

Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

2 - 8 Minimum Square Error

Find the trigonometric polynomial $F(x)$ of the form (2) for which the square error with respect to the given $f(x)$ on the interval $-\pi < x < \pi$ is minimum. Compare the minimum value for $N=1,2,\dots,5$ (or also for larger values if you have a CAS). (Note: The form (2) referred to is the first mention of minimum square error on p. 495, but a more usable one is form (6) on p. 496.)

$$3. f(x) = |x| \quad (-\pi < x < \pi)$$

The first thing to say is that this problem was worked after problem 5, which has the advantage of a completely worked-out section in the solutions manual. Here again the minimum square error (MSE) will be of high importance. Its expression is: $E^* = \int_{-\pi}^{\pi} f^2 dx - \pi[2a_0^2 + a_n^2]$.

```
Clear["Global`*"]
```

```
f[x_] = Piecewise[{ {Abs[x], -π < x < π} }, x]
{ Abs[x] -π < x < π
  x      True
```

The absolute value function is an even function. It has no natural period, so the period P will be considered to be the finite domain assigned by the problem. Then $L = \pi$. On p. 487 such a series is identified with a Fourier cosine series, and a template for such an equation is given as $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$. As can be seen, the b_n factors have dropped out. In order to assemble an E^* I must first get an integral of f^2 .

```
Integrate[f[x]^2, {x, -π, π}]
2 π^3
3
```

This quantity will be set aside until the final assembly of E^* . Next, the summary box on p. 487 gives the form of the remaining a_n factors which I must seek out. These are:

```
azero = 1/π Integrate[f[x], {x, 0, π}]
π/2
```

and,

```
asuvN = 2/π Integrate[f[x] Cos[n x], {x, 0, π}]
2 (-1 + Cos[n π] + n π Sin[n π])
n^2 π
```

The sine term will drop out, since $\sin n\pi = 0$ for all n .

$$\text{asuvNF} = \text{asuvN} / . \text{Sin}[n \pi] \rightarrow 0$$

$$\frac{2(-1 + \text{Cos}[n \pi])}{n^2 \pi}$$

The above expression for a_n flip-flops depending on whether n is positive or negative.

$$\text{asuvNE} = \text{asuvNF} / . \text{Cos}[n \pi] \rightarrow 1 \text{ (*for multiples of } 2\pi \text{ *)}$$

$$0$$

$$\text{asuvNO} = \text{asuvNF} / . \text{Cos}[n \pi] \rightarrow -1 \text{ (* for multiples of } \pi \text{ *)}$$

$$-\frac{4}{n^2 \pi}$$

At this point it is possible to assemble F .

$$\text{bigF1} = \frac{\pi}{2} - \frac{4}{\pi} \text{Sum}\left[\frac{1}{n^2} \text{Cos}[n x], \{n, 1, 5, 2\}\right]$$

$$\frac{\pi}{2} - \frac{1}{\pi} 4 \left(\text{Cos}[x] + \frac{1}{9} \text{Cos}[3x] + \frac{1}{25} \text{Cos}[5x] \right)$$

The above shows what $F(x)$ would look like with $N=5$. It matches the text answer. However, for calculating E^* , it is best to organize things a little differently. Below are some series which capture values of E^* . Notice that in the below cells, all the $\cos nx$ terms have disappeared, each replaced by the value $= 1$.

$$\text{estar1} = N \left[\frac{2}{3} \pi^3 - \pi \left(2 \left(\frac{\pi}{2} \right)^2 + \text{Sum} \left[\left(\frac{-4}{\pi} \frac{1}{n^2} \right)^2, \{n, 1, 1, 2\} \right] \right) \right]$$

0.0747546

$$\text{estar2} = N \left[\frac{2}{3} \pi^3 - \pi \left(2 \left(\frac{\pi}{2} \right)^2 + \text{Sum} \left[\left(\frac{-4}{\pi} \frac{1}{n^2} \right)^2, \{n, 1, 2, 2\} \right] \right) \right]$$

0.0747546

$$\text{estar3} = N \left[\frac{2}{3} \pi^3 - \pi \left(2 \left(\frac{\pi}{2} \right)^2 + \text{Sum} \left[\left(\frac{-4}{\pi} \frac{1}{n^2} \right)^2, \{n, 1, 3, 2\} \right] \right) \right]$$

0.0118786

$$\text{estar4} = N \left[\frac{2}{3} \pi^3 - \pi \left(2 \left(\frac{\pi}{2} \right)^2 + \text{Sum} \left[\left(\frac{-4}{\pi} \frac{1}{n^2} \right)^2, \{n, 1, 4, 2\} \right] \right) \right]$$

0.0118786

$$\text{estar5} = \mathbf{N} \left[\frac{2}{3} \pi^3 - \pi \left(2 \left(\frac{\pi}{2} \right)^2 + \text{Sum} \left[\left(\frac{-4}{\pi} \frac{1}{n^2} \right)^2, \{n, 1, 5, 2\} \right] \right) \right]$$

0.00372984

The above green cells have answers agreeing with those of the text.

$$5. f(x) = \begin{cases} -1 & \text{if } -\pi < x < 0 \\ 1 & \text{if } 0 < x < \pi \end{cases}$$

This problem is worked out in the s.m., so it will be worked first, before problem 3. The important expression $E^* = \int_{-\pi}^{\pi} f^2 dx - \pi[2a_0^2 + b_n^2]$ represents the MSE. In the below cells it will be chopped up.

```
Clear["Global`*"]
```

```
f[x_] = Piecewise[{{-1, -π < x < 0}, {1, 0 < x < π}}, x]
```

$$\begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \\ x & \text{True} \end{cases}$$

```
Integrate[f[x]^2, {x, -π, 0}] + Integrate[f^2, {x, 0, π}]
```

```
π + f^2 π
```

The above is the best I can do at this time to force Mathematica to admit that the integral = 2π . (True since f^2 always equals 1.) By its definition, $f(x)$ is an odd function. The s.m. points to p. 486 - 487 as authority for making the a_n factors equal to zero, leaving only the b_n factors. Also that the summary box on p. 487 gives the formula for the remaining b_n , which is

$$\text{beeN} = \frac{2}{\pi} \text{Integrate}[f[x] \text{Sin}[n x], \{x, 0, \pi\}]$$

$$\frac{2 (1 - \text{Cos}[n \pi])}{n \pi}$$

```
beeNO = beeN /. Cos[n π] → -1 (* n odd*)
```

$$\frac{4}{n \pi}$$

```
beeNE = beeN /. Cos[n π] → 1 (* n even *)
```

```
0
```

So that only terms with odd n exist. And according to the setup equation for Fourier approximation series, F will look like:

$$F(x) = \frac{4}{\pi} \left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots + \frac{1}{N} \sin Nx \right) \text{ for } N \text{ odd}$$

$$\mathbf{F}[\mathbf{x_}] = \mathbf{Simplify}\left[\frac{4}{\pi} \mathbf{Sum}\left[\frac{1}{\mathbf{nn}} \mathbf{Sin}[\mathbf{nn\ x}], \{\mathbf{nn}, 1, 7, 2\}\right], \right. \\ \left. \mathbf{Assumptions} \rightarrow \mathbf{nn} \in \mathbf{OddQ} \ \&\& \ \mathbf{nn} > 0\right]$$

$$\frac{1}{\pi} 4 \left(\mathbf{Sin}[\mathbf{x}] + \frac{1}{3} \mathbf{Sin}[3 \mathbf{x}] + \frac{1}{5} \mathbf{Sin}[5 \mathbf{x}] + \frac{1}{7} \mathbf{Sin}[7 \mathbf{x}] \right)$$

The 'big F' function is found, above. It agrees with the answer in the text. $E^* = 2\pi$ $-\pi(\sum_{n=1}^N b_n^2)$, by the way, is the expression for E^* .

With a somewhat compressed formula, I came up with a way to calculate E^* , below. Notice that all reference to sine functions is missing, each replaced by value of 1.

$$\mathbf{Estar1} = \mathbf{N}\left[\left(2\pi - \pi \left(\mathbf{Sum}\left[\left(\frac{4}{\pi} \frac{1}{\mathbf{n}}\right)^2, \{\mathbf{n}, 1, 1, 2\}\right]\right)\right)\right]$$

1.19023

$$\mathbf{Estar2} = \mathbf{N}\left[\left(2\pi - \pi \left(\mathbf{Sum}\left[\left(\frac{4}{\pi} \frac{1}{\mathbf{n}}\right)^2, \{\mathbf{n}, 1, 2, 2\}\right]\right)\right)\right]$$

1.19023

$$\mathbf{Estar3} = \mathbf{N}\left[\left(2\pi - \pi \left(\mathbf{Sum}\left[\left(\frac{4}{\pi} \frac{1}{\mathbf{n}}\right)^2, \{\mathbf{n}, 1, 3, 2\}\right]\right)\right)\right]$$

0.624343

$$\mathbf{Estar4} = \mathbf{N}\left[\left(2\pi - \pi \left(\mathbf{Sum}\left[\left(\frac{4}{\pi} \frac{1}{\mathbf{n}}\right)^2, \{\mathbf{n}, 1, 4, 2\}\right]\right)\right)\right]$$

0.624343

$$\mathbf{Estar5} = \mathbf{N}\left[\left(2\pi - \pi \left(\mathbf{Sum}\left[\left(\frac{4}{\pi} \frac{1}{\mathbf{n}}\right)^2, \{\mathbf{n}, 1, 5, 2\}\right]\right)\right)\right]$$

0.420625

$$\mathbf{Estar20} = \mathbf{N}\left[\left(2\pi - \pi \left(\mathbf{Sum}\left[\left(\frac{4}{\pi} \frac{1}{\mathbf{n}}\right)^2, \{\mathbf{n}, 1, 20, 2\}\right]\right)\right)\right]$$

0.127218

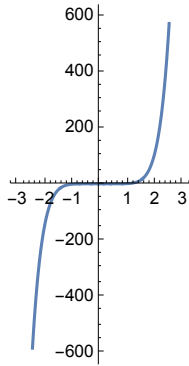
The above answers match those of the text.

$$7. f(x) = x^3 \quad (-\pi < x < \pi)$$

This problem is not covered in the s.m. The function is odd.

```
Clear["Global`*"]
```

```
Plot[x^7, {x, -π, π}, AspectRatio → 2]
```



```
f[x_] = Piecewise[{{x^3, -π < x < π}}, x]
```

```
{ x^3 -π < x < π  
{ x True
```

Piecewise is a pretty good way to handle a restricted domain, even if there is only one piece. At this point, as before, I need to get the value of function-squared-integrated.

```
Integrate[f[x]^2, {x, -π, π}]
```

$$\frac{2\pi^7}{7}$$

With odd function, again I am dealing with the b_n factors, the a_n factors dropping out as before. This means all terms in F will be sine terms, it does not mean either all odd or all even coefficients.

```
beeN = 2/π Integrate[f[x] Sin[n x], {x, 0, π}]
```

$$\frac{1}{n^4 \pi} 2 \left(6 n \pi \cos[n \pi] - n^3 \pi^3 \cos[n \pi] - 6 \sin[n \pi] + 3 n^2 \pi^2 \sin[n \pi] \right)$$

```
beeNO = beeN /. {Cos[n π] → -1, Sin[n π] → 0} (* n odd*)
```

$$\frac{2 \left(-6 n \pi + n^3 \pi^3 \right)}{n^4 \pi}$$

```
beeNO1 = Simplify[beeNO]
```

$$\frac{2 \left(-6 + n^2 \pi^2 \right)}{n^3}$$

$$\text{beeNE} = \text{beeN} /. \{\text{Cos}[n \pi] \rightarrow 1, \text{Sin}[n \pi] \rightarrow 0\} \quad (* \text{ n even } *)$$

$$\frac{2 (6 n \pi - n^3 \pi^3)}{n^4 \pi}$$

$$\text{beeNE1} = \text{Simplify}[\text{beeNE}]$$

$$\frac{12 - 2 n^2 \pi^2}{n^3}$$

The only difference in b_n factors between odd and even terms is that it makes the terms alternate in sign.

So according to the setup equation for Fourier approximation series, F will look like:

$$F(x) = 2 (b_1 \sin x - b_2 \sin 2x + b_3 \sin 3x + \dots + b_n \sin Nx) \quad \text{for } N \text{ odd and even}$$

$$F[x] = 2 \left(\frac{(-6 + 1^2 \pi^2)}{1^3} \text{Sin}[x] - \frac{(-6 + 2^2 \pi^2)}{2^3} \text{Sin}[2x] + \frac{(-6 + 3^2 \pi^2)}{3^3} \text{Sin}[3x] \right);$$

The green cell above matches the text answer (except for ellipsis).

For a change, I will try to calculate the E^* values directly, using the (6) on p. 496 as a template. Recalling that a_0 has already dropped out:

$$\text{Estar1} = N \left[\frac{2 \pi^7}{7} - \pi \left(\text{Sum} \left[2 \left(\frac{(-1)^{n-1} (-6 + n^2 \pi^2)}{n^3} \right) \right]^2, \{n, 1, 1\} \right] \right]$$

674.774

$$\text{Estar2} = N \left[\frac{2 \pi^7}{7} - \pi \left(\text{Sum} \left[2 \left(\frac{(-1)^{n-1} (-6 + n^2 \pi^2)}{n^3} \right) \right]^2, \{n, 1, 2\} \right] \right]$$

454.705

$$\text{Estar3} = N \left[\frac{2 \pi^7}{7} - \pi \left(\text{Sum} \left[2 \left(\frac{(-1)^{n-1} (-6 + n^2 \pi^2)}{n^3} \right) \right]^2, \{n, 1, 3\} \right] \right]$$

336.449

$$\text{Estar4} = N \left[\frac{2 \pi^7}{7} - \pi \left(\text{Sum} \left[2 \left(\frac{(-1)^{n-1} (-6 + n^2 \pi^2)}{n^3} \right) \right]^2, \{n, 1, 4\} \right] \right]$$

265.648

$$\mathbf{Estar5} = \mathbf{N} \left[\frac{2 \pi^7}{7} - \pi \left(\mathbf{Sum} \left[2 \left(\frac{(-1)^{n-1} (-6 + n^2 \pi^2)}{n^3} \right) \right]^2, \{\mathbf{n}, 1, 5\} \right] \right]$$

219.037

The green cells match the text calculation of E^* for $N=1 - 5$. The problem presentation ends with a question: why is E^* so large? My answer is that the alternating sign nature of the series defeats progress on narrowing the gap between $f(x)$ and $F(x)$.