Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

```
Clear["Global`*"]
```

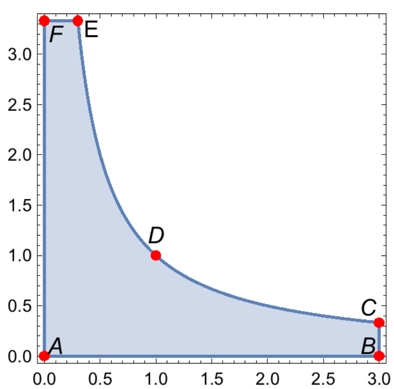
3 - 5 Application of theorem 1

3. Find the potential $\Phi$ in the region R in the first quadrant of the z-plane bounded by the axes (having potential $U_1$) and the hyperbola $y = \frac{1}{x}$ (having potential $U_2$) by mapping R onto a suitable infinite strip. Show that $\Phi$ is harmonic. What are its boundary values?

```
Clear["Global`*"]
```

The region plot is of an infinite region, and the 6 sample points are for trying to keep track of what maps where.

```
RegionPlot[0 < y && 0 < x && y < 1/x, {x, 0, 3}, {y, 0, 10/3},
 ImageSize → 200, Epilog → {{Red, PointSize[0.03], Point[{0, 0}]},
   {Red, PointSize[0.03], Point[{3, 0}]}, {Red, PointSize[0.03],
    Point[{3, 1/3}]}, {Red, PointSize[0.03], Point[{1, 1}]},
   {Red, PointSize[0.03], Point[{0.3, 1/0.3}]}, {Red, PointSize[0.03],
    Point[{0, 10/3}]}, {Text[Style[A, Medium], {0.1, 0.1}]},
   {Text[Style[B, Medium], {2.9, 0.1}]},
   {Text[Style[C, Medium], {2.9, 0.48}]},
   {Text[Style[D, Medium], {1, 1.2}]},
   {Text[Style["E", Medium], {0.42, 3.25}]},
   {Text[Style[F, Medium], {0.1, 3.2}]}}]
```




One useful way to set up what the text answer seems to refer to as t-space is to establish an ImplicitRegion.

```
d1 = ImplicitRegion[0 < x ∧ 0 < y ∧ y < 1/x, {x, y}];
```

Then the implicit region can be mapped onto a semi-infinite strip using the squaring function. Having played with this before, I recognize that the squaring function by itself will give a horizontal strip (below right), or I could agree with the rotation contained in the text answer and throw in the $i$ for a vertical strip (below left). Either way it is a semi-infinite strip, not an infinite one. First I will try it my way. Note that the domain of the mapping is taken directly off the implicit region.

(If I make an array of sample points and calculate them in advance it will save space.)

```
sx = {{0, 0}, {3, 0}, {3, 1 / 3}, {1, 1}, {0.3, 1 / 0.3}, {0, 10 / 3}}
```

$\{\{0,0\},\{3,0\},\{3,\frac{1}{3}\},\{1,1\},\{0.3,3.33333\},\{0,\frac{10}{3}\}\}$

Since the points are for plotting, I don’t need super high accuracy.

```
gp[{x_, y_}] = {N[Re[(x + ⅈ y)^2]], N[Im[(x + ⅈ y)^2]]}
```

$\{\mathrm{Re}[(x+(0.+1.\,i)\,y)^2],\ \mathrm{Im}[(x+(0.+1.\,i)\,y)^2]\}$

```
Thread[gp[ sx]]
```

```
{{0., 0.}, {9., 0.}, {8.88889, 2.},
 {0., 2.}, {-11.0211, 2.}, {-11.1111, 0.}}
```

Then a version of the points for the mapping according to the approach implied by the text answer.

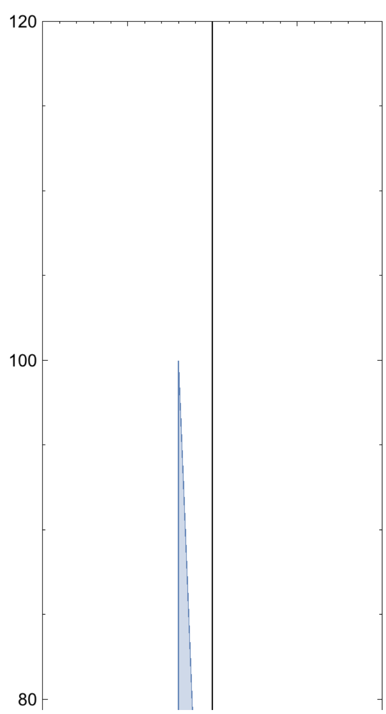
```
tp[{x_, y_}] = {N[Re[ⅈ (x + ⅈ y)^2]], N[Im[ⅈ (x + ⅈ y)^2]]}
```

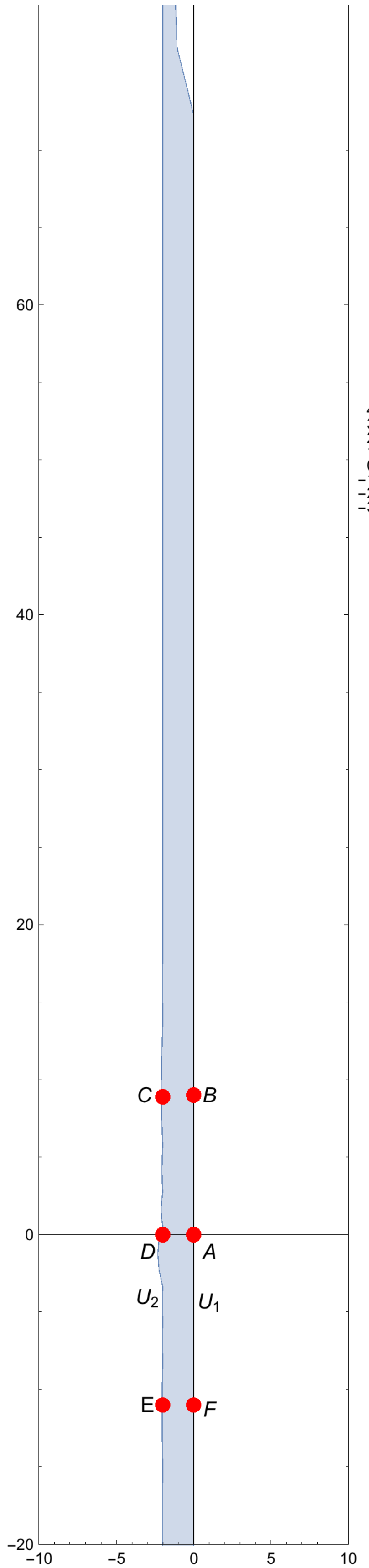
$\{-1.\,\mathrm{Im}[(x+(0.+1.\,i)\,y)^2],\ \mathrm{Re}[(x+(0.+1.\,i)\,y)^2]\}$

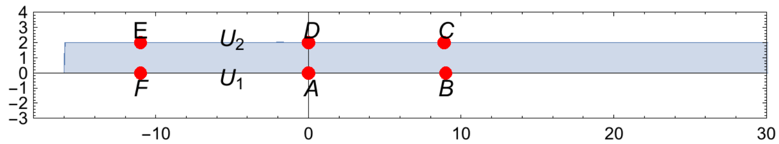
```
Thread[tp[ sx]]
```

```
{{0., 0.}, {0., 9.}, {-2., 8.88889},
 {-2., 0.}, {-2., -11.0211}, {0., -11.1111}}
```

```
Row[{ParametricPlot[Through[{Re, Im}[ⅈ (x + ⅈ y)^2]],
    {x, y} ∈ d1, PlotRange → {{-10, 10}, {-20, 120}}, Frame → True,
    ImageSize → 200, Epilog → {{Red, PointSize[0.05], Point[{0, 0}]},
      {Text[Style[A, Medium], {1, -1.1}]}, {Red, PointSize[0.05],
       Point[{0, 9}]}, {Text[Style[B, Medium], {1, 9}]},
      {Red, PointSize[0.05], Point[{-2, 8.9}]},
      {Text[Style[C, Medium], {-3.2, 9}]}, {Red, PointSize[0.05],
       Point[{-2, 0}]}, {Text[Style[D, Medium], {-3, -1.1}]},
      {Red, PointSize[0.05], Point[{-2, -11}]},
      {Text[Style["E", Medium], {-3, -11}]}, {Red, PointSize[0.05],
       Point[{0, -11}]}, {Text[Style[F, Medium], {1, -11.3}]},
      {Text[Style[ U1, Medium], {1, -4.3}]},
      {Text[Style[ U2, Medium], {-3, -4}]}}],
  ParametricPlot[Through[{Re, Im}[(x + ⅈ y)^2]], {x, y} ∈ d1,
    PlotRange → {{-18, 30}, {-3, 4}},
    Frame → True, ImageSize → 400,
    Epilog → {{Red, PointSize[0.018], Point[{0, 0}]},
      {Text[Style[A, Medium], {0.2, -1}]}, {Red, PointSize[0.018],
       Point[{9, 0}]}, {Text[Style[B, Medium], {9, -1}]},
      {Red, PointSize[0.018], Point[{8.88, 2}]},
      {Text[Style[C, Medium], {9, 2.8}]}, {Red, PointSize[0.018],
       Point[{0, 2}]}, {Text[Style[D, Medium], {0.2, 2.8}]},
      {Red, PointSize[0.018], Point[{-11, 2}]},
      {Text[Style["E", Medium], {-11, 2.8}]}, {Red, PointSize[0.018],
       Point[{-11, 0}]}, {Text[Style[F, Medium], {-11, -1}]},
      {Text[Style[ U1, Medium], {-5, -0.3}]},
      {Text[Style[ U2, Medium], {-5, 2.3}]}}]}]
```




60
40
20
0
−20
−10
−5
0
5
10
C
B
D
A
$U_2$
$U_1$
E
F


4
3
2
1
0
−1
−2
−3
E
$U_2$
D
C
F
$U_1$
A
B
−10
0
10
20
30

Problem 17 in section 17.4 looks a lot like this, except the equation of the hyperbola is different. After doing some research, almost everything I found on line deals with double mapping, because a semi-infinite strip is evidently normally mapped to a half plane for analysis. Then there are three planes, z-plane, Z-plane, and w-plane. To model the w-plane, I need to access the array of once-mapped sample points.

```
ss = {{0, 0}, {9, 0}, {80/9, 2},
  {0, 2}, {-11.021111111111113`, 2.`}, {-100/9, 0}}
```

```
{{0, 0}, {9, 0}, {80/9, 2}, {0, 2}, {-11.0211, 2.}, {-100/9, 0}}
```

So I can re-map them.

```
gs2[{x_, y_}] = {N[Re[Exp[x + π/2 ⅈ y]], 2], N[Im[Exp[x + π/2 ⅈ y]], 2]}
```

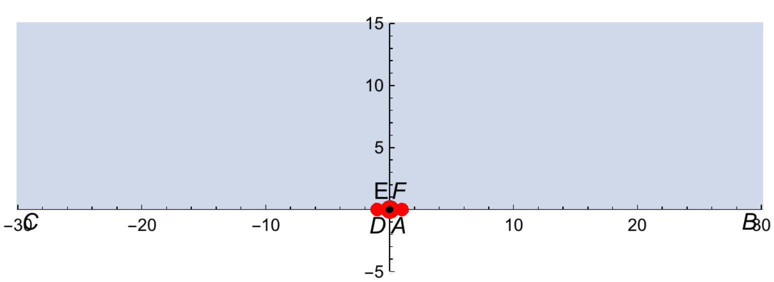
$\{\mathrm{Re}[2.7^{x+(0.\times10^{-3}+1.57\,i)\,y}],\ \mathrm{Im}[2.7^{x+(0.\times10^{-3}+1.57\,i)\,y}]\}$

```
Thread[gs2[ss]]
```

$\{\{1, 0\}, \{8.\times10^{3}, 0\}, \{-7.\times10^{3}, \boxed{0.\times10^{2}}\}, \{-1.0, 0.\times10^{-2}\}, \{-0.0000163528, 2.00264\times10^{-21}\}, \{0.00001, 0\}\}$

And now to map the semi-infinite strip to a half plane. In this case it happens to be horizontal. I don't see an advantage to rotating it.

```
ParametricPlot[Through[{Re, Im}[Exp[x + π/2 ⅈ y]]], {x, -25, 25},
 {y, 0, 2}, PlotRange → {{-30, 30}, {-5, 15}}, Frame → False,
 ImageSize → 400, Epilog → {{Red, PointSize[0.018], Point[{1, 0}]},
   {Text[Style[A, Medium], {0.7, -1.3}]}, {Red, PointSize[0.018],
    Point[{8000, 0}]}, {Text[Style[B, Medium], {29, -1}]},
   {Red, PointSize[0.018], Point[{-7000, 0}]},
   {Text[Style[C, Medium], {-29, -1}]}, {Red, PointSize[0.018],
    Point[{-1, 0}]}, {Text[Style[D, Medium], {-1, -1.3}]},
   {Red, PointSize[0.025], Point[{0, 0}]},
   {Text[Style["E", Medium], {-0.7, 1.5}]}, {Black, PointSize[0.01],
    Point[{0, 0}]}, {Text[Style[F, Medium], {0.7, 1.5}]}}]
```



I see that the disposition of sample points is sort of problematic. Points B and C are off the plot. But right now, my focus is on groups of points. F, A, B, associated with potential $U_1$, can be interpreted as being on the right, and points C, D, E, associated with potential $U_2$, are on the left. And, noting that the semi-plane is infinite makes it seem like a natural for example 3 on p. 760, the potential of an angular sector, with the sector $\alpha=\pi$.

This looks okay so far, but it is overly complicated with two successive mappings. The way to go would be a composition function that would do it all in one step. So, how about

```
girt[{x_, y_}] = {Re[(x + ⅈ y)^2], Im[(x + ⅈ y)^2]}
```

$\{\mathrm{Re}[(x + i\, y)^2], \mathrm{Im}[(x + i\, y)^2]\}$

for the hyperbola-to-strip, and

```
toside[{x_, y_}] = {Re[ⅇ^(x + ⅈ π y/2)], Im[ⅇ^(x + ⅈ π y/2)]}
```

$\{\mathrm{Re}[e^{x+\frac{i \pi y}{2}}], \mathrm{Im}[e^{x+\frac{i \pi y}{2}}]\}$

for the strip to semi-plane, the two rolled up into

```
combi[{x_, y_}] = Composition[toside, girt][{x, y}]
```

$\{\mathrm{Re}[e^{\frac{1}{2} i \pi \mathrm{Im}[(x+i y)^2]+\mathrm{Re}[(x+i y)^2]}], \mathrm{Im}[e^{\frac{1}{2} i \pi \mathrm{Im}[(x+i y)^2]+\mathrm{Re}[(x+i y)^2]}]\}$

The following test shows that the composition function handles all the sample points correctly.

```
N[Thread[combi[sx]]]
```

$\{\{1., 0.\}, \{8103.08, 0.\}, \{-7250.96, 0.\}, \{-1., 0.\}, \{-0.0000163528, 2.00264\times 10^{-21}\}, \{0.0000149453, 0.\}\}$

On the combi declaration out-line [xxx] I can see under the function's hood to tell what it really is.

$e^{\frac{1}{2} i \pi z^2}$

$e^{\frac{1}{2} i \pi z^2}$

Expanding shows off some simple components

```
ComplexExpand[%]
```

$\mathrm{Cos}\left[\frac{\pi z^2}{2}\right] + i\, \mathrm{Sin}\left[\frac{\pi z^2}{2}\right]$

Bringing in the punchline of example 3 on p. 760. (Note: the $\frac{v}{u}$ instead of $\frac{y}{x}$ below was Murray Spiegel's idea.) Doing some reorganizing, and subject as always to my mistakes,

$\Phi[x, y] = \frac{1}{2}(\Phi_2 + \Phi_1) + \frac{1}{\pi}(\Phi_1 - \Phi_2)\,\mathrm{ArcTan}\left[\frac{v}{u}\right] \Longrightarrow$

$$\Phi[x, y] = \frac{1}{2}(\Phi_2 + \Phi_1) + \frac{1}{\pi}(\Phi_1 - \Phi_2)\,\text{ArcTan}\left[\text{Tan}\left[\frac{\pi z^2}{2}\right]\right] \Longrightarrow$$

$$\Phi[x, y] = \frac{1}{2}(\Phi_2 + \Phi_1) + \frac{1}{\pi}(\Phi_1 - \Phi_2)\,\frac{\pi z^2}{2} \Longrightarrow$$

$$\Phi[x, y] = \frac{1}{2}(\Phi_2 + \Phi_1) + \frac{1}{2}(\Phi_1 - \Phi_2)\,(x + i\,y)^2 \Longrightarrow$$

$$(U_1 - U_2)\left(\frac{1}{2}u\right) + \frac{1}{2}(U_1 + U_2)$$

The above cell is not particularly close to the text answer, but so far it’s the best I can do. Just for the record,

$$\text{PossibleZeroQ}\left[\left((U_1 - U_2)\left(\frac{1}{2}u\right) + \frac{1}{2}(U_1 + U_2)\right) - \left(U_2 + (U_1 - U_2)\left(1 + \frac{1}{2}u\right)\right)\right]$$

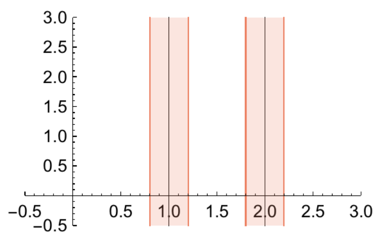
```
False
```

5. CAS Project. Graphing potential fields. Graph equipotential lines (a) in example 1 of the text, (b) if the complex potential is F[z] = $z^2$, $i\,z^2$, $e^z$. (c) Graph the equipotential surfaces for F[x] = Log[z] as cylinders in space.

```
Clear["Global`*"]

cran = RGBColor[0.925, 0.498, 0.376];
grap = RGBColor[0.529, 0.474, 0.694];
cin = RGBColor[0.764, 0.431, 0.153]; pur = RGBColor[0.729, 0.361, 0.502];
grn = RGBColor[0.298, 0.709, 0.439] ; gra = RGBColor[0.7, 0.7, 0.7];
```

5.1 Plot of example 1. With plates at x=1 and x=2, and $\Phi_2 = 0.2$, $\Phi_1 = 0.05$

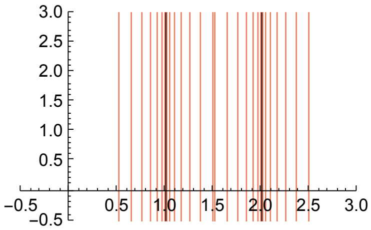
```
p1 = ListLinePlot[Table[{x, y}, {x, {(1 + .195/2 + .205/2), (1 - .195/2 - .205/2),
      (2 + .195/2 + .205/2), (2 - .195/2 - .205/2)}}, {y, -0.5, 3, 0.1}],
  ImageSize → 190, PlotStyle → {{cran, Thickness[0.004]}},
  PlotRange → {{-0.5, 3}, {-0.5, 3}},
  Epilog → {{Line[{{2, -0.5}, {2, 3}}]}, {Line[{{1, -0.5}, {1, 3}}]},
    {Opacity[0.2], cran, Rectangle[{1 - .195/2 - .205/2, -0.5},
     {1 + .195/2 + .205/2, 3}]}, {Opacity[0.2], cran,
     Rectangle[{2 - .195/2 - .205/2, -0.5}, {2 + .195/2 + .205/2, 3}]}}]
```




## 5.2 Plot of example 1 with potential on both sides equal to $z^2$.

```
inc = Flatten[Table[{1 + x^2}, {x, 0.1, √0.5, .1}]];

incm = Flatten[Table[{1 - x^2}, {x, 0.1, √0.5, .1}]];

incn = Flatten[Table[{2 + x^2}, {x, 0.1, √0.5, .1}]];

inco = Flatten[Table[{2 - x^2}, {x, 0.1, √0.5, .1}]];

p1 = ListLinePlot[Table[{x, y}, {x, inc}, {y, -0.5, 3, 0.1}],
   ImageSize → 190, PlotStyle → {{cran, Thickness[0.004]}},
   PlotRange → {{-0.5, 3}, {-0.5, 3}},
   Epilog → {{Line[{{2, -0.5}, {2, 3}}]}, {Line[{{1, -0.5}, {1, 3}}]}}];

p2 = ListLinePlot[Table[{x, y}, {x, incm}, {y, -0.5, 3, 0.1}],
   ImageSize → 190, PlotStyle → {{cran, Thickness[0.004]}},
   PlotRange → {{-0.5, 3}, {-0.5, 3}}];
p3 = ListLinePlot[Table[{x, y}, {x, incn}, {y, -0.5, 3, 0.1}],
   ImageSize → 190, PlotStyle → {{cran, Thickness[0.004]}},
   PlotRange → {{-0.5, 3}, {-0.5, 3}}];
p4 = ListLinePlot[Table[{x, y}, {x, inco}, {y, -0.5, 3, 0.1}],
   ImageSize → 190, PlotStyle → {{cran, Thickness[0.004]}},
   PlotRange → {{-0.5, 3}, {-0.5, 3}}];
```

```
Show[p1, p2, p3, p4]
```




## 5.3 Plot of example 1 with potential on both sides equal to $i\,z^2$.

If this z were not squared, the $i$ would have the effect of flipping the Re and Im parts of the points. As it is

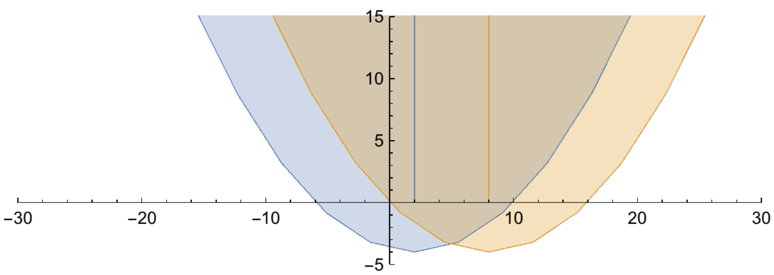
```
Clear["Global`*"]
```

```
ComplexExpand[ⅈ (x + ⅈ y)^2]
```

$-2\,x\,y + i\,(x^2 - y^2)$

It’s a little more complicated than simple flipping. I will suppose that my plates are at x=2 and x=8. Then I can take a look at

```
ParametricPlot[{Through[{Re, Im}[(-2 x y + ⅈ (x^2 - y^2)) + 2]],
  Through[{Re, Im}[(-2 x y + ⅈ (x^2 - y^2)) + 8]]}, {x, -25, 25}, {y, 0, 2},
 PlotRange → {{-30, 30}, {-5, 15}}, Frame → False, ImageSize → 400]
```




It looks like Mathematica is plotting the poles of the functions in the location where the plates would be. If a plate happens to occupy the space of a pole, I suppose that wouldn’t affect the propagation of the potential field. To be shown right, this plot should show the interaction of the intersecting fields, but that would require *way* more expertise than I have.

## 5.4 Plot of example 1 with potential on both sides (both plates) equal to $e^z$.

Showing only a handful of possible tracks.

```
Clear["Global`*"]
```

$$\text{p1} = \text{Plot}\Big[\big\{e^{x-4}, e^{2(x-4)}, e^{3(x-4)}, e^{4(x-4)}, 1+e^{x-4}, 1+e^{2(x-4)}, 1+e^{3(x-4)}, 1+e^{4(x-4)}, 2+e^{x-4}, 2+e^{2(x-4)}, 2+e^{3(x-4)}, 2+e^{4(x-4)}, 3+e^{x-4}, 3+e^{2(x-4)}, 3+e^{3(x-4)}, 3+e^{4(x-4)}\big\},$$

```
    {x, 4, 8}, PlotRange → {{0, 10}, {0, 10}},
    Epilog → {{Thick, Line[{{4, 0}, {4, 10}}]},
      {Thick, Line[{{8, 0}, {8, 10}}]}, {},}];
```

$$\text{p2} = \text{Plot}\Big[\Big\{\left(\frac{1}{e}\right)^{x-4}, \left(\frac{1}{e}\right)^{2(x-4)}, \left(\frac{1}{e}\right)^{3(x-4)}, \left(\frac{1}{e}\right)^{4(x-4)}, 1+\left(\frac{1}{e}\right)^{x-4}, 1+\left(\frac{1}{e}\right)^{2(x-4)}, 1+\left(\frac{1}{e}\right)^{3(x-4)}, 1+\left(\frac{1}{e}\right)^{4(x-4)}, 2+\left(\frac{1}{e}\right)^{x-4}, 2+\left(\frac{1}{e}\right)^{2(x-4)}, 2+\left(\frac{1}{e}\right)^{3(x-4)}, 2+\left(\frac{1}{e}\right)^{4(x-4)}, 3+\left(\frac{1}{e}\right)^{x-4}, 3+\left(\frac{1}{e}\right)^{2(x-4)}, 3+\left(\frac{1}{e}\right)^{3(x-4)}, 3+\left(\frac{1}{e}\right)^{4(x-4)}\Big\},$$

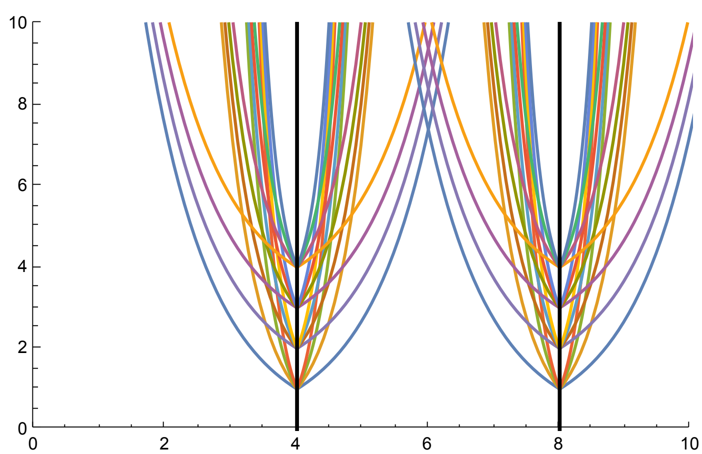
```
    {x, 0, 4}, PlotRange → {{0, 10}, {0, 10}}];
```

$$\text{p3} = \text{Plot}\Big[\big\{e^{x-8}, e^{2(x-8)}, e^{3(x-8)}, e^{4(x-8)}, 1+e^{x-8}, 1+e^{2(x-8)}, 1+e^{3(x-8)}, 1+e^{4(x-8)}, 2+e^{x-8}, 2+e^{2(x-8)}, 2+e^{3(x-8)}, 2+e^{4(x-8)}, 3+e^{x-8}, 3+e^{2(x-8)}, 3+e^{3(x-8)}, 3+e^{4(x-8)}\big\},$$

```
    {x, 8, 12}, PlotRange → {{0, 10}, {0, 10}}];
```

$$\text{p4} = \text{Plot}\Big[\Big\{\left(\frac{1}{e}\right)^{x-8}, \left(\frac{1}{e}\right)^{2(x-8)}, \left(\frac{1}{e}\right)^{3(x-8)}, \left(\frac{1}{e}\right)^{4(x-8)}, 1+\left(\frac{1}{e}\right)^{x-8}, 1+\left(\frac{1}{e}\right)^{2(x-8)}, 1+\left(\frac{1}{e}\right)^{3(x-8)}, 1+\left(\frac{1}{e}\right)^{4(x-8)}, 2+\left(\frac{1}{e}\right)^{x-8}, 2+\left(\frac{1}{e}\right)^{2(x-8)}, 2+\left(\frac{1}{e}\right)^{3(x-8)}, 2+\left(\frac{1}{e}\right)^{4(x-8)}, 3+\left(\frac{1}{e}\right)^{x-8}, 3+\left(\frac{1}{e}\right)^{2(x-8)}, 3+\left(\frac{1}{e}\right)^{3(x-8)}, 3+\left(\frac{1}{e}\right)^{4(x-8)}\Big\},$$

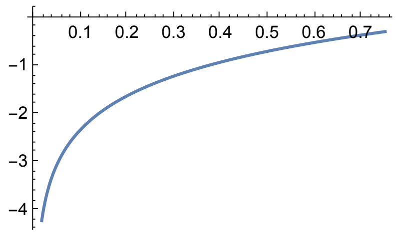
```
    {x, 4, 8}, PlotRange → {{0, 12}, {0, 10}}];

Show[p1, p2, p3, p4]
```




5.5 3D plot of the F[x] = Log[z] function as electrostatic fields around two cylinders in space.

Adapted from *Mathematica* contour3D documentation. First a look-see at the plot of intervals.

```
Plot[Log[r], {r, 0, 0.75}, ImageSize → 200]
```




Now I have a plot of the log function, but what I need are the y intervals, so

```
x7 = Flatten[Table[N[Solve[Exp[x] == i]], {i, 0.01, 0.75, 0.1}]]
```

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

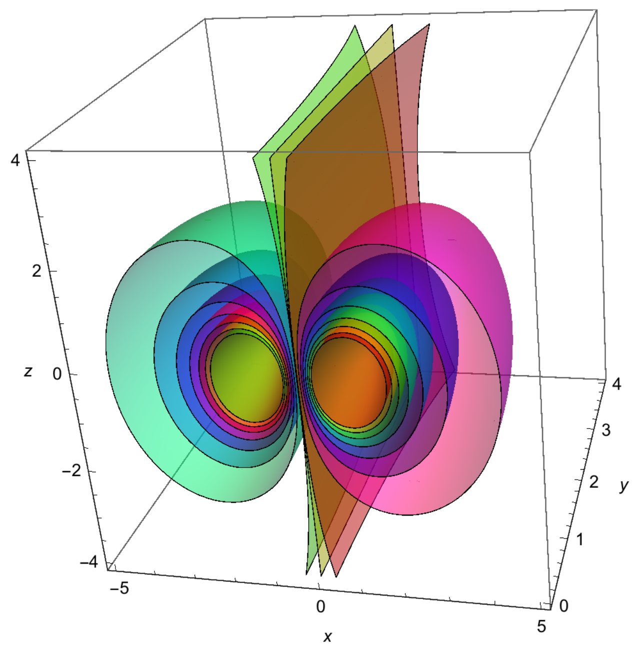
Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

General::stop: Further output of Solve::ifun will be suppressed during this calculation. >>

```
{x → -4.60517, x → -2.20727, x → -1.56065, x → -1.17118,
 x → -0.891598, x → -0.673345, x → -0.494296, x → -0.34249}
```

And I can use the Log (actually Exp) intervals to inform the Contour3D plot.

```
ContourPlot3D[Evaluate[
  electroStaticPotential[{1, -1}, {{-1, 0, 0}, {1, 0, 0}}, {x, y, z}]],
 {x, -5, 5}, {y, 0, 4}, {z, -4, 4}, Contours → {-Exp[-0.34],
   -Exp[-0.49], -Exp[-0.67], -Exp[-0.89], -Exp[-1.17], -Exp[-1.56],
   -Exp[-2.2], -Exp[-4.6], 0, Exp[-4.6], Exp[-2.2], Exp[-1.56],
   Exp[-1.17], Exp[-0.89], Exp[-0.67], Exp[-0.49], Exp[-0.34]},
 ContourStyle → Table[Hue[i/7, 1, 1, 0.5], {i, 0, 6}],
 Mesh → None, AxesLabel → Automatic]
```




7. Rectangle, Sin[z]. Let D: $0 \le x \le \frac{\pi}{2}$, $0 \le y \le 1$; D* the image of D under w = Sin[z]; and $\Phi^* = u^2 - v^2$. What is the corresponding potential Φ in D? What are its boundary values? Sketch D and D*.

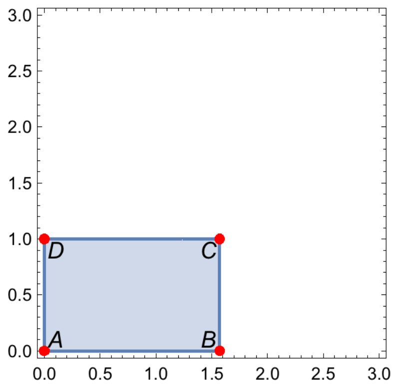
```
Clear["Global`*"]
```

```
d2 = ImplicitRegion[0 ≤ y ≤ 1 && 0 ≤ x ≤ π/2, {x, y}];
```

```
oliv = RGBColor[.572, .58, .109];
```

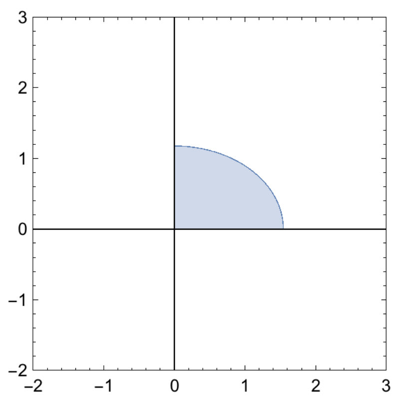
A plot of the z-plane could be helpful.

```
p3 = RegionPlot[{x, y} ∈ d2, {x, 0, 3}, {y, 0, 3},
  ImageSize → 200, Epilog → {{Red, PointSize[0.03], Point[{0, 0}]},
    {Red, PointSize[0.03], Point[{π/2, 0}]},
    {Red, PointSize[0.03], Point[{π/2, 1}]}, {Red, PointSize[0.03],
     Point[{0, 1}]}, {Text[Style[A, Medium], {0.1, 0.1}]},
    {Text[Style[B, Medium], {π/2 - 0.1, 0.1}]},
    {Text[Style[C, Medium], {π/2 - 0.1, 0.9}]},
    {Text[Style[D, Medium], {0.1, 0.9}]}}]
```




A plot of the z-plane mapped onto the w-plane by the problem's designated mapping function is below.

```
p7 = ParametricPlot[Through[{Re, Im}[Sin[(x + ⅈ y)]]], {x, y} ∈ d2,
  PlotRange → {{-2, 3}, {-2, 3}}, Frame → True, ImageSize → 200]
```



Putting the plots next to each other could make things easier to think about.

```
p1 = ListLinePlot[Table[{Re[Sin[(x + ⅈ y)]], Im[Sin[(x + ⅈ y)]]},
     {x, 0, π/2, 0.1}, {y, 0, 1, 0.1}], ImageSize → 300,
   PlotStyle → {{oliv, Thickness[0.004]}}];
```

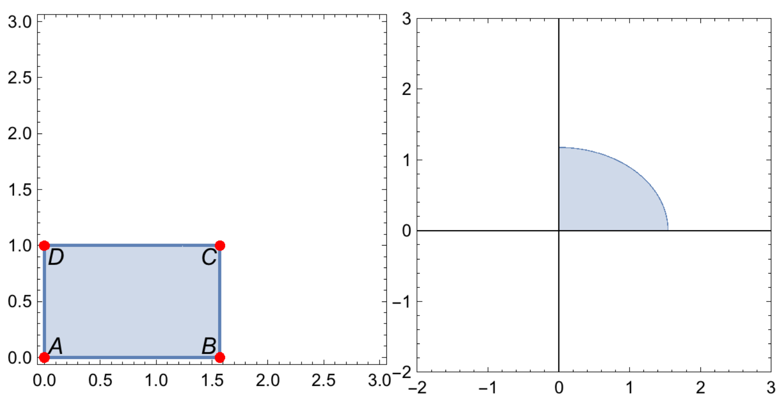
```
p4 = ListLinePlot[Table[Table[{Re[Sin[(x + ⅈ y)]], Im[Sin[(x + ⅈ y)]]},
     {x, 0, 1.59, 0.0365}], {y, 0.1, 1.0, 0.1}], ImageSize → 300,
   PlotStyle → {{oliv, Thickness[0.004]}, {oliv, Thickness[0.004]}}];
```

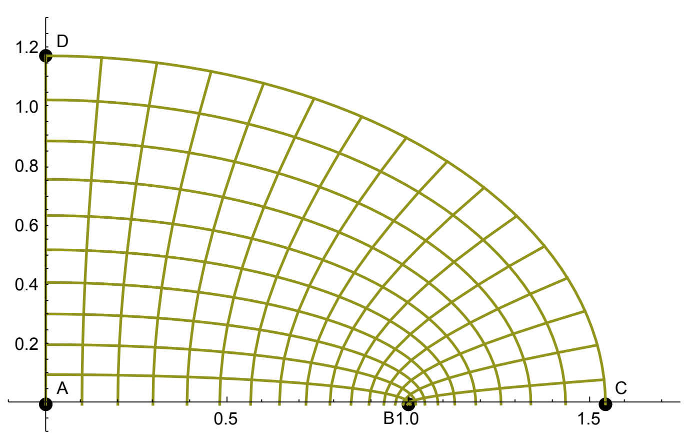
Here is something strange. Below is a Listplot with one point coordinate commented out. Yet in Mathematica 10.3 the final two mapped points show up simultaneously when the third input coordinate $\{\frac{\pi}{2}, 1\}$ is exposed. (The order of traverse was verified from the s.m.)

```
p6 = ListPlot[Table[{Re[Sin[(x + ⅈ y)]], Im[Sin[(x + ⅈ y)]]},
     {x, {0, π/2, π/2 (*,0*)}}, {y, {0, 0, 1 (*,1*)}}],
   PlotStyle → {{Black, PointSize[0.02]}},
   Epilog → {{Text["A", {0.05, 0.05}]}, {Text["B", {0.95, -0.05}]},
     {Text["C", {1.59, 0.05}]}, {Text["D", {0.05, 1.22}]}},
   PlotRange → {{-0.1, 1.75}, {-0.1, 1.3}}, ImageSize → 350];
```

Trivial comment. It is necessary that the first Show item be p6 for the point labels to show up.

```
Row[{p3, p7, Show[p6, p1, p4]}]
```




Now that I've done the plots, maybe I should take care of some math. The problem gives that $\Phi^* = u^2 - v^2$. I am prompted to recall that the sine conformal transform is covered on p.

750-751. There talk deals with the w-plane ellipse and provides the equations

```
u[x_, y_] = Sin[x] Cosh[y]
Cosh[y] Sin[x]
```

and

```
v[x_, y_] = Cos[x] Sinh[y]
Cos[x] Sinh[y]
```

I am given that in w

```
Φ*[u_, v_] = u[x, y]^2 - v[x, y]^2
Cosh[y]^2 Sin[x]^2 - Cos[x]^2 Sinh[y]^2
```

and left to ponder how the potential will be distributed along the mapped boundary. I need to consider the corner points

```
Table[{Cosh[y]^2 Sin[x]^2, -Cos[x]^2 Sinh[y]^2}, {x, {0, π/2}}, {y, {0, 1}}]
{{{0, 0}, {0, -Sinh[1]^2}}, {{1, 0}, {Cosh[1]^2, 0}}}
```

```
N[-Sinh[1]^2]
-1.3811
```

```
N[Cosh[1]^2]
2.3811
```

and looking at a grid may help organize

```
data = {{"point A", "x=0 y=0", 0, 0}, {"point B", "x=π/2 y=0", 1, 0},
  {"point C", "x=π/2 y=1", 2.3811, 0}, {"point D", "x=0 y=1", 0, -1.3811}}
{{point A, x=0 y=0, 0, 0}, {point B, x=π/2 y=0, 1, 0},
 {point C, x=π/2 y=1, 2.3811, 0}, {point D, x=0 y=1, 0, -1.3811}}
```

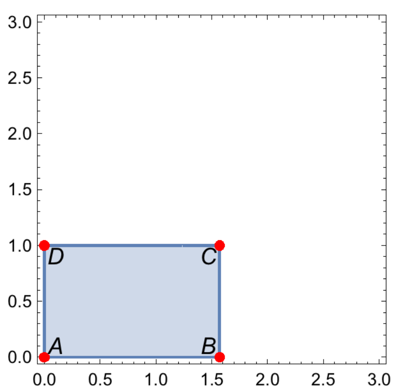
```
Text@Grid[Prepend[data,
   {" ", " ", "Cosh[y]^2Sin[x]^2", "-Cos[x]^2Sinh[y]^2"}], Frame → All]
```

| | | $\text{Cosh}[y]^2\text{Sin}[x]^2$ | $-\text{Cos}[x]^2\text{Sinh}[y]^2$ |
|---|---|---|---|
| point A | x=0 y=0 | 0 | 0 |
| point B | $x=\frac{\pi}{2}$ y=0 | 1 | 0 |
| point C | $x=\frac{\pi}{2}$ y=1 | 2.3811 | 0 |
| point D | x=0 y=1 | 0 | −1.3811 |

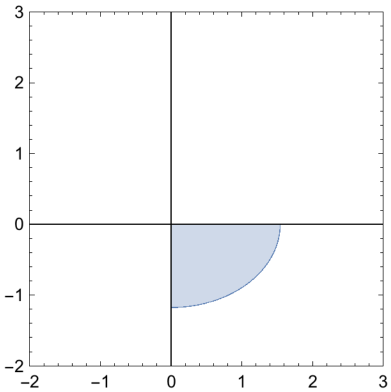
The functions $\Phi^*$ and $\Phi$ behave in the same way, which is what the mapping trick is all about. So considering curves (in w) or sides (in z), the potential will equal the sum of the right columns in my grid. I see that for A – B the potential increases from 0 to 1, then on B – C the potential increases from 1 to 2.38, then on C – D the potential decreases from 2.38 to –1.38, then on D – A the potential increases from –1.38 to 0.

9. Translation. What happens in problem 7 if we replace Sin[z] by Cos[z] = Sin[z+ $\frac{\pi}{2}$]?

```
Clear["Global`*"]

d2 = ImplicitRegion[0 ≤ y ≤ 1 && 0 ≤ x ≤ π/2, {x, y}];

p3 = RegionPlot[{x, y} ∈ d2, {x, 0, 3}, {y, 0, 3},
  ImageSize → 200, Epilog → {{Red, PointSize[0.03], Point[{0, 0}]},
    {Red, PointSize[0.03], Point[{π/2, 0}]},
    {Red, PointSize[0.03], Point[{π/2, 1}]}, {Red, PointSize[0.03],
     Point[{0, 1}]}, {Text[Style[A, Medium], {0.1, 0.1}]},
    {Text[Style[B, Medium], {π/2 - 0.1, 0.1}]},
    {Text[Style[C, Medium], {π/2 - 0.1, 0.9}]},
    {Text[Style[D, Medium], {0.1, 0.9}]}}]
```

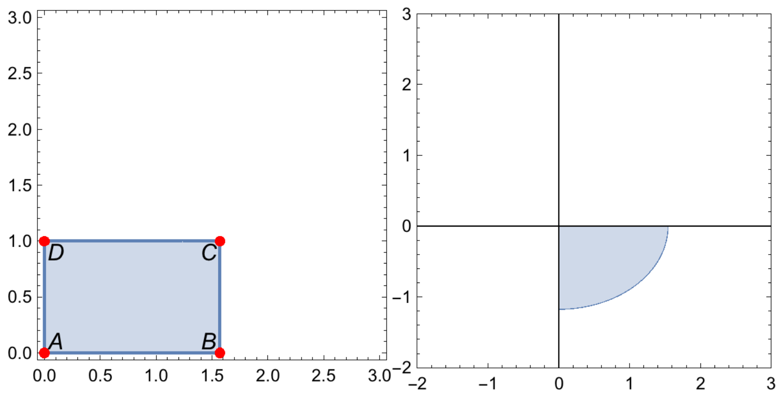


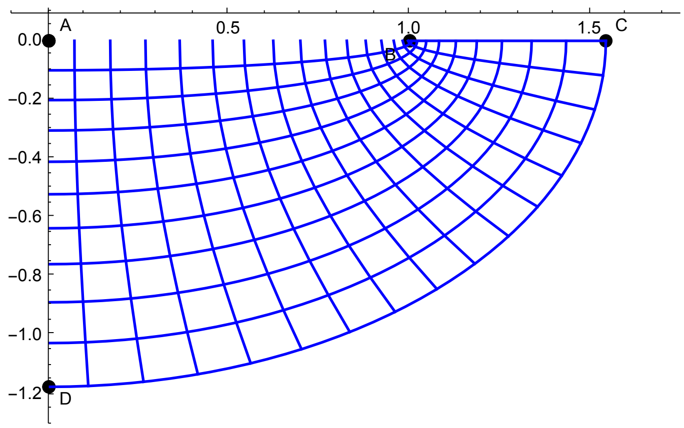
```
p7 = ParametricPlot[Through[{Re, Im}[Cos[(x + ⅈ y)]]], {x, y} ∈ d2,
  PlotRange → {{-2, 3}, {-2, 3}}, Frame → True, ImageSize → 200]
```




```
p1 = ListLinePlot[Table[{Re[Cos[(x + ⅈ y)]], Im[Cos[(x + ⅈ y)]]},
    {x, 0, π/2, 0.1}, {y, 0, 1, 0.1}], ImageSize → 300,
   PlotStyle → {{Blue, Thickness[0.004]}}];

p4 = ListLinePlot[Table[Table[{Re[Cos[(x + ⅈ y)]], Im[Cos[(x + ⅈ y)]]},
     {x, 0, 1.59, 0.0365}], {y, 0.1, 1.0, 0.1}], ImageSize → 300,
   PlotStyle → {{Blue, Thickness[0.004]}, {Blue, Thickness[0.004]}}];

p6 = ListPlot[Table[{Re[Cos[(x + ⅈ y)]], Im[Cos[(x + ⅈ y)]]},
    {x, {0, π/2, π/2(*,0*)}}, {y, {0, 0, 1(*,1*)}}],
   PlotStyle → {{Black, PointSize[0.02]}},
   Epilog → {{Text["A", {0.05, 0.05}]}, {Text["B", {0.95, -0.05}]},
     {Text["C", {1.59, 0.05}]}, {Text["D", {0.05, -1.22}]}},
   PlotRange → {{-0.1, 1.75}, {0.1, -1.3}}, ImageSize → 350];
```

```
Row[{p3, p7, Show[p6, p1, p4]}]
```




Down to this level, I do not see any differences between the Sin and Cos cases, except for the obvious change in quadrant. I expect the assignment of bands of potential would be analogous to the Sin case.

13. At $z = \pm 1$ in figure 405 the tangents to the equipotential lines as shown make equal angles. Why?

The potentials in the two semicircular plates are mirror images, therefore the equipotential curves should be and are drawn as mirror images, thus equal tangent angles. The text answer also mentions conformal attributes, which of course do include preserved angles, but to me it looks like the two semicircular plates in figure 405 are in a pre-mapped state.