

Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

```
Clear["Global`*"]
```

3 - 12 Residues

Find all the singularities in the finite plane and the corresponding residues.

$$3. \frac{\sin[2z]}{z^6}$$

```
Clear["Global`*"]
```

$$f1[z_] = \frac{\sin[2z]}{z^6}$$

$$\frac{\sin[2z]}{z^6}$$

```
Residue[f1[z], 0] (* singularity at 0 *)
```

$$\text{Residue}\left[\frac{\sin[2z]}{z^6}, \{z, 0\}\right]$$

$$\frac{4}{15}$$

$$5. \frac{8}{1+z^2}$$

```
Clear["Global`*"]
```

$$f2[z_] = \frac{8}{1+z^2}$$

$$\frac{8}{1+z^2}$$

The singularity is at $\pm i$.

I tried a few ways to get Mathematica to calculate both signs in a single step, but was unable to.

$$-4i$$

```
Residue[f2[z], {z, -i}]
```

$$4i$$

7. $\text{Cot}[\pi z]$

```

Clear["Global`*"]

f3[z_] = Cot[ $\pi z$ ]
Cot[ $\pi z$ ]

exec[N_] = TableForm[Table[{n, f3[n]}, {n, -N, N}]]
TableIterate[Table[{n, f3[n]}, {n, -N, N}]] does not have appropriate bounds >>
Table[{n, f3[n]}, {n, -N, N}]

```

The problem function has singularities at multiples of $z=1$

```

exec[5]
-5    ComplexInfinity
-4    ComplexInfinity
-3    ComplexInfinity
-2    ComplexInfinity
-1    ComplexInfinity
0     ComplexInfinity
1     ComplexInfinity
2     ComplexInfinity
3     ComplexInfinity
4     ComplexInfinity
5     ComplexInfinity

Residue[f3[z], {z, 1}]

```

$$\frac{1}{\pi}$$

$$9. \frac{1}{1 - e^z}$$

```

Clear["Global`*"]

f4[z_] =  $\frac{1}{1 - e^z}$ 
 $\frac{1}{1 - e^z}$ 

```

The problem function has a singularity as described below.

```
Solve[ $1 - e^z == 0$ , z]
```

```
{ {z -> ConditionalExpression[ $2 \pi i C[1]$ , C[1] ∈ Integers] } }
```

```
Residue[f4[z], {z, 2 π i}]
```

```
- 1
```

$$11. \frac{e^z}{(z - \pi i)^3}$$

```
Clear["Global`*"]
```

$$f5[z_] = \frac{e^z}{(z - \pi i)^3}$$

$$\frac{e^z}{(-i\pi + z)^3}$$

The problem function has a singularity at the multiple root shown below,

```
Solve[(-i π + z)^3 == 0, z]
```

```
{{z -> i π}, {z -> i π}, {z -> i π}}
```

```
Residue[ $\frac{e^z}{(z - \pi i)^3}$ , {z, π i}]
```

```
-  $\frac{1}{2}$ 
```

14 - 25 Residue integration

Evaluate (counterclockwise).

$$15. \oint_C \tan[2\pi z] dz, \quad C: |z - 0.2| = 0.2$$

```
Clear["Global`*"]
```

First I will investigate the singularities.

```
Solve[Cos[2 π z] == 0, z]
```

```
{{z -> ConditionalExpression[ $-\frac{\pi}{2} + 2\pi C[1]$ , C[1] ∈ Integers]},  
{z -> ConditionalExpression[ $\frac{\pi}{2} + 2\pi C[1]$ , C[1] ∈ Integers]}}
```

I can put together a plot of the path of the integral.

```
p4 = ParametricPlot[{0.2 Cos[t] + 0.2, 0.2 Sin[t]}, {t, -π, π}, ImageSize ->  
200, Epilog -> {PointSize[0.013], Point[{0.2, 0}, {1, 3}]}],  
AxesLabel -> {"Re", "Im"}, PlotRange -> {-1, 1};
```

And generate a plot of the singularities.

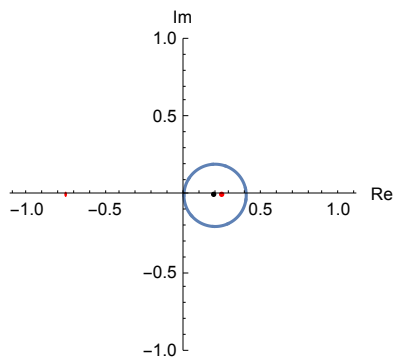
```
p1p = ListPlot[
  Table[{Re[ $\frac{\pi + 2 \pi n}{2 \pi}$ ], Im[ $\frac{\pi + 2 \pi n}{2 \pi}$ ]}, {n, -5, 5}], PlotStyle -> {Red}];
```

Since in this case the singularities have only real components, the two list plots have the same output, and it is not necessary to deal with them both.

```
p2p = ListPlot[Table[{Re[ $\frac{\pi + 2 \pi n}{2 \pi}$ ], Im[ $\frac{-\pi + 2 \pi n}{2 \pi}$ ]}, {n, -5, 5}]];
```

There is only one singularity inside the integral's path, the one at $z=1/4+0i$. I can show both plots together.

```
Show[p4, p1p]
```



```
Residue[Tan[2 π z], {z, 1 / 4}]
```

$$-\frac{1}{2\pi}$$

According to numbered line (6) on p. 723,

$$\oint_C f[z] dz = 2\pi i \sum_{j=1}^k \text{Residue}[f[z], z=z_j] \text{ where } j \text{ to } k \text{ covers the number of relevant singularities}$$

So in this case it is just a matter of

$$2\pi i \left(-\frac{1}{2\pi}\right)$$

$$-i$$

$$17. \oint_C \frac{e^z}{\cos[z]} dz, \quad C: \left| z - \frac{\pi i}{2} \right| = 4.5$$

```
Clear["Global`*"]
```

First I will investigate the singularities.

```
Solve[Cos[z] == 0, z]
```

```
{ {z -> ConditionalExpression[-\frac{\pi}{2} + 2 \pi C[1], C[1] \in Integers] },
  {z -> ConditionalExpression[\frac{\pi}{2} + 2 \pi C[1], C[1] \in Integers] } }
```

I can put together a plot of the path of the integral.

```
p5 = ParametricPlot[ {4.5 Cos[t], 4.5 Sin[t] + \frac{\pi}{2}}, {t, -\pi, \pi},
  ImageSize -> 200, Epilog -> {PointSize[0.013], Point[{0, \frac{\pi}{2}}]}},
  AxesLabel -> {"Re", "Im"}, PlotRange -> {-10, 10},
  PlotStyle -> {Thickness[0.004]}];
```

And generate a plot of the singularities.

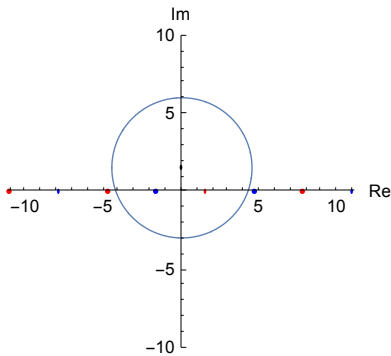
```
p1p = ListPlot[
  Table[{Re[\frac{\pi}{2} + 2 n \pi], Im[\frac{\pi}{2} + 2 n \pi]}, {n, -10, 10}], PlotStyle -> {Red}];
```

This time, since the plotted points differ, it is necessary to look at both plots.

```
p2p = ListPlot[Table[{Re[-\frac{\pi}{2} + 2 n \pi], Im[-\frac{\pi}{2} + 2 n \pi]}, {n, -10, 10}],
  PlotStyle -> {Blue}];
```

And show both plots together.

```
Show[p5, p1p, p2p]
```



The relevant singularities occur when $n = 0$, thus at $-\frac{\pi}{2} + 0i$ and $\frac{\pi}{2} + 0i$

```
Residue[\frac{e^z}{Cos[z]}, {z, -\frac{\pi}{2}}]
```

```
e^{-\pi/2}
```

$$\text{Residue}\left[\frac{e^z}{\cos[z]}, \left\{z, \frac{\pi}{2}\right\}\right]$$

$$-e^{\pi/2}$$

So this time the assemblage will be equal to

$$2\pi i \left(e^{-\pi/2} + -e^{\pi/2}\right)$$

$$2i \left(e^{-\pi/2} - e^{\pi/2}\right) \pi$$

FullSimplify[%]

$$-4i\pi \sinh\left[\frac{\pi}{2}\right]$$

$$19. \oint_C \frac{\sinh[z]}{2z - i} dz, \quad C: |z - 2i| = 2$$

Clear["Global`*"]

First I will investigate the singularities.

Solve[2 z - i == 0, z]

$$\left\{\left\{z \rightarrow \frac{i}{2}\right\}\right\}$$

I can put together a plot of the path of the integral.

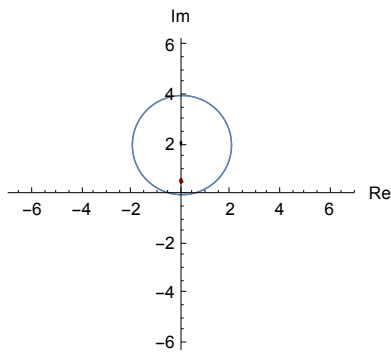
```
p7 = ParametricPlot[{2 Cos[t], 2 Sin[t] + 2}, {t, -2 π, 2 π},
  ImageSize → 200, Epilog → {PointSize[0.013], Point[{0, 2}]},
  AxesLabel → {"Re", "Im"}, PlotRange → {-2 π, 2 π},
  PlotStyle → {Thickness[0.004]}];
```

And generate a plot of the singularities.

```
p1p = ListPlot[Table[{Re[i/2], Im[i/2]}, {n, -10, 10}], PlotStyle → {Red}];
```

And show both plots together.

Show[p7, p1p]



The sole singularity is relevant.

Residue $\left[\frac{\text{Sinh}[z]}{2z - i}, \{z, \frac{i}{2}\}\right]$

$$\frac{1}{2} i \sin\left[\frac{1}{2}\right]$$

Oddly, the residue calculated by Mathematica does not agree with the text answer. The text answer looks the same, except $\sin\left[\frac{1}{2}\right]$ is shown as $\sinh\left[\frac{1}{2}\right]$. Comparing the two for a possible typo,

N[Sin[1 / 2]]

0.479426

N[Sinh[1 / 2]]

0.521095

Hmm. Anyway, this time the developed answer will be equal to

$$2\pi i \left(\frac{1}{2} i \sin\left[\frac{1}{2}\right] \right)$$

$$-\pi \sin\left[\frac{1}{2}\right]$$

And in this answer, agreement is found with that of the text.

$$21. \oint_C \frac{\cos[\pi z]}{z^5} dz, \quad C: |z| = \frac{1}{2}$$

Clear["Global`*"]

First I will investigate the singularities.

Solve[z⁵ == 0, z]

{{z -> 0}, {z -> 0}, {z -> 0}, {z -> 0}, {z -> 0}}

I can put together a plot of the path of the integral.

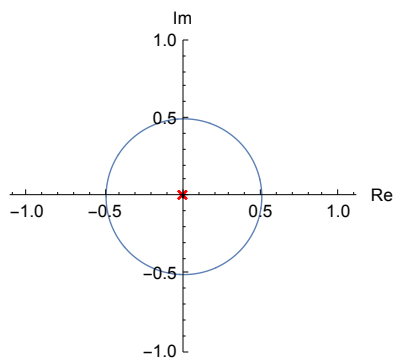
```
p8 = ParametricPlot[{1/2 Cos[t], 1/2 Sin[t]}, {t, -π, π},
  ImageSize → 200, Epilog -> {PointSize[0.013], Point[{0, 0}]},
  AxesLabel → {"Re", "Im"}, PlotRange → {-1, 1},
  PlotStyle → {Thickness[0.004]}];
```

And generate a plot of the singularities.

```
p1p = ListPlot[Table[{Re[0], Im[0]}, {n, -2, 2}],
  PlotStyle → {Red}, PlotMarkers → {"x", 12}];
```

And show both plots together.

```
Show[p8, p1p]
```



The sole singularity is relevant.

```
Residue[Cos[π z], {z, 0}]
```

$$\frac{\pi^4}{24}$$

Another case where the residue as calculated by Mathematica does not match exactly with the text answer. In this case the text answer shows a z factor in the denominator. Ignoring this snag, the final assemblage will be equal to

$$2\pi i \left(\frac{\pi^4}{24} \right)$$

$$\frac{i\pi^5}{12}$$

The above answer matches that of the text.

$$23. \oint_C \frac{30z^2 - 23z + 5}{(2z - 1)^2(3z - 1)} dz, \quad C: \text{the unit circle}$$


```
Clear["Global`*"]
```

First I will investigate the singularities.

```
Solve[(2 z - 1)^2 (3 z - 1) == 0, z]
{{z -> 1/3}, {z -> 1/2}, {z -> 1/2}}
```

I can put together a plot of the path of the integral.

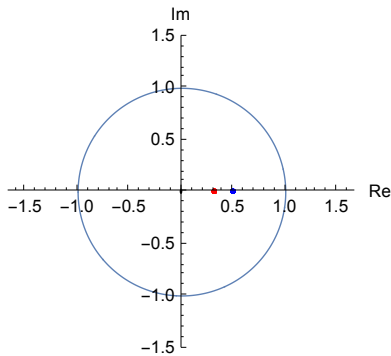
```
p9 = ParametricPlot[{1 Cos[t], 1 Sin[t]}, {t, -π, π},
  ImageSize -> 200, Epilog -> {PointSize[0.013], Point[{0, 0}]},
  AxesLabel -> {"Re", "Im"}, PlotRange -> {-1.5, 1.5},
  PlotStyle -> {Thickness[0.004]}];
```

And generate a plot of the singularities.

```
p1p = ListPlot[Table[{Re[1/3], Im[1/3]}, {n, -10, 10}], PlotStyle -> {Red}];
p2p = ListPlot[Table[{Re[1/2], Im[1/2]}, {n, -10, 10}], PlotStyle -> {Blue}];
```

And show three plots together.

```
Show[p9, p1p, p2p]
```



Evidently two singularities are relevant.

```
Residue[30 z^2 - 23 z + 5 / ((2 z - 1)^2 (3 z - 1)), {z, 1/3}]
```

2

```
Residue[30 z^2 - 23 z + 5 / ((2 z - 1)^2 (3 z - 1)), {z, 1/2}]
```

$\frac{1}{2}$

The consolidated answer is

$$2\pi i \left(2 + \frac{1}{2}\right)$$

$$5i\pi$$

Green cells match text answers.

$$25. \oint_C \frac{z \cosh[\pi z]}{z^4 + 13z^2 + 36} dz, \quad C: |z| = \pi$$

```
Clear["Global`*"]
```

First I will investigate the singularities.

```
Solve[z^4 + 13 z^2 + 36 == 0, z]
{{z -> -2 i}, {z -> 2 i}, {z -> -3 i}, {z -> 3 i}}
```

I can put together a plot of the path of the integral.

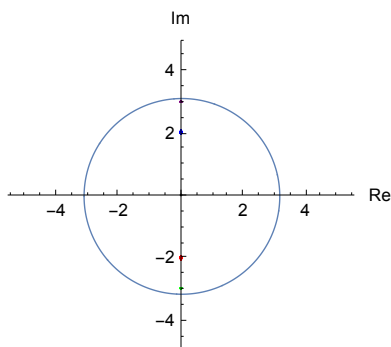
```
p10 = ParametricPlot[{π Cos[t], π Sin[t]}, {t, -π, π},
  ImageSize -> 200, Epilog -> {PointSize[0.013], Point[{0, 0}]},
  AxesLabel -> {"Re", "Im"}, PlotRange -> {-5, 5},
  PlotStyle -> {Thickness[0.004]};
```

And generate plots of the singularities.

```
p1p =
  ListPlot[Table[{Re[-2 i], Im[-2 i]}, {n, -10, 10}], PlotStyle -> {Red}];
p2p = ListPlot[Table[{Re[2 i], Im[2 i]}, {n, -10, 10}], PlotStyle -> {Blue}];
p3p = ListPlot[
  Table[{Re[-3 i], Im[-3 i]}, {n, -10, 10}], PlotStyle -> {Green}];
p4p =
  ListPlot[Table[{Re[3 i], Im[3 i]}, {n, -10, 10}], PlotStyle -> {Purple}];
```

And show five plots together.

```
Show[p10, p1p, p2p, p3p, p4p]
```



The combined plot shows four relevant singularities.

$$\text{Residue}\left[\frac{z \cosh[\pi z]}{z^4 + 13 z^2 + 36}, \{z, -2 \text{ i}\}\right]$$

$$\frac{1}{10}$$

$$\text{Residue}\left[\frac{z \cosh[\pi z]}{z^4 + 13 z^2 + 36}, \{z, 2 \text{ i}\}\right]$$

$$\frac{1}{10}$$

$$\text{Residue}\left[\frac{z \cosh[\pi z]}{z^4 + 13 z^2 + 36}, \{z, -3 \text{ i}\}\right]$$

$$\frac{1}{10}$$

$$\text{Residue}\left[\frac{z \cosh[\pi z]}{z^4 + 13 z^2 + 36}, \{z, 3 \text{ i}\}\right]$$

$$\frac{1}{10}$$

So the consolidated answer will be

$$2 \pi \text{ i} \left(\frac{4}{10} \right)$$

$$\frac{4 \text{ i} \pi}{5}$$