Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

Comment. Mathematica has a dedicated function called **`LinearFractionalTransform`**, the main subject of this section. However, this polynomial-centric function did not seem suited to the following problems.

```
Clear["Global`*"]
```

7 - 10 Inverse

Find the inverse z=z(w). Check by solving z(w) for w.

7. $w = \dfrac{i}{2 z - 1}$

```
Clear["Global`*"]
```

w is the transform. So it is necessary to express it as

$w == \dfrac{i}{2 z - 1} == \dfrac{0 z + i}{2 z - 1} == \dfrac{a z + b}{c z + d}$ and then $a == 0,\ b == i,\ c == 2,$ and $d == -1$

Numbered line (4) on p. 745 gives the general form of the inverse mapping

$z = z(w) = \dfrac{d w - b}{-c w + a}$

Which in this case is

$$\frac{-1 w - i}{-2 w + 0}$$

$$-\frac{-i - w}{2 w}$$

```
Simplify[%]
```

$$\frac{i + w}{2 w}$$

9. $w = \dfrac{z - i}{3 i z + 4}$

```
Clear["Global`*"]
```

Following the same procedure as in problem 7, $z = z(w) = \frac{d w - b}{-c w + a}$ required the following replacements.

$a == 1,\ b == -i,\ c == 3\,i,\ d == 4$

$$\frac{4 w + i}{-3 i w + 1}$$

$$\frac{i + 4 w}{1 - 3 i w}$$

11 - 16 Fixed Points
Find the fixed points.

11. $w = (a + i b) z^2$

```
Clear["Global`*"]
```

By the definition of fixed point, I want a solution of z such that

Solve$\left[(a + i b) z^2 == z, z\right]$

$$\left\{\{z \to 0\}, \left\{z \to \frac{1}{a + i b}\right\}\right\}$$

Which Mathematica can find easily.

13. $w = 16 z^5$

```
Clear["Global`*"]
```

Solve$\left[16 z^5 - z == 0, z\right]$

$$\left\{\left\{z \to -\frac{1}{2}\right\}, \{z \to 0\}, \left\{z \to -\frac{i}{2}\right\}, \left\{z \to \frac{i}{2}\right\}, \left\{z \to \frac{1}{2}\right\}\right\}$$

15. $w = \dfrac{i z + 4}{2 z - 5 i}$

```
Clear["Global`*"]
```

Solve$\left[\dfrac{i z + 4}{2 z - 5 i} == z, z\right]$

$$\{\{z \to i\}, \{z \to 2 i\}\}$$

17 - 20 Fixed Points
Find all LFTs with fixed point(s).

17. $z = 0$

This problem is worked according to the general method of the s.m. In the next problem, very similar, my approach is more direct.

```
Clear["Global`*"]
```

$$\frac{a\,z+b}{c\,z+d} == z$$

$$\frac{b+a\,z}{d+c\,z} == z$$

This is the general form of the transformation w, but now I want an expression which yields zero

$$\frac{b+a\,z}{d+c\,z} - z == 0$$

$$-z + \frac{b+a\,z}{d+c\,z} == 0$$

I can factor

$$\text{Factor}\left[-z + \frac{b+a\,z}{d+c\,z}\right]$$

$$-\frac{-b-a\,z+d\,z+c\,z^2}{d+c\,z}$$

The expression above is still equal to zero. If I want to, I can multiply both sides by $\frac{-(d+c\,z)}{-(d+c\,z)}$, giving

$$-b-a\,z+d\,z+c\,z^2 == 0$$

$$-b-a\,z+d\,z+c\,z^2 == 0$$

or

$$-a\,z+d\,z+c\,z^2 == b$$

$$-a\,z+d\,z+c\,z^2 == b$$

I can factor out a z on lhs, and since the objective is to find an expression where z can equal zero if it gets a chance, this implies that b=0 is a solution. So taking the remaining coefficients back to the transformation definition gives me

$$w == \frac{a\,z+b}{c\,z+d} == \frac{a\,z+0}{c\,z+d} == \frac{a\,z}{c\,z+d}$$

The final fractional expression above does not match the text answer. However, it is the answer shown and checked in the s.m., and works. (The text answer also works and checks as correct.)

19. $z = \pm i$

```
Clear["Global`*"]
```

A slightly different procedure than in the last problem, and looking first at z=i,

$$\frac{a\,z + b}{c\,z + d} - i == 0$$

$$-i + \frac{b + a\,z}{d + c\,z} == 0$$

Since I am simply trying to make a fixed point out of 0+1 i, I might as well stick it in there.

```
Solve[(a i + b)/(c i + d) - i == 0, {a, b, c, d}]
```

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

```
{{d → a - i b - i c}}
```

I insert the found value of d by hand into the previous expression.

```
rau = (a i + b)/(c i + a - i b - i c)
```

$$\frac{i\,a + b}{a - i\,b}$$

And make substitutions. Two, for some reason, are required.

```
rau1 = rau /. {i → z, -i b → -z b}
```

$$\frac{b + a\,z}{a - b\,z}$$

The above cell matches the text answer for the positive case.

Testing,

```
Simplify[(b + a z)/(a - b z) /. z → i]
```

i

Now trying to solve the negative case

```
Clear["Global`*"]
```

```
Solve[(-a i + b)/(-c i + d) + i == 0, {a, b, c, d}]
```

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

```
{{d → a + i b + i c}}
```

```
rau2 = (-a i + b)/(-c i + a + i b + i c)
```

$$\frac{-i\,a + b}{a + i\,b}$$

```
rau3 = rau2 /. {i → -z, -i → z}
```

$$\frac{b + a\,z}{a - b\,z}$$

Testing,

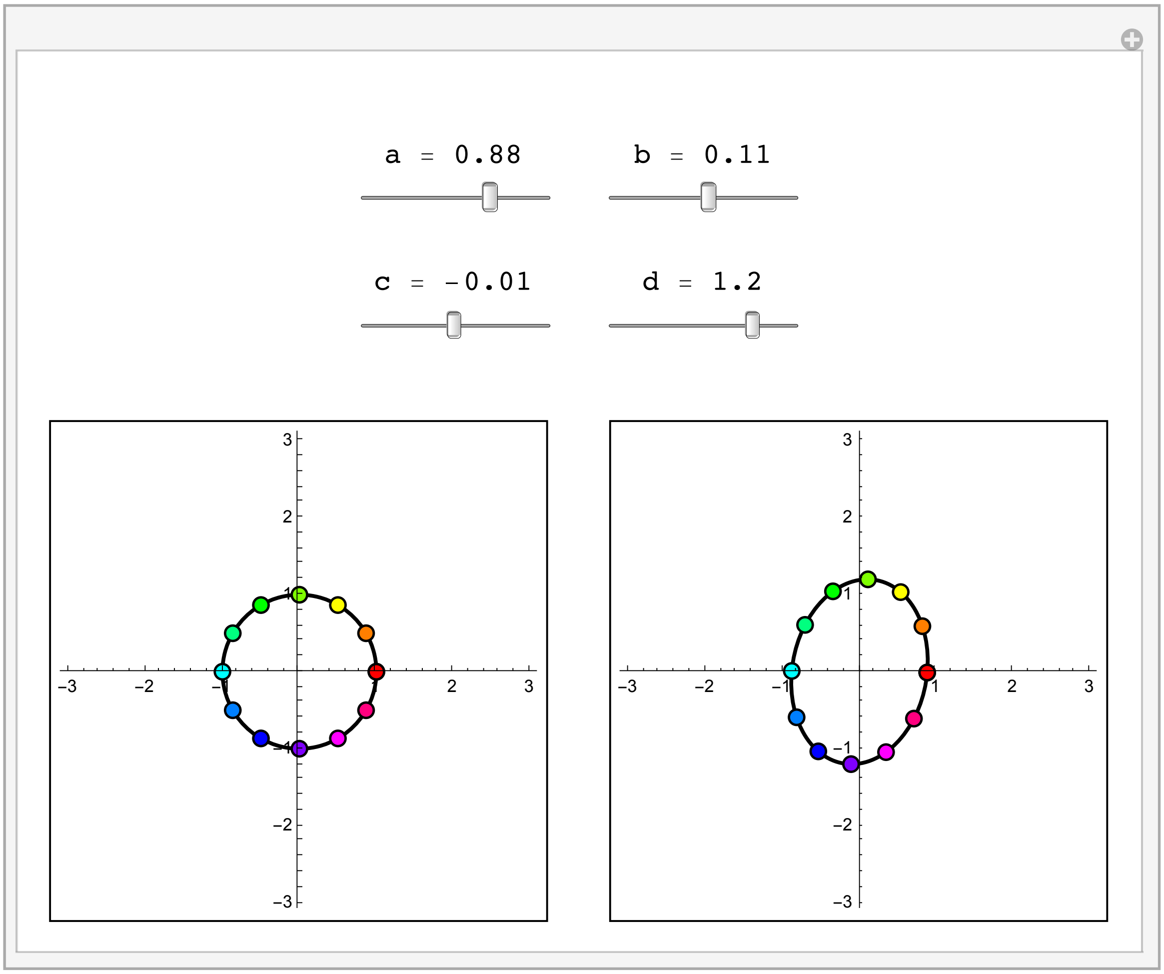
```
Simplify[(b + a z)/(a - b z) /. z → -i]
```

```
-i
```

****

Though not used directly in this section, the following WolframDemonstration project, authored by Michael Ford, is interesting, and uses the conventional a,b,c,d coefficients of the standard LRT equation.

```
Manipulate[
 Column[{" ", " ", Grid[{{Labeled[Slider[Dynamic[a], {-2, 2, 0.01},
        ImageSize → Small], Row[{"a = ", Dynamic[a]}], Top],
      Labeled[Slider[Dynamic[b], {-2, 2, 0.01}, ImageSize → Small],
       Row[{"b = ", Dynamic[b]}], Top]},
     {Labeled[Slider[Dynamic[c], {-2, 2, 0.01}, ImageSize → Small],
       Row[{"c = ", Dynamic[c]}], Top],
      Labeled[Slider[Dynamic[d], {-2, 2, 0.01}, ImageSize → Small],
       Row[{"d = ", Dynamic[d]}], Top]}},
    Spacings → {2, 2}], " ", " ", Deploy[
    Grid[{{Framed[Dynamic[Graphics[{AbsoluteThickness[2], (*Blue,*)
            Dynamic[Circle[]], AbsolutePointSize[8], EdgeForm[
             {Black, Thickness[0.005]}], Table[{Hue[θ / (2 π)], Disk[
               {Cos[θ], Sin[θ]}, 0.1]}, {θ, 0, 2 π, π / 6}]}, PlotRange →
            {{-3.1, 3.1}, {-3.1, 3.1}}, Axes → True, ImageSize → 250]]],
       " ", Framed[Dynamic[Graphics[{AbsoluteThickness[2],
            (*Blue,*) Dynamic[GeometricTransformation[
              Circle[{0, 0}, 1, {0, 2 π}], {{a, b}, {c, d}}]],
            AbsolutePointSize[8], EdgeForm[{Black, Thickness[0.005]}],
            Table[{Hue[θ / (2 π)], Disk[{{a, b}, {c, d}}.{Cos[θ], Sin[θ]},
               0.1]}, {θ, 0, 2 π, π / 6}]}, PlotRange → {{-3.1, 3.1},
             {-3.1, 3.1}}, Axes → True, ImageSize → 250]]]}}]]},
  Center], {{a, 1}, -2, 2, ControlType → None},
 {{b,
   1},
  -2, 2,
  ControlType →
   None}, {{c,
   0},
  -2, 2, ControlType →
   None}, {{d,
   1},
  -2, 2, ControlType →
   None}]
```


a = 0.88
b = 0.11
c = -0.01
d = 1.2
3
2
1
-1
-2
-3
-3
-2
-1
1
2
3
3
2
1
-1
-2
-3
-3
-2
-1
1
2
3