1 - 9 Further ODEs reducible to Bessel's ODE

Find a general solution in terms of J_ν and Y_ν . Indicate whether you could also use $J_{-\nu}$ instead of Y_ν. Use the indicated substitution.

1. $x^2 y'' + x y' + (x^2 - 16) y = 0$

```
Clear["Global`*"]
```

```
e1 = {x^2 y''[x] + x y'[x] + (x^2 - 16) y[x] == 0}
e2 = DSolve[e1, y, x]
```

$\{(-16 + x^2)\, y[x] + x\, y'[x] + x^2\, y''[x] == 0\}$

```
{{y → Function[{x}, BesselJ[4, x] C[1] + BesselY[4, x] C[2]]}}
```

```
e1 /. e2 // FullSimplify
```

```
{{True}}
```

The above answer matches the text's. I don't know how to check for equivalence with BesselJ(-ν). I believe that **FullSimplify** is needed to check **DSolve** in this case because Bessels are special functions.

3. $9 x^2 y'' + 9 x y' + (36 x^4 - 16) y = 0 \quad (x^2 = z)$

```
Clear["Global`*"]
```

```
e1 = {9 x^2 y''[x] + 9 x y'[x] + (36 x^4 - 16) y[x] == 0}
e2 = DSolve[e1, y[x], x, Assumptions → x^2 → z]
```

$\{(-16 + 36 x^4)\, y[x] + 9 x\, y'[x] + 9 x^2\, y''[x] == 0\}$

$\{\{y[x] \to \text{BesselJ}[-\frac{2}{3}, x^2]\, C[1]\, \text{Gamma}[\frac{1}{3}] + \text{BesselJ}[\frac{2}{3}, x^2]\, C[2]\, \text{Gamma}[\frac{5}{3}]\}\}$

$\text{PossibleZeroQ}[\text{BesselJ}[-\frac{2}{3}, x^2]\, C[1]\, \text{Gamma}[\frac{1}{3}] - \text{BesselY}[\frac{2}{3}, x^2]\, C[1]\, \text{Gamma}[\frac{1}{3}]]$

```
False
```

It appears that the yellow answer above does not agree with the text answer.

5. $4 x y'' + 4 y' + y = 0 \quad (\sqrt{x} = z)$

```
Clear["Global`*"]
```

```
e1 = {4 x y''[x] + 4 y'[x] + y[x] == 0}
e2 = DSolve[e1, y[x], x, Assumptions → √x → z]
{y[x] + 4 y′[x] + 4 x y″[x] == 0}
```

```
{{y[x] → BesselJ[0, √x] C[1] + 2 BesselY[0, √x] C[2]}}
```

The answer above agrees with the text answer, as I interpret it. It appears that C[2] and the 2 factor in the second term need to be combined.

7. $y'' + k^2 x^2 y = 0 \quad \left(y = u\sqrt{x},\ \frac{1}{2} k x^2 = z\right)$

```
Clear["Global`*"]
```

```
e1 = {y''[x] + k^2 x^2 y[x] == 0}
e2 = DSolve[e1, y[x], x, Assumptions → {y[x] → u √x, k x^2 → z}]
{k^2 x^2 y[x] + y″[x] == 0}
```

$$\left\{\left\{y[x] \to C[1]\ \text{ParabolicCylinderD}\left[-\frac{1}{2},\ (-1)^{1/4}\sqrt{2}\sqrt{k}\ x\right] + C[2]\ \text{ParabolicCylinderD}\left[-\frac{1}{2},\ (-1)^{3/4}\sqrt{2}\sqrt{k}\ x\right]\right\}\right\}$$

$$\text{PossibleZeroQ}\left[\text{ParabolicCylinderD}\left[-\frac{1}{2},\ (-1)^{1/4}\sqrt{2}\sqrt{k}\ x\right] + \text{ParabolicCylinderD}\left[-\frac{1}{2},\ (-1)^{3/4}\sqrt{2}\sqrt{k}\ x\right] - \sqrt{x}\ \text{BesselJ}\left[\frac{1}{4},\ \frac{1}{2} k x^2\right] - \text{BesselY}\left[\frac{1}{4},\ \frac{1}{2} k x^2\right]\right]$$

```
False
```

It appears that Mathematica's answer does not equal that of the text.

9. $x y'' - 5 y' + x y = 0 \quad \left(y = x^3 u\right)$

```
Clear["Global`*"]
```

```
e1 = {x y''[x] - 5 y'[x] + x y[x] == 0}
e2 = DSolve[e1, y, x, Assumptions → y[x] → x^3 u]
{x y[x] - 5 y′[x] + x y″[x] == 0}
```

```
{{y → Function[{x}, x^3 BesselJ[3, x] C[1] + x^3 BesselY[3, x] C[2]]}}
```

```
e1 /. e2 // FullSimplify
{{True}}
```

The above answer agrees with the text’s.

11 - 15 Hankel and modified Bessel functions

11. Hankel functions. Show that the Hankel functions (10) form a basis of solutions of Bessel’s equation for any ν.

```
Clear["Global`*"]
```

$$H_\nu^{(1)}(x) = J_\nu(x) + i Y_\nu(x)$$
$$H_\nu^{(2)}(x) = J_\nu(x) - i Y_\nu(x)$$

```
e1 = c1 (jν + i yν) + c2 (jν - i yν) == 0
c2 (jν - i yν) + c1 (jν + i yν) == 0
```

Above: inserted definitions. It is necessary to change the symbols, I suppose Mathematica recognzied the traditional forms of the Bessels.

```
e2 = Expand[e1]
c1 jν + c2 jν + i c1 yν - i c2 yν == 0

e3 = Collect[e2, {jν, yν}]
(c1 + c2) jν + (i c1 - i c2) yν == 0

e4 = e3 /. (i c1 - i c2) → i (c1 - c2)
(c1 + c2) jν + i (c1 - c2) yν == 0
```

jν and **yν** are known to be linearly independent. (Multiplying one of them by i will not change their linear independence.) That means that the above equation can only be true if (c1 + c2) and (c1 – c2) are both zero.

```
Solve[(c1 + c2) == 0 && (c1 - c2) == 0, {c1, c2}]
{{c1 → 0, c2 → 0}}
```

The above tells me that the two expressions, which were definitions of the Hankel functions, are linearly independent.