

3 - 10 Reduction of order

Reduce to first order and solve, showing each step in detail.

$$3. y'' + y' = 0$$

Reduction of order is something that Mathematica does not generally need to do.

```
eqn = y'[x] + y''[x] == 0
```

```
y'[x] + y''[x] == 0
```

```
sol = DSolve[eqn, y, x]
```

```
{ {y -> Function[{x}, -e-x C[1] + C[2]] } }
```

```
eqn /. sol // Simplify
```

```
{True}
```

$$5. y y' = 3 (y')^2$$

```
eqn = y[x] y'[x] == 3 y'[x]^2
```

```
y[x] y''[x] == 3 y'[x]^2
```

```
sol = DSolve[eqn, y, x]
```

```
{ {y -> Function[{x},  $\frac{C[2]}{\sqrt{2x + C[1]}}$ ] } }
```

```
eqn /. sol // Simplify
```

```
{True}
```

The text answer is $1/\sqrt{c_1 x + c_2}$. So Mathematica and the text answer each have assigned a value to one of their three constants. This leaves leeway for the remaining assignments to be made in such a way that the two solutions become equivalent.

$$7. y'' + y'^3 \sin[y] = 0$$

```
ClearAll["Global`*"]
```

This problem is a topsy-turvy little trip with an inverted domain. The substitution $z = y'[x]$ is made. Afterwards there is the form

```
eqn2 = z'[y] z[y] == -z[y]^3 Sin[y]
```

```
z[y] z'[y] == -Sin[y] z[y]^3
```

Which can be processed by DSolve into the solution

```
sol2 = DSolve[eqn2, z, y]
```

```
{ {z -> Function[{y}, 0]}, {z -> Function[{y},  $\frac{1}{-C[1] - \cos[y]}$ ]}}
```

The above green cell agrees with the text, though the text uses the inverted form of the fractional expression, calling it $\frac{dx}{dy}$. Using the terms of the substitution, the solution checks out.

```
eqn2 /. sol2 // Simplify
{True, True}
```

The next step is to reverse the substitution level by solving again.

```
eqn3 = -x'[y] == C[1] + Cos[y]
-x'[y] == C[1] + Cos[y]
```

```
sol3 = DSolve[eqn3, x, y]
```

```
{ {x -> Function[{y}, -y C[1] + C[2] - Sin[y]]}}
```

The green cell above matches the final answer in the text, with the provision that the sign on the constant -C[1] is opposite to the constant c_1 in the text. The second use of DSolve also checks out true.

```
eqn3 /. sol3
{True}
```

9. $x^2 y'' - 5x y' + 9y = 0, y_1 = x^3$

```
ClearAll["Global`*"]
```

The substitution $y_1 = x^3$ works as advertised as a singular solution. If it is ignored,

```
eqn = x^2 y''[x] - 5 x y'[x] + 9 y[x] == 0
9 y[x] - 5 x y'[x] + x^2 y''[x] == 0
```

then Mathematica comes up with an equivalent solution, so long as C[1] is assigned the value 0 and C[2] is assigned the value $\frac{1}{3}$.

```
sol = DSolve[eqn, y, x]
```

```
{ {y -> Function[{x}, x^3 C[1] + 3 x^3 C[2] Log[x]]}}
```

The Mathematica solution, neither more nor less general than the text, checks out.

```
eqn /. sol // Simplify
{True}
```

11 - 14 Applications of reducible ODEs

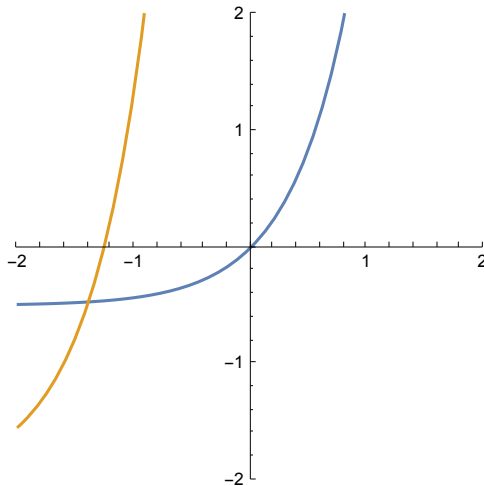
11. Curve. Find the curve through the origin in the xy -plane which satisfies $y'' = 2y'$ and whose tangent at the origin has slope 1.

```
ClearAll["Global`*"]
eqn = y''[x] == 2 y'[x]
y''[x] == 2 y'[x]
sol = DSolve[{eqn, y'[0] == 1, y[0] == 0}, y, x]
```

```
{ {y -> Function[{x},  $\frac{1}{2}(-1 + e^{2x})$ ] ] }
```

The plot below shows that the text answer meets neither of the two requirements stated for the solution. The function in the yellow cell above meets both.

```
Plot[ {  $\frac{1}{2}(-1 + e^{2x})$ ,  $-2 + 25e^{2x}$  }, {x, -2, 2}, AspectRatio -> Automatic,
PlotRange -> {{-2, 2}, {-2, 2}}, ImageSize -> 250]
```



13. Motion. If, in the motion of a small body on a straight line, the sum of the velocity and acceleration equals a positive constant, how will the distance $y[t]$ depend on the initial velocity and position?

```
ClearAll["Global`*"]
```

First, there is an objection against the statement that the sum of velocity and acceleration equals a constant. The two quantities have different units, so they can't be added. The

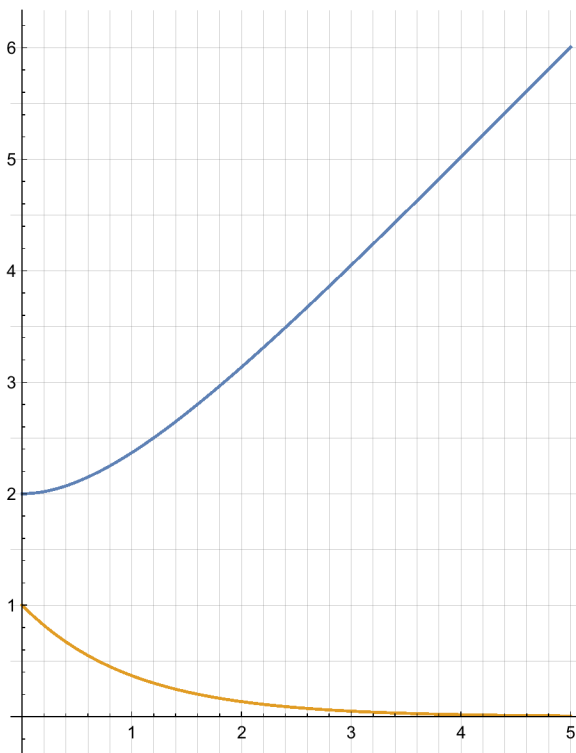
problem must mean to stipulate that the sum of the coefficients of acceleration and velocity add to a constant. To try to understand this a little bit, I will plot the text answer.

$$y[t_] = c_1 e^{-t} + k t + c_2$$

$$k t + e^{-t} c_1 + c_2$$

The grid squares do not appear as squares, but the axes's major ticks seem to be about equal. The problem is supposed to be about travel along a straight line; here the straight line must be the y-axis. With my choice of c_1 , c_2 , and $k = 1$, the starting point must be $y = 2$, and sum of acceleration and velocity must be 1, and the starting velocity must be 1.

```
Plot[{e^-t + t + 1, e^-t}, {t, 0, 5},
  AspectRatio -> 1.3, ImageSize -> 300, GridLines -> All]
```



```
tid = N[Table[{t, e^-t + t + 1}, {t, 0, 15}]]
{{0., 2.}, {1., 2.36788}, {2., 3.13534},
 {3., 4.04979}, {4., 5.01832}, {5., 6.00674}, {6., 7.00248},
 {7., 8.00091}, {8., 9.00034}, {9., 10.0001}, {10., 11.},
 {11., 12.}, {12., 13.}, {13., 14.}, {14., 15.}, {15., 16.}}
```

What can be seen from the two cells below is that by the time $t=14$, acceleration has nearly disappeared, which means that added velocity is also nearly gone, and the travel velocity is at the rate of the starting velocity.

```
tir = Table[tid[[n]][[2]] - tid[[n]][[1]], {n, 15}]
{2., 1.36788, 1.13534, 1.04979, 1.01832, 1.00674, 1.00248,
 1.00091, 1.00034, 1.00012, 1.00005, 1.00002, 1.00001, 1., 1.}
```

```
N[e-14]
8.31529 × 10-7
```

```
y'[t]
k - e-t c1
```

15 - 19 General solution. Initial value problem (IVP)
(More in the next set.) (a) Verify that the given functions are linearly independent and form a basis of solutions of the given ODE. (b) Solve the IVP. Graph or sketch the solution.

15. $4y'' + 25y = 0$, $y[0] = 3.0$, $y'[0] = -2.5$, $\text{Cos}[2.5x]$, $\text{Sin}[2.5x]$

```
ClearAll["Global`*"]
```

By inspection, the two trig expressions are independent. To test whether they are solutions,

```
eqn = 4 y''[x] + 25 y[x] == 0
```

```
25 y[x] + 4 y''[x] == 0
```

```
sol = DSolve[{eqn, y[0] == 3.0, y'[0] == -2.5}, y, x]
```

```
{ {y -> Function[{x}, 3. Cos[ $\frac{5x}{2}$ ] - 1. Sin[ $\frac{5x}{2}$ ]] } }
```

The solution checks.

```
eqn /. sol // Simplify
```

```
{True}
```

The two proposed solutions check.

```
eqn /. Cos[2.5 x] // Simplify
```

ReplaceAllreps:

{Cos[2.5x]} is neither a list of replacement rules nor a valid dispatch table and so cannot be used for replacing>>

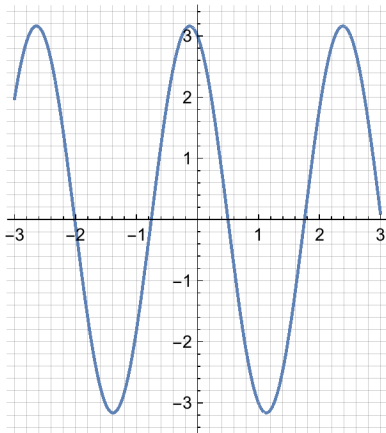
```
True /. Cos[2.5 x]
```

```
eqn /. Sin[2.5 x] // Simplify
```

ReplaceAllreps: {Sin[2.5x]} is neither a list of replacement rules nor a valid dispatch table and so cannot be used for replacing>>

```
True /. Sin[2.5 x]
```

```
Plot[3. Cos[ $\frac{5x}{2}$ ] - 1. Sin[ $\frac{5x}{2}$ ], {x, -3, 3},
  AspectRatio -> Automatic, ImageSize -> 200, GridLines -> All]
```



17. $4x^2 y'' - 3y = 0, y(1) = -3, y'(1) = 0, x^{3/2}, x^{-1/2}$

```
ClearAll["Global`*"]
```

```
eqn = 4 x^2 y''[x] - 3 y[x] == 0
```

```
- 3 y[x] + 4 x^2 y''[x] == 0
```

```
sol = DSolve[{eqn, y[1] == -3, y'[1] == 0}, y, x]
```

```
{ {y -> Function[{x}, - $\frac{3(3+x^2)}{4\sqrt{x}}$ ] ] }
```

```
eqn /. sol
```

```
{True}
```

Although they look a little different due to their format, the green cell above and the text answer are equivalent.

```
PossibleZeroQ[- $\frac{3(3+x^2)}{4\sqrt{x}}$  - (-0.75 x3/2 - 2.25 x-1/2)]
```

```
True
```

Checking the proposed solutions is a little more complicated than usual.

```
d2 = D[x3/2, {x, 2}]
```

```
 $\frac{3}{4\sqrt{x}}$ 
```

```
eqn /. {y[x] -> x3/2, y''[x] -> d2}
```

```
True
```

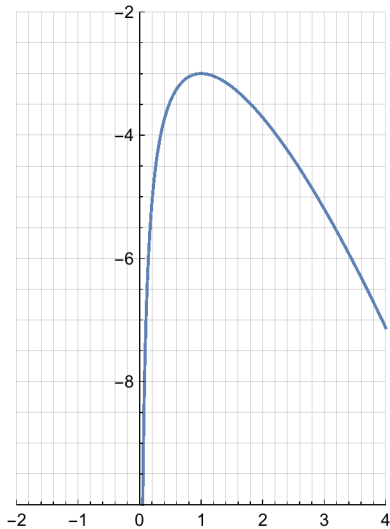
```
d22 = D[x-1/2, {x, 2}]
```

$$\frac{3}{4 x^{5/2}}$$

```
eqn /. {y[x] -> x-1/2, y''[x] -> d22}
```

```
True
```

```
Plot[- $\frac{3(3+x^2)}{4\sqrt{x}}$ , {x, -2, 4}, AspectRatio -> Automatic,  
ImageSize -> 200, GridLines -> All, PlotRange -> {{-2, 4}, {-10, -2}}]
```



19. $y'' + 2y' + 2y = 0$, $y(0) = 0$, $y'(0) = 15$, $e^{-x} \cos[x]$, $e^{-x} \sin[x]$

```
ClearAll[eqn, sol]
```

```
eqn = y''[x] + 2 y'[x] + 2 y[x] == 0;
```

```
sol = DSolve[{eqn, y[0] == 0, y'[0] == 15}, y, x]
```

```
{{y -> Function[{x}, 15 e-x Sin[x]]}}
```

```
eqn /. sol // Simplify
```

```
{True}
```

```
f1[x_] = e-x Cos[x]
```

```
e-x Cos[x]
```

```

d1 = D[f1[x], x]
-e-x Cos[x] - e-x Sin[x]

d2 = D[f1[x], {x, 2}]
2 e-x Sin[x]

eqn /. {y[x] → f1[x], y'[x] → d1, y''[x] → d2} // Simplify
True

f2[x_] = e-x Sin[x]
e-x Sin[x]

d11 = D[f2[x], x]
e-x Cos[x] - e-x Sin[x]

d22 = D[f2[x], {x, 2}]
-2 e-x Cos[x]

eqn /. {y[x] → f2[x], y'[x] → d11, y''[x] → d22} // Simplify
True

Plot[15 e-x Sin[x], {x, -1, 2}, AspectRatio → Automatic,
  ImageSize → 150, GridLines → All, PlotRange → {{-1, 2}, {-1, 5}}]

```

