3. $y'' + 4 y = \delta (t - \pi)$, $y[0] = 8$, $y'[0] = 0$

```
Clear["Global`*"]

e1 = LaplaceTransform[y''[t] + 4 y[t] == DiracDelta[ (t - π)], t, s]
```

$4 \, \text{LaplaceTransform}[y[t], t, s] +$
$\quad s^2 \, \text{LaplaceTransform}[y[t], t, s] - s \, y[0] - y'[0] == e^{-\pi s}$

```
Clear["Global`*"]

e1 = LaplaceTransform[y''[t] + 4 y[t] == DiracDelta[ (t - π)], t, s]
```

$4 \, \text{LaplaceTransform}[y[t], t, s] +$
$\quad s^2 \, \text{LaplaceTransform}[y[t], t, s] - s \, y[0] - y'[0] == e^{-\pi s}$

```
e2 = e1 /. {y[0] → 8, y'[0] → 0, LaplaceTransform[y[t], t, s] → bigY}
```

$4 \, \text{bigY} - 8 \, s + \text{bigY} \, s^2 == e^{-\pi s}$

```
e3 = Solve[e2, bigY]
```

$\left\{ \left\{ \text{bigY} \to \dfrac{e^{-\pi s} \, (1 + 8 \, e^{\pi s} \, s)}{4 + s^2} \right\} \right\}$

```
e4 = e3[[1, 1, 2]]
```

$\dfrac{e^{-\pi s} \, (1 + 8 \, e^{\pi s} \, s)}{4 + s^2}$

```
e5 = InverseLaplaceTransform[e4, s, t]
```

> $8 \, \text{Cos}[2 \, t] + \text{Cos}[t] \, \text{HeavisideTheta}[-\pi + t] \, \text{Sin}[t]$

```
PossibleZeroQ[Cos[t] Sin[t] - 1/2 Sin[2 t]]
```

True

I showed in section 6.3 that **HeavisideTheta** is equivalent to **UnitStep**. Combined with the PZQ above, it makes the green cell equivalent to the text answer.

```
plot1 = Plot[e5, {t, 0, 3 π}, PlotRange → Automatic,
    PlotStyle → {Yellow, Thickness[0.003]}, ImageSize → 250];
plot2 = Plot[8 Cos[2 t] + 1/2 UnitStep[t - π] Sin[2 t], {t, 0, 3 π}, PlotRange →
    Automatic, PlotStyle → {Red, Thickness[0.01]}, ImageSize → 250];
```

Show[plot2, plot1]
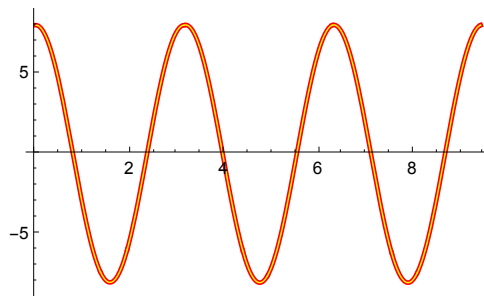
Above: The solution tracks well with that of the text.

5. $y'' + y = \delta(t - \pi) - \delta(t - 2\pi),\ y[0] = 0,\ y'[0] = 1$

Clear["Global`*"]

e1 = LaplaceTransform[
   y''[t] + y[t] == DiracDelta[t - π] - DiracDelta[t - 2 π], t, s]

LaplaceTransform[y[t], t, s] +
   s^2 LaplaceTransform[y[t], t, s] - s y[0] - y'[0] == $-e^{-2\pi s} + e^{-\pi s}$

e2 = e1 /. {y[0] → 0, y'[0] → 1, LaplaceTransform[y[t], t, s] → bigY}

$-1 + \text{bigY} + \text{bigY}\ s^2 == -e^{-2\pi s} + e^{-\pi s}$

e3 = Solve[e2, bigY]

$$\left\{\left\{\text{bigY} \to \frac{e^{-2\pi s}\left(-1 + e^{\pi s} + e^{2\pi s}\right)}{1 + s^2}\right\}\right\}$$

e4 = e3[[1, 1, 2]]

$$\frac{e^{-2\pi s}\left(-1 + e^{\pi s} + e^{2\pi s}\right)}{1 + s^2}$$

e5 = InverseLaplaceTransform[e4, s, t]

$-(-1 + \text{HeavisideTheta}[-2\pi + t] + \text{HeavisideTheta}[-\pi + t])\ \text{Sin}[t]$

e6 = e5 /. {HeavisideTheta[-2 π + t] → 0, HeavisideTheta[-π + t] → 0}

Sin[t]

Above: The answer agrees with the text for the subinterval $t < \pi$.

e7 = e5 /. {HeavisideTheta[-2 π + t] → 0, HeavisideTheta[-π + t] → 1}

0

Above: The answer agrees with the text for the subinterval $\pi < t < 2\pi$.

```
e8 = e5 /. {HeavisideTheta[-2 π + t] → 1, HeavisideTheta[-π + t] → 1}
```

$$-\text{Sin}[t]$$

Above: The answer agrees with the text for the subinterval $t > 2\pi$.

7. $4 y'' + 24 y' + 37 y = 17 e^{-t} + \delta\left(t - \dfrac{1}{2}\right)$, $y[0] = 1$, $y'[0] = 1$

```
Clear["Global`*"]
```

```
e1 = LaplaceTransform[
   4 y''[t] + 24 y'[t] + 37 y[t] == 17 e^-t + DiracDelta[t - 1/2], t, s]
```

$$37 \, \text{LaplaceTransform}[y[t], t, s] +$$
$$24 \, (s \, \text{LaplaceTransform}[y[t], t, s] - y[0]) +$$
$$4 \left(s^2 \, \text{LaplaceTransform}[y[t], t, s] - s \, y[0] - y'[0]\right) == e^{-s/2} + \frac{17}{1 + s}$$

```
e2 = e1 /. {y[0] → 1, y'[0] → 1, LaplaceTransform[y[t], t, s] → bigY}
```

$$37 \, \text{bigY} + 24 \, (-1 + \text{bigY} \, s) + 4 \left(-1 - s + \text{bigY} \, s^2\right) == e^{-s/2} + \frac{17}{1 + s}$$

```
e3 = Solve[e2, bigY]
```

$$\left\{\left\{\text{bigY} \to \frac{28 + e^{-s/2} + 4 \, s + \frac{17}{1+s}}{37 + 24 \, s + 4 \, s^2}\right\}\right\}$$

```
e4 = e3[[1, 1, 2]]
```

$$\frac{28 + e^{-s/2} + 4 \, s + \frac{17}{1+s}}{37 + 24 \, s + 4 \, s^2}$$

```
e5 = InverseLaplaceTransform[e4, s, t]
```

$$\frac{1}{4} \, e^{-\frac{i}{4} - \left(3 + \frac{i}{2}\right) t}$$
$$\left(4 \, e^{\frac{i}{4}} \left(2 \, i - 2 \, i \, e^{i \, t} + e^{\left(2 + \frac{i}{2}\right) t}\right) + i \, e^{3/2} \left(e^{\frac{i}{2}} - e^{i \, t}\right) \text{HeavisideTheta}\left[-\frac{1}{2} + t\right]\right)$$

```
e6 = FullSimplify[e5]
```

$$\frac{1}{2} \, e^{-3 \, t} \left(2 \, e^{2 \, t} - e^{3/2} \, \text{HeavisideTheta}\left[-\frac{1}{2} + t\right] \text{Sin}\left[\frac{1}{4} \, (1 - 2 \, t)\right] + 8 \, \text{Sin}\left[\frac{t}{2}\right]\right)$$

```
e7 = Expand[e6]
```

$$e^{-t} - \frac{1}{2} \, e^{\frac{3}{2} - 3 \, t} \, \text{HeavisideTheta}\left[-\frac{1}{2} + t\right] \text{Sin}\left[\frac{1}{4} \, (1 - 2 \, t)\right] + 4 \, e^{-3 \, t} \, \text{Sin}\left[\frac{t}{2}\right]$$

$\texttt{PossibleZeroQ}\Big[\Big(e^{-t} - \frac{1}{2} e^{\frac{3}{2}-3t} \texttt{Sin}\Big[\frac{1}{4}(1-2t)\Big] + 4 e^{-3t} \texttt{Sin}\Big[\frac{t}{2}\Big]\Big) -$

$\quad \Big(e^{-t} + 4 e^{-3t} \texttt{Sin}\Big[\frac{1}{2}t\Big] + \frac{1}{2}\Big(e^{-3(t-1/2)} \texttt{Sin}\Big[\frac{1}{2}t - \frac{1}{4}\Big]\Big)\Big)\Big]$

True

Above: By comparison of plots in section 6.3 I decided that **HeavisideTheta** is equivalent to **UnitStep**, the function the text prefers. Granted that, the PZQ above confirms that the green cell is equivalent to the text answer.

plot1 = Plot[e7, {t, 0, 1}, PlotRange → Automatic,
PlotStyle → {Yellow, Thickness[0.002]}, ImageSize → 250];

$\texttt{plot2 = Plot}\Big[e^{-t} + 4 e^{-3t} \texttt{Sin}\Big[\frac{t}{2}\Big] + \frac{1}{2} \texttt{UnitStep}\Big[t - \frac{1}{2}\Big] e^{-3\left(t-\frac{1}{2}\right)} \texttt{Sin}\Big[\frac{t}{2} - \frac{1}{4}\Big],$

{t, 0, 1}, PlotRange → Automatic,
$\quad \texttt{PlotStyle → {Red, Thickness[0.008]}, ImageSize → 250}\Big];$

Show[plot2, plot1]
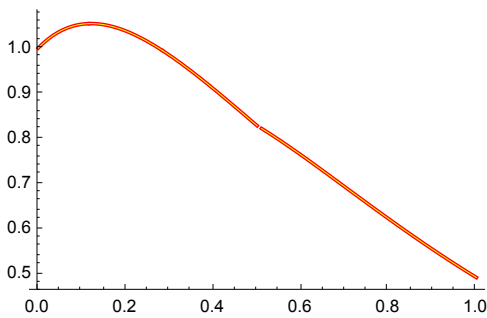
Note the interesting little gap which exists in both plots above.

9. $y'' + 4 y' + 5 y = (1 - u(t - 10)) e^t - e^{10} \delta(t - 10),$
$y[0] = 0, \ y'[0] = 1$

Clear["Global`*"]

$\texttt{e1 = LaplaceTransform}\Big[y''[t] + 4 y'[t] + 5 y[t] ==$
$\quad (1 - \texttt{UnitStep}[t - 10]) e^t - e^{10} \texttt{DiracDelta}[t - 10], t, s\Big]$

5 LaplaceTransform[y[t], t, s] + s² LaplaceTransform[y[t], t, s] +
4 (s LaplaceTransform[y[t], t, s] − y[0]) − s y[0] − y′[0] ==

$-e^{10-10s} + \frac{1}{-1+s} - \frac{e^{-10(-1+s)}}{-1+s}$

Above: Here I get thrown for a loop. The $-1 + s$ denominators are due to the exponential e^t; there doesn't seem to be any extra s added, so none needs to be removed. There is only one s that needs removing.

```
e2 = e1 /. {y[0] → 0, y'[0] → 1, LaplaceTransform[y[t], t, s] → bigY}
```

$$-1 + 5 \, bigY + 4 \, bigY \, s + bigY \, s^2 == -e^{10-10\,s} + \frac{1}{-1+s} - \frac{e^{-10\,(-1+s)}}{-1+s}$$

```
e3 = Solve[e2, bigY]
```

$$\left\{\left\{bigY \to \frac{e^{-10\,s}\left(-e^{10} + e^{10\,s}\right)s}{(-1+s)\left(5+4\,s+s^2\right)}\right\}\right\}$$

```
e4 = e3[[1, 1, 2]]
```

$$\frac{e^{-10\,s}\left(-e^{10} + e^{10\,s}\right)s}{(-1+s)\left(5+4\,s+s^2\right)}$$

```
e5 = InverseLaplaceTransform[e4, s, t]
```

$$\frac{1}{20} e^{(-2-i)\,((2+4\,i)+t)}\left((-1-i)\,e^{10\,i}\left((-3-4\,i)+(4+3\,i)\,e^{2\,i\,t}-(1-i)\,e^{(3+i)\,t}\right)+\right.$$
$$\left.\left((1-7\,i)\,e^{30+20\,i}+(1+7\,i)\,e^{30+2\,i\,t}-2\,e^{(3+i)\,((1+3\,i)+t)}\right)\right.$$
$$\left.\textsf{HeavisideTheta}[-10+t]\right)$$

```
e6 = ComplexExpand[Re[e5]];
```

Above: Marking the first time I have tried this command. It's Mr. Clean for imaginary atomics.

```
e7 = FullSimplify[e6]
```

$$\frac{1}{10} e^{-2\,t}\left(e^{3\,t} - \text{Cos}[t] + 7\,\text{Sin}[t] + \right.$$
$$\left.\left(-e^{3\,t} + e^{30}\left(\text{Cos}[10-t] + 7\,\text{Sin}[10-t]\right)\right)\text{UnitStep}[-10+t]\right)$$

Above: This looks very close to the text answer; even the UnitStep is there. However, there is a taint of suspicion. Note: Below: In entering the text answer I changed 0.1 to $\frac{1}{10}$ (two occurrences).
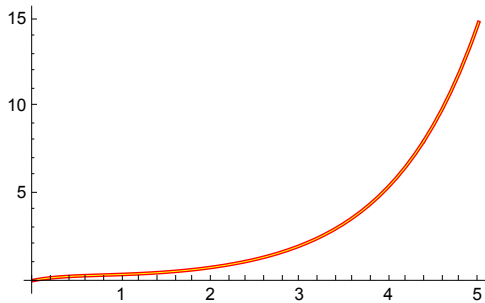
$$e8 = \frac{1}{10}\left(e^t + e^{-2\,t}\left(-\text{Cos}[t] + 7\,\text{Sin}[t]\right)\right) +$$
$$\frac{1}{10}\,\text{UnitStep}[t-10]\left(-e^{-t} + e^{-2\,t+30}\left(\text{Cos}[t-10] - 7\,\text{Sin}[t-10]\right)\right)$$
$$\frac{1}{10}\left(e^t + e^{-2\,t}\left(-\text{Cos}[t] + 7\,\text{Sin}[t]\right)\right) +$$
$$\frac{1}{10}\left(-e^{-t} + e^{30-2\,t}\left(\text{Cos}[10-t] + 7\,\text{Sin}[10-t]\right)\right)\text{UnitStep}[-10+t]$$

```
plot1 = Plot[e7, {t, 0, 5}, PlotRange → Automatic,
    PlotStyle → {Yellow, Thickness[0.002]}, ImageSize → 250];
plot2 = Plot[e8, {t, 0, 5}, PlotRange → Automatic,
    PlotStyle → {Red, Thickness[0.008]}, ImageSize → 250];
```

```
Show[plot2, plot1]
```


The two plots suggest equality on a limited range.

```
PossibleZeroQ[Chop[(0.1 e^{-2 t} (e^{3 t} - Cos[t] + 7 Sin[t] +
        (-e^{3 t} + e^{30} (Cos[10 - t] + 7 Sin[10 - t])) UnitStep[-10 + t])), 10^{-10}] -
  Chop[(0.1 (e^t + e^{-2 t} (-Cos[t] + 7 Sin[t])) + 0.1 UnitStep[-10 + t]
        (-e^{-t} + e^{-2 t+30} (Cos[t - 10] - 7 Sin[t - 10]))), 10^{-10}]]
```
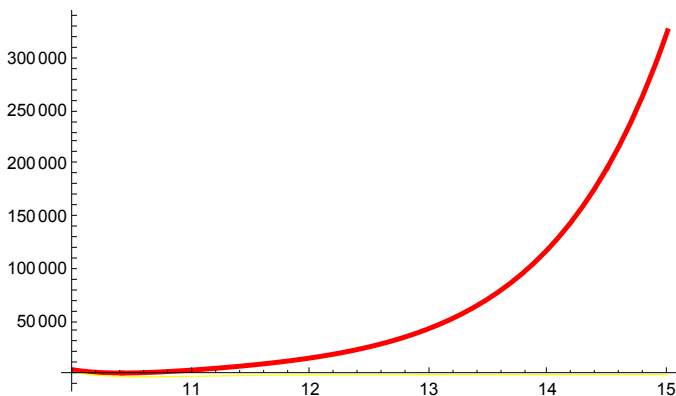
```
False
```

```
plot3 = Plot[e7, {t, 10, 15}, PlotRange → Automatic,
    PlotStyle → {Yellow, Thickness[0.002]}, ImageSize → 350];
plot4 = Plot[0.1 (e^t + e^{-2 t} (-Cos[t] + 7 Sin[t])) +
    0.1 UnitStep[-10 + t] (-e^{-t} + e^{-2 t+30} (Cos[t - 10] - 7 Sin[t - 10])),
    {t, 10, 15}, PlotRange → Automatic,
    PlotStyle → {Red, Thickness[0.008]}, ImageSize → 350];
```

```
Show[plot4, plot3]
```


The PZQ and the plots above show that the answer I got does not match the answer in the text. I will have to look into this further.

11. $y'' + 5 y' + 6 y = u (t - 1) + \delta (t - 2)$, $y[0] = 0$, $y'[0] = 1$

```
Clear["Global`*"]
```

```
e1 = LaplaceTransform[
    y''[t] + 5 y'[t] + 6 y[t] == UnitStep[t - 1] + DiracDelta[t - 2], t, s]
```

$$6 \text{ LaplaceTransform}[y[t], t, s] + s^2 \text{ LaplaceTransform}[y[t], t, s] +$$
$$5 \ (s \text{ LaplaceTransform}[y[t], t, s] - y[0]) - s \ y[0] - y'[0] == e^{-2s} + \frac{e^{-s}}{s}$$

```
e2 = e1 /. {y[0] → 0, y'[0] → 1, LaplaceTransform[y[t], t, s] → bigY}
```

$$-1 + 6 \text{ bigY} + 5 \text{ bigY} \ s + \text{bigY} \ s^2 == e^{-2s} + \frac{e^{-s}}{s}$$

```
e3 = Solve[e2, bigY]
```

$$\left\{\left\{\text{bigY} \rightarrow \frac{e^{-2s} \left(e^s + s + e^{2s} s\right)}{s \left(6 + 5 s + s^2\right)}\right\}\right\}$$

```
e4 = e3[[1, 1, 2]]
```

$$\frac{e^{-2s} \left(e^s + s + e^{2s} s\right)}{s \left(6 + 5 s + s^2\right)}$$

```
e5 = InverseLaplaceTransform[e4, s, t]
```

$$\frac{1}{6} \ e^{-3t} \left(6 \left(-1 + e^t\right) + 6 e^4 \left(-e^2 + e^t\right) \text{ HeavisideTheta}[-2 + t] + \left(e - e^t\right)^2 \left(2 e + e^t\right) \text{ HeavisideTheta}[-1 + t]\right)$$

```
e6 = -e^{-3t} + e^{-2t} + \frac{1}{6} UnitStep[t - 1] (1 - 3 e^{-2(t-1)} + 2 e^{-3(t-1)}) +
    UnitStep[t - 2] (e^{-2(t-2)} - e^{-3(t-2)})
```

$$-e^{-3t} + e^{-2t} + \left(-e^{-3(-2+t)} + e^{-2(-2+t)}\right) \text{ UnitStep}[-2 + t] +$$
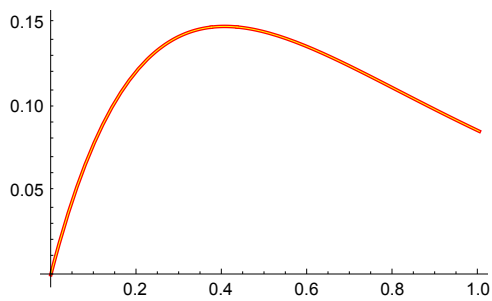$$\frac{1}{6} \left(1 + 2 e^{-3(-1+t)} - 3 e^{-2(-1+t)}\right) \text{ UnitStep}[-1 + t]$$

Above: The text answer is entered.

```
plot1 = Plot[e5, {t, 0, 1}, PlotRange → Automatic,
    PlotStyle → {Yellow, Thickness[0.002]}, ImageSize → 250];
plot2 = Plot[e6, {t, 0, 1}, PlotRange → Automatic,
    PlotStyle → {Red, Thickness[0.008]}, ImageSize → 250];
```

Show[plot2, plot1]

Above: the two plots suggest equality.

e7 = e6 /. UnitStep → HeavisideTheta

$$-e^{-3t} + e^{-2t} + \left(-e^{-3(-2+t)} + e^{-2(-2+t)}\right) \text{HeavisideTheta}[-2+t] +$$
$$\frac{1}{6}\left(1 + 2e^{-3(-1+t)} - 3e^{-2(-1+t)}\right) \text{HeavisideTheta}[-1+t]$$

FullSimplify[e5 == e7]

True

Above: So: If the UnitSteps are exchanged for Heavisides, the answers match.