3 - 12 Effect of delta (impulse) on vibrating systems
Find and graph or sketch the solution of the IVP.

3. y'' + 4 y = δ (t - π), y[0] = 8, y'[0] = 0

```
Clear["Global`*"]

e1 = LaplaceTransform[y''[t] + 4 y[t] == DiracDelta[ (t - π)], t, s]
```

$$4\,\text{LaplaceTransform}[y[t],t,s] + s^2\,\text{LaplaceTransform}[y[t],t,s] - s\,y[0] - y'[0] == e^{-\pi s}$$

```
e2 = e1 /. {y[0] → 8, y'[0] → 0, LaplaceTransform[y[t], t, s] → bigY}
```

$$4\,\text{bigY} - 8\,s + \text{bigY}\,s^2 == e^{-\pi s}$$

```
e3 = Solve[e2, bigY]
```

$$\left\{\left\{\text{bigY} \to \frac{e^{-\pi s}\,(1 + 8\,e^{\pi s}\,s)}{4 + s^2}\right\}\right\}$$

```
e4 = e3[[1, 1, 2]]
```

$$\frac{e^{-\pi s}\,(1 + 8\,e^{\pi s}\,s)}{4 + s^2}$$

```
e5 = InverseLaplaceTransform[e4, s, t]
```

```
8 Cos[2 t] + Cos[t] HeavisideTheta[-π + t] Sin[t]
```

$$\text{PossibleZeroQ}\left[\text{Cos}[t]\,\text{Sin}[t] - \frac{1}{2}\,\text{Sin}[2\,t]\right]$$

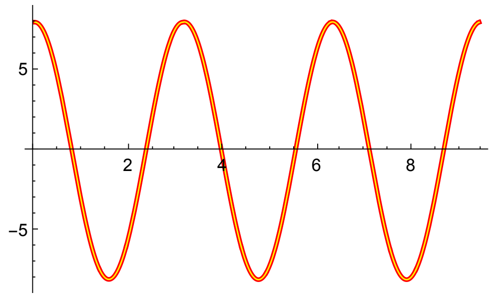
```
True
```

I showed in section 6.3 that **HeavisideTheta** is equivalent to **UnitStep**. Combined with the PZQ above, it makes the green cell equivalent to the text answer.

```
plot1 = Plot[e5, {t, 0, 3 π}, PlotRange → Automatic,
   PlotStyle → {Yellow, Thickness[0.003]}, ImageSize → 250];
```

$$\text{plot2} = \text{Plot}\left[8\,\text{Cos}[2\,t] + \frac{1}{2}\,\text{UnitStep}[t - \pi]\,\text{Sin}[2\,t],\ \{t, 0, 3\,\pi\},\ \text{PlotRange} \to \text{Automatic},\ \text{PlotStyle} \to \{\text{Red}, \text{Thickness}[0.01]\},\ \text{ImageSize} \to 250\right];$$

```
Show[plot2, plot1]
```



Above: The solution tracks well with that of the text.

5. y'' + y = δ (t - π) - δ (t - 2 π), y[0] = 0, y'[0] = 1

```
Clear["Global`*"]
```

```
e1 = LaplaceTransform[
  y''[t] + y[t] == DiracDelta[t - π] - DiracDelta[t - 2 π], t, s]
```

$$\text{LaplaceTransform}[y[t], t, s] + s^2\,\text{LaplaceTransform}[y[t], t, s] - s\,y[0] - y'[0] == -e^{-2\pi s} + e^{-\pi s}$$

```
e2 = e1 /. {y[0] → 0, y'[0] → 1, LaplaceTransform[y[t], t, s] → bigY}
```

$$-1 + \text{bigY} + \text{bigY}\,s^2 == -e^{-2\pi s} + e^{-\pi s}$$

```
e3 = Solve[e2, bigY]
```

$$\left\{\left\{\text{bigY} \to \frac{e^{-2\pi s}\left(-1 + e^{\pi s} + e^{2\pi s}\right)}{1 + s^2}\right\}\right\}$$

```
e4 = e3[[1, 1, 2]]
```

$$\frac{e^{-2\pi s}\left(-1 + e^{\pi s} + e^{2\pi s}\right)}{1 + s^2}$$

```
e5 = InverseLaplaceTransform[e4, s, t]
```

$$-(-1 + \text{HeavisideTheta}[-2\pi + t] + \text{HeavisideTheta}[-\pi + t])\,\text{Sin}[t]$$

```
e6 = e5 /. {HeavisideTheta[-2 π + t] → 0, HeavisideTheta[-π + t] → 0}
```

```
Sin[t]
```

Above: The answer agrees with the text for the subinterval $t < \pi$.

```
e7 = e5 /. {HeavisideTheta[-2 π + t] → 0, HeavisideTheta[-π + t] → 1}
```

```
0
```

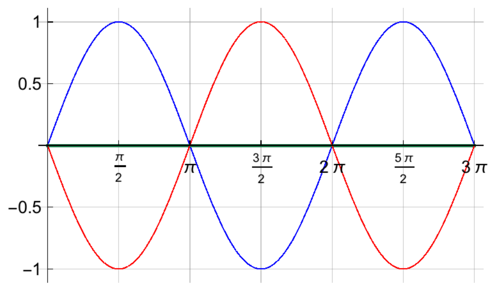
Above: The answer agrees with the text for the subinterval $\pi < t < 2\pi$.

```
e8 = e5 /. {HeavisideTheta[-2 π + t] → 1, HeavisideTheta[-π + t] → 1}
```

```
-Sin[t]
```

Above: The answer agrees with the text for the subinterval $t > 2\pi$.

```
plot1 = Plot[{e6, e7, e8}, {t, 0, 3 π}, PlotRange → Automatic,
  PlotStyle → {{Blue, Thickness[0.003]}, {RGBColor[0.1, 0.5, 0.3],
     Thickness[0.007]}, {Red, Thickness[0.003]}}, ImageSize → 250,
  Ticks → {{π/2, π, 3 π/2, 2 π, 5 π/2, 3 π}, {-1, -.5, .5, 1}},
  GridLines -> {{π/2, π, 3 π/2, 2 π, 5 π/2, 3 π}, {-1, -.5, .5, 1}}]
```



7. $4y'' + 24y' + 37y = 17e^{-t} + \delta\left(t - \frac{1}{2}\right)$, $y[0] = 1$, $y'[0] = 1$

```
Clear["Global`*"]
```

```
e1 = LaplaceTransform[
  4 y''[t] + 24 y'[t] + 37 y[t] == 17 e^-t + DiracDelta[t - 1/2], t, s]
```

$$37\,\text{LaplaceTransform}[y[t], t, s] + 24\,(s\,\text{LaplaceTransform}[y[t], t, s] - y[0]) + 4\left(s^2\,\text{LaplaceTransform}[y[t], t, s] - s\,y[0] - y'[0]\right) == e^{-s/2} + \frac{17}{1+s}$$

```
e2 = e1 /. {y[0] → 1, y'[0] → 1, LaplaceTransform[y[t], t, s] → bigY}
```

$$37\,\text{bigY} + 24\,(-1 + \text{bigY}\,s) + 4\left(-1 - s + \text{bigY}\,s^2\right) == e^{-s/2} + \frac{17}{1+s}$$

```
e3 = Solve[e2, bigY]
```

$$\left\{\left\{\text{bigY} \to \frac{28 + e^{-s/2} + 4s + \frac{17}{1+s}}{37 + 24s + 4s^2}\right\}\right\}$$

```
e4 = e3[[1, 1, 2]]
```

$$\frac{28 + e^{-s/2} + 4s + \frac{17}{1+s}}{37 + 24s + 4s^2}$$

```
e5 = InverseLaplaceTransform[e4, s, t]
```

$$\frac{1}{4} e^{-\frac{i}{4}-\left(3+\frac{i}{2}\right) t} \left(4 e^{\frac{i}{4}} \left(2 i - 2 i e^{i t} + e^{\left(2+\frac{i}{2}\right) t}\right) + i e^{3/2} \left(e^{\frac{i}{2}} - e^{i t}\right) \text{HeavisideTheta}\left[-\frac{1}{2} + t\right]\right)$$

```
e6 = FullSimplify[e5]
```

$$\frac{1}{2} e^{-3 t} \left(2 e^{2 t} - e^{3/2} \text{HeavisideTheta}\left[-\frac{1}{2} + t\right] \text{Sin}\left[\frac{1}{4} (1 - 2 t)\right] + 8 \text{Sin}\left[\frac{t}{2}\right]\right)$$

```
e7 = Expand[e6]
```

$$e^{-t} - \frac{1}{2} e^{\frac{3}{2}-3 t} \text{HeavisideTheta}\left[-\frac{1}{2} + t\right] \text{Sin}\left[\frac{1}{4} (1 - 2 t)\right] + 4 e^{-3 t} \text{Sin}\left[\frac{t}{2}\right]$$

$$\text{PossibleZeroQ}\left[\left(e^{-t} - \frac{1}{2} e^{\frac{3}{2}-3 t} \text{Sin}\left[\frac{1}{4} (1 - 2 t)\right] + 4 e^{-3 t} \text{Sin}\left[\frac{t}{2}\right]\right) - \left(e^{-t} + 4 e^{-3 t} \text{Sin}\left[\frac{1}{2} t\right] + \frac{1}{2} \left(e^{-3 (t-1/2)} \text{Sin}\left[\frac{1}{2} t - \frac{1}{4}\right]\right)\right)\right]$$

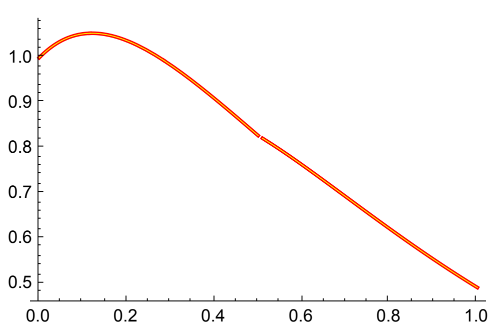
```
True
```

Above: By comparison of plots in section 6.3 I decided that **HeavisideTheta** is equivalent to **UnitStep**, the function the text prefers to use. Granted that equivalence, the PZQ above confirms that the green cell is equivalent to the text answer.

```
plot1 = Plot[e7, {t, 0, 1}, PlotRange → Automatic,
   PlotStyle → {Yellow, Thickness[0.002]}, ImageSize → 250];
```

$$\text{plot2} = \text{Plot}\left[e^{-t} + 4 e^{-3 t} \text{Sin}\left[\frac{t}{2}\right] + \frac{1}{2} \text{UnitStep}\left[t - \frac{1}{2}\right] e^{-3 \left(t-\frac{1}{2}\right)} \text{Sin}\left[\frac{t}{2} - \frac{1}{4}\right], \{t, 0, 1\}, \text{PlotRange} \to \text{Automatic}, \text{PlotStyle} \to \{\text{Red}, \text{Thickness}[0.008]\}, \text{ImageSize} \to 250\right];$$

```
Show[plot2, plot1]
```



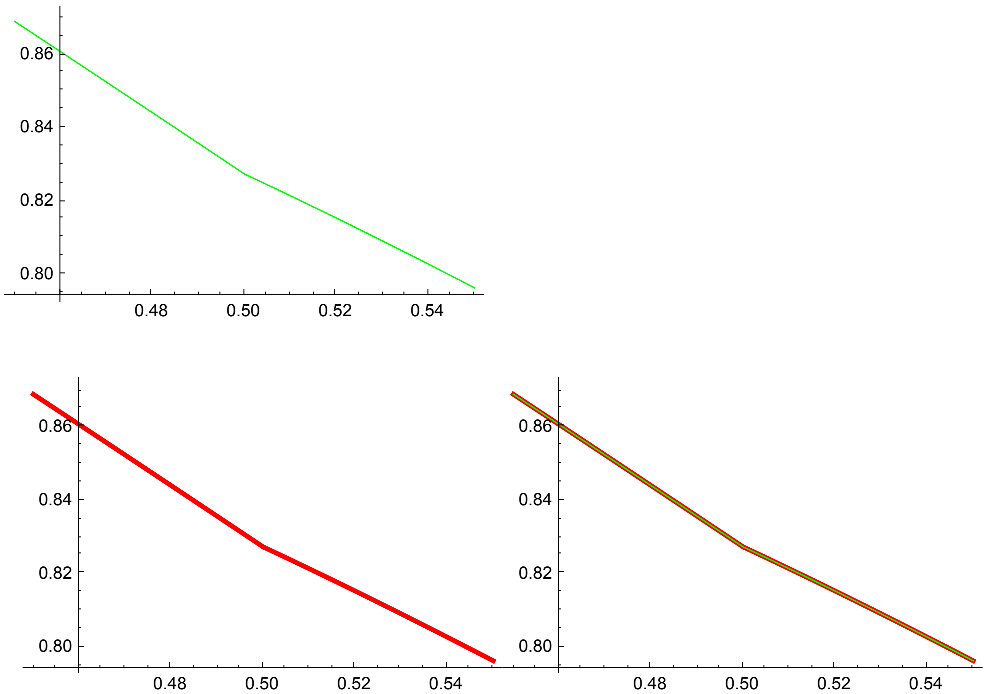
Note the interesting little gap which seems to exist in the combined plot above.

```
plot3 = Plot[e7, {t, 0.45, 0.55}, PlotRange → Automatic,
   PlotStyle → {Green, Thickness[0.003]}, ImageSize → 250];
```

```
plot5 = Plot[e^-t + 4 e^-3 t Sin[t/2] + 1/2 UnitStep[t - 1/2] e^-3 (t-1/2) Sin[t/2 - 1/4],
   {t, 0.45, 0.55}, PlotRange → Automatic,
   PlotStyle → {Red, Thickness[0.01]}, ImageSize → 250];
```

In zoomed view, there is a slight dogleg bend, but no gap. WolframAlpha rules that e7 is continuous on ℝ, so I don't know what the problem is with plotting the superposition.

```
Row[{plot3, plot5, Show[plot5, plot3]}]
```



9. $y'' + 4y' + 5y = (1 - u(t - 10))\, e^t - e^{10}\, \delta(t - 10)$,
$y[0] = 0$, $y'[0] = 1$

```
In[37]:= Clear["Global`*"]
```

I try again with the method of the last two problems, but this one is harder.

```
In[38]:= e1 = LaplaceTransform[y''[t] + 4 y'[t] + 5 y[t] ==
     (1 - UnitStep[t - 10]) e^t - e^10 DiracDelta[t - 10], t, s]

Out[38]= 5 LaplaceTransform[y[t], t, s] + s^2 LaplaceTransform[y[t], t, s] +
   4 (s LaplaceTransform[y[t], t, s] - y[0]) - s y[0] - y′[0] ==
  -e^(10-10 s) + 1/(-1 + s) - e^(-10 (-1+s))/(-1 + s)
```

Above: The Laplace transform is similar to the one in the last problem, as a term containing s has been placed in the denominator of the rhs. I use the same ploy as in the past to isolate the expression I need for the reverse transform.

In[39]:= `e2 = e1 /. {y[0] → 0, y'[0] → 1, LaplaceTransform[y[t], t, s] → bigY}`

Out[39]= $-1 + 5\,\text{bigY} + 4\,\text{bigY}\,s + \text{bigY}\,s^2 == -e^{10-10\,s} + \frac{1}{-1+s} - \frac{e^{-10\,(-1+s)}}{-1+s}$

In[40]:= `e3 = Solve[e2, bigY]`

Out[40]= $\left\{\left\{\text{bigY} \to \frac{e^{-10\,s}\left(-e^{10} + e^{10\,s}\right)s}{(-1+s)\left(5+4\,s+s^2\right)}\right\}\right\}$

In[41]:= `e4 = e3[[1, 1, 2]]`

Out[41]= $\frac{e^{-10\,s}\left(-e^{10} + e^{10\,s}\right)s}{(-1+s)\left(5+4\,s+s^2\right)}$

I try to get a reverse transfrom from the bigY object, in which all subexpressions are real.

In[42]:= `e5 = InverseLaplaceTransform[e4, s, t]`

Out[42]= $\frac{1}{20}\,e^{(-2-i)\,((2+4\,i)+t)}\left((-1-i)\,e^{10\,i}\left((-3-4\,i) + (4+3\,i)\,e^{2\,i\,t} - (1-i)\,e^{(3+i)\,t}\right) + \left((1-7\,i)\,e^{30+20\,i} + (1+7\,i)\,e^{30+2\,i\,t} - 2\,e^{(3+i)\,((1+3\,i)+t)}\right)\text{HeavisideTheta}[-10+t]\right)$

But in the result I see there are imaginaries, which, unlike in previous cases, do not disappear after using FullSimplify.

In[43]:= `e17 = FullSimplify[e5]`

Out[43]= $\frac{1}{20}\,e^{(-2-i)\,((2+4\,i)+t)}\left((-1-i)\,e^{10\,i}\left((-3-4\,i) + (4+3\,i)\,e^{2\,i\,t} - (1-i)\,e^{(3+i)\,t}\right) + \left((1-7\,i)\,e^{30+20\,i} + (1+7\,i)\,e^{30+2\,i\,t} - 2\,e^{(3+i)\,((1+3\,i)+t)}\right)\text{HeavisideTheta}[-10+t]\right)$

So I take a side step to get rid of the imaginaries. Maybe later I can judge whether this is a wise step.

In[44]:= `e6 = ComplexExpand[Re[e5]];`

In[45]:= `e7 = FullSimplify[e6]`

Out[45]= $\frac{1}{10}\,e^{-2\,t}\left(e^{3\,t} - \text{Cos}[t] + 7\,\text{Sin}[t] + \left(-e^{3\,t} + e^{30}\left(\text{Cos}[10-t] + 7\,\text{Sin}[10-t]\right)\right)\text{UnitStep}[-10+t]\right)$

Time to bring in the text answer. (In entering the text answer I changed 0.1 to $\frac{1}{10}$ (two occurrences).)

In[46]:=
```
e8 = 1/10 (ⅇ^t + ⅇ^(-2 t) (-Cos[t] + 7 Sin[t])) +
  1/10 UnitStep[t - 10] (-ⅇ^(-t) + ⅇ^(-2 t+30) (Cos[t - 10] - 7 Sin[t - 10]))
```

Out[46]=
$$\frac{1}{10}\left(e^{t}+e^{-2t}(-\mathrm{Cos}[t]+7\,\mathrm{Sin}[t])\right)+\frac{1}{10}\left(-e^{-t}+e^{30-2t}(\mathrm{Cos}[10-t]+7\,\mathrm{Sin}[10-t])\right)\mathrm{UnitStep}[-10+t]$$

I see that the text answer comes up with the correct result for one of the initial conditions. The Mathematica answer also gets past this hurdle.

In[47]:=
```
e8t = e8 /. t → 0
```

Out[47]= 0

In[48]:=
```
e7t = e7 /. t → 0
```

Out[48]= 0

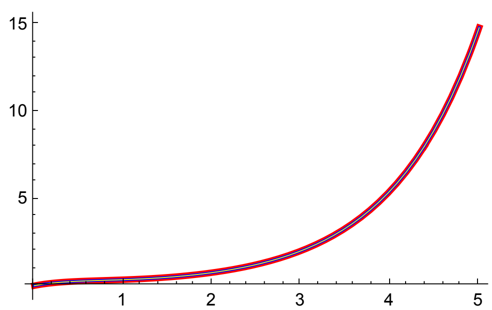
In[62]:=
```
N[e7t10 = e7 /. t → 11]
```

Out[62]= -1594.81

In[49]:=
```
plot1 = Plot[e7, {t, 0, 5}, PlotRange → Automatic,
   PlotStyle → {Yellow, Thickness[0.002]}, ImageSize → 250];
plot2 = Plot[e8, {t, 0, 5}, PlotRange → Automatic,
   PlotStyle → {Red, Thickness[0.014]}, ImageSize → 250];
plot3 = Plot[e17, {t, 0, 5}, PlotRange → Automatic,
   PlotStyle → {Blue, Thickness[0.006]}, ImageSize → 250];
```

Plotting all three of the proposed solutions. On the selected interval they all track one other well.

In[52]:=
```
Show[plot2, plot3, plot1]
```

Out[52]=


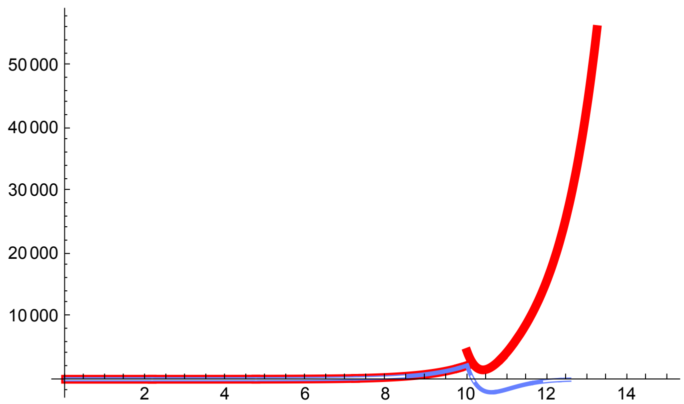
I try subtractive tests but the text answer is not the same as the Mathematica answer. I move on to looking at some more plots.

```
In[63]:= plot3 = Plot[e17, {t, 0, 15},
    PlotRange → {{0, 13}, {-50 000, 50 000}}, PlotStyle →
     {RGBColor[0.4, 0.5, 1], Thickness[0.007]}, ImageSize → 350];
plot4 = Plot[e8, {t, 0, 15}, PlotRange → Automatic,
    PlotStyle → {Red, Thickness[0.014]}, ImageSize → 350];
plot7 = Plot[e7, {t, 0, 15}, PlotRange → Automatic,
    PlotStyle → {White, Thickness[0.003]}, ImageSize → 350];
```

Plotting a slightly longer interval. It seems I have three different functions. The one that has discarded imaginary elements seems to have, for some reason, a slightly smaller range. However, Wolfram Alpha judges it to be continuous on ℝ. In contrast the text function has a jump discontinuity at t=10.

```
In[66]:= Show[plot4, plot3, plot7]
```

Out[66]=



Both the Mathematica (real) solution and the text solution meet the second initial condition.

```
In[74]:= dp = D[e8, t];

In[69]:= dp /. t → 0

Out[69]= 1

In[73]:= dpm = D[e7, t];

In[72]:= dpm /. t → 0

Out[72]= 1
```

So if the Mathematica solution meets both initial conditions, is it considered correct?

11. y'' + 5 y' + 6 y = u (t - 1) + δ (t - 2), y[0] = 0, y'[0] = 1

```
Clear["Global`*"]

e1 = LaplaceTransform[
  y''[t] + 5 y'[t] + 6 y[t] == UnitStep[t - 1] + DiracDelta[t - 2], t, s]
```

$$6\,\mathrm{LaplaceTransform}[y[t], t, s] + s^2\,\mathrm{LaplaceTransform}[y[t], t, s] + 5\,(s\,\mathrm{LaplaceTransform}[y[t], t, s] - y[0]) - s\,y[0] - y'[0] == e^{-2 s} + \frac{e^{-s}}{s}$$

```
e2 = e1 /. {y[0] → 0, y'[0] → 1, LaplaceTransform[y[t], t, s] → bigY}
```

$$-1 + 6\,\text{bigY} + 5\,\text{bigY}\,s + \text{bigY}\,s^2 == e^{-2s} + \frac{e^{-s}}{s}$$

```
e3 = Solve[e2, bigY]
```

$$\left\{\left\{\text{bigY} \to \frac{e^{-2s}\left(e^{s} + s + e^{2s}s\right)}{s\left(6 + 5s + s^2\right)}\right\}\right\}$$

```
e4 = e3[[1, 1, 2]]
```

$$\frac{e^{-2s}\left(e^{s} + s + e^{2s}s\right)}{s\left(6 + 5s + s^2\right)}$$

```
e5 = InverseLaplaceTransform[e4, s, t]
```

$$\frac{1}{6}e^{-3t}\left(6\left(-1 + e^{t}\right) + 6\,e^{4}\left(-e^{2} + e^{t}\right)\text{HeavisideTheta}[-2 + t] + \left(e - e^{t}\right)^{2}\left(2\,e + e^{t}\right)\text{HeavisideTheta}[-1 + t]\right)$$

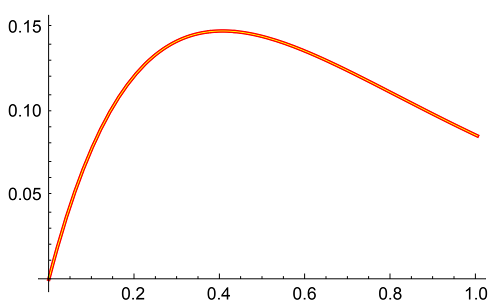
$$\text{e6} = -e^{-3t} + e^{-2t} + \frac{1}{6}\text{UnitStep}[t - 1]\left(1 - 3\,e^{-2(t-1)} + 2\,e^{-3(t-1)}\right) + \text{UnitStep}[t - 2]\left(e^{-2(t-2)} - e^{-3(t-2)}\right)$$

$$-e^{-3t} + e^{-2t} + \left(-e^{-3(-2+t)} + e^{-2(-2+t)}\right)\text{UnitStep}[-2 + t] + \frac{1}{6}\left(1 + 2\,e^{-3(-1+t)} - 3\,e^{-2(-1+t)}\right)\text{UnitStep}[-1 + t]$$

Above: The text answer is entered.

```
plot1 = Plot[e5, {t, 0, 1}, PlotRange → Automatic,
   PlotStyle → {Yellow, Thickness[0.002]}, ImageSize → 250];
plot2 = Plot[e6, {t, 0, 1}, PlotRange → Automatic,
   PlotStyle → {Red, Thickness[0.008]}, ImageSize → 250];
```

```
Show[plot2, plot1]
```



Above: the two plots suggest equality.

```
e7 = e6 /. UnitStep → HeavisideTheta
```

$$-e^{-3t} + e^{-2t} + \left(-e^{-3(-2+t)} + e^{-2(-2+t)}\right) \text{HeavisideTheta}[-2+t] + \frac{1}{6}\left(1 + 2e^{-3(-1+t)} - 3e^{-2(-1+t)}\right) \text{HeavisideTheta}[-1+t]$$

```
FullSimplify[e5 == e7]
True
```

Above: So: If the UnitSteps are exchanged for Heavisides, the answers match.