2 - 5 Review: radius of convergence

3. $\sum_{m=0}^{\infty}\left(\frac{-1}{k}\right)^{m} x^{2m}$

```
Clear["Global`*"]
```

$$\text{Sum}\left[\frac{(-1)^m}{k^m} x^{2m}, \{m, 0, \infty\}, \text{GenerateConditions} \to \text{True}\right]$$

$$\text{ConditionalExpression}\left[\frac{k}{k + x^2}, \text{Abs}[k] > \text{Abs}[x]^2 \;\&\&\; k \neq 0 \;\&\&\; k + x^2 \neq 0 \;\&\&\; 1 + \frac{x^2}{k} \neq 0\right]$$

$$\text{SumConvergence}\left[\frac{(-1)^m}{k^m} x^{2m}, m\right]$$

$$\text{Abs}[k] > \text{Abs}[x]^2$$

1. Above: According to *MathWorld*, $|x|$ is the standard expression for a radius of convergence, also shown as $|x| < R$, where (-R, R) is the interval of convergence, R being the radius of convergence. Dropping in the text answer as the radius of convergence would make it $|x| < \sqrt{|k|} \Rightarrow |x|^2 < (\sqrt{|k|})^2 \Rightarrow |x|^2 < |k|$. This is equivalent to the above green cell.

5. $\sum_{m=0}^{\infty}\left(\frac{2}{3}\right)^{m} x^{2m}$

```
Clear["Global`*"]
```

$$\text{SumConvergence}\left[\left(\frac{2}{3}\right)^m x^{2m}, m\right]$$

$$\text{Abs}[x] < \sqrt{\frac{3}{2}}$$

The answer in the green cells above match the answers in the text.

6 - 9 Series solutions by hand
Apply the power series method. Do this by hand, not by a CAS, to get a feel for the method, e.g. why a series may terminate, or has even powers only, etc.

7. $y' = -2\,x\,y$

```
Clear["Global`*"]
```

```
e1 = DSolve[y'[x] == -2 x y[x], y[x], x]
```

$\{\{y[x] \to e^{-x^2} C[1]\}\}$

```
e2 = e1 /. C[1] → a0
```

$\{\{y[x] \to e^{-x^2} a_0\}\}$

e3 =

$\text{Series}[a_0 e^{-x^2}, \{x, 0, 8\}]$

$$a_0 - a_0 x^2 + \frac{a_0 x^4}{2} - \frac{a_0 x^6}{6} + \frac{a_0 x^8}{24} + O[x]^9$$

$\text{e4} = \text{Collect}[\text{e3}, a_0]$

$$\left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24}\right) a_0$$

The answer in the green cells above match the answers in the text.

9. y" + y = 0

```
Clear["Global`*"]
```

```
e1 = DSolve[y''[x] + y[x] == 0, y[x], x]
```

```
{{y[x] → C[1] Cos[x] + C[2] Sin[x]}}
```

$\text{e2} = \text{e1} /. \{C[1] \to a_0, C[2] \to a_1\}$

$\{\{y[x] \to \text{Cos}[x]\, a_0 + \text{Sin}[x]\, a_1\}\}$

```
e3 = e2[[1, 1, 2]]
```

$\text{Cos}[x]\, a_0 + \text{Sin}[x]\, a_1$

```
e4 = Series[e3, {x, 0, 8}]
```

$$a_0 + a_1 x - \frac{a_0 x^2}{2} - \frac{a_1 x^3}{6} + \frac{a_0 x^4}{24} + \frac{a_1 x^5}{120} - \frac{a_0 x^6}{720} - \frac{a_1 x^7}{5040} + \frac{a_0 x^8}{40320} + O[x]^9$$

The answer in the green cells above match the answers in the text.

10 - 14 Series solutions

Find a power series solution in powers of x.

11. $y'' - y' + x^2 y = 0$

```
Clear["Global`*"]
```

```
e1 = y[x_] = Sum[a_m x^m, {m, 0, 6}]
```

$a_0 + x\, a_1 + x^2\, a_2 + x^3\, a_3 + x^4\, a_4 + x^5\, a_5 + x^6\, a_6$

```
e2 = y'[x]
```

$a_1 + 2\, x\, a_2 + 3\, x^2\, a_3 + 4\, x^3\, a_4 + 5\, x^4\, a_5 + 6\, x^5\, a_6$

```
e3 = y''[x]
```

$2\, a_2 + 6\, x\, a_3 + 12\, x^2\, a_4 + 20\, x^3\, a_5 + 30\, x^4\, a_6$

Now for the assembly of the staged components.

```
e6 = y''[x] - y'[x] + x^2 y[x] == 0
```

$-a_1 + 2\, a_2 - 2\, x\, a_2 + 6\, x\, a_3 - 3\, x^2\, a_3 + 12\, x^2\, a_4 - 4\, x^3\, a_4 + 20\, x^3\, a_5 - 5\, x^4\, a_5 + 30\, x^4\, a_6 - 6\, x^5\, a_6 + x^2 \left(a_0 + x\, a_1 + x^2\, a_2 + x^3\, a_3 + x^4\, a_4 + x^5\, a_5 + x^6\, a_6\right) == 0$

And rearranging

```
e7 = Expand[e6]
```

$x^2\, a_0 - a_1 + x^3\, a_1 + 2\, a_2 - 2\, x\, a_2 + x^4\, a_2 + 6\, x\, a_3 - 3\, x^2\, a_3 + x^5\, a_3 + 12\, x^2\, a_4 - 4\, x^3\, a_4 + x^6\, a_4 + 20\, x^3\, a_5 - 5\, x^4\, a_5 + x^7\, a_5 + 30\, x^4\, a_6 - 6\, x^5\, a_6 + x^8\, a_6 == 0$

And more rearranging

```
e8 = Collect[e7, x]
```

$-a_1 + 2\, a_2 + x\, (-2\, a_2 + 6\, a_3) + x^6\, a_4 + x^2\, (a_0 - 3\, a_3 + 12\, a_4) + x^7\, a_5 + x^3\, (a_1 - 4\, a_4 + 20\, a_5) + x^5\, (a_3 - 6\, a_6) + x^8\, a_6 + x^4\, (a_2 - 5\, a_5 + 30\, a_6) == 0$

```
e9 = Solve[2 a_2 == a_1, a_2]
```

$\left\{\left\{a_2 \to \frac{a_1}{2}\right\}\right\}$

Above: x^0

```
e11 = Solve[2 a_2 == 6 a_3, a_3] /. a_2 -> a_1/2
```

$\left\{\left\{a_3 \to \frac{a_1}{6}\right\}\right\}$

Above: x^1

```
e13 = Expand[Solve[a_0 - 3 a_3 + 12 a_4 == 0, a_4] /. a_3 → a_1/6]
```

$\left\{\left\{a_4 \to -\frac{a_0}{12} + \frac{a_1}{24}\right\}\right\}$

Above: x^2

$$e14 = \text{Simplify}\left[\text{Solve}[a_1 - 4\,a_4 + 20\,a_5 == 0,\ a_5]\ /.\ a_4 \to \frac{1}{12}\left(-a_0 + \frac{a_1}{2}\right)\right]$$

$$\left\{\left\{a_5 \to \frac{1}{120}\left(-2\,a_0 - 5\,a_1\right)\right\}\right\}$$

Above: x^3

$$e15 = \text{Simplify}\left[\text{Solve}[a_2 - 5\,a_5 + 30\,a_6 == 0,\ a_6]\ /.\ \left\{a_5 \to \frac{1}{120}\left(-2\,a_0 - 5\,a_1\right),\ a_2 \to \frac{a_1}{2}\right\}\right]$$

$$\left\{\left\{a_6 \to \frac{1}{720}\left(-2\,a_0 - 17\,a_1\right)\right\}\right\}$$

$$e16 = y[x]\ /.\ \left\{a_2 \to \frac{a_1}{2},\ a_3 \to \frac{a_1}{6},\ a_4 \to -\frac{a_0}{12} + \frac{a_1}{24},\ a_5 \to \frac{1}{120}\left(-2\,a_0 - 5\,a_1\right),\ a_6 \to \frac{1}{720}\left(-2\,a_0 - 17\,a_1\right)\right\}$$

$$a_0 + \frac{1}{720}x^6\left(-2\,a_0 - 17\,a_1\right) + \frac{1}{120}x^5\left(-2\,a_0 - 5\,a_1\right) + x^4\left(-\frac{a_0}{12} + \frac{a_1}{24}\right) + x\,a_1 + \frac{x^2\,a_1}{2} + \frac{x^3\,a_1}{6}$$

$$e17 = \text{Collect}[e16,\ \{a_0,\ a_1\}]$$

$$\left(1 - \frac{x^4}{12} - \frac{x^5}{60} - \frac{x^6}{360}\right)a_0 + \left(x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{24} - \frac{17\,x^6}{720}\right)a_1$$

Above: The answer in the green cell matches the text answer.

13. $y'' + \left(1 + x^2\right) y = 0$

```
Clear["Global`*"]
```

$$e1 = y[x\_] = \text{Sum}[a_m\,x^m,\ \{m,\ 0,\ 7\}]$$

$$a_0 + x\,a_1 + x^2\,a_2 + x^3\,a_3 + x^4\,a_4 + x^5\,a_5 + x^6\,a_6 + x^7\,a_7$$

$$e2 = y'[x]$$

$$a_1 + 2\,x\,a_2 + 3\,x^2\,a_3 + 4\,x^3\,a_4 + 5\,x^4\,a_5 + 6\,x^5\,a_6 + 7\,x^6\,a_7$$

$$e3 = y''[x]$$

$$2\,a_2 + 6\,x\,a_3 + 12\,x^2\,a_4 + 20\,x^3\,a_5 + 30\,x^4\,a_6 + 42\,x^5\,a_7$$

$$e4 = y''[x] + \left(1 + x^2\right) y[x] == 0$$

$$2\,a_2 + 6\,x\,a_3 + 12\,x^2\,a_4 + 20\,x^3\,a_5 + 30\,x^4\,a_6 + 42\,x^5\,a_7 + \left(1 + x^2\right)\left(a_0 + x\,a_1 + x^2\,a_2 + x^3\,a_3 + x^4\,a_4 + x^5\,a_5 + x^6\,a_6 + x^7\,a_7\right) == 0$$

e5 = Expand[e4]

$$a_0 + x^2 a_0 + x a_1 + x^3 a_1 + 2 a_2 + x^2 a_2 + x^4 a_2 + 6 x a_3 + x^3 a_3 + x^5 a_3 + 12 x^2 a_4 + x^4 a_4 + x^6 a_4 + 20 x^3 a_5 + x^5 a_5 + x^7 a_5 + 30 x^4 a_6 + x^6 a_6 + x^8 a_6 + 42 x^5 a_7 + x^7 a_7 + x^9 a_7 == 0$$

e6 = Collect[e5, x]

$$a_0 + 2 a_2 + x (a_1 + 6 a_3) + x^2 (a_0 + a_2 + 12 a_4) + x^3 (a_1 + a_3 + 20 a_5) + x^8 a_6 + x^6 (a_4 + a_6) + x^4 (a_2 + a_4 + 30 a_6) + x^9 a_7 + x^7 (a_5 + a_7) + x^5 (a_3 + a_5 + 42 a_7) == 0$$

e7 = Solve[$a_0 + 2 a_2 == 0, a_2$]

$$\left\{\left\{a_2 \to -\frac{a_0}{2}\right\}\right\}$$

e8 = Solve[$a_1 + 6 a_3 == 0, a_3$]

$$\left\{\left\{a_3 \to -\frac{a_1}{6}\right\}\right\}$$

e9 = Solve[$a_0 + a_2 + 12 a_4 == 0, a_4$] /. $a_2 \to -\frac{a_0}{2}$

$$\left\{\left\{a_4 \to -\frac{a_0}{24}\right\}\right\}$$

Above: x^2

e10 = Solve[$a_1 + a_3 + 20 a_5 == 0, a_5$] /. $a_3 \to -\frac{a_1}{6}$

$$\left\{\left\{a_5 \to -\frac{a_1}{24}\right\}\right\}$$

Above: x^3

e11 = Solve[$a_2 + a_4 + 30 a_6 == 0, a_6$] /. $\left\{a_2 \to -\frac{a_0}{2}, a_4 \to -\frac{a_0}{24}\right\}$

$$\left\{\left\{a_6 \to \frac{13 a_0}{720}\right\}\right\}$$

Above: x^4

e12 = Solve[$a_3 + a_5 + 42 a_7 == 0, a_7$] /. $\left\{a_3 \to -\frac{a_1}{6}, a_5 \to -\frac{a_1}{24}\right\}$

$$\left\{\left\{a_7 \to \frac{5 a_1}{1008}\right\}\right\}$$

Above: x^5

e12 = y[x] /. $\left\{a_2 \to -\frac{a_0}{2}, a_3 \to -\frac{a_1}{6}, a_4 \to -\frac{a_0}{24}, a_5 \to -\frac{a_1}{24}, a_6 \to \frac{13 a_0}{720}, a_7 \to \frac{5 a_1}{1008}\right\}$

$$a_0 - \frac{x^2 a_0}{2} - \frac{x^4 a_0}{24} + \frac{13 x^6 a_0}{720} + x a_1 - \frac{x^3 a_1}{6} - \frac{x^5 a_1}{24} + \frac{5 x^7 a_1}{1008}$$

```
e13 = Collect[e12, {a0, a1}]
```

$$\left(1-\frac{x^2}{2}-\frac{x^4}{24}+\frac{13 x^6}{720}\right) a_0+\left(x-\frac{x^3}{6}-\frac{x^5}{24}+\frac{5 x^7}{1008}\right) a_1$$

$$e14 = \text{Normal}\left[\left(1-\frac{x^2}{2}-\frac{x^4}{24}+\frac{13 x^6}{720}\right) a_0+\left(x-\frac{x^3}{6}-\frac{x^5}{24}+\frac{5 x^7}{1008}\right) a_1\right] /.\ x\to 1$$

$$\frac{343 a_0}{720}+\frac{803 a_1}{1008}$$

Above: The answer in the green cell matches the text answer. The cell below the answer is an experiment for doing IVP.

16 - 19 CAS problems. IVPs

Solve the initial value problem by a power series. Graph the partial sums of the powers up to and including x^5. Find the value of the sum s (5 digits) at x_1.

17. y'' + 3 x y' + 2 y = 0, y[0] = 1, y'[0] = 1, x = 0.5

```
Clear["Global`*"]
```

$$e1 = y[x\_] = \text{Sum}[a_m x^m, \{m, 0, 5\}]$$

$$a_0+x a_1+x^2 a_2+x^3 a_3+x^4 a_4+x^5 a_5$$

```
e4 = y''[x] + 3 x y'[x] + 2 y[x] == 0
```

$$2 a_2+6 x a_3+12 x^2 a_4+20 x^3 a_5+3 x \left(a_1+2 x a_2+3 x^2 a_3+4 x^3 a_4+5 x^4 a_5\right)+2 \left(a_0+x a_1+x^2 a_2+x^3 a_3+x^4 a_4+x^5 a_5\right)=0$$

```
e5 = Expand[e4]
```

$$2 a_0+5 x a_1+2 a_2+8 x^2 a_2+6 x a_3+11 x^3 a_3+12 x^2 a_4+14 x^4 a_4+20 x^3 a_5+17 x^5 a_5=0$$

```
e6 = Collect[e5, x]
```

$$2 a_0+2 a_2+x (5 a_1+6 a_3)+14 x^4 a_4+x^2 (8 a_2+12 a_4)+17 x^5 a_5+x^3 (11 a_3+20 a_5)=0$$

$$e7 = \text{Solve}[2 a_0+2 a_2 = 0, a_2]$$

$$\{\{a_2\to -a_0\}\}$$

$$e8 = \text{Solve}[5 a_1+6 a_3 = 0, a_3]$$

$$\left\{\left\{a_3\to -\frac{5 a_1}{6}\right\}\right\}$$

Above: x^1

```
e9 = Solve[8 a2 + 12 a4 == 0, a4] /. a2 → -a0
```

$$\left\{\left\{a_4 \to \frac{2\, a_0}{3}\right\}\right\}$$

Above: x^2

```
e10 = Solve[11 a3 + 20 a5 == 0, a5] /. a3 → -5 a1/6
```

$$\left\{\left\{a_5 \to \frac{11\, a_1}{24}\right\}\right\}$$

Above: x^3

Above: With discovery of a_5, all the coefficient values for calculation of s have been found.

```
e19 = y[x] /. {a2 → -a0, a3 → -5 a1/6, a4 → 2 a0/3, a5 → 11 a1/24}
```

$$a_0 - x^2 a_0 + \frac{2\, x^4 a_0}{3} + x\, a_1 - \frac{5\, x^3 a_1}{6} + \frac{11\, x^5 a_1}{24}$$

Above. This is the general solution. The initial value condition of $y(0) = 1$ will make $a_0 = 1$, and the other initial value condition of $y'(0) = 1$ will make $a_1 = 1$.

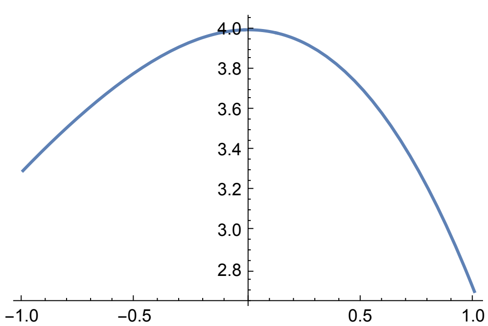
```
e20 = s[x_] = e19 /. {a0 → 1, a1 → 1}
```

$$1 + x - x^2 - \frac{5\, x^3}{6} + \frac{2\, x^4}{3} + \frac{11\, x^5}{24}$$

```
s[1/2]
```

$$\frac{923}{768}$$

```
Plot[s[x], {x, -1, 1}, PlotRange → Automatic, ImageSize → 250]
```



The answers in the green cells above match the answers in the text.

19. $(x - 2)\, y' = x\, y$, $y[0] = 4$, $x_1 = 2$

```
Clear["Global`*"]
```

```
e1 = y[x_] = Sum[a_m x^m, {m, 0, 5}]
```

$a_0 + x\, a_1 + x^2\, a_2 + x^3\, a_3 + x^4\, a_4 + x^5\, a_5$

```
e2 = (x - 2) y'[x] - x y[x] == 0
```

$$(-2 + x)\,\left(a_1 + 2\,x\,a_2 + 3\,x^2\,a_3 + 4\,x^3\,a_4 + 5\,x^4\,a_5\right) - x\,\left(a_0 + x\,a_1 + x^2\,a_2 + x^3\,a_3 + x^4\,a_4 + x^5\,a_5\right) == 0$$

```
e3 = Expand[e2]
```

$$-x\,a_0 - 2\,a_1 + x\,a_1 - x^2\,a_1 - 4\,x\,a_2 + 2\,x^2\,a_2 - x^3\,a_2 - 6\,x^2\,a_3 + 3\,x^3\,a_3 - x^4\,a_3 - 8\,x^3\,a_4 + 4\,x^4\,a_4 - x^5\,a_4 - 10\,x^4\,a_5 + 5\,x^5\,a_5 - x^6\,a_5 == 0$$

```
e4 = Collect[e3, x]
```

$$-2\,a_1 + x\,(-a_0 + a_1 - 4\,a_2) + x^2\,(-a_1 + 2\,a_2 - 6\,a_3) + x^3\,(-a_2 + 3\,a_3 - 8\,a_4) + x^4\,(-a_3 + 4\,a_4 - 10\,a_5) - x^6\,a_5 + x^5\,(-a_4 + 5\,a_5) == 0$$

Below: a_1 , which will be the coefficient of x in the final equation, has no business sticking out by itself.

```
e5 = Solve[-2 a_1 == 0, a_1]
```

$\{\{a_1 \to 0\}\}$

Below: This value of a_0 was set with the belief that it is necessary for the initial condition, $y(0) = 4$.

```
e6 = Solve[-a_0 + a_1 - 4 a_2 == 0, a_2] /. {a_0 -> 4, a_1 -> 0}
```

$\{\{a_2 \to -1\}\}$

```
e7 = Simplify[Solve[-a_1 + 2 a_2 - 6 a_3 == 0, a_3] /. {a_2 -> -1, a_1 -> 0}]
```

$\left\{\left\{a_3 \to -\frac{1}{3}\right\}\right\}$

```
e8 = Simplify[Solve[-a_2 + 3 a_3 - 8 a_4 == 0, a_4] /. {a_2 -> -1, a_3 -> -1/3}]
```

$\{\{a_4 \to 0\}\}$

```
e9 = Simplify[Solve[-a_3 + 4 a_4 - 10 a_5 == 0, a_5] /. {a_3 -> -1/3, a_4 -> 0}]
```

$\left\{\left\{a_5 \to \frac{1}{30}\right\}\right\}$

Above: Discovery of a_5 gives all the coefficients necessary to express s up to fifth power of x.

```
e10 = y[x] /. {a_0 -> 4, a_1 -> 0, a_2 -> -1, a_3 -> -1/3, a_4 -> 0, a_5 -> 1/30}
```

$$4 - x^2 - \frac{x^3}{3} + \frac{x^5}{30}$$

```
e11 = s[x_] = e10
```

$$4 - x^2 - \frac{x^3}{3} + \frac{x^5}{30}$$

```
s[0]
```

4

```
s[2]
```

$$-\frac{8}{5}$$

```
Plot[s[x], {x, -1, 1}, PlotRange → Automatic, ImageSize → 250]
```