

Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

```
Clear["Global`*"]
```

1 - 9 Integrals involving sine and cosine

Evaluate the following integrals.

$$1. \int_0^{\pi} \frac{2}{k - \cos[\theta]} d\theta$$

```
Clear["Global`*"]
```

First I will search for singularities.

```
Reduce[k - Cos[θ] == 0, {k, θ}]
```

```
C[1] ∈ Integers && (θ == -ArcCos[k] + 2 π C[1] || θ == ArcCos[k] + 2 π C[1])
```

The cells both above and below show the pattern for multiples of ArcCos[k], and allow me to use simply ArcCos[k]+2 π as the root. However, looking at the interval of evaluation for the integral, only C[1]=0 will be available. For real k, this root will be on the x axis, constituting a simple pole, and that permits use of theorem 1 on p. 731.

The above mentioned theorem 1 sets forth the answer to the integral as

$$\pi i \operatorname{Residue}\left[\frac{2}{k - \cos[\theta]}, \{\theta, \operatorname{ArcCos}[k]\}\right]$$

$$\frac{2 i \pi}{\sqrt{1 - k^2}}$$

Or,

$$\frac{2 i \pi}{\sqrt{1 - k^2}} == \frac{2 i \pi}{\sqrt{k^2 - 1} i} == \frac{2 \pi}{\sqrt{k^2 - 1}}$$

$$3. \int_0^{2\pi} \frac{1 + \sin[\theta]}{3 + \cos[\theta]} d\theta$$

```
Clear["Global`*"]
```

```
Integrate[1 + Sin[θ] / (3 + Cos[θ]), {θ, 0, 2 π}]
```

$$\frac{\pi}{\sqrt{2}}$$

I tried this a long way first, involving residue, but did not get the right answer. Just by pushing the integrate button, the right answer pops out.

$$5. \int_0^{2\pi} \frac{\cos[\theta]^2}{5 - 4 \cos[\theta]} d\theta$$

```
Clear["Global`*"]
```

```
Integrate[ $\frac{\cos[\theta]^2}{5 - 4 \cos[\theta]}$ , { $\theta$ , 0, 2  $\pi$ }]
```

$$\frac{5\pi}{12}$$

Another one matches the text answer without any preparation or application.

$$7. \int_0^{2\pi} \frac{a}{a - \sin[\theta]} d\theta$$

```
Clear["Global`*"]
```

```
Integrate[ $\frac{a}{a - \sin[\theta]}$ , { $\theta$ , 0, 2  $\pi$ }]
```

$$2 \sqrt{\frac{a^2}{-1 + a^2}} \pi = \frac{2 \sqrt{a^2}}{\sqrt{1 - a^2}} \pi = \frac{-2 a \pi}{\sqrt{a^2 - 1}}$$

Mathematica gives an answer with reversed sign, as compared to the text answer.

(Mathematica took a long think on this.) Since this is the only problem in this section where there is disagreement in answers, I will attempt to work it out the long way.

$$\int_0^{2\pi} \frac{a}{a - \sin[\theta]} d\theta = a \int_0^{2\pi} \frac{1}{a - \sin[\theta]} d\theta$$

True

Numbered line (2) on p. 726 has a way to modify the appearance of the denominator,

$$a - \sin[\theta] = a - \frac{1}{2i} \left(z - \frac{1}{z} \right)$$

and in the text below numbered line (2),

$$d\theta = \frac{dz}{iz}$$

Implying that

$$\int_0^{2\pi} \frac{1}{a - \sin[\theta]} d\theta = \oint_C \frac{dz}{iz \left(a - \frac{1}{2i} \left(z - \frac{1}{z} \right) \right)}$$

where C is the unit circle.

Operating on the denominator above,

$$i z \left(a - \frac{1}{2 i} \left(z - \frac{1}{z} \right) \right) == i z a - \frac{i z}{2 i} z + \frac{1}{2 i} \frac{i z}{z} ==$$

$$i z a - \frac{z^2}{2} + \frac{1}{2} == -\frac{1}{2} (z^2 - 2 a i z - 1)$$

making the last integral equal to

$$-2 \oint_C \frac{dz}{z^2 - 2 a i z - 1}$$

The denominator is a quadratic equation, with $a=1$, $b=-2 a i$, and $c=-1$. Solving,

$$\text{Solve}[z^2 - 2 a i z - 1 == 0, z]$$

$$\left\{ \left\{ z \rightarrow i a - \sqrt{1 - a^2} \right\}, \left\{ z \rightarrow i a + \sqrt{1 - a^2} \right\} \right\}$$

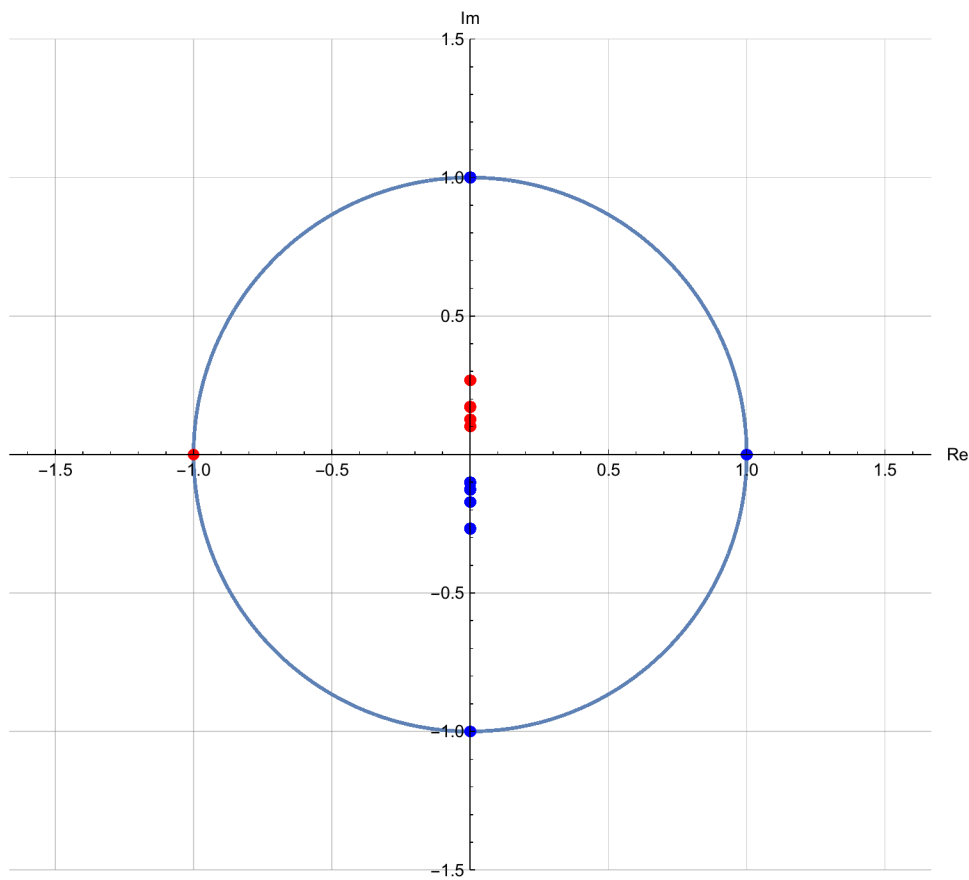
The roots will be simple poles of the problem function. A plot would be appropriate at this point.

```
p1 = ParametricPlot[{1 Cos[t], 1 Sin[t]}, {t, -π, π},
  ImageSize → 500, AxesLabel → {"Re", "Im"}, PlotRange → {-1.5, 1.5},
  PlotStyle → {Thickness[0.004]}, GridLines -> Automatic];

p1p = ListPlot[Table[{Re[i a - √(1 - a²)], Im[i a - √(1 - a²)]}, {a, -5, 5}],
  PlotStyle → {Red}];

p2p = ListPlot[Table[{Re[i a + √(1 - a²)], Im[i a + √(1 - a²)]}, {a, -5, 5}],
  PlotStyle → {Blue}];
```

Show[p1, p1p, p2p]



There appear to be 8 points inside the unit circle. For the first table below (red points), these are $n=2,3,4,5$.

```
TableForm[Table[{a, N[Re[i a - Sqrt[1 - a^2]]], N[Im[i a - Sqrt[1 - a^2]]],
  Re[i a - Sqrt[1 - a^2]], Im[i a - Sqrt[1 - a^2]]}, {a, -5, 5}]]
```

-5	0.	-9.89898	0	$-5 - 2\sqrt{6}$
-4	0.	-7.87298	0	$-4 - \sqrt{15}$
-3	0.	-5.82843	0	$-3 - 2\sqrt{2}$
-2	0.	-3.73205	0	$-2 - \sqrt{3}$
-1	0.	-1.	0	-1
0	-1.	0.	-1	0
1	0.	1.	0	1
2	0.	0.267949	0	$2 - \sqrt{3}$
3	0.	0.171573	0	$3 - 2\sqrt{2}$
4	0.	0.127017	0	$4 - \sqrt{15}$
5	0.	0.101021	0	$5 - 2\sqrt{6}$

A series like the below may not be too smart, but it is possible.

Series $\left[\frac{a}{a - \sin[\theta]}, \{a, 5 - 2\sqrt{6}, 4\}\right]$

Which I take to mean that I could examine residues in terms of a if I wanted to.

Residue $\left[\frac{a}{a - \sin[\theta]}, \{a, 5 - 2\sqrt{6}\}\right]$

0

Residue $\left[\frac{a}{a - \sin[\theta]}, \{a, 4 - 2\sqrt{6}\}\right]$

0

Residue $\left[\frac{a}{a - \sin[\theta]}, \{a, 3 - 2\sqrt{2}\}\right]$

0

Residue $\left[\frac{a}{a - \sin[\theta]}, \{a, 2 - \sqrt{3}\}\right]$

0

The cells above are probably an abuse of the Residue function. In any case, all are zero.

Next are the blue points, $n=-5,-4,-3,-2$.

TableForm $\left[\text{Table}\left[\left\{a, \text{N}\left[\text{Re}\left[i a + \sqrt{1 - a^2}\right]\right], \text{N}\left[\text{Im}\left[i a + \sqrt{1 - a^2}\right]\right], \right.\right.\right.$
 $\left.\left.\text{Re}\left[i a + \sqrt{1 - a^2}\right], \text{Im}\left[i a + \sqrt{1 - a^2}\right]\right\}, \{a, -5, 5\}\right]\right]$

-5	0.	-0.101021	0	$-5 + 2\sqrt{6}$
-4	0.	-0.127017	0	$-4 + \sqrt{15}$
-3	0.	-0.171573	0	$-3 + 2\sqrt{2}$
-2	0.	-0.267949	0	$-2 + \sqrt{3}$
-1	0.	-1.	0	-1
0	1.	0.	1	0
1	0.	1.	0	1
2	0.	3.73205	0	$2 + \sqrt{3}$
3	0.	5.82843	0	$3 + 2\sqrt{2}$
4	0.	7.87298	0	$4 + \sqrt{15}$
5	0.	9.89898	0	$5 + 2\sqrt{6}$

Residue $\left[\frac{a}{a - \sin[\theta]}, \{a, -5 + 2\sqrt{6}\}\right]$

0

```
Residue[ $\frac{a}{a - \text{Sin}[\theta]}$ , {a, -4 +  $\sqrt{15}$ }]
```

0

```
Residue[ $\frac{a}{a - \text{Sin}[\theta]}$ , {a, -3 + 2  $\sqrt{2}$ }]
```

0

```
Residue[ $\frac{a}{a - \text{Sin}[\theta]}$ , {a, -2 +  $\sqrt{3}$ }]
```

0

The residues of all the blue points also equal zero. What has been accomplished, at least, is to find eight values of a which, when inserted into the root formula, produce roots inside the unit circle. So returning to the problem function, I can ask for a table of a values, like so

```
TableForm[Table[{a, Integrate[ $\frac{a}{a - \text{Sin}[\theta]}$ , { $\theta$ , 0, 2  $\pi$ }]},
  {a, {-5, -4, -3, -2, 2, 3, 4, 5}}]]
```

-5	$\frac{5 \pi}{\sqrt{6}}$
-4	$\frac{8 \pi}{\sqrt{15}}$
-3	$\frac{3 \pi}{\sqrt{2}}$
-2	$\frac{4 \pi}{\sqrt{3}}$
2	$\frac{4 \pi}{\sqrt{3}}$
3	$\frac{3 \pi}{\sqrt{2}}$
4	$\frac{8 \pi}{\sqrt{15}}$
5	$\frac{5 \pi}{\sqrt{6}}$

All of the resulting values conform to the text answer, $\frac{2 a \pi}{\sqrt{a^2 - 1}}$, and these test cases carry a positive sign. Maybe this suggests that *Mathematica's* answer was in error. In any case it is puzzling that no minus sign is generated when evaluating integer tokens, but a minus sign is generated when evaluating symbolic tokens. What if the integral is presented in a different way?

```
Integrate[ $\frac{a}{a - \text{Sin}[\theta]}$ , { $\theta$ , 0, 2  $\pi$ }, Assumptions  $\rightarrow a \in \text{Integers}$ ]
```

```
ConditionalExpression[ $\frac{2 a \pi \text{Sign}[a]}{\sqrt{-1 + a^2}}$ ,
```

```
Im[ArcSin[a]]  $\neq$  0 || Re[ArcSin[a]]  $>$   $\pi$  ||  $\pi + \text{Re}[ \text{ArcSin}[a] ] < 0$ ]
```

The numerator of the conditional expression is structured to remain positive. This could be used as a definitive result, but I'll continue, next with a try at splitting up the integral into two pieces.

$$\text{Integrate}\left[\frac{a}{a - \sin[\theta]}, \{\theta, 0, 2\pi\}, \text{Assumptions} \rightarrow -5 \leq a \leq -2\right]$$

$$- \frac{2 a \pi}{\sqrt{-1 + a^2}}$$

The above cell agrees with the test results for specific negative values of a .

$$\text{Integrate}\left[\frac{a}{a - \sin[\theta]}, \{\theta, 0, 2\pi\}, \text{Assumptions} \rightarrow 2 \leq a \leq 5\right]$$

$$\frac{2 a \pi}{\sqrt{-1 + a^2}}$$

The above cell agrees with the test results for specific positive values of a . At this point it looks like Mathematica may have a better adapted solution, whereas the text answer may fail on blue points. (Checking again, I cannot see that negative values of a are prohibited.) The case may be better defined, but it is still a case of disagreement.

$$9. \int_0^{2\pi} \frac{\cos[\theta]}{13 - 12 \cos[2\theta]} d\theta$$

`Clear["Global`*"]`

$$\text{Integrate}\left[\frac{\cos[\theta]}{13 - 12 \cos[2\theta]}, \{\theta, 0, 2\pi\}\right]$$

0

10 - 22 Improper integrals: Infinite interval of integration

Evaluate the following integrals.

$$11. \int_{-\infty}^{\infty} \frac{1}{(1 + x^2)^2} dx$$

`Clear["Global`*"]`

$$\text{Integrate}\left[\frac{1}{(1 + x^2)^2}, \{x, -\infty, \infty\}\right]$$

$\frac{\pi}{2}$

$$13. \int_{-\infty}^{\infty} \frac{x}{(1+x^2)(x^2+4)} dx$$

```
Clear["Global`*"]
```

$$\text{Integrate}\left[\frac{x}{(1+x^2)(x^2+4)}, \{x, -\infty, \infty\}\right]$$

0

$$15. \int_{-\infty}^{\infty} \frac{x^2}{x^6+1} dx$$

```
Clear["Global`*"]
```

$$\text{Integrate}\left[\frac{x^2}{x^6+1}, \{x, -\infty, \infty\}\right]$$

$\frac{\pi}{3}$

$$17. \int_{-\infty}^{\infty} \frac{\sin[3x]}{x^4+1} dx$$

```
Clear["Global`*"]
```

$$\int_{-\infty}^{\infty} \frac{\sin[3x]}{x^4+1} dx$$

0

$$19. \int_{-\infty}^{\infty} \frac{1}{x^4-1} dx$$

```
Clear["Global`*"]
```

$$\text{Integrate}\left[\frac{1}{x^4-1}, \{x, -\infty, \infty\}, \text{PrincipalValue} \rightarrow \text{True}\right]$$

$-\frac{\pi}{2}$

In this problem, I got a non-converge warning until I added the request for principal value.

$$21. \int_{-\infty}^{\infty} \frac{\sin[x]}{(x-1)(x^2+4)} dx$$

```
Clear["Global`*"]
```



```
Integrate[ $\frac{\text{Sin}[x]}{(x-1)(x^2+4)}$ , {x, -∞, ∞}, PrincipalValue → True]
```

$$\frac{1}{5} \pi \left(-\frac{1}{e^2} + \text{Cos}[1] \right)$$

In this problem, I got a non-converge warning until I added the request for principal value.

23 - 26 Improper integrals: Poles on the real axis

Find the Cauchy principal value.

$$23. \int_{-\infty}^{\infty} \frac{1}{x^4 - 1} dx$$

```
Clear["Global`*"]
```

```
Integrate[ $\frac{1}{x^4 - 1}$ , {x, -∞, ∞}, PrincipalValue → True]
```

$$-\frac{\pi}{2}$$

$$25. \int_{-\infty}^{\infty} \frac{x+5}{x^3 - x} dx$$

```
Clear["Global`*"]
```

```
Integrate[ $\frac{x+5}{x^3 - x}$ , {x, -∞, ∞}, PrincipalValue → True]
```

$$0$$

All of the green cells in this section contain expressions which agree with the text answers.