2 - 10 ODEs reducible to Bessel's ODE

This is just a sample of such ODEs; some more follow in the next problem set. Find a general solution in terms of J_ν and $J_{-\nu}$ or indicate when this is not possible. Use the indicated substitutions.

3. $x y'' + y' + \frac{1}{4} y = 0 \quad \left(\sqrt{x} = z\right)$

```
Clear["Global`*"]
```

```
e1 = {x y''[x] + y'[x] + 1/4 y[x] == 0}
e2 = DSolve[e1, y[x], x, Assumptions → {√x → z}]
```

$\left\{\frac{y[x]}{4} + y'[x] + x\, y''[x] == 0\right\}$

$\{\{y[x] \to \text{BesselJ}[0, \sqrt{x}]\, C[1] + 2\, \text{BesselY}[0, \sqrt{x}]\, C[2]\}\}$

The yellow expression above seems to include the text answer, but also a BesselY, which the text answer does not mention.

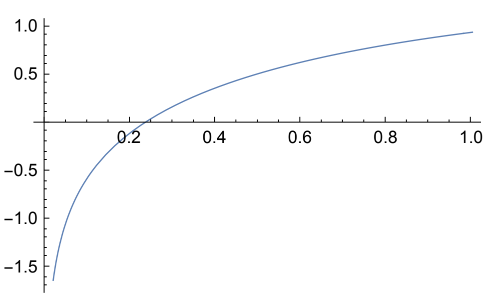
```
e2[[1, 1, 2]]
```

$\text{BesselJ}[0, \sqrt{x}]\, C[1] + 2\, \text{BesselY}[0, \sqrt{x}]\, C[2]$

```
gad = e2[[1, 1, 2]] /. {C[1] → 1, C[2] → 1}
```

$\text{BesselJ}[0, \sqrt{x}] + 2\, \text{BesselY}[0, \sqrt{x}]$

```
Plot[gad, {x, 0, 1}, PlotRange → Automatic,
 PlotStyle → Thickness[0.003], ImageSize → 250]
```



5. Two - parameter ODE
$x^2 y'' + x y' + \left(\lambda^2 x^2 - \nu^2\right) y = 0 \quad (\lambda x = z)$

```
Clear["Global`*"]
```

```
e1 = {x² y''[x] + x y'[x] + (λ² x² - ν²) y[x] == 0}
```

$\{(x^2 \lambda^2 - \nu^2)\, y[x] + x\, y'[x] + x^2\, y''[x] == 0\}$

```
e2 = DSolve[e1, y, x, Assumptions → {λ x → z}]
```

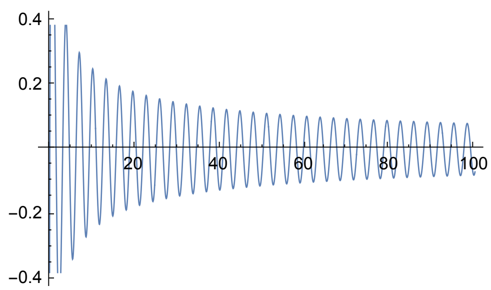
```
{{y → Function[{x}, BesselJ[ν, x λ] C[1] + BesselY[ν, x λ] C[2]]}}
```

Maybe including the assumptions keyword prevents the solution from being checked. The yellow cell above contains a Bessel function of the first kind and one of the second kind. The text answer contains the same Bessel of the first kind, but the other half is expressed as one independent from the first, not, that I can see, that it is one of the second kind.

```
glif = e2[[1, 1]] /. {C[1] → 1, C[2] → 1, λ → 2, ν → 0}
```

```
y[x] → BesselJ[0, 2 x] + BesselY[0, 2 x]
```

```
Plot[y[x] /. glif, {x, -100, 100}, PlotRange → Automatic,
 PlotStyle → Thickness[0.003], ImageSize → 250]
```



7. $x^2 y'' + x y' + \frac{1}{4}(x^2 - 1)\, y = 0 \quad (x = 2z)$

```
Clear["Global`*"]
```

$e1 = \{x^2 y''[x] + x y'[x] + \frac{1}{4}(x^2 - 1)\, y[x] == 0\}$

$\{\frac{1}{4}(-1 + x^2)\, y[x] + x\, y'[x] + x^2\, y''[x] == 0\}$

```
e2 = DSolve[e1, y[x], x, Assumptions → {x → 2 z, x ∈ Reals}]
```

$\{\{y[x] \to \frac{e^{-\frac{i x}{2}} C[1]}{\sqrt{x}} - \frac{i\, e^{\frac{i x}{2}} C[2]}{\sqrt{x}}\}\}$

```
e3 = ExpToTrig[e2]
```

$\{\{y[x] \to \frac{C[1] \cos[\frac{x}{2}]}{\sqrt{x}} - \frac{i\, C[2] \cos[\frac{x}{2}]}{\sqrt{x}} - \frac{i\, C[1] \sin[\frac{x}{2}]}{\sqrt{x}} + \frac{C[2] \sin[\frac{x}{2}]}{\sqrt{x}}\}\}$

Mathematica gives the right answer, but includes imaginary elements, which, looking at the text answer, are not necessary.

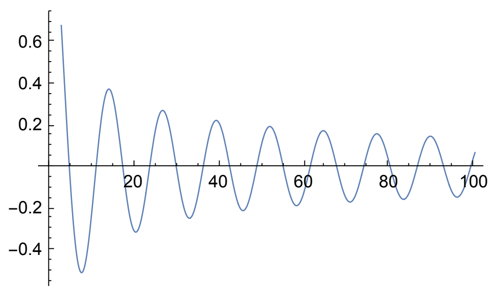
$$e4 = e3\ /.\ \left\{-\frac{i\, C[2]\, Cos\left[\frac{x}{2}\right]}{\sqrt{x}} \to 0,\ -\frac{i\, C[1]\, Sin\left[\frac{x}{2}\right]}{\sqrt{x}} \to 0\right\}$$

$$\left\{\left\{y[x] \to \frac{C[1]\, Cos\left[\frac{x}{2}\right]}{\sqrt{x}} + \frac{C[2]\, Sin\left[\frac{x}{2}\right]}{\sqrt{x}}\right\}\right\}$$

```
e5 = e4 /. {C[1] → 1, C[2] → 1}
```

$$\left\{\left\{y[x] \to \frac{Cos\left[\frac{x}{2}\right]}{\sqrt{x}} + \frac{Sin\left[\frac{x}{2}\right]}{\sqrt{x}}\right\}\right\}$$

```
Plot[y[x] /. e5, {x, -100, 100}, PlotRange → Automatic,
 PlotStyle → Thickness[0.003], ImageSize → 250]
```



9. $x y'' + (2\nu + 1) y' + x y = 0 \quad (y = x^{-\nu} u)$

```
Clear["Global`*"]
```

```
e1 = {x y''[x] + (2 ν + 1) y'[x] + x y[x] == 0}
e2 = DSolve[e1, y[x], x, Assumptions → {y[x] → x^-ν u}]
```

$\{x\, y[x] + (1 + 2\nu)\, y'[x] + x\, y''[x] == 0\}$

$\{\{y[x] \to x^{-\nu}\, BesselJ[\nu, x]\, C[1] + x^{-\nu}\, BesselY[\nu, x]\, C[2]\}\}$

The form of the above answer is not exactly the same as the text answer. I doubt they are equal.

```
e3 = e2 /. {C[1] → 1, C[2] → 1}
```

$\{\{y[x] \to x^{-\nu}\, BesselJ[\nu, x] + x^{-\nu}\, BesselY[\nu, x]\}\}$

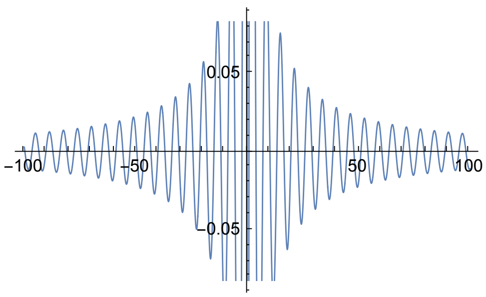
```
e4 = Simplify[e3, Assumptions → ν ∉ Integers]
```

$\{\{y[x] \to x^{-\nu}\, (BesselJ[\nu, x] + BesselY[\nu, x])\}\}$

```
e5 = e4 /. ν → .5
```

$$\left\{\left\{y[x] \to \frac{1}{x^{0.5}}\left(\frac{1}{\sqrt{x}}0.7978845608028654\,\mathrm{Cos}[1.5708 - x] - \frac{1}{\sqrt{x}}0.7978845608028654\,\mathrm{Sin}[1.5708 - x]\right)\right\}\right\}$$

```
Plot[y[x] /. e5, {x, -100, 100}, PlotRange → Automatic,
 PlotStyle → Thickness[0.003], ImageSize → 250]
```



19 - 25 Application of (21): deriviatives, integrals
Use the powerful formulas (21) to do problems 19 - 25.

23. Integration. Show that Integrate$[x^2 J_0, x] = x^2 J_1[x] + x J_0[x] -$ Integrate$[J_0[x], x]$. (The last integral is nonelementary; tables exist, e.g. in reference [A13] in appendix 1.)