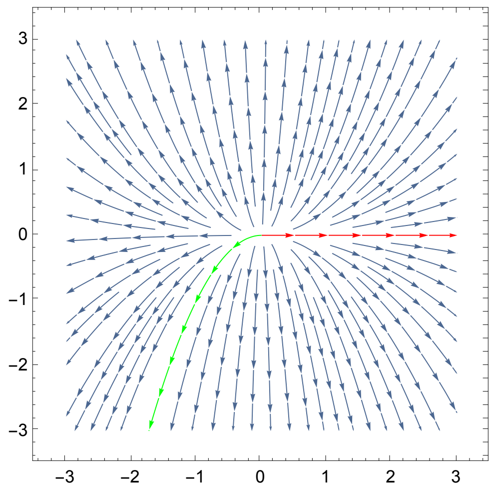
1 - 10 Type and stability of critical point

Determine the type and stability of the critical point. Then find a real general solution and sketch or graph some of the trajectories in the phase plane.

1. $y_1' = y_1$
$y_2' = 2 y_2$

```
StreamPlot[{y1, 2 y2}, {y1, -3, 3}, {y2, -3, 3}, StreamPoints →
  {{{{1, 0}, Red}, {{-1, -1}, Green}, Automatic}}, ImageSize → 250]
```



```
Clear["Global`*"]
```

```
e1 = {y1'[t] == y1[t], y2'[t] == 2 y2[t]}
e2 = DSolve[e1, {y1, y2}, t]
```

```
{y1′[t] == y1[t], y2′[t] == 2 y2[t]}
```

$\{\{y1 \to \text{Function}[\{t\}, e^t C[1]], y2 \to \text{Function}[\{t\}, e^{2t} C[2]]\}\}$

1. Above: the general, real sol’ns.

```
te = e2[[1, 1, 2, 2]]
```

$e^t C[1]$

```
fe = te /. C[1] → c1
```

$c1\, e^t$

```
e3 = Eigensystem[{{1, 0}, {0, 2}}]
```

```
{{2, 1}, {{0, 1}, {1, 0}}}
```

$\lambda_1 = 2$

2

$\lambda_2 = 1$

1

$p = \lambda_1 + \lambda_2$

3

$q = \lambda_1 \lambda_2$

2

$\Delta = (\lambda_1 - \lambda_2)^2$

1

1. Because p>0, the critical point is unstable according to Table 4-2.

```
TableForm[Table[{t, c1, fe}, {t, 4}, {c1, -1, 1}],
 TableHeadings → {{}, {"t", "c1 ", "fe "}}]
```

t	c1	fe
1	1	1
-1	0	1
$-e$	0	e
2	2	2
-1	0	1
$-e^2$	0	e^2
3	3	3
-1	0	1
$-e^3$	0	e^3
4	4	4
-1	0	1
$-e^4$	0	e^4

```
fifo = Table[{t, fe}, {t, 4}, {c1, -1, 1}]
```

$\{\{\{1, -e\}, \{1, 0\}, \{1, e\}\}, \{\{2, -e^2\}, \{2, 0\}, \{2, e^2\}\}, \{\{3, -e^3\}, \{3, 0\}, \{3, e^3\}\}, \{\{4, -e^4\}, \{4, 0\}, \{4, e^4\}\}\}$

```
hiu[c1_, t_] := fe

plot1 = Plot[Evaluate[Table[hiu[c1, t], {c1, -8, 8}]],
   {t, -3, 3}, PlotRange → {-50, 50}, PlotStyle → Thickness[0.003]];
```

3. Above: This is a plot of the first sol'n, with trajectories of various constant values.

```
f[c1_, t_] := c1 ⅇ^t

VectorPlot[{1, f[c1, t]}, {c1, -3, 3}, {t, -1, 1},
  Axes → True, Frame → False, VectorScale → {Tiny, Tiny, None},
  BaseStyle → AbsoluteThickness[0.4`], PlotTheme → None, ImageSize → 250];
```

```
plot2 = VectorPlot[{1, f[t, c1]}, {c1, -3, 3}, {t, -8, 8},
   Axes → True, Frame → False, VectorScale → {Tiny, Tiny, None},
   BaseStyle → AbsoluteThickness[0.4], PlotTheme → None, ImageSize → 350];

Show[plot1, plot2];

fi = e2[[1, 2, 2, 2]]
```

$e^{2t}\,C[2]$

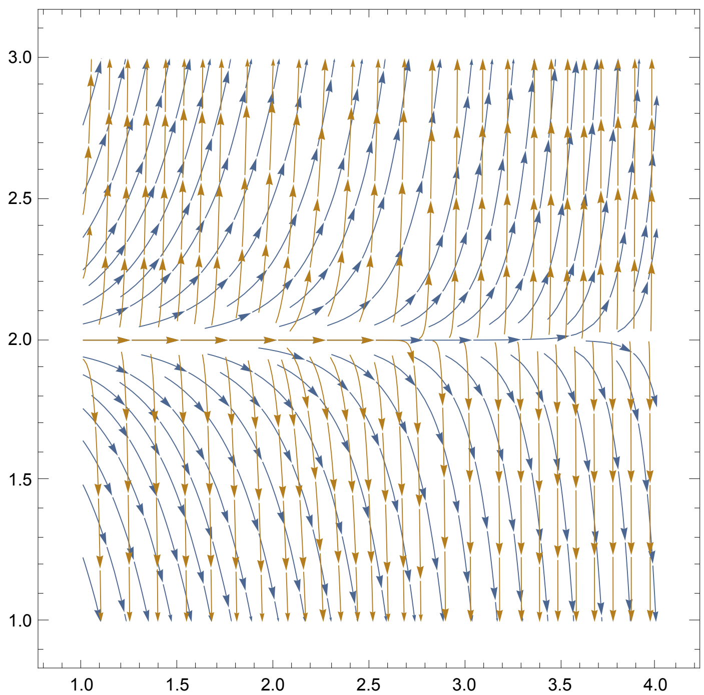
```
fif = fi /. C[2] → c2
```

$c2\, e^{2t}$

```
fifi = Table[{t, fif}, {t, 4}, {c2, -1, 1}]
```

$\{\{\{1, -e^2\}, \{1, 0\}, \{1, e^2\}\}, \{\{2, -e^4\}, \{2, 0\}, \{2, e^4\}\}, \{\{3, -e^6\}, \{3, 0\}, \{3, e^6\}\}, \{\{4, -e^8\}, \{4, 0\}, \{4, e^8\}\}\}$

```
ListStreamPlot[{fifo, fifi}]
```




```
Clear["Global`*"]
```

3. $y_1' = y_2$

$y_2' = -9\, y_1$

```
e1 = {y1'[t] == y2[t], y2'[t] == -9 y1[t]}
e2 = DSolve[e1, {y1, y2}, t]
```

{y1′[t] == y2[t], y2′[t] == -9 y1[t]}

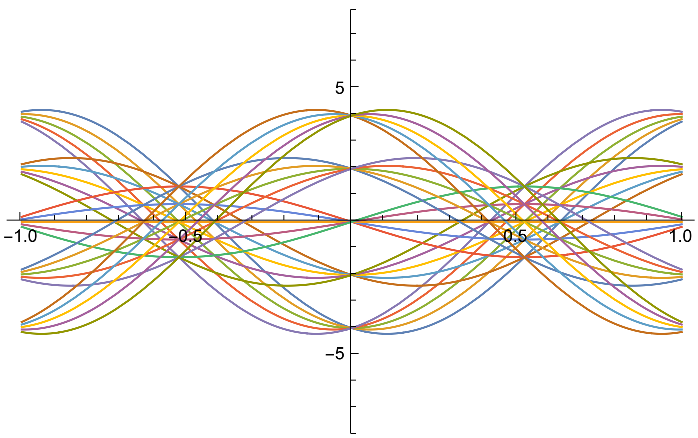
$$\left\{\left\{y1 \to \text{Function}\left[\{t\}, C[1]\,\text{Cos}[3\,t] + \frac{1}{3} C[2]\,\text{Sin}[3\,t]\right],\right.\right.$$
$$\left.\left. y2 \to \text{Function}[\{t\}, C[2]\,\text{Cos}[3\,t] - 3\,C[1]\,\text{Sin}[3\,t]]\right\}\right\}$$

```
e3 = e2[[1, 1, 2, 2]]
```

$$C[1]\,\text{Cos}[3\,t] + \frac{1}{3} C[2]\,\text{Sin}[3\,t]$$

$$\text{hiy}[c1\_, c2\_, t\_] := c1\,\text{Cos}[3\,t] + \frac{1}{3} c2\,\text{Sin}[3\,t]$$

```
plot1 =
 Plot[Evaluate[Table[hiy[c1, c2, t], {c1, -4, 4, 2}, {c2, -4, 4, 2}]],
  {t, -1, 1}, PlotRange → {-8, 8}, PlotStyle → Thickness[0.003]]
```



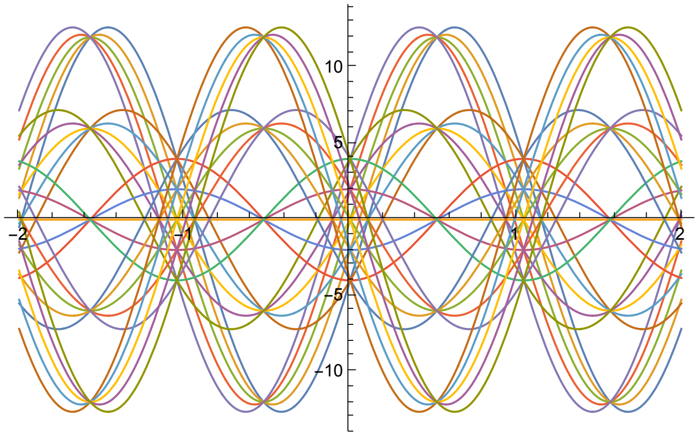
1. Above: Some trajectories of the first sol’n.

```
e4 = e2[[1, 2, 2, 2]]
```

C[2] Cos[3 t] - 3 C[1] Sin[3 t]

```
hiz[c1_, c2_, t_] := c2 Cos[3 t] - 3 c1 Sin[3 t]
```

```
plot1 =
 Plot[Evaluate[Table[hiz[c1, c2, t], {c1, -4, 4, 2}, {c2, -4, 4, 2}]],
  {t, -2, 2}, PlotRange → Automatic, PlotStyle → Thickness[0.003]]
```



2. Above: Some trajectories of the second sol’n.

```
e5 = Eigensystem[{{0, 1}, {-9, 0}}]
{{3 ⅈ, -3 ⅈ}, {{-ⅈ, 3}, {ⅈ, 3}}}

p = 3 ⅈ - 3 ⅈ
0

q = 3 ⅈ (-3 ⅈ)
9

Δ = (3 ⅈ - (-3 ⅈ))^2
-36
```

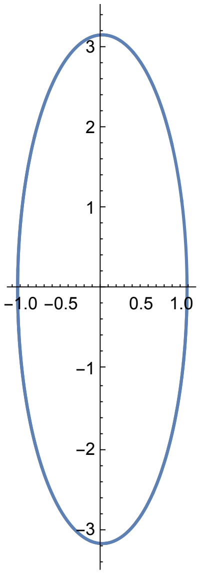
3. The system’s critical point is center. According to Table 4-2, it is stable.

```
e3p = e3 /. {C[1] → 1, C[2] → 1}
```

$$\mathrm{Cos}[3\,t] + \frac{1}{3}\,\mathrm{Sin}[3\,t]$$

```
e4p = e4 /. {C[1] → 1, C[2] → 1}
Cos[3 t] - 3 Sin[3 t]
```

```
ParametricPlot[{e3p, e4p}, {t, -2, 2}, ImageSize → 100]
```



5. $y_1' = -2\,y_1 + 2\,y_2$
$y_2' = -2\,y_1 - 2\,y_2$

```
Clear["Global`*"]

e1 = {y1'[t] == -2 y1[t] + 2 y2[t], y2'[t] == -2 y1[t] - 2 y2[t]}
e2 = DSolve[e1, {y1, y2}, t]
{y1′[t] == -2 y1[t] + 2 y2[t], y2′[t] == -2 y1[t] - 2 y2[t]}
```

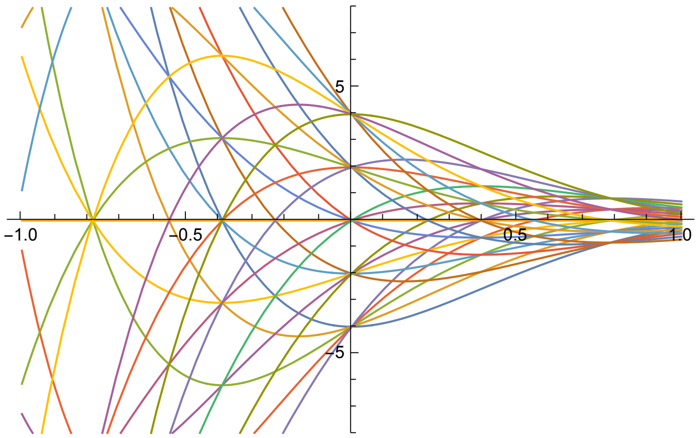
$\{\{y1 \to \text{Function}[\{t\}, e^{-2t} C[1] \text{Cos}[2t] + e^{-2t} C[2] \text{Sin}[2t]],$
$y2 \to \text{Function}[\{t\}, e^{-2t} C[2] \text{Cos}[2t] - e^{-2t} C[1] \text{Sin}[2t]]\}\}$

```
e3 = e2[[1, 1, 2, 2]]
```

$e^{-2t} C[1] \text{Cos}[2t] + e^{-2t} C[2] \text{Sin}[2t]$

hiy[c1_, c2_, t_] := e^{-2t} c1 Cos[2 t] + e^{-2t} c2 Sin[2 t]

```
plot1 =
 Plot[Evaluate[Table[hiy[c1, c2, t], {c1, -4, 4, 2}, {c2, -4, 4, 2}]],
  {t, -1, 1}, PlotRange → {-8, 8}, PlotStyle → Thickness[0.003]]
```

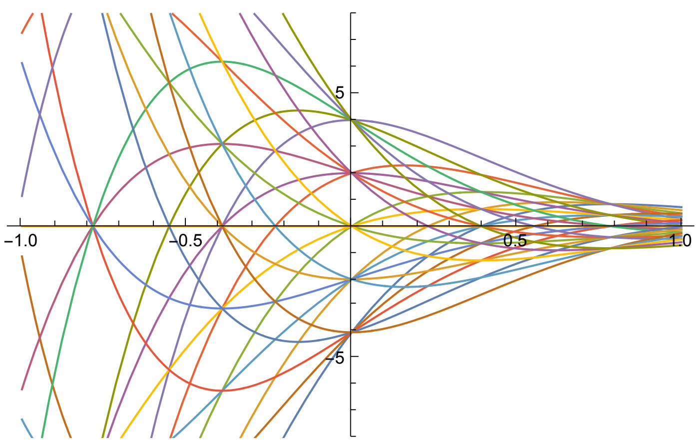



```
e4 = e2[[1, 2, 2, 2]]
```

e^{-2t} C[2] Cos[2 t] - e^{-2t} C[1] Sin[2 t]

hiz[c1_, c2_, t_] := e^{-2t} c2 Cos[2 t] - e^{-2t} c1 Sin[2 t]

```
plot2 =
 Plot[Evaluate[Table[hiz[c1, c2, t], {c1, -4, 4, 2}, {c2, -4, 4, 2}]],
  {t, -1, 1}, PlotRange → {-8, 8}, PlotStyle → Thickness[0.003]]
```




```
e5 = Eigensystem[{{-2, 2}, {-2, -2}}]
{{-2 + 2 i, -2 - 2 i}, {{-i, 1}, {i, 1}}}
```

```
p = -2 + 2 i + (-2 - 2 i)
```

-4

```
q = -2 + 2 i (-2 - 2 i)
```

2 - 4 i

```
Δ = ((-2 + 2 ⅈ) - (-2 - 2 ⅈ))^2
```

-16

According to Table 4-1, the critical point is a spiral point. If the presence of imaginary part does not matter, the point is stable.

7. $y_1{}' = y_1 + 2\,y_2$
$y_2{}' = 2\,y_1 + y_2$

```
Clear["Global`*"]
```

```
e1 = {y1'[t] == y1[t] + 2 y2[t], y2'[t] == 2 y1[t] + y2[t]}
e2 = DSolve[e1, {y1, y2}, t]
```

{y1′[t] == y1[t] + 2 y2[t], y2′[t] == 2 y1[t] + y2[t]}

$$\left\{\left\{y1 \to \text{Function}\left[\{t\}, \frac{1}{2} e^{-t} \left(1 + e^{4 t}\right) C[1] + \frac{1}{2} e^{-t} \left(-1 + e^{4 t}\right) C[2]\right],\right.\right.$$
$$\left.\left. y2 \to \text{Function}\left[\{t\}, \frac{1}{2} e^{-t} \left(-1 + e^{4 t}\right) C[1] + \frac{1}{2} e^{-t} \left(1 + e^{4 t}\right) C[2]\right]\right\}\right\}$$

```
e3 = e2[[1, 1, 2, 2]]
```

$$\frac{1}{2} e^{-t} \left(1 + e^{4 t}\right) C[1] + \frac{1}{2} e^{-t} \left(-1 + e^{4 t}\right) C[2]$$

```
e5 = Expand[e3]
```

$$\frac{1}{2} e^{-t} C[1] + \frac{1}{2} e^{3 t} C[1] - \frac{1}{2} e^{-t} C[2] + \frac{1}{2} e^{3 t} C[2]$$

$$\text{e6} = \text{Collect}\left[\text{e5}, e^{3 t}\right]$$

$$e^{-t} \left(\frac{C[1]}{2} - \frac{C[2]}{2}\right) + e^{3 t} \left(\frac{C[1]}{2} + \frac{C[2]}{2}\right)$$

$$\text{e7} = \text{e6} \,/.\, \left\{\left(\frac{C[1]}{2} - \frac{C[2]}{2}\right) \to \text{c1}, \left(\frac{C[1]}{2} + \frac{C[2]}{2}\right) \to \text{c2}\right\}$$

$$\text{c1}\, e^{-t} + \text{c2}\, e^{3 t}$$

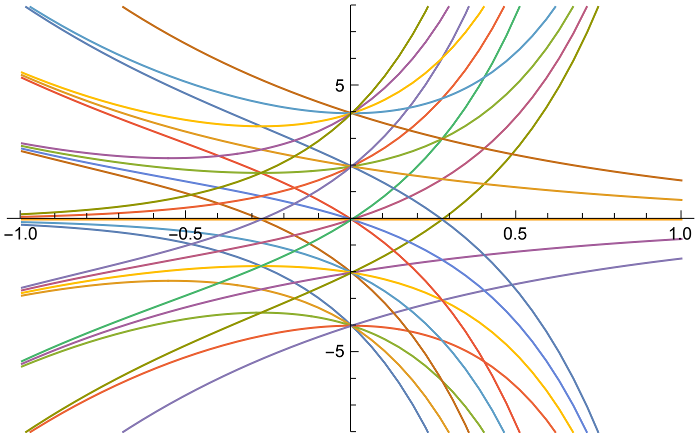
Above: y1, matching the text answer.

$$\text{Solve}\left[\left(\frac{C[1]}{2} - \frac{C[2]}{2}\right) == \text{c1} \;\&\&\; \left(\frac{C[1]}{2} + \frac{C[2]}{2}\right) == \text{c2}, \{\text{c1}, \text{c2}\}\right]$$

$$\left\{\left\{\text{c1} \to \frac{1}{2} (C[1] - C[2]), \text{c2} \to \frac{1}{2} (C[1] + C[2])\right\}\right\}$$

$$\text{hiy}[\text{c1\_}, \text{c2\_}, \text{t\_}] := \frac{1}{2} e^{-t} \left(1 + e^{4 t}\right) \text{c1} + \frac{1}{2} e^{-t} \left(-1 + e^{4 t}\right) \text{c2}$$

```
plot1 =
 Plot[Evaluate[Table[hiy[c1, c2, t], {c1, -4, 4, 2}, {c2, -4, 4, 2}]],
  {t, -1, 1}, PlotRange → {-8, 8}, PlotStyle → Thickness[0.003]]
```




```
e4 = e2[[1, 2, 2, 2]]
```

$$\frac{1}{2} e^{-t} \left(-1 + e^{4t}\right) C[1] + \frac{1}{2} e^{-t} \left(1 + e^{4t}\right) C[2]$$

```
e8 = Expand[e4]
```

$$-\frac{1}{2} e^{-t} C[1] + \frac{1}{2} e^{3t} C[1] + \frac{1}{2} e^{-t} C[2] + \frac{1}{2} e^{3t} C[2]$$

```
e9 = Collect[e8, ⅇ^(3 t)]
```

$$e^{-t} \left(-\frac{C[1]}{2} + \frac{C[2]}{2}\right) + e^{3t} \left(\frac{C[1]}{2} + \frac{C[2]}{2}\right)$$

$$\text{e10} = \text{e9} \;/.\; \left\{\left(-\frac{C[1]}{2} + \frac{C[2]}{2}\right) \to -\text{c1},\; \left(\frac{C[1]}{2} + \frac{C[2]}{2}\right) \to \text{c2}\right\}$$

$$-\text{c1}\, e^{-t} + \text{c2}\, e^{3t}$$

Above: y2, matching the text answer.

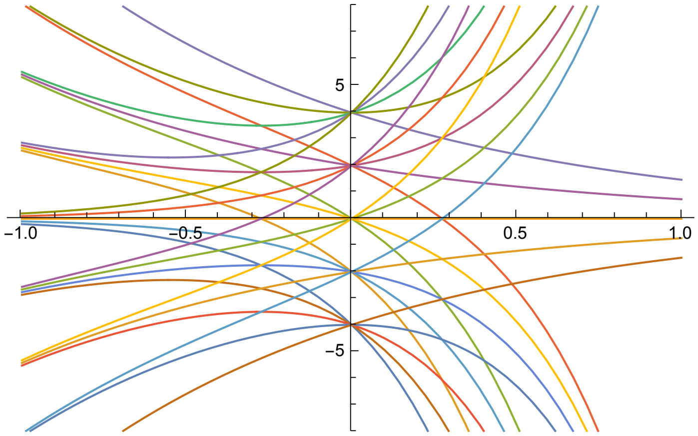
$$\text{Solve}\left[\left(-\frac{C[1]}{2} + \frac{C[2]}{2}\right) == -\text{c1} \,\&\&\, \left(\frac{C[1]}{2} + \frac{C[2]}{2}\right) == \text{c2},\; \{\text{c1}, \text{c2}\}\right]$$

$$\left\{\left\{\text{c1} \to \frac{1}{2} (C[1] - C[2]),\; \text{c2} \to \frac{1}{2} (C[1] + C[2])\right\}\right\}$$

The system of constants used in this problem is consistent with the text's, as demonstrated in pink cells.

$$\text{hiz}[\text{c1\_}, \text{c2\_}, \text{t\_}] := \frac{1}{2} e^{-t} \left(-1 + e^{4t}\right) \text{c1} + \frac{1}{2} e^{-t} \left(1 + e^{4t}\right) \text{c2}$$

```
plot2 =
 Plot[Evaluate[Table[hiz[c1, c2, t], {c1, -4, 4, 2}, {c2, -4, 4, 2}]],
  {t, -1, 1}, PlotRange → {-8, 8}, PlotStyle → Thickness[0.003]]
```




```
Eigensystem[{{1, 2}, {2, 1}}]
{{3, -1}, {{1, 1}, {-1, 1}}}
```

```
p = 3 - 1
```

```
2
```

```
q = 3 (-1)
```

```
-3
```

```
Δ = (3 - (-1))^2
```

```
16
```

According to Table 4-1, the critical point is a saddle point. According to Table 4-2, it is unstable.

9. $y_1' = 4\ y_1 + y_2$
$y_2' = 4\ y_1 + 4\ y_2$

```
Clear["Global`*"]
```

```
e1 = {y1'[t] == 4 y1[t] + y2[t], y2'[t] == 4 y1[t] + 4 y2[t]}
e2 = DSolve[e1, {y1, y2}, t]
```

```
{y1′[t] == 4 y1[t] + y2[t], y2′[t] == 4 y1[t] + 4 y2[t]}
```

$$\left\{\left\{y1 \to \text{Function}\left[\{t\},\ \frac{1}{2} e^{2t}\left(1+e^{4t}\right) C[1] + \frac{1}{4} e^{2t}\left(-1+e^{4t}\right) C[2]\right],\right.\right.$$
$$\left.\left. y2 \to \text{Function}\left[\{t\},\ e^{2t}\left(-1+e^{4t}\right) C[1] + \frac{1}{2} e^{2t}\left(1+e^{4t}\right) C[2]\right]\right\}\right\}$$

```
e3 = e2[[1, 1, 2, 2]]
```

$$\frac{1}{2} e^{2t} \left(1 + e^{4t}\right) C[1] + \frac{1}{4} e^{2t} \left(-1 + e^{4t}\right) C[2]$$

```
e4 = Expand[e3]
```

$$\frac{1}{2} e^{2t} C[1] + \frac{1}{2} e^{6t} C[1] - \frac{1}{4} e^{2t} C[2] + \frac{1}{4} e^{6t} C[2]$$

$$\text{e5 = Collect}\left[\text{e4}, e^{6t}\right]$$

$$e^{2t} \left(\frac{C[1]}{2} - \frac{C[2]}{4}\right) + e^{6t} \left(\frac{C[1]}{2} + \frac{C[2]}{4}\right)$$

$$\text{e6 = e5 /. } \left\{\left(\frac{C[1]}{2} - \frac{C[2]}{4}\right) \to \text{c2}, \left(\frac{C[1]}{2} + \frac{C[2]}{4}\right) \to \text{c1}\right\}$$

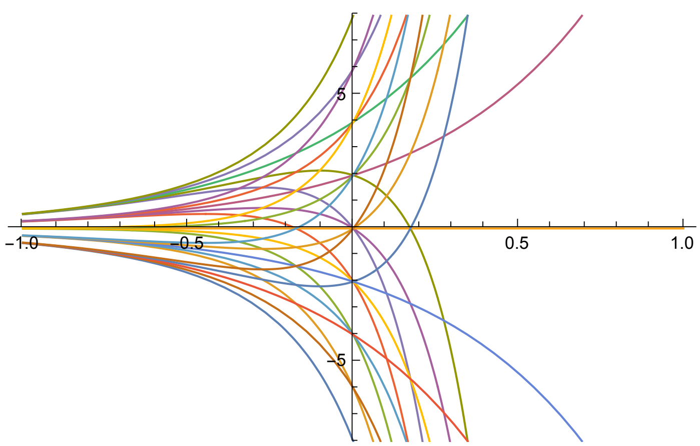
$$\text{c2}\, e^{2t} + \text{c1}\, e^{6t}$$

$$\text{Solve}\left[\left(\frac{C[1]}{2} - \frac{C[2]}{4}\right) == \text{c2} \,\&\&\, \left(\frac{C[1]}{2} + \frac{C[2]}{4}\right) == \text{c1}, \{\text{c1}, \text{c2}\}\right]$$

$$\left\{\left\{\text{c1} \to \frac{1}{4} (2\, C[1] + C[2]), \text{c2} \to \frac{1}{4} (2\, C[1] - C[2])\right\}\right\}$$

$$\text{e7[c1\_, c2\_, t\_] := c2}\, e^{2t} + \text{c1}\, e^{6t}$$

```
plot1 =
 Plot[Evaluate[Table[e7[c1, c2, t], {c1, -4, 4, 2}, {c2, -4, 4, 2}]],
  {t, -1, 1}, PlotRange → {-8, 8}, PlotStyle → Thickness[0.003]]
```




```
e8 = e2[[1, 2, 2, 2]]
```

$$e^{2t} \left(-1 + e^{4t}\right) C[1] + \frac{1}{2} e^{2t} \left(1 + e^{4t}\right) C[2]$$

```
e9 = Expand[e8]
```

$$-e^{2t}\,C[1] + e^{6t}\,C[1] + \frac{1}{2}e^{2t}\,C[2] + \frac{1}{2}e^{6t}\,C[2]$$

```
e10 = Collect[e9, e^(6 t)]
```

$$e^{2t}\left(-C[1] + \frac{C[2]}{2}\right) + e^{6t}\left(C[1] + \frac{C[2]}{2}\right)$$

$$e11 = e10\ /.\ \left\{\left(-C[1] + \frac{C[2]}{2}\right) \to -2\,c2,\ \left(C[1] + \frac{C[2]}{2}\right) \to 2\,c1\right\}$$

$$-2\,c2\,e^{2t} + 2\,c1\,e^{6t}$$

$$\text{Solve}\left[\left(-C[1] + \frac{C[2]}{2}\right) == -2\,c2\ \&\&\ \left(C[1] + \frac{C[2]}{2}\right) == 2\,c1,\ \{c1, c2\}\right]$$

$$\left\{\left\{c1 \to \frac{1}{4}(2\,C[1] + C[2]),\ c2 \to \frac{1}{4}(2\,C[1] - C[2])\right\}\right\}$$

```
Eigensystem[{{4, 1}, {4, 4}}]
{{6, 2}, {{1, 2}, {-1, 2}}}
```

```
p = 6 + 2
```

8

```
q = 6 × 2
```

12

$$\Delta = (6-2)^2$$

16

According to Table 4-1, the critical point is a node. According to Table 4-2, it is unstable.

11 - 18 Trajectories of systems and second-order ODEs. Critical points.

11. Damped oscillations. Solve **y'' + 2 y' + 2 y = 0**. What kind of curves are the trajectories?

17. Perturbation. The system in example 4 in section 4.3 has a center as its critical point. Replace each a_{jk} in example 4, section 4.3, by a_{jk} + b. Find values of b such that you get (a) a saddle point, (b) a stable and attractive node, (c) a stable and attractive spiral, (d) an unstable spiral, (e) an unstable node.