t) = \sin(t)$

(%i43) atvalue(z(t), t=0, −1);

(%o43) −1

(%i44) atvalue('diff(z(t), t), t=0, −1);

(%o44) −1

(%i46) atvalue(y(t), t=0, 1);

(%o46) 1

(%i45) atvalue('diff(y(t), t), t=0, 0);

(%o45) 0

(%i47) lap_ode2: laplace(ode2, t, s);

(%o47) $-\operatorname{laplace}(z(t), t, s) + s^2 \operatorname{laplace}(y(t), t, s) - s = \frac{1}{s^2 + 1}$

25.6 Solve the system:

$$w'' - y + 2z = 3\,e^{-x}$$
$$-2w' + 2y' + z = 0$$
$$2w' - 2y + z' + 2z'' = 0;$$
$$w(0) = 1, \quad w'(0) = 1, \quad y(0) = 2, \quad z(0) = 2, \quad z'(0) = -2$$

There is no sense in even considering Wolfram Alpha to work on this system; the number of prime symbols in the second and third equations would overwhelm it. So it will have to be Maxima again. The Laplaceland equivalent equations are described on output lines 27, 40, and 47 of the following three cells. In Laplacelandese, they are:

$$2\,Z(s) - Y(s) + s^2\,W(s) - s - 1 = \frac{3}{s + \log(e)}$$
$$Z(s) + 2\,(s\,Y(s) - 2) - 2\,(s^2\,W(s) - s - 1) = 0$$
$$2\,(s^2\,Z(s) - 2s + 2) + s\,Z(s) - 2\,Y(s) + 2\,(s\,W(s) - 1) - 2 = 0$$

There is no disputing the fact that the above three equations could benefit from some collecting and simplifying. However, since the simultaneous equations will be solved just as quickly in a slightly disorganized syntax, nothing about appearance will be done at this time.

The entry that will be tried for the simultaneous equation calculation is:

!| solve[2 z -y + s^2 w - s-1 = 3/(s + 1), z + 2 (s y - 2)- 2 (s^2 w -s -1) = 0, 2 (s^2 z - 2 s + 2) + s z - 2 y + 2 (s w - 1) - 2=0],w, y, z |!

Not only was Wolfram Alpha able to process this complex system of simultaneous equations, it provided 5 different forms of solution sets. The most intuitive one of these was checked against the text solution. It is:

$$w = \frac{1}{s - 1}$$
$$y = \frac{2s}{s^2 - 1}$$
$$z = \frac{2}{s + 1}$$

The problem is almost but not quite finished. The equations above have to be inverse transformed back to Regularland. The first entry line is:

!| inverse laplace transform[1/(s -1), s, t] |!

and the solution is: e^t . The second entry line is:

!| inverse laplace transform[(2 s)/(s^2 -1), s, t] |!

and the solution is: $e^{-t}(e^{2t} + 1)$. The third entry line is:

!| inverse laplace transform[2/(s + 1), s, t] |!

and the solution is: $2\,e^{-t}$

In Regularland the t symbols are replaced with x, which concludes the answer.

(%i21) ode: 'diff(w(t), t, 2) − y(t) + 2·z(t) = 3·e^(−t);

(%o21) $\frac{d^2}{d\,t^2}\,w(t)+2\,z(t)-y(t)=\frac{3}{e^t}$

(%i22) atvalue(w(t), t=0, 1);

(%o22) 1

(%i23) atvalue('diff(w(t), t), t=0, 1);

(%o23) 1

(%i24) atvalue(y(t), t=0, 2);

(%o24) 2

(%i25) atvalue(z(t), t=0, 2);

(%o25) 2

(%i26) atvalue('diff(z(t), t), t=0, −2);

(%o26) -2

(%i27) lap_ode: laplace(ode, t, s);

(%o27) $2\ \mathrm{laplace}(z(t),t,s)-\mathrm{laplace}(y(t),t,s)+s^2\ \mathrm{laplace}(w(t),t,s)-s-1=\frac{3}{s+\log(e)}$

(%i34) ode2: −2·'diff(w(t), t, 2) +2·'diff(y(t), t) + z(t) = 0;

(%o34) $2\left(\frac{d}{d\,t}\,y(t)\right)-2\left(\frac{d^2}{d\,t^2}\,w(t)\right)+z(t)=0$

(%i35) atvalue(w(t), t=0, 1);

(%o35) 1

(%i36) atvalue('diff(w(t), t), t=0, 1);

(%o36) 1

(%i37) atvalue(y(t), t=0, 2);

(%o37) 2

(%i38) atvalue(z(t), t=0, 2);

(%o38) 2

(%i39) atvalue('diff(z(t), t), t=0, −2);

(%o39) -2

(%i40) lap_ode2: laplace(ode2, t, s);

(%o40) $\mathrm{laplace}(z(t),t,s)+2\ (s\ \mathrm{laplace}(y(t),t,s)-2)-2\ (s^2\ \mathrm{laplace}(w(t),t,s)-s-1)=0$

(%i41) ode3: 2·'diff(w(t), t) − 2·y(t) + 'diff(z(t), t) + 2·'diff(z(t), t, 2) = 0;

(%o41) $2\left(\frac{d^2}{d\,t^2}\,z(t)\right)+\frac{d}{d\,t}\,z(t)+2\left(\frac{d}{d\,t}\,w(t)\right)-2\,y(t)=0$

(%i42) atvalue(w(t), t=0, 1);

(%o42) 1

(%i43) atvalue('diff(w(t), t), t=0, 1);

(%o43) 1

(%i44) atvalue(y(t), t=0, 2);

(%o44) 2

(%i45) atvalue(z(t), t=0, 2);

(%o45) 2

(%i46) atvalue('diff(z(t), t), t=0, −2);

(%o46) -2

(%i47) lap_ode3: laplace(ode3, t, s);

(%o47) $2\ (s^2\ \mathrm{laplace}(z(t),t,s)-2\ s+2)+s\ \mathrm{laplace}(z(t),t,s)-2\ \mathrm{laplace}(y(t),t,s)+2$
$(s\ \mathrm{laplace}(w(t),t,s)-1)-2=0$

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