Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

Problems in the workbook: 5, 17, 21.

1 - 8 Application: mass distribution

Find the total mass of a mass distribution of density σ in a region T in space.

1. $\sigma = x^2 + y^2 + z^2$, T the box $|x| \le 4$, $|y| \le 1$, $0 \le z \le 2$

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Clear["Global`*"]
```

$$\int_{-4}^{4}\int_{-1}^{1}\int_{0}^{2}\left(x^2 + y^2 + z^2\right) dz\, dy\, dx$$

224

The answer above matches the text's.

3. $\sigma = e^{-x-y-z}$, T : $0 \le x \le 1 - y$, $0 \le y \le 1$, $0 \le z \le 2$

```
Clear["Global`*"]
```

$$\text{outt} = \int_{0}^{2}\int_{0}^{1}\int_{0}^{1-y} e^{-x-y-z}\, dx\, dy\, dz$$

$$\frac{(-2 + e)\ (-1 + e)\ (1 + e)}{e^3}$$

This problem is perplexing. Why and how is the answer in the form of a vector? And the exponents in the answer slots retain the original variables. Don't understand.

5. $\sigma = \text{Sin}[2\,x]\ \text{Cos}[2\,y]$, T : $0 \le x \le \frac{1}{4}\pi$, $\frac{1}{4}\pi - x \le y \le \frac{1}{4}\pi$, $0 \le z \le 6$

```
Clear["Global`*"]
```

$$\int_{0}^{6}\int_{0}^{\pi/4}\int_{(\pi/4)-x}^{\pi/4} \text{Sin}[2\,x]\ \text{Cos}[2\,y]\, dy\, dx\, dz$$

$$\frac{3}{4}$$

This problem was worked in the s.m.. The answer is found without the 2 - 3 pages shown there.

7. $\sigma = \text{ArcTan}\left[\frac{y}{x}\right]$, T : $x^2 + y^2 + z^2 \le a^2$, $z \ge 0$

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Clear["Global`*"]
```

$$\int_0^6 \int_0^{\pi/4} \int_{(\pi/4)-x}^{\pi/4} \text{ArcTan}\left[\frac{y}{x}\right] dy\, dx\, dz$$

$$\frac{3\,\pi^3}{64}$$

I need to work on this some more. At least Mathematica can do the integral. The text answer is $\frac{2\,\pi^2\,a^3}{3}$.

9 - 18 Application of the divergence theorem

Evaluate the surface integral $\int_S\int \mathbf{F.n}\; dA$ by the divergence theorem.

9. $F = \{x^2, 0, z^2\}$, S the surface of the box $|x| \le 1$, $|y| \le 3$, $0 \le z \le 2$

```
Clear["Global`*"]
```

```
divv = Div[{x^2, 0, z^2}, {x, y, z}]
```

$2\,x + 2\,z$

$$\text{outt} = \int_{-1}^{1}\int_{-3}^{3}\int_{0}^{2} (2\,x + 2\,z)\, dz\, dy\, dx$$

48

The above answer matches the text’s.

11. $F = \{e^x, e^y, e^z\}$,
S the surface of the cube $|x| \le 1$, $|y| \le 1$, $|z| \le 1$

```
Clear["Global`*"]
```

```
divv = Div[{e^x, e^y, e^z}, {x, y, z}]
```

$e^x + e^y + e^z$

$$\text{luco} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (e^x + e^y + e^z)\, dz\, dy\, dx$$

$$\frac{12\,\left(-1 + e^2\right)}{e}$$

$$\text{PossibleZeroQ}\left[\frac{12\,\left(-1 + e^2\right)}{e} - 12\left(\frac{e - 1}{e}\right)\right]$$

False

```
PossibleZeroQ[12 (-1 + ⅇ^2)/ⅇ - 24 Sinh[1]]
```

```
True
```

Apparently there is a typo in the first version of the text answer (blue cell) which, however, is corrected in the alternate expression (green cell), showing agreement with Mathematica's answer.

13. F = {Sin[y], Cos[x], Cos[z]}, S, the surface of $x^2 + y^2 \leq 4$, $| z | \leq 2$ (a cylinder and two disks)

```
Clear["Global`*"]
```

```
divv = Div[{Sin[y], Cos[x], Cos[z]}, {x, y, z}]
```

```
-Sin[z]
```

$$\text{luco} = \int_0^{2\pi}\int_0^{2\pi}\int_{-2}^{2} (-\text{Sin}[z])\, dz\, dy\, dx$$

```
0
```

The answer above matches the text's.

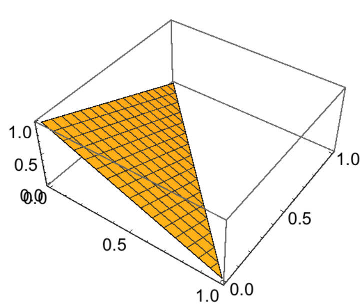
15. F = $\left\{2 x^2, \frac{1}{2} y^2, \text{Sin}[\pi z]\right\}$, S the surface of the tetrahedron with vertices {0, 0, 0}, {1, 0, 0}, {0, 0, 1}

```
Clear["Global`*"]
```

```
mylist = {{0, 0, 0}, {1, 0, 0}, {0, 1, 0}, {0, 0, 1}}
```

```
{{0, 0, 0}, {1, 0, 0}, {0, 1, 0}, {0, 0, 1}}
```

```
ListPlot3D[{{1, 0, 0}, {0, 0, 1}, {0, 1, 0}}]
```



```
divv = Div[{2 x^2, y^2/2, Sin[π z]}, {x, y, z}]
```

$4 x + y + \pi \,\mathrm{Cos}[\pi z]$

$$\text{luco} = \int_0^1 \int_0^{1-x} \int_0^{1-y-x} (4 x + y + \pi \,\mathrm{Cos}[\pi z])\, dz\, dy\, dx$$

$$\frac{5}{24} + \frac{1}{\pi}$$

The above value agrees with the text’s answer. The integration limits were tricky. I had to find the equation of the plane, x+y+z=1, and then play around with that until I found the right combination of limits.

17. $F = \{x^2, y^2, z^2\}$, S the surface of the cone $x^2 + y^2 \le z^2$, $0 \le z \le h$

```
Clear["Global`*"]
divv = Div[{x^2, y^2, z^2}, {x, y, z}]
```

$2 x + 2 y + 2 z$

I need to find a parametric formula for a cone. The s.m. has it:

```
pa[r_, u_, v_] = {r Cos[v], r Sin[v], u}
```

{r Cos[v], r Sin[v], u}

The parametric representation is a little odd in that it has three variables; therefore in this case there is not a reduction in the number of active variables, as there usually is.

The s.m. gives the general integration as $\iiint_T (2 x + 2 y + 2 z)\, dV$. Then it explains that the ‘volume element’,
dV, is equal to r dr du dv, and that the addition of the ‘r’ is due to the action of the Jacobian upon a change of variables.

The parametrization, $r^2 = x^2 + y^2 \le z^2 = u^2$. By multiplying this out, I see that it is true. It explains why r goes to u. Why does it start at 0? The original problem description said that $0 \le z \le h$ and in the parametrization $z = u$, so $0 \le u$ and $0 \le r \le u$. This does not seem like an airtight case to have r start at 0, but hey, why not? In the parametrization v is the variable that makes the circular cone and so in recognition of its circular nature its limits go from 0 to 2π. I said that the problem statement gives
$0 \le z \le h$, and in the parametrization u is h, so it makes sense to have u go from 0 to h.

$$\text{luco2} = \int_0^{2\pi}\int_0^{h}\int_0^{u}\left(2\, r^2 \text{Cos}[v] + 2\, r^2 \text{Sin}[v] + 2\, u\, r\right) dr\, du\, dv$$

$$\frac{h^4 \pi}{2}$$

The above answer matches the text's. The extra r is prominently visible in the integral argument.

19 - 23 Application: moment of inertia

Given a mass of density 1 in a region T of space, find the moment of inertia about the x-axis

$$I_x = \iiint_T \left(y^2 + z^2\right) dx\, dy\, dz$$

19. The box - a ≤ x ≤ a, -b ≤ y ≤ b, -c ≤ z ≤ c

```
Clear["Global`*"]
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$$\text{loco2} = \int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}\left(y^2 + z^2\right) dx\, dy\, dz$$

$$\frac{8}{3}\, a\, b\, c\left(b^2 + c^2\right)$$

The quantity in the above line matches the answer in the text.

21. The cylinder $y^2 + z^2 \le a^2$, $0 \le x \le h$

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Clear["Global`*"]
```

In an interesting twist, the s.m. parametrizes the circle but leaves the height dimension, x, unparametrized.

$$F[u\_, v\_] = (u\, \text{Cos}[v])^2 + (u\, \text{Sin}[v])^2 == u^2$$

$$u^2 \text{Cos}[v]^2 + u^2 \text{Sin}[v]^2 == u^2$$

The integral will look like: $I_x = \int_0^h \int_0^a \int_0^{2\pi} u^2\, u\, dv\, du\, dx$

Note that in the above, and extra u came in from the Jacobian thing. On limits. The x limits are self-explanatory. As for u, since it is a stand-in for a, that will be its upper limit. As for v, it is the circularity variable, I guess, and that is why it takes the 0 to 2 π limits.

$$\text{luco2} = \int_0^{h}\int_0^{a}\int_0^{2\pi}\left(u^3\right) dv\, du\, dx$$

$$\frac{1}{2}\, a^4\, h\, \pi$$

The quantity above matches the answer in the text.

23. The cone $y^2 + z^2 \leq x^2$, $0 \leq x \leq h$

```
Clear["Global`*"]
```

$$\text{loco3} = \int_0^h \int_0^x \int_0^{2\pi} \left(u^3\right) \, dv \, du \, dx$$

$$\frac{h^5 \pi}{10}$$

The answer above matches the text's. This problem is exactly like the last except that the cone's radius is given by x instead of a.

25. Show that for a solid of revolution $I_x = \frac{\pi}{2} \int_0^h r^4 (x) \, dx$. Solve problems 20 - 23 by this formula.