Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

3 - 9 Path independent integrals

Show that the form under the integral sign is exact in the plane (problems 3-4) or in space (problems 5-9) and evaluate the integral.

3. $\int_{(\pi/2,\pi)}^{(\pi,0)} \left(\frac{1}{2}\operatorname{Cos}\left[\frac{1}{2}x\right]\operatorname{Cos}[2y]\,dx - 2\operatorname{Sin}\left[\frac{1}{2}x\right]\operatorname{Sin}[2y]\,dy\right)$

```
Clear["Global`*"]
```

After trying a couple ways, I decided to follow the procedure in the s.m.. I am looking for f, the function which has as its Grad the given function. Call the given function **F**. I can, following s.m., express **F** as

$$\mathbf{F} = \left\{\tfrac{1}{2}\cos\tfrac{1}{2}x\cos 2y, -2\sin\tfrac{1}{2}x\sin 2y\right\}$$

Also,

$$e1 = f_x[x\_] = \frac{1}{2}\operatorname{Cos}\left[\frac{1}{2}x\right]\operatorname{Cos}[2y]$$

$$\frac{1}{2}\operatorname{Cos}\left[\frac{x}{2}\right]\operatorname{Cos}[2y]$$

And this integrated with respect to x is,

$$e2 = f = \operatorname{Integrate}\left[\frac{1}{2}\operatorname{Cos}\left[\frac{x}{2}\right]\operatorname{Cos}[2y], x\right]$$

$$\operatorname{Cos}[2y]\operatorname{Sin}\left[\frac{x}{2}\right]$$

Above: Mathematica does not put in an integration constant. One item that would behave as a constant under differentiation and which I am interested in here is a function of y. So I follow the s.m. and put it in.

```
e3 = e2 + g[y_]
```

$$g[y\_] + \operatorname{Cos}[2y]\operatorname{Sin}\left[\frac{x}{2}\right]$$

Below: now I do the same procedure with the other half of the given expression:

$$e4 = f_y[y\_] = -2\operatorname{Sin}\left[\frac{1}{2}x\right]\operatorname{Sin}[2y]$$

$$-2\operatorname{Sin}\left[\frac{x}{2}\right]\operatorname{Sin}[2y]$$

Integrating this time with respect to y,

```
e5 = f = Integrate[-2 Sin[1/2 x] Sin[2 y], y]
```

$\text{Cos}[2\,y]\ \text{Sin}\left[\frac{x}{2}\right]$

And this time adding a self-destructing function of *x*,

```
e6 = e5 + h[x_]
```

$h[x\_] + \text{Cos}[2\,y]\ \text{Sin}\left[\frac{x}{2}\right]$

And here comparing e6 with e3, I see they are already equal without any balancing functions. Therefore I choose **g** and **h** to be zero, leaving

```
e7 = f = Cos[2 y] Sin[x/2]
```

$\text{Cos}[2\,y]\ \text{Sin}\left[\frac{x}{2}\right]$

as the candidate potential function I was looking for. Now I test it,

```
e8 = Grad[e7, {x, y}]
```

$\left\{\frac{1}{2}\text{Cos}\left[\frac{x}{2}\right]\text{Cos}[2\,y],\ -2\,\text{Sin}\left[\frac{x}{2}\right]\text{Sin}[2\,y]\right\}$

and the test is successful. There remains the problem of evaluating the answer function at the integration limits.

```
e9 = upperlimit = e7 /. {x → π, y → 0}
```

1

```
e10 = lowerlimit = e7 /. {x → π/2, y → π}
```

$\frac{1}{\sqrt{2}}$

```
e11 = e9 - e10
```

$1 - \frac{1}{\sqrt{2}}$

5. $\int_{(0,0,\pi)}^{(2,1/2,\pi/2)} e^{x\,y}\ (y\ \text{Sin}[z]\ dz + x\ \text{Sin}[z]\ dy + \text{Cos}[z]\ dz)$

```
Clear["Global`*"]
```

```
e1 = e^(x y) y Sin[z]
```

$e^{x\,y}\ y\ \text{Sin}[z]$

```
e2 = xcomponent = Integrate[e1, x]
```

$e^{x y}$ Sin[z]

```
e3 = e^(x y) x Sin[z]
```

$e^{x y}$ x Sin[z]

```
e4 = ycomponent = Integrate[e3, y]
```

$e^{x y}$ Sin[z]

```
e5 = e^(x y) Cos[z]
```

$e^{x y}$ Cos[z]

```
e6 = zcomponent = Integrate[e5, z]
```

$e^{x y}$ Sin[z]

No integration constant accompanies either e2, e4, or e6. But seeing that the integrals are equal, I will call the 'virtual' constants all zero. Though e6 looks good, I should test it,

```
e7 = Grad[e6, {x, y, z}]
```

{$e^{x y}$ y Sin[z], $e^{x y}$ x Sin[z], $e^{x y}$ Cos[z]}

The test is successful, producing the vector function form of the problem, and demonstrating that e6 is the potential function I need to solve the problem. Evaluating this function at the integration limits,

e8 = upperlimit = e6 /. $\left\{x \to 2,\ y \to \frac{1}{2},\ z \to \frac{\pi}{2}\right\}$

e

e9 = lowerlimit = e6 /. $\{x \to 0,\ y \to 0,\ z \to \pi\}$

0

```
e10 = finalanswer = e8 - e9
```

e

7. $\int_{(0,2,3)}^{(1,1,1)} (yz\ \mathrm{Sinh}[x\,z]\,dx + \mathrm{Cosh}[x\,z]\,dy + x\,t\ \mathrm{Sinh}[x\,z]\,dz)$

```
Clear["Global`*"]
```

```
e1 = y z Sinh[x z]
```

y z Sinh[x z]

```
e2 = xcomponent = Integrate[e1, x]
y Cosh[x z]

e3 = Cosh[x z]
Cosh[x z]

e4 = ycomponent = Integrate[e3, y]
y Cosh[x z]

e5 = x y Sinh[x z]
x y Sinh[x z]

e6 = zcomponent = Integrate[e5, z]
y Cosh[x z]
```

Below: the test

```
e7 = thetest = Grad[e6, {x, y, z}]
{y z Sinh[x z], Cosh[x z], x y Sinh[x z]}
```

It passes the test, identifying it as the potential function I was looking for. Evaluating this function at the integration limits,

```
e8 = upperlimit = e6 /. {x → 1, y → 1, z → 1}
Cosh[1]

e9 = lowerlimit = e6 /. {x → 0, y → 2, z → 3}
2

e10 = finalanswer = Cosh[1] - 2
-2 + Cosh[1]
```

9. $\int_{(0,1,0)}^{(1,0,1)} \left(e^x \operatorname{Cosh}[y]\, dx + \left(e^x \operatorname{Sinh}[y] + e^z \operatorname{Cosh}[y]\right) dy + e^z \operatorname{Sinh}[y]\, dz\right)$

```
Clear["Global`*"]

e1 = ⅇ^x Cosh[y]
ⅇ^x Cosh[y]

e2 = xcomponent = Integrate[e1, x]
ⅇ^x Cosh[y]

e3 = ⅇ^x Sinh[y] + ⅇ^z Cosh[y]
ⅇ^z Cosh[y] + ⅇ^x Sinh[y]
```

```
e4 = ycomponent = Integrate[e3, y]
```

```
ⅇ^x Cosh[y] + ⅇ^z Sinh[y]
```

```
e5 = ⅇ^z Sinh[y]
ⅇ^z Sinh[y]
```

```
e6 = zcomponent = Integrate[e5, z]
ⅇ^z Sinh[y]
```

The potential function is not so easy to find as in the last three problems. I need to find what to add to e2 to make it equal to e4, because e4 is my candidate potential function.

```
e7 = xtoy = Solve[e2 + r == e4, r]
```

```
{{r → ⅇ^z Sinh[y]}}
```

And, likewise what to add to e6 to make it equal to e4.

```
e8 = ztoy = Solve[e6 + s == e4, s]
```

```
{{s → ⅇ^x Cosh[y]}}
```

Okay, so there are simple factors which will make f_x as well as f_z turn into f_y. I can therefore test f_y:

```
e9 = Grad[ⅇ^x Cosh[y] + ⅇ^z Sinh[y], {x, y, z}]
{ⅇ^x Cosh[y], ⅇ^z Cosh[y] + ⅇ^x Sinh[y], ⅇ^z Sinh[y]}
```

The test is successful. e4 is the potential function. All I need to do is evaluate it at the limits of integration.

```
e10 = upperlimit = e4 /. {x → 1, y → 0, z → 1}
ⅇ
```

```
e11 = lowerlimit = e4 /. {x → 0, y → 1, z → 0}
Cosh[1] + Sinh[1]
```

```
e12 = e10 - e11
```

```
ⅇ - Cosh[1] - Sinh[1]
```

```
e13 = FullSimplify[e12]
```

```
0
```

13 - 19 Path independence?

Check, and if independent, integrate from (0,0,0) to (a,b,c).

13. $2\, e^{x^2}$ (x Cos[2 y] dx - Sin[2 y] dy)

```
Clear["Global`*"]
```

The way the problem instructions are written, it seems assumed that this one will be path dependent.

```
e1 = Curl[{2 e^(x^2) x Cos[2 y], -2 e^(x^2) Sin[2 y]}, {x, y}]
0
```

So it is independent as to path after all. I guess I have to modify the instructions slightly, and integrate from {0,0} to {a,b}.

```
e2 = Integrate[2 e^(x^2) x Cos[2 y], x]
e^(x^2) Cos[2 y]

e3 = Integrate[-2 e^(x^2) Sin[2 y], y]
e^(x^2) Cos[2 y]
```

Do the easy check:

```
e4 = Grad[e3, {x, y}]
{2 e^(x^2) x Cos[2 y], -2 e^(x^2) Sin[2 y]}
```

Do the integration:

```
e5 = upperlimit = e3 /. {x → a, y → b}
e^(a^2) Cos[2 b]

e6 = lowerlimit = e3 /. {x → 0, y → 0}
1

e7 = finalanswer = e5 - e6
-1 + e^(a^2) Cos[2 b]
```

The above answer disagrees with the text. However, I don't completely understand. The lower limit involves the cosine of 0 not the sine, and would not therefore disappear. Maybe I'm not looking at it right. The text answer is e^{a^2} Cos[2 b].

15. x^2 y dx - 4 x y^2 dy + 8 z^2 x dz

```
Clear["Global`*"]

e1 = Curl[{x^2 y, -4 x y^2, 8 z^2 x}, {x, y, z}]
{0, -8 z^2, -x^2 - 4 y^2}
```

The above is not equal to zero except in a special case; therefore the function is not path independent.

17. $4\,y\,dx + z\,dy + (y - 2\,z)\,dz$

```
Clear["Global`*"]
```

```
e1 = Curl[{4 y, z, y - 2 z}, {x, y, z}]
{0, 0, -4}
```

The above is not equal to zero except in a special case; therefore the function is not path independent.

19. $\left(\mathrm{Cos}\left[x^2 + 2\,y^2 + z^2\right]\right)\,(2\,x\,dx + 4\,y\,dy + 2\,z\,dz)$

```
Clear["Global`*"]
```

```
e1 = Curl[{Cos[x^2 + 2 y^2 + z^2] 2 x,
   Cos[x^2 + 2 y^2 + z^2] 4 y, Cos[x^2 + 2 y^2 + z^2] 2 z}, {x, y, z}]
{0, 0, 0}
```

The function is path independent.

```
e2 = xcomponent = Integrate[Cos[x^2 + 2 y^2 + z^2] 2 x, x] // Simplify
Sin[x^2 + 2 y^2 + z^2]
```

```
e3 = ycomponent = Integrate[Cos[x^2 + 2 y^2 + z^2] 4 y, y] // Simplify
Sin[x^2 + 2 y^2 + z^2]
```

```
e4 = zcomponent = Integrate[Cos[x^2 + 2 y^2 + z^2] 2 z, z] // Simplify
```

```
Sin[x^2 + 2 y^2 + z^2]
```

It’s a unanimous decision, we have a function, folks.

```
e5 = upperlimit = e4 /. {x → a, y → b, z → c}
Sin[a^2 + 2 b^2 + c^2]
```

```
e6 = lowerlimit = e4 /. {x → 0, y → 0, z → 0}
0
```

```
e7 = finalanswer = e5 - e6
```

```
Sin[a^2 + 2 b^2 + c^2]
```