Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

2 - 13 Verification of Solutions

Verify (by substitution) that the given function is a solution of the PDE. Sketch or graph the solution as a surface in space.

2 - 5 Wave Equation (1) with suitable c

3. u = cos 4t sin 2x

```
Clear["Global`*"]
```

```
u[x_, t_] = Cos[4 t] Sin[2 x]
```

```
Cos[4 t] Sin[2 x]
```

```
d1 = D[u[x, t], {t, 2}]
```

```
-16 Cos[4 t] Sin[2 x]
```

```
d2 = D[u[x, t], {x, 2}]
```

```
-4 Cos[4 t] Sin[2 x]
```

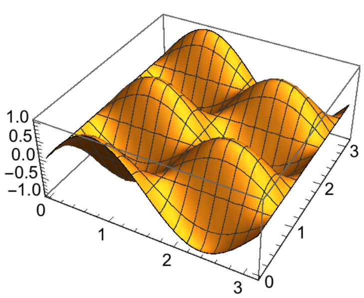
```
d1 == c^2 d2 (* 1D wave equation *)
```

```
-16 Cos[4 t] Sin[2 x] == -4 c^2 Cos[4 t] Sin[2 x]
```

```
Solve[-16 Cos[4 t] Sin[2 x] == -4 c^2 Cos[4 t] Sin[2 x], {c}]
```

```
{{c → -2}, {c → 2}}
```

```
Plot3D[Cos[4 t] Sin[2 x], {x, 0, Pi}, {t, 0, Pi}]
```



The value of the constant, c, is the key to the description of the particular solution **u**.

5. u=sin at sin bx

```
Clear["Global`*"]
```

```
u[x_, t_] = Sin[a t] Sin[b x]
```

```
Sin[a t] Sin[b x]
```

```
d1 = D[u[x, t], {t, 2}]
```
$-a^2$ Sin[a t] Sin[b x]

```
d2  = D[u[x, t], {x, 2}]
```
$-b^2$ Sin[a t] Sin[b x]

$d1 == c^2\ d2$ (* 1D wave equation *)

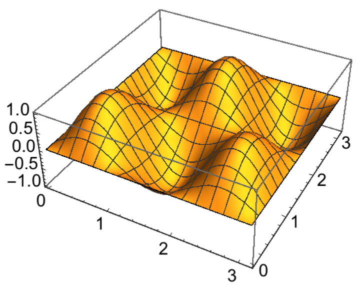
$-a^2$ Sin[a t] Sin[b x] == $-b^2 c^2$ Sin[a t] Sin[b x]

Solve[$-a^2$ Sin[a t] Sin[b x] == $-b^2 c^2$ Sin[a t] Sin[b x], {c}]

$\{\{c \to -\frac{a}{b}\}, \{c \to \frac{a}{b}\}\}$

```
subeq = u[x, t] /. {a → 2, b → 3}
Sin[2 t] Sin[3 x]
```

```
Plot3D[subeq, {x, 0, Pi}, {t, 0, Pi}]
```



6 - 9 Heat Equation (2) with suitable c

7. $u = e^{-\omega^2 c^2 t} \sin x$

```
Clear["Global`*"]
```

u[x_, t_] = $e^{-\omega^2 c^2 t}$ Sin[x]

$e^{-c^2 t \omega^2}$ Sin[x]

```
d1 = D[u[x, t], {t}]
```
$-c^2 e^{-c^2 t \omega^2} \omega^2$ Sin[x]

```
d2 = D[u[x, t], {x, 2}]
```
$-e^{-c^2 t \omega^2}$ Sin[x]

$d1 == c^2\ d2$ (* 1D heat equation *)

$-c^2 e^{-c^2 t \omega^2} \omega^2$ Sin[x] == $-c^2 e^{-c^2 t \omega^2}$ Sin[x]

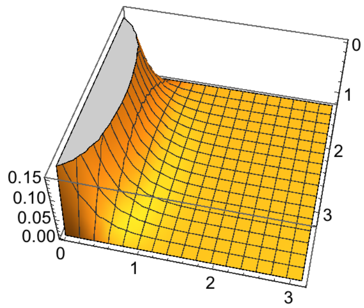
I can't get Solve to give me what I want here. By inspection, c can take on any value, with ω

=1 or -1.

```
subeq = u[x, t] /. {c → 2, ω → 1}
```

e^{-4t} **Sin[x]**

```
Plot3D[subeq, {x, 0, Pi}, {t, 0, Pi}]
```



9. u=$e^{-\pi^2 t} \cos 25x$

```
Clear["Global`*"]
```

u[x_, t_] = $e^{-\pi^2 t}$ **Cos[25 x]**

$e^{-\pi^2 t}$ **Cos[25 x]**

```
d1 = D[u[x, t], {t}]
```

$-e^{-\pi^2 t} \pi^2$ **Cos[25 x]**

```
d2 = D[u[x, t], {x, 2}]
```

$-625\, e^{-\pi^2 t}$ **Cos[25 x]**

```
d1 == c^2 d2 (* 1D heat equation *)
```

$-e^{-\pi^2 t} \pi^2$ **Cos[25 x] ==** $-625\, c^2 e^{-\pi^2 t}$ **Cos[25 x]**

Solve[$-e^{-\pi^2 t} \pi^2$ **Cos[25 x] ==** $-625\, c^2 e^{-\pi^2 t}$ **Cos[25 x], {c}]**

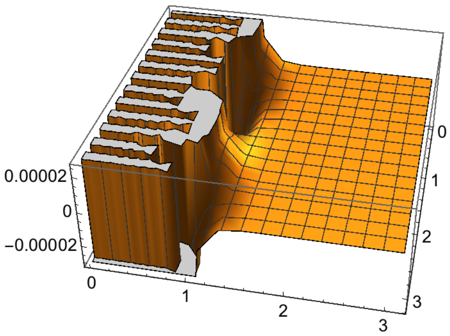
$\{\{c \to -\frac{\pi}{25}\}, \{c \to \frac{\pi}{25}\}\}$

One value of c agrees with the text answer. Mathematica adds the negative value, perhaps overlooked by the text.

```
subeq = u[x, t] /. {c → π / 25}
```

$e^{-\pi^2 t}$ **Cos[25 x]**

```
Plot3D[subeq, {x, 0, Pi}, {t, 0, Pi}]
```



10 - 13 Laplace Equation (3)

10. u=$e^x \cos y$, $e^x \sin y$

```
Clear["Global`*"]
```

```
ucos[x_, y_] = ⅇ^x Cos[y]
```

ⅇ^x Cos[y]

```
d1 = D[ucos[x, y], {x, 2}]
```

ⅇ^x Cos[y]

```
d2 = D[ucos[x, y], {y, 2}]
```

-ⅇ^x Cos[y]

```
eqc = d1 + d2 (* 2D Laplace equation *)
```

0

```
usin[x_, y_] = ⅇ^x Sin[y]
```

ⅇ^x Sin[y]

```
d3 = D[usin[x, y], {x, 2}]
```

ⅇ^x Sin[y]

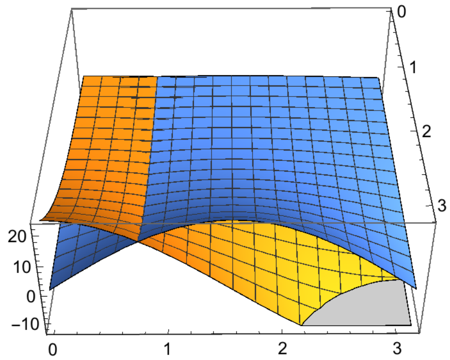
```
d4 = D[usin[x, y], {y, 2}]
```

-ⅇ^x Sin[y]

```
eqs = d3 + d4 (* 2D Laplace equation *)
```

0

```
Plot3D[{ucos[x, y], usin[x, y]}, {x, 0, Pi}, {y, 0, Pi}]
```



I wasn't supposed to work this even-numbered problem, but in view of difficulties encountered in the next one, I'll leave this one in for now.

11. u=arctan(y/x)

```
Clear["Global`*"]
u[x_, y_] = ArcTan[y / x]
```

$$\text{ArcTan}\left[\frac{y}{x}\right]$$

```
d1 = D[u[x, y], {x, 2}]
```

$$-\frac{2\,y^3}{x^5\left(1+\frac{y^2}{x^2}\right)^2}+\frac{2\,y}{x^3\left(1+\frac{y^2}{x^2}\right)}$$

```
d2 = D[u[x, y], {y, 2}]
```

$$-\frac{2\,y}{x^3\left(1+\frac{y^2}{x^2}\right)^2}$$

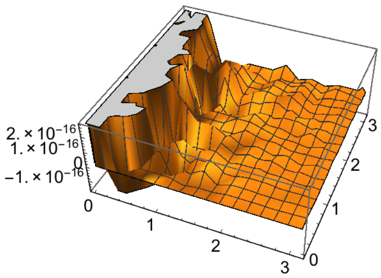
```
eq2 = d1 + d2 == 0 (* 2D Laplace equation *)
```

$$-\frac{2\,y}{x^3\left(1+\frac{y^2}{x^2}\right)^2}-\frac{2\,y^3}{x^5\left(1+\frac{y^2}{x^2}\right)^2}+\frac{2\,y}{x^3\left(1+\frac{y^2}{x^2}\right)}==0$$

An answer to this problem is omitted in the text. I can try to plot it (the Laplace surface).

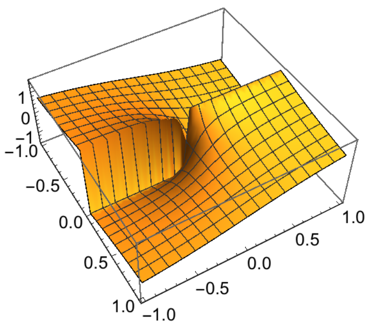
```
Plot3D[-2 y/(x^3 (1 + y^2/x^2)^2) - 2 y^3/(x^5 (1 + y^2/x^2)^2) + 2 y/(x^3 (1 + y^2/x^2)), {x, 0, Pi}, {y, 0, Pi}]
```

$$\text{Plot3D}\left[-\frac{2\,y}{x^3\left(1+\frac{y^2}{x^2}\right)^2}-\frac{2\,y^3}{x^5\left(1+\frac{y^2}{x^2}\right)^2}+\frac{2\,y}{x^3\left(1+\frac{y^2}{x^2}\right)},\ \{x,0,\text{Pi}\},\ \{y,0,\text{Pi}\}\right]$$



The one that was supposed to be plotted is the solution, i.e. the given function:

```
Plot3D[ArcTan[y / x], {x, -1, 1}, {y, -1, 1}]
```



13. u=x/(x^2+y^2), $y/(x^2+y^2)$

```
Clear["Global`*"]
```

```
ux[x_, y_] = x / (x^2 + y^2)
```

$$\frac{x}{x^2+y^2}$$

```
d1 = D[ux[x, y], {x, 2}]
```

$$-\frac{4\,x}{\left(x^2+y^2\right)^2}+x\left(\frac{8\,x^2}{\left(x^2+y^2\right)^3}-\frac{2}{\left(x^2+y^2\right)^2}\right)$$

```
d2 = D[ux[x, y], {y, 2}]
```

$$x\left(\frac{8\,y^2}{\left(x^2+y^2\right)^3}-\frac{2}{\left(x^2+y^2\right)^2}\right)$$

```
eq2 = d1 + d2 == 0 (* 2D Laplace equation, the sum = 0 *)
```

$$-\frac{4\,x}{\left(x^2+y^2\right)^2}+x\left(\frac{8\,x^2}{\left(x^2+y^2\right)^3}-\frac{2}{\left(x^2+y^2\right)^2}\right)+x\left(\frac{8\,y^2}{\left(x^2+y^2\right)^3}-\frac{2}{\left(x^2+y^2\right)^2}\right)==0$$

```
uy[x_, y_] = y / (x^2 + y^2)
```

$$\frac{y}{x^2 + y^2}$$

```
d3 = D[uy[x, y], {x, 2}]
```

$$y\left(\frac{8\,x^2}{\left(x^2 + y^2\right)^3} - \frac{2}{\left(x^2 + y^2\right)^2}\right)$$

```
d4 = D[uy[x, y], {y, 2}]
```

$$-\frac{4\,y}{\left(x^2 + y^2\right)^2} + y\left(\frac{8\,y^2}{\left(x^2 + y^2\right)^3} - \frac{2}{\left(x^2 + y^2\right)^2}\right)$$

```
eq3 = d3 + d4 == 0 (* 2D Laplace equation, the sum = 0 *)
```

$$-\frac{4\,y}{\left(x^2 + y^2\right)^2} + y\left(\frac{8\,x^2}{\left(x^2 + y^2\right)^3} - \frac{2}{\left(x^2 + y^2\right)^2}\right) + y\left(\frac{8\,y^2}{\left(x^2 + y^2\right)^3} - \frac{2}{\left(x^2 + y^2\right)^2}\right) == 0$$

To have a look at the surfaces that makes the Laplace equation true:

plot1 = Plot3D[

$$-\frac{4\,x}{\left(x^2 + y^2\right)^2} + x\left(\frac{8\,x^2}{\left(x^2 + y^2\right)^3} - \frac{2}{\left(x^2 + y^2\right)^2}\right) + x\left(\frac{8\,y^2}{\left(x^2 + y^2\right)^3} - \frac{2}{\left(x^2 + y^2\right)^2}\right) == 0,$$

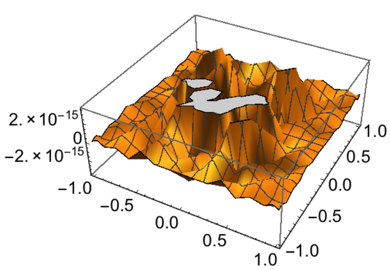
{x, -1, 1}, {y, -1, 1}];

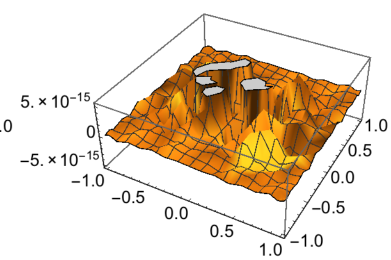
plot2 = Plot3D[

$$-\frac{4\,y}{\left(x^2 + y^2\right)^2} + y\left(\frac{8\,x^2}{\left(x^2 + y^2\right)^3} - \frac{2}{\left(x^2 + y^2\right)^2}\right) + y\left(\frac{8\,y^2}{\left(x^2 + y^2\right)^3} - \frac{2}{\left(x^2 + y^2\right)^2}\right) == 0,$$

{x, -1, 1}, {y, -1, 1}];

```
Show[plot1] Show[plot2]
```



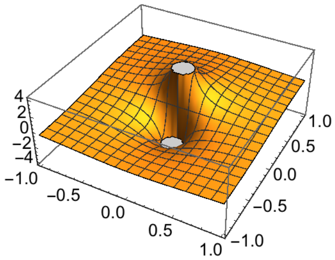


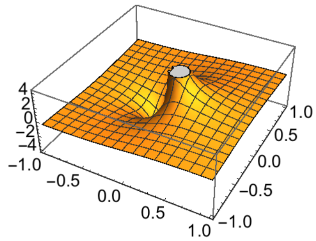
plot3 = Plot3D[$\frac{x}{x^2 + y^2}$, {x, -1, 1}, {y, -1, 1}];

plot4 = Plot3D[$\frac{y}{x^2 + y^2}$, {x, -1, 1}, {y, -1, 1}];

```
Show[plot3] Show[plot4]
```

And at the given functions:





No answer to this problem appears in the text's answer appendix.

15. Boundary value problem

Verify that the function u(x,y)= a $\log(x^2+y^2)+b$ satisfies Laplace's equation (3) and determine a and b so that u satisfies the boundary conditions u=110 on the circle $x^2+y^2=100$.

```
Clear["Global`*"]
```

This one is worked in the s.m.

```
u[x_, y_] = a Log[x^2 + y^2] + b
```

$b + a \, \mathrm{Log}\left[x^2 + y^2\right]$

```
d1 = D[u[x, y], {x, 2}]
```

$$a\left(-\frac{4x^2}{\left(x^2+y^2\right)^2}+\frac{2}{x^2+y^2}\right)$$

```
d2 = D[u[x, y], {y, 2}]
```

$$a\left(-\frac{4y^2}{\left(x^2+y^2\right)^2}+\frac{2}{x^2+y^2}\right)$$

```
FullSimplify[d1 + d2]
```

0

The Laplace equation equality is verified. The function u is a solution. Now for the boundary values.

```
Solve[a Log[100] + b == 110, {b}]
```

{{b → 110 - a Log[100]}}

```
Solve[a Log[100] + b == 110, {a}]
```

$$\left\{\left\{a \to \frac{110 - b}{\text{Log}[100]}\right\}\right\}$$

I was not overly pleased with the way the discovery of the constants a and b needed to be done. I could not find a way to do it in one step.

16 - 23 PDEs Solvable as ODEs

This happens if a PDE involves derivatives with respect to one variable only (or can be transformed to such a form), so that the other variable(s) can be treated as parameter(s). Solve for u = u(x,y):

17. $u_{xx} + 16\pi^2 u = 0$

```
Clear["Global`*"]
eqn = D[u[x, y], {x, 2}] + 16 π^2 u[x, y] == 0
```

$16\pi^2 u[x, y] + u^{(2,0)}[x, y] == 0$

```
sol = DSolve[eqn, u[x, y], {x, y}]
```

```
{{u[x, y] → Cos[4 π x] C[1][y] + Sin[4 π x] C[2][y]}}
```

Even though the independent variable y does not make an active appearance, its presence must be directly acknowledged in order to get its representation shown in the solution. The answer matches the text's.

19. $u_y + y^2 u = 0$

```
Clear["Global`*"]
eqn = D[u[x, y], {y}] + y^2 u[x, y] == 0
```

$y^2 u[x, y] + u^{(0,1)}[x, y] == 0$

```
sol = DSolve[eqn, u[x, y], {x, y}]
```

$$\left\{\left\{u[x, y] \to e^{-\frac{y^3}{3}} C[1][x]\right\}\right\}$$

The above answer matches the text's.

21. $u_{yy} + 6u_y + 13u = 4e^{3y}$

```
Clear["Global`*"]
eqn = D[u[x, y], {y, 2}] + 6 D[u[x, y], {y}] + 13 u[x, y] - 4 e^(3 y) == 0
```

$-4 e^{3y} + 13 u[x, y] + 6 u^{(0,1)}[x, y] + u^{(0,2)}[x, y] == 0$

```
sol = Simplify[DSolve[eqn, u[x, y], {x, y}]]
```

$$\left\{\left\{u[x, y] \to \frac{1}{10} e^{-3 y} \left(e^{6 y} + 10 \operatorname{Sin}[2 y] C[1][x] + 10 \operatorname{Cos}[2 y] C[2][x]\right)\right\}\right\}$$

The above answer matches the text’s.

23. $x^2 u_{xx} + 2 x u_x - 2 u = 0$

```
Clear["Global`*"]
```

```
eqn = x^2 D[u[x, y], {x, 2}] + 2 x D[u[x, y], {x}] - 2 u[x, y] == 0
```

$-2 u[x, y] + 2 x u^{(1,0)}[x, y] + x^2 u^{(2,0)}[x, y] == 0$

```
sol = Simplify[DSolve[eqn, u[x, y], {x, y}]]
```

$$\left\{\left\{u[x, y] \to x C[1][y] + \frac{C[2][y]}{x^2}\right\}\right\}$$

The above answer matches the text’s.

25. System of PDEs

Solve $u_{xx} = 0,\ u_{yy} = 0$

```
Clear["Global`*"]
```

```
eqn1 = D[u[x, y], {x, 2}] == 0
```

$u^{(2,0)}[x, y] == 0$

```
eqn2 = D[u[x, y], {y, 2}] == 0
```

$u^{(0,2)}[x, y] == 0$

$\text{DSolve}\left[\left\{\left\{u^{(2,0)}[x, y] == 0\right\}, \left\{u^{(0,2)}[x, y] == 0\right\}\right\}, u[x, y], \{x, y\}\right]$

$\text{DSolve}\left[\left\{\left\{u^{(2,0)}[x, y] == 0\right\}, \left\{u^{(0,2)}[x, y] == 0\right\}\right\}, u[x, y], \{x, y\}\right]$

After trying several variations in formatting, I find that Mathematica 10 will not do this differential equation system. I find that Mathematica 11 won’t do it either, and neither will WolframAlpha.

```
h1 = DSolve[eqn1, u[x, y], {x, y}]
{{u[x, y] → C[1][y] + x C[2][y]}}
```

```
h2 = DSolve[eqn2, u[x, y], {x, y}]
{{u[x, y] → C[1][x] + y C[2][x]}}
```

```
tot = C[1][y] + x C[2][y] + C[3][x] + y C[4][x]
```

The simplicity of the system allows it to be done by hand, by adding the partial solutions. In the above yellow cell the C[2] and C[4] terms need to be combined, and there is no isolated arbitrary constant. With these modifications, it would match the text answer.