

Chapter 31-10 Method of Lines.

The method of lines (MOL, NMOL, NUMOL) is a technique for solving partial differential equations (PDEs) in which all but one dimension is discretized. By reducing a PDE to a single continuous dimension, the method of lines allows solutions to be computed via methods and software developed for the numerical integration of ordinary differential equations (ODEs) and differential-algebraic systems of equations (DAEs). Integration routines have been developed over the years in numerous programming languages.

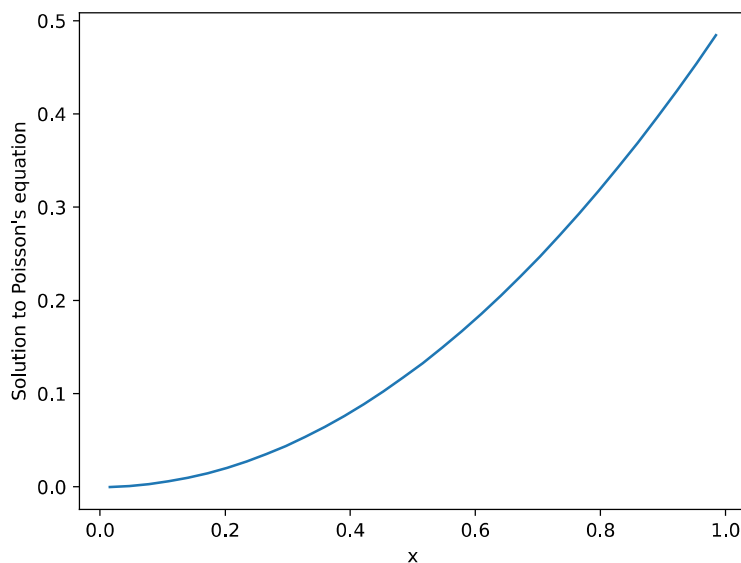
The method of lines most often refers to the construction or analysis of numerical methods for partial differential equations that proceeds by first discretizing the spatial derivatives only and leaving the time variable continuous. This leads to a system of ordinary differential equations to which a numerical method for initial value ordinary equations can be applied. The method of lines in this context dates back to at least the early 1960s.

The `pde` module `py-pde` is a quick Python add-in that is convenient in many situations. Its presence is assumed from the import statement in the example. From the `py-pde` docs:

3.1. Mathematical basics

To solve partial differential equations (PDEs), the `py-pde` package provides differential operators to express spatial derivatives. These operators are implemented using the finite difference method to support various boundary conditions. The time evolution of the PDE is then calculated using the *method of lines* by explicitly discretizing space using the grid classes. This reduces the PDEs to a set of ordinary differential equations, which can be solved using standard methods as described below.

```
In [1]: 1 from pde import CartesianGrid, ScalarField, solve_poisson_equation
2 %config InlineBackend.figure_formats = ['svg']
3
4 grid = CartesianGrid([[0, 1]], 32, periodic=False)
5 field = ScalarField(grid, 1)
6 result = solve_poisson_equation(field, bc=[{"value": 0}, {"derivative": 1}])
```



```
Out[1]: <pde.tools.plotting.PlotReference at 0xldb62700100>
```

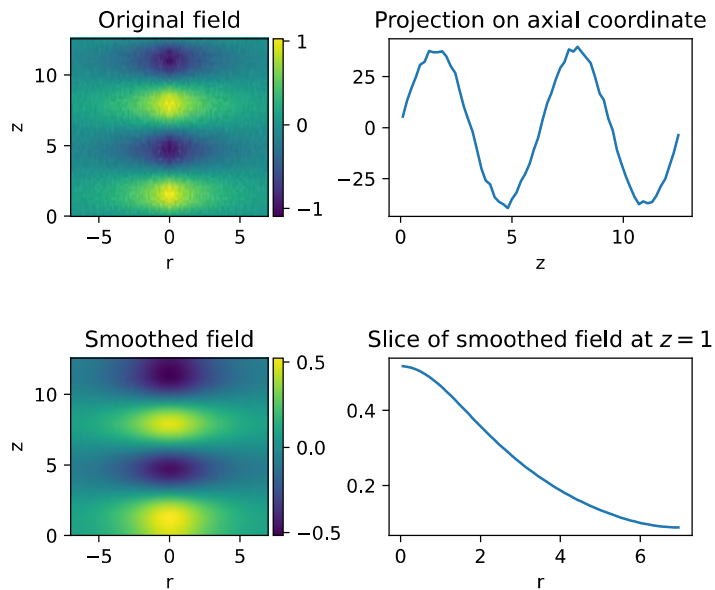
A second `py-pde` example is shown below.

```
In [2]: 1 import matplotlib.pyplot as plt
2 import numpy as np
3 %config InlineBackend.figure_formats = ['svg']
4
5 from pde import CylindricalSymGrid, ScalarField
```

```

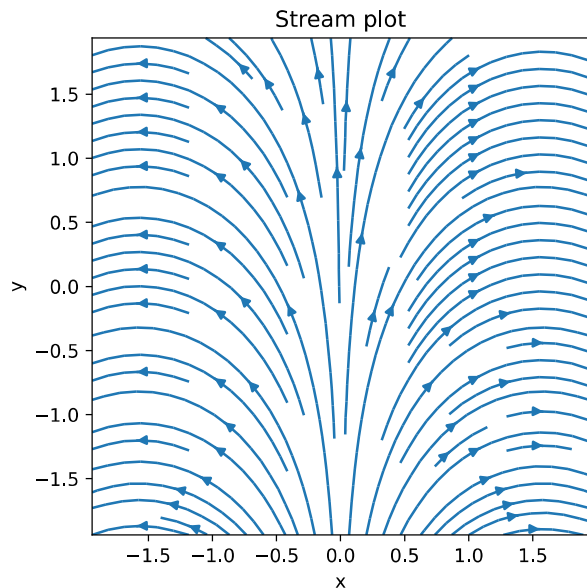
6
7 # create a scalar field with some noise
8 grid = CylindricalSymGrid(7, [0, 4 * np.pi], 64)
9 data = ScalarField.from_expression(grid, "sin(z) * exp(-r / 3)")
10 data += 0.05 * ScalarField.random_normal(grid)
11
12 # manipulate the field
13 smoothed = data.smooth() # Gaussian smoothing to get rid of the noise
14 projected = data.project("r") # integrate along the radial direction
15 sliced = smoothed.slice({"z": 1}) # slice the smoothed data
16
17 # create four plots of the field and the modifications
18 fig, axes = plt.subplots(nrows=2, ncols=2)
19 data.plot(ax=axes[0, 0], title="Original field")
20 smoothed.plot(ax=axes[1, 0], title="Smoothed field")
21 projected.plot(ax=axes[0, 1], title="Projection on axial coordinate")
22 sliced.plot(ax=axes[1, 1], title="Slice of smoothed field at $z=1$")
23 plt.subplots_adjust(hspace=0.8)

```



And a third example:

```
In [3]: 1 from pde import CartesianGrid, VectorField
2 %config InlineBackend.figure_formats = ['svg']
3
4 grid = CartesianGrid([[-2, 2], [-2, 2]], 32)
5 field = VectorField.from_expression(grid, ["sin(x)", "cos(x)"])
```



```
Out[3]: <pde.tools.plotting.PlotReference at 0x1aaaf591600>
```

Method of Lines example. An evolution equation such as

$$u_t = -uu_x + \mu u_{xx}$$

(referred to below as 'ref_eq,') can be recast as the expression

$$u_t(t, x) = S[u],$$

where $S[u]$ is a functional of $u(t, x)$, which does not include t-derivatives of u . Here it could be written explicitly as

$$S[u] = F(u(t, x), u_x(t, x), u_{xx}(t, x)) = -uu_x + \mu u_{xx}$$

The main point is that for any given t the right-hand side of 'ref_eq' can be computed if we know the values of $u(t, x)$ for the same t and all x .

The 'ref_eq' can be regarded as an infinite collection of ordinary differential equations, one for each value of x , with t as the independent variable and initial data furnished by

$$u(0, x) = f(x) \quad a \leq x \leq b,$$

Further, the infinite set of x is replaced with $a \leq x \leq b$, by a representative finite set, and the spatial derivatives by some discrete approximation. This is the method of lines (MoL). This means that we can utilize the experience and techniques that have already been acquired in the study of initial value problems for ordinary differential equations.

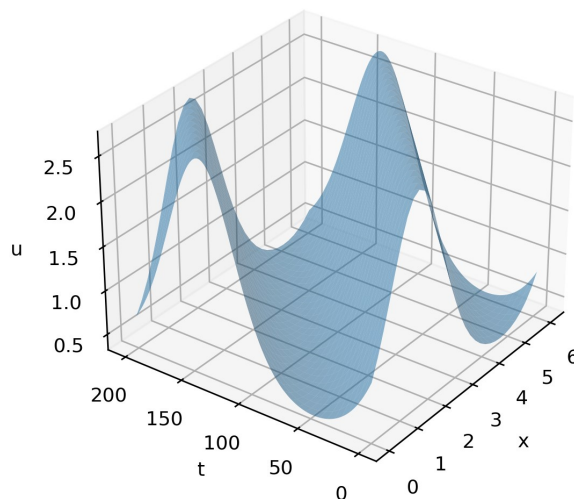
The example plot below is interactive immediately following the execution of its script.

```
In [1]: 1
2 import matplotlib.pyplot as plt
3 from mpl_toolkits.mplot3d import Axes3D
4 %matplotlib notebook
5 import numpy as np
6 from scipy.fftpack import diff
7 from scipy.integrate import odeint
8 import matplotlib.pyplot as plt
9
```

```

10  %%config InlineBackend.figure_formats = ['svg']
11
12  def u_exact(t,x):
13      """ Exact solution. """
14      return np.exp(np.sin(x-2*np.pi*t))
15
16
17  def rhs(u, t):
18      """ Return rhs. """
19      return -2.0*np.pi*diff(u)
20
21  N=32
22  x=np.linspace(0,2*np.pi,N,endpoint=False)
23  u0=u_exact(0,x)
24  t_initial=0.0
25  t_final=64*np.pi
26  t=np.linspace(t_initial,t_final,101)
27  sol=odeint(rhs,u0,t,mxstep=5000)
28
29  fig=plt.figure()
30  ax=fig.add_subplot(111,projection='3d')
31  t_gr,x_gr=np.meshgrid(x,t)
32  ax.plot_surface(t_gr,x_gr,sol,alpha=0.5)
33  ax.elev,ax.azim=47,-137
34  ax.set_xlabel('x')
35  ax.set_ylabel('t')
36  ax.set_zlabel('u')
37
38
39  u_ex=u_exact(t[-1],x)
40  err=np.abs(np.max(sol[-1,:]-u_ex))
41  print("With %d Fourier nodes the final error = %g" % (N, err))

```



With 32 Fourier nodes the final error = 0.000106904

In []: