2 - 7 General solution

Find a general solution.

3. $y_1{}' = y_2 + e^{3t}$

$y_2{}' = y_1 - 3\, e^{3t}$

```
Clear["Global`*"]
```

As in the last section, there will be rearranging, recasting, and substitutions after the solution appears in order to make its form like the text answer.

```
e1 = {y1'[t] == y2[t] + e^(3 t), y2'[t] == y1[t] - 3 e^(3 t)}
e2 = DSolve[e1, {y1, y2}, t]
```

$$\{y1'[t] == e^{3t} + y2[t],\ y2'[t] == -3\, e^{3t} + y1[t]\}$$

$$\{\{y1 \to \text{Function}[\{t\},\ \frac{1}{2} e^{-t} (1 + e^{2t})\, C[1] + \frac{1}{2} e^{-t} (-1 + e^{2t})\, C[2]],\ y2 \to \text{Function}[\{t\},\ \frac{1}{4} e^{t} (-1 + e^{2t})^2 - \frac{1}{4} e^{t} (1 + e^{2t})^2 + \frac{1}{2} e^{-t} (-1 + e^{2t})\, C[1] + \frac{1}{2} e^{-t} (1 + e^{2t})\, C[2]]\}\}$$

```
e3 = e2[[1, 1, 2, 2]]
```

$$\frac{1}{2} e^{-t} (1 + e^{2t})\, C[1] + \frac{1}{2} e^{-t} (-1 + e^{2t})\, C[2]$$

```
e4 = Expand[e3]
```

$$\frac{1}{2} e^{-t} C[1] + \frac{1}{2} e^{t} C[1] - \frac{1}{2} e^{-t} C[2] + \frac{1}{2} e^{t} C[2]$$

```
e5 = Collect[e4, e^-t]
```

$$e^{-t} \left(\frac{C[1]}{2} - \frac{C[2]}{2}\right) + e^{t} \left(\frac{C[1]}{2} + \frac{C[2]}{2}\right)$$

$$e6 = e5\ /.\ \left\{\left(\frac{C[1]}{2} - \frac{C[2]}{2}\right) \to c1,\ \left(\frac{C[1]}{2} + \frac{C[2]}{2}\right) \to c2\right\}$$

$$c1\, e^{-t} + c2\, e^{t}$$

```
e7 = e2[[1, 2, 2, 2]]
```

$$\frac{1}{4} e^{t} (-1 + e^{2t})^2 - \frac{1}{4} e^{t} (1 + e^{2t})^2 + \frac{1}{2} e^{-t} (-1 + e^{2t})\, C[1] + \frac{1}{2} e^{-t} (1 + e^{2t})\, C[2]$$

```
e8 = Expand[e7]
```

$$-e^{3t} - \frac{1}{2} e^{-t} C[1] + \frac{1}{2} e^{t} C[1] + \frac{1}{2} e^{-t} C[2] + \frac{1}{2} e^{t} C[2]$$

```
e9 = Collect[e8, e^t]
```

$$-e^{3t} + e^{-t}\left(-\frac{C[1]}{2} + \frac{C[2]}{2}\right) + e^{t}\left(\frac{C[1]}{2} + \frac{C[2]}{2}\right)$$

$$e10 = e9 /. \left\{\left(-\frac{C[1]}{2} + \frac{C[2]}{2}\right) \to -c1, \left(\frac{C[1]}{2} + \frac{C[2]}{2}\right) \to c2\right\}$$

$$-c1\, e^{-t} + c2\, e^{t} - e^{3t}$$

1. Above: The expressions in the green cells match the text answers for y1 and y2 respectively. The substitutions of symbolic constants shown in the yellow cells match values of constants between the two function expressions, demonstrating that the constant substitution system is self-consistent and consistent with the text.

5. $y_1' = 4 y_1 + y_2 + 0.6\, t$
$y_2' = 2 y_1 + 3 y_2 - 2.5\, t$

```
Clear["Global`*"]
```

```
e1 = {y1'[t] == 4 y1[t] + y2[t] + 0.6 t, y2'[t] == 2 y1[t] + 3 y2[t] - 2.5 t}
e2 = DSolve[e1, {y1, y2}, t]
```

$$\{y1'[t] == 0.6\, t + 4\, y1[t] + y2[t],\ y2'[t] == -2.5\, t + 2\, y1[t] + 3\, y2[t]\}$$

$$\{\{y1 \to \text{Function}[\{t\},\ -0.333333\left(1.\, e^{2.t} - 1.\, e^{5.t}\right)\left(-2.06667\, e^{-2.t}(-0.25 - 0.5\,t) - 0.433333\, e^{-5.t}(-0.04 - 0.2\,t)\right) + 0.333333\left(1.\, e^{2.t} + 2.\, e^{5.t}\right)\left(1.03333\, e^{-2.t}(-0.25 - 0.5\,t) - 0.433333\, e^{-5.t}(-0.04 - 0.2\,t)\right) + 0.333333\left(1.\, e^{2.t} + 2.\, e^{5.t}\right) C[1] - 0.333333\left(1.\, e^{2.t} - 1.\, e^{5.t}\right) C[2]],$$

$$y2 \to \text{Function}[\{t\},\ 0.666667\left(1.\, e^{2.t} + 0.5\, e^{5.t}\right)\left(-2.06667\, e^{-2.t}(-0.25 - 0.5\,t) - 0.433333\, e^{-5.t}(-0.04 - 0.2\,t)\right) - 0.666667\left(1.\, e^{2.t} - 1.\, e^{5.t}\right)\left(1.03333\, e^{-2.t}(-0.25 - 0.5\,t) - 0.433333\, e^{-5.t}(-0.04 - 0.2\,t)\right) - 0.666667\left(1.\, e^{2.t} - 1.\, e^{5.t}\right) C[1] + 0.666667\left(1.\, e^{2.t} + 0.5\, e^{5.t}\right) C[2]]\}\}$$

```
e3 = e2[[1, 1, 2, 2]]
```

$$-0.333333\left(1.\,e^{2.t}-1.\,e^{5.t}\right)\left(-2.06667\,e^{-2.t}(-0.25-0.5t)-0.433333\,e^{-5.t}(-0.04-0.2t)\right)+0.333333\left(1.\,e^{2.t}+2.\,e^{5.t}\right)\left(1.03333\,e^{-2.t}(-0.25-0.5t)-0.433333\,e^{-5.t}(-0.04-0.2t)\right)+0.333333\left(1.\,e^{2.t}+2.\,e^{5.t}\right)C[1]-0.333333\left(1.\,e^{2.t}-1.\,e^{5.t}\right)C[2]$$

```
e4 = Simplify[e3]
```

$$e^{-3.t}\left(-8.67362\times10^{-19}+e^{3.t}(-0.241-0.43t)+e^{6.t}\left(-2.77556\times10^{-17}-5.55112\times10^{-17}t\right)+e^{5.t}(0.333333\,C[1]-0.333333\,C[2])+e^{8.t}(0.666667\,C[1]+0.333333\,C[2])\right)$$

```
e5 = Expand[e4]
```

$$-0.241-8.67362\times10^{-19}e^{-3.t}-2.77556\times10^{-17}e^{3.t}-0.43t-5.55112\times10^{-17}e^{3.t}t+0.333333\,e^{2.t}C[1]+0.666667\,e^{5.t}C[1]-0.333333\,e^{2.t}C[2]+0.333333\,e^{5.t}C[2]$$

```
e6 = Chop[e5, 10^-16]
```

$$-0.241-0.43t+0.333333\,e^{2.t}C[1]+0.666667\,e^{5.t}C[1]-0.333333\,e^{2.t}C[2]+0.333333\,e^{5.t}C[2]$$

```
e7 = Collect[e6, {e^(2. t), e^(5. t)}]
```

$$-0.241-0.43t+e^{2.t}(0.333333\,C[1]-0.333333\,C[2])+e^{5.t}(0.666667\,C[1]+0.333333\,C[2])$$

```
e8 = e7 /. {(0.3333333333333334` C[1] - 0.3333333333333333` C[2]) → c2,
    (0.6666666666666666` C[1] + 0.3333333333333333` C[2]) → c1}
```

$$-0.241+c2\,e^{2.t}+c1\,e^{5.t}-0.43t$$

```
e9 = e2[[1, 2, 2, 2]]
```

$$0.666667\left(1.\,e^{2.t}+0.5\,e^{5.t}\right)\left(-2.06667\,e^{-2.t}(-0.25-0.5t)-0.433333\,e^{-5.t}(-0.04-0.2t)\right)-0.666667\left(1.\,e^{2.t}-1.\,e^{5.t}\right)\left(1.03333\,e^{-2.t}(-0.25-0.5t)-0.433333\,e^{-5.t}(-0.04-0.2t)\right)-0.666667\left(1.\,e^{2.t}-1.\,e^{5.t}\right)C[1]+0.666667\left(1.\,e^{2.t}+0.5\,e^{5.t}\right)C[2]$$

```
e10 = Simplify[e9]
```

$$0.534+1.73472\times10^{-18}e^{-3.t}+1.12t+e^{5.t}(0.666667\,C[1]+0.333333\,C[2])+e^{2.t}(-0.666667\,C[1]+0.666667\,C[2])$$

```
e11 = Chop[e10, 10^-16]
```

$0.534 + 1.12\,t + e^{5.\,t}\,(0.666667\,C[1] + 0.333333\,C[2]) + e^{2.\,t}\,(-0.666667\,C[1] + 0.666667\,C[2])$

```
e12 = e11 /. {(0.6666666666666669` C[1] + 0.3333333333333334` C[2]) → c1,
    (-0.6666666666666669` C[1] + 0.6666666666666667` C[2]) → -2 c2}
```

$0.534 - 2\,c2\,e^{2.\,t} + c1\,e^{5.\,t} + 1.12\,t$

1. Above: The expressions in the green cells match the text answers for y1 and y2 respectively. The substitutions of symbolic constants shown in the yellow cells match values of constants between the two function expressions, demonstrating that the constant substitution system is self-consistent and consistent with the text.

7. $y_1{}' = -3\,y_1 - 4\,y_2 + 11\,t + 15$
$y_2{}' = 5\,y_1 + 6\,y_2 + 3\,e^{-t} - 15\,t - 20$

```
Clear["Global`*"]
```

```
e1 = {y1'[t] == -3 y1[t] - 4 y2[t] + 11 t + 15,
  y2'[t] == 5 y1[t] + 6 y2[t] + 3 e^-t - 15 t - 20}
e2 = DSolve[e1, {y1, y2}, t]
```

$\{y1'[t] == 15 + 11\,t - 3\,y1[t] - 4\,y2[t],\; y2'[t] == -20 + 3\,e^{-t} - 15\,t + 5\,y1[t] + 6\,y2[t]\}$

$\{\{y1 \to \text{Function}[\{t\}, -e^{t}\,(-5 + 4\,e^{t})\,(4\,e^{-3\,t} + e^{-2\,t}\,(-20 - 8\,t) + e^{-t}\,(10 + 5\,t)) - 4\,e^{t}\,(-1 + e^{t})\,\left(-5\,e^{-3\,t} + e^{-t}\,(-10 - 5\,t) + e^{-2\,t}\left(\frac{47}{2} + 10\,t\right)\right) - e^{t}\,(-5 + 4\,e^{t})\,C[1] - 4\,e^{t}\,(-1 + e^{t})\,C[2]],$
$y2 \to \text{Function}[\{t\}, 5\,e^{t}\,(-1 + e^{t})\,(4\,e^{-3\,t} + e^{-2\,t}\,(-20 - 8\,t) + e^{-t}\,(10 + 5\,t)) + e^{t}\,(-4 + 5\,e^{t})\,\left(-5\,e^{-3\,t} + e^{-t}\,(-10 - 5\,t) + e^{-2\,t}\left(\frac{47}{2} + 10\,t\right)\right) + 5\,e^{t}\,(-1 + e^{t})\,C[1] + e^{t}\,(-4 + 5\,e^{t})\,C[2]]\}\}$

```
e3 = e2[[1, 1, 2, 2]]
```

$-e^{t}\,(-5 + 4\,e^{t})\,(4\,e^{-3\,t} + e^{-2\,t}\,(-20 - 8\,t) + e^{-t}\,(10 + 5\,t)) - 4\,e^{t}\,(-1 + e^{t})\,\left(-5\,e^{-3\,t} + e^{-t}\,(-10 - 5\,t) + e^{-2\,t}\left(\frac{47}{2} + 10\,t\right)\right) - e^{t}\,(-5 + 4\,e^{t})\,C[1] - 4\,e^{t}\,(-1 + e^{t})\,C[2]$

```
e4 = Simplify[e3]
```

$e^{-t}\,(-2 - e^{t}\,(4 + 3\,t) - 4\,e^{3\,t}\,(C[1] + C[2]) + e^{2\,t}\,(5\,C[1] + 4\,C[2]))$

```
e5 = Expand[e4]
```

$-4 - 2 e^{-t} - 3 t + 5 e^{t} C[1] - 4 e^{2t} C[1] + 4 e^{t} C[2] - 4 e^{2t} C[2]$

```
e6 = Collect[e5, {e^(2 t), e^t}]
```

$-4 - 2 e^{-t} - 3 t + e^{2t} (-4 C[1] - 4 C[2]) + e^{t} (5 C[1] + 4 C[2])$

```
e7 = e6 /. {(-4 C[1] - 4 C[2]) → 4 c2, (5 C[1] + 4 C[2]) → c1}
```

$-4 - 2 e^{-t} + c1\, e^{t} + 4\, c2\, e^{2t} - 3 t$

```
e8 = e2[[1, 2, 2, 2]]
```

$$5 e^{t} \left(-1 + e^{t}\right) \left(4 e^{-3t} + e^{-2t} (-20 - 8 t) + e^{-t} (10 + 5 t)\right) + e^{t} \left(-4 + 5 e^{t}\right) \left(-5 e^{-3t} + e^{-t} (-10 - 5 t) + e^{-2t} \left(\frac{47}{2} + 10 t\right)\right) + 5 e^{t} \left(-1 + e^{t}\right) C[1] + e^{t} \left(-4 + 5 e^{t}\right) C[2]$$

```
e9 = Simplify[e8]
```

$$\frac{15}{2} + e^{-t} + 5 t + 5 e^{2t} (C[1] + C[2]) - e^{t} (5 C[1] + 4 C[2])$$

```
e10 = e9 /. {(C[1] + C[2]) → -c2, (5 C[1] + 4 C[2]) → c1}
```

$$\frac{15}{2} + e^{-t} - c1\, e^{t} - 5\, c2\, e^{2t} + 5 t$$

1. Above: The expressions in the green cells match the text answers for y1 and y2 respectively. The substitutions of symbolic constants shown in the yellow cells match values of constants between the two function expressions, demonstrating that the constant substitution system is self-consistent and consistent with the text.

10 - 15 Initial value problem
Solve, showing details.

11. $y_1' = y_2 + 6 e^{2t}$
$y_2' = y_1 - e^{2t}$
$y_1[0] = 1,\ y_2[0] = 0$

```
Clear["Global`*"]
```

```
e1 = {y1'[t] == y2[t] + 6 ⅇ^(2 t), y2'[t] == y1[t] - ⅇ^(2 t), y1[0] == 1, y2[0] == 0}
e2 = DSolve[e1, {y1, y2}, t]
```

$\{y1'[t] == 6\, e^{2t} + y2[t],\ y2'[t] == -e^{2t} + y1[t],\ y1[0] == 1,\ y2[0] == 0\}$

$\{\{y1 \to \text{Function}[\{t\}, \frac{1}{3} e^{-t} (-2 - 6\, e^{2t} + 11\, e^{3t})],$
$y2 \to \text{Function}[\{t\}, \frac{2}{3} e^{-t} (-1 + e^{t})^2 (1 + 2\, e^{t})]\}\}$

```
e3 = e2[[1, 1, 2, 2]]
```

$\frac{1}{3} e^{-t} (-2 - 6\, e^{2t} + 11\, e^{3t})$

```
e4 = Expand[e3]
```

$-\frac{2\, e^{-t}}{3} - 2\, e^{t} + \frac{11\, e^{2t}}{3}$

```
e5 = e4 /. (-2 ⅇ^-t/3 - 2 ⅇ^t) → ExpToTrig[-2 ⅇ^-t/3 - 2 ⅇ^t]
```

$\frac{11\, e^{2t}}{3} - \frac{8\, \text{Cosh}[t]}{3} - \frac{4\, \text{Sinh}[t]}{3}$

```
e6 = e2[[1, 2, 2, 2]]
```

$\frac{2}{3} e^{-t} (-1 + e^{t})^2 (1 + 2\, e^{t})$

```
e7 = Expand[e6]
```

$\frac{2\, e^{-t}}{3} - 2\, e^{t} + \frac{4\, e^{2t}}{3}$

```
e8 = e7 /. (2 ⅇ^-t/3 - 2 ⅇ^t) → ExpToTrig[2 ⅇ^-t/3 - 2 ⅇ^t]
```

$\frac{4\, e^{2t}}{3} - \frac{4\, \text{Cosh}[t]}{3} - \frac{8\, \text{Sinh}[t]}{3}$

1. Above: The top and bottom green cell expressions match the text answers for y1 and y2 respectively.

13. $y_1' = y_2 - 5\, \text{Sin}[t]$
$y_2' = -4\, y_1 + 17\, \text{Cos}[t]$
$y_1[0] = 5,\ y_2[0] = 2$

```
Clear["Global`*"]
```

```
e1 = {y1'[t] == y2[t] - 5 Sin[t],
  y2'[t] == -4 y1[t] + 17 Cos[t], y1[0] == 5, y2[0] == 2}
e2 = DSolve[e1, {y1, y2}, t]
```

{y1′[t] == -5 Sin[t] + y2[t], y2′[t] == 17 Cos[t] - 4 y1[t], y1[0] == 5, y2[0] == 2}

$$\{\{y1 \to \text{Function}[\{t\}, \frac{1}{4} (4 \text{Cos}[2 t] + 7 \text{Cos}[t] \text{Cos}[2 t] + 9 \text{Cos}[2 t] \text{Cos}[3 t] + 4 \text{Sin}[2 t] + 7 \text{Sin}[t] \text{Sin}[2 t] + 9 \text{Sin}[2 t] \text{Sin}[3 t])],$$
$$y2 \to \text{Function}[\{t\}, \frac{1}{2} (4 \text{Cos}[2 t] + 7 \text{Cos}[2 t] \text{Sin}[t] - 4 \text{Sin}[2 t] - 7 \text{Cos}[t] \text{Sin}[2 t] - 9 \text{Cos}[3 t] \text{Sin}[2 t] + 9 \text{Cos}[2 t] \text{Sin}[3 t])]\}\}$$

```
e3 = e2[[1, 1, 2, 2]]
```

$$\frac{1}{4} (4 \text{Cos}[2 t] + 7 \text{Cos}[t] \text{Cos}[2 t] + 9 \text{Cos}[2 t] \text{Cos}[3 t] + 4 \text{Sin}[2 t] + 7 \text{Sin}[t] \text{Sin}[2 t] + 9 \text{Sin}[2 t] \text{Sin}[3 t])$$

```
e4 = Simplify[e3]
```

4 Cos[t] + Cos[2 t] + Sin[2 t]

```
e5 = e2[[1, 2, 2, 2]]
```

$$\frac{1}{2} (4 \text{Cos}[2 t] + 7 \text{Cos}[2 t] \text{Sin}[t] - 4 \text{Sin}[2 t] - 7 \text{Cos}[t] \text{Sin}[2 t] - 9 \text{Cos}[3 t] \text{Sin}[2 t] + 9 \text{Cos}[2 t] \text{Sin}[3 t])$$

```
e6 = Simplify[e5]
```

2 Cos[2 t] + Sin[t] - 2 Sin[2 t]

1. Above: The top and bottom blue cell expressions match the text answers for y1 and y2 respectively.

15. $y_1' = y_1 + 2 y_2 + e^{2t} - 2t$
$y_2' = -y_2 + 1 + t$
$y_1[0] = 1,\ y_2[0] = -4$

```
Clear["Global`*"]
```

```
e1 = {y1'[t] == y1[t] + 2 y2[t] + e^(2 t) - 2 t,
   y2'[t] == -y2[t] + 1 + t, y1[0] == 1, y2[0] == -4}
e2 = DSolve[e1, {y1, y2}, t]
```

$$\{y1'[t] == e^{2t} - 2t + y1[t] + 2\,y2[t],\; y2'[t] == 1 + t - y2[t],\; y1[0] == 1,\; y2[0] == -4\}$$

$$\{\{y1 \to \text{Function}[\{t\}, -((e^{-t}(4 - e^{2t}e^{t} - 2e^{2t} - 8\,\text{Log}[e] + 6e^{2t}\,\text{Log}[e]))/(-1 + 2\,\text{Log}[e]))],\; y2 \to \text{Function}[\{t\}, e^{-t}(-4 + e^{t}t)]\}\}$$

```
e3 = e2[[1, 1, 2, 2]]
```

$$-((e^{-t}(4 - e^{2t}e^{t} - 2e^{2t} - 8\,\text{Log}[e] + 6e^{2t}\,\text{Log}[e]))/(-1 + 2\,\text{Log}[e]))$$

```
e4 = e3 /. (e^-t (4 - e^(2 t) e^t - 2 e^(2 t) - 8 Log[e] + 6 e^(2 t) Log[e])) →
    Expand[e^-t (4 - e^(2 t) e^t - 2 e^(2 t) - 8 Log[e] + 6 e^(2 t) Log[e])]
```

$$-((-e^{2t} + 4e^{-t} - 2e^{t} - 8e^{-t}\,\text{Log}[e] + 6e^{t}\,\text{Log}[e])/(-1 + 2\,\text{Log}[e]))$$

```
e5 = FullSimplify[e4]
```

$$(e^{2t} + 2\,\text{Cosh}[t]\,(-1 + \text{Log}[e]) + (6 - 14\,\text{Log}[e])\,\text{Sinh}[t])/(-1 + 2\,\text{Log}[e])$$

```
e6 = e5 /. Log[e] → 1
```

$$e^{2t} - 8\,\text{Sinh}[t]$$

```
e7 = e6 /. (-8 Sinh[t]) → TrigToExp[-8 Sinh[t]]
```

$$e^{2t} + 4e^{-t} - 4e^{t}$$

```
e8 = e2[[1, 2, 2, 2]]
```

$$e^{-t}(-4 + e^{t}t)$$

```
e9 = Expand[e8]
```

$$-4e^{-t} + t$$

1. Above: The top and bottom green cell expressions match the text answers for y1 and y2 respectively.