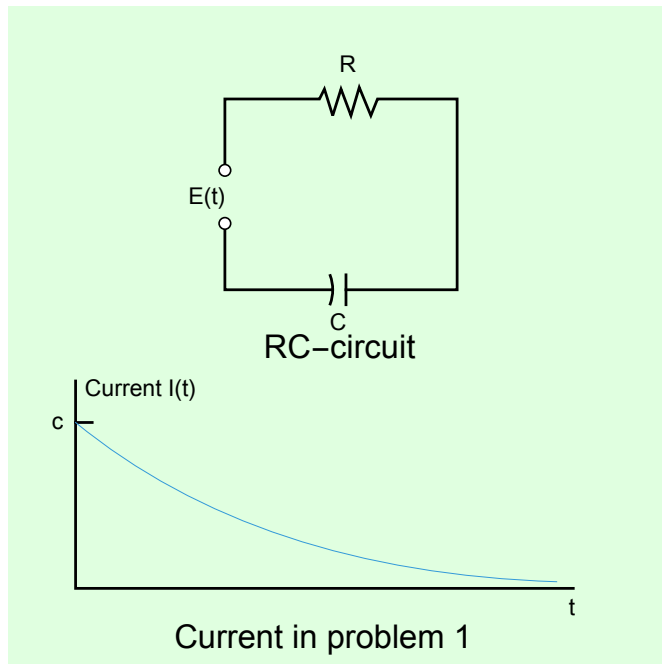


1 - 6 RLC-Circuits: special cases

1. RC-Circuit. Model the RC-Circuit in the figure below. Find the current due to a constant E.



```
In[9]:= Clear["Global`*"]
```

The problem is asking for a look at RC circuit, not RLC.

The site <https://www.intmath.com/differential-equations/6-rc-circuits.php> assumes a constant voltage source, just what the problem specifies. Below: There is no inductance here, only R and C.

```
In[99]:= eqnw = rR (D[eye[t], t]) + eye[t] / cC == 0
```

```
Out[99]:= 
$$\frac{\text{eye}[t]}{cC} + rR \text{eye}'[t] == 0$$

```

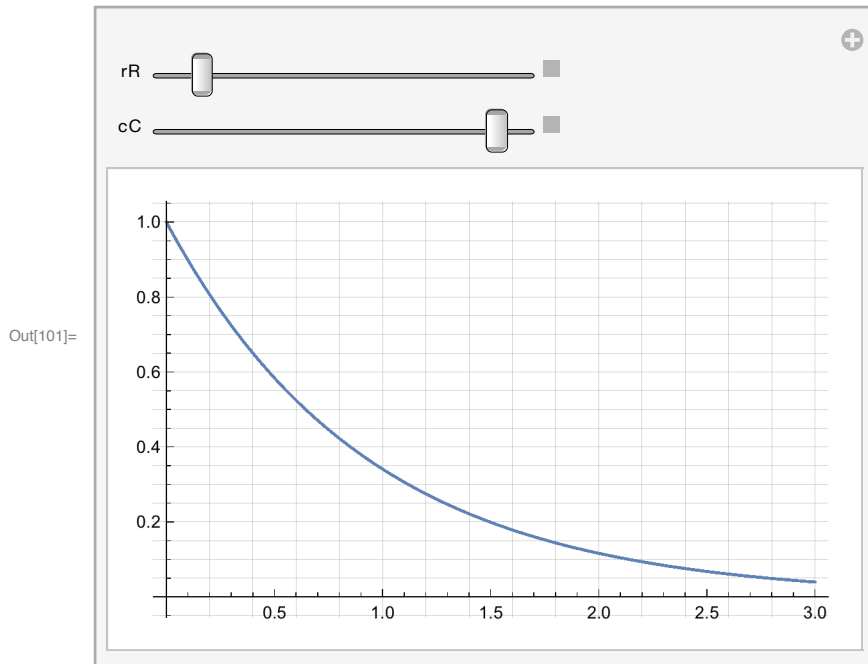
Within a certain range of capacitance and resistance, the plot resembles the one in the problem description, and can be manipulated to imitate changing parameters, with the voltage remaining constant.

```
In[100]:= sol2 = DSolve[eqnw, eye, t]
```

```
Out[100]:= {{eye -> Function[{t}, e- $\frac{t}{cC rR}$  C[1]]}}
```

It looks like the current is normalized to 1 at $t=0$, and the fraction of its max value at a given time needs to be estimated from the underlying grid.

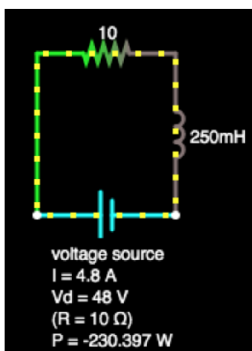
```
In[101]:= Manipulate[Plot[E^(-t/(cC rR)), {t, 0, 3}, PlotRange -> All, GridLines -> All], {rR, 0.2, 10}, {cC, 0.01, 1}]
```



A random scrap from a different perspective, kept as interesting junk.

```
{ind, cap, res} = {l i'[t] == v_l[t], v_c'[t] == 1/c i[t], r i[t] == v_r[t]};
kirchhoff = v_l[t] + v_c[t] + v_r[t] == v_s[t];
```

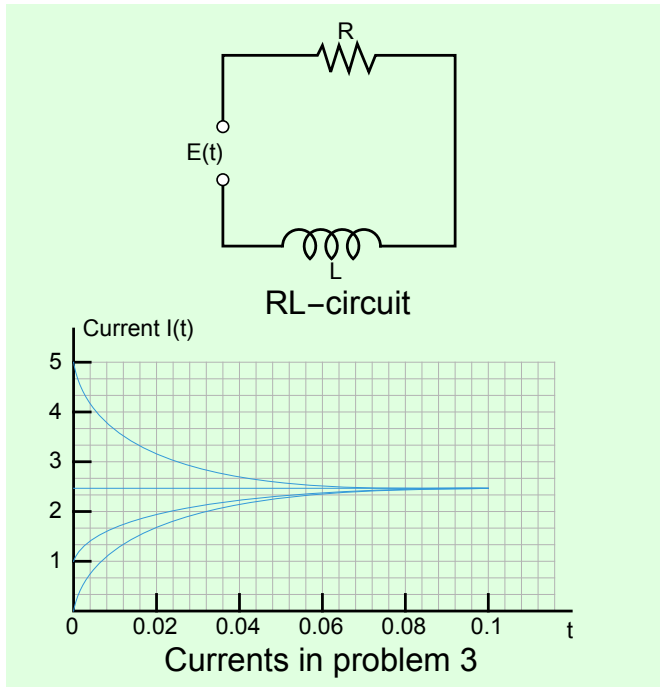
3. RL-Circuit. Model the RL-circuit in the figure below. Find a general solution when R , L , E are any constants. Graph or sketch solutions when $L = 0.25$ H, $R = 10 \Omega$, and $E = 48$ V.



The above screenshot came from the online app at <https://falstad.com/circuit/>. The current it shows agrees with the old formula for current, $I=E/R$, and was captured after the resistance had plenty of time to decay. And that's all it is, except that there is a time constant to apply. The time constant becomes ever smaller as the operation time increases. Since the

problem description talks in terms of a constant state, it seems the time constant would become vanishingly small, leaving merely $I = E/R = 4.8$ amps.

In[205]:= **Clear["Global`*"]**



When there are a lot of variables to watch, the Manipulate command is the only way I know to get an overview. The box below is based on the material at <https://www.electronics-tutorials.ws/inductor/lr-circuits.html> and may not agree with the text in detail.

In[206]:= **eye[vee_, are_, ell_, tee_] = $\frac{\text{vee}}{\text{are}} \left(1 - e^{-\frac{\text{are tee}}{\text{ell}}} \right)$**

Out[206]=
$$\frac{\left(1 - e^{-\frac{\text{are tee}}{\text{ell}}} \right) \text{vee}}{\text{are}}$$

It takes some time for the current to reach its max value. From $t=0.4$ on in the green grid below, the circuit current is nominal.

In[207]:= **Grid[Table[{tee, eye[48, 10, 0.25, tee]}, {tee, 0, 0.6, 0.1}], Frame → All]**

Out[207]=

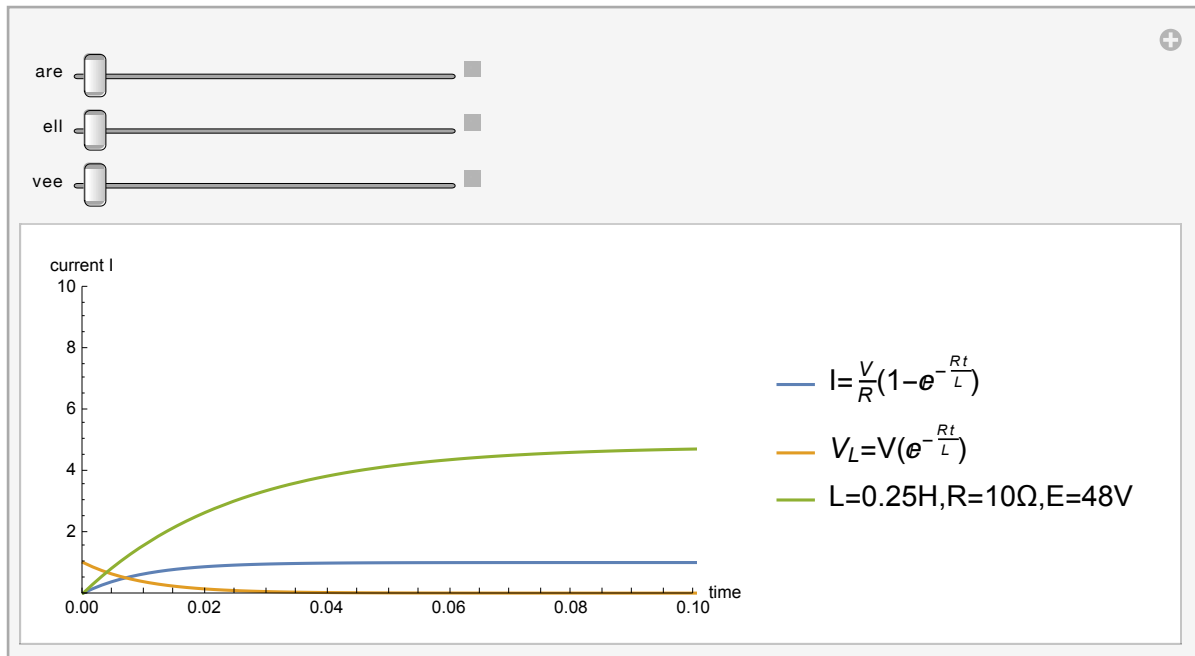
0.	0.
0.1	4.71208
0.2	4.79839
0.3	4.79997
0.4	4.8
0.5	4.8
0.6	4.8

In[208]:= `veel[vee_, are_, ell_, tee_] = vee (e- $\frac{are\ tee}{ell}$)`

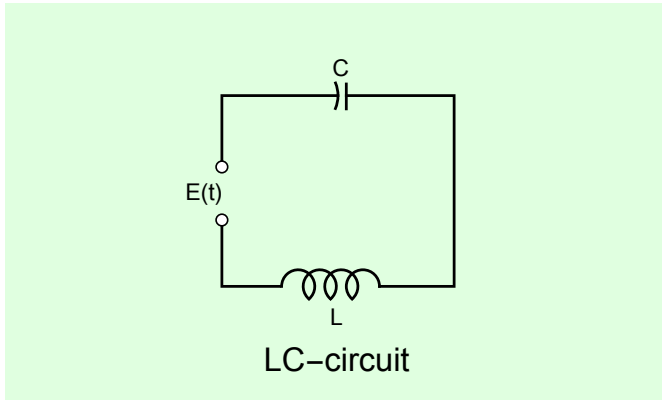
Out[208]= $e^{-\frac{are\ tee}{ell}} vee$

In[209]:= `Manipulate[
 Plot[{Abs[eye[vee, are, ell, tee]], Abs[veel[vee, are, ell, tee]],
 Abs[eye[48, 10, 0.25, tee]]}, {tee, 0, 5},
 PlotLegends → {"I = $\frac{V}{R}(1 - e^{-\frac{Rt}{L}}$ ", " $V_L = V(e^{-\frac{Rt}{L}})$ ", "L=0.25H, R=10Ω, E=48V"},
 PlotRange → {{0, 0.1}, {0, 10}}, AxesLabel → {"time", "current I"},
 AspectRatio → 0.5], {are, 1, 200}, {ell, 0.01, 10}, {vee, 1, 50}]`

Out[209]=



5. LC-Circuit. This is an RLC-circuit with negligibly small R (analog of an undamped mass-spring system). Find the current when $L=0.5$ H, $C = 0.005$ F, and $E = \sin[t$ V], assuming zero initial current and charge.



I ran across a couple of snippets, including one from the *Mathematica* documentation, suggesting that state space modeling would be a good way to look at circuits in Mathematica. I use it here.

```
In[102]:= Clear["Global`*"]
```

```
In[103]:= eqns = {eL q''[t] + aR q'[t] + 1/cC q[t] == Vee[t]};
```

```
In[104]:= m1 = StateSpaceModel[eqns,
  {{q[t], 0}, {q'[t], 0}}, {{Vee[t], 0}}, {q'[t]}, t]
```

```
Out[104]= 
$$\left( \begin{array}{cc|c} 0 & 1 & 0 \\ -\frac{1}{cC eL} & -\frac{aR}{eL} & \frac{1}{eL} \\ \hline 0 & 1 & 0 \end{array} \right) S$$

```

Here I put in the given parameters, taking the opportunity to equate the resistance with zero.

```
In[115]:= ms = m1 /. {cC -> 0.005, eL -> 0.5, aR -> 0}
```

```
Out[115]= 
$$\left( \begin{array}{cc|c} 0 & 1 & 0 \\ -400. & 0. & 2. \\ \hline 0 & 1 & 0 \end{array} \right) S$$

```

The way to get output from a state space model is to use the command **OutputResponse**. Since the voltage depends on a periodic function, I drop the V for the input field, the voltage, following the lead of the s.m.. (Although I'm starting to think that the "V" in the problem description may have just been a label.)

```
In[116]:= outz = OutputResponse[{ms}, Sin[t], t]
Out[116]= { (1.46082 × 10-17 + 0.0526316 i)
  ( (0. + 0.0952381 i) Cos[20. t] - (0. + 1. i) Cos[19. t] Cos[20. t] +
    (0. + 0.904762 i) Cos[20. t] Cos[21. t] -
    (1.66533 × 10-16 - 7.21645 × 10-17 i) Cos[20. t] Sin[19. t] -
    (2.24688 × 10-17 + 6.60847 × 10-19 i) Sin[20. t] +
    (2.35922 × 10-16 + 6.93889 × 10-18 i) Cos[19. t] Sin[20. t] -
    (2.13454 × 10-16 + 6.27805 × 10-18 i) Cos[21. t] Sin[20. t] +
    (5.96745 × 10-17 - 1. i) Sin[19. t] Sin[20. t] +
    (1.50673 × 10-16 - 6.52917 × 10-17 i) Cos[20. t] Sin[21. t] -
    (5.39912 × 10-17 - 0.904762 i) Sin[20. t] Sin[21. t]) }
```

It is necessary to clean up the result with a small **Chop**.

```
In[140]:= outt = Chop[ComplexExpand[Re[outz]], 10-16] // FullSimplify
Out[140]= {0.00501253 Cos[1. t] - 0.00501253 Cos[20. t] + 3.46945 × 10-18 Cos[39. t]}
```

Recognizing the periodic value of cosine, I can get the expression ready for a second chop by doing

```
In[127]:= outtff = outt /. Cos[39. t] → 1
Out[127]= {3.46945 × 10-18 + 0.00501253 Cos[1. t] - 0.00501253 Cos[20. t]}
```

And then the **Chop**.

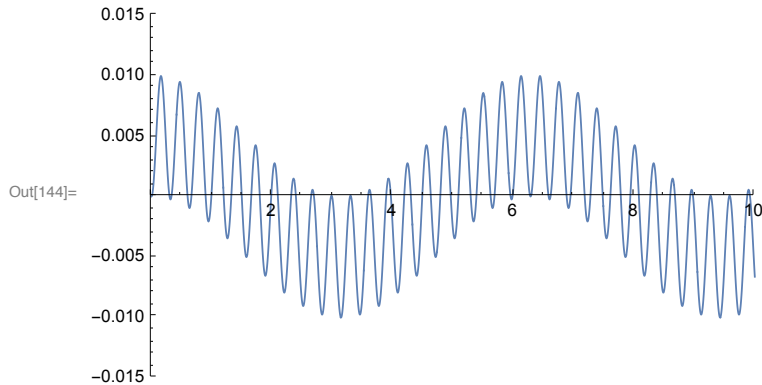
```
In[129]:= outtfff = Chop[%, 10-17]
Out[129]= {0.00501253 Cos[1. t] - 0.00501253 Cos[20. t]}
```

Testing the identity of those coefficients

```
In[118]:= 1 / 0.005012531328320802`
Out[118]= 199.5
```

I find that the answer matches the text answer, justifying the green coloration above.
The plot is interesting.

```
In[144]:= Plot[outtff, {t, 0, 10}, ImageSize → 350, AspectRatio → 0.6,
  PlotRange → {{-0.01, 10}, {-0.015, 0.015}}, PlotStyle → Thickness[0.003]]
```



7 - 18 General RLC-circuits

7. Tuning. In tuning a stereo system to a radio station, we adjust the tuning control (turn a knob) that changes C (or perhaps L) in an RLC-circuit so that the amplitude of the steady-state current, numbered line (5), p. 95 becomes maximum. For what C will this happen?

It is where the particular solution of the homogeneous equation is maximized. Numbered line (5) looks like

$$I_p(t) = I_0 \sin[\omega t - \theta]$$

The quantity θ is known as the phase lag, and, I suppose, the signal is best, I_p maximized, when θ equals zero.

8 - 14 Find the steady-state current in the RLC-circuit in the figure below for the given data.

9. $R = 4 \Omega$, $L = 0.1 \text{ H}$, $C = 0.05 \text{ F}$, $E = 110 \text{ V}$

$$L D[q[t], \{t, 2\}] + R D[q[t], t] - \frac{1}{C} q[t] = v[t]$$

$$\text{eqn} = 0.1 q''[t] + 4 q'[t] - \frac{1}{0.05} q[t] == 110$$

$$-20. q[t] + 4 q'[t] + 0.1 q''[t] == 110$$

$$\text{sol} = \text{DSolve}[\text{eqn}, q, t]$$

$$\left\{ \left\{ q \rightarrow \text{Function}[t, -5.5 + e^{-44.4949 t} C[1] + e^{4.4949 t} C[2]] \right\} \right\}$$

If $C[1]=C[2]=0$, then the green cell above matches the text answer.

$$11. R = 12 \, \Omega, L = 0.4 \, \text{H}, C = \frac{1}{80} \text{F}, E = 220 \sin[10 t] \, \text{V}$$

The state space method has been working where former methods I tried did not, so it makes sense to stick with it.

```
In[102]:= Clear["Global`*"]
```

```
In[103]:= eqns = {eL q'[t] + aR q[t] + 1/cC q[t] == Vee[t]};
```

```
In[104]:= m1 = StateSpaceModel[eqns,
  {{q[t], 0}, {q'[t], 0}}, {{Vee[t], 0}}, {q'[t]}, t]
```

$$\text{Out[104]} = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -\frac{1}{cC} & -\frac{aR}{eL} & \frac{1}{eL} \\ 0 & 1 & 0 \end{array} \right) S$$

Here I put in the given parameters.

```
In[145]:= ms = m1 /. {cC -> 1/80, eL -> 0.4, aR -> 12}
```

$$\text{Out[145]} = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -200. & -30. & 2.5 \\ 0 & 1 & 0 \end{array} \right) S$$

The way to get output from a state space model is to use the command **OutputResponse**.

```
In[146]:= outz = OutputResponse[{ms}, 220 Sin[10 t], t]
```

$$\text{Out[146]} = \left\{ 0. + e^{-30. t} \left(22. e^{10. t} - 27.5 e^{20. t} - 7.10543 \times 10^{-15} e^{20. t} \cos[10. t] + 5.5 e^{30. t} \cos[10. t] + 7.10543 \times 10^{-15} e^{20. t} \sin[10. t] + 16.5 e^{30. t} \sin[10. t] + 7.10543 \times 10^{-15} e^{40. t} \sin[10. t] \right) \right\}$$

It is necessary to clean up the result with a **Chop**.

```
In[153]:= outt = Chop[outz, 10^-14] // FullSimplify
```

$$\text{Out[153]} = \left\{ 22. e^{-20. t} - 27.5 e^{-10. t} + 5.5 \cos[10. t] + 16.5 \sin[10. t] \right\}$$

I guess the e factors can be dropped if they are small enough, say, at 3 seconds.

```
In[150]:= N[-27.500000000000007 e^{-10. t}] /. t -> 3
```

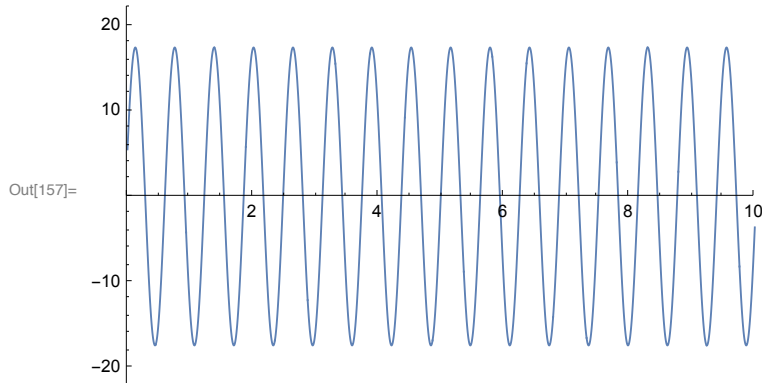
$$\text{Out[150]} = -2.57335 \times 10^{-12}$$

Evidently the text considers that size to be negligible, leaving

$$5.5 \cos[10. t] + 16.5 \sin[10. t]$$

as the answer. The plot looks routine.


```
In[157]:= Plot[5.5 Cos[10. t] + 16.5 Sin[10. t],
  {t, 0, 10}, ImageSize → 350, AspectRatio → 0.6,
  PlotRange → {{-0.01, 10}, {-22, 22}}, PlotStyle → Thickness[0.003]]
```



13. $R = 12$, $L = 1.2$ H, $C = \frac{20}{3} \times 10^{-3}$ F, $E = 12,000 \sin[25 t]$ V

$$C = \frac{20}{3} * \frac{1}{1000} = \frac{20}{3000} = \frac{2}{300}$$

```
In[185]:= Clear["Global`*"]
```

```
In[186]:= eqns = {eL q''[t] + aR q'[t] + 1/cC q[t] == Vee[t]};
```

```
In[187]:= m1 = StateSpaceModel[eqns,
  {{q[t], 0}, {q'[t], 0}}, {{Vee[t], 0}}, {q'[t]}, t]
```

Out[187]=

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -\frac{1}{cC} & -\frac{aR}{eL} & \frac{1}{eL} \\ 0 & 1 & 0 \end{array} \right) S$$

Here I put in the given parameters.

```
In[188]:= ms = m1 /. {cC → 20/3 * 10^-3, eL → 1.2, aR → 12}
```

Out[188]=

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -125. & -10. & 0.833333 \\ 0 & 1 & 0 \end{array} \right) S$$

The way to get output from a state space model is to use the command **OutputResponse**.

```
In[189]:= outz = OutputResponse[{ms}, 12 000 Sin[25 t], t]
Out[189]= { (0. + 0. i) - (400. + 1.56319 × 10-14 i) e-5. t
  ((-1. + 0. i) Cos[10. t] + (1. + 0. i) e5. t Cos[10. t]2 Cos[25. t] +
  (0.75 - 4.80505 × 10-16 i) Sin[10. t] -
  (3.19744 × 10-16 - 3.21521 × 10-16 i) e5. t Cos[10. t] Cos[25. t]
  Sin[10. t] + (1. + 2.45581 × 10-16 i) e5. t Cos[25. t] Sin[10. t]2 -
  (0.5 - 7.49623 × 10-17 i) e5. t Cos[10. t]2 Sin[25. t] +
  (1.42109 × 10-16 - 9.97247 × 10-17 i) e5. t Cos[10. t] Sin[10. t]
  Sin[25. t] - (0.5 - 1.39035 × 10-16 i) e5. t Sin[10. t]2 Sin[25. t]) }
```

It is necessary to clean up the result with a **Chop**.

```
In[192]:= outt = Chop[ComplexExpand[Re[outz]], 10-15] // Simplify
Out[192]= {-300. e-5. t Sin[10. t] +
  Cos[10. t] (400. e-5. t + 1.27898 × 10-13 Cos[25. t] Sin[10. t]) -
  2.84217 × 10-14 Sin[20. t] Sin[25. t] +
  Cos[10. t]2 (-400. Cos[25. t] + 200. Sin[25. t]) +
  Sin[10. t]2 (-400. Cos[25. t] + 200. Sin[25. t]) }
```

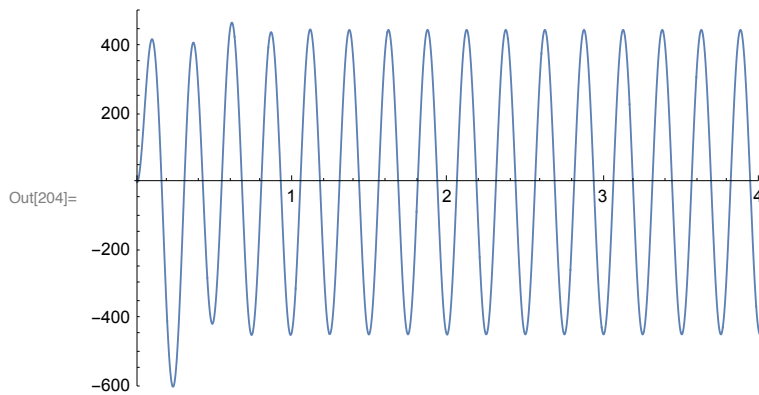
There is a $\sin^2 + \cos^2$ trig identity in the above, but I'm going to have to pull it out by hand.

```
In[194]:= outhnd = -300. e-5. t Sin[10. t] +
  Cos[10. t] (400. e-5. t) + (-400. Cos[25. t] + 200. Sin[25. t])
Out[194]= 400. e-5. t Cos[10. t] - 400. Cos[25. t] - 300. e-5. t Sin[10. t] + 200. Sin[25. t]

In[195]:= outhnd2 = Collect[outhnd, e-5. t]
Out[195]= -400. Cos[25. t] + e-5. t (400. Cos[10. t] - 300. Sin[10. t]) + 200. Sin[25. t]
```

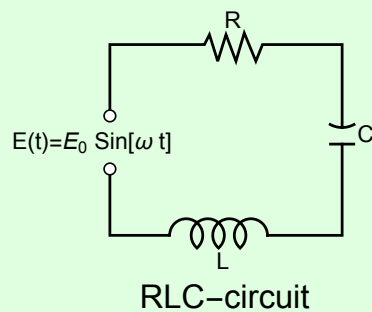
While I was pulling things out by hand, I pulled out a choppable term. The text constant B is equal to -300. The text constant A is equal to 1 in one position and 400 in another position. That makes my answer wrong, technically. I guess I should make it yellow, though I don't feel it is a just action to do so. I feel like it is correct.

```
In[204]:= Plot[-400. Cos[25. t] + e-5. t (400. Cos[10. t] - 300. Sin[10. t]) +
  200. Sin[25. t], {t, 0, 4}, ImageSize -> 350, AspectRatio -> 0.6,
  PlotRange -> {{-0.01, 4}, {-600, 500}}, PlotStyle -> Thickness[0.003]]
```



15. Cases of damping. What are the conditions for an RLC-circuit to be (I) overdamped, (II) critically damped, (III) underdamped? What is the critical resistance R_{crit} (the analog of the critical damping constant $2\sqrt{mk}$)

16 - 18 Solve the initial value problem for the RLC-circuit shown below, with the given data, assuming zero initial current and charge. Graph or sketch the solution.



17. $R = 6 \, \Omega$, $L = 1 \, \text{H}$, $C = 0.04 \, \text{F}$, $E = 600(\text{Cos}[t] + 4 \text{Sin}[t])\text{V}$

```
In[158]:= Clear["Global`*"]
```

```
In[159]:= eqns = {eL q''[t] + aR q'[t] + 1/cC q[t] == Vee[t]};
```

```
In[160]:= m1 = StateSpaceModel[eqns,
  {{q[t], 0}, {q'[t], 0}}, {{Vee[t], 0}}, {q'[t]}, t]
```

$$\text{Out[160]} = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -\frac{1}{cC eL} & -\frac{aR}{eL} & \frac{1}{eL} \\ 0 & 1 & 0 \end{array} \right) S$$

Here I put in the given parameters.

```
In[174]:= ms = m1 /. {cC → 0.04, eL → 1, aR → 6}
```

$$\text{Out[174]} = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -25. & -6 & 1 \\ 0 & 1 & 0 \end{array} \right) S$$

The way to get output from a state space model is to use the command **OutputResponse**.

```
In[175]:= outz = OutputResponse[{ms}, 600 (Cos[t] + 4 Sin[t]), t]
```

$$\begin{aligned} \text{Out[175]} = & \left\{ (0. + 0. i) + \right. \\ & e^{-3. t} \left((-100. - 1.11022 \times 10^{-14} i) \cos[4. t] + (100. + 1.11022 \times 10^{-14} i) \right. \\ & e^{3. t} \cos[t] \cos[4. t]^2 - (1.87214 \times 10^{-14} - 1.65445 \times 10^{-14} i) \\ & e^{3. t} \cos[4. t]^2 \sin[t] + (75. + 1.52656 \times 10^{-14} i) \sin[4. t] - \\ & (8.65974 \times 10^{-15} + 1.80411 \times 10^{-14} i) e^{3. t} \cos[t] \cos[4. t] \sin[4. t] + \\ & (2.27374 \times 10^{-13} + 2.91161 \times 10^{-14} i) e^{3. t} \cos[4. t] \sin[t] \sin[4. t] + \\ & (100. - 1.14492 \times 10^{-14} i) e^{3. t} \cos[t] \sin[4. t]^2 - \\ & \left. \left. (0. + 7.91555 \times 10^{-14} i) e^{3. t} \sin[t] \sin[4. t]^2 \right) \right\} \end{aligned}$$

```
In[176]:= outt = Chop[ComplexExpand[Re[outz]]] // Simplify
```

$$\text{Out[176]} = \left\{ -100. e^{-3. t} \cos[4. t] + 100. \cos[t] \cos[4. t]^2 + \right. \\ \left. \sin[4. t] (75. e^{-3. t} + 100. \cos[t] \sin[4. t]) \right\}$$

```
In[177]:= outtf = Collect[outt, e^{-3. t}]
```

$$\text{Out[177]} = \left\{ 100. \cos[t] \cos[4. t]^2 + 100. \cos[t] \sin[4. t]^2 + \right. \\ \left. e^{-3. t} (-100. \cos[4. t] + 75. \sin[4. t]) \right\}$$

I can see the $\sin^2 + \cos^2$ identity in the above, but will have to take it out by hand.

$$100. \cos[t] + e^{-3. t} (-100. \cos[4. t] + 75. \sin[4. t])$$

And with that, the above cell matches the text answer.

In[184]:= **Plot**[$100 \cdot \cos[t] + e^{-3 \cdot t} (-100 \cdot \cos[4 \cdot t] + 75 \cdot \sin[4 \cdot t])$,
{**t**, 0, 10}, **ImageSize** → 350, **AspectRatio** → 0.6,
PlotRange → {{-0.01, 10}, {-125, 125}}, **PlotStyle** → **Thickness**[0.003]]

