4 - 8 Critical points. Linearization.

Find the location and type of all critical points by linearization.

5. $y_1' = y_2$

$y_2' = -y_1 + \frac{1}{2} y_1^2$

```
Clear["Global`*"]
```

The other way. The Jacobian would have the form: $\begin{pmatrix} 0 & 1 \\ -1 + y1 & 0 \end{pmatrix}$. So for the point (0, 0)

```
Eigensystem[{{0, 1}, {-1, 0}}]
```

```
{{i, -i}, {{-i, 1}, {i, 1}}}
```

```
ev1 = i
```

```
i
```

```
ev2 = -i
```

```
-i
```

```
p = ev1 + ev2
```

```
0
```

```
q = ev1 ev2
```

```
1
```

```
Δ = (ev1 - ev2)^2
```

```
-4
```

This would be a center point. Equals text answer. For the point (2, 0)

```
Eigensystem[{{0, 1}, {1, 0}}]
```

```
{{-1, 1}, {{-1, 1}, {1, 1}}}
```

```
ev1 = -1
```

```
-1
```

```
ev2 = 1
```

```
1
```

```
p = ev1 + ev2
```

```
0
```

```
q = ev1 ev2
```

```
-1
```

$\Delta = (\text{ev1} - \text{ev2})^2$

```
4
```

This would be a saddle point. Equals text answer.

7. $y_1{}' = -y_1 + y_2 - y_2{}^2$
$y_2{}' = -y_1 - y_2$

```
Clear["Global`*"]
```

```
Eigensystem[{{-1, 1}, {-1, -1}}]
{{-1 + i, -1 - i}, {{-i, 1}, {i, 1}}}
```

The eigenvalues are not equal, and they are not ***pure*** imaginary. So I guess the theorem holds.

$\begin{pmatrix} -1 & 1 - 2\,y2 \\ -1 & -1 \end{pmatrix}$ is the general form of the Jacobian.

```
e1 = -1 + i
e2 = -1 - i
-1 + i

-1 - i
```

```
p == e1 + e2
p == -2
```

```
q == e1 e2
q == 2
```

$\Delta = (\text{e1} - \text{e2})^2$

```
-4
```

According to Tables 4-1 and 4-2, the critical point under consideration is a spiral point, and which is stable and attractive. $p = -2,\ q = 2,\ \Delta = -4.$

An interesting implication of the answer is that in finding critical points, the derivatives of all factors count.

```
Solve[-a + b - 2 b == 0 && -a - b == 0, {a, b}]
```

Solve::svars: Equations may not give solutions for all "solve" variables≫

```
{{b → -a}}
```

Using the Jacobian system for the point $(-2, 2)$

```
Eigensystem[{{-1, -3}, {-1, -1}}]
```

$\{\{-1 - \sqrt{3}, -1 + \sqrt{3}\}, \{\{\sqrt{3}, 1\}, \{-\sqrt{3}, 1\}\}\}$

$\text{ev1} = -1 - \sqrt{3}$

$-1 - \sqrt{3}$

$\text{ev2} = -1 + \sqrt{3}$

$-1 + \sqrt{3}$

```
p = ev1 + ev2
-2
```

```
q = ev1 ev2
```

$(-1 - \sqrt{3})(-1 + \sqrt{3})$ // N

```
-2.
```

$\Delta = (\text{ev1} - \text{ev2})^2$

```
12
```

This would be a saddle point. Equals text answer.

9 - 13 Critical points of ODEs

Find the location and type of all critical points by first converting the ODE to a system and then linearizing it.

9. $y'' - 9y + y^3 = 0$

```
Clear["Global`*"]
```

$y'' = 9y - y^3$

$9y - y^3$

$y1' = y2$

$y2$

$y2' = 9\,y1 - y1^3$

$9\,y1 - y1^3$

```
Eigensystem[{{0, 1}, {9, 0}}]
{{-3, 3}, {{-1, 3}, {1, 3}}}
```

Eigenvalues not imaginary, but not equal.

```
e1 = -3
e2 = 3
-3

3

p = e1 + e2
0

q = e1 e2
-9
```

$\Delta = (e1 - e2)^2$

36

So for the critical point (0, 0) I have a saddle point by Table 4-1, and it is unstable by Table 4-2.

The text is right, for points (3, 0) and (−3, 0) the ratio of derivatives is undetermined, meaning there are also critical points there.

Stepping in here with the Jacobian system, the prototype matrix is $\begin{pmatrix} 0 & 1 \\ 9 - 3\,y1^2 & 0 \end{pmatrix}$. So for the point (3, 0),

```
Eigensystem[{{0, 1}, {-18, 0}}]
```

$\left\{\left\{3\,i\,\sqrt{2},\ -3\,i\,\sqrt{2}\right\},\ \left\{\left\{-\frac{i}{3\sqrt{2}},\ 1\right\},\ \left\{\frac{i}{3\sqrt{2}},\ 1\right\}\right\}\right\}$

```
ee1 = 3 ⅈ √2
3 ⅈ √2

ee2 = -3 ⅈ √2
-3 ⅈ √2

p = Simplify[ee1 + ee2]
0

q = Simplify[ee1 ee2]
18

Δ = Simplify[(ee1 - ee2)^2]
-72
```

This would be a center point. Agrees with text. The point $(-3, 0)$ would give the same results, also in agreement with the text.

11. y'' + Cos[y] = 0

```
Clear["Global`*"]
```

This problem is similar to Example 1 in Sec 4.5, where the sol'n is based on small angle formula for $\sin x \approx x$. Looking at the answer, it is seen that a peculiarity of the problem is that (0, 0) is not a critical point, since cos x is not zero there. cos x equals zero at $\frac{\pi}{2}$ and multiples of it.

```
y'' = -Cos[y]
-Cos[y]

y1' = y2
y2

y2' = -Cos[y1]
-Cos[y1]
```

Using the suggestion of the text answer,

$y2' = -\mathrm{Cos}[y1] = -\mathrm{Cos}\left[\pm\frac{\pi}{2} + \tilde{y}1\right] = \mathrm{Sin}\left[\pm\tilde{y}1\right] = \pm\tilde{y}1$

```
y2' = ±ỹ1
±ỹ1
```

What is $\tilde{y}1$? It is a point, something like $\left(\frac{\pi}{2}, 0\right)$. The second value (for y2) will be zero.

```
Eigensystem[{{0, 1}, {π/2, 0}}]
```

$$\left\{\left\{-\sqrt{\frac{\pi}{2}}, \sqrt{\frac{\pi}{2}}\right\}, \left\{\left\{-\sqrt{\frac{2}{\pi}}, 1\right\}, \left\{\sqrt{\frac{2}{\pi}}, 1\right\}\right\}\right\}$$

$$e1 = -\sqrt{\frac{\pi}{2}}$$

$$-\sqrt{\frac{\pi}{2}}$$

$$e2 = \sqrt{\frac{\pi}{2}}$$

$$\sqrt{\frac{\pi}{2}}$$

```
p = e1 + e2
```

0

```
q = e1 e2
```

$$-\frac{\pi}{2}$$

$$\Delta = (e1 - e2)^2$$

$$2\pi$$

So for the point $\tilde{y1} = \left(\frac{\pi}{2}, 0\right)$ I get a saddle point, just as the text said.

```
Eigensystem[{{0, 1}, {-π/2, 0}}]
```

$$\left\{\left\{i\sqrt{\frac{\pi}{2}}, -i\sqrt{\frac{\pi}{2}}\right\}, \left\{\left\{-i\sqrt{\frac{2}{\pi}}, 1\right\}, \left\{i\sqrt{\frac{2}{\pi}}, 1\right\}\right\}\right\}$$

$$b1 = i\sqrt{\frac{\pi}{2}}$$

$$i\sqrt{\frac{\pi}{2}}$$

$$b2 = -i\sqrt{\frac{\pi}{2}}$$

$$-i\sqrt{\frac{\pi}{2}}$$

```
p = b1 + b2
```

```
0
```

```
q = b1 b2
```

$$\frac{\pi}{2}$$

And for the point $\left(-\frac{\pi}{2}, 0\right)$ I get a center, again just as the text predicted.

$$\text{Cos}\left[\frac{\pi}{2} + x\right] == -\text{Sin}[x]$$

```
True
```

Checking what seemed reasonable.

```
TableForm[Table[{x, N[Sin[x], 22]},
  {x, -.010000000000, .010000000000, 0.001000000000}], 32]
```

-0.01	-0.00999983
-0.009	-0.00899988
-0.008	-0.00799991
-0.007	-0.00699994
-0.006	-0.00599996
-0.005	-0.00499998
-0.004	-0.00399999
-0.003	-0.003
-0.002	-0.002
-0.001	-0.001
0.	0.
0.001	0.001
0.002	0.002
0.003	0.003
0.004	0.00399999
0.005	0.00499998
0.006	0.00599996
0.007	0.00699994
0.008	0.00799991
0.009	0.00899988
0.01	0.00999983

Below is the answer for sin 0.001 which Mathematica is holding in memory:

```
0.0009999998333333408`
```

This is still a approximation.

13. y" + Sin[y] = 0

```
Clear["Global`*"]
```

This one looks just like the last one.

```
y'' = -Sin[y]
-Sin[y]
```

```
y1' = y2
y2
```

```
y2' = -Sin[y1]
```

The difference from the last problem may consist in the fact that sin is 0 at (0, 0).

Trying to use the Jacobian approach, $\begin{pmatrix} 0 & 1 \\ -\cos x & 0 \end{pmatrix}$ would be the Jacobian standard matrix, I believe. So for $x = \pm 2\, n\, \pi$, it should be

```
Eigensystem[{{0, 1}, {-1, 0}}]
{{ⅈ, -ⅈ}, {{-ⅈ, 1}, {ⅈ, 1}}}
```

```
e1 = ⅈ
ⅈ

e2 = -ⅈ
-ⅈ

p = e1 + e2
0

q = e1 e2
1

Δ = (e1 - e2)^2
-4
```

This would be a center point, in agreement with the text. For $x = \pi \pm 2\, n\, \pi$,

```
Eigensystem[{{0, 1}, {1, 0}}]
{{-1, 1}, {{-1, 1}, {1, 1}}}
```

This would be a saddle point, in agreement with the text. (p=0, q<0).

15. Trajectories. Write the ODE **y '' - 4 y + y³ = 0** as a system, solve it for y_2 as a function of y_1, and sketch or graph some of the trajectories in the phase plane.

I do not follow the problem's instructions to make a system.

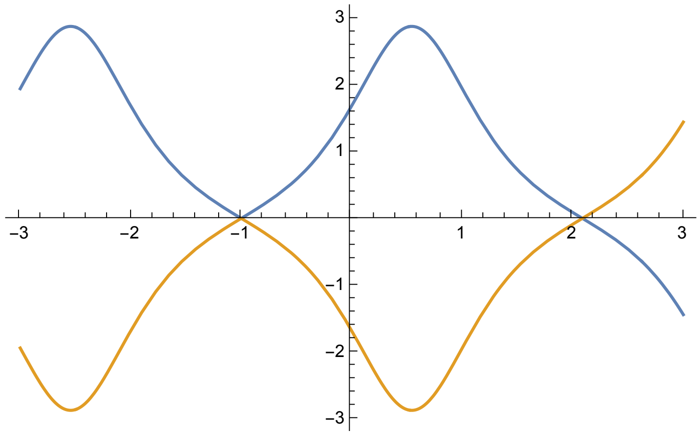
```
eqn = y''[x] - 4 y[x] + y[x]^3 == 0
-4 y[x] + y[x]^3 + y''[x] == 0

sol = DSolve[eqn, y, x];
```

Solve::ifun:
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

The solution is complex, and fairly dense. There is some success in plotting (both halves of) the solution in regular x-y space.

```
Plot[Evaluate[y[x] /. sol /. {C[1] → 1, C[2] → 1}],
 {x, -3, 3}, PlotRange → All]
```



Trying for phase space is not very successful. I am able to show one half of the solution, the 'negative' half. But it doesn't look like the text, or like what I would expect.

$$\text{StreamPlot}\Big[\Big\{y,\ -3\,\mathrm{i}\sqrt{\frac{2}{-4+3\sqrt{2}}}\ \text{JacobiSN}\Big[\frac{1}{\sqrt{2}}\left(\sqrt{\left(-4-3\sqrt{2}-8\,x-6\sqrt{2}\,x-4\,x^2-3\sqrt{2}\,x^2\right)}\right),\ \frac{4-3\sqrt{2}}{4+3\sqrt{2}}\Big]+\left(4\,\mathrm{i}\,\text{JacobiSN}\Big[\frac{1}{\sqrt{2}}\left(\sqrt{\left(-4-3\sqrt{2}-8\,x-6\sqrt{2}\,x-4\,x^2-3\sqrt{2}\,x^2\right)}\right),\ \frac{4-3\sqrt{2}}{4+3\sqrt{2}}\Big]\right)\Big/\left(\sqrt{-4+3\sqrt{2}}\right)\Big\},\ \{x,-10,10\},\ \{y,-10,10\},\ \text{Frame}\rightarrow\text{True}\Big]$$