Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

1 - 8 Parametric surface representation
Familiarize yourself with parametric representations of important surfaces by deriving a representation (1), by finding the **parameter curves** (curves u = const and v = const) of the surface and a normal vector $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$ of the surface.

1. *xy*-plane $\mathbf{r}$(u, v) = (u, v) (thus u $\mathbf{i}$ + v $\mathbf{j}$; similarly in problems 2 - 8).

3. Cone $\mathbf{r}$(u, v) = {u Cos[v], u Sin[v], c u}

```
Clear["Global`*"]
```

I am given that

```
x = u Cos[v];
y = u Sin[v];
z = u c
```

c u

If I happen to have the brainwave to look at $x^2 + y^2$,

```
x² + y²
```

u² Cos[v]² + u² Sin[v]²

I see that it is equal to u^2. At this point I can find one sort of parameter curve, by setting u =const. If u=const.
then u^2 = const. and $x^2 + y^2$ = const., that is,   What about when v=const.?
        equals *a* circle. So circles are one of the parameter curves.
Then by the definition of the coordinates, the quantity y/x=some constant, and y=kx. That describes a straight line. So straight lines are the other parameter curve.

```
yip = Simplify[Solve[x² + y² == z²/c², {z}]]
```

$\left\{\left\{z \to -\sqrt{c^2 \left(x^2 + y^2\right)} \right\}, \left\{z \to \sqrt{c^2 \left(x^2 + y^2\right)} \right\}\right\}$
Or,

$z = c \sqrt{x^2 + y^2}$

z=f(x,y) is the formula for a cone, if you can recognize it, according to the s.m.. Now I need to find a
surface normal on this surface. To do this I will probably need both partial derivatives.

```
ru = D[{u Cos[v], u Sin[v], c u}, u]
{Cos[v], Sin[v], c}

rv = D[{u Cos[v], u Sin[v], c u}, v]
{-u Sin[v], u Cos[v], 0}

cprod = Cross[ru, rv]
{-c u Cos[v], -c u Sin[v], u Cos[v]² + u Sin[v]²}

TrigReduce[u Cos[v]² + u Sin[v]²]
u

cprodf = cprod /. u Cos[v]² + u Sin[v]² → u
```

```
{-c u Cos[v], -c u Sin[v], u}
```

The above line holds the normal vector the problem was looking for.

5. Paraboloid of revolution $\mathbf{r}(u, v) = \{u\ \mathrm{Cos}[v],\ u\ \mathrm{Sin}[v],\ u^2\}$

```
Clear["Global`*"]

x = u Cos[v]
u Cos[v]

y = u Sin[v]
u Sin[v]

z = u²
u²

TrigReduce[x² + y²]
u²
```

So $x^2 + y^2 = z$. What about the parameter curves? If u = const., then z is const., and $x^2 + y^2 =$const., in other words, circles again. If v is const., then the coordinates of the parameter curve look like (c1 u, c2 u, u^2), which will describe a parabola. Now I am ready to go on to the normal vector calculation.

```
ru = D[{u Cos[v], u Sin[v], u²}, u]
{Cos[v], Sin[v], 2 u}

rv = D[{u Cos[v], u Sin[v], u²}, v]
{-u Sin[v], u Cos[v], 0}
```

```
cprod = Cross[ru, rv]
```

$$\left\{-2\,u^2\,Cos[v],\ -2\,u^2\,Sin[v],\ u\,Cos[v]^2 + u\,Sin[v]^2\right\}$$

```
cprodf = cprod /. u Cos[v]^2 + u Sin[v]^2 → u
```

$$\left\{-2\,u^2\,Cos[v],\ -2\,u^2\,Sin[v],\ u\right\}$$

The above line contains the normal vector expression, which agrees with the text's answer.

7. Ellipsoid r(u, v) = {a Cos[v] Cos[u], b Cos[v] Sin[u], c Sin[v]}

```
Clear["Global`*"]
```

```
x = a Cos[v] Cos[u]
y = b Cos[v] Sin[u]
z = c Sin[v]
```

a Cos[u] Cos[v]

b Cos[v] Sin[u]

c Sin[v]

```
firs = Cos[v] Cos[u];
seco = Cos[v] Sin[u];
thir = Sin[v];
FullSimplify[firs^2 + seco^2 + thir^2]
```

1

The somewhat amazing result stated on the above line allows the ellipsoid equation to become $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

The ratio of the coefficients a and b (and c) describe the eccentricity of the ellipsoid. If u = const., then a surface incorporating the still-active v is created, an ellipse. If v = const., then a different kind of ellipse is expressed, still dependent on the ratio between a and b. I can try to find the normal vector.

```
ru = D[{a Cos[v] Cos[u], b Cos[v] Sin[u], c Sin[v]}, u]
```

{-a Cos[v] Sin[u], b Cos[u] Cos[v], 0}

```
rv = D[{a Cos[v] Cos[u], b Cos[v] Sin[u], c Sin[v]}, v]
```

{-a Cos[u] Sin[v], -b Sin[u] Sin[v], c Cos[v]}

```
cprod = Cross[ru, rv]
```

$$\left\{b\,c\,Cos[u]\,Cos[v]^2,\ a\,c\,Cos[v]^2\,Sin[u],\right.$$
$$\left.a\,b\,Cos[u]^2\,Cos[v]\,Sin[v] + a\,b\,Cos[v]\,Sin[u]^2\,Sin[v]\right\}$$

```
EqualTo[a b Cos[u]² Cos[v] Sin[v] + a b Cos[v] Sin[u]² Sin[v]][
  a b Cos[v] Sin[v]]
```

```
a b Cos[v] Sin[v] == a b Cos[u]² Cos[v] Sin[v] + a b Cos[v] Sin[u]² Sin[v]
```

MMAflips the compared expressions around, its indication that it concurs in the equality.

```
cprodf = cprod /.
  a b Cos[u]² Cos[v] Sin[v] + a b Cos[v] Sin[u]² Sin[v] -> a b Cos[v] Sin[v]
```

$$\{b\ c\ Cos[u]\ Cos[v]^2,\ a\ c\ Cos[v]^2\ Sin[u],\ a\ b\ Cos[v]\ Sin[v]\}$$

The above line contains the text answer for the normal vector.

11. Satisfying numbered line (4), p. 441. Represent the paraboloid in problem 5 so that $\tilde{N}(0, 0) \neq 0$ and show $\tilde{N}$.

13. Representation $z = f(x, y)$. Show that $z = f(x, y)$ or $g = z - f(x, y) = 0$ can be written $\left(f_u = \dfrac{\partial f}{\partial u},\ etc.\right)$

Even with treatment in the s.m., I don't understand what is supposed to happen with the above problem.

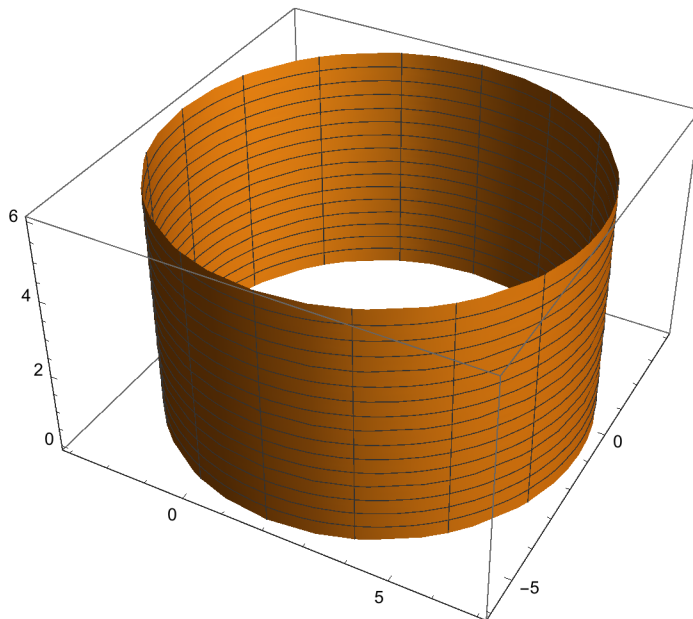**14 - 19 Derive a parametric representation**
Find a normal vector. The answer gives one representation; there are many. Sketch the surface and parameter curves.

15. Cylinder of revolution $(x - 2)^2 + (y + 1)^2 = 25$

```
Clear["Global`*"]
```

The constants just specify the location of the center of the cylinder.

```
ParametricPlot3D[{2 + 5 Cos[u], -1 + 5 Sin[u], v}, {u, 0, 2 π}, {v, 0, 6}]
```


The parametric curves are shown in the figure. In the constant-u plane they are circles. In the constant-v plane they are straight lines.

```
par[u_, v_] = {2 + 5 Cos[u], -1 + 5 Sin[u], v}
{2 + 5 Cos[u], -1 + 5 Sin[u], v}
```

Above is the parametric equation for the cylinder. Below I will take the partial derivatives so I can find a normal vector by the cross product of them.

```
fir = D[{2 + 5 Cos[u], -1 + 5 Sin[u], v}, {u}]
{-5 Sin[u], 5 Cos[u], 0}
```

```
sec = D[{2 + 5 Cos[u], -1 + 5 Sin[u], v}, {v}]
{0, 0, 1}
```

```
norm = Cross[fir, sec]
```

```
{5 Cos[u], 5 Sin[u], 0}
```

The vector shown in the line above is a normal vector, and agrees with the text's answer.

17. Sphere $x^2 + (y + 2.8)^2 + (z - 3.2)^2 = 2.25$

```
Clear["Global`*"]
```

Again, the constants 2.8 and -3.2 merely locate the center of the sphere. In the cartesian formula for the sphere, the quantity 2.25 above is the square of the radius.

```
radd = (2.25).5
```

> 1.5

And so my parametric version will look like

```
parsph[u_, v_] =
 {1.5 Sin[u] Cos[v], 1.5 Sin[u] Sin[v] - 2.8, 1.5 Cos[u] + 3.2}
```

```
{1.5` Cos[v] Sin[u], -2.8` + 1.5` Sin[u] Sin[v], 3.2` + 1.5` Cos[u]}
```

```
ParametricPlot3D[{1.5 Sin[u] Cos[v],
  1.5 Sin[u] Sin[v] - 2.8, 1.5 Cos[u] + 3.2}, {u, -7, 7}, {v, -7, 7}]
```
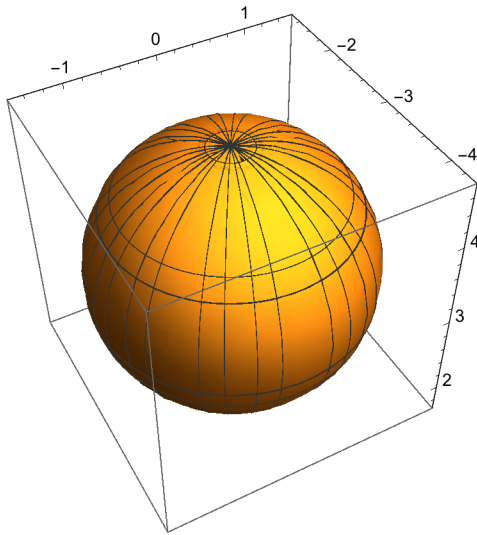

As for parametric curves, they are circles in both the constant-u and constant-v planes, and are shown in the figure. And the center of the sphere seems to be in the same location as in the cartesian one.

```
fir = D[{1.5 Sin[u] Cos[v], 1.5 Sin[u] Sin[v] - 2.8, 1.5 Cos[u] + 3.2}, {u}]
```

```
{1.5 Cos[u] Cos[v], 1.5 Cos[u] Sin[v], -1.5 Sin[u]}
```

```
sec = D[{1.5 Sin[u] Cos[v], 1.5 Sin[u] Sin[v] - 2.8, 1.5 Cos[u] + 3.2}, {v}]
```

```
{-1.5 Sin[u] Sin[v], 1.5 Cos[v] Sin[u], 0}
```

```
norm = Simplify[Cross[fir, sec]]
```

$$\left\{2.25 \, Cos[v] \, Sin[u]^2, \ 2.25 \, Sin[u]^2 \, Sin[v], \ 2.25 \, Cos[u] \, Sin[u]\right\}$$

The above answer does not match the text's, but maybe it is equivalent. My Sin[u] terms match the Cos[v] terms in the answer, and my Cos[u] matches the Sin[v] terms. Anyway, the structure of the normal vector is consistent with that in the answer.

> **19.** Hyperbolic cylinder $x^2 - y^2 = 1$

```
Clear["Global`*"]
```

I found a reference to a hyperbolic identity which is analogous to $\sin^2 x + \cos^2 x = 1$. It is $\cosh^2 x - \sinh^2 x = 1$. Since the squared exponent is already in the problem expression, there is no need to repeat it.

```
parhyp[u_, v_] = {Cosh[u], Sinh[u], v}
```

```
{Cosh[u], Sinh[u], v}
```

```
ParametricPlot3D[{Cosh[u], Sinh[u], v}, {u, -3, 3}, {v, -7, 7}]
```
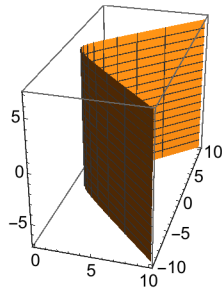

As for parameter curves, I think the ones in constant-u planes are hyperbolas (or at least half hyperbolas). The ones in constant-v planes are straight lines.

```
fir = D[{Cosh[u], Sinh[u], v}, {u}]
```
```
{Sinh[u], Cosh[u], 0}
```

```
sec = D[{Cosh[u], Sinh[u], v}, {v}]
```
```
{0, 0, 1}
```

```
norm = Cross[fir, sec]
```

```
{Cosh[u], -Sinh[u], 0}
```

The above line agrees with the normal vector contained in the text's answer.