

Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

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2 - 9 Power series

Where does the power series converge uniformly?

$$3. \sum_{n=0}^{\infty} \left(\frac{1}{3^n} \right)^n (z + i)^{2n}$$

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By theorem 1, p. 699, a power series in powers of $z - z_0$ converges uniformly in the closed disk $|z - z_0| \leq r$, where $r < R$ and R is the radius of convergence of the series. In other words, look for the radius of convergence.

$$\text{Series}\left[\left(\frac{1}{3^n}\right)^n (z + i)^{2n}, \{n, 1, 4\}\right];$$

The power series is in terms of $Z = (z + i)^2$, and has the form $\sum_{n=0}^{\infty} a_n Z^n$ with coefficients $\frac{1}{3^n}$. So

$$a_n = 3^{-n}$$

$$3^{-n}$$

$$a_{n+1} = 3^{-(n+1)}$$

$$3^{-1-n}$$

and

$$\frac{a_n}{a_{n+1}}$$

$$3$$

$$3$$

Since the power of the power term is $2n$, the radius of convergence R is

$$3^{1/2}$$

$$\sqrt{3}$$

The disk of uniform convergence is less than R , so a δ must be allowed so that $|z + i| \leq \sqrt{3} - \delta$, with $\delta > 0$.

$$5. \sum_{n=2}^{\infty} \text{Binomial}[n, 2] (4z + 2i)^n$$

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The form of the series can be changed to

$$\text{Sum}\left[\text{Binomial}[n, 2] 4^n \left(z + \frac{i}{2}\right)^n, \{n, 2, 8\}\right]$$

$$16 \left(\frac{i}{2} + z\right)^2 + 192 \left(\frac{i}{2} + z\right)^3 + 1536 \left(\frac{i}{2} + z\right)^4 + 10240 \left(\frac{i}{2} + z\right)^5 +$$

$$61440 \left(\frac{i}{2} + z\right)^6 + 344064 \left(\frac{i}{2} + z\right)^7 + 1835008 \left(\frac{i}{2} + z\right)^8$$

And to find the general sequence of coefficients,

$$\text{FindSequenceFunction}[\{16, 192, 1536, 10240, 61440, 344064, 1835008\}, n]$$

$$2^{1+2n} n (1+n)$$

Reaching back to get the Cauchy-Hadamard criterion, I can find the raw radius,

$$\text{Limit}\left[\text{Abs}\left[\frac{2^{1+2n} n (1+n)}{2^{3+2n} (n+1) (2+n)}\right], n \rightarrow \infty\right]$$

$$\frac{1}{4}$$

And to convert the raw radius to the actual radius of convergence, I apply the 1/n factor of the power term,

$$\left(\frac{1}{4}\right)^{1/1}$$

$$\frac{1}{4}$$

As the radius of convergence is $R = \frac{1}{4}$, I now need r such that $r + \delta = \frac{1}{4}$, where $\delta > 0$ and $\text{Abs}[z + \frac{i}{2}] \leq r$.

$$7. \sum_{n=1}^{\infty} \frac{n!}{n^2} \left(z + \frac{i}{2}\right)^n$$

`Clear["Global`*"]`

I see that the Maclaurin series does not converge, but the Taylor series does,

$$\text{Sum}\left[\frac{n!}{n^2} \left(z + \frac{i}{2}\right)^n, \{n, 1, 8\}\right]$$

$$\frac{i}{2} + z + \frac{1}{2} \left(\frac{i}{2} + z\right)^2 + \frac{2}{3} \left(\frac{i}{2} + z\right)^3 + \frac{3}{2} \left(\frac{i}{2} + z\right)^4 +$$

$$\frac{24}{5} \left(\frac{i}{2} + z\right)^5 + 20 \left(\frac{i}{2} + z\right)^6 + \frac{720}{7} \left(\frac{i}{2} + z\right)^7 + 630 \left(\frac{i}{2} + z\right)^8$$

The z and n parts of the series are already set up nicely. I can try to find the coefficients,

```
FindSequenceFunction[{1, 1/2, 2/3, 3/2, 24/5, 20, 720/7, 630}, n]
Pochhammer[1, -1 + n]
n
FullSimplify[%]
Gamma[n]
n
```

Using the Cauchy-Hadamard criterion, I can try to find the radius,

```
Limit[Abs[Gamma[n]/n * (n+1/Gamma[n+1])], n -> Infinity]
0
```

The radius of convergence is zero. The disk of uniform convergence must be strictly less than the radius of convergence, which is impossible. Therefore the series is uniformly convergent nowhere.

$$9. \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n^2} (z - 2i)^n$$

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Clear["Global`*"]
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The Maclaurin series does not converge, but the Taylor series does,

$$\text{Sum}\left[\frac{(-1)^n}{2^n n^2} (z - 2i)^n, \{n, 1, 10\}\right]$$

$$\frac{1}{2} (2i - z) + \frac{1}{16} (-2i + z)^2 - \frac{1}{72} (-2i + z)^3 + \frac{1}{256} (-2i + z)^4 - \frac{1}{800} (-2i + z)^5 +$$

$$\frac{(-2i + z)^6}{2304} - \frac{(-2i + z)^7}{6272} + \frac{(-2i + z)^8}{16384} - \frac{(-2i + z)^9}{41472} + \frac{(-2i + z)^{10}}{102400}$$

The sign needs to be adjusted on the first term in order to match the others. The z and n parts of the series seem neat and orderly. I can try to find the coefficients,

```
FindSequenceFunction[
{-1/2, 1/16, -1/72, 1/256, -1/800, 1/2304, -1/6272, 1/16384}, n]
(-1/2)^n
n^2
```

Using the Cauchy - Hadamard criterion, I can try to find the radius,

$$\text{Limit}\left[\text{Abs}\left[\frac{\left(-\frac{1}{2}\right)^n}{n^2}\left(\frac{(n+1)^2}{\left(-\frac{1}{2}\right)^{n+1}}\right)\right], n \rightarrow \infty\right]$$

2

And to convert the raw radius to the actual radius of convergence, I apply the $1/n$ factor of the power term,

$$2^{1/1}$$

2

As the radius of convergence is $R=2$, I now need r such that $r+\delta=2$, where $\delta>0$ and $\text{Abs}[z-2] \leq r$.

10 - 17 Uniform convergence

Prove that the series converges uniformly in the indicated region.

$$11. \sum_{n=1}^{\infty} \frac{z^n}{n^2}, \text{Abs}[z] \leq 1$$

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Clear["Global`*"]
```

```
SumConvergence[ $\frac{z^n}{n^2}$ , n]
```

$$\text{Abs}[z] \leq 1$$

Mathematica indicates the region of convergence, which is exactly the region I am interested in. (However, since I want to enforce r and not R , I don't think I can do it on the response by Mathematica.) The Weierstrass M-test is very easy to apply in this case. For the domain of interest, for any z^n with z from that domain,

$$\frac{z^n}{n^2} \leq \frac{1}{n^2} \quad (* \text{ for all positive } n \in \mathbb{N} *)$$

And the series $\frac{1}{n^2}$ can be used as the Weierstrass comparison series. Example 4 on p. 682 remarks that the series $\frac{1}{n^2}$ converges. Therefore by the Weierstrass M-test, the problem series converges uniformly in the indicated region.

$$13. \sum_{n=1}^{\infty} \frac{\text{Sin}[\text{Abs}[z]]^n}{n^2}, \text{ all } z$$

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Clear["Global`*"]
```

This problem is very similar to the last. The numerator of the function must either equal 1 or be less than 1, for all z . In either case

$$\frac{\text{Sin}[\text{Abs}[z]]^n}{n^2} \leq \frac{1}{n^2} \quad (* \text{ for all positive } n \in \mathbb{N} *)$$

And again the series $\frac{1}{n^2}$ can be used as the Weierstrass comparison series. Example 4 on p. 682 remarks that the series $\frac{1}{n^2}$ converges. Therefore by the Weierstrass M - test, the series converges uniformly in the indicated region.

$$15. \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} z^n, \quad \text{Abs}[z] \leq 3$$

```
Clear["Global`*"]
```

$$\text{SumConvergence}\left[\frac{(n!)^2}{(2n)!} z^n, n\right]$$

$$\text{Abs}[z] < 4$$

Mathematica tells me that the radius of convergence, R, is equal to 4. Therefore, since the problem series is a power series, the radius of uniform convergence, r, is anything smaller than R. As 3 is certainly less than 4, the series converges uniformly in the indicated region.

$$17. \sum_{n=1}^{\infty} \frac{\pi^n}{n^4} z^{2n}, \quad \text{Abs}[z] \leq 0.56$$

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Clear["Global`*"]
```

$$\text{ver} = \frac{\pi^n}{n^4} z^{2n}$$

$$\frac{\pi^n z^{2n}}{n^4}$$

```
ver1 = ver /. z -> 0.56; ver2 = ver /. z -> 0.1; ver3 = ver /. z -> 0.0;
ver4 = ver /. z -> -0.1; ver5 = ver /. z -> -0.56;
```

```
TableForm[Table[{n, NumberForm[ver1, 3], NumberForm[ver2, 3],
  ver3, NumberForm[ver4, 3], NumberForm[ver5, 3],
  NumberForm[N[1/n^2], 3]}, {n, 1, 10}], TableHeadings -> {{},
  {"", "z→0.56", "z→0.1", "z→0.0", "z→-0.1", "z→-0.56", "N[1/n^2]"}]]
```

| | z→0.56 | z→0.1 | z→0.0 | z→-0.1 | z→-0.56 | N[1/n^2] |
|----|-----------|------------------------|-------|------------------------|-----------|----------|
| 1 | 0.985 | 0.0314 | 0. | 0.0314 | 0.985 | 1. |
| 2 | 0.0607 | 0.0000617 | 0. | 0.0000617 | 0.0607 | 0.25 |
| 3 | 0.0118 | 3.83×10^{-7} | 0. | 3.83×10^{-7} | 0.0118 | 0.111 |
| 4 | 0.00368 | 3.81×10^{-9} | 0. | 3.81×10^{-9} | 0.00368 | 0.0625 |
| 5 | 0.00149 | 4.9×10^{-11} | 0. | 4.9×10^{-11} | 0.00149 | 0.04 |
| 6 | 0.000706 | 7.42×10^{-13} | 0. | 7.42×10^{-13} | 0.000706 | 0.0278 |
| 7 | 0.000375 | 1.26×10^{-14} | 0. | 1.26×10^{-14} | 0.000375 | 0.0179 |
| 8 | 0.000217 | 2.32×10^{-16} | 0. | 2.32×10^{-16} | 0.000217 | 0.0156 |
| 9 | 0.000133 | 4.54×10^{-18} | 0. | 4.54×10^{-18} | 0.000133 | 0.0123 |
| 10 | 0.0000862 | 9.36×10^{-20} | 0. | 9.36×10^{-20} | 0.0000862 | 0.01 |

The familiar series $\frac{1}{n^2}$ can be used as the Weierstrass comparison series. Example 4 on p.

682 remarks that the series $\frac{1}{n^2}$ converges. Using this series, the Weierstrass M - test demonstrates convincingly that the series

$$\frac{\pi^n}{n^4} z^{2^n} < \frac{1}{n^2}$$

by use of a sequence of successive values, with difference gap opening on increasing n, and that it therefore converges uniformly in the indicated region. (Note: I wanted to use **Solve** or **Reduce** to make a better case, but neither was able to come through with something useful.)