Note: In this problem set, there are no green cells, because no answers match the text answers.

```
Clear["Global`*"]
```

2 - 9 Verifications. Derivations. Comparisons.

9. Natural spline condition. Using the given coefficients, verify that the spline in example 2 satisfies $g''[x] = 0$ at the ends.

```
Clear["Global`*"]
```

The top part of this section does not really apply to the problem. The points are from the example referred to by the problem.

```
data = {{0, 3.9}, {-0.8, 3.5}, {-1.5, 2.7},
  {-2.5, 2.2}, {-4.0, 1.8}, {-5., 1.5}, {-5.8, 0}}
```

```
{{0, 3.9}, {-0.8, 3.5}, {-1.5, 2.7},
 {-2.5, 2.2}, {-4., 1.8}, {-5., 1.5}, {-5.8, 0}}
```

Some examples of use of interpolating polynomials was done in the last section. I'm keeping this in here for now, because it shows up in the plot.

```
inpp[x_] = InterpolatingPolynomial[data, x]
```

```
(5.8 + x)
 (0.672414 + x (0.00229885 + (2.5 + x) (0.0561303 + (4. + x) (-0.024302 +
             (0.8 + x) (0.00397479 - 0.00635968 (5. + x))))))
```

```
Simplify[%]
```

$$3.9 - 1.2009\,x - 3.66164\,x^2 - 2.4553\,x^3 - 0.754728\,x^4 - 0.111135\,x^5 - 0.00635968\,x^6$$

I take a derivative out of curiosity, but it doesn't relate to the direction the problem should be going.

```
dinpp = Simplify[D[inpp[x], {x, 2}]]
```

$$-7.32327 - 14.7318\,x - 9.05673\,x^2 - 2.22271\,x^3 - 0.190791\,x^4$$

With the two cells below, I find that the interpolated spline I came up with is clamped at the ends.

```
dinpp /. x → 0
```

```
-7.32327
```

```
dinpp /. x → 6
```

```
-1149.13
```

Next is the polynomial which the text says accompanies the figure on example 2, p. 825.

```
tex[x_] = 3.900 - 0.65083 x^2 + 0.033858 x^4 +
   0.011041 x^6 - 0.0014010 x^8 + 0.000055595 x^10 - 0.00000071867 x^12
```

$$3.9 - 0.65083\,x^2 + 0.033858\,x^4 + 0.011041\,x^6 - 0.001401\,x^8 + 0.000055595\,x^{10} - 7.1867\times 10^{-7}\,x^{12}$$

Also without much purpose, I take a derivative of this function.

```
dtex = Simplify[D[tex[x], {x, 2}]]
```

$$-1.30166 + 0.406296\,x^2 + 0.33123\,x^4 - 0.078456\,x^6 + 0.00500355\,x^8 - 0.0000948644\,x^{10}$$

```
dtex /. x → 0
```

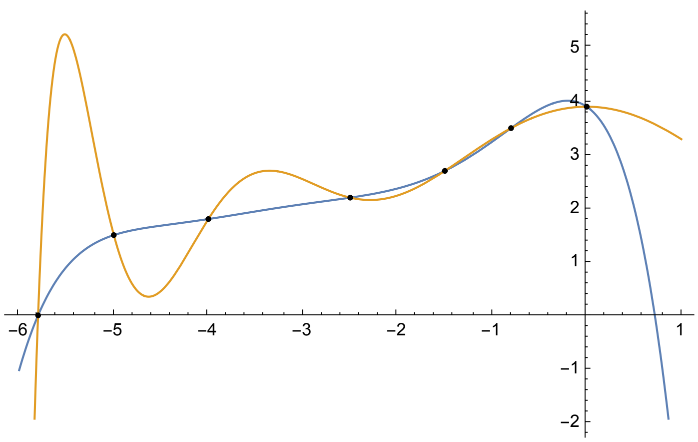
-1.30166

```
dtex /. x → 6
```

-549.891

Now I plot the two polynomials, but neither is g[x], or really relates to what was asked.

```
Plot[{inpp[x], tex[x]}, {x, -6, 1},
 PlotStyle → Thickness[0.003], Epilog → Map[Point, data]]
```



A formula from the text, meant to calculate q_0 is apparently not right

```
q0[x_] = 3.9 + 0 (x + 5.8) - 0.61 (x + 5.8)^2 - 0.015 (x + 5.8)^3
```

$$3.9 - 0.61\,(5.8 + x)^2 - 0.015\,(5.8 + x)^3$$

or I am applying it wrong

```
Solve[q0''[x] == 0, x]
```

{{x → -19.3556}}

which is demonstrated by the next two cells not equaling zero.

```
q0''[.8]
```

-1.814

```
q₀''[0]
-1.742
```

Now switching to the text answer to ponder it. First, in working it out, I see that the below cell produces the value the text refers to as its near-zero. The −1.39 and 0.58 coefficients show that the text is looking at the last two columns of the final row of the table at the bottom of p. 825. But why are there no terms for a_{j0} and a_{j1}? As in the calculation of g[x] in example 1 on p. 824, it appears that something made the zeroth and first terms drop out, even though values for a_{j0} and a_{j1} are shown in the table on p. 825. Maybe some day I will come back to it and try again to figure out how that happened.

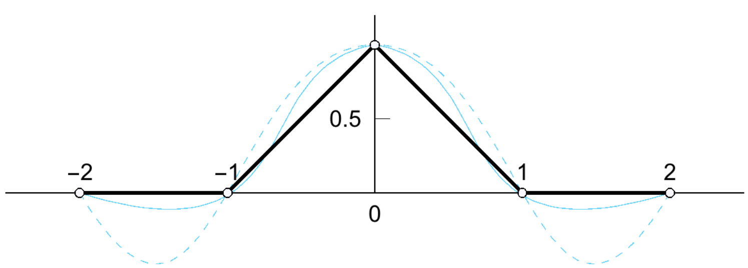
```
q5[x_] = -1.39 (x - 5)^2 + 0.58 (x - 5)^3
-1.39 (-5 + x)^2 + 0.58 (-5 + x)^3
```

```
q5''[5.8]
0.004
```

10 - 16 Determination of splines.
Find the cubic spline g[x] for the given data with k_0 and k_n as given.

11. If we started from the piecewise linear function in figure 438, we would obtain g[x] in problem 10 as the spline satisfying $g'[-2] = f'[-2] = 0$, $g'[2] = f'[2] = 0$. Find and sketch or graph the corresponding interpolation polynomial of 4th degree and compare it with the spline. Comment.

```
kru = RGBColor[0.392, 0.823, 0.98];
innerbw = RGBColor[.97, .97, .994];
pts = {{-2, 0}, {-0.4, -0.5}, {-0.9, 0.85}, {0, 1}};
ptsR = {{2, 0}, {0.4, -0.5}, {0.9, 0.85}, {0, 1}};
ptsL = {{-2, 0}, {-1.1, -1.5}, {-1, 1}, {0, 1}};
ptsR2 = {{2, 0}, {1.1, -1.5}, {1, 1}, {0, 1}};
```



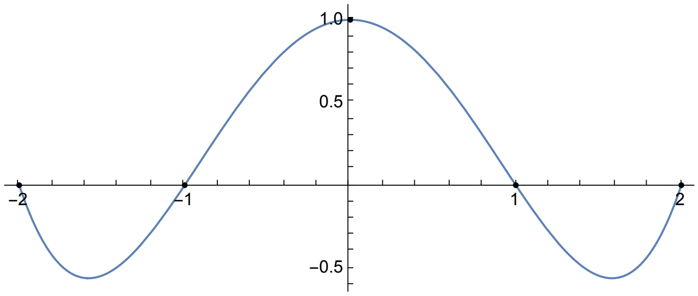

```
Clear["Global`*"]
```

```
data = {{-2, 0}, {-1, 0}, {0, 1}, {1, 0}, {2, 0}}
{{-2, 0}, {-1, 0}, {0, 1}, {1, 0}, {2, 0}}
```

```
inpp[x_] = InterpolatingPolynomial[data, x]
```

$$(1 + x)\ (2 + x) \left(\frac{1}{2} + \left(-\frac{1}{2} + \frac{1}{4}\ (-1 + x)\right) x\right)$$

```
p1 = Plot[{inpp[x]}, {x, -2, 2}, PlotStyle → Thickness[0.003],
  AspectRatio → Automatic, Epilog → Map[Point, data]]
```



The plot above happens to be of 4th degree, as requested. It looks a lot like the dashed curve above. Using the method of piecewise poly described in problem 13 and 15, I could get something pretty close to the solid curve in the multi-curve plot, I think.

13. f_0 = f[0] = 1, f_1 = f[1] =0, f_2 = f[2] =−1
f_3 = f[3] = 0, k_0 = 0, k_3 = −6

```
Clear["Global`*"]
```

```
Needs["Splines`"]
```

Bounced around for awhile looking for an easy way to work with splines. The text wants to give only control points, plus the first derivatives at the ends. This is enough descriptive info to plot a solution, but within *Mathematica* documentation, and considering most of what I found on-line, the talk was about basis splines, knots, and weights as well as control points. I finally found an excellent resource in Ray Koopman's answer to SEMma question *https://mathematica.stackexchange.com/questions/35405/extracting-polynomials-from-a-cubic-spline-function*. It goes straight for the piecewise poly equation without messing around with that other spline stuff. I like splines, but as J.M. says in the question site just referenced, *"Unfortunately, as it stands, BSplineFunction[] objects are immune to exposure from PiecewiseExpand[]."*

A comment about the first section. It seems that Solve cannot handle more than three control points at a time. I commented out some second derivative conditions that I don't think are necessary, in order to make room for some crucial first derivative conditions, without overloading Solve. I had to play with it a little, because the join at x=2, though smooth, was lopsided at first. With yellow below, I simply picked out a first derivative that I thought would look good, and then matched it with the calculation of f3.

```
dat = {{0, 1}, {1, 0}, {2, -1}, {3, 0}};
```

```
f1[x_] = a1 + b1 * x + c1 * x^2 + d1 * x^3;
f2[x_] = a2 + b2 * x + c2 * x^2 + d2 * x^3;
s = Solve[{f1@dat[[1, 1]] == dat[[1, 2]],
   f1@dat[[2, 1]] == dat[[2, 2]], f2@dat[[2, 1]] == dat[[2, 2]],
   f2@dat[[3, 1]] == dat[[3, 2]], f1'@dat[[2, 1]] == f2'@dat[[2, 1]],
   (*f1''@dat[[2,1]]==f2''@dat[[2,1]],*)f1''@dat[[1, 1]] == 0,
   (*f2''@dat[[3,1]]==0,*)f1'@dat[[1, 1]] == 0,
   f2'@dat[[3, 1]] == 1.5}, {a1, b1, c1, d1, a2, b2, c2, d2}]
```

$$\{\{a1 \to 1., b1 \to 0., c1 \to 0., d1 \to -1., a2 \to 4., b2 \to -4.5, c2 \to 5.32907\times10^{-15}, d2 \to 0.5\}\}$$

After taking care of the first two pieces, I have to do the last one. With pink, I requested a smooth join at x=2. I found out I could not get that join smooth and have the second derivative equal zero on the right end. So I guess that means the right end is “clamped”, and not “free” or “natural” in the terms of the text, p. 823. However, I do not see that it violates the problem description to have it clamped.

```
f1[x_] = a1 + b1 * x + c1 * x^2 + d1 * x^3 /. {a1 → 1, b1 → 0, c1 → 0, d1 → -1}
f2[x_] = a2 + b2 * x + c2 * x^2 + d2 * x^3 /. {a2 → 4, b2 → -4.5, c2 → 0, d2 → 0.5}
f3[x_] = a3 + b3 * x + c3 * x^2 + d3 * x^3;

s2 = N[Solve[{f3@dat[[3, 1]] == dat[[3, 2]],
    f3@dat[[4, 1]] == dat[[4, 2]], f3'@dat[[3, 1]] == 1.5,
    f3'@dat[[4, 1]] == -6(*,f3''@dat[[4,1]]  ==  0*)}, {a3, b3, c3, d3}]]
```

$1 - x^3$ (* text solution $1-x^2$ *)

$4 - 4.5` x + 0.5` x^3$ (* text solution $-2(x-1)-(x-1)^2+2(x-1)^3$ *)

$\{\{a3 \to 72., b3 \to -100.5, c3 \to 45., d3 \to -6.5\}\}$

```
f3[x_] =
 a3 + b3 * x + c3 * x^2 + d3 * x^3 /. {a3 → 72, b3 → -100.5, c3 → 45, d3 → -6.5}
```

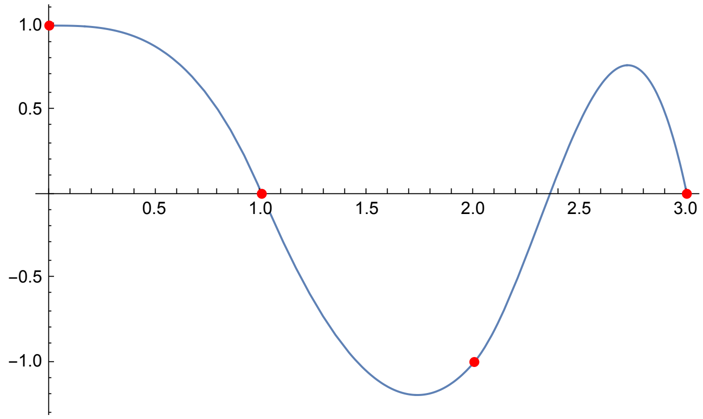
$72 - 100.5` x + 45 x^2 - 6.5` x^3$ (* text solution $-1 +2(x-2)+5(x-2)^2-6(x-2)^3$ *)

```
f[x_] = Piecewise[{{f1[x], 0 ≤ x ≤ 1}, {f2[x], 1 ≤ x ≤ 2}, {f3[x], 2 ≤ x ≤ 3}}]
```

$$\begin{cases} 1 - x^3 & 0 \le x \le 1 \\ 4 - 4.5 x + 0.5 x^3 & 1 \le x \le 2 \\ 72 - 100.5 x + 45 x^2 - 6.5 x^3 & 2 \le x \le 3 \\ 0 & \text{True} \end{cases}$$

Incidentally, none of the poly pieces matches the equations listed in the text answer.

```
Plot[f[x], {x, 0, 3}, PlotStyle → Thickness[0.003],
 Epilog → {Red, PointSize[0.015], Point /@ dat}]
```



I feel fairly satisfied with this problem. It seems like a fairly general way to build piecewise cubic polys when they are needed.

15. f_0 = f[0] = 4, f_1 = f[2] =0, f_2 = f[4] =−4
f_3 = f[6] = 78, k_0 = 0, k_3 = 0

```
Clear["Global`*"]
```

```
dat = {{0, 4}, {2, 0}, {4, -4}, {6, 78}};
```

I didn't ask for the Splines package here, and I didn't need it. I was able to copy everything from the last problem, change the designation of control points, and do a minor tweak on the join at x=4, choosing a reasonable value of the first derivative.

```
f1[x_] = a1 + b1 * x + c1 * x^2 + d1 * x^3;
f2[x_] = a2 + b2 * x + c2 * x^2 + d2 * x^3;
s = Solve[{f1@dat[[1, 1]] == dat[[1, 2]],
    f1@dat[[2, 1]] == dat[[2, 2]], f2@dat[[2, 1]] == dat[[2, 2]],
    f2@dat[[3, 1]] == dat[[3, 2]], f1'@dat[[2, 1]] == f2'@dat[[2, 1]],
    (*f1''@dat[[2,1]]==f2''@dat[[2,1]],*)f1''@dat[[1, 1]] == 0,
    (*f2''@dat[[3,1]]==0,*)f1'@dat[[1, 1]] == 0,
    f2'@dat[[3, 1]] == 1.0}, {a1, b1, c1, d1, a2, b2, c2, d2}]
```

```
{{a1 → 4., b1 → 0., c1 → 0.,
  d1 → -0.5, a2 → 24., b2 → -19., c2 → 4., d2 → -0.25}}
```

$$f1[x\_] = a1 + b1 * x + c1 * x\^{}2 + d1 * x\^{}3 \text{ /. } \left\{a1 \to 4,\ b1 \to 0,\ c1 \to 0,\ d1 \to -\frac{1}{2}\right\}$$

```
f2[x_] =
 a2 + b2 * x + c2 * x^2 + d2 * x^3 /. {a2 → 24., b2 → -19., c2 → 4., d2 → -0.25}
f3[x_] = a3 + b3 * x + c3 * x^2 + d3 * x^3;
```

$4 - \frac{x^3}{2}$ (* text solution $4+x^2-x^3$ *)

```
24.` - 19.` x + 4.` x^2 -
 0.25` x^3 (* text solution -8(x-2)-5(x-2)^2+5(x-2)^3 *)

s2 = N[Solve[{f3@dat[[3, 1]] == dat[[3, 2]],
    f3@dat[[4, 1]] == dat[[4, 2]], f3'@dat[[3, 1]] == 1.0,
    f3'@dat[[4, 1]] == 0(*,f3''@dat[[4,1]] == 0*)}, {a3, b3, c3, d3}]]

{{a3 → 2256., b3 → -1455., c3 → 303.5, d3 → -20.25}}

f3[x_] = a3 + b3 * x + c3 * x^2 + d3 * x^3 /.
  {a3 → 2256, b3 → -1455, c3 → 303.5, d3 → -20.25}

2256 - 1455 x + 303.5` x^2 -
 20.25` x^3 (* text solution 4+32(x-4)+25(x-4)^2-11(x-4)^3 *)

f[x_] = Piecewise[{{f1[x], 0 ≤ x ≤ 2}, {f2[x], 2 ≤ x ≤ 4}, {f3[x], 4 ≤ x ≤ 6}}]
```

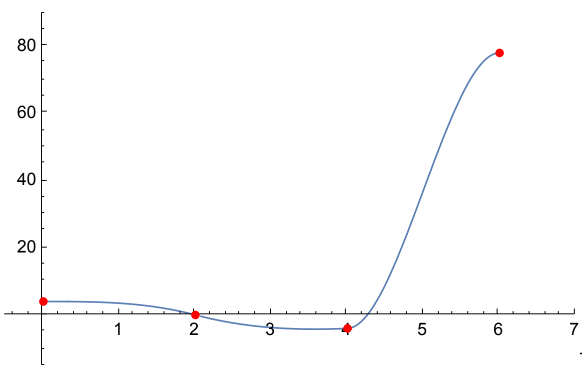
$$\begin{cases} 4 - \frac{x^3}{2} & 0 \le x \le 2 \\ 24. - 19.\,x + 4.\,x^2 - 0.25\,x^3 & 2 \le x \le 4 \\ 2256 - 1455\,x + 303.5\,x^2 - 20.25\,x^3 & 4 \le x \le 6 \\ 0 & \text{True} \end{cases}$$

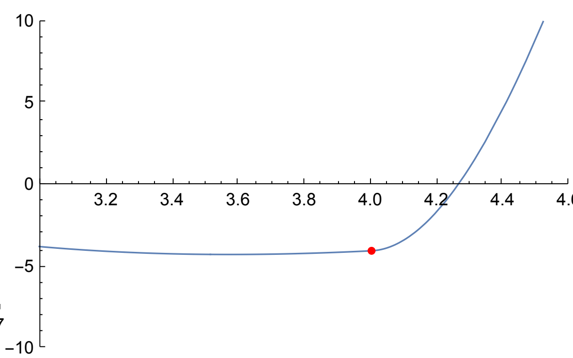
```
p1 = Plot[f[x], {x, 0, 6}, PlotStyle → Thickness[0.003],
   PlotRange → {{-0.5, 7}, {-15, 90}}, ImageSize → 300,
   Epilog → {Red, PointSize[0.015], Point /@ dat}];

p2 = Plot[f[x], {x, 0, 6}, PlotStyle → Thickness[0.003],
   PlotRange → {{3, 4.6}, {-10, 10}}, ImageSize → 300,
   Epilog → {Red, PointSize[0.015], Point /@ dat}];
```

The plots below show that all is okay at x=4. However, none of the piecewise equations match the answer in the text. Again, I feel this approach could be useful.

```
Row[{p1, p2}]
```





17. If a cubic spline is three times continuously differentiable (that is, it has continuous first, second, and third derivatives), show that it must be a single polynomial.

Hmm, sounds slightly intriguing.