Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

2 - 11 Line integral. Work.

Calculate $\int_C \mathbf{F}\,(\mathbf{r}) . d\mathbf{r}$ for the given data. If F is a force, this gives the work done by the force in the displacement along C.

2. F = $\{y^2, -x^2\}$, C : $y = 4 x^2$ from $\{0, 0\}$ to $\{1, 4\}$

```
Clear["Global`*"]
```

Above: on line I found that the standard parameterization of the parabola $x^2 = 4 a y$ is $x = 2 a t$, $y = a t^2$.

The first task is parameterization of the path.

$$\text{Solve}\left[\frac{1}{4} y == 4 a y\right]$$

$$\left\{\left\{a \to \frac{1}{16}\right\}, \{y \to 0\}\right\}$$

$$p[t\_] = \left\{\frac{t}{8}, \frac{t^2}{16}\right\}$$

$$\left\{\frac{t}{8}, \frac{t^2}{16}\right\}$$

Above: it can be seen that t will run from 0 to 8. Now to define the vector field:

$$ff[\{x\_, y\_\}] = \{y^2, -x^2\}$$

$$\{y^2, -x^2\}$$

Below: then evaluate the field along the path:

```
e1 = ff[p[t]]
```

$$\left\{\frac{t^4}{256}, -\frac{t^2}{64}\right\}$$

Below: dot the last vector russian doll with the derivative of the position function.

```
e2 = e1.p'[t]
```

$$-\frac{t^3}{512} + \frac{t^4}{2048}$$

Below: and then do the integration,

```
e3 = Integrate[e2, {t, 0, 8}]
```

$$\frac{6}{5}$$

Problem 2 is not odd, so there is no answer in the appendix.

3. F as in problem 2. C from {0, 0} straight to {1, 4}. Compare.

```
Clear["Global`*"]
```

Again, the first step is parameterization. To parameterize a straight line from P to Q means a function $r(t) = (1 - t)\,P + t\,Q$. In the present case that would be

```
r[t_] = (1 - t) {0, 0} + t {1, 4}
{t, 4 t}
```

Above: it can be seen that t will run from 0 to 1. Now to define the vector field:

```
ff[{x_, y_}] = {y^2, -x^2}
{y^2, -x^2}
```

Above: using the same vector field equation as in the last problem. Below: then evaluate the field along the path:

```
e1 = ff[r[t]]
{16 t^2, -t^2}
```

Below: dot the last chinese fortune cookie with the derivative of the position function.

```
e2 = e1.r'[t]
12 t^2
```

Below: and then do the integration,

```
e3 = Integrate[e2, {t, 0, 1}]
```

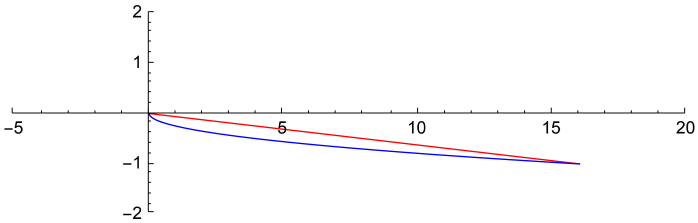
```
4
```

The problem description wanted to have problem 2 and 3 compared. Problem 2 is blue, problem 3 is red.

```
plot1 =
  ParametricPlot[{16 t^2, -t^2}, {t, 0, 1}, PlotRange → {{-5, 20}, {-2, 2}},
   PlotStyle → {Red, Thickness[0.002]}, AspectRatio → .3];

plot2 =
  ParametricPlot[{t^4/256, -t^2/64}, {t, 0, 8}, PlotRange → {{-5, 20}, {-2, 2}},
   PlotStyle → {Blue, Thickness[0.002]}, AspectRatio → .3];
```

```
Show[plot1, plot2]
```



Above: something odd here. It seems obvious that blue is longer than red, but the integral comes out smaller. Knowing that the line integral does not measure the *length* of the line, it is still a hard concept to accept.

4. F = $\{xy, x^2 y^2\}$, C from {2, 0} straight to {0, 2}

5. F as in problem 4. C the quarter-circle from {2, 0} to {0, 2} with center {0, 0}

```
Clear["Global`*"]
```

First the parameterization, which should be easy.

```
r[t_] = {2 Cos[t], 2 Sin[t]}
```

```
{2 Cos[t], 2 Sin[t]}
```

Above: t will run from 0 to $\frac{\pi}{2}$. Now to define the vector field:

```
ff[{x_, y_}] = {x y, x^2 y^2}
```

$\{x\,y, x^2\,y^2\}$

Below: then evaluate the field along the path:

```
e1 = ff[r[t]]
```

$\{4\,\text{Cos}[t]\,\text{Sin}[t], 16\,\text{Cos}[t]^2\,\text{Sin}[t]^2\}$

Below: then dot the multi-level function just calculated with the derivative of the position function,

```
e2 = e1.r'[t]
```

$-8\,\text{Cos}[t]\,\text{Sin}[t]^2 + 32\,\text{Cos}[t]^3\,\text{Sin}[t]^2$

Below: and then do the integration:

$\text{e3} = \text{Integrate}\left[\text{e2}, \left\{t, 0, \frac{\pi}{2}\right\}\right]$

$\frac{8}{5}$

7. F = $\{x^2, y^2, z^2\}$,

C : r = $\{Cos[t], Sin[t], e^t\}$ from {1, 0, 1} to $\{1, 0, e^{2\pi}\}$. Sketch C.

```
Clear["Global`*"]
```

Here the function r is given. It is apparent that t will run from 0 to 2π.

```
r[t_] = {Cos[t], Sin[t], e^t}
```

$\{Cos[t], Sin[t], e^t\}$

Below: and the vector field:

```
ff[{x_, y_, z_}] = {x^2, y^2, z^2}
```

$\{x^2, y^2, z^2\}$

Below: I need to run the field function along the path:

```
e1 = ff[r[t]]
```

$\{Cos[t]^2, Sin[t]^2, e^{2t}\}$

Below: and then dot the multi-level with the derivative of the position function:

```
e2 = e1.r'[t]
```

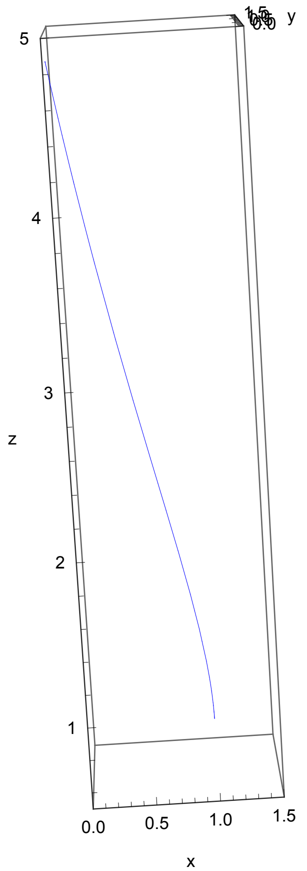
$e^{3t} - Cos[t]^2 Sin[t] + Cos[t] Sin[t]^2$

Below: And then do the integration:

```
Integrate[e2, {t, 0, 2 π}]
```

$\frac{1}{3}\left(-1 + e^{6\pi}\right)$

```
plot2 = ParametricPlot3D[{Cos[t], Sin[t], e^t},
  {t, 0, 2 π}, PlotRange → {{0, 1.5}, {0, 1.5}, {.5, 5}},
  PlotStyle → {Blue, Thickness[0.002]}, BoxRatios → {1, 1, 4},
  ImageSize → 150, AxesLabel → {"x", "y", "z"}]
```



9. **F** = {x + y, y + z, z + x}, C : **r** = {2 t, 5 t, t} from t = 0 to 1. Also from t = -1 to 1.

```
Clear["Global`*"]
```

Another one where the parameterization is already done for me.

```
r[t_] = {2 t, 5 t, t}
{2 t, 5 t, t}
```

The limits on *t* are set in the problem.

```
ff[{x_, y_, z_}] = {x + y, y + z, z + x}
{x + y, y + z, x + z}
```

Below: run the field function along the path:

```
e1 = ff[r[t]]
{7 t, 6 t, 3 t}
```

Below: and dot the result with the derivative of the position function:

```
e2 = e1.r'[t]
47 t
```

Below: and then do the integration:

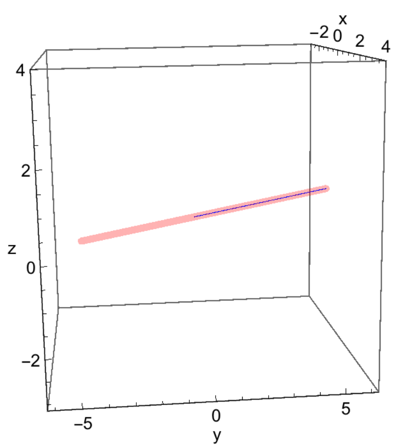
```
e3 = Integrate[e2, {t, 0, 1}]
```

$$\frac{47}{2}$$

```
e4 = Integrate[e2, {t, -1, 1}]
```

0

```
plot2 = ParametricPlot3D[{2 t, 5 t, t},
   {t, 0, 1}, PlotRange → {{-3, 4}, {-2 π, 2 π}, {-π, 4}},
   PlotStyle → {Blue, Thickness[0.002]}, BoxRatios → {1, 1, 1},
   ImageSize → 200, AxesLabel → {"x", "y", "z"}];
plot3 = ParametricPlot3D[{2 t, 5 t, t},
   {t, -1, 1}, PlotRange → {{-3, 4}, {-2 π, 2 π}, {-π, 4}},
   PlotStyle → {Red, Thickness[0.02], Opacity[.3]},
   BoxRatios → {1, 1, 1}, ImageSize → 200, AxesLabel → {"x", "y", "z"}];
Show[plot2, plot3]
```



Above: the shorter range of *t* is within.

11. $F = \{e^{-x}, e^{-y}, e^{-z}\}$,
$C : r = \{t, t^2, t\}$ from $\{0, 0, 0\}$ to $\{2, 4, 2\}$. Sketch C.

```
Clear["Global`*"]
```

Here again, the parameterization is taken care of in the problem statement. The position function:

```
r[t_] = {t, t^2, t}
```

$\{t, t^2, t\}$

t will go from 0 to 2. Defining the vector field:

```
ff[{x_, y_, z_}] = {e^-x, e^-y, e^-z}
```

$\{e^{-x}, e^{-y}, e^{-z}\}$

feeding the vector field through the position function:

```
e1 = ff[r[t]]
```

$\{e^{-t}, e^{-t^2}, e^{-t}\}$

dotting the previous step with the derivative of the position function:

```
e2 = e1.r'[t]
```

$2\,e^{-t} + 2\,e^{-t^2}\,t$

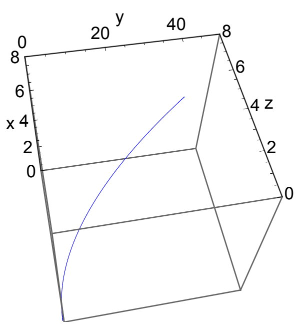
performing the integration:

```
e3 = Integrate[e2, {t, 0, 2}]
```

$$3 - \frac{1}{e^4} - \frac{2}{e^2}$$

Above: this answer matches the second part of the text answer. It is the line length.

```
plot2 = ParametricPlot3D[{t, t^2, t},
  {t, 0, 2 π}, PlotRange → {{0, 8}, {0, 50}, {0, 8}},
  PlotStyle → {Blue, Thickness[0.002]}, BoxRatios → {1, 1, 1},
  ImageSize → 150, AxesLabel → {"x", "y", "z"}]
```



15 - 20 Integrals (8) and (8*)

These would refer to numbered lines (8) and (8*) on p. 417. Evaluate them with **F** or f and C as follows.

15. F = $\{y^2, z^2, x^2\}$, C : r = {3 Cos[t], 3 Sin[t], 2 t}, 0 ≤ t ≤ 4 π

```
Clear["Global`*"]
```

```
r[t_] = {3 Cos[t], 3 Sin[t], 2 t}
{3 Cos[t], 3 Sin[t], 2 t}
```

The problem gives the limits on *t*. Now to define the vector field:

```
ff[{x_, y_, z_}] = {y^2, z^2, x^2}
{y^2, z^2, x^2}
```

. . . and run it through the position function:

```
e1 = ff[r[t]]
{9 Sin[t]^2, 4 t^2, 9 Cos[t]^2}
```

Now to dot the composite above with the derivative of the position function:

```
e2 = e1.r'[t]
```

```
12 t^2 Cos[t] + 18 Cos[t]^2 - 27 Sin[t]^3
```

Now to do the integration:

```
e3 = Integrate[e1, {t, 0, 4 π}]
```

$$\left\{18\,\pi,\ \frac{256\,\pi^3}{3},\ 18\,\pi\right\}$$

The above works by skipping the dot product step (purple), and just integrating the previous step with the integration limits set for the parameterized variable. However, I don’t understand which functions qualify for this treatment.

17. F = {x + y, y + z, z + x}, C : r = {4 Cos[t], Sin[t], 0}, 0 <= t <= π

```
Clear["Global`*"]
```

```
r[t_] = {4 Cos[t], Sin[t], 0}
{4 Cos[t], Sin[t], 0}
```

```
ff[{x_, y_, z_}] = {x + y, y + z, z + x}
{x + y, y + z, x + z}
```

```
e1 = ff[r[t]]
```

```
{4 Cos[t] + Sin[t], Sin[t], 4 Cos[t]}
```

```
e2 = Integrate[e1, {t, 0, π}]
```

```
{2, 2, 0}
```

19. $f = x\,y\,z$, $C : r = \{4\,t,\ 3\,t^2,\ 12\,t\}$, $-2 \le t \le 2$. Sketch C.

```
Clear["Global`*"]
```

```
r[t_] = {4 t, 3 t^2, 12 t}
```

$\{4\,t,\ 3\,t^2,\ 12\,t\}$

```
ff[{x_, y_, z_}] = x y z
```

```
x y z
```

```
e1 = ff[r[t]]
```

$144\,t^4$

```
e2 = Integrate[e1, {t, -2, 2}]
```

$\frac{9216}{5}$

```
e3 = e2 // N
```

```
1843.2
```

I'm not clear about the circumstances when this type of line integral applies.