Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

2 - 8 Minimum Square Error

Find the trigonometric polynomial F(x) of the form (2) for which the square error with respect to the given f(x) on the interval $-\pi < x < \pi$ is minimum. Compare the minimum value for N=1,2,...,5 (or also for larger values if you have a CAS). *(Note: The form (2) referred to is the first mention of minimum square error on p. 495, but a more usable one is form (6) on p. 496.)*

3. f(x) = |x| $(-\pi < x < \pi)$

The first thing to say is that this problem was worked after problem 5, which has the advantage of a completely worked-out section in the solutions manual. Here again the minimum square error (MSE) will be of high importance. Its expression is: $E^* = \int_{-\pi}^{\pi} f^2\, dx - \pi[2a_0{}^2 + a_n{}^2)]$.

```
Clear["Global`*"]
```

```
f[x_] = Piecewise[{{Abs[x], -π < x < π}}, x]
```

$$\begin{cases} \text{Abs}[x] & -\pi < x < \pi \\ x & \text{True} \end{cases}$$

The absolute value function is an even function. It has no natural period, so the period P will be considered to be the finite domain assigned by the problem. Then L = π. On p. 487 such a series is identified with a Fourier cosine series, and a template for such an equation is given as $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos n x$. As can be seen, the b_n factors have dropped out. In order to assemble an E* I must first get an integral of f^2.

```
Integrate[f[x]^2, {x, -π, π}]
```

$$\frac{2\pi^3}{3}$$

This quantity will be set aside until the final assembly of E*. Next, the summary box on p. 487 gives the form of the remaining a_n factors which I must seek out. These are:

```
azero = 1/π Integrate[f[x], {x, 0, π}]
```

$$\frac{\pi}{2}$$

and,

$$\text{asuvN} = \frac{2}{\pi}\,\text{Integrate}[\text{f}[\text{x}]\,\text{Cos}[\text{n x}],\ \{\text{x}, 0, \pi\}]$$

$$\frac{2\,(-1 + \text{Cos}[\text{n}\,\pi] + \text{n}\,\pi\,\text{Sin}[\text{n}\,\pi])}{\text{n}^2\,\pi}$$

The sine term will drop out, since sin nπ = 0 for all n.

```
asuvNF = asuvN /. Sin[n π] → 0
```

$$\frac{2\,(-1 + \text{Cos}[\text{n}\,\pi])}{\text{n}^2\,\pi}$$

The above expression for a_n flip-flops depending on whether n is positive or negative.

```
asuvNE = asuvNF /. Cos[n π] → 1 (*for multiples of 2π  *)
```

0

```
asuvNO = asuvNF /. Cos[n π] → -1 (* for multiples of π *)
```

$$-\frac{4}{\text{n}^2\,\pi}$$

At this point it is possible to assemble F.

$$\text{bigF1} = \frac{\pi}{2} - \frac{4}{\pi}\,\text{Sum}\left[\frac{1}{\text{n}^2}\,\text{Cos}[\text{n x}],\ \{\text{n}, 1, 5, 2\}\right]$$

$$\frac{\pi}{2} - \frac{4\left(\text{Cos}[\text{x}] + \frac{1}{9}\text{Cos}[3\,\text{x}] + \frac{1}{25}\text{Cos}[5\,\text{x}]\right)}{\pi}$$

The above shows what F(x) would look like with N=5. It matches the text answer. However, for calculating E*, it is best to organize things a little differently. Below are some series which capture values of E*. Notice that in the below cells, all the cos nx terms have disappeared, each replaced by the value = 1.

$$\text{estar1} = \text{N}\left[\frac{2}{3}\pi^3 - \pi\left(2\left(\frac{\pi}{2}\right)^2 + \text{Sum}\left[\left(\frac{-4}{\pi}\,\frac{1}{\text{n}^2}\right)^2,\ \{\text{n}, 1, 1, 2\}\right]\right)\right]$$

0.0747546

$$\text{estar2} = \text{N}\left[\frac{2}{3}\pi^3 - \pi\left(2\left(\frac{\pi}{2}\right)^2 + \text{Sum}\left[\left(\frac{-4}{\pi}\,\frac{1}{\text{n}^2}\right)^2,\ \{\text{n}, 1, 2, 2\}\right]\right)\right]$$

0.0747546

$$\text{estar3} = \text{N}\left[\frac{2}{3}\pi^3 - \pi\left(2\left(\frac{\pi}{2}\right)^2 + \text{Sum}\left[\left(\frac{-4}{\pi}\,\frac{1}{\text{n}^2}\right)^2,\ \{\text{n}, 1, 3, 2\}\right]\right)\right]$$

0.0118786

```
estar4 = N[2/3 π^3 - π (2 (π/2)^2 + Sum[(-4/π 1/n^2)^2, {n, 1, 4, 2}])]
```

0.0118786

```
estar5 = N[2/3 π^3 - π (2 (π/2)^2 + Sum[(-4/π 1/n^2)^2, {n, 1, 5, 2}])]
```

0.00372984

The above green cells have answers agreeing with those of the text.

5. f(x) $= \begin{cases} -1 & \text{if } -\pi < x < 0 \\ 1 & \text{if } 0 < x < \pi \end{cases}$

This problem is worked out in the s.m., so it will be worked first, before problem 3. The important expression E* $= \int_{-\pi}^{\pi} f^2\, dx$ - $\pi[2a_0^2 + b_n^2)]$ represents the MSE. In the below cells it will be chopped up.

```
Clear["Global`*"]
```

```
f[x_] = Piecewise[{{-1, -π < x < 0}, {1, 0 < x < π}}, x]
```

$\begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \\ x & \text{True} \end{cases}$

```
Integrate[f[x]^2, {x, -π, 0}] + Integrate[f^2, {x, 0, π}]
```

$\pi + f^2 \pi$

The above is the best I can do at this time to force Mathematica to admit that the integral = 2π. (True since f^2 always equals 1.) By its definition, f(x) is an odd function. The s.m. points to p. 486 - 487 as authority for making the a_n factors equal to zero, leaving only the b_n factors. Also that the summary box on p. 487 gives the formula for the remaining b_n, which is

```
beeN = 2/π Integrate[f[x] Sin[n x], {x, 0, π}]
```

$\frac{2\,(1 - \text{Cos}[n\,\pi])}{n\,\pi}$

```
beeNO = beeN /. Cos[n π] → -1 (* n odd*)
```

$\frac{4}{n\,\pi}$

```
beeNE = beeN /. Cos[n π] → 1 (* n even *)
```

0

So that only terms with odd n exist. And according to the setup equation for Fourier approximation series, F will look like:
F(x) $= \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots + \frac{1}{N}\sin \mathrm{N}x\right)$ for N odd

```
F[x_] = Simplify[4/π Sum[1/nn Sin[nn x], {nn, 1, 7, 2}],
  Assumptions → nn ∈ OddQ && nn > 0]
```

$$\frac{4\left(\mathrm{Sin}[x] + \frac{1}{3}\mathrm{Sin}[3\,x] + \frac{1}{5}\mathrm{Sin}[5\,x] + \frac{1}{7}\mathrm{Sin}[7\,x]\right)}{\pi}$$

The ‘big F’ function is found, above. It agrees with the answer in the text. E* $= 2\pi$ $-\pi(\sum_{n=1}^{N} b_n{}^2)$, by the way, is the expression for E*.

With a somewhat compressed formula, I came up with a way to calculate E*, below. Notice that all reference to sine functions is missing, each replaced by value of 1.

```
Estar1 = N[(2 π - π (Sum[(4/π 1/n)^2, {n, 1, 1, 2}]))]
```

1.19023

```
Estar2 = N[(2 π - π (Sum[(4/π 1/n)^2, {n, 1, 2, 2}]))]
```

1.19023

```
Estar3 = N[(2 π - π (Sum[(4/π 1/n)^2, {n, 1, 3, 2}]))]
```

0.624343

```
Estar4 = N[(2 π - π (Sum[(4/π 1/n)^2, {n, 1, 4, 2}]))]
```

0.624343

```
Estar5 = N[(2 π - π (Sum[(4/π 1/n)^2, {n, 1, 5, 2}]))]
```

0.420625

```
Estar20 = N[(2 π - π (Sum[(4/π 1/n)^2, {n, 1, 20, 2}]))]
```

0.127218

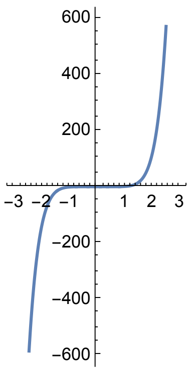
The above answers match those of the text.

7. f(x) = x^3 (-π < x < π)

This problem is not covered in the s.m. The function is odd.

```
Clear["Global`*"]
```

```
Plot[x^7, {x, -π, π}, AspectRatio → 2]
```




```
f[x_] = Piecewise[{{x^3, -π < x < π}}, x]
```

$$\begin{cases} x^3 & -\pi < x < \pi \\ x & \text{True} \end{cases}$$

Piecewise is a pretty good way to handle a restricted domain, even if there is only one piece. At this point, as before, I need to get the value of function-squared-integrated.

```
Integrate[f[x]^2, {x, -π, π}]
```

$$\frac{2\pi^7}{7}$$

With odd function, again I am dealing with the b_n factors, the a_n factors dropping out as before. This means all terms in F will be sine terms, it does not mean either all odd or all even coefficients.

```
beeN = 2/π Integrate[f[x] Sin[n x], {x, 0, π}]
```

$$\frac{2\left(6\,n\,\pi\,\text{Cos}[n\,\pi] - n^3\,\pi^3\,\text{Cos}[n\,\pi] - 6\,\text{Sin}[n\,\pi] + 3\,n^2\,\pi^2\,\text{Sin}[n\,\pi]\right)}{n^4\,\pi}$$

```
beeNO = beeN /. {Cos[n π] → -1, Sin[n π] → 0} (* n odd*)
```

$$\frac{2\left(-6\,n\,\pi + n^3\,\pi^3\right)}{n^4\,\pi}$$

```
beeNO1 = Simplify[beeNO]
```

$$\frac{2\left(-6+n^2\pi^2\right)}{n^3}$$

```
beeNE = beeN /. {Cos[n π] → 1, Sin[n π] → 0} (* n even *)
```

$$\frac{2\left(6\,n\,\pi-n^3\pi^3\right)}{n^4\,\pi}$$

```
beeNE1 = Simplify[beeNE]
```

$$\frac{12-2\,n^2\pi^2}{n^3}$$

The only difference in b_n factors between odd and even terms is that it makes the terms alternate in sign.

So according to the setup equation for Fourier approximation series, F will look like:
$F(x) = 2\,(b_1 \sin x - b_2 \sin 2x + b_3 \sin 3x + \ldots + b_n \sin Nx)$ for N odd and even

$$F[x] = 2\left(\frac{\left(-6+1^2\pi^2\right)}{1^3}\,\mathrm{Sin}[x]-\frac{\left(-6+2^2\pi^2\right)}{2^3}\,\mathrm{Sin}[2\,x]+\frac{\left(-6+3^2\pi^2\right)}{3^3}\,\mathrm{Sin}[3\,x]\right);$$

The green cell above matches the text answer (except for ellipsis).

For a change, I will try to calculate the E* values directly, using the (6) on p. 496 as a template. Recalling that a_0 has already dropped out:

$$\mathrm{Estar1} = N\left[\frac{2\,\pi^7}{7}-\pi\left(\mathrm{Sum}\left[\left(2\left(\frac{(-1)^{n-1}\left(-6+n^2\pi^2\right)}{n^3}\right)\right)^2,\ \{n,1,1\}\right]\right)\right]$$

```
674.774
```

$$\mathrm{Estar2} = N\left[\frac{2\,\pi^7}{7}-\pi\left(\mathrm{Sum}\left[\left(2\left(\frac{(-1)^{n-1}\left(-6+n^2\pi^2\right)}{n^3}\right)\right)^2,\ \{n,1,2\}\right]\right)\right]$$

```
454.705
```

$$\mathrm{Estar3} = N\left[\frac{2\,\pi^7}{7}-\pi\left(\mathrm{Sum}\left[\left(2\left(\frac{(-1)^{n-1}\left(-6+n^2\pi^2\right)}{n^3}\right)\right)^2,\ \{n,1,3\}\right]\right)\right]$$

```
336.449
```

$$\text{Estar4} = N\left[\frac{2\,\pi^7}{7} - \pi\left(\text{Sum}\left[\left(2\left(\frac{(-1)^{n-1}\left(-6 + n^2\,\pi^2\right)}{n^3}\right)\right)^2, \{n, 1, 4\}\right]\right)\right]$$

265.648

$$\text{Estar5} = N\left[\frac{2\,\pi^7}{7} - \pi\left(\text{Sum}\left[\left(2\left(\frac{(-1)^{n-1}\left(-6 + n^2\,\pi^2\right)}{n^3}\right)\right)^2, \{n, 1, 5\}\right]\right)\right]$$

219.037

The green cells match the text calculation of E* for N=1 - 5. The problem presentation ends with a question: why is E* so large? My answer is that the alternating sign nature of the series defeats progress on narrowing the gap between f(x) and F(x).