Lots of buzz and energy about Frobenius and his method. Probably more than the average person can put up with. Problems 5, 7, 9, 11, 15, 17 and 19 were worked straightforwardly without resorting to Frobenius, (and 13 could have been).

The following example is found at *https : // wps.prenhall.com/wps/media/objects/884/905578/Assignment8_ 2 _V5.pdf.* It seems to work for certain equations, and I keep it here for general interest. It does not work on problem 9, however. Reading the included comments may give insight on where it can be expected to work.

```
Clear["Global`*"]
```

```
eqn = 2 x^2 (x + 1) y''[x] + 3 x (x + 1)^3 y'[x] - (1 - x^2) y[x] == 0;
```

The first step is to verify that x=0 is in fact a regular singular point of this equation. This is done by dividing through by the leading coefficient, which is $2\,x^2\,(x+1)$. This will put the equation in standard form.

$$y''[x] + \frac{3\,x\,(x+1)^3}{2\,x^2\,(x+1)}\,y'[x] - \frac{(1-x^2)}{2\,x^2\,(x+1)}\,y[x] == 0;$$

The standard form, shown in numbered line (1) on p. 180, includes a $\frac{1}{x}$sub-term for the y'[x] term and a $\frac{1}{x^2}$sub-term for the y[x] term. Thus of the remainder of the terms, called p[x] and q[x], the p[x] will be the coefficient for the y' and the q[x] will be the coefficient for the y.

$$q[x\_] = \text{Simplify}\left[-\frac{(1-x^2)}{2\,(x+1)}\right]$$

$$\frac{1}{2}\,(-1+x)$$

$$p[x\_] = \text{Simplify}\left[\frac{3\,(1+x)^2}{2}\right]$$

$$\frac{3}{2}\,(1+x)^2$$

Taking a good look at the two declarations above, I see that neither one has any x^{-m} factor in it. This may be necessary for the present example to work. Now to set the number of terms in the series.

```
n = 7;
```

Expand p[x] in a Maclaurin series

$$\text{pseries} = \text{Series}\left[\frac{p[x]}{x},\ \{x,\ 0,\ n\}\right]\ //\ \text{Normal}$$

$$3 + \frac{3}{2\,x} + \frac{3\,x}{2}$$

and also q[x]

```
qseries = Series[q[x]/x^2, {x, 0, n}] // Normal
```

$$-\frac{1}{2\,x^{2}}+\frac{1}{2\,x}$$

The first step in making a generic Frobenius series is the following

```
coeffs = Array[c, n, 0];
```

Another component is determined by

```
lotsofxpowers = Table[x^(r+j), {j, 0, 2 n}];
```

And a third necessary component comes from

```
xpowers = Table[x^(r+j), {j, 0, n - 1}];
```

Stirring and gelling the series is accomplished by

```
y = coeffs.xpowers;
```

The first derivative of the series will be needed

```
yprime = D[y, x]
```

$$r\,x^{-1+r}\,c[0]+(1+r)\,x^{r}\,c[1]+(2+r)\,x^{1+r}\,c[2]+(3+r)\,x^{2+r}\,c[3]+(4+r)\,x^{3+r}\,c[4]+(5+r)\,x^{4+r}\,c[5]+(6+r)\,x^{5+r}\,c[6]$$

Note above that the largest minus exponent in yprime is x^{-1}. Now to find the second derivative also

```
y2prime = D[y, {x, 2}]
```

$$(-1+r)\,r\,x^{-2+r}\,c[0]+r\,(1+r)\,x^{-1+r}\,c[1]+(1+r)\,(2+r)\,x^{r}\,c[2]+(2+r)\,(3+r)\,x^{1+r}\,c[3]+(3+r)\,(4+r)\,x^{2+r}\,c[4]+(4+r)\,(5+r)\,x^{3+r}\,c[5]+(5+r)\,(6+r)\,x^{4+r}\,c[6]$$

Note above that the largest minus exponent in y2prime is x^{-2}. The differential equation itself is the next to be created. Because of the limited presence of minus exponents on x, multiplying through by x^2 here normalizes the series so that the lowest order term appearing will be x^r.

```
x^2 * (y2prime + pseries * yprime + qseries * y) // Expand;
```

The series will adopt an appearance of order after executing the following

```
lhs = Collect[%, lotsofxpowers]
```

$$x^r\left(-\frac{c[0]}{2}+\frac{1}{2}r\,c[0]+r^2\,c[0]\right)+$$
$$x^{1+r}\left(\frac{c[0]}{2}+3\,r\,c[0]+c[1]+\frac{5}{2}r\,c[1]+r^2\,c[1]\right)+$$
$$x^{2+r}\left(\frac{3}{2}r\,c[0]+\frac{7\,c[1]}{2}+3\,r\,c[1]+\frac{9\,c[2]}{2}+\frac{9}{2}r\,c[2]+r^2\,c[2]\right)+$$
$$x^{3+r}\left(\frac{3\,c[1]}{2}+\frac{3}{2}r\,c[1]+\frac{13\,c[2]}{2}+3\,r\,c[2]+10\,c[3]+\frac{13}{2}r\,c[3]+r^2\,c[3]\right)+$$
$$x^{4+r}\left(3\,c[2]+\frac{3}{2}r\,c[2]+\frac{19\,c[3]}{2}+3\,r\,c[3]+\frac{35\,c[4]}{2}+\frac{17}{2}r\,c[4]+r^2\,c[4]\right)+$$
$$x^{5+r}\left(\frac{9\,c[3]}{2}+\frac{3}{2}r\,c[3]+\frac{25\,c[4]}{2}+3\,r\,c[4]+27\,c[5]+\frac{21}{2}r\,c[5]+r^2\,c[5]\right)+$$
$$x^{8+r}\left(9\,c[6]+\frac{3}{2}r\,c[6]\right)+x^{7+r}\left(\frac{15\,c[5]}{2}+\frac{3}{2}r\,c[5]+\frac{37\,c[6]}{2}+3\,r\,c[6]\right)+$$
$$x^{6+r}\left(6\,c[4]+\frac{3}{2}r\,c[4]+\frac{31\,c[5]}{2}+3\,r\,c[5]+\frac{77\,c[6]}{2}+\frac{25}{2}r\,c[6]+r^2\,c[6]\right)$$

Note above that **Collect** has ordered the terms so that the x^r term comes first. This may not always be the case, and if it is not, it will affect the following steps. A necessary preliminary step to finding the indicial roots is

```
firstcoeff = lhs[[1, 2]]
```

$$-\frac{c[0]}{2}+\frac{1}{2}r\,c[0]+r^2\,c[0]$$

Note two things about firstcoeff above. It contains elements of r, and it contains a single constant parameter, here c[0]. Both of these characteristics are needed to make the example work. To reveal the roots, set the above expression to zero and solve as follows.

```
indroots = Solve[firstcoeff == 0, r];
```

Order the roots by size

```
maxroot = Max[r /. indroots]
```

$$\frac{1}{2}$$

```
minroot = Min[r /. indroots]
```

$$-1$$

The next step is the first in a series of six steps to find the first solution, y_1.

```
lhsy1 = lhs /. r → maxroot;
```

To determine the values of the arbitrary coefficients, a preliminary step is

$$\text{y1series} = \frac{\text{lhsy1}}{x^{\text{maxroot}}}\ //\ \text{Distribute;}$$

followed by the next step, which makes a list of coefficients.

```
y1seriescoeffs = Take[CoefficientList[y1series, x], n - 1];
```

The higher-degree coefficients in terms of C[0] will be found by

```
y1coeffs = Solve[{y1seriescoeffs == 0, c[0] == 1}][[1]];
```

At this point the first solution, $y_{1,}$ can be created by

```
y = Drop[coeffs, -1].Drop[xpowers, -1]
```

$$x^r c[0] + x^{1+r} c[1] + x^{2+r} c[2] + x^{3+r} c[3] + x^{4+r} c[4] + x^{5+r} c[5]$$

followed by the final necessary step

```
y1 = Collect[y /. r → maxroot /. y1coeffs, x]
```

$$\sqrt{x} - \frac{4 x^{3/2}}{5} + \frac{13 x^{5/2}}{28} - \frac{134 x^{7/2}}{945} - \frac{2741 x^{9/2}}{332\,640} + \frac{20\,429 x^{11/2}}{772\,200}$$

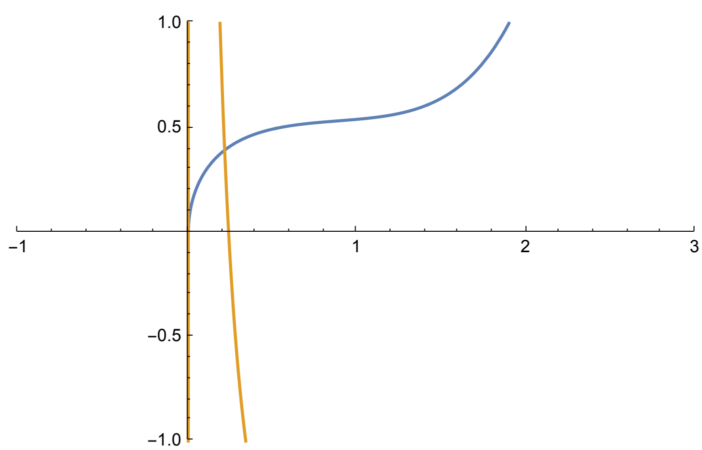
To find the second solution, y_2, requires a similar set of six steps.

```
lhsy2 = lhs /. r → minroot;
y2series = lhsy2 / x^minroot // Distribute;
y2seriescoeffs = Take[CoefficientList[y2series, x], n - 1];
y2coeffs = Solve[{y2seriescoeffs == 0, c[0] == 1}][[1]];
y = Drop[coeffs, -1].Drop[xpowers, -1];
y2 = Collect[y /. r → minroot /. y2coeffs, x]
```

$$-5 + \frac{1}{x} + 4 x - \frac{28 x^2}{9} + \frac{64 x^3}{45} - \frac{376 x^4}{1575}$$

And now the two solutions, y_1 and y_2, are plotted. Note that the roots to the equations are not those found above. Those roots were indicial roots, not roots to the actual ODE solution equation.

```
Plot[{y1, y2}, {x, -1, 3}, PlotRange → {{-1, 3}, {-1, 1}}]
```



If I want to find the actual roots to the two solutions, I can.

```
FindRoot[y1, {x, 0}]
{x → 0.}
```

```
FindRoot[y2, {x, 0.2}]
{x → 0.237723}
```

If I task WolframAlpha with the problem, it tells me it is Sturm-Liouville, but doesn't solve it, only gives some possible plots. Taking this as a cue, I could do (for y1)

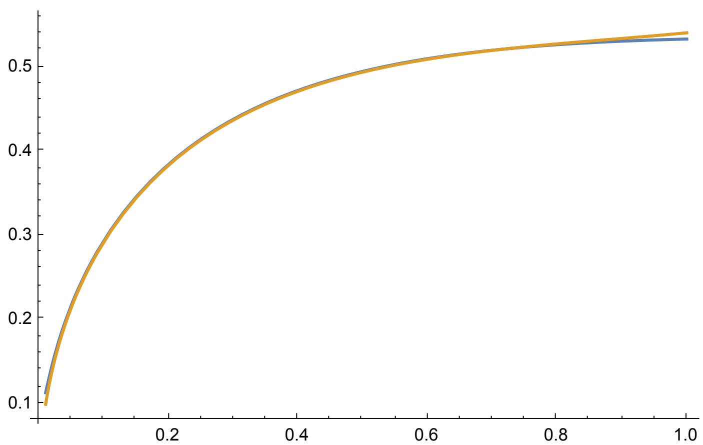
```
Clear["Global`*"]
```

$$\text{eqn2} = 2\,x^2\,(x+1)\,y''[x] + 3\,x\,(x+1)^3\,y'[x] - \left(1 - x^2\right)\,y[x] == 0;$$

```
ics = {y[0.6] == 0.51, y'[0.13] == 1};
num = First[y /. NDSolve[{eqn2, ics}, y, {x, 0.01, 1}]]
```

```
InterpolatingFunction[ Domain: {{0.01, 1.}} Output: scalar ]
```

$$\text{Plot}\left[\left\{\text{num}[x],\ \sqrt{x} - \frac{4\,x^{3/2}}{5} + \frac{13\,x^{5/2}}{28} - \frac{134\,x^{7/2}}{945} - \frac{2741\,x^{9/2}}{332\,640} + \frac{20\,429\,x^{11/2}}{772\,200}\right\},\ \{x,\ 0.01,\ 1\}\right]$$



showing the y1 solution just starting to peel away above the interpolated function at x=1. I have not been successful in checking the solutions y1 and y2, though I have tried a couple of different ways.
And the equation seems resistant to DSolve. For instance, the straightforward attempt below only returns an uninformative briar patch of formal characters.

```
Clear["Global`*"]
```

$$\text{eqn2} = 2\,x^2\,(x+1)\,y''[x] + 3\,x\,(x+1)^3\,y'[x] - \left(1 - x^2\right)\,y[x] == 0;$$

```
sol = DSolve[eqn2, y, x]
```

$\{\{y \to \text{DifferentialRoot}[$
$\quad \text{Function}[\{\mathring{y}, \mathring{x}\}, \{(-1+\mathring{x})\,\mathring{y}[\mathring{x}] + 3\,\mathring{x}\,(1+\mathring{x})^2\,\mathring{y}'[\mathring{x}] + 2\,\mathring{x}^2\,\mathring{y}''[\mathring{x}] == 0,$
$\quad\quad \mathring{y}[1] == C[1],\ \mathring{y}'[1] == C[2]\}],\ \ll\gg]\}\}$

2 - 13 Frobenius method
Find a basis of solutions by the Frobenius method. Try to identify the series as expansions of known functions.

3. x y" + 2 y' + x y = 0

```
Clear["Global`*"]
```

$1. Working through the long and involved Frobenius method described in the text and s.m.. The **Hold** command does not seem to work as I would like, it gets in the way. I don't have nomenclature to designate a Mathematica pseudosum.

$e7 = f[x\_] = a_m\, x^{m+r}$

$x^{m+r}\, a_m$

$2. What is meant by above is the **Sum[a_m x^{m+r},{m,0,∞}]**

```
e8 = f'[x]
```

$(m+r)\, x^{-1+m+r}\, a_m$

```
e9 = f''[x]
```

$(-1+m+r)\,(m+r)\, x^{-2+m+r}\, a_m$

$e10 = x^2 f''[x] + 2\, x\, f'[x] + x^2 f[x] == 0$

$2\,(m+r)\, x^{m+r}\, a_m + (-1+m+r)\,(m+r)\, x^{m+r}\, a_m + x^{2+m+r}\, a_m == 0$

$3. Above: The exponential powers of the first two 'x' factors are equal, but the third is two higher. Since these represent infinite sums, it would not affect their individual (or collective) summations if the indices were adjusted to all match. As the s.m. suggested, this can be accomplished by effectively subtracting 2 from the power of *x* in the third occurence of that variable. This would make the third pseudosum start at 2 (instead of 0) to compensate. As for a_m, that element would be changed to a_{m-2} in order to have it start at the same place as before the change.

$e11 = e10\ /.\ (x^{2+m+r}\, a_m) \to (x^{m+r}\, a_{m-2})$

$x^{m+r}\, a_{-2+m} + 2\,(m+r)\, x^{m+r}\, a_m + (-1+m+r)\,(m+r)\, x^{m+r}\, a_m == 0$

$4. Above: Last reminder of the form change. Having all the factors mix together should not invalidate anything. However, if I find myself in a position where I want to assign 0 or 1 to the index m, the coefficient m_{-2+m} will drop out.

```
e12 = Expand[e11]
```

$x^{m+r}\, a_{-2+m} + m\, x^{m+r}\, a_m + m^2\, x^{m+r}\, a_m + r\, x^{m+r}\, a_m + 2\, m\, r\, x^{m+r}\, a_m + r^2\, x^{m+r}\, a_m == 0$

```
e13 = Simplify[e12]
```

$x^{m+r}\left(a_{-2+m} + \left(m + m^2 + r + 2\, m\, r + r^2\right) a_m\right) == 0$

\$5. Below: The indicial equation. (What makes it the indicial equation is setting $m = 0$.) In transferring from above, the factor $\mathbf{a_{-2+m}}$ was ignored.

```
e14 = Solve[m + m^2 + r + 2 m r + r^2 == 0, r] /. m → 0
```

{{r → -1}, {r → 0}}

\$6. Above: The sol'n of the indicial equation. The s.m. wants to look at the larger root first, $r = 0$. Where am I? After I equalized powers of x, expanded, and simplified, I got e13. Now I will look at e13 again, but with r evaluated.

```
e15 = e13 /. {r → 0}
```

$x^m\left(a_{-2+m} + \left(m + m^2\right) a_m\right) == 0$

```
e16 = Solve[(a_{-2+m} + (m + m^2) a_m) == 0, a_m] /. {m → 1, a_{-2+m} → 0}
```

$\{\{a_1 \to 0\}\}$

\$7. Making $m = 1$ in the above allows the finding of a_1. Since $m < 2$, the coefficient a_{-2+m} will be zero.

```
e17 = e13 /. {a_{-2+m} → 0, r → 0}
```

$\left(m + m^2\right) x^m\, a_m == 0$

\$8. Above: Getting a look at the heart of the equation updated.

```
A = {};
Do[A = Union[A, Solve[(a_{-2+m} + (m + m^2) a_m) == 0, a_m]], {m, 2, 7}];
A
```

$\left\{\left\{a_2 \to -\frac{a_0}{6}\right\}, \left\{a_3 \to -\frac{a_1}{12}\right\}, \left\{a_4 \to -\frac{a_2}{20}\right\}, \left\{a_5 \to -\frac{a_3}{30}\right\}, \left\{a_6 \to -\frac{a_4}{42}\right\}, \left\{a_7 \to -\frac{a_5}{56}\right\}\right\}$

\$9. What the above set A does not include is the factor $a_0\, x^0$. The s.m recommends assigning the value 1 to a_0, and this will be added when the opportunity presents. The starting value of m is 2, because that is the lowest value for which a_{-2+m} has meaning. There is no assigned value for a_1 yet. But now everything has a value based on either a_0 or a_1.

```
B = {};
```

$\text{Eliminate}\left[\left\{a_4 == -\frac{a_2}{20},\ a_2 == -\frac{a_0}{6}\right\},\ a_2\right]$

$120\, a_4 == a_0$

$$\text{Eliminate}\left[\left\{a_6 == -\frac{a_4}{42},\ 120\, a_4 == a_0\right\},\ a_4\right]$$

$$-5040\, a_6 == a_0$$

$$\text{Eliminate}\left[\left\{a_5 == -\frac{a_3}{30},\ a_3 == -\frac{a_1}{12}\right\},\ a_3\right]$$

$$360\, a_5 == a_1$$

$$\text{Eliminate}\left[\left\{a_7 == -\frac{a_5}{56},\ 360\, a_5 == a_1\right\},\ a_5\right]$$

$$-20\,160\, a_7 == a_1$$

$$B = \left\{\{a_0 \to 1\},\ \left\{a_2 \to -\frac{a_0}{6}\right\},\ \left\{a_3 \to -\frac{a_1}{12}\right\},\ \left\{a_4 \to \frac{a_0}{120}\right\},\ \left\{a_5 \to \frac{a_1}{360}\right\},\ \left\{a_6 \to -\frac{a_0}{5040}\right\},\ \left\{a_7 \to -\frac{a_1}{20\,160}\right\}\right\}$$

$$\left\{\{a_0 \to 1\},\ \left\{a_2 \to -\frac{a_0}{6}\right\},\ \left\{a_3 \to -\frac{a_1}{12}\right\},\ \left\{a_4 \to \frac{a_0}{120}\right\},\ \left\{a_5 \to \frac{a_1}{360}\right\},\ \left\{a_6 \to -\frac{a_0}{5040}\right\},\ \left\{a_7 \to -\frac{a_1}{20\,160}\right\}\right\}$$

```
cs = {2!, 3!, 4!, 5!, 6!, 7!, 8!}
```

{2, 6, 24, 120, 720, 5040, 40 320}

$$B[[3]] = B[[3]]\ /.\ \frac{a_2}{120} \to \frac{a_1}{24}$$

$$\left\{a_3 \to -\frac{a_1}{12}\right\}$$

$$B[[4]] = B[[4]]\ /.\ a_2 \to a_0$$

$$\left\{a_4 \to \frac{a_0}{120}\right\}$$

$$B[[5]] = B[[5]]\ /.\ a_3 \to a_1$$

$$\left\{a_5 \to \frac{a_1}{360}\right\}$$

$$B[[6]] = B[[6]]\ /.\ a_4 \to a_0$$

$$\left\{a_6 \to -\frac{a_0}{5040}\right\}$$

$$B[[7]] = B[[7]]\ /.\ a_5 \to a_1$$

$$\left\{a_7 \to -\frac{a_1}{20\,160}\right\}$$

```
e19 = TableForm[Table[{m, a_m, B[[m]]}, {m, 2, 7}],
  TableHeadings → {{}, {"m", "a_m ", "B[m] ", "a_1"}}]
```

m	a_m	B[m]	a_1
2	a_2	$a_2 \to -\frac{a_0}{6}$	
3	a_3	$a_3 \to -\frac{a_1}{12}$	
4	a_4	$a_4 \to \frac{a_0}{120}$	
5	a_5	$a_5 \to \frac{a_1}{360}$	
6	a_6	$a_6 \to -\frac{a_0}{5040}$	
7	a_7	$a_7 \to -\frac{a_1}{20160}$	

$10. There are two series. Let me see if I can separate them:

```
B1 = {B[[3, 1, 2]], B[[5, 1, 2]], B[[7, 1, 2]]}
```

$$\left\{-\frac{a_1}{12}, \frac{a_1}{360}, -\frac{a_1}{20160}\right\}$$

$11. There is a problem with B1. However, since it will not be used, there is no need to investigate it.

```
B2 = {B[[1, 1, 2]], B[[2, 1, 2]], B[[4, 1, 2]], B[[6, 1, 2]]}
```

$$\left\{1, -\frac{a_0}{6}, \frac{a_0}{120}, -\frac{a_0}{5040}\right\}$$

```
e20 = TableForm[Table[{m, B2[[m]]}, {m, 1, 4}],
  TableHeadings → {{}, {"term", "B2[m] "}}]
```

term	B2[m]
1	1
2	$-\frac{a_0}{6}$
3	$\frac{a_0}{120}$
4	$-\frac{a_0}{5040}$

```
y1 = Sum[B2[[s]] x^(2 (s-1)), {s, 1, 4}]
```

$$1 - \frac{x^2 a_0}{6} + \frac{x^4 a_0}{120} - \frac{x^6 a_0}{5040}$$

$12. To do as s.m., a_0 was assigned a value of 1. Then y1 = $1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!}$

$13. Below is shown the definition version of what will now be called y1. It agrees with the text answer for y_1.

```
Series[Sin[x]/x, {x, 0, 4}]
```

$$1 - \frac{x^2}{6} + \frac{x^4}{120} + O[x]^5$$

$14. To get the second sol’n in the basis, y2, it is recommended by the s.m. to march off and do reduction of order, covered in Sec 2.1 of the text. From that perspective it is deemed important to put the original equation into standard form,

```
e21 = {y''[x] + 2/x y'[x] + y[x] == 0}
```

$\left\{y[x] + \frac{2\,y'[x]}{x} + y''[x] == 0\right\}$

```
p[x_] = 2/x
```

$\frac{2}{x}$

```
-Integrate[p[x], x, GenerateConditions → True]
```

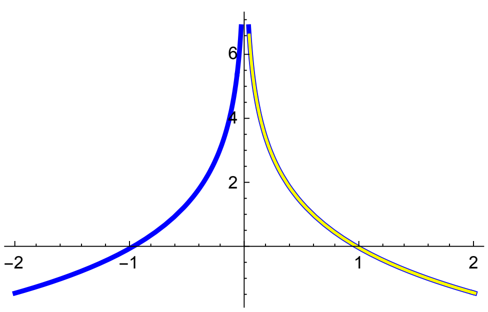
-2 Log[x]

$15. Above: Mathematica does not bother to show that the correct answer involves Abs[x]. The plot below shows the difference. This doesn’t seem to affect this particular answer, but the omission is somewhat disturbing.

```
plot1 = Plot[-2 Log[x], {x, -2, 2}, PlotStyle → Yellow, ImageSize → 250];

plot2 =
  Plot[-2 Log[Abs[x]], {x, -2, 2}, PlotStyle → {Blue, Thickness[0.01]}];

Show[plot2, plot1]
```



$16. Putting the integral into another form.

```
Exp[-Integrate[p[x], x]]
```

$\frac{1}{x^2}$

$17. The expression: $U = \frac{1}{y_1^2}\, e^{-\int p\, dx}$ is from section 2.1, p.52, where one sol’n to a homogeneous linear ODE with constant coefficents is already known and you are tracking down the other part of the basis. Putting it to use,

```
capU = Exp[-Integrate[p[x], x]] / (Sin[x]/x)^2
```

$\mathrm{Csc}[x]^2$

```
smallu = Integrate[capU, x]
```

-Cot[x]

```
y2 = smallu Sin[x]/x
```

$-\frac{\mathrm{Cos}[x]}{x}$

$18. The s.m. points out that since it is a combo of y1 and y2 and there are arbitrary coefficients involved, the minus sign on y2 is not necessary, and the version shown in the answer section (no minus sign) can be claimed.

5. x y'' + (2 x + 1) y' + (x + 1) y = 0

```
Clear["Global`*"]
```

```
e1 = {x y''[x] + (2 x + 1) y'[x] + (x + 1) y[x] == 0}
```

{(1 + x) y[x] + (1 + 2 x) y′[x] + x y′′[x] == 0}

```
sol = DSolve[e1, y, x]
```

{{y → Function[{x}, e^{-x} C[1] + e^{-x} C[2] Log[x]]}}

```
e1 /. sol // Simplify
```

{{True}}

Mathematica can solve this one without all the Frobenius gingerbread. It should be noted that the two parts (terms) of the sol'n shown in green above are considered two separate sol'ns, y1 and y2. This was the same case with the first problem, no. 3. In the text answer C[1] and C[2] are equal to 1. The answer agrees with that of the text.

7. y'' + (x - 1) y = 0

Also with this problem I attempt to skip the Frobenius method in favor of a direct assault.

```
Clear["Global`*"]
```

```
eqn = y''[x] + (x - 1) y[x] == 0
```

(-1 + x) y[x] + y′′[x] == 0

```
sol = DSolve[eqn, y, x]
{{y → Function[{x},
    AiryAi[-(-1)^(1/3) (1 - x)] C[1] + AiryBi[-(-1)^(1/3) (1 - x)] C[2]]}}
```

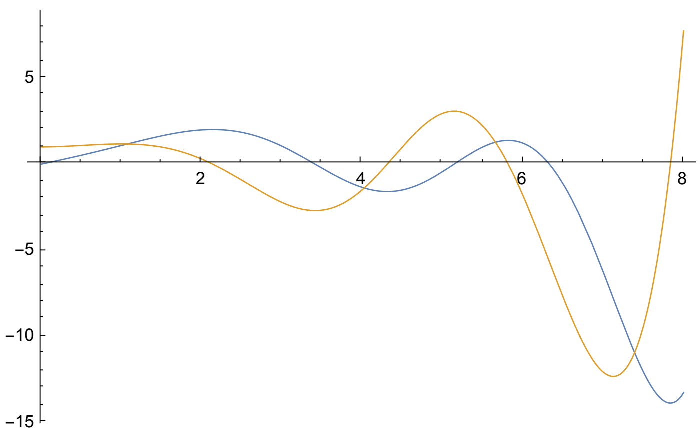
```
eqn /. sol // Simplify
{True}
```

```
FindRoot[AiryAi[-(-1)^(1/3) (1 - x)] + AiryBi[-(-1)^(1/3) (1 - x)], {x, .1}]
{x → -0.845332 + 3.19621 i}
```

Though Mathematica finds a pair of apparently viable solutions, dealing with Airy is not that easy for me. To plot it, some kind of numericalization seems required, one example being

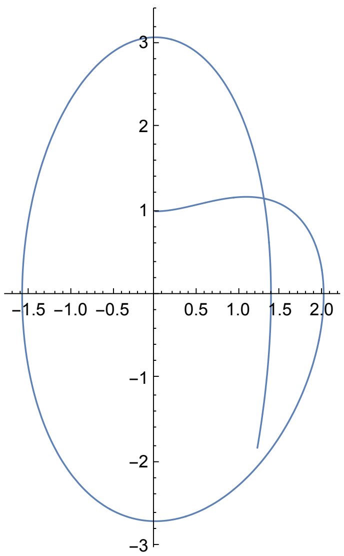
```
solN = First[y /. NDSolve[{eqn, y[0] == 0, y'[0] == 1}, y, {x, 0, 5}]]
InterpolatingFunction[ Domain: {{0., 5.}} Output: scalar ]
```

```
Plot[{solN[x], solN'[x]}, {x, 0, 8}, PlotStyle → Thickness[0.002]]
```



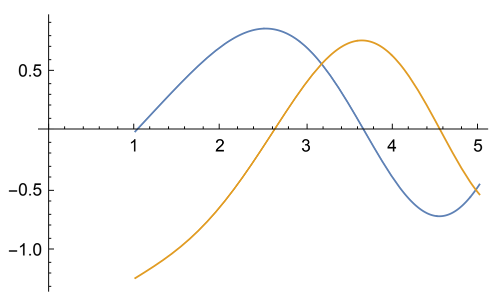
And a parametric plot suggested by the WolframAlpha result,

```
ParametricPlot[{solN[x], solN'[x]}, {x, 0, 6},
 ImageSize → 175, PlotStyle → Thickness[0.005]
]
```



As a matter of interest, I experimented with the Cocalc site, (https://cocalc.com), where Sage leverages Maxima to get a friendlier Bessel solution with the command: **sage: desolve(diff(y,x,2)+(x-1)*y==0,y,contrib_ode=True,show_method=True)**

$$\text{Plot}\left[\left\{\sqrt{x-1}\ \text{BesselJ}\left[\frac{1}{3}, \frac{2}{3}(x-1)^{3/2}\right], \sqrt{x-1}\ \text{BesselY}\left[\frac{1}{3}, \frac{2}{3}(x-1)^{3/2}\right]\right\}, \{x, 0, 5\}, \text{ImageSize} \to 250, \text{PlotStyle} \to \text{Thickness}[0.004]\right]$$



One obvious advantage of the Bessel version is that it is not necessary to particularize it in order to get a plot. At the site *https://mathematica.stackexchange.com/questions/183164/ndeigenvalues-vs-findroot-for-finding-the-eigenvalues-of-airy-equation/183276#183276*, in the answer of Bill Watts, are some functions that convert Airy to Bessel. P is for positive x, and M is for minus x.

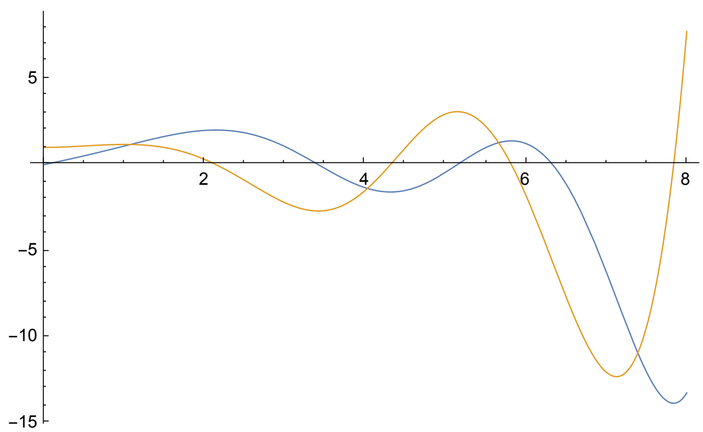
```
AiryToBesP =
  {AiryAi[x_] → (1 / 3) * Sqrt[x] * (BesselI[-3^(-1), (2 / 3) * x^(3 / 2)] -
       BesselI[1 / 3, (2 / 3) * x^(3 / 2)]),
   AiryBi[x_] → Sqrt[x / 3] * (BesselI[-3^(-1), (2 / 3) * x^(3 / 2)] +
       BesselI[1 / 3, (2 / 3) * x^(3 / 2)])};
AiryToBesM = {AiryAi[x_] → (1 / 3) * Sqrt[-x] * (BesselJ[-3^(-1),
        (2 / 3) * Sqrt[-x]^3] + BesselJ[1 / 3, (2 / 3) * Sqrt[-x]^3]),
   AiryBi[x_] → Sqrt[-x / 3] * (BesselJ[-3^(-1), (2 / 3) * Sqrt[-x]^3] -
       BesselJ[1 / 3, (2 / 3) * Sqrt[-x]^3])};
```

Carrying on with the above choice of positive x,

```
y[x_] = solN[x] /. AiryToBesP // Simplify
```

```
InterpolatingFunction[ Domain: {{0., 5.}} Output: scalar ][x]
```

```
Plot[{y[x], y'[x]}, {x, 0, 8}, PlotStyle → Thickness[0.002]]
```



The above looks like the previous plot of solN, except that it was not necessary to use NDSolve, which could be an advantage, as it seems to have greater generality.

9. 2 x (x - 1) y'' - (x + 1) y' + y = 0

```
Clear["Global`*"]
```

```
eqn = 2 x (x - 1) y''[x] - (x + 1) y'[x] + y[x] == 0
```

$y[x] - (1 + x)\, y'[x] + 2\,(-1 + x)\, x\, y''[x] == 0$

```
sol = DSolve[eqn, y, x]
```

$\{\{y \to \text{Function}[\{x\}, \sqrt{x}\, C[1] - 2\,(1 + x)\, C[2]]\}\}$

```
eqn /. sol // Simplify
```

{True}

It appears that Frobenius's method is not necessary with this problem. In order to make the green cell match the text answer, I choose C[1]=1 and C[2]=-$\frac{1}{2}$.

11. x y" + (2 - 2 x) y' + (x - 2) y = 0

```
Clear["Global`*"]
```

```
eqn = x y''[x] + (2 - 2 x) y'[x] + (x - 2) y[x] == 0
(-2 + x) y[x] + (2 - 2 x) y′[x] + x y″[x] == 0
```

```
sol = DSolve[eqn, y, x]
```

$$\left\{\left\{y \to \text{Function}\left[\{x\}, \frac{e^x C[1]}{x} + e^x C[2]\right]\right\}\right\}$$

```
eqn /. sol // Simplify
{True}
```

Again the Frobenius method is not necessary. To make the green cell match the text answer, I choose C[1] = C[2]=1.

13. x y" + (1 - 2 x) y' + (x - 1) y = 0

This is one problem where Frobenius works smoothly. However, I notice now that Frobenius is not necessary, and the solutions pop right out using plain old **DSolve**.

```
Clear["Global`*"]
```

```
e1 = x y''[x] + (1 - 2 x) y'[x] + (x - 1) y[x] == 0
(-1 + x) y[x] + (1 - 2 x) y′[x] + x y″[x] == 0
```

```
e2 = x e1
x ((-1 + x) y[x] + (1 - 2 x) y′[x] + x y″[x] == 0)
```

```
e3 = x ((-1 + x) y[x] + (1 - 2 x) y′[x] + x y″[x]) == 0
x ((-1 + x) y[x] + (1 - 2 x) y′[x] + x y″[x]) == 0
```

```
e4 = Expand[e3]
```

$$-x\, y[x] + x^2\, y[x] + x\, y'[x] - 2\, x^2\, y'[x] + x^2\, y''[x] == 0$$

```
e5 = Collect[e4, {y''[x], y'[x], y[x]}]
```

$$\left(-x + x^2\right) y[x] + \left(x - 2\, x^2\right) y'[x] + x^2\, y''[x] == 0$$

$x\, b(x)$ must equal $\left(x - 2\, x^2\right)$ and $c(x)\, y$ must equal $\left(-x + x^2\right)$. So $b(x)$ equals $(1 - 2\, x)$. To find out b_0 and c_0, it is necessary to expand them.

```
e6 = Series[1 - 2 x, {x, 0, 2}]
```

$$1 - 2\, x + O[x]^3$$

So $b_0 = 1$.

```
e7 = Series[-x + x^2, {x, 0, 2}]
```

$-x + x^2 + O[x]^3$

Because there is no constant term in the expansion of $c(x)$, the s.m. tells me that $c_0 = 0$.

The sol'n series will look like:

```
y1[x_] = a_m x^(m+r)
```

$x^{m+r} a_m$

```
y1'[x]
```

$(m + r)\, x^{-1+m+r} a_m$

```
y1''[x]
```

$(-1 + m + r)\,(m + r)\, x^{-2+m+r} a_m$

Since I know $b(x)$ and $c(x)$ and b_0 and c_0, I can write the indicial equation.

```
e8 = Solve[r (r - 1) + 1 r + 0 == 0, r]
```

$\{\{r \to 0\}, \{r \to 0\}\}$

Mathematica is telling me it is a **double root**. Now I can write the original equation,

```
e9 = x y1''[x] + (1 - 2 x) y1'[x] + (x - 1) y1[x] == 0
```

$(-1 + m + r)\,(m + r)\, x^{-1+m+r} a_m + (m + r)\,(1 - 2\,x)\, x^{-1+m+r} a_m + (-1 + x)\, x^{m+r} a_m == 0$

```
e10 = e9 /. {r → 0}
```

$(-1 + m)\, m\, x^{-1+m} a_m + m\,(1 - 2\,x)\, x^{-1+m} a_m + (-1 + x)\, x^m a_m == 0$

```
e11 = Collect[e10, (-1 + m)]
```

$m\, x^{-1+m} a_m + (-1 + m)\, m\, x^{-1+m} a_m - x^m a_m - 2\, m\, x^m a_m + x^{1+m} a_m == 0$

The five factors above are the same factors as in the s.m.

```
e12 = Collect[e11, {x^(-1+m), x^m, x^(1+m)}]
```

$m^2 x^{-1+m} a_m + x^{1+m} a_m + x^m (-a_m - 2\, m\, a_m) == 0$

This also matches, at the place where the powers are adjusted.

```
e13 = e12 /. {(m^2 x^(-1+m) a_m) → ((s + 1)^2 a_(s+1) x^s),
    (x^(1+m) a_m) → (a_(s-1) x^s), (x^m (-a_m - 2 m a_m)) → (-(2 s + 1) a_s x^s)}
```

$x^s a_{-1+s} + (-1 - 2\,s)\, x^s a_s + (1 + s)^2 x^s a_{1+s} == 0$

The powers are adjusted. If shown as real sums, the third factor (sum) would be $\{s, -1, \infty\}$, the second $\{s, 0, \infty\}$, and the first $\{s, 1, \infty\}$.

For the case of $s = -1$, only the third factor would work, fitting into that index range.

```
e14 = Solve[(1 + s)^2 x^s a_{1+s} == 0, a_{s+1}] /. s → -1
```

$\{\{a_0 \to 0\}\}$

In the case of $s = 0$, both second and third factors can accommodate.

```
e15 = Solve[(-1 - 2 s) x^s a_s + (1 + s)^2 x^s a_{1+s} == 0, a_s] /. s → 0
```

$\{\{a_0 \to a_1\}\}$

In the case of $s > 0$, all three factors can accommodate.

```
e16 = Solve[x^s a_{-1+s} + (-1 - 2 s) x^s a_s + (1 + s)^2 x^s a_{1+s} == 0, a_s] /. s → 1
```

$\left\{\left\{a_1 \to \frac{1}{3}(a_0 + 4 a_2)\right\}\right\}$

```
e17 = Solve[a_1 == 1/3 (a_0 + 4 a_2), a_2] /. a_1 → a_0
```

$\left\{\left\{a_2 \to \frac{a_0}{2}\right\}\right\}$

```
e18 = Solve[x^s a_{-1+s} + (-1 - 2 s) x^s a_s + (1 + s)^2 x^s a_{1+s} == 0, a_{s+1}]
```

$\left\{\left\{a_{1+s} \to \frac{-a_{-1+s} + a_s + 2 s a_s}{(1+s)^2}\right\}\right\}$

```
e19 = e18 /. {s → 1, a_0 → 1}
```

$\left\{\left\{a_2 \to \frac{1}{4}(-a_0 + 3 a_1)\right\}\right\}$

```
loc = {};
e20 =
 Do[loc = Union[loc, Solve[x^s a_{-1+s} + (-1 - 2 s) x^s a_s + (1 + s)^2 x^s a_{1+s} == 0,
       a_{s+1}] /. {a_0 → 1, a_1 → 1}], {s, 1, 4}];
loc
```

This marks the point where values for both a_0 and a_1 are shown as assigned.

$\left\{\left\{a_2 \to \frac{1}{2}\right\}, \left\{a_3 \to \frac{1}{9}(-1 + 5 a_2)\right\}, \left\{a_4 \to \frac{1}{16}(-a_2 + 7 a_3)\right\}, \left\{a_5 \to \frac{1}{25}(-a_3 + 9 a_4)\right\}\right\}$

```
e21 = Solve[a_3 == 1/9 (-1 + 5 a_2), a_3] /. a_2 → 1/2
```

$\left\{\left\{a_3 \to \frac{1}{6}\right\}\right\}$

```
e22 = Solve[a_4 == 1/16 (-a_2 + 7 a_3), a_4] /. {a_2 → 1/2, a_3 → 1/6}
```

$\left\{\left\{a_4 \to \frac{1}{24}\right\}\right\}$

```
e23 = Solve[a5 == 1/25 (-a3 + 9 a4), a5] /. {a2 → 1/2, a3 → 1/6, a4 → 1/24}
```

$\{\{a_5 \to \frac{1}{120}\}\}$

```
e24 = cs = {2!, 3!, 4!, 5!, 6!, 7!, 8!}
```

{2, 6, 24, 120, 720, 5040, 40 320}

```
e25 = y1[x_] = Sum[a_m x^m, {m, 0, 4}]
```

$a_0 + x\, a_1 + x^2 a_2 + x^3 a_3 + x^4 a_4$

In e20 I said that $a_0 = a_1 = 1$. Thus y1 $= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdot\cdot\cdot$ Looks like e^x.

```
y1 = Series[ⅇ^x, {x, 0, 4}]
```

$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + O[x]^5$

The green cell above matches the text answer for y_1. It is still necessary to get a second solution. Again, the method is reduction of order. The first step is to put the equation into standard form.

```
e26 = 1/x (-y[x] + x y[x] + y'[x] - 2 x y'[x] + x y''[x]) == 0
```

$\frac{-y[x] + x\, y[x] + y'[x] - 2\, x\, y'[x] + x\, y''[x]}{x} == 0$

```
e28 = Collect[e26, {y''[x], y'[x], y[x]}]
```

$\frac{(-1 + x)\, y[x]}{x} + \frac{(1 - 2\, x)\, y'[x]}{x} + y''[x] == 0$

```
e29 = p[x_] = (1 - 2 x)/x
```

$\frac{1 - 2\, x}{x}$

Following the procedure for reduction of order,

```
e30 = -Integrate[p[x], x]
```

2 x - Log[x]

Using log identity, this is

```
e31 = e30 /. -Log[x] → Log[1/x]
```

$2\, x + \mathrm{Log}\left[\frac{1}{x}\right]$

Again, Mathematica forgot to use the **Abs** function when integrating a fraction.

```
e32 = Exp[e31]
```

$$\frac{e^{2x}}{x}$$

Continuing to follow the reduction recipe, we have our big U and little u. As shown in s.m.,

```
e33 = bigU[x_] = 1/(e^x)^2 (e^(2 x)/x)
```

$$\frac{1}{x}$$

And as for little u,

```
e34 = u[x_] = Integrate[bigU[x], x]
Log[x]
```

Again, should be **Abs**

```
y2 = u[x] e^x
```

```
e^x Log[x]
```

The green cell above matches the text answer for y_2.

15 - 20 Hypergeometric ODE
Find a general solution in terms of hypergeometric functions.

15. 2 x (1 - x) y" - (1 + 6 x) y' - 2 y = 0

```
Clear["Global`*"]
```

Sometimes Mathematica expresses its output as Hypergeometric. But if it doesn't, I'm not going to feel bad about it. In fact the plain dealing output below is better, in my opinion. Anyway, I need to try to implement a translation of Hypergeometric in assessing the answer.

```
eqn = 2 x (1 - x) y''[x] - (1 + 6 x) y'[x] - 2 y[x] == 0
-2 y[x] - (1 + 6 x) y′[x] + 2 (1 - x) x y′′[x] == 0
```

```
sol = DSolve[eqn, y, x]
```

$$\left\{\left\{y \to \text{Function}\left[\{x\}, \frac{x^{3/2}\,C[1]}{(1-x)^{5/2}} + \frac{2\left(-\sqrt{1-x} + 4\sqrt{1-x}\;x + 3\,x^{3/2}\,\text{ArcSin}\left[\sqrt{x}\right]\right)C[2]}{3\sqrt{1-x}\;(-1+x)^2}\right]\right\}\right\}$$

```
eqn /. sol // Simplify
{True}
```

Though the format differs slightly, it is not too hard to interpret the text answer in the syntax native to Mathematica.

$$x^{3/2}\ \text{Hypergeometric2F1}\left[\frac{5}{2}, \frac{5}{2}, \frac{5}{2}, x\right]$$

$$\frac{x^{3/2}}{(1-x)^{5/2}}$$

$$\text{Hypergeometric2F1}\left[1, 1, -\frac{1}{2}, x\right]$$

$$\text{Hypergeometric2F1}\left[1, 1, -\frac{1}{2}, x\right]$$

Mathematica returns what looks like a valid solution. The top pink cell contents, contained in the text answer, matches with the first function, assuming C[1] is assigned the value of 1, but the bottom pink expression, also from the text solution, is returned unevaluated by Mathematica. As an emergency resource, I turn to WolframAlpha for output:

$$\text{Hypergeometric2F1}\left[1, 1, -\frac{1}{2}, x\right] = \frac{1-4x}{(x-1)^2} - \frac{3\,x^{3/2}\,\text{ArcSin}\left[\sqrt{x}\right]}{\sqrt{1-x}\ (x-1)^2}$$

So with the W|A output I can try

$$\text{Solve}\left[\left(\frac{2\left(-\sqrt{1-x} + 4\sqrt{1-x}\ x + 3\,x^{3/2}\,\text{ArcSin}\left[\sqrt{x}\right]\right) C[2]}{3\sqrt{1-x}\ (-1+x)^2}\right) - \left(\frac{1-4x}{(x-1)^2} - \frac{3\,x^{3/2}\,\text{ArcSin}\left[\sqrt{x}\right]}{\sqrt{1-x}\ (x-1)^2}\right) == 0,\ C[2]\right]$$

$$\left\{\left\{C[2] \to \frac{3\sqrt{1-x}\ (-1+x)^2\left(\frac{1-4x}{(-1+x)^2} - \frac{3\,x^{3/2}\,\text{ArcSin}\left[\sqrt{x}\right]}{\sqrt{1-x}\ (-1+x)^2}\right)}{2\left(-\sqrt{1-x} + 4\sqrt{1-x}\ x + 3\,x^{3/2}\,\text{ArcSin}\left[\sqrt{x}\right]\right)}\right\}\right\}$$

$$\text{FullSimplify}\left[\frac{3\sqrt{1-x}\ (-1+x)^2\left(\frac{1-4x}{(-1+x)^2} - \frac{3\,x^{3/2}\,\text{ArcSin}\left[\sqrt{x}\right]}{\sqrt{1-x}\ (-1+x)^2}\right)}{2\left(-\sqrt{1-x} + 4\sqrt{1-x}\ x + 3\,x^{3/2}\,\text{ArcSin}\left[\sqrt{x}\right]\right)}\right]$$

$$-\frac{3}{2}$$

So simply by assigning the value -3/2 to the constant C[2], the answers match.

17. 4 x (1 - x) y'' + y' + 8 y = 0

```
Clear["Global`*"]
eqn = 4 x (1 - x) y''[x] + y'[x] + 8 y[x] == 0
```

$8\,y[x] + y'[x] + 4\,(1 - x)\,x\,y''[x] == 0$

```
sol = DSolve[eqn, y, x]
```

$\{\{y \to \text{Function}[\{x\}, (1 - x)^{5/4} x^{3/4} C[1] - \frac{4}{15}(5 - 40 x + 32 x^2) C[2]]\}\}$

```
eqn /. sol // Simplify
```

{True}

To check the answer, I need to convert the text answer to non-Hyper.

$x^{3/4}\,\text{Hypergeometric2F1}[\frac{7}{4}, -\frac{5}{4}, \frac{7}{4}, x]$

$(1 - x)^{5/4} x^{3/4}$

The green cell above matches the text answer, with C[1]=1 and C[2]=-$\frac{3}{4}$, (assuming the text constants A and B both equal 1).

19. $2\,(t^2 - 5t + 6)\,\ddot{y} + (2t - 3)\,\dot{y} - 8y = 0$

```
Clear["Global`*"]
eqn = 2 (t^2 - 5 t + 6) y''[t] + (2 t - 3) y'[t] - 8 y[t] == 0
```

$-8\,y[t] + (-3 + 2t)\,y'[t] + 2\,(6 - 5t + t^2)\,y''[t] == 0$

```
sol = DSolve[eqn, y, t]
```

$\{\{y \to \text{Function}[\{t\}, \frac{(2 - t)^{1/4} (-3 + t)^{1/4} (-2 + t)^{5/4} (-17 + 6t) C[1]}{6 (3 - t)^{3/4}} + \frac{4 (2 - t)^{1/4} (-3 + t)^{3/4} (111 - 104 t + 24 t^2) C[2]}{5 (3 - t)^{3/4} (-2 + t)^{1/4}}]\}\}$

```
eqn /. sol // Simplify
```

{True}

Trying to re-express the text answer into a non-Hyper form,

$(t - 2)^{3/2}\,\text{Hypergeometric2F1}[\frac{7}{2}, -\frac{1}{2}, \frac{5}{2}, t - 2]$

$\frac{(17 - 6t)(-2 + t)^{3/2}}{5\sqrt{3 - t}}$

$$\text{Hypergeometric2F1}\left[2, -2, -\frac{1}{2}, t-2\right]$$

$$-111 + 104\,t - 24\,t^2$$

I need to try to reconcile Mathematica's answer with that of the text.

$$\text{Solve}\Big[\frac{(2-t)^{1/4}\,(-3+t)^{1/4}\,(-2+t)^{5/4}\,(-17+6\,t)\,\text{C}[1]}{6\,(3-t)^{3/4}} - \frac{(17-6\,t)\,(-2+t)^{3/2}}{5\,\sqrt{3-t}} == 0,\ \text{C}[1]\Big]\ //\ \text{Simplify}$$

$$\left\{\left\{\text{C}[1] \to -\frac{6\,(-(-3+t)\,(-2+t))^{1/4}}{5\,(2-t)^{1/4}\,(-3+t)^{1/4}}\right\}\right\}$$

$$\text{int} = -\frac{6\,(-(-3+t)\,(-2+t))^{1/4}}{5\,(2-t)^{1/4}\,(-3+t)^{1/4}}$$

$$-\frac{6\,((3-t)\,(-2+t))^{1/4}}{5\,(2-t)^{1/4}\,(-3+t)^{1/4}}$$

$$\text{Simplify}\left[\frac{(3-t)\,(-2+t)}{(2-t)\,(-3+t)}\right]$$

$$1$$

From the above, I see that the first pair of functions differs only by the arbitrary constant C[1], and that this constant should receive the value of 1.

$$\text{Solve}\Big[\frac{4\,(2-t)^{1/4}\,(-3+t)^{3/4}\,\left(111-104\,t+24\,t^2\right)\,\text{C}[2]}{5\,(3-t)^{3/4}\,(-2+t)^{1/4}} - \left(-111+104\,t-24\,t^2\right) == 0,\ \text{C}[2]\Big]$$

$$\left\{\left\{\text{C}[2] \to \frac{5\,(3-t)^{3/4}\,(-2+t)^{1/4}\,\left(-111+104\,t-24\,t^2\right)}{4\,(2-t)^{1/4}\,(-3+t)^{3/4}\,\left(111-104\,t+24\,t^2\right)}\right\}\right\}$$

$$\text{Simplify}\left[\frac{(3-t)}{(-3+t)}\right]$$

$$-1$$

$$\text{Simplify}\left[\frac{(-2+t)}{(2-t)}\right]$$

$$-1$$

```
Simplify[(-111 + 104 t - 24 t^2)/(111 - 104 t + 24 t^2)]
```

```
-1
```

From the above four output cells, I see that the second pair of functions differs only by the assigned value of C[2], which looks like it should get the value of -1, stemming from (-1*-1*-1) = -1.

I therefore conclude that yellow cell equals the sum of the two pinks, and therefore yellow is equivalent to the text answer.