1 - 7 General solution

Solve the following ODEs, showing the details of your work.

```
1. y''' + 3 y' + y = e^x - x - 1
```

```
Clear["Global`*"]
```

```
dapple = y'''[x] + 3 y''[x] + 3 y'[x] + y[x] == e^x - x - 1
apple = DSolve[dapple, y[x], x]
```

$$y[x] + 3\,y'[x] + 3\,y''[x] + y^{(3)}[x] == -1 + e^{x} - x$$

$$\left\{\left\{y[x] \to \frac{1}{8}\,(16 + e^{x} - 8\,x) + e^{-x}\,C[1] + e^{-x}\,x\,C[2] + e^{-x}\,x^{2}\,C[3]\right\}\right\}$$

```
Collect[apple, e^-x]
```

$$\left\{\left\{y[x] \to 2 + \frac{e^{x}}{8} - x + e^{-x}\left(C[1] + x\,C[2] + x^{2}\,C[3]\right)\right\}\right\}$$

1. Above: The expression matches the answer in the text.

$$3.\ \left(D^{4} + 10\,D^{2} + 9\,I\right)\,y = 6.5\,\mathrm{Sinh}[2\,x]$$

```
Clear["Global`*"]
```

```
prank = y''''[x] + 10 y''[x] + 9 y[x] == 6.5 Sinh[2 x]
dank = DSolve[prank, y[x], x]
```

$$9\,y[x] + 10\,y''[x] + y^{(4)}[x] == 6.5\,\mathrm{Sinh}[2\,x]$$

$$\{\{y[x] \to 1.\,C[3]\,\mathrm{Cos}[1.\,x] + 1.\,C[1]\,\mathrm{Cos}[3.\,x] + 1.\,C[4]\,\mathrm{Sin}[1.\,x] + 1.\,C[2]\,\mathrm{Sin}[3.\,x] + 0.1625\,\big(0. + 1.\,\mathrm{Cos}[1.\,x]^{2}\,\mathrm{Sinh}[2.\,x] - 0.384615\,\mathrm{Cos}[3.\,x]^{2}\,\mathrm{Sinh}[2.\,x] + 1.\,\mathrm{Sin}[1.\,x]^{2}\,\mathrm{Sinh}[2.\,x] - \left(0.384615 + 2.31296\times 10^{-17}\,i\right)\,\mathrm{Sin}[3.\,x]^{2}\,\mathrm{Sinh}[2.\,x]\big)\}\}$$

```
bank = Chop[dank, 10^-16]
```

$$\{\{y[x] \to 1.\,C[3]\,\mathrm{Cos}[1.\,x] + 1.\,C[1]\,\mathrm{Cos}[3.\,x] + 1.\,C[4]\,\mathrm{Sin}[1.\,x] + 1.\,C[2]\,\mathrm{Sin}[3.\,x] + 0.1625\,\big(1.\,\mathrm{Cos}[1.\,x]^{2}\,\mathrm{Sinh}[2.\,x] - 0.384615\,\mathrm{Cos}[3.\,x]^{2}\,\mathrm{Sinh}[2.\,x] + 1.\,\mathrm{Sin}[1.\,x]^{2}\,\mathrm{Sinh}[2.\,x] - 0.384615\,\mathrm{Sin}[3.\,x]^{2}\,\mathrm{Sinh}[2.\,x]\big)\}\}$$

```
sank = Simplify[bank]
```

$$\{\{y[x] \to 1.\,C[3]\,\mathrm{Cos}[(1. + 0.\,i)\,x] + 1.\,C[1]\,\mathrm{Cos}[(3. + 0.\,i)\,x] + 1.\,C[4]\,\mathrm{Sin}[(1. + 0.\,i)\,x] + 1.\,C[2]\,\mathrm{Sin}[(3. + 0.\,i)\,x] + 0.1\,\mathrm{Sinh}[2.\,x]\}\}$$

```
Chop[sank, 10^-15]
```

$$\{\{y[x] \to 1.\,C[3]\,\mathrm{Cos}[1.\,x] + 1.\,C[1]\,\mathrm{Cos}[3.\,x] + 1.\,C[4]\,\mathrm{Sin}[1.\,x] + 1.\,C[2]\,\mathrm{Sin}[3.\,x] + 0.1\,\mathrm{Sinh}[2.\,x]\}\}$$

1. Above: The expression matches the text’s answer.

5. $\left(x^3\,D^3 + x^2\,D^2 - 2\,x\,D + 2\,I\right)\,y = x^{-2}$

```
Clear["Global`*"]
```

```
plow = x^3 y'''[x] + x^2 y''[x] - 2 x y'[x] + 2 y[x] == x^-2
cow = DSolve[plow, y[x], x]
```

$$2\,y[x] - 2\,x\,y'[x] + x^2\,y''[x] + x^3\,y^{(3)}[x] == \frac{1}{x^2}$$

$$\left\{\left\{y[x] \to -\frac{1}{12\,x^2} + \frac{C[1]}{x} + x\,C[2] + x^2\,C[3]\right\}\right\}$$

1. Above: The answer matches the text’s.

7. $\left(D^3 - 9\,D^2 + 27\,D - 27\,I\right)\,y = 27\,\mathrm{Sin}[3\,x]$

```
boat = y'''[x] - 9 y''[x] + 27 y'[x] - 27 y[x] == 27 Sin[3 x]
coat = DSolve[boat, y[x], x]
```

$$-27\,y[x] + 27\,y'[x] - 9\,y''[x] + y^{(3)}[x] == 27\,\mathrm{Sin}[3\,x]$$

$$\left\{\left\{y[x] \to e^{3\,x}\,C[1] + e^{3\,x}\,x\,C[2] + e^{3\,x}\,x^2\,C[3] + \frac{1}{4}\,(-\mathrm{Cos}[3\,x] + \mathrm{Sin}[3\,x])\right\}\right\}$$

$$\text{goat} = \mathrm{Collect}\left[\text{coat},\, e^{3\,x}\right]$$

$$\left\{\left\{y[x] \to e^{3\,x}\,\left(C[1] + x\,C[2] + x^2\,C[3]\right) - \frac{1}{4}\,\mathrm{Cos}[3\,x] + \frac{1}{4}\,\mathrm{Sin}[3\,x]\right\}\right\}$$

1. Above: The answer matches the text’s.

8 - 13 Initial value problem
Solve the given IVP.

9. $y^{iv} + 5\,y'' + 4\,y = 90\,\mathrm{Sin}[x],\ y[0] = 1,$
$y'[0] = 2,\ y''[0] = -1,\ y'''[0] = -32$

```
Clear["Global`*"]
```

```
sing = {y''''[x] + 5 y''[x] + 4 y[x] == 90 Sin[4 x],
  y[0] == 1, y'[0] == 2, y''[0] == -1, y'''[0] == -32}
ring = DSolve[sing, y[x], x]
```

$$\{4\,y[x] + 5\,y''[x] + y^{(4)}[x] == 90\,\mathrm{Sin}[4\,x],\ y[0] == 1,\ y'[0] == 2,\ y''[0] == -1,\ y^{(3)}[0] == -32\}$$

$$\{\{y[x] \to \frac{1}{4}\big(4\,\mathrm{Cos}[x] + 80\,\mathrm{Cos}[x]^3\,\mathrm{Sin}[x] - 40\,\mathrm{Cos}[3\,x]\,\mathrm{Sin}[x] - 12\,\mathrm{Cos}[5\,x]\,\mathrm{Sin}[x] - 80\,\mathrm{Cos}[x]\,\mathrm{Sin}[x]^3 + 15\,\mathrm{Cos}[2\,x]\,\mathrm{Sin}[2\,x] + 20\,\mathrm{Cos}[2\,x]^3\,\mathrm{Sin}[2\,x] - 40\,\mathrm{Cos}[x]\,\mathrm{Sin}[3\,x] + 12\,\mathrm{Cos}[x]\,\mathrm{Sin}[5\,x] - 5\,\mathrm{Cos}[2\,x]\,\mathrm{Sin}[6\,x]\big)\}\}$$

```
thing = Simplify[ring]
{{y[x] → Cos[x] (1 - Sin[x] + Sin[3 x])}}
```

1. Below: To see what I need to make equal to $\frac{1}{2}\mathrm{Sin}[4\,x]$.

```
TrigExpand[-Cos[x] Sin[x] + Cos[x] Sin[3 x]]
```

$2\,\mathrm{Cos}[x]^3\,\mathrm{Sin}[x] - 2\,\mathrm{Cos}[x]\,\mathrm{Sin}[x]^3$

```
bling = thing /.
  (Cos[x] (1 - Sin[x] + Sin[3 x])) → (Cos[x] - Cos[x] Sin[x] + Cos[x] Sin[3 x])
{{y[x] → Cos[x] - Cos[x] Sin[x] + Cos[x] Sin[3 x]}}
```

---

2. Below: Put together some indents to use.

```
TrigExpand[Sin[2 x]]
2 Cos[x] Sin[x]
```

```
TrigExpand[Sin[3 x]]
```

$3\,\mathrm{Cos}[x]^2\,\mathrm{Sin}[x] - \mathrm{Sin}[x]^3$

```
TrigExpand[Cos[2 x]]
```

$\mathrm{Cos}[x]^2 - \mathrm{Sin}[x]^2$

3. Therefore $\mathrm{Sin}[4\,x] == 2\,\mathrm{Cos}[2\,x]\,\mathrm{Sin}[2\,x] == 2\left(\left(\mathrm{Cos}[x]^2 - \mathrm{Sin}[x]^2\right)(2\,\mathrm{Cos}[x]\,\mathrm{Sin}[x])\right)$

```
sling = bling /. (Sin[3 x]) → (3 Cos[x]^2 Sin[x] - Sin[x]^3)
```

$$\{\{y[x] \to \mathrm{Cos}[x] - \mathrm{Cos}[x]\,\mathrm{Sin}[x] + \mathrm{Cos}[x]\,\left(3\,\mathrm{Cos}[x]^2\,\mathrm{Sin}[x] - \mathrm{Sin}[x]^3\right)\}\}$$

```
string = sling /. (Cos[x] (3 Cos[x]^2 Sin[x] - Sin[x]^3)) ->
    (Cos[x] Sin[x] (3 Cos[x]^2 - Sin[x]^2))
```

$\{\{y[x] \to \text{Cos}[x] - \text{Cos}[x]\ \text{Sin}[x] + \text{Cos}[x]\ \text{Sin}[x]\ (3\ \text{Cos}[x]^2 - \text{Sin}[x]^2)\}\}$

```
zing = string /. (3 Cos[x]^2 - Sin[x]^2) → ( 2 Cos[x]^2 + Cos[2 x])
```

$\{\{y[x] \to \text{Cos}[x] - \text{Cos}[x]\ \text{Sin}[x] + \text{Cos}[x]\ (2\ \text{Cos}[x]^2 + \text{Cos}[2\ x])\ \text{Sin}[x]\}\}$

```
fling = zing /.
   (Cos[x] (2 Cos[x]^2 + Cos[2 x]) Sin[x]) → (1/2 Sin[2 x] (Cos[2 x] + 2 Cos[x]^2))
```

$\{\{y[x] \to \text{Cos}[x] - \text{Cos}[x]\ \text{Sin}[x] + \frac{1}{2}\ (2\ \text{Cos}[x]^2 + \text{Cos}[2\ x])\ \text{Sin}[2\ x]\}\}$

```
ping = fling /.
   (1/2 (2 Cos[x]^2 + Cos[2 x]) Sin[2 x]) → (1/2 (2 Cos[x]^2 Sin[2 x] + 1/2 Sin[4 x]))
```

$\{\{y[x] \to \text{Cos}[x] - \text{Cos}[x]\ \text{Sin}[x] + \frac{1}{2}\left(2\ \text{Cos}[x]^2\ \text{Sin}[2\ x] + \frac{1}{2}\ \text{Sin}[4\ x]\right)\}\}$

4. Above: This is looking worse and worse. Time to swing for the fence.

```
p1 = Cos[x]
```

Cos[x]

```
p2 = Simplify[-Cos[x] Sin[x] + 1/2 (2 Cos[x]^2 Sin[2 x] + 1/2 Sin[4 x])]
```

$\frac{1}{2}\ \text{Sin}[4\ x]$

5. Above: A miraculous hit.

```
out = p1 + p2
```

$\text{Cos}[x] + \frac{1}{2}\ \text{Sin}[4\ x]$

6. Above: The answer does match the text answer.

11. $(D^3 - 2\ D^2 - 3\ D)\ y = 74\ e^{-3\ x}\ \text{Sin}[x]$,
$y[0] = -1.4,\ y'[0] = 3.2,\ y''[0] = -5.2$

```
Clear["Global`*"]
```

```
alt = {y'''[x] - 2 y''[x] - 3 y'[x] == 74 e^(-3 x) Sin[x],
   y[0] == -1.4, y'[0] == 3.2, y''[0] == -5.2}
kalt = DSolve[alt, y[x], x]
```

$\{-3\, y'[x] - 2\, y''[x] + y^{(3)}[x] == 74\, e^{-3x}\, \mathrm{Sin}[x],$
$y[0] == -1.4,\ y'[0] == 3.2,\ y''[0] == -5.2\}$

$\left\{\left\{y[x] \to -\frac{1}{5} e^{-3x} (7\, \mathrm{Cos}[x] + 5\, \mathrm{Sin}[x])\right\}\right\}$

$\mathrm{salt} = \mathrm{kalt}\ /.\ \left(-\frac{1}{5} e^{-3x} (7\, \mathrm{Cos}[x] + 5\, \mathrm{Sin}[x])\right) \to \left(e^{-3x} \left(-\frac{7}{5} \mathrm{Cos}[x] - \frac{5}{5} \mathrm{Sin}[x]\right)\right)$

$\left\{\left\{y[x] \to e^{-3x} \left(-\frac{7\, \mathrm{Cos}[x]}{5} - \mathrm{Sin}[x]\right)\right\}\right\}$

1. Above: Substitution by hand results in the text's answer.

13. $(D^3 - 4\, D)\, y = 10\, \mathrm{Cos}[x] + 5\, \mathrm{Sin}[x],\ y[0] = 3,\ y'[0] = -2,\ y''[0] = -1$

```
Clear["Global`*"]
```

```
rog = {y'''[x] - 4 y'[x] == 10 Cos[x] + 5 Sin[x],
   y[0] == 3, y'[0] == -2, y''[0] == -1}
dog = DSolve[rog, y[x], x]
```

$\{-4\, y'[x] + y^{(3)}[x] == 10\, \mathrm{Cos}[x] + 5\, \mathrm{Sin}[x],\ y[0] == 3,\ y'[0] == -2,\ y''[0] == -1\}$

$\{\{y[x] \to 2 + \mathrm{Cos}[x] - 2\, \mathrm{Sin}[x]\}\}$

1. Above: The answer matches the text's.