Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

3. Eigenfunctions

Sketch or graph and compare the first three eigenfunctions (8) with $B_n = 1$, c = 1, and L = π for t = 0, 0.1, 0.2, ⋯, 1.0.

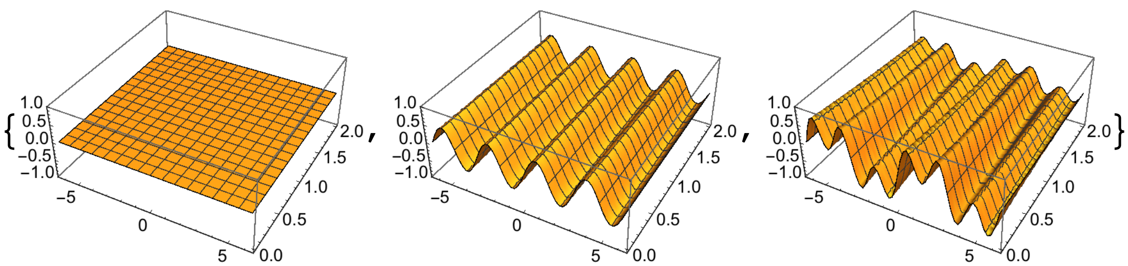
```
Clear["Global`*"]
```

$$B_n = 1;\ c = 1;\ L = \pi;\ \lambda = \frac{c\,n\,\pi}{L};$$

$$un[x\_,\ t\_] = B_n\ Sin\left[\frac{n\,\pi\,x}{L}\right]\ e^{-\lambda^2\,t\,(.1)}$$

$$Table\left[Plot3D\left[B_n\ Cos[x]\ Sin\left[\frac{n\,\pi\,x}{L}\right]\ e^{-\lambda^2\,t\,(.1)},\ \{x, -2\,Pi, 2\,Pi\}, \{t, 0, 2\}, PlotRange \to \{-1, 1\}\right], \{n, 0, 2\}\right]$$

$$e^{-0.1\,n^2\,t}\ Sin[n\,x]$$



In order to let t run in integer values in its list expression, I altered the exponent of e by adding a factor of .1, thereby getting the t= 0, 0.1, 0.2 required by the problem description.

5 - 7 Laterally Insulated Bar

Find the temperature u(x,t) in a bar of silver of length 10 cm and constant cross section of area 1 cm^2 (density 10.6 g/cm^3, thermal conductivity 1.04 cal/(cm sec deg-C), specific heat 0.056 cal/(g deg-C)) that is perfectly insulated laterally, with ends kept at temperature 0 deg-C and initial temperature f(x) deg-C, where

5. f(x) = sin 0.1 π x

```
Clear["Global`*"]

f[x_] = Sin[0.1 π x]
Sin[0.314159 x]

ρ = 10.6; c = 0.056; K₀ = 1.04; L = 10; k = K₀ / (c ρ);
B[n_] = (2 / L) Integrate[f[x] Sin[n Pi x / L], {x, 0, L}]
```

$$\left(-3.89817\times10^{-16}\ n\ Cos[(3.14159 + 0.\,i)\ n] - 3.1831\ Sin[(3.14159 + 0.\,i)\ n]\right) / \left(5\left(-1. + n^2\right)\right)$$

```
Assuming[n ∈ Integers && n > 0, FullSimplify[%]]
```

$$\frac{1}{-1. + n^2}\left(-7.79634\times 10^{-17}\, n\, \mathrm{Cos}[3.14159\, n] - 0.63662\, \mathrm{Sin}[3.14159\, n]\right)$$

Since the expression will be undefined for n=1, n must start at 2:

```
u[x_, t_, N_] := Sum[B[n] Sin[n Pi x / L] Exp[-k (n Pi / L) ^2 t], {n, 2, N}]
Simplify[u[x, t, n]]
```

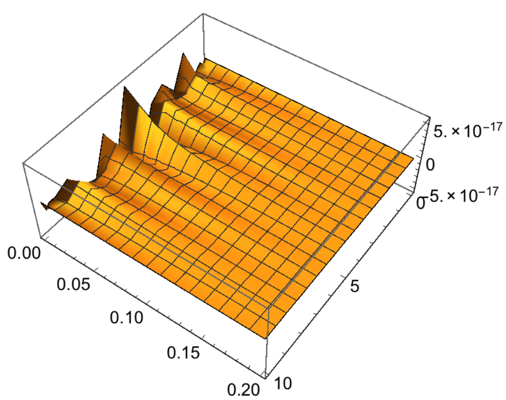
$$\sum_{n=2}^{n} -\frac{1}{n^3 \pi^3} 200\, e^{-0.172918\, n^2 t}\, (-2 + 2\,\mathrm{Cos}[n\pi] + n\pi\,\mathrm{Sin}[n\pi])\,\mathrm{Sin}\left[\frac{n\pi x}{10}\right]$$

The above expression agrees with the text expression for u, after B_n and n have been removed. (p. 561) Note that even though the cross section area is given in the problem description, I have treated it as a 1D heat problem. The s.m. takes this approach also.

```
u[x, t, 12]
```

$$\frac{800\, e^{-1.55626\, t}\,\mathrm{Sin}\left[\frac{3\pi x}{10}\right]}{27\pi^3} + \frac{32\, e^{-4.32294\, t}\,\mathrm{Sin}\left[\frac{\pi x}{2}\right]}{5\pi^3} + \frac{800\, e^{-8.47296\, t}\,\mathrm{Sin}\left[\frac{7\pi x}{10}\right]}{343\pi^3} + \frac{800\, e^{-14.0063\, t}\,\mathrm{Sin}\left[\frac{9\pi x}{10}\right]}{729\pi^3} + \frac{800\, e^{-20.923\, t}\,\mathrm{Sin}\left[\frac{11\pi x}{10}\right]}{1331\pi^3}$$

```
Plot3D[u[x, t, 1000], {x, 0, L}, {t, 0, 0.2}, PlotRange → Full]
```



7. f(x) = x(10 - x)

```
Clear["Global`*"]
f[x_] = x (10 - x)
```

(10 - x) x

```
ρ = 10.6; c = 0.056; K0 = 1.04; L = 10; k = K0 / (c ρ);
B[n_] = (2 / L) Integrate[f[x] Sin[n Pi x / L], {x, 0, L}]
```

$$-\frac{1}{n^3 \pi^3} 200 \, (-2 + 2 \operatorname{Cos}[n \pi] + n \pi \operatorname{Sin}[n \pi])$$

```
Assuming[n ∈ Integers, FullSimplify[%]]
```

$$-\frac{400 \, (-1 + (-1)^n)}{n^3 \pi^3}$$

```
u[x_, t_, N_] := Sum[B[n] Sin[n Pi x / L] Exp[-k (n Pi / L) ^2 t], {n, 1, N}]
u[x, t, 5]
```

$$\frac{800 \, e^{-0.172918 \, t} \operatorname{Sin}\left[\frac{\pi x}{10}\right]}{\pi^3} + \frac{800 \, e^{-1.55626 \, t} \operatorname{Sin}\left[\frac{3 \pi x}{10}\right]}{27 \pi^3} + \frac{32 \, e^{-4.32294 \, t} \operatorname{Sin}\left[\frac{\pi x}{2}\right]}{5 \pi^3}$$

```
N[π^2]
9.8696

-1.5562583759130117` / 9.869604401089358`
-0.157682

-0.1576819407008086` / 9
-0.0175202

-0.172917597323668` / 9.869604401089358`
-0.0175202
```

After verifying the alteration in the e exponent due to consolidation, the green cell above matches the text answer. There is still the potential snag of the apparent 2D nature of the problem, but I am ignoring it due to the match of the answer to text's, and the separate section dedicated to 2D, starting with problem 18.

9. If the ends x=0 and x=L of the bar (in problem 5 and 7 above) are kept at constant temperatures U_1 and U_2, respectively, what is the temperature in the bar at any time?

```
Clear["Global`*"]
```

Model the flow of heat in a bar of length 1 using the heat equation:

```
heqn = D[u[x, t], t] == D[u[x, t], {x, 2}];
```

Specify the fixed temperature at both ends of the bar:

```
bc = {u[0, t] == u1, u[L, t] == u2};
```

Specify an initial condition:

```
ic = u[x, 0] == x (10 - x);
```

Solve the heat equation subject to these conditions:

```
sol = DSolve[{heqn, bc, ic}, u[x, t], {x, t}];
```

DSolve::deqn: Equation or list of equations expected instead of heqn in the first argument {heqn, bc, ic}. >>

```
soln = sol /. K[1] → n;
```

The light gray cells above give a reasonable answer. However, they are not what the text is aiming for. Reverse engineering the answer suggests the more general approach below is the one desired:

```
Clear["Global`*"]

g[x_] = f[x] - u1[x]
f[x] - u1[x]

B[n_] = (2 / L) Integrate[g[x] Sin[n π x / L], {x, 0, L}]
```

$$\frac{2 \int_0^L \mathrm{Sin}\left[\frac{n \pi x}{L}\right] (f[x] - u1[x])\, dx}{L}$$

```
u2[x_, t_, N_] := Sum[B[n] Sin[n π x / L] Exp[-c (n π / L)^2 t], {n, 1, N}]

u2[x, t, 2]
```

$$\frac{1}{L} 2\, e^{-\frac{c \pi^2 t}{L^2}} \left(\int_0^L \mathrm{Sin}\left[\frac{\pi x}{L}\right] (f[x] - u1[x])\, dx\right) \mathrm{Sin}\left[\frac{\pi x}{L}\right] + \frac{1}{L} 2\, e^{-\frac{4 c \pi^2 t}{L^2}} \left(\int_0^L \mathrm{Sin}\left[\frac{2 \pi x}{L}\right] (f[x] - u1[x])\, dx\right) \mathrm{Sin}\left[\frac{2 \pi x}{L}\right]$$

The green cell above matches the first two terms of the text answer.

11. For a completely insulated bar (adiabatic),

$u_x(0, t) = 0,\ u_x(L, t) = 0,\ u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{cn\pi}{L}\right)^2 t}.$

The text development of the steady state problem starts on p. 559. Numbered line (7) on p. 560 gives the general form of the expected answer. Example 4 on p. 564 discusses a bar with insulated ends and eigenvalue of 0, and numbered line (12) gives a form of answer with Fourier cosine series rather than sine.

12 -15 Find the temperature in problem 11 with L=π, c=1, and

13. f(x) = 1

```
Clear["Global`*"]
```

$L = \pi;\ c = 1;\ f[x\_] = 1;$

$A0 = \frac{1}{L}\text{Integrate}[f[x], \{x, 0, L\}]$

1

$An = \frac{2}{L}\int_0^L f[x]\,\text{Cos}\left[\frac{n\,\pi\,x}{L}\right]dx$

$\frac{2\,\text{Sin}[n\,\pi]}{n\,\pi}$

$bui = An\left(\text{Cos}\left[\frac{n\,\pi\,x}{\pi}\right]e^{-\lambda^2 t}\right)$

$\frac{2\,e^{-t\,\lambda^2}\,\text{Cos}[n\,x]\,\text{Sin}[n\,\pi]}{n\,\pi}$

This can be reduced to a simple form very quickly with a substitution:

$bui2 = bui\ /.\ \text{Sin}[n\,\pi] \to 0$

0

$u[x\_, t\_] = A0 + bui2$

1

The above green cell matches the answer in the text.

15. $f(x) = 1 - \frac{x}{\pi}$

```
Clear["Global`*"]
```

$L = \pi;\ c = 1;\ f[x\_] = 1 - \frac{x}{\pi};$

$A0 = \frac{1}{L}\text{Integrate}[f[x], \{x, 0, L\}]$

$\frac{1}{2}$

$An = \frac{2}{L}\int_0^L f[x]\,\text{Cos}\left[\frac{n\,\pi\,x}{L}\right]dx$

$\frac{2\,(1 - \text{Cos}[n\,\pi])}{n^2\,\pi^2}$

In the above cell I lost my x, but it’s okay.

```
bui = An (Cos[n π x/π] e^(-λ^2 t))
```

$$\frac{2\, e^{-t\,\lambda^2}\,(1 - \mathrm{Cos}[n\,\pi])\,\mathrm{Cos}[n\,x]}{n^2\,\pi^2}$$

The above cell follows the form of numbered line (12) on p. 563 of text.

```
bui2 = bui /. Cos[n π] → (-1)^n
```

$$\frac{2\left(1 + (-1)^{1+n}\right) e^{-t\,\lambda^2}\,\mathrm{Cos}[n\,x]}{n^2\,\pi^2}$$

The substitution in the above cell is useful.

```
bui3 = bui2 /. λ → c n π/L
```

$$\frac{2\left(1 + (-1)^{1+n}\right) e^{-n^2\,t}\,\mathrm{Cos}[n\,x]}{n^2\,\pi^2}$$

The substitution in the above cell makes things more explicit.

```
bui4[n_, N_] = Sum[2 (1 + (-1)^(1+n)) e^(-n^2 t) Cos[n x]/(n^2 π^2), {n, 1, N}]
```

$$\sum_{n=1}^{N} \frac{2\left(1 + (-1)^{1+n}\right) e^{-n^2\,t}\,\mathrm{Cos}[n\,x]}{n^2\,\pi^2}$$

The above cell is looking good, but it took an unexpectedly long time to calculate.

```
bui4[n, 4]
```

$$\frac{4\, e^{-t}\,\mathrm{Cos}[x]}{\pi^2} + \frac{4\, e^{-9\,t}\,\mathrm{Cos}[3\,x]}{9\,\pi^2}$$

```
u[x_, t_] = A0 + bui4[n, 5]
```

$$\frac{1}{2} + \frac{4\, e^{-t}\,\mathrm{Cos}[x]}{\pi^2} + \frac{4\, e^{-9\,t}\,\mathrm{Cos}[3\,x]}{9\,\pi^2} + \frac{4\, e^{-25\,t}\,\mathrm{Cos}[5\,x]}{25\,\pi^2}$$

The above cell matches the answer in the text.

17. The heat flux of a solution u(x,t) across x = 0 is defined by $\phi(t) = -\mathrm{K}u_x(0, t)$. Find $\phi(t)$ for the solution (9).

After looking at this for awhile, I believe the ‘solution (9)’ does not refer to problem 9, but to numbered line (9) on p. 560. This is where it says that $u(x, t) = \sum_{n=0}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda^2 t}$

```
Clear["Global`*"]
```

First build the static armature of u(x,t):

```
u[x_, t_] = Bn Sin[n π x / L] e^(-λ^2 t)
```

$Bn\, e^{-t \lambda^2} \operatorname{Sin}\left[\frac{n \pi x}{L}\right]$

I am informed that the desired function ϕ(t) is based on the x-derivative of u(x,t):

```
fir = D[u[x, t], x]
```

$$\frac{Bn\, e^{-t \lambda^2} n \pi \operatorname{Cos}\left[\frac{n \pi x}{L}\right]}{L}$$

...and is decorated with a -K factor:

```
firk = -K fir
```

$$-\frac{Bn\, e^{-t \lambda^2} K n \pi \operatorname{Cos}\left[\frac{n \pi x}{L}\right]}{L}$$

a substitution simplifies things:

```
firk1 = firk /. Cos[n π x / L] → 1
```

$$-\frac{Bn\, e^{-t \lambda^2} K n \pi}{L}$$

and now to cast as a summation, pulling out from the summation symbol all which can be pulled out:

```
firk2[n_, N_] = -(K π / L) Sum[Bn e^(-t λ^2) n , {n, 1, N}];
```

The above cell matches the text answer.

18 - 25 Two-Dimensional Problems

19. Find the potential in the square 0≤x≤2, 0≤y≤2 if the upper side is kept at the potential 1000 sin$\left(\frac{1}{2}\pi x\right)$ and the other sides are grounded.

This problem was worked after problem 21, which is used as a guide for it. See that problem for further comments.

```
Clear["Global`*"]
```

```
brus =
  Simplify[(1 / bigF[x]) D[bigF[x], {x, 2}] == (-1 / bigG[y]) D[bigG[y], {y, 2}] == -k]
```

$$\frac{\text{bigF}''[x]}{\text{bigF}[x]} == -\frac{\text{bigG}''[y]}{\text{bigG}[y]} == -k$$

I take grounded side as analogous to the 0 degC sides on the heated plate. The size of the plate is different than in problem 21, and the function for the top edge is different.

```
brusF = {bigF''[x] + k bigF[x] == 0}
{k bigF[x] + bigF''[x] == 0}

brusF2 = DSolve[{brusF, bigF[0] == 0, bigF[a] == 0}, bigF, x]
```

$$\left\{\left\{\text{bigF} \to \text{Function}\left[\{x\},\ \begin{cases} C[1]\,\text{Sin}\left[\sqrt{k}\ x\right] & n \in \text{Integers}\ \&\&\ n \geq 1\ \&\&\ k == \frac{n^2 \pi^2}{a^2}\ \&\&\ a > 0 \\ 0 & \text{True} \end{cases}\right]\right\}\right\}$$

$$k = \left(\frac{n\,\pi}{a}\right)^2$$

$$\frac{n^2 \pi^2}{a^2}$$

$$\text{bigF2}[x\_,\ n\_,\ N\_] = \text{Sum}\left[\text{Sin}\left[\frac{n\,\pi}{a}\,x\right],\ \{n,\ 1,\ N\}\right];$$

$$\text{brusG} = \text{bigG}''[y] - \left(\frac{\pi\,n}{a}\right)^2 \text{bigG}[y] == 0$$

$$-\frac{n^2 \pi^2\,\text{bigG}[y]}{a^2} + \text{bigG}''[y] == 0$$

```
brusG2 = DSolve[brusG, bigG, y]
```

$$\left\{\left\{\text{bigG} \to \text{Function}\left[\{y\},\ e^{\frac{n \pi y}{a}}\,C[1] + e^{-\frac{n \pi y}{a}}\,C[2]\right]\right\}\right\}$$

$$\text{bigGN}[y\_] = A_n\left(e^{n \pi y/a} - e^{-n \pi y/a}\right)$$

$$\left(-e^{-\frac{n \pi y}{a}} + e^{\frac{n \pi y}{a}}\right) A_n$$

```
FullSimplify[bigGN[y]]
```

$$2\,\text{Sinh}\left[\frac{n\,\pi\,y}{a}\right] A_n$$

$$\text{uN}[x\_,\ y\_] = \text{Simplify}\left[\text{Sin}\left[\frac{n\,\pi}{a}\,x\right]\ \text{Sinh}\left[\frac{n\,\pi\,y}{a}\right]\ \text{aN2}\right]$$

$$\text{aN2}\ \text{Sin}\left[\frac{n\,\pi\,x}{a}\right]\ \text{Sinh}\left[\frac{n\,\pi\,y}{a}\right]$$

$$u[x\_,\ b\_,\ N\_] = \text{Sum}\left[\text{aN2}\ \text{Sin}\left[\frac{n\,\pi\,x}{a}\right]\ \text{Sinh}\left[\frac{n\,\pi\,b}{a}\right],\ \{n,\ 1,\ N\}\right];$$

$$\text{uf}[x\_,\ b\_,\ N\_] = \sum_{n=1}^{N}\left(\text{aN2}\ \text{Sinh}\left[\frac{n\,\pi\,b}{a}\right]\right)\ \text{Sin}\left[\frac{n\,\pi\,x}{a}\right];$$

```
aN2 = 2/(a Sinh[n π b / a]) ∫_0^a f[x] Sin[n π x / a] dx
```

$$\frac{2\,\text{Csch}\left[\frac{b\,n\,\pi}{a}\right]\int_0^a f[x]\,\text{Sin}\left[\frac{n\,\pi\,x}{a}\right]dx}{a}$$

```
aN2F = FullSimplify[aN2 /. {a → 2, b → 2, f[x] → 1000 Sin[π x / 2]}]
```

$$\frac{2000\,\text{Csch}[n\,\pi]\,\text{Sin}[n\,\pi]}{\pi - n^2\,\pi}$$

```
ufF = Sum[aN2F Sin[n π x / 2] Sinh[n π y / 2], {n, 1, ∞, 2}]
```

$$1000\,\text{Csch}[\pi]\,\text{Sin}\left[\frac{\pi\,x}{2}\right]\,\text{Sinh}\left[\frac{\pi\,y}{2}\right]$$

The green cell above agrees with the answer in the text. As the series converges, the n factor disappears.

21. Heat flow in a plate. The faces of a thin square plate with sides a=24 are perfectly insulated. The upper side is kept at 25 deg-C and the other sides are kept at 0 deg-C. Find the steady-state temperature u(x,y) in the plate.

```
Clear["Global`*"]
```

From the top of p. 565 the expressions can be found:

```
crus =
 Simplify[1/bigF[x] D[bigF[x], {x, 2}] == -1/bigG[y] D[bigG[y], {y, 2}] == -k]
```

$$\frac{\text{bigF}''[x]}{\text{bigF}[x]} == -\frac{\text{bigG}''[y]}{\text{bigG}[y]} == -k$$

The bigF function will be associated with the left and right edges; the bigG function will be associated with the bottom edge. The top edge will be calculated separately.

With two equal signs, I can deal with subsets of the terms if I wish. Thus I can write:

```
crusF = {bigF''[x] + k bigF[x] == 0}
```

$$\{k\,\text{bigF}[x] + \text{bigF}''[x] == 0\}$$

The left and right boundary conditions of the problem imply that bigF(0)=0 and bigF(a) =0.

```
crusF2 = DSolve[{crusF, bigF[0] == 0, bigF[a] == 0}, bigF, x]
```

$$\left\{\left\{\text{bigF} \to \text{Function}\left[\{x\}, \begin{cases} C[1]\,\text{Sin}\left[\sqrt{k}\;x\right] & \mathbb{n} \in \text{Integers}\;\&\&\;\mathbb{n} \geq 1\;\&\&\;k == \frac{\mathbb{n}^2 \pi^2}{a^2}\;\&\&\;a > 0 \\ 0 & \text{True} \end{cases}\right]\right\}\right\}$$

If I take the formal n without its dots, I then have:

$$k = \left(\frac{n\,\pi}{a}\right)^2$$

$$\frac{n^2\,\pi^2}{a^2}$$

Also, taking C[1] to be 1, the bigF function is now a series in n:

$$\text{bigF2}[x\_,\, n\_,\, N\_] = \text{Sum}\left[\text{Sin}\left[\frac{n\,\pi}{a}\,x\right],\, \{n,\, 1,\, N\}\right];$$

Having found a value for k, I can take it back to the triple equal sign expression and start looking for bigG:

$$\frac{\text{bigG}''[y]}{\text{bigG}[y]} == -k$$

$$\text{crusG} = \text{bigG}''[y] - \left(\frac{\pi\,n}{a}\right)^2 \text{bigG}[y] == 0$$

$$-\frac{n^2\,\pi^2\,\text{bigG}[y]}{a^2} + \text{bigG}''[y] == 0$$

```
crusG2 = DSolve[crusG, bigG, y]
```

$$\left\{\left\{\text{bigG} \to \text{Function}\left[\{y\},\, e^{\frac{n \pi y}{a}}\, C[1] + e^{-\frac{n \pi y}{a}}\, C[2]\right]\right\}\right\}$$

The text points out that having the boundary condition on the bottom edge of the plate equal to zero implies that bigG(0)=0, which can only happen if B_n=- A_n. This gives:

$$\text{bigGN}[y\_] = A_n\left(e^{n \pi y/a} - e^{-n \pi y/a}\right)$$

$$\left(-e^{-\frac{n \pi y}{a}} + e^{\frac{n \pi y}{a}}\right) A_n$$

The expression of the coefficient looks familiar:

```
FullSimplify[bigGN[y]]
```

$$2\,\text{Sinh}\left[\frac{n\,\pi\,y}{a}\right] A_n$$

Looking back at bigF2, the text reminds me that this is close to revealing the eigenfunctions of the problem, and all that is needed is to combine the A_n and the 2 in bigGN, and *m*ultiply it by bigF2 to get the eigenfunctions:

```
uN[x_, y_] = Simplify[Sin[n π/a x] Sinh[n π y/a] aN2]
```

$$aN2\, Sin\left[\frac{n\,\pi\,x}{a}\right]\, Sinh\left[\frac{n\,\pi\,y}{a}\right]$$

The text states that these functions just described satisfy the zero boundary conditions on left, right, and bottom edges of the plate. As for the top edge, the function uN morphs into a new function:

$$u[x\_,\, b\_,\, N\_] = Sum\left[aN2\, Sin\left[\frac{n\,\pi\,x}{a}\right]\, Sinh\left[\frac{n\,\pi\,b}{a}\right],\, \{n,\, 1,\, N\}\right];$$

Or, choosing a slightly different form:

$$uf[x\_,\, b\_,\, N\_] = \sum_{n=1}^{N}\left(aN2\, Sinh\left[\frac{n\,\pi\,b}{a}\right]\right)\, Sin\left[\frac{n\,\pi\,x}{a}\right];$$

In the above form, the text wants me to recognize that the expression in parenthesis is the Fourier coefficient a_n of the function f(x). And so the main task remaining is to evaluate this a_n, amounting to the evaluation:

$$aN2 = \frac{2}{a\, Sinh[n\,\pi\,b\,/\,a]}\int_0^a f[x]\, Sin\left[\frac{n\,\pi\,x}{a}\right]\, dx$$

$$\frac{2\, Csch\left[\frac{b\,n\,\pi}{a}\right]\int_0^a f[x]\, Sin\left[\frac{n\,\pi\,x}{a}\right]\, dx}{a}$$

$$aN2F = FullSimplify[aN2\, /.\, \{a \to 24,\, b \to 24,\, f[x] \to 25\}]$$

$$-\frac{50\,(-1 + Cos[n\,\pi])\, Csch[n\,\pi]}{n\,\pi}$$

Mathematica cannot simplify the expression of aN2F further, but the s.m. does so admirably, finding that

$$aN2F = -\frac{50}{n\,\pi\, Sinh[n\,\pi]}\,((-1)^n - 1)$$

$$-\frac{50\,(-1 + (-1)^n)\, Csch[n\,\pi]}{n\,\pi}$$

So that, even more simply,

$$aN2F = \frac{100}{n\,\pi\, Sinh[n\,\pi]};\ (*\ n\ odd*)$$

And then placing the calculated aN2F into the following formula for u:

$$ufF = Sum\left[aN2F\, Sin\left[\frac{n\,\pi\,x}{24}\right]\, Sinh\left[\frac{n\,\pi\,y}{24}\right],\, \{n,\, 1,\, \infty,\, 2\}\right]$$

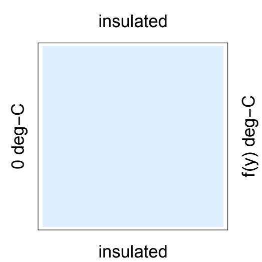
$$Sum\left[\frac{1}{n\,\pi}100\, Csch[n\,\pi]\, Sin\left[\frac{n\,\pi\,x}{24}\right]\, Sinh\left[\frac{n\,\pi\,y}{24}\right],\, \{n,\, 1,\, \infty,\, 2\}\right]$$

Although the above is slightly different in form, it is equivalent to the text answer, only expressed in Mathematica's preferred style.

23. Mixed boundary value problem. Find the steady-state temperature in the plate in problem 21 with the upper and lower sides perfectly insulated, the left side kept at 0 deg-C, and the right side kept at f(y) deg-C.

The rectangle for this problem is the same as in problem 21. The fact that the heat is emitting from the right edge, in the horizontal direction, instead of upward, vertically from the top edge, changes everything about the problem. I got lost trying to transpose the details in the text, as shown in problem 21. Instead, I found that *https : // www.math.tamu.edu/~yvorobet/Math412/Home4solved.pdf,* pertaining to a rectangle, uses the same process, except that all x and y occurrences need to be swapped. The a_0 and a_n factors in the text disagree with the use of b_0 and b_n in yvorobet, but I tend to go along with the latter. The problem is problem 1, the first one on the first page.

```
Graphics[{LightBlue, Rectangle[{1, 1}]},
 ImageSize → 130, FrameTicks → False, Frame → True,
 FrameLabel → {{"0 deg-C", "f(y) deg-C"}, {insulated, insulated}}]
```




```
Clear["Global`*"]
```

Showing only the answer. The problem is fully worked in the source.

$$u[x\_, y\_] = b_0 \frac{x}{L} + \sum_{n=1}^{\infty} b_n \left(\text{Sinh}\left[\frac{n \pi L}{H}\right]\right)^{-1} \text{Sinh}\left[\frac{n \pi x}{H}\right] \text{Cos}\left[\frac{n \pi y}{H}\right]$$

where

$$b_0 + \sum_{n=1}^{\infty} b_n \text{Cos}\left[\frac{n \pi y}{H}\right]$$

$$\frac{\int_0^H f[y]\, dy}{H} + \sum_{n=1}^{\infty} \frac{2\, \text{Cos}\left[\frac{n \pi y}{H}\right] \int_0^H \text{Cos}\left[\frac{n \pi y}{H}\right] f[y]\, dy}{H}$$

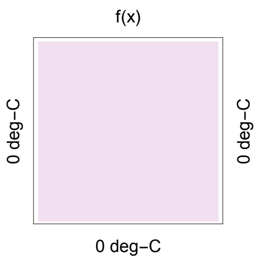
is the Fourier cosine series of the function f(y) on [0,H], that is,

$$b_0 = \frac{1}{H} \int_0^H f[y]\, dy;$$

$$b_n = \frac{2}{H} \int_0^H f[y]\, Cos\left[\frac{n \pi y}{H}\right] dy$$

The green cell matches the text answers for a_n. There is an anomaly in text a_0 versus yvorobet b_0. In the text, an extra H (or L) is included in the formula for a_0 (denominator), whereas in yvorobet this shows up as a divisor in the first term of u(x,y). So the formulas for u(x,y) agree. This discrepancy is noted by the yellow instead of green cells above.

```
Graphics[{LightPurple, Rectangle[{1, 1}]},
 ImageSize → 130, FrameTicks → False, Frame → True,
 FrameLabel → {{"0 deg-C", "0 deg-C"}, {"0 deg-C", "f(x)"}}]
```



Just to show the exact circumstance of the online resource, yvorobet, the problem is depicted and repeated here.

$$u[x\_, y\_] = b_0 \frac{y}{H} + \sum_{n=1}^{\infty} b_n \left(Sinh\left[\frac{n \pi H}{L}\right]\right)^{-1} Sinh\left[\frac{n \pi y}{L}\right] Cos\left[\frac{n \pi x}{L}\right];$$

where

$$b_0 + \sum_{n=1}^{\infty} b_n\, Cos\left[\frac{n \pi x}{L}\right]$$

is the Fourier cosine series of the function f(x) on [0,L], that is,

$$b_0 = \frac{1}{L} \int_0^L f[x]\, dx;$$

$$b_n = \frac{2}{L} \int_0^L f[x]\, Cos\left[\frac{n \pi x}{L}\right] dx$$