3 - 12 Effect of delta (impulse) on vibrating systems
Find and graph or sketch the solution of the IVP.

3. $y'' + 4y = \delta(t - \pi)$, $y[0] = 8$, $y'[0] = 0$

```
Clear["Global`*"]

e1 = LaplaceTransform[y''[t] + 4 y[t] == DiracDelta[ (t - π)], t, s]

4 LaplaceTransform[y[t], t, s] +
  s^2 LaplaceTransform[y[t], t, s] - s y[0] - y′[0] == e^(-π s)

Clear["Global`*"]

e1 = LaplaceTransform[y''[t] + 4 y[t] == DiracDelta[ (t - π)], t, s]

4 LaplaceTransform[y[t], t, s] +
  s^2 LaplaceTransform[y[t], t, s] - s y[0] - y′[0] == e^(-π s)

e2 = e1 /. {y[0] → 8, y'[0] → 0, LaplaceTransform[y[t], t, s] → bigY}
4 bigY - 8 s + bigY s^2 == e^(-π s)

e3 = Solve[e2, bigY]
```

$$\left\{\left\{\text{bigY} \to \frac{e^{-\pi s}\,(1 + 8\,e^{\pi s}\,s)}{4 + s^2}\right\}\right\}$$

```
e4 = e3[[1, 1, 2]]
```

$$\frac{e^{-\pi s}\,(1 + 8\,e^{\pi s}\,s)}{4 + s^2}$$

```
e5 = InverseLaplaceTransform[e4, s, t]

8 Cos[2 t] + Cos[t] HeavisideTheta[-π + t] Sin[t]
```

$$\text{PossibleZeroQ}\left[\text{Cos}[t]\ \text{Sin}[t] - \frac{1}{2}\,\text{Sin}[2\,t]\right]$$

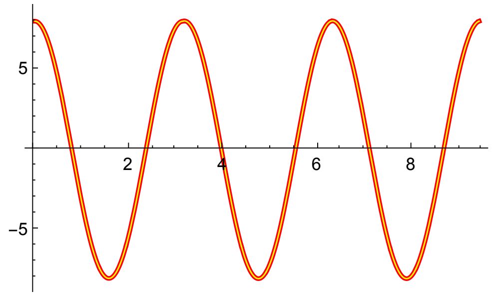
```
True
```

I showed in section 6.3 that **HeavisideTheta** is equivalent to **UnitStep**. Combined with the PZQ above, it makes the green cell equivalent to the text answer.

```
plot1 = Plot[e5, {t, 0, 3 π}, PlotRange → Automatic,
   PlotStyle → {Yellow, Thickness[0.003]}, ImageSize → 250];
```

$$\text{plot2} = \text{Plot}\left[8\,\text{Cos}[2\,t] + \frac{1}{2}\,\text{UnitStep}[t - \pi]\ \text{Sin}[2\,t],\ \{t, 0, 3\,\pi\},\ \text{PlotRange} \to \text{Automatic},\ \text{PlotStyle} \to \{\text{Red}, \text{Thickness}[0.01]\},\ \text{ImageSize} \to 250\right];$$

```
Show[plot2, plot1]
```



Above: The solution tracks well with that of the text.

5. y'' + y = δ (t - π) - δ (t - 2 π), y[0] = 0, y'[0] = 1

```
Clear["Global`*"]
```

```
e1 = LaplaceTransform[
  y''[t] + y[t] == DiracDelta[t - π] - DiracDelta[t - 2 π], t, s]
```

$\text{LaplaceTransform}[y[t], t, s] + s^2\, \text{LaplaceTransform}[y[t], t, s] - s\, y[0] - y'[0] == -e^{-2\pi s} + e^{-\pi s}$

```
e2 = e1 /. {y[0] → 0, y'[0] → 1, LaplaceTransform[y[t], t, s] → bigY}
```

$-1 + \text{bigY} + \text{bigY}\, s^2 == -e^{-2\pi s} + e^{-\pi s}$

```
e3 = Solve[e2, bigY]
```

$\left\{\left\{\text{bigY} \to \frac{e^{-2\pi s}\left(-1 + e^{\pi s} + e^{2\pi s}\right)}{1 + s^2}\right\}\right\}$

```
e4 = e3[[1, 1, 2]]
```

$\frac{e^{-2\pi s}\left(-1 + e^{\pi s} + e^{2\pi s}\right)}{1 + s^2}$

```
e5 = InverseLaplaceTransform[e4, s, t]
```

```
-(-1 + HeavisideTheta[-2 π + t] + HeavisideTheta[-π + t]) Sin[t]
```

```
e6 = e5 /. {HeavisideTheta[-2 π + t] → 0, HeavisideTheta[-π + t] → 0}
```

```
Sin[t]
```

Above: The answer agrees with the text for the subinterval $t < \pi$.

```
e7 = e5 /. {HeavisideTheta[-2 π + t] → 0, HeavisideTheta[-π + t] → 1}
```

```
0
```

Above: The answer agrees with the text for the subinterval $\pi < t < 2\pi$.

```
e8 = e5 /. {HeavisideTheta[-2 π + t] → 1, HeavisideTheta[-π + t] → 1}
```

$-\text{Sin}[t]$

Above: The answer agrees with the text for the subinterval $t > 2\pi$.

7. $4 y'' + 24 y' + 37 y = 17 e^{-t} + \delta\left(t - \frac{1}{2}\right)$, y[0] = 1, y'[0] = 1

```
Clear["Global`*"]
```

$$\text{e1} = \text{LaplaceTransform}\left[4 y''[t] + 24 y'[t] + 37 y[t] == 17 e^{-t} + \text{DiracDelta}\left[t - \frac{1}{2}\right], t, s\right]$$

$$37\,\text{LaplaceTransform}[y[t], t, s] + 24\,(s\,\text{LaplaceTransform}[y[t], t, s] - y[0]) + 4\left(s^2\,\text{LaplaceTransform}[y[t], t, s] - s\,y[0] - y'[0]\right) == e^{-s/2} + \frac{17}{1+s}$$

```
e2 = e1 /. {y[0] → 1, y'[0] → 1, LaplaceTransform[y[t], t, s] → bigY}
```

$$37\,\text{bigY} + 24\,(-1 + \text{bigY}\,s) + 4\left(-1 - s + \text{bigY}\,s^2\right) == e^{-s/2} + \frac{17}{1+s}$$

```
e3 = Solve[e2, bigY]
```

$$\left\{\left\{\text{bigY} \to \frac{28 + e^{-s/2} + 4 s + \frac{17}{1+s}}{37 + 24 s + 4 s^2}\right\}\right\}$$

```
e4 = e3[[1, 1, 2]]
```

$$\frac{28 + e^{-s/2} + 4 s + \frac{17}{1+s}}{37 + 24 s + 4 s^2}$$

```
e5 = InverseLaplaceTransform[e4, s, t]
```

$$\frac{1}{4} e^{-\frac{i}{4} - \left(3 + \frac{i}{2}\right) t}\left(4 e^{\frac{i}{4}}\left(2 i - 2 i e^{i t} + e^{\left(2 + \frac{i}{2}\right) t}\right) + i e^{3/2}\left(e^{\frac{i}{2}} - e^{i t}\right)\text{HeavisideTheta}\left[-\frac{1}{2} + t\right]\right)$$

```
e6 = FullSimplify[e5]
```

$$\frac{1}{2} e^{-3 t}\left(2 e^{2 t} - e^{3/2}\,\text{HeavisideTheta}\left[-\frac{1}{2} + t\right]\text{Sin}\left[\frac{1}{4}(1 - 2 t)\right] + 8\,\text{Sin}\left[\frac{t}{2}\right]\right)$$

```
e7 = Expand[e6]
```

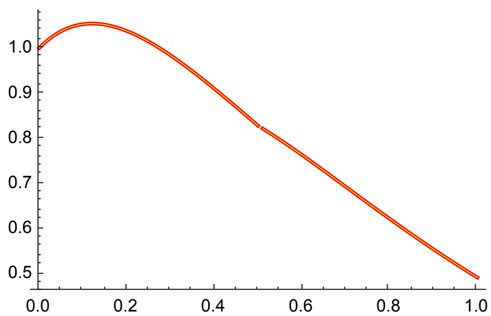
$$e^{-t} - \frac{1}{2} e^{\frac{3}{2} - 3 t}\,\text{HeavisideTheta}\left[-\frac{1}{2} + t\right]\text{Sin}\left[\frac{1}{4}(1 - 2 t)\right] + 4 e^{-3 t}\,\text{Sin}\left[\frac{t}{2}\right]$$

```
PossibleZeroQ[(e^-t - 1/2 e^(3/2-3 t) Sin[1/4 (1 - 2 t)] + 4 e^-3 t Sin[t/2]) -
  (e^-t + 4 e^-3 t Sin[1/2 t] + 1/2 (e^-3 (t-1/2) Sin[1/2 t - 1/4]))]
True
```

Above: By comparison of plots in section 6.3 I decided that **HeavisideTheta** is equivalent to **UnitStep**, the function the text prefers. Granted that, the PZQ above confirms that the green cell is equivalent to the text answer.

```
plot1 = Plot[e7, {t, 0, 1}, PlotRange → Automatic,
   PlotStyle → {Yellow, Thickness[0.002]}, ImageSize → 250];
plot2 = Plot[e^-t + 4 e^-3 t Sin[t/2] + 1/2 UnitStep[t - 1/2] e^-3 (t-1/2) Sin[t/2 - 1/4],
   {t, 0, 1}, PlotRange → Automatic,
   PlotStyle → {Red, Thickness[0.008]}, ImageSize → 250];
```

```
Show[plot2, plot1]
```



Note the interesting little gap which exists in both plots above.

9. $y'' + 4y' + 5y = (1 - u(t-10))\, e^t - e^{10}\,\delta(t-10)$,
$y[0] = 0$, $y'[0] = 1$

```
Clear["Global`*"]
```

```
e1 = LaplaceTransform[y''[t] + 4 y'[t] + 5 y[t] ==
   (1 - UnitStep[t - 10]) e^t - e^10 DiracDelta[t - 10], t, s]
5 LaplaceTransform[y[t], t, s] + s^2 LaplaceTransform[y[t], t, s] +
  4 (s LaplaceTransform[y[t], t, s] - y[0]) - s y[0] - y′[0] ==
 -e^(10-10 s) + 1/(-1 + s) - e^(-10 (-1+s))/(-1 + s)
```

Above: Here I get thrown for a loop. The $-1+s$ denominators are due to the exponential e^t; there doesn't seem to be any extra s added, so none needs to be removed. There is only one s that needs removing.

```
e2 = e1 /. {y[0] → 0, y'[0] → 1, LaplaceTransform[y[t], t, s] → bigY}
```

$$-1 + 5\,\text{bigY} + 4\,\text{bigY}\,s + \text{bigY}\,s^2 == -e^{10-10\,s} + \frac{1}{-1+s} - \frac{e^{-10\,(-1+s)}}{-1+s}$$

```
e3 = Solve[e2, bigY]
```

$$\left\{\left\{\text{bigY} \to \frac{e^{-10\,s}\left(-e^{10} + e^{10\,s}\right)s}{(-1+s)\left(5+4\,s+s^2\right)}\right\}\right\}$$

```
e4 = e3[[1, 1, 2]]
```

$$\frac{e^{-10\,s}\left(-e^{10} + e^{10\,s}\right)s}{(-1+s)\left(5+4\,s+s^2\right)}$$

```
e5 = InverseLaplaceTransform[e4, s, t]
```

$$\frac{1}{20}\,e^{(-2-i)\,((2+4\,i)+t)}\left((-1-i)\,e^{10\,i}\left((-3-4\,i) + (4+3\,i)\,e^{2\,i\,t} - (1-i)\,e^{(3+i)\,t}\right) + \left((1-7\,i)\,e^{30+20\,i} + (1+7\,i)\,e^{30+2\,i\,t} - 2\,e^{(3+i)\,((1+3\,i)+t)}\right)\text{HeavisideTheta}[-10+t]\right)$$

```
e6 = ComplexExpand[Re[e5]];
```

Above: Marking the first time I have tried this command. It's Mr. Clean for imaginary atomics.

```
e7 = FullSimplify[e6]
```

$$\frac{1}{10}\,e^{-2\,t}\left(e^{3\,t} - \text{Cos}[t] + 7\,\text{Sin}[t] + \left(-e^{3\,t} + e^{30}\,(\text{Cos}[10-t] + 7\,\text{Sin}[10-t])\right)\text{UnitStep}[-10+t]\right)$$

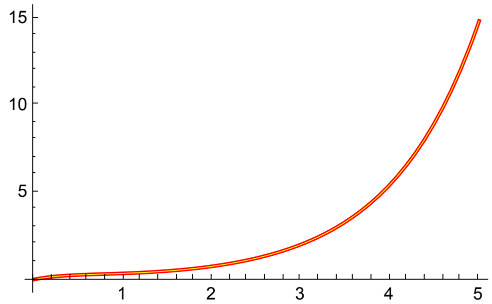
Above: This looks very close to the text answer; even the UnitStep is there. However, there is a taint of suspicion. Note: Below: In entering the text answer I changed 0.1 to $\frac{1}{10}$ (two occurrences).

$$\text{e8} = \frac{1}{10}\left(e^{t} + e^{-2\,t}\,(-\text{Cos}[t] + 7\,\text{Sin}[t])\right) + \frac{1}{10}\,\text{UnitStep}[t-10]\left(-e^{-t} + e^{-2\,t+30}\,(\text{Cos}[t-10] - 7\,\text{Sin}[t-10])\right)$$

$$\frac{1}{10}\left(e^{t} + e^{-2\,t}\,(-\text{Cos}[t] + 7\,\text{Sin}[t])\right) + \frac{1}{10}\left(-e^{-t} + e^{30-2\,t}\,(\text{Cos}[10-t] + 7\,\text{Sin}[10-t])\right)\text{UnitStep}[-10+t]$$

```
plot1 = Plot[e7, {t, 0, 5}, PlotRange → Automatic,
   PlotStyle → {Yellow, Thickness[0.002]}, ImageSize → 250];
plot2 = Plot[e8, {t, 0, 5}, PlotRange → Automatic,
   PlotStyle → {Red, Thickness[0.008]}, ImageSize → 250];
```

```
Show[plot2, plot1]
```



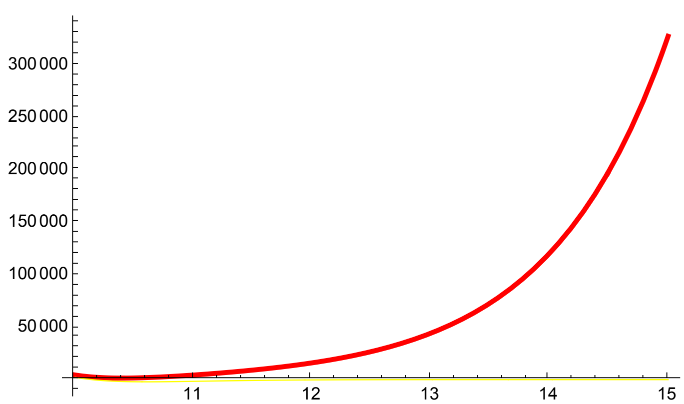
The two plots suggest equality on a limited range.

```
PossibleZeroQ[Chop[(0.1 ⅇ^(-2 t) (ⅇ^(3 t) - Cos[t] + 7 Sin[t] +
        (-ⅇ^(3 t) + ⅇ^30 (Cos[10 - t] + 7 Sin[10 - t])) UnitStep[-10 + t])), 10^-10] -
  Chop[(0.1 (ⅇ^t + ⅇ^(-2 t) (-Cos[t] + 7 Sin[t])) + 0.1 UnitStep[-10 + t]
       (-ⅇ^-t + ⅇ^(-2 t+30) (Cos[t - 10] - 7 Sin[t - 10]))), 10^-10]]
```

```
False
```

```
plot3 = Plot[e7, {t, 10, 15}, PlotRange → Automatic,
   PlotStyle → {Yellow, Thickness[0.002]}, ImageSize → 350];
plot4 = Plot[0.1 (ⅇ^t + ⅇ^(-2 t) (-Cos[t] + 7 Sin[t])) +
    0.1 UnitStep[-10 + t] (-ⅇ^-t + ⅇ^(-2 t+30) (Cos[t - 10] - 7 Sin[t - 10])),
   {t, 10, 15}, PlotRange → Automatic,
   PlotStyle → {Red, Thickness[0.008]}, ImageSize → 350];
```

```
Show[plot4, plot3]
```



The PZQ and the plots above show that the answer I got does not match the answer in the text. I will have to look into this further.

11. $y'' + 5y' + 6y = u(t - 1) + \delta(t - 2)$, $y[0] = 0$, $y'[0] = 1$

```
Clear["Global`*"]
```

```
e1 = LaplaceTransform[
  y''[t] + 5 y'[t] + 6 y[t] == UnitStep[t - 1] + DiracDelta[t - 2], t, s]
```

$$6\,\text{LaplaceTransform}[y[t], t, s] + s^2\,\text{LaplaceTransform}[y[t], t, s] + 5\,(s\,\text{LaplaceTransform}[y[t], t, s] - y[0]) - s\,y[0] - y'[0] == e^{-2s} + \frac{e^{-s}}{s}$$

```
e2 = e1 /. {y[0] → 0, y'[0] → 1, LaplaceTransform[y[t], t, s] → bigY}
```

$$-1 + 6\,\text{bigY} + 5\,\text{bigY}\,s + \text{bigY}\,s^2 == e^{-2s} + \frac{e^{-s}}{s}$$

```
e3 = Solve[e2, bigY]
```

$$\left\{\left\{\text{bigY} \to \frac{e^{-2s}\left(e^{s} + s + e^{2s}\,s\right)}{s\left(6 + 5s + s^2\right)}\right\}\right\}$$

```
e4 = e3[[1, 1, 2]]
```

$$\frac{e^{-2s}\left(e^{s} + s + e^{2s}\,s\right)}{s\left(6 + 5s + s^2\right)}$$

```
e5 = InverseLaplaceTransform[e4, s, t]
```

$$\frac{1}{6}e^{-3t}\left(6\left(-1 + e^{t}\right) + 6\,e^{4}\left(-e^{2} + e^{t}\right)\text{HeavisideTheta}[-2 + t] + \left(e - e^{t}\right)^2\left(2\,e + e^{t}\right)\text{HeavisideTheta}[-1 + t]\right)$$

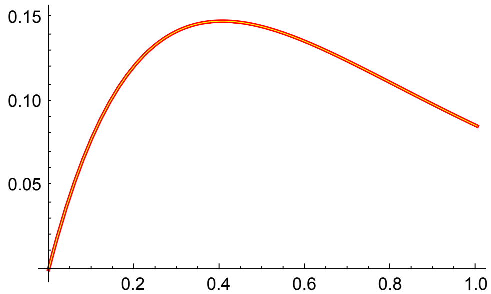
$$e6 = -e^{-3t} + e^{-2t} + \frac{1}{6}\,\text{UnitStep}[t - 1]\left(1 - 3\,e^{-2(t-1)} + 2\,e^{-3(t-1)}\right) + \text{UnitStep}[t - 2]\left(e^{-2(t-2)} - e^{-3(t-2)}\right)$$

$$-e^{-3t} + e^{-2t} + \left(-e^{-3(-2+t)} + e^{-2(-2+t)}\right)\text{UnitStep}[-2 + t] + \frac{1}{6}\left(1 + 2\,e^{-3(-1+t)} - 3\,e^{-2(-1+t)}\right)\text{UnitStep}[-1 + t]$$

Above: The text answer is entered.

```
plot1 = Plot[e5, {t, 0, 1}, PlotRange → Automatic,
   PlotStyle → {Yellow, Thickness[0.002]}, ImageSize → 250];
plot2 = Plot[e6, {t, 0, 1}, PlotRange → Automatic,
   PlotStyle → {Red, Thickness[0.008]}, ImageSize → 250];
```

```
Show[plot2, plot1]
```



Above: the two plots suggest equality.

```
e7 = e6 /. UnitStep → HeavisideTheta
```

$$-e^{-3t} + e^{-2t} + \left(-e^{-3(-2+t)} + e^{-2(-2+t)}\right) \text{HeavisideTheta}[-2+t] + \frac{1}{6}\left(1 + 2e^{-3(-1+t)} - 3e^{-2(-1+t)}\right) \text{HeavisideTheta}[-1+t]$$

```
FullSimplify[e5 == e7]
True
```

Above: So: If the UnitSteps are exchanged for Heavisides, the answers match.