## 1 - 6 Eigenvalues and vectors

Is the given matrix Hermitian? Skew-Hermitian? Unitary?

1. 
$$\begin{pmatrix} 6 & \mathbb{1} \\ -\mathbb{1} & 6 \end{pmatrix}$$

Clear["Global`\*"]

$$e1 = \begin{pmatrix} 6 & \dot{n} \\ -\dot{n} & 6 \end{pmatrix}$$

$$\{ \{6, \dot{n}\}, \{-\dot{n}, 6\} \}$$

e2 = Transpose[e1] // MatrixForm

e3 = Inverse[e1] // MatrixForm

$$\left(\begin{array}{ccc}
\frac{6}{35} & -\frac{\dot{n}}{35} \\
\frac{\dot{n}}{35} & \frac{6}{35}
\end{array}\right)$$

Above: It is Hermitian. It is not skew-Hermitian. It is not Unitary.

e4 = {vals, vecs} = Eigensystem[e1]

$$\{\{7, 5\}, \{\{\dot{n}, 1\}, \{-\dot{n}, 1\}\}\}$$

The green cells above, containing eigenvalues, eigenvectors, and matrix ID, agree with the text answer.

$$3. \left(\begin{array}{ccc} \frac{1}{2} & \text{if } \sqrt{\frac{3}{4}} \\ & \sqrt{\frac{3}{4}} & \frac{1}{2} \end{array}\right)$$

Clear["Global`\*"]

e1 = 
$$\begin{pmatrix} \frac{1}{2} & \dot{n} \sqrt{\frac{3}{4}} \\ \dot{n} \sqrt{\frac{3}{4}} & \frac{1}{2} \end{pmatrix}$$
  
 $\left\{ \left\{ \frac{1}{2}, \frac{\dot{n} \sqrt{3}}{2} \right\}, \left\{ \frac{\dot{n} \sqrt{3}}{2}, \frac{1}{2} \right\} \right\}$ 

e2 = Transpose[e1] // MatrixForm

$$\left(\begin{array}{ccc} \frac{1}{2} & \frac{\dot{\underline{n}} \cdot \sqrt{3}}{2} \\ \frac{\dot{\underline{n}} \cdot \sqrt{3}}{2} & \frac{1}{2} \end{array}\right)$$

e3 = Inverse[e1] // MatrixForm

$$\left(\begin{array}{ccc} \frac{1}{2} & -\frac{\dot{\mathbf{n}} \sqrt{3}}{2} \\ -\frac{\dot{\mathbf{n}} \sqrt{3}}{2} & \frac{1}{2} \end{array}\right)$$

It is not Hermitian. It is not skew-Hermitian. It is Unitary.

e4 = {vals, vecs} = Eigensystem[e1]

$$\left\{\left\{\frac{1}{2}\left(1+i\sqrt{3}\right), \frac{1}{2}\left(1-i\sqrt{3}\right)\right\}, \left\{\left\{1, 1\right\}, \left\{-1, 1\right\}\right\}\right\}$$

The green cells above, containing eigenvalues, eigenvectors, and matrix ID, agree with the text answer.

5. 
$$\begin{pmatrix} \dot{1} & 0 & 0 \\ 0 & 0 & \dot{1} \\ 0 & \dot{1} & 0 \end{pmatrix}$$

Clear["Global`\*"]

$$e1 = \begin{pmatrix} \dot{n} & 0 & 0 \\ 0 & 0 & \dot{n} \\ 0 & \dot{n} & 0 \end{pmatrix}$$
{{\dar{n}, 0, 0}, {\dar{0}, 0, \dar{n}}, {\dar{0}, \dar{n}, 0}}

e2 = Transpose[e1] // MatrixForm

e3 = Inverse[e1] // MatrixForm

$$\left(\begin{array}{cccc}
-i & 0 & 0 \\
0 & 0 & -i \\
0 & -i & 0
\end{array}\right)$$

Above: the matrix is both Unitary and Skew-Hermitian.

$$\{\{\dot{\mathbf{n}}, \dot{\mathbf{n}}, -\dot{\mathbf{n}}\}, \{\{0, 1, 1\}, \{1, 0, 0\}, \{0, -1, 1\}\}\}$$

The green cells above, containing eigenvalues, eigenvectors, and matrix ID, agree with the text answer.

7. Pauli spin matrices. Find the eigenvalues and eigenvectors of the so-called Pauli spin matrices and show that

$$\mathbf{S}_{\mathbf{x}} \; \mathbf{S}_{\mathbf{y}} = \dot{\mathbf{1}} \; \mathbf{S}_{\mathbf{z}}, \quad \mathbf{S}_{\mathbf{y}} \; \mathbf{S}_{\mathbf{x}} = -\dot{\mathbf{1}} \; \mathbf{S}_{\mathbf{z}}, \quad \mathbf{S}_{\mathbf{x}}^2 = \mathbf{S}_{\mathbf{y}}^2 = \mathbf{S}_{\mathbf{z}}^2 = \mathbf{I}, \quad \text{where}$$

$$\mathbf{S}_{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{S}_{\mathbf{y}} = \begin{pmatrix} 0 & -\dot{\mathbf{n}} \\ \dot{\mathbf{n}} & 0 \end{pmatrix}, \quad \mathbf{S}_{\mathbf{z}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Clear["Global`\*"]

$$\mathbf{s}\mathbf{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
 {{0, 1}, {1, 0}}

$$\mathbf{s}\mathbf{y} = \begin{pmatrix} \mathbf{0} & -\dot{\mathbf{n}} \\ \dot{\mathbf{n}} & \mathbf{0} \end{pmatrix}$$
$$\left\{ \left\{ \mathbf{0}, -\dot{\mathbf{n}} \right\}, \left\{ \dot{\mathbf{n}}, \mathbf{0} \right\} \right\}$$

$$\mathbf{sz} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}$$

$$e1 = FullSimplify[sx.sy = isz]$$

True

True

## 9 - 12 Complex forms

Is the matrix A Hermitian or skew-Hermitian? Find  $\mathbf{x}^{\mathsf{T}} \mathbf{A} \mathbf{x}$ 

9. 
$$\begin{pmatrix} 4 & 3-2 i \\ 3+2 i & -4 \end{pmatrix}$$

Clear["Global`\*"]

$$e1 = \begin{pmatrix} 4 & 3 - 2 \dot{n} \\ 3 + 2 \dot{n} & -4 \end{pmatrix}$$
$$\{\{4, 3 - 2 \dot{n}\}, \{3 + 2 \dot{n}, -4\}\}$$

$$e2 = xx = \{-4 i, 2 + 2 i\}$$
  
 $\{-4 i, 2 + 2 i\}$ 

e3 = Transpose[e1] // MatrixForm

$$\begin{pmatrix} 4 & 3 + 2 i \\ 3 - 2 i & -4 \end{pmatrix}$$

e4 = Inverse[e1] // MatrixForm

$$\left(\begin{array}{ccc} \frac{4}{29} & \frac{3}{29} - \frac{2\,\dot{\mathrm{n}}}{29} \\ \frac{3}{29} + \frac{2\,\dot{\mathrm{n}}}{29} & -\frac{4}{29} \end{array}\right)$$

The matrix is Hermitian.

e6 = xbar.e1.xx

16

The two green cells above match the text answer for matrix ID and the value of  $\mathbf{x}^{\mathsf{T}} \mathbf{A} \mathbf{x}$ .

11. 
$$A = \begin{pmatrix} \dot{1} & 1 & 2 + \dot{1} \\ -1 & 0 & 3 \dot{1} \\ -2 + \dot{1} & 3 \dot{1} & \dot{1} \end{pmatrix}$$
,  $x = \begin{pmatrix} 1 \\ \dot{1} \\ -\dot{1} \end{pmatrix}$ 

Clear["Global`\*"]

$$e1 = \begin{pmatrix} \dot{n} & 1 & 2 + \dot{n} \\ -1 & 0 & 3 \dot{n} \\ -2 + \dot{n} & 3 \dot{n} & \dot{n} \end{pmatrix}$$

$$\{\{\dot{\mathtt{n}}\,,\,\,1\,,\,\,2\,+\,\dot{\mathtt{n}}\}\,,\,\,\{-\,1\,,\,\,0\,,\,\,3\,\,\dot{\mathtt{n}}\}\,,\,\,\{-\,2\,+\,\dot{\mathtt{n}}\,,\,\,3\,\,\dot{\mathtt{n}}\,,\,\,\dot{\mathtt{n}}\}\}$$

$$xx = \{1, i, -i\}$$

$$\{1, \dot{n}, -\dot{n}\}$$

$$xbar = \{1, -i, i\}$$

$$\{1, -i, i\}$$

e2 = Transpose[e1] // MatrixForm

$$\begin{pmatrix} \dot{n} & -1 & -2 + \dot{n} \\ 1 & 0 & 3 \dot{n} \\ 2 + \dot{n} & 3 \dot{n} & \dot{n} \end{pmatrix}$$

e3 = Inverse[e1] // MatrixForm

$$\begin{pmatrix}
\frac{9 \,\dot{\mathbf{n}}}{2} & -\frac{5}{2} - \frac{3 \,\dot{\mathbf{n}}}{2} & -\frac{3}{2} \\
\frac{5}{2} - \frac{3 \,\dot{\mathbf{n}}}{2} & 2 \,\dot{\mathbf{n}} & \frac{1}{2} + \frac{\dot{\mathbf{n}}}{2} \\
\frac{3}{2} & -\frac{1}{2} + \frac{\dot{\mathbf{n}}}{2} & \frac{\dot{\mathbf{n}}}{2}
\end{pmatrix}$$

The matrix is skew-Hermitian.

e4 = xbar.e1.xx

-6 i

The two green cells above match the text answer for matrix ID and the value of  $\mathbf{x}^{\mathsf{T}} \mathbf{A} \mathbf{x}$ .