1 - 7 General solution

Solve the following ODEs, showing the details of your work.

1. $y''' + 3y' + y = e^x - x - 1$

```
Clear["Global`*"]
```

```
dapple = y'''[x] + 3 y''[x] + 3 y'[x] + y[x] == e^x - x - 1
apple = DSolve[dapple, y[x], x]
```

$$y[x] + 3y'[x] + 3y''[x] + y^{(3)}[x] == -1 + e^x - x$$

$$\left\{\left\{y[x] \to \frac{1}{8}(16 + e^x - 8x) + e^{-x}C[1] + e^{-x}xC[2] + e^{-x}x^2C[3]\right\}\right\}$$

```
Collect[apple, e^-x]
```

$$\left\{\left\{y[x] \to 2 + \frac{e^x}{8} - x + e^{-x}\left(C[1] + xC[2] + x^2C[3]\right)\right\}\right\}$$

1. Above: The expression matches the answer in the text.

3. $\left(D^4 + 10D^2 + 9I\right)y = 6.5\,\text{Sinh}[2x]$

```
Clear["Global`*"]
```

```
prank = y''''[x] + 10 y''[x] + 9 y[x] == 6.5 Sinh[2 x]
dank = DSolve[prank, y[x], x]
```

$$9y[x] + 10y''[x] + y^{(4)}[x] == 6.5\,\text{Sinh}[2x]$$

$$\left\{\left\{y[x] \to 1.C[3]\,\text{Cos}[1.x] + 1.C[1]\,\text{Cos}[3.x] + 1.C[4]\,\text{Sin}[1.x] + 1.C[2]\,\text{Sin}[3.x] + 0.1625\left(0. + 1.\,\text{Cos}[1.x]^2\,\text{Sinh}[2.x] - 0.384615\,\text{Cos}[3.x]^2\,\text{Sinh}[2.x] + 1.\,\text{Sin}[1.x]^2\,\text{Sinh}[2.x] - \left(0.384615 + 2.31296 \times 10^{-17}\,i\right)\text{Sin}[3.x]^2\,\text{Sinh}[2.x]\right)\right\}\right\}$$

```
bank = Chop[dank, 10^-16]
```

$$\left\{\left\{y[x] \to 1.C[3]\,\text{Cos}[1.x] + 1.C[1]\,\text{Cos}[3.x] + 1.C[4]\,\text{Sin}[1.x] + 1.C[2]\,\text{Sin}[3.x] + 0.1625\left(1.\,\text{Cos}[1.x]^2\,\text{Sinh}[2.x] - 0.384615\,\text{Cos}[3.x]^2\,\text{Sinh}[2.x] + 1.\,\text{Sin}[1.x]^2\,\text{Sinh}[2.x] - 0.384615\,\text{Sin}[3.x]^2\,\text{Sinh}[2.x]\right)\right\}\right\}$$

```
sank = Simplify[bank]
```

$$\{\{y[x] \to 1.C[3]\,\text{Cos}[(1. + 0.i)x] + 1.C[1]\,\text{Cos}[(3. + 0.i)x] + 1.C[4]\,\text{Sin}[(1. + 0.i)x] + 1.C[2]\,\text{Sin}[(3. + 0.i)x] + 0.1\,\text{Sinh}[2x]\}\}$$

```
Chop[sank, 10^-15]
```

```
{{y[x] → 1. C[3] Cos[1. x] + 1. C[1] Cos[3. x] +
     1. C[4] Sin[1. x] + 1. C[2] Sin[3. x] + 0.1 Sinh[2. x]}}
```

1. Above: The expression matches the text's answer.

5. $\left(x^3 D^3 + x^2 D^2 - 2 x D + 2 I\right) y = x^{-2}$

```
Clear["Global`*"]
```

```
plow = x^3 y'''[x] + x^2 y''[x] - 2 x y'[x] + 2 y[x] == x^-2
cow = DSolve[plow, y[x], x]
```

$$2\,y[x] - 2\,x\,y'[x] + x^2\,y''[x] + x^3\,y^{(3)}[x] == \frac{1}{x^2}$$

$$\left\{\left\{y[x] \to -\frac{1}{12\,x^2} + \frac{C[1]}{x} + x\,C[2] + x^2\,C[3]\right\}\right\}$$

1. Above: The answer matches the text's.

7. $\left(D^3 - 9 D^2 + 27 D - 27 I\right) y = 27 \operatorname{Sin}[3 x]$

```
boat = y'''[x] - 9 y''[x] + 27 y'[x] - 27 y[x] == 27 Sin[3 x]
coat = DSolve[boat, y[x], x]
```

$$-27\,y[x] + 27\,y'[x] - 9\,y''[x] + y^{(3)}[x] == 27\,\text{Sin}[3\,x]$$

$$\left\{\left\{y[x] \to e^{3x} C[1] + e^{3x} x\,C[2] + e^{3x} x^2\,C[3] + \frac{1}{4}\left(-\text{Cos}[3\,x] + \text{Sin}[3\,x]\right)\right\}\right\}$$

```
goat = Collect[coat, e^3x]
```

$$\left\{\left\{y[x] \to e^{3x}\left(C[1] + x\,C[2] + x^2\,C[3]\right) - \frac{1}{4}\text{Cos}[3\,x] + \frac{1}{4}\text{Sin}[3\,x]\right\}\right\}$$

1. Above: The answer matches the text's.

8 - 13 Initial value problem
Solve the given IVP.

9. $y^{iv} + 5\,y'' + 4\,y = 90 \operatorname{Sin}[x],\ y[0] = 1,$
$y'[0] = 2,\ y''[0] = -1,\ y'''[0] = -32$

```
Clear["Global`*"]
```

```
sing = {y''''[x] + 5 y''[x] + 4 y[x] == 90 Sin[4 x],
   y[0] == 1, y'[0] == 2, y''[0] == -1, y'''[0] == -32}
ring = DSolve[sing, y[x], x]
```

$$\left\{4\, y[x] + 5\, y''[x] + y^{(4)}[x] == 90\, \text{Sin}[4\, x],\ y[0] == 1,\ y'[0] == 2,\ y''[0] == -1,\ y^{(3)}[0] == -32\right\}$$

$$\left\{\left\{y[x] \to \frac{1}{4}\left(4\, \text{Cos}[x] + 80\, \text{Cos}[x]^3\, \text{Sin}[x] - 40\, \text{Cos}[3\, x]\, \text{Sin}[x] - 12\, \text{Cos}[5\, x]\, \text{Sin}[x] - 80\, \text{Cos}[x]\, \text{Sin}[x]^3 + 15\, \text{Cos}[2\, x]\, \text{Sin}[2\, x] + 20\, \text{Cos}[2\, x]^3\, \text{Sin}[2\, x] - 40\, \text{Cos}[x]\, \text{Sin}[3\, x] + 12\, \text{Cos}[x]\, \text{Sin}[5\, x] - 5\, \text{Cos}[2\, x]\, \text{Sin}[6\, x]\right)\right\}\right\}$$

Below I do some hammering to try to get the Mathematica solution into the same form as the text answer.

```
thing = Simplify[ring]
```

$$\{\{y[x] \to \text{Cos}[x]\ (1 - \text{Sin}[x] + \text{Sin}[3\, x])\}\}$$

1. Below: To see what I need to make equal to $\frac{1}{2}\text{Sin}[4\,x]$.

```
TrigExpand[-Cos[x] Sin[x] + Cos[x] Sin[3 x]]
```

$$2\, \text{Cos}[x]^3\, \text{Sin}[x] - 2\, \text{Cos}[x]\, \text{Sin}[x]^3$$

```
bling = thing /.
   (Cos[x] (1 - Sin[x] + Sin[3 x])) → (Cos[x] - Cos[x] Sin[x] + Cos[x] Sin[3 x])
```

$$\{\{y[x] \to \text{Cos}[x] - \text{Cos}[x]\, \text{Sin}[x] + \text{Cos}[x]\, \text{Sin}[3\, x]\}\}$$

---

2. Below: Put together some indents to use.

```
TrigExpand[Sin[2 x]]
```

$$2\, \text{Cos}[x]\, \text{Sin}[x]$$

```
TrigExpand[Sin[3 x]]
```

$$3\, \text{Cos}[x]^2\, \text{Sin}[x] - \text{Sin}[x]^3$$

```
TrigExpand[Cos[2 x]]
```

$$\text{Cos}[x]^2 - \text{Sin}[x]^2$$

3. Therefore $\text{Sin}[4\,x] == 2\,\text{Cos}[2\,x]\,\text{Sin}[2\,x] == 2\left(\left(\text{Cos}[x]^2 - \text{Sin}[x]^2\right)(2\,\text{Cos}[x]\,\text{Sin}[x])\right)$

```
sling = bling /. (Sin[3 x]) → (3 Cos[x]^2 Sin[x] - Sin[x]^3)
```

$$\left\{\left\{y[x] \to \text{Cos}[x] - \text{Cos}[x]\, \text{Sin}[x] + \text{Cos}[x]\ \left(3\, \text{Cos}[x]^2\, \text{Sin}[x] - \text{Sin}[x]^3\right)\right\}\right\}$$

```
string = sling /. (Cos[x] (3 Cos[x]^2 Sin[x] - Sin[x]^3)) ->
    (Cos[x] Sin[x] (3 Cos[x]^2 - Sin[x]^2))
```

$$\{\{y[x] \to \text{Cos}[x] - \text{Cos}[x]\ \text{Sin}[x] + \text{Cos}[x]\ \text{Sin}[x]\ \left(3\ \text{Cos}[x]^2 - \text{Sin}[x]^2\right)\}\}$$

```
zing = string /. (3 Cos[x]^2 - Sin[x]^2) -> (2 Cos[x]^2 + Cos[2 x])
```

$$\{\{y[x] \to \text{Cos}[x] - \text{Cos}[x]\ \text{Sin}[x] + \text{Cos}[x]\ \left(2\ \text{Cos}[x]^2 + \text{Cos}[2\,x]\right)\ \text{Sin}[x]\}\}$$

```
fling = zing /.
   (Cos[x] (2 Cos[x]^2 + Cos[2 x]) Sin[x]) -> ((1/2) Sin[2 x] (Cos[2 x] + 2 Cos[x]^2))
```

$$\{\{y[x] \to \text{Cos}[x] - \text{Cos}[x]\ \text{Sin}[x] + \frac{1}{2}\left(2\ \text{Cos}[x]^2 + \text{Cos}[2\,x]\right)\ \text{Sin}[2\,x]\}\}$$

```
ping = fling /.
   ((1/2) (2 Cos[x]^2 + Cos[2 x]) Sin[2 x]) -> ((1/2) (2 Cos[x]^2 Sin[2 x] + (1/2) Sin[4 x]))
```

$$\{\{y[x] \to \text{Cos}[x] - \text{Cos}[x]\ \text{Sin}[x] + \frac{1}{2}\left(2\ \text{Cos}[x]^2\ \text{Sin}[2\,x] + \frac{1}{2}\ \text{Sin}[4\,x]\right)\}\}$$

4. Above: This is looking worse and worse. Time to swing for the fence.

```
p1 = Cos[x]
```

```
Cos[x]
```

```
p2 = Simplify[-Cos[x] Sin[x] + (1/2) (2 Cos[x]^2 Sin[2 x] + (1/2) Sin[4 x])]
```

$$\frac{1}{2}\ \text{Sin}[4\,x]$$

5. Above: A miraculous hit.

```
out = p1 + p2
```

$$\text{Cos}[x] + \frac{1}{2}\ \text{Sin}[4\,x]$$

6. Above: The answer does match the text answer.

11. $\left(D^3 - 2\,D^2 - 3\,D\right)\,y = 74\,e^{-3\,x}\,\text{Sin}[x]$,
$y[0] = -1.4,\ y'[0] = 3.2,\ y''[0] = -5.2$

```
Clear["Global`*"]
```

```
alt = {y'''[x] - 2 y''[x] - 3 y'[x] == 74 e^(-3 x) Sin[x],
   y[0] == -1.4, y'[0] == 3.2, y''[0] == -5.2}
kalt = DSolve[alt, y[x], x]
```

$$\left\{-3\,y'[x]-2\,y''[x]+y^{(3)}[x]==74\,e^{-3\,x}\,\text{Sin}[x],\ y[0]==-1.4,\ y'[0]==3.2,\ y''[0]==-5.2\right\}$$

$$\left\{\left\{y[x]\to -\frac{1}{5}e^{-3\,x}\,(7\,\text{Cos}[x]+5\,\text{Sin}[x])\right\}\right\}$$

```
salt = kalt /. (-1/5 e^(-3 x) (7 Cos[x] + 5 Sin[x])) -> (e^(-3 x) (-7/5 Cos[x] - 5/5 Sin[x]))
```

$$\left\{\left\{y[x]\to e^{-3\,x}\left(-\frac{7\,\text{Cos}[x]}{5}-\text{Sin}[x]\right)\right\}\right\}$$

1. Above: Substitution by hand results in the text's answer.

13. $\left(D^3-4\,D\right)y=10\,\text{Cos}[x]+5\,\text{Sin}[x],\ y[0]=3,\ y'[0]=-2,\ y''[0]=-1$

```
Clear["Global`*"]
rog = {y'''[x] - 4 y'[x] == 10 Cos[x] + 5 Sin[x],
   y[0] == 3, y'[0] == -2, y''[0] == -1}
dog = DSolve[rog, y[x], x]
```

$$\left\{-4\,y'[x]+y^{(3)}[x]==10\,\text{Cos}[x]+5\,\text{Sin}[x],\ y[0]==3,\ y'[0]==-2,\ y''[0]==-1\right\}$$

```
{{y[x] -> 2 + Cos[x] - 2 Sin[x]}}
```

1. Above: The answer matches the text's.