```
SetOptions[EvaluationNotebook[], StyleHints → {"CodeFont" → "Courier"}]
```

Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

7 - 15 Sturm-Liouville Problems

Find the eigenvalues and eigenfunctions. Verify orthogonality. Start by writing the ODE in the form of numbered line (1), p. 499, using Prob. 6.

7. y”+λy=0, y(0)=0, y(10)=0

```
Clear["Global`*"]
sol = DSolve[{y''[x] + λ y[x] == 0, y[0] == 0, y[10] == 0}, y[x], x]
```

$$\left\{\left\{y[x] \to \begin{cases} C[1]\,\mathrm{Sin}\left[x\sqrt{\lambda}\right] & n \in \mathrm{Integers}\ \&\&\ n \geq 1\ \&\&\ \lambda == \frac{n^2 \pi^2}{100} \\ 0 & \mathrm{True} \end{cases}\right\}\right\}$$

After reformulation and assiduous comparison, I found that the above answer matches the text’s.

$$\mathrm{eigfuns} = \mathrm{Table}\left[y[x]\ /.\ \mathrm{sol}[[1]]\ //.\ \left\{n \geq i,\ \lambda \to \frac{n^2 \pi^2}{100}\right\}\ /.\ \{C[1] \to 1\},\ \{i, 2\}\right]$$

9. y”+λy=0, y(0)=0, y’(L)=0

```
Clear["Global`*"]
eqn = {y''[x] + λ y[x] == 0, y[0] == 0, y'[ell] == 0}
{λ y[x] + y''[x] == 0, y[0] == 0, y'[ell] == 0}
sol = DSolve[{eqn}, y[x], x]
```

$$\left\{\left\{y[x] \to \begin{cases} C[1]\,\mathrm{Sin}\left[x\sqrt{\lambda}\right] & n \in \mathrm{Integers}\ \&\&\ n \geq 1\ \&\&\ \lambda == \frac{\left(-\frac{1}{2}+n\right)^2 \pi^2}{\mathrm{ell}^2}\ \&\&\ \mathrm{ell} > 0 \\ 0 & \mathrm{True} \end{cases}\right\}\right\}$$

Try to verify the solution:

```
FullSimplify[eqn /. sol]
```

$$\left\{\left\{\lambda \left(\begin{cases} C[1]\,\mathrm{Sin}\left[x\sqrt{\lambda}\right] & n \in \mathrm{Integers}\ \&\&\ n \geq 1\ \&\&\ \lambda == \frac{\left(-\frac{1}{2}+n\right)^2 \pi^2}{\mathrm{ell}^2}\ \&\&\ \mathrm{ell} > 0 \\ 0 & \mathrm{True} \end{cases}\right) + y''[x] == 0,\ y[0] == 0,\ y'[\mathrm{ell}] == 0\right\}\right\}$$

Mathematica, in the above cell, recreated the orig eqn and inserted, in the place of y[x], its solution. But for my part, I expect *Mathematica* to print {true} if it intends to signal verification of the correctness of the solution.

$D\left[C[1]\ \mathrm{Sin}\left[x\sqrt{\lambda}\right], \{x, 2\}\right]$

$-\lambda\, C[1]\ \mathrm{Sin}\left[x\sqrt{\lambda}\right]$

$-\lambda\, C[1]\ \mathrm{Sin}\left[x\sqrt{\lambda}\right] + \lambda \left(C[1]\ \mathrm{Sin}\left[x\sqrt{\lambda}\right]\right)$

0

Above cell verifies the solution of y[x].

$\mathrm{try0} = \left(C[1]\ \mathrm{Sin}\left[x\sqrt{\lambda}\right]\right)$

$C[1]\ \mathrm{Sin}\left[x\sqrt{\lambda}\right]$

$\mathrm{try0ver} = \mathrm{try0}\ /.\ x \to 0$

0

Above cell verifies the first boundary point.

$\mathrm{sbp} = D\left[C[1]\ \mathrm{Sin}\left[x\sqrt{\lambda}\right], x\right]$

$\sqrt{\lambda}\ C[1]\ \mathrm{Cos}\left[x\sqrt{\lambda}\right]$

$\mathrm{subs} = \mathrm{sbp}\ /.\ \lambda \to \frac{\left(-\frac{1}{2}+n\right)^2 \pi^2}{L^2}$

$\sqrt{\frac{\left(-\frac{1}{2}+n\right)^2}{L^2}}\ \pi\, C[1]\ \mathrm{Cos}\left[\sqrt{\frac{\left(-\frac{1}{2}+n\right)^2}{L^2}}\ \pi\, x\right]$

$\mathrm{subssol} = \mathrm{FullSimplify}[\mathrm{subs}\ /.\ x \to L,\ \mathrm{Assumptions} \to n \in \mathrm{Integers}]$

0

Above cell verifies the second boundary point. This validates the Mathematica solution to y[x].

There is some doubt as to whether sol above contains equal expressions with the answer in the text, so I evaluate numerically to check. The two λ expressions:

$N\left[\frac{\left(-\frac{1}{2}+n\right)^2 \pi^2}{L^2}\ /.\ \{L \to 1, n \to 1\}\right]$ (*calculated by MMA *)

2.4674

$N\left[\left(\frac{(2\,n+1)\,\pi}{2\,L}\right)^2\ /.\ \{L \to 1, n \to 1\}\right]$ (* from the text's answer appendix *)

22.2066

Nope, not even close to equal. So I'll have to take back my green background on the answer cell above, the answer does not agree with the text. However, since it is possible to have multiple solutions to ODEs, it is possible that both answers, both eigenvalues, are correct.

11. $(y'/x)' + (\lambda+1)y/x^3=0$, $y(1)=0$, $y(e^\pi)=0$. (Set $x=e^t$.)

```
Clear["Global`*"]
```

Mathematica will not do anything with this until the first term is untangled through use of the quotient rule for differentiation.

```
eqn = {y''[x]/x - y'[x]/x^2 + (λ + 1) y[x]/x^3 == 0}
y[1] == 0
y[e^π] == 0
```

$$\left\{\frac{(1+\lambda)\, y[x]}{x^3} - \frac{y'[x]}{x^2} + \frac{y''[x]}{x} == 0\right\}$$

$y[1] == 0$

$y[e^\pi] == 0$

```
(*eqnnox = eqn/.{x^3→e^(3 t),x^2→e^(2 t),x→e^t}
```

```
(*{e^(-3 t) (1+λ) y[e^t]-e^(-2 t) y'[e^t]+e^(-t) y''[e^t]==0,y[e^0]==0,y[e^π]==0}
```

$$\{e^{-3t}(1+\lambda)\, y[e^t] - e^{-2t} y'[e^t] + e^{-t} y''[e^t] == 0,\ y[1] == 0,\ y[e^\pi] == 0\}$$

```
(*y[t_] =e^t
```

```
(*eqnet={(y[t])^-3 (1+λ)y[t]-y[t]^-2 y'[t]+y[t]^-1 y''[t]==0,y[0]==0,y[π]==0}
```

```
(*sol=DSolve[{eqnet}, Flatten@{y[t]},t]
```

```
(*sol=DSolve[eqnnox, y[e^t],e^t]
```

```
(*DSolve[
 {e^(-3 t) (1+λ) y[e^t]-e^(-2 t) y'[e^t]+e^(-t) y''[e^t]==0,y[1]==0,y[e^π]==0},y[e^t],e^t]*)
```

Like problem 13 below, this one has a sol which includes an eigenvalue, but in implicit form.

```
y[t_] = e^t
```

e^t

```
eqne = {y''[t]/t + - y'[t]/t^2 + (λ + 1) y[t]/t^3 == 0}
```

$$\left\{-\frac{e^t}{t^2} + \frac{e^t}{t} + \frac{e^t(1+\lambda)}{t^3} == 0\right\}$$

```
sol2 = DSolve[{eqne}, y[t], t]
```

DSolve::dsfun: e^t cannot be used as a function≫

$$\text{DSolve}\left[\left\{\left\{-\frac{e^t}{t^2} + \frac{e^t}{t} + \frac{e^t(1+\lambda)}{t^3} == 0\right\}\right\},\ e^t,\ t\right]$$

$\{-e^{-4+t}+e^{-2+t}+e^{-6+t}(1+\lambda)==0\}$

$\{-e^{-4+t}+e^{-2+t}+e^{-6+t}(1+\lambda)==0\}$

$$\text{soll} = \text{Solve}\left[x^{1-\frac{\sqrt{\lambda}\sqrt{1+\lambda}}{\sqrt{-1-\lambda}}}\left(-1+x^{\frac{2\sqrt{\lambda}\sqrt{1+\lambda}}{\sqrt{-1-\lambda}}}\right)C[1]-e^{\pi\sqrt{1+\lambda}\left(-\frac{\sqrt{\lambda}}{\sqrt{-1-\lambda}}+\frac{1}{\sqrt{1+\lambda}}\right)}+e^{\pi\sqrt{1+\lambda}\left(\frac{\sqrt{\lambda}}{\sqrt{-1-\lambda}}+\frac{1}{\sqrt{1+\lambda}}\right)}==0,\{\lambda\}\right]$$

$$\text{Solve}\left[-e^{\pi\sqrt{1+\lambda}\left(-\frac{\sqrt{\lambda}}{\sqrt{-1-\lambda}}+\frac{1}{\sqrt{1+\lambda}}\right)}+e^{\pi\sqrt{1+\lambda}\left(\frac{\sqrt{\lambda}}{\sqrt{-1-\lambda}}+\frac{1}{\sqrt{1+\lambda}}\right)}+x^{1-\frac{\sqrt{\lambda}\sqrt{1+\lambda}}{\sqrt{-1-\lambda}}}\left(-1+x^{\frac{2\sqrt{\lambda}\sqrt{1+\lambda}}{\sqrt{-1-\lambda}}}\right)C[1]==0,\{\lambda\}\right]$$

```
y[t_] = e^t
```
e^t

```
y'[t]
```
e^t

```
y''[t]
```
e^t

```
DSolve[
 {y''[x]/x + - y'[x]/x^2 + (λ + 1) y[x]/x^3 == 0, y[1] == 0, y[e^π] == 0}, y[x], x]
```

MessageTemplate[DSolve, dsfun, e^x cannot be used as a function, 2, 30, 12, 27280790681456479611, Local]

$$\text{DSolve}\left[\left\{-\frac{e^x}{x^2}+\frac{e^x}{x}+\frac{e^x(1+\lambda)}{x^3}==0,\text{False},\text{False}\right\},e^x,x\right]$$

```
tag = y''[x]/x + - y'[x]/x^2 + (λ + 1) y[x]/x^3 == 0
```

$$-\frac{e^x}{x^2}+\frac{e^x}{x}+\frac{e^x(1+\lambda)}{x^3}==0$$

```
tage = tag /. x → e^t
```

$-e^{e^t-2t}+e^{e^t-t}+e^{e^t-3t}(1+\lambda)==0$

$\text{DSolve}\left[\left\{e^{-3t}(1+\lambda)\,y[e^t]-e^{-2t}y'[e^t]+e^{-t}y''[e^t]==0\right\},y[t],t\right]$

MessageTemplate[DSolve, dsfun, e^t cannot be used as a function, 2, 33, 13, 27280790681456479611, Local]

$\text{DSolve}\left[\left\{-e^{e^t-2t}+e^{e^t-t}+e^{e^t-3t}(1+\lambda)==0\right\},e^t,t\right]$

13. $y''+8y'+(\lambda+16)y=0$, $y(0)=0$, $y(\pi)=0$

```
Clear["Global`*"]
eqn = {y''[x] + 8 y'[x] + (λ + 16) y[x] == 0, y[0] == 0, y[π] == 0}
```

$\{(16+\lambda)\, y[x] + 8\, y'[x] + y''[x] == 0,\ y[0] == 0,\ y[\pi] == 0\}$

```
sol = DSolve[eqn, y[x], x]
```

$\left\{\left\{y[x] \to \begin{cases} 2\, e^{-4x}\, C[1]\, \mathrm{Sinh}\left[x\sqrt{-\lambda}\right] & \mathrm{Sinh}\left[\pi\sqrt{-\lambda}\right] == 0 \\ 0 & \mathrm{True} \end{cases}\right\}\right\}$

If there is only one boundary condition Mathematica will not report a eigenvalue. In the case of this problem, it reports one, but in implicit form, and I can't see a way to reduce it.

"Compute the first three terms in the eigenfunction expansion of the function $f(x)$ with respect to the basis provided by a 1D Laplacian with a Dirichlet condition on the interval $\{0, \pi\}$:" This description comes directly from the documentation for DEigensystem. What it looks like it is doing is to first establish a basis for the region, then find Fourier coefficients for three eigenfunctions, then determine those functions by an expansion of the given function. The complexity of the expressions makes them borderline useless, and it is quite a contrast to compare them with the answer in the text.

```
basis = DEigensystem[
   {-Laplacian[u[x], {x}] + u[x], DirichletCondition[u[x] == 0, True]},
   u[x], {x, 0, π}, 3, Method → "Normalize"][[2]]
```

$\left\{\sqrt{\frac{2}{\pi}}\, \mathrm{Sin}[x],\ \sqrt{\frac{2}{\pi}}\, \mathrm{Sin}[2x],\ \sqrt{\frac{2}{\pi}}\, \mathrm{Sin}[3x]\right\}$

```
(*f[x_]:=x^2(π-x)^3*)

f[x_] = 2 e^(-4 x) C[1] Sinh[x Sqrt[-λ]]
```

$2\, e^{-4x}\, C[1]\, \mathrm{Sinh}\left[x\sqrt{-\lambda}\right]$

```
coeffs = Table[Integrate[f[x] basis[[i]], {x, 0, Pi}], {i, 3}]
```

$$\left\{\frac{2\, e^{-4\pi}\sqrt{\frac{2}{\pi}}\, C[1]\left(8\sqrt{-\lambda}\left(e^{4\pi} + \mathrm{Cos}\left[\pi\sqrt{\lambda}\right]\right) - (-17+\lambda)\, \mathrm{Sinh}\left[\pi\sqrt{-\lambda}\right]\right)}{289 + 30\lambda + \lambda^2},\right.$$
$$\frac{4\, e^{-4\pi}\sqrt{\frac{2}{\pi}}\, C[1]\left(8\sqrt{-\lambda}\left(e^{4\pi} - \mathrm{Cos}\left[\pi\sqrt{\lambda}\right]\right) + (-20+\lambda)\, \mathrm{Sinh}\left[\pi\sqrt{-\lambda}\right]\right)}{400 + 24\lambda + \lambda^2},$$
$$\left.\frac{6\, e^{-4\pi}\sqrt{\frac{2}{\pi}}\sqrt{-\lambda}\, C[1]\left(8\, e^{4\pi} + 8\, \mathrm{Cos}\left[\pi\sqrt{\lambda}\right] - \frac{(-25+\lambda)\, \mathrm{Sin}\left[\pi\sqrt{\lambda}\right]}{\sqrt{\lambda}}\right)}{625 + \lambda(14+\lambda)}\right\}$$

```
eigexp[x_] = Sum[coeffs[[i]] basis[[i]], {i, 3}]
```

$$\frac{12\, e^{-4\pi}\sqrt{-\lambda}\, C[1]\, \mathrm{Sin}[3x]\left(8\, e^{4\pi} + 8\,\mathrm{Cos}\left[\pi\sqrt{\lambda}\right] - \frac{(-25+\lambda)\,\mathrm{Sin}\left[\pi\sqrt{\lambda}\right]}{\sqrt{\lambda}}\right)}{\pi\,(625 + \lambda\,(14 + \lambda))} +$$
$$\frac{8\, e^{-4\pi}\, C[1]\,\mathrm{Sin}[2x]\left(8\sqrt{-\lambda}\left(e^{4\pi} - \mathrm{Cos}\left[\pi\sqrt{\lambda}\right]\right) + (-20+\lambda)\,\mathrm{Sinh}\left[\pi\sqrt{-\lambda}\right]\right)}{\pi\left(400 + 24\lambda + \lambda^2\right)} +$$
$$\frac{4\, e^{-4\pi}\, C[1]\,\mathrm{Sin}[x]\left(8\sqrt{-\lambda}\left(e^{4\pi} + \mathrm{Cos}\left[\pi\sqrt{\lambda}\right]\right) - (-17+\lambda)\,\mathrm{Sinh}\left[\pi\sqrt{-\lambda}\right]\right)}{\pi\left(289 + 30\lambda + \lambda^2\right)}$$

The cells below show how to run a DSolve function after dropping boundary values, then employing the **Reduce** command to force the boundary values to reveal the eigenvalues. However, in this case it does not work. However, it did work on the Stack Exchange page where I saw it. (search for eigenfunction).

```
Reduce[y'[0] == 0 && y'[1] == 0 && a ≠ 0 && (C[1] ≠ 0 || C[2] ≠ 0) /.
   DSolve[y''[x] + a^2 y[x] == 0, y, x], a] // FullSimplify
```

```
C[3] ∈ Integers && C[2] == 0 && C[1] ≠ 0 &&
 ((a == 2 π C[3] && a ≠ 0) || π + 2 π C[3] == a)
```

```
DSolve[{y''[x] + a^2 y[x] == 0, y'[1] == 0, y'[0] == 0}, y[x], x] //
 FullSimplify
```

$$\left\{\left\{y[x] \to \begin{cases} C[1]\,\mathrm{Cos}[a\,x] & n \in \mathrm{Integers}\ \&\&\ n \geq 1\ \&\&\ a^2 == n^2\pi^2 \\ 0 & \mathrm{True} \end{cases}\right\}\right\}$$