Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

1 - 8 Fourier Cosine Transform

1. Find the cosine transform $\hat{f}_c(w)$ of f(x) $= \begin{cases} 1 & 0 < x < 1 \\ 0 & x > 2 \\ -1 & 1 < x < 2 \end{cases}$

Here the integer 1 is both a limit and a function description. Panels (1a) and (1b) on p. 518 give a format. Cheating a look at the answer, I see that (1a) is what's wanted.

$$\textbf{fsubchat} = \textbf{Simplify}\left[\sqrt{\frac{2}{\pi}} \int_0^1 \textbf{Cos}[\textbf{w x}]\, d\textbf{x} + \sqrt{\frac{2}{\pi}} \int_1^2 -\textbf{Cos}[\textbf{w x}]\, d\textbf{x}\right]$$

$$-\frac{2\sqrt{\frac{2}{\pi}}\,(-1 + \textbf{Cos}[\textbf{w}])\,\textbf{Sin}[\textbf{w}]}{\textbf{w}}$$

$$\textbf{N}\left[-\frac{2\sqrt{\frac{2}{\pi}}\,(-1 + \textbf{Cos}[\textbf{w}])\,\textbf{Sin}[\textbf{w}]}{\textbf{w}} \;/.\; \textbf{w} \to 2,\ 16\right]$$

1.027434891333624

$$\textbf{N}\left[\sqrt{\frac{2}{\pi}}\,\frac{(2\,\textbf{Sin}[\textbf{w}] - \textbf{Sin}[2\,\textbf{w}])}{\textbf{w}} \;/.\; \textbf{w} \to 2,\ 16\right]$$

1.027434891333624

The green cell above is the text answer for $\hat{f}_c(w)$, modified by trig identity sin 2x = 2 sin x cos x, as demonstrated.

3. Find $\hat{f}_c(w)$ for f(x) $= \begin{cases} x & 0 < x < 2 \\ 0 & x > 2 \end{cases}$

Again making use of panel (1a)

$$\textbf{fsubchat3} = \sqrt{\frac{2}{\pi}} \int_0^2 \textbf{x}\,\textbf{Cos}[\textbf{w x}]\, d\textbf{x}$$

$$\frac{1}{\textbf{w}^2} 2\sqrt{\frac{2}{\pi}}\,(2\,\textbf{w}\,\textbf{Cos}[\textbf{w}] - \textbf{Sin}[\textbf{w}])\,\textbf{Sin}[\textbf{w}]$$

$$\textbf{N}\left[\frac{1}{\textbf{w}^2} 2\sqrt{\frac{2}{\pi}}\,(2\,\textbf{w}\,\textbf{Cos}[\textbf{w}] - \textbf{Sin}[\textbf{w}])\,\textbf{Sin}[\textbf{w}] \;/.\; \textbf{w} \to 2,\ 16\right]$$

-0.9336952051225846

$$N\left[\sqrt{\frac{2}{\pi}}\left(\frac{1}{w^2}(\text{Cos}[2\,w] + 2\,w\,(\text{Sin}[2\,w]) - 1)\right) /.\ w \to 2,\ 16\right]$$

-0.9336952051225846

Again $\hat{f}_c(w)$ is a little tangled in a trig identity, it looks like possibly the same one. The green cell is demonstrated to equal the text answer.

5. Find $\hat{f}_c(w)$ for f(x) = $\begin{cases} x^2 & 0 < x < 1 \\ 0 & x > 1 \end{cases}$

$$\text{fsubchat5} = \sqrt{\frac{2}{\pi}} \int_0^1 x^2\,\text{Cos}[w\,x]\,dx$$

$$\frac{1}{w^3}\sqrt{\frac{2}{\pi}}\left(2\,w\,\text{Cos}[w] + \left(-2 + w^2\right)\text{Sin}[w]\right)$$

This time Mathematica agrees with the text on format.

9 - 15 Fourier Sine Transform

9. Find $\mathcal{F}_s(e^{-ax})$, $a > 0$, by integration.

The template for the Fourier Sine Transform is shown in text panel (2 a) on p. 518.

$$\text{fsubswha} = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-a\,x}\,\text{Sin}[w\,x]\,dx$$

$$\text{ConditionalExpression}\left[\frac{\sqrt{\frac{2}{\pi}}\,w}{a^2 + w^2},\ \text{Abs}[\text{Im}[w]] < \text{Re}[a]\right]$$

Mathematica took a fairly long think before coming up with this one, which matches the text.

11. Find $f_s(w)$ for $f(x)$ = $\begin{cases} x^2 & 0 < x < 1 \\ 0 & x > 1 \end{cases}$

The template for the Fourier Sine Transform is shown in text panel (2a) on p. 518.

$$\text{fsubswhat} = \sqrt{\frac{2}{\pi}} \int_0^1 x^2\,\text{Sin}[w\,x]\,dx$$

$$\frac{1}{w^3}\sqrt{\frac{2}{\pi}}\left(-2 - \left(-2 + w^2\right)\text{Cos}[w] + 2\,w\,\text{Sin}[w]\right)$$

$$\text{N}\left[\frac{1}{w^3}\sqrt{\frac{2}{\pi}}\left(-2-\left(-2+w^2\right)\text{Cos}[w]+2\,w\,\text{Sin}[w]\right)\text{ /. }w\to 2,\ 16\right]$$

0.2462953327972556

$$\text{N}\left[\frac{1}{w^3}\sqrt{\frac{2}{\pi}}\left(\left(2-w^2\right)\text{Cos}[w]+2\,w\,\text{Sin}[w]-2\right)\text{ /. }w\to 2,\ 16\right]$$

0.2462953327972556

It proves to be a case of simply plugging it in. Green cell above is equivalent to text answer for $f_s(w)$, as demonstrated numerically.

13. Find $\mathscr{F}_s\ (e^{-x})$ from (4a) and formula 3 of Table 1 in Sec. 11.10.

As for the references, (4a) goes like this: $\mathscr{F}_c\{f'(x)\} = w\mathscr{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}}\ f(0)$

and formula 3 of Table 1 goes like this:

e^{-ax} $(a > 0)$ goes with $\sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+w^2}\right) = \mathscr{F}_c\{f'(x)\}$

First I need to find that right-hand term (while leaving the minus sign alone):

$$\text{rhterm} = e^{-0}\left(\sqrt{\frac{2}{\pi}}\right)$$

$$\sqrt{\frac{2}{\pi}}$$

Then from Table 1 I should have:

$$\text{fcfprimex} = \sqrt{\frac{2}{\pi}}\left(\frac{1}{1+w^2}\right)$$

$$\frac{\sqrt{\frac{2}{\pi}}}{1+w^2}$$

And rearranging 4a, taking advantage of the minus sign to make an additive :

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fsfx = Simplify[(fcfprimex + rhterm) / w]
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$$\frac{\sqrt{\frac{2}{\pi}}\left(2+w^2\right)}{w+w^3}$$

$$\frac{\sqrt{\frac{2}{\pi}}\left(2+w^2\right)}{w+w^3} \text{ /. } w \to 2$$

$$\frac{3\sqrt{\frac{2}{\pi}}}{5}$$

$$\sqrt{\frac{2}{\pi}}\left(\frac{w}{w^2+1}\right) \text{ /. } w \to 2$$

$$\frac{2\sqrt{\frac{2}{\pi}}}{5}$$

$$\text{Simplify}\left[\frac{1}{w}\left(\sqrt{\frac{2}{\pi}}\left(\frac{1}{w^2+1}\right)+\sqrt{\frac{2}{\pi}}\right)\right]$$

$$\frac{\sqrt{\frac{2}{\pi}}\left(2+w^2\right)}{w+w^3}$$

$$\% \text{ /. } w \to 2$$

$$\frac{3\sqrt{\frac{2}{\pi}}}{5}$$

As straightforward as this problem seemed, there was a difficulty matching answers with the text. In checking, first yellow and gray cells do not match. But the text answer contains intermediate forms, yielding purple cell and agreement with first yellow, and pointing to an error in text answer final form.