

1 - 10 Type and stability of critical point

Determine the type and stability of the critical point. Then find a real general solution and sketch or graph some of the trajectories in the phase plane.

I'm going to need to bring Tables 4.1

Name	$p=\lambda_1+\lambda_2$	$q=\lambda_1\lambda_2$	$\Delta=(\lambda_1-\lambda_2)^2$	Comments on λ_1, λ_2
(a) Node		$q>0$	$\Delta\geq 0$	Real, same sign
(b) Saddle point		$q<0$		Real, opposite signs
(c) Center	$p=0$	$q>0$		Pure imaginary
(d) Spiral point	$p\neq 0$		$\Delta<0$	Complex, not pure imaginary

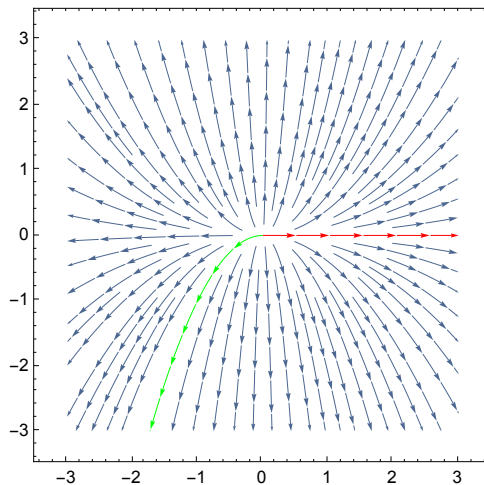
and 4.2 in here for consultation.

Type of Stability	$p=\lambda_1+\lambda_2$	$q=\lambda_1\lambda_2$
(a) Stable and attractive	$p<0$	$q>0$
(b) Stable	$p\leq 0$	$q>0$
(c) Unstable	$p>0$ OR	OR $q<0$

$$1. y_1' = y_1$$

$$y_2' = 2 y_2$$

```
StreamPlot[{y1, 2 y2}, {y1, -3, 3}, {y2, -3, 3}, StreamPoints ->
  {{{{1, 0}, Red}, {{-1, -1}, Green}, Automatic}}, ImageSize -> 250]
```



```
ClearAll["Global`*"]
```

```
e1 = {y1'[t] == y1[t], y2'[t] == 2 y2[t]}
```

```
e2 = DSolve[e1, {y1, y2}, t]
```

```
{y1'[t] == y1[t], y2'[t] == 2 y2[t]}
```

```
{ {y1 -> Function[{t}, e^t C[1]], y2 -> Function[{t}, e^2 t C[2]] }
```

1. Above: the general, real sol'ns.

```
te = e2[[1, 1, 2, 2]]
```

```
e^t C[1]
```

The solution for y1, below, matches the text.

```
fe = te /. C[1] -> c1
```

```
c1 e^t
```

```
e3 = Eigensystem[{{1, 0}, {0, 2}}]
```

```
{{2, 1}, {{0, 1}, {1, 0}}}
```

```
 $\lambda_1 = 2$ 
```

```
2
```

```
 $\lambda_2 = 1$ 
```

```
1
```

```
p =  $\lambda_1 + \lambda_2$ 
```

```
3
```

```
q =  $\lambda_1 \lambda_2$ 
```

```
2
```

```
 $\Delta = (\lambda_1 - \lambda_2)^2$ 
```

```
1
```

1. Because $p > 0$, the critical point is unstable according to Table 4-2.

```
TableForm[Table[{t, c1, fe}, {t, 4}, {c1, -1, 1}],  
TableHeadings -> {{}, {"t", "c1 ", "fe "}}]
```

t	c1	fe
1	1	1
-1	0	1
-e	0	e
2	2	2
-1	0	1
-e ²	0	e ²
3	3	3
-1	0	1
-e ³	0	e ³
4	4	4
-1	0	1
-e ⁴	0	e ⁴

```
fifo = Table[{t, fe}, {t, 4}, {c1, -1, 1}]
{{{1, -e}, {1, 0}, {1, e}}, {{2, -e^2}, {2, 0}, {2, e^2}},
 {{3, -e^3}, {3, 0}, {3, e^3}}, {{4, -e^4}, {4, 0}, {4, e^4}}}

hiu[c1_, t_] := fe

plot1 = Plot[Evaluate[Table[hiu[c1, t], {c1, -8, 8}]],
 {t, -3, 3}, PlotRange → {-50, 50}, PlotStyle → Thickness[0.003]];
```

3. Above: This is a plot of the first sol'n, with trajectories of various constant values.

```
f[c1_, t_] := c1 e^t

VectorPlot[{1, f[c1, t]}, {c1, -3, 3}, {t, -1, 1},
 Axes → True, Frame → False, VectorScale → {Tiny, Tiny, None},
 BaseStyle → AbsoluteThickness[0.4], PlotTheme → None, ImageSize → 250];

plot2 = VectorPlot[{1, f[t, c1]}, {c1, -3, 3}, {t, -8, 8},
 Axes → True, Frame → False, VectorScale → {Tiny, Tiny, None},
 BaseStyle → AbsoluteThickness[0.4], PlotTheme → None, ImageSize → 350];

Show[plot1, plot2];

fi = e2[[1, 2, 2, 2]]
e^2 t C[2]
```

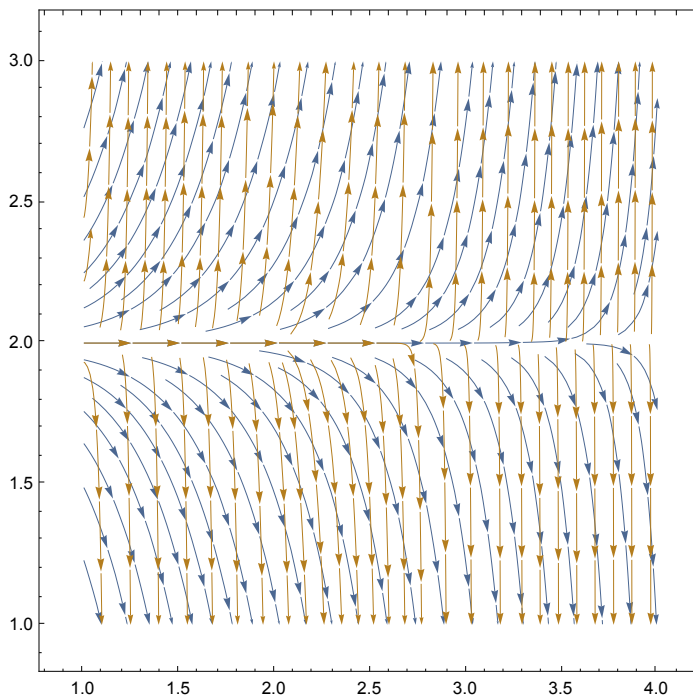
The solution for y2, below, agrees with the text.

```
fif = fi /. C[2] → c2
```

```
c2 e^2 t
```

```
fifi = Table[{t, fif}, {t, 4}, {c2, -1, 1}]
{{{1, -e^2}, {1, 0}, {1, e^2}}, {{2, -e^4}, {2, 0}, {2, e^4}},
 {{3, -e^6}, {3, 0}, {3, e^6}}, {{4, -e^8}, {4, 0}, {4, e^8}}}
```

```
ListStreamPlot[{f1fo, f1fi}]
```



```
3. y1' = y2
   y2' = -9 y1
```

```
ClearAll["Global`*"]
```

```
e1 = {y1'[t] == y2[t], y2'[t] == -9 y1[t]}
```

```
e2 = DSolve[e1, {y1, y2}, t]
```

```
{y1'[t] == y2[t], y2'[t] == -9 y1[t]}
```

```
{ {y1 -> Function[{t}, C[1] Cos[3 t] + 1/3 C[2] Sin[3 t]],
    y2 -> Function[{t}, C[2] Cos[3 t] - 3 C[1] Sin[3 t]] } }
```

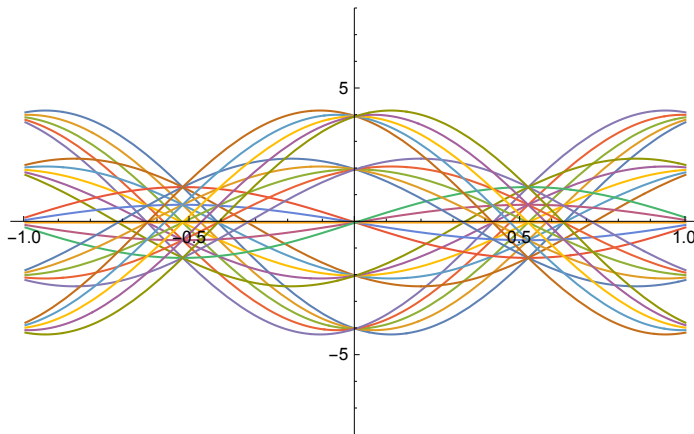
```
e3 = e2[[1, 1, 2, 2]]
```

```
C[1] Cos[3 t] + 1/3 C[2] Sin[3 t]
```

The solution for y_1 , below, agrees with the text, provided that text constant A is assigned the value of $C[1]$, and text constant B is assigned the value of $\frac{1}{3}C[2]$.

```
hiy[c1_, c2_, t_] := c1 Cos[3 t] + 1/3 c2 Sin[3 t]
```

```
plot1 =
  Plot[Evaluate[Table[h1y[c1, c2, t], {c1, -4, 4, 2}, {c2, -4, 4, 2}]],
    {t, -1, 1}, PlotRange → {-8, 8}, PlotStyle → Thickness[0.003]]
```



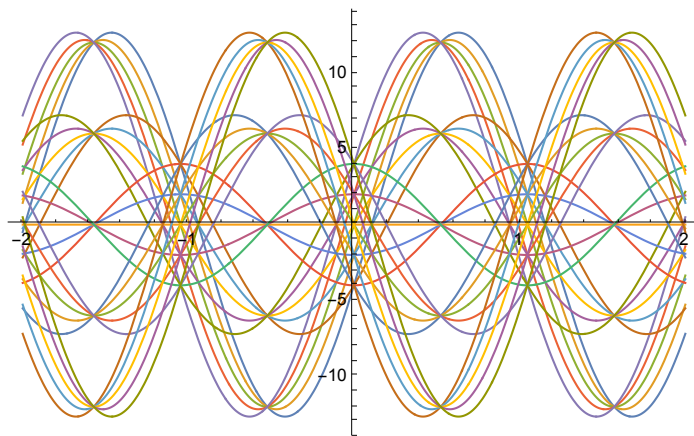
1. Above: Some trajectories of the first sol'n. Below: the solution for y_2 agrees with the text, with appropriate constant assignments.

```
e4 = e2[[1, 2, 2, 2]]
```

```
C[2] Cos[3 t] - 3 C[1] Sin[3 t]
```

```
hiz[c1_, c2_, t_] := c2 Cos[3 t] - 3 c1 Sin[3 t]
```

```
plot1 =
  Plot[Evaluate[Table[hiz[c1, c2, t], {c1, -4, 4, 2}, {c2, -4, 4, 2}]],
    {t, -2, 2}, PlotRange → Automatic, PlotStyle → Thickness[0.003]]
```



2. Above: Some trajectories of the second sol'n.

```
e5 = Eigensystem[{{0, 1}, {-9, 0}}]
```

```
{{3 i, -3 i}, {{-i, 3}, {i, 3}}}
```

$$p = 3i - 3i$$

$$0$$

$$q = 3i(-3i)$$

$$9$$

$$\Delta = (3i - (-3i))^2$$

$$-36$$

3. The system's critical point is center. According to Table 4-2, it is stable.

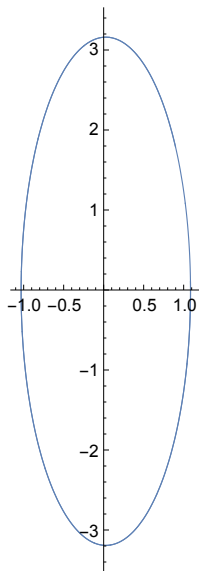
$$e3p = e3 /. \{C[1] \rightarrow 1, C[2] \rightarrow 1\}$$

$$\cos[3t] + \frac{1}{3}\sin[3t]$$

$$e4p = e4 /. \{C[1] \rightarrow 1, C[2] \rightarrow 1\}$$

$$\cos[3t] - 3\sin[3t]$$

```
ParametricPlot[{e3p, e4p}, {t, -2, 2},
  ImageSize -> 100, PlotStyle -> Thickness[0.006]]
```



$$5. \quad y_1' = -2y_1 + 2y_2$$

$$y_2' = -2y_1 - 2y_2$$

```
ClearAll["Global`*"]
```

```

e1 = {y1'[t] == -2 y1[t] + 2 y2[t], y2'[t] == -2 y1[t] - 2 y2[t]}
e2 = DSolve[e1, {y1, y2}, t]
{y1'[t] == -2 y1[t] + 2 y2[t], y2'[t] == -2 y1[t] - 2 y2[t]}

{{y1 -> Function[{t}, e-2 t C[1] Cos[2 t] + e-2 t C[2] Sin[2 t]],
  y2 -> Function[{t}, e-2 t C[2] Cos[2 t] - e-2 t C[1] Sin[2 t]]}}

e3 = e2[[1, 1, 2, 2]]
e-2 t C[1] Cos[2 t] + e-2 t C[2] Sin[2 t]

```

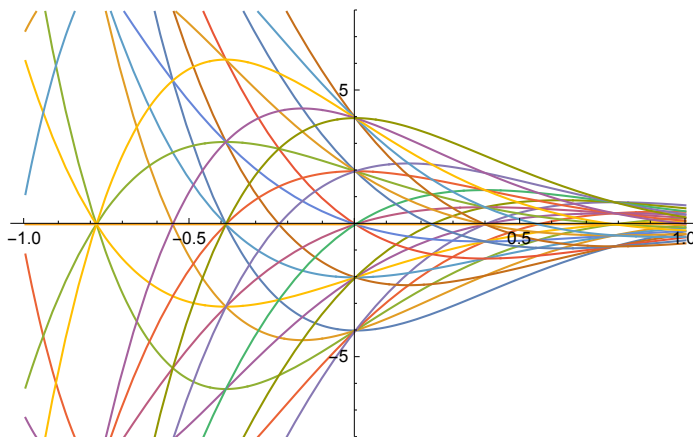
```
hiy[c1_, c2_, t_] := e-2 t c1 Cos[2 t] + e-2 t c2 Sin[2 t]
```

Above: The green cell matches the answer in the text for y_1 , assuming appropriate assignment of constants.

```

plot1 =
Plot[Evaluate[Table[hiy[c1, c2, t], {c1, -4, 4, 2}, {c2, -4, 4, 2}]],
{t, -1, 1}, PlotRange -> {-8, 8}, PlotStyle -> Thickness[0.003]]

```



```

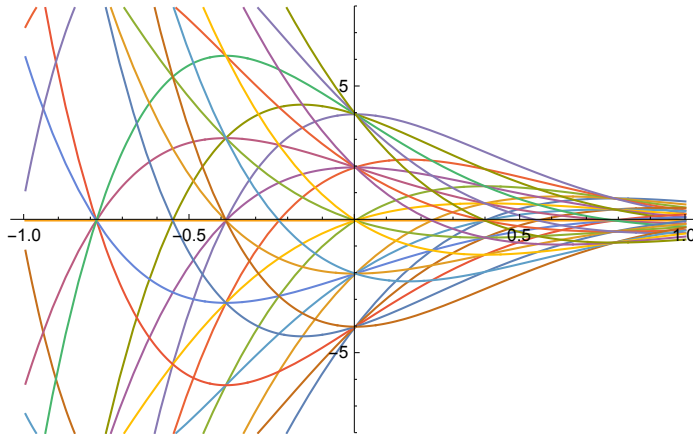
e4 = e2[[1, 2, 2, 2]]
e-2 t C[2] Cos[2 t] - e-2 t C[1] Sin[2 t]

```

```
hiz[c1_, c2_, t_] := e-2 t c2 Cos[2 t] - e-2 t c1 Sin[2 t]
```

Above: The green cell matches the answer in the text for y_2 , assuming appropriate assignment of constants.

```
plot2 =
  Plot[Evaluate[Table[hiz[c1, c2, t], {c1, -4, 4, 2}, {c2, -4, 4, 2}]],
    {t, -1, 1}, PlotRange → {-8, 8}, PlotStyle → Thickness[0.003]]
```



```
e5 = Eigensystem[{{-2, 2}, {-2, -2}}]
{{-2 + 2 i, -2 - 2 i}, {{-i, 1}, {i, 1}}}
```

```
p = -2 + 2 i + (-2 - 2 i)
```

```
-4
```

```
q = -2 + 2 i (-2 - 2 i)
```

```
2 - 4 i
```

```
Δ = ((-2 + 2 i) - (-2 - 2 i))^2
```

```
-16
```

According to Table 4-1, the critical point is a spiral point. If the presence of imaginary part does not matter, the point is stable.

```
7. y1' = y1 + 2 y2
y2' = 2 y1 + y2
```

```
ClearAll["Global`*"]
```

```
e1 = {y1'[t] == y1[t] + 2 y2[t], y2'[t] == 2 y1[t] + y2[t]}
```

```
e2 = DSolve[e1, {y1, y2}, t]
```

```
{y1'[t] == y1[t] + 2 y2[t], y2'[t] == 2 y1[t] + y2[t]}
```

```
{ {y1 → Function[{t}, 1/2 e^-t (1 + e^4 t) C[1] + 1/2 e^-t (-1 + e^4 t) C[2]],
  y2 → Function[{t}, 1/2 e^-t (-1 + e^4 t) C[1] + 1/2 e^-t (1 + e^4 t) C[2]] }
```



```
e3 = e2[[1, 1, 2, 2]]
```

$$\frac{1}{2} e^{-t} (1 + e^{4t}) C[1] + \frac{1}{2} e^{-t} (-1 + e^{4t}) C[2]$$

```
e5 = Expand[e3]
```

$$\frac{1}{2} e^{-t} C[1] + \frac{1}{2} e^{3t} C[1] - \frac{1}{2} e^{-t} C[2] + \frac{1}{2} e^{3t} C[2]$$

```
e6 = Collect[e5, e^{3t}]
```

$$e^{-t} \left(\frac{C[1]}{2} - \frac{C[2]}{2} \right) + e^{3t} \left(\frac{C[1]}{2} + \frac{C[2]}{2} \right)$$

$$e7 = e6 /. \left\{ \left(\frac{C[1]}{2} - \frac{C[2]}{2} \right) \rightarrow c1, \left(\frac{C[1]}{2} + \frac{C[2]}{2} \right) \rightarrow c2 \right\}$$

$$c1 e^{-t} + c2 e^{3t}$$

Above: y1, matching the text answer.

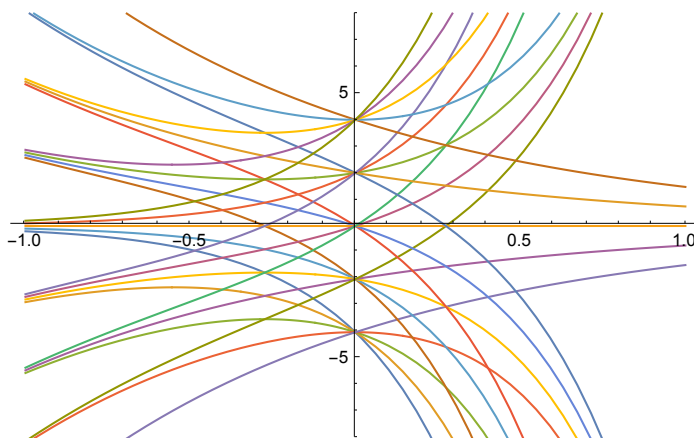
$$\text{Solve} \left[\left(\frac{C[1]}{2} - \frac{C[2]}{2} \right) == c1 \ \&\& \ \left(\frac{C[1]}{2} + \frac{C[2]}{2} \right) == c2, \{c1, c2\} \right]$$

$$\left\{ \left\{ c1 \rightarrow \frac{1}{2} (C[1] - C[2]), c2 \rightarrow \frac{1}{2} (C[1] + C[2]) \right\} \right\}$$

$$\text{hiy}[c1_, c2_, t_] := \frac{1}{2} e^{-t} (1 + e^{4t}) c1 + \frac{1}{2} e^{-t} (-1 + e^{4t}) c2$$

```
plot1 =
```

```
Plot[Evaluate[Table[hiy[c1, c2, t], {c1, -4, 4, 2}, {c2, -4, 4, 2}]],  
{t, -1, 1}, PlotRange -> {-8, 8}, PlotStyle -> Thickness[0.003]]
```



```
e4 = e2[[1, 2, 2, 2]]
```

$$\frac{1}{2} e^{-t} (-1 + e^{4t}) C[1] + \frac{1}{2} e^{-t} (1 + e^{4t}) C[2]$$

```
e8 = Expand[e4]
```

$$-\frac{1}{2}e^{-t}C[1] + \frac{1}{2}e^{3t}C[1] + \frac{1}{2}e^{-t}C[2] + \frac{1}{2}e^{3t}C[2]$$

```
e9 = Collect[e8, e^3 t]
```

$$e^{-t} \left(-\frac{C[1]}{2} + \frac{C[2]}{2} \right) + e^{3t} \left(\frac{C[1]}{2} + \frac{C[2]}{2} \right)$$

$$e10 = e9 /. \left\{ \left(-\frac{C[1]}{2} + \frac{C[2]}{2} \right) \rightarrow -c1, \left(\frac{C[1]}{2} + \frac{C[2]}{2} \right) \rightarrow c2 \right\}$$

$$-c1 e^{-t} + c2 e^{3t}$$

Above: y2, matching the text answer.

$$\text{Solve} \left[\left(-\frac{C[1]}{2} + \frac{C[2]}{2} \right) == -c1 \ \&\& \ \left(\frac{C[1]}{2} + \frac{C[2]}{2} \right) == c2, \{c1, c2\} \right]$$

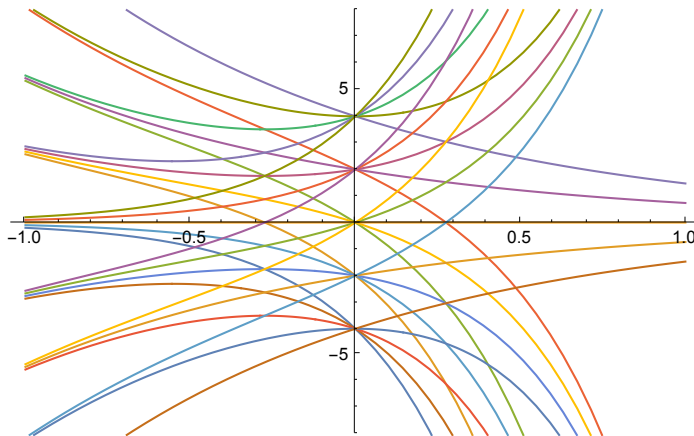
$$\left\{ \left\{ c1 \rightarrow \frac{1}{2} (C[1] - C[2]), c2 \rightarrow \frac{1}{2} (C[1] + C[2]) \right\} \right\}$$

The system of constants used in this problem is consistent with the text's, as demonstrated in pink cells.

$$\text{hiz}[c1_, c2_, t_] := \frac{1}{2}e^{-t}(-1 + e^{4t})c1 + \frac{1}{2}e^{-t}(1 + e^{4t})c2$$

```
plot2 =
```

```
Plot[Evaluate[Table[hiz[c1, c2, t], {c1, -4, 4, 2}, {c2, -4, 4, 2}]],  
{t, -1, 1}, PlotRange -> {-8, 8}, PlotStyle -> Thickness[0.003]]
```



```
Eigensystem[{{1, 2}, {2, 1}}]
```

```
{{3, -1}, {{1, 1}, {-1, 1}}}
```

$$p = 3 - 1$$

$$2$$

$$q = 3 - (-1)$$

$$-3$$

$$\Delta = (3 - (-1))^2$$

$$16$$

According to Table 4-1, the critical point is a saddle point. According to Table 4-2, it is unstable.

$$9. \ y_1' = 4 y_1 + y_2$$

$$y_2' = 4 y_1 + 4 y_2$$

```
ClearAll["Global`*"]
```

```
e1 = {y1'[t] == 4 y1[t] + y2[t], y2'[t] == 4 y1[t] + 4 y2[t]}
```

```
e2 = DSolve[e1, {y1, y2}, t]
```

```
{y1'[t] == 4 y1[t] + y2[t], y2'[t] == 4 y1[t] + 4 y2[t]}
```

```
{ {y1 -> Function[{t},  $\frac{1}{2} e^{2t} (1 + e^{4t}) C[1] + \frac{1}{4} e^{2t} (-1 + e^{4t}) C[2]$ ],  
  y2 -> Function[{t},  $e^{2t} (-1 + e^{4t}) C[1] + \frac{1}{2} e^{2t} (1 + e^{4t}) C[2]$ ] } }
```

```
e3 = e2[[1, 1, 2, 2]]
```

$$\frac{1}{2} e^{2t} (1 + e^{4t}) C[1] + \frac{1}{4} e^{2t} (-1 + e^{4t}) C[2]$$

```
e4 = Expand[e3]
```

$$\frac{1}{2} e^{2t} C[1] + \frac{1}{2} e^{6t} C[1] - \frac{1}{4} e^{2t} C[2] + \frac{1}{4} e^{6t} C[2]$$

```
e5 = Collect[e4, e^{6t}]
```

$$e^{2t} \left(\frac{C[1]}{2} - \frac{C[2]}{4} \right) + e^{6t} \left(\frac{C[1]}{2} + \frac{C[2]}{4} \right)$$

$$e6 = e5 /. \left\{ \left(\frac{C[1]}{2} - \frac{C[2]}{4} \right) \rightarrow c2, \left(\frac{C[1]}{2} + \frac{C[2]}{4} \right) \rightarrow c1 \right\}$$

$$c2 e^{2t} + c1 e^{6t}$$

Above: the text answer for y_1 .

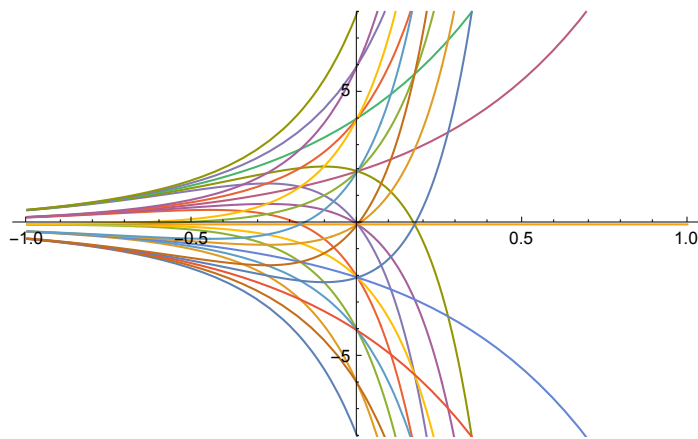
$$\text{Solve}\left[\left(\frac{C[1]}{2} - \frac{C[2]}{4}\right) == c2 \ \&\& \ \left(\frac{C[1]}{2} + \frac{C[2]}{4}\right) == c1, \{c1, c2\}\right]$$

$$\left\{\left\{c1 \rightarrow \frac{1}{4} (2 C[1] + C[2]), c2 \rightarrow \frac{1}{4} (2 C[1] - C[2])\right\}\right\}$$

$$e7[c1_, c2_, t_] := c2 e^{2t} + c1 e^{6t}$$

plot1 =

```
Plot[Evaluate[Table[e7[c1, c2, t], {c1, -4, 4, 2}, {c2, -4, 4, 2}]],
{t, -1, 1}, PlotRange -> {-8, 8}, PlotStyle -> Thickness[0.003]]
```



$$e8 = e2[[1, 2, 2, 2]]$$

$$e^{2t} (-1 + e^{4t}) C[1] + \frac{1}{2} e^{2t} (1 + e^{4t}) C[2]$$

$$e9 = \text{Expand}[e8]$$

$$-e^{2t} C[1] + e^{6t} C[1] + \frac{1}{2} e^{2t} C[2] + \frac{1}{2} e^{6t} C[2]$$

$$e10 = \text{Collect}[e9, e^{6t}]$$

$$e^{2t} \left(-C[1] + \frac{C[2]}{2}\right) + e^{6t} \left(C[1] + \frac{C[2]}{2}\right)$$

$$e11 = e10 /. \left\{\left(-C[1] + \frac{C[2]}{2}\right) \rightarrow -2 c2, \left(C[1] + \frac{C[2]}{2}\right) \rightarrow 2 c1\right\}$$

$$-2 c2 e^{2t} + 2 c1 e^{6t}$$

Above: the text answer for y_2 .

```
Solve[(-C[1] + C[2]/2) == -2 c2 && (C[1] + C[2]/2) == 2 c1, {c1, c2}]
```

```
{ {c1 -> 1/4 (2 C[1] + C[2]), c2 -> 1/4 (2 C[1] - C[2]) } }
```

```
Eigensystem[{ {4, 1}, {4, 4} }
```

```
{ {6, 2}, { {1, 2}, {-1, 2} } }
```

```
p = 6 + 2
```

```
8
```

```
q = 6 × 2
```

```
12
```

```
Δ = (6 - 2)2
```

```
16
```

According to Table 4.1, the critical point is a node. According to Table 4.2, it is unstable.

11 - 18 Trajectories of systems and second-order ODEs. Critical points.

11. Damped oscillations. Solve $y'' + 2y' + 2y = 0$. What kind of curves are the trajectories?

```
ClearAll["Global`*"]
```

```
eqn = y'[x] + 2 y'[x] + 2 y[x] == 0
```

```
2 y[x] + 2 y'[x] + y''[x] == 0
```

```
sol = DSolve[eqn, y, x]
```

```
{ {y -> Function[{x}, e-x C[2] Cos[x] + e-x C[1] Sin[x]] } }
```

The above green cell matches the answer in the text.

```
eqn /. sol // Simplify
```

```
{True}
```

In order to find the eigensystem, I need to make this equation into a system, using numbered lines (9) and (10) on p. 135. So I will have $y_1 = y$, and $y_2 = y'$, and $y_3 = y''$. And the arrangement will be adopted whereby $y_1' = y_2$, and $y_2' = y_3$. Going by the text examples, the rows of the system matrix will be formed of the coefficients of the equations (lhs) of y_1' and y_2' . This will be

$$y_1' = y_2 \quad \text{by definition}$$

$$y_2' = y_3 = y_2' = -2y_1 - 2y_2 \quad \text{by problem equation description}$$

What are the critical points? From the first expression, the first coordinate will be zero. From the second expression, the coordinates will be equal. This means that $\{0,0\}$ will be the only critical point.

```
A = ( 0  1
      -2 -2 )
{{0, 1}, {-2, -2}}

{vals, vecs} = Eigensystem[A]
{{-1 + I, -1 - I}, {-1 - I, 2}, {-1 + I, 2}}

p = vals[[1]] + vals[[2]]
-2

q = vals[[1]] * vals[[2]]
2

Δ = (vals[[1]] - vals[[2]])^2
-4
```

According to Table 4.1, the critical point is a spiral point, and according to Table 4.2 it is stable.

17. Perturbation. The system in example 4 in section 4.3, p. 144, has a center as its critical point. Replace each a_{jk} in example 4 by $a_{jk} + b$. Find values of b such that you get (a) a saddle point, (b) a stable and attractive node, (c) a stable and attractive spiral, (d) an unstable spiral, (e) an unstable node.

```
ClearAll["Global`*"]
```

The characteristic matrix for this problem, given in the example, is like this, (but without the added 'b' characters).

```
y' = ( 0 + b  1 + b
      -4 + b  0 + b )
{{b, 1 + b}, {-4 + b, b}}

beig =
Table[Eigenvalues[y'], {b, {-π, -e, -2, -1.5, -1, -0.3, 0.1, 3, π, 4}}];

Table[{ {beig[[n, 1]] + beig[[n, 2]]}, {beig[[n, 1]] * beig[[n, 2]]},
  {(beig[[n, 1]] - beig[[n, 2]])^2}}, {n, 1, 10}];
```

```

g2 = Grid[
  N[Table[{n - 6, beig[[n, 1]] + beig[[n, 2]], beig[[n, 1]] * beig[[n, 2]],
    (beig[[n, 1]] - beig[[n, 2]])^2}, {n, 1, 10}]], Frame → All];

g1 = Grid[{" n ", "      p      ", "      q      ", "      Δ      "},
  Frame → All];

Column[{g1, g2}]

```

n	p	q	Δ
-5.	-6.28319	-5.42478	61.1775
-4.	-5.43656	-4.15485	46.1756
-3.	-4.	-2.	24.
-2.	-3.	-0.5	11.
-1.	-2.	1.	0.
0.	-0.6 + 0. i	3.1 + 0. i	-12.04 + 0. i
1.	0.2 + 0. i	4.3 + 0. i	-17.16 + 0. i
2.	6.	13.	-16.
3.	6.28319 + 0. i	13.4248 + 0. i	-14.2207 + 0. i
4.	8.	16.	0.

The grid below identifies the 'n' number critical points with the required characteristics, based on Table 4.1 and 4.2.

-3	unstable saddle point
-1	stable and attrac node
0	stable and attrac spiral
2	unstable spiral
4	unstable node