

2 - 7 General solution

Find a general solution.

$$\begin{aligned} 3. \quad y_1' &= y_2 + e^{3t} \\ y_2' &= y_1 - 3e^{3t} \end{aligned}$$

`Clear["Global`*"]`

As in the last section, there will be rearranging, recasting, and substitutions after the solution appears in order to make its form like the text answer.

$$e1 = \{y1'[t] == y2[t] + e^{3t}, y2'[t] == y1[t] - 3e^{3t}\}$$

$$e2 = \text{DSolve}[e1, \{y1, y2\}, t]$$

$$\{y1'[t] == e^{3t} + y2[t], y2'[t] == -3e^{3t} + y1[t]\}$$

$$\left\{ \left\{ y1 \rightarrow \text{Function}[\{t\}, \frac{1}{2}e^{-t}(1 + e^{2t})C[1] + \frac{1}{2}e^{-t}(-1 + e^{2t})C[2]], \right. \right. \\ \left. y2 \rightarrow \text{Function}[\{t\}, \frac{1}{4}e^t(-1 + e^{2t})^2 - \frac{1}{4}e^t(1 + e^{2t})^2 + \right. \\ \left. \left. \frac{1}{2}e^{-t}(-1 + e^{2t})C[1] + \frac{1}{2}e^{-t}(1 + e^{2t})C[2] \right\} \right\}$$

$$e3 = e2[[1, 1, 2, 2]]$$

$$\frac{1}{2}e^{-t}(1 + e^{2t})C[1] + \frac{1}{2}e^{-t}(-1 + e^{2t})C[2]$$

$$e4 = \text{Expand}[e3]$$

$$\frac{1}{2}e^{-t}C[1] + \frac{1}{2}e^tC[1] - \frac{1}{2}e^{-t}C[2] + \frac{1}{2}e^tC[2]$$

$$e5 = \text{Collect}[e4, e^{-t}]$$

$$e^{-t} \left(\frac{C[1]}{2} - \frac{C[2]}{2} \right) + e^t \left(\frac{C[1]}{2} + \frac{C[2]}{2} \right)$$

$$e6 = e5 /. \left\{ \left(\frac{C[1]}{2} - \frac{C[2]}{2} \right) \rightarrow c1, \left(\frac{C[1]}{2} + \frac{C[2]}{2} \right) \rightarrow c2 \right\}$$

$$c1 e^{-t} + c2 e^t$$

$$e7 = e2[[1, 2, 2, 2]]$$

$$\frac{1}{4}e^t(-1 + e^{2t})^2 - \frac{1}{4}e^t(1 + e^{2t})^2 + \frac{1}{2}e^{-t}(-1 + e^{2t})C[1] + \frac{1}{2}e^{-t}(1 + e^{2t})C[2]$$

```
e8 = Expand[e7]
```

$$-e^{3t} - \frac{1}{2} e^{-t} C[1] + \frac{1}{2} e^t C[1] + \frac{1}{2} e^{-t} C[2] + \frac{1}{2} e^t C[2]$$

```
e9 = Collect[e8, e^t]
```

$$-e^{3t} + e^{-t} \left(-\frac{C[1]}{2} + \frac{C[2]}{2} \right) + e^t \left(\frac{C[1]}{2} + \frac{C[2]}{2} \right)$$

$$e10 = e9 /. \left\{ \left(-\frac{C[1]}{2} + \frac{C[2]}{2} \right) \rightarrow -c1, \left(\frac{C[1]}{2} + \frac{C[2]}{2} \right) \rightarrow c2 \right\}$$

$$-c1 e^{-t} + c2 e^t - e^{3t}$$

1. Above: The expressions in the green cells match the text answers for y_1 and y_2 respectively. The substitutions of symbolic constants shown in the yellow cells match values of constants between the two function expressions, demonstrating that the constant substitution system is self-consistent and consistent with the text.

$$\begin{aligned} 5. \quad y_1' &= 4 y_1 + y_2 + 0.6 t \\ y_2' &= 2 y_1 + 3 y_2 - 2.5 t \end{aligned}$$

```
Clear["Global`*"]
```

```
e1 = {y1'[t] == 4 y1[t] + y2[t] + 0.6 t, y2'[t] == 2 y1[t] + 3 y2[t] - 2.5 t}
```

```
e2 = DSolve[e1, {y1, y2}, t]
```

```
{y1'[t] == 0.6 t + 4 y1[t] + y2[t], y2'[t] == -2.5 t + 2 y1[t] + 3 y2[t]}
```

```
{ {y1 → Function[{t},
  -0.333333 (1. e^{2. t} - 1. e^{5. t}) (-2.06667 e^{-2. t} (-0.25 - 0.5 t) -
    0.433333 e^{-5. t} (-0.04 - 0.2 t)) + 0.333333 (1. e^{2. t} + 2. e^{5. t})
    (1.03333 e^{-2. t} (-0.25 - 0.5 t) - 0.433333 e^{-5. t} (-0.04 - 0.2 t)) +
    0.333333 (1. e^{2. t} + 2. e^{5. t}) C[1] - 0.333333 (1. e^{2. t} - 1. e^{5. t}) C[2] ],
  y2 → Function[{t}, 0.666667 (1. e^{2. t} + 0.5 e^{5. t})
    (-2.06667 e^{-2. t} (-0.25 - 0.5 t) - 0.433333 e^{-5. t} (-0.04 - 0.2 t)) -
    0.666667 (1. e^{2. t} - 1. e^{5. t})
    (1.03333 e^{-2. t} (-0.25 - 0.5 t) - 0.433333 e^{-5. t} (-0.04 - 0.2 t)) -
    0.666667 (1. e^{2. t} - 1. e^{5. t}) C[1] + 0.666667 (1. e^{2. t} + 0.5 e^{5. t}) C[2] ] } }
```

```

e3 = e2[[1, 1, 2, 2]]
-0.333333 (1. e2. t - 1. e5. t)
  (-2.06667 e-2. t (-0.25 - 0.5 t) - 0.433333 e-5. t (-0.04 - 0.2 t)) +
0.333333 (1. e2. t + 2. e5. t)
  (1.03333 e-2. t (-0.25 - 0.5 t) - 0.433333 e-5. t (-0.04 - 0.2 t)) +
0.333333 (1. e2. t + 2. e5. t) C[1] - 0.333333 (1. e2. t - 1. e5. t) C[2]

e4 = Simplify[e3]
e-3. t (-8.67362 × 10-19 + e3. t (-0.241 - 0.43 t) +
  e6. t (-2.77556 × 10-17 - 5.55112 × 10-17 t) + e5. t
  (0.333333 C[1] - 0.333333 C[2]) + e8. t (0.666667 C[1] + 0.333333 C[2]))

e5 = Expand[e4]
-0.241 - 8.67362 × 10-19 e-3. t - 2.77556 × 10-17 e3. t -
  0.43 t - 5.55112 × 10-17 e3. t t + 0.333333 e2. t C[1] +
  0.666667 e5. t C[1] - 0.333333 e2. t C[2] + 0.333333 e5. t C[2]

e6 = Chop[e5, 10-16]
-0.241 - 0.43 t + 0.333333 e2. t C[1] +
  0.666667 e5. t C[1] - 0.333333 e2. t C[2] + 0.333333 e5. t C[2]

e7 = Collect[e6, {e2. t, e5. t}]
-0.241 - 0.43 t + e2. t (0.333333 C[1] - 0.333333 C[2]) +
  e5. t (0.666667 C[1] + 0.333333 C[2])

e8 = e7 /. {(0.3333333333333334` C[1] - 0.3333333333333333` C[2]) → c2,
  (0.6666666666666666` C[1] + 0.3333333333333333` C[2]) → c1}

-0.241 + c2 e2. t + c1 e5. t - 0.43 t

e9 = e2[[1, 2, 2, 2]]
0.666667 (1. e2. t + 0.5 e5. t)
  (-2.06667 e-2. t (-0.25 - 0.5 t) - 0.433333 e-5. t (-0.04 - 0.2 t)) -
0.666667 (1. e2. t - 1. e5. t)
  (1.03333 e-2. t (-0.25 - 0.5 t) - 0.433333 e-5. t (-0.04 - 0.2 t)) -
0.666667 (1. e2. t - 1. e5. t) C[1] + 0.666667 (1. e2. t + 0.5 e5. t) C[2]

e10 = Simplify[e9]
0.534 + 1.73472 × 10-18 e-3. t + 1.12 t +
  e5. t (0.666667 C[1] + 0.333333 C[2]) + e2. t (-0.666667 C[1] + 0.666667 C[2])

```

```
e11 = Chop[e10, 10^-16]
```

```
0.534 + 1.12 t + e^5 t (0.666667 C[1] + 0.333333 C[2]) +  
e^2 t (-0.666667 C[1] + 0.666667 C[2])
```

```
e12 = e11 /. {(0.6666666666666669` C[1] + 0.3333333333333334` C[2]) -> c1,  
(-0.6666666666666669` C[1] + 0.6666666666666667` C[2]) -> -2 c2}
```

```
0.534 - 2 c2 e^2 t + c1 e^5 t + 1.12 t
```

1. Above: The expressions in the green cells match the text answers for y_1 and y_2 respectively. The substitutions of symbolic constants shown in the yellow cells match values of constants between the two function expressions, demonstrating that the constant substitution system is self-consistent and consistent with the text.

```
7. y1' = -3 y1 - 4 y2 + 11 t + 15  
y2' = 5 y1 + 6 y2 + 3 e^-t - 15 t - 20
```

```
Clear["Global`*"]
```

```
e1 = {y1'[t] == -3 y1[t] - 4 y2[t] + 11 t + 15,  
y2'[t] == 5 y1[t] + 6 y2[t] + 3 e^-t - 15 t - 20}
```

```
e2 = DSolve[e1, {y1, y2}, t]
```

```
{y1'[t] == 15 + 11 t - 3 y1[t] - 4 y2[t],  
y2'[t] == -20 + 3 e^-t - 15 t + 5 y1[t] + 6 y2[t]}
```

```
{ {y1 -> Function[{t}, -e^t (-5 + 4 e^t) (4 e^-3 t + e^-2 t (-20 - 8 t) + e^-t (10 + 5 t)) -  
4 e^t (-1 + e^t) (-5 e^-3 t + e^-t (-10 - 5 t) + e^-2 t (47/2 + 10 t)) -  
e^t (-5 + 4 e^t) C[1] - 4 e^t (-1 + e^t) C[2]],  
y2 -> Function[{t}, 5 e^t (-1 + e^t) (4 e^-3 t + e^-2 t (-20 - 8 t) + e^-t (10 + 5 t)) +  
e^t (-4 + 5 e^t) (-5 e^-3 t + e^-t (-10 - 5 t) + e^-2 t (47/2 + 10 t)) +  
5 e^t (-1 + e^t) C[1] + e^t (-4 + 5 e^t) C[2]] }
```

```
e3 = e2[[1, 1, 2, 2]]
```

```
-e^t (-5 + 4 e^t) (4 e^-3 t + e^-2 t (-20 - 8 t) + e^-t (10 + 5 t)) -  
4 e^t (-1 + e^t) (-5 e^-3 t + e^-t (-10 - 5 t) + e^-2 t (47/2 + 10 t)) -  
e^t (-5 + 4 e^t) C[1] - 4 e^t (-1 + e^t) C[2]
```

```
e4 = Simplify[e3]
```

```
e^-t (-2 - e^t (4 + 3 t) - 4 e^3 t (C[1] + C[2]) + e^2 t (5 C[1] + 4 C[2]))
```

```
e5 = Expand[e4]
```

$$-4 - 2 e^{-t} - 3 t + 5 e^t C[1] - 4 e^{2t} C[1] + 4 e^t C[2] - 4 e^{2t} C[2]$$

```
e6 = Collect[e5, {e^{2t}, e^t}]
```

$$-4 - 2 e^{-t} - 3 t + e^{2t} (-4 C[1] - 4 C[2]) + e^t (5 C[1] + 4 C[2])$$

```
e7 = e6 /. {(-4 C[1] - 4 C[2]) -> 4 c2, (5 C[1] + 4 C[2]) -> c1}
```

$$-4 - 2 e^{-t} + c1 e^t + 4 c2 e^{2t} - 3 t$$

```
e8 = e2[[1, 2, 2, 2]]
```

$$\begin{aligned} & 5 e^t (-1 + e^t) (4 e^{-3t} + e^{-2t} (-20 - 8 t) + e^{-t} (10 + 5 t)) + \\ & e^t (-4 + 5 e^t) \left(-5 e^{-3t} + e^{-t} (-10 - 5 t) + e^{-2t} \left(\frac{47}{2} + 10 t \right) \right) + \\ & 5 e^t (-1 + e^t) C[1] + e^t (-4 + 5 e^t) C[2] \end{aligned}$$

```
e9 = Simplify[e8]
```

$$\frac{15}{2} + e^{-t} + 5 t + 5 e^{2t} (C[1] + C[2]) - e^t (5 C[1] + 4 C[2])$$

```
e10 = e9 /. {(C[1] + C[2]) -> -c2, (5 C[1] + 4 C[2]) -> c1}
```

$$\frac{15}{2} + e^{-t} - c1 e^t - 5 c2 e^{2t} + 5 t$$

1. Above: The expressions in the green cells match the text answers for y_1 and y_2 respectively. The substitutions of symbolic constants shown in the yellow cells match values of constants between the two function expressions, demonstrating that the constant substitution system is self-consistent and consistent with the text.

10 - 15 Initial value problem
Solve, showing details.

$$\begin{aligned} 11. \quad & y_1' = y_2 + 6 e^{2t} \\ & y_2' = y_1 - e^{2t} \\ & y_1[0] = 1, \quad y_2[0] = 0 \end{aligned}$$

```
Clear["Global`*"]
```

```
e1 = {y1'[t] == y2[t] + 6 e^2 t, y2'[t] == y1[t] - e^2 t, y1[0] == 1, y2[0] == 0}
```

```
e2 = DSolve[e1, {y1, y2}, t]
```

```
{y1'[t] == 6 e^2 t + y2[t], y2'[t] == -e^2 t + y1[t], y1[0] == 1, y2[0] == 0}
```

```
{ {y1 -> Function[{t},  $\frac{1}{3} e^{-t} (-2 - 6 e^{2 t} + 11 e^{3 t})$ ],  
  y2 -> Function[{t},  $\frac{2}{3} e^{-t} (-1 + e^t)^2 (1 + 2 e^t)$ ] } }
```

```
e3 = e2[[1, 1, 2, 2]]
```

```
 $\frac{1}{3} e^{-t} (-2 - 6 e^{2 t} + 11 e^{3 t})$ 
```

```
e4 = Expand[e3]
```

```
 $-\frac{2 e^{-t}}{3} - 2 e^t + \frac{11 e^{2 t}}{3}$ 
```

```
e5 = e4 /.  $\left(-\frac{2 e^{-t}}{3} - 2 e^t\right) \rightarrow \text{ExpToTrig}\left[-\frac{2 e^{-t}}{3} - 2 e^t\right]$ 
```

```
 $\frac{11 e^{2 t}}{3} - \frac{8 \text{Cosh}[t]}{3} - \frac{4 \text{Sinh}[t]}{3}$ 
```

```
e6 = e2[[1, 2, 2, 2]]
```

```
 $\frac{2}{3} e^{-t} (-1 + e^t)^2 (1 + 2 e^t)$ 
```

```
e7 = Expand[e6]
```

```
 $\frac{2 e^{-t}}{3} - 2 e^t + \frac{4 e^{2 t}}{3}$ 
```

```
e8 = e7 /.  $\left(\frac{2 e^{-t}}{3} - 2 e^t\right) \rightarrow \text{ExpToTrig}\left[\frac{2 e^{-t}}{3} - 2 e^t\right]$ 
```

```
 $\frac{4 e^{2 t}}{3} - \frac{4 \text{Cosh}[t]}{3} - \frac{8 \text{Sinh}[t]}{3}$ 
```

1. Above: The top and bottom green cell expressions match the text answers for y_1 and y_2 respectively.

```
13. y1' = y2 - 5 Sin[t]  
y2' = -4 y1 + 17 Cos[t]  
y1[0] = 5, y2[0] = 2
```

```
Clear["Global`*"]
```

```

e1 = {y1'[t] == y2[t] - 5 Sin[t],
      y2'[t] == -4 y1[t] + 17 Cos[t], y1[0] == 5, y2[0] == 2}
e2 = DSolve[e1, {y1, y2}, t]
{y1'[t] == -5 Sin[t] + y2[t], y2'[t] == 17 Cos[t] - 4 y1[t], y1[0] == 5, y2[0] == 2}

{{y1 -> Function[{t},  $\frac{1}{4} (4 \cos[2 t] + 7 \cos[t] \cos[2 t] + 9 \cos[2 t] \cos[3 t] +$ 
 $4 \sin[2 t] + 7 \sin[t] \sin[2 t] + 9 \sin[2 t] \sin[3 t])$ ],
  y2 -> Function[{t},  $\frac{1}{2} (4 \cos[2 t] + 7 \cos[2 t] \sin[t] - 4 \sin[2 t] -$ 
 $7 \cos[t] \sin[2 t] - 9 \cos[3 t] \sin[2 t] + 9 \cos[2 t] \sin[3 t])$ ]}]}

e3 = e2[[1, 1, 2, 2]]
 $\frac{1}{4} (4 \cos[2 t] + 7 \cos[t] \cos[2 t] + 9 \cos[2 t] \cos[3 t] +$ 
 $4 \sin[2 t] + 7 \sin[t] \sin[2 t] + 9 \sin[2 t] \sin[3 t])$ 

e4 = Simplify[e3]
4 Cos[t] + Cos[2 t] + Sin[2 t]

e5 = e2[[1, 2, 2, 2]]
 $\frac{1}{2} (4 \cos[2 t] + 7 \cos[2 t] \sin[t] - 4 \sin[2 t] -$ 
 $7 \cos[t] \sin[2 t] - 9 \cos[3 t] \sin[2 t] + 9 \cos[2 t] \sin[3 t])$ 

e6 = Simplify[e5]
2 Cos[2 t] + Sin[t] - 2 Sin[2 t]

```

1. Above: The top and bottom blue cell expressions match the text answers for y_1 and y_2 respectively.

```

15. y1' = y1 + 2 y2 + e2 t - 2 t
y2' = -y2 + 1 + t
y1[0] = 1, y2[0] = -4

```

```

Clear["Global`*"]

```

```

e1 = {y1'[t] == y1[t] + 2 y2[t] + e2 t - 2 t,
      y2'[t] == -y2[t] + 1 + t, y1[0] == 1, y2[0] == -4}
e2 = DSolve[e1, {y1, y2}, t]
{y1'[t] == e2 t - 2 t + y1[t] + 2 y2[t],
 y2'[t] == 1 + t - y2[t], y1[0] == 1, y2[0] == -4}

{{y1 -> Function[{t}, -  $\frac{e^{-t} (4 - e^{2 t} e^t - 2 e^{2 t} - 8 \text{Log}[e] + 6 e^{2 t} \text{Log}[e])}{-1 + 2 \text{Log}[e]}$ ],
  y2 -> Function[{t}, e-t (-4 + et t)]}}
```

```

e3 = e2[[1, 1, 2, 2]]
-  $\frac{e^{-t} (4 - e^{2 t} e^t - 2 e^{2 t} - 8 \text{Log}[e] + 6 e^{2 t} \text{Log}[e])}{-1 + 2 \text{Log}[e]}$ 

e4 = e3 /. (e-t (4 - e2 t et - 2 e2 t - 8 Log[e] + 6 e2 t Log[e])) ->
Expand[e-t (4 - e2 t et - 2 e2 t - 8 Log[e] + 6 e2 t Log[e])]
-  $\frac{e^{2 t} + 4 e^{-t} - 2 e^t - 8 e^{-t} \text{Log}[e] + 6 e^t \text{Log}[e]}{-1 + 2 \text{Log}[e]}$ 

e5 = FullSimplify[e4]
 $\frac{e^{2 t} + 2 \text{Cosh}[t] (-1 + \text{Log}[e]) + (6 - 14 \text{Log}[e]) \text{Sinh}[t]}{-1 + 2 \text{Log}[e]}$ 

e6 = e5 /. Log[e] -> 1
e2 t - 8 Sinh[t]

e7 = e6 /. (-8 Sinh[t]) -> TrigToExp[-8 Sinh[t]]
e2 t + 4 e-t - 4 et

e8 = e2[[1, 2, 2, 2]]
e-t (-4 + et t)

e9 = Expand[e8]
-4 e-t + t
```

1. Above: The top and bottom green cell expressions match the text answers for y_1 and y_2 respectively.