

1 - 9 Further ODEs reducible to Bessel's ODE

Find a general solution in terms of J_ν and Y_ν . Indicate whether you could also use $J_{-\nu}$ instead of Y_ν . Use the indicated substitution.

$$1. \quad x^2 y'' + x y' + (x^2 - 16) y = 0$$

```
Clear["Global`*"]
```

```
e1 = {x^2 y''[x] + x y'[x] + (x^2 - 16) y[x] == 0}
```

```
e2 = DSolve[e1, y, x]
```

```
{(-16 + x^2) y[x] + x y'[x] + x^2 y''[x] == 0}
```

```
{ {y -> Function[{x}, BesselJ[4, x] C[1] + BesselY[4, x] C[2]] }
```

```
e1 /. e2 // FullSimplify
```

```
{ {True} }
```

The above answer matches the text's. I believe that **FullSimplify** is needed to check **DSolve** in this case because Bessels are special functions.

$$3. \quad 9 x^2 y'' + 9 x y' + (36 x^4 - 16) y = 0 \quad (x^2 = z)$$

```
Clear["Global`*"]
```

```
e1 = {9 x^2 y''[x] + 9 x y'[x] + (36 x^4 - 16) y[x] == 0}
```

```
e2 = DSolve[e1, y[x], x, Assumptions -> x^2 -> z]
```

```
{(-16 + 36 x^4) y[x] + 9 x y'[x] + 9 x^2 y''[x] == 0}
```

```
{ {y[x] -> BesselJ[-2/3, x^2] C[1] Gamma[1/3] + BesselJ[2/3, x^2] C[2] Gamma[5/3] } }
```

```
In[7]:= PossibleZeroQ[BesselJ[-2/3, x^2] Gamma[1/3] - BesselY[2/3, x^2] ]
```

```
Out[7]= False
```

It appears that the yellow answer above does not agree with the text answer.

$$5. \quad 4 x y'' + 4 y' + y = 0 \quad (\sqrt{x} = z)$$

```
Clear["Global`*"]
```

```
e1 = {4 x y''[x] + 4 y'[x] + y[x] == 0}
e2 = DSolve[e1, y[x], x, Assumptions -> Sqrt[x] -> z]
{y[x] + 4 y'[x] + 4 x y''[x] == 0}

{{y[x] -> BesselJ[0, Sqrt[x]] C[1] + 2 BesselY[0, Sqrt[x]] C[2]}}
```

The answer above agrees with the text answer, as I interpret it. It appears that C[2] and the 2 factor in the second term need to be combined.

$$7. \quad y'' + k^2 x^2 y = 0 \quad \left(y = u \sqrt{x}, \quad \frac{1}{2} k x^2 = z \right)$$

```
Clear["Global`*"]
e1 = {y''[x] + k^2 x^2 y[x] == 0}
e2 = DSolve[e1, y[x], x, Assumptions -> {y[x] -> u Sqrt[x], k x^2 -> z}]
{k^2 x^2 y[x] + y''[x] == 0}
```

$$\left\{ \left\{ y[x] \rightarrow C[1] \text{ParabolicCylinderD}\left[-\frac{1}{2}, (-1)^{1/4} \sqrt{2} \sqrt{k} x\right] + \right. \right. \\ \left. \left. C[2] \text{ParabolicCylinderD}\left[-\frac{1}{2}, (-1)^{3/4} \sqrt{2} \sqrt{k} x\right] \right\} \right\}$$

```
PossibleZeroQ[ParabolicCylinderD[-1/2, (-1)^(1/4) Sqrt[2] Sqrt[k] x] +
  ParabolicCylinderD[-1/2, (-1)^(3/4) Sqrt[2] Sqrt[k] x] -
  Sqrt[x] BesselJ[1/4, 1/2 k x^2] - BesselY[1/4, 1/2 k x^2]]
False
```

It appears that Mathematica's answer does not equal that of the text.

$$9. \quad x y'' - 5 y' + x y = 0 \quad (y = x^3 u)$$

```
Clear["Global`*"]
e1 = {x y''[x] - 5 y'[x] + x y[x] == 0}
e2 = DSolve[e1, y, x, Assumptions -> y[x] -> x^3 u]
{x y[x] - 5 y'[x] + x y''[x] == 0}

{{y -> Function[{x}, x^3 BesselJ[3, x] C[1] + x^3 BesselY[3, x] C[2]]}}
```

```
e1 /. e2 // FullSimplify
{{True}}
```

The above answer agrees with the text's.

11 - 15 Hankel and modified Bessel functions

11. Hankel functions. Show that the Hankel functions (10) form a basis of solutions of Bessel's equation for any ν .

```
Clear["Global`*"]
```

$$H_{\nu}^{(1)}(x) = J_{\nu}(x) + iY_{\nu}(x)$$

$$H_{\nu}^{(2)}(x) = J_{\nu}(x) - iY_{\nu}(x)$$

$$e1 = c1 (j\nu + i y\nu) + c2 (j\nu - i y\nu) == 0$$

$$c2 (j\nu - i y\nu) + c1 (j\nu + i y\nu) == 0$$

Above: inserted definitions. It is necessary to change the symbols, I suppose Mathematica recognized the traditional forms of the Bessels.

```
e2 = Expand[e1]
```

$$c1 j\nu + c2 j\nu + i c1 y\nu - i c2 y\nu == 0$$

```
e3 = Collect[e2, {j\nu, y\nu}]
```

$$(c1 + c2) j\nu + (i c1 - i c2) y\nu == 0$$

$$e4 = e3 /. (i c1 - i c2) \rightarrow i (c1 - c2)$$

$$(c1 + c2) j\nu + i (c1 - c2) y\nu == 0$$

$j\nu$ and $y\nu$ are known to be linearly independent. (Multiplying one of them by i will not change their linear independence.) That means that the above equation can only be true if $(c1 + c2)$ and $(c1 - c2)$ are both zero.

```
Solve[(c1 + c2) == 0 && (c1 - c2) == 0, {c1, c2}]
```

```
{{c1 -> 0, c2 -> 0}}
```

The above tells me that the two expressions, which were definitions of the Hankel functions, are linearly independent.