Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

6 - 11 General Solution

Find a general solution of the ODE y”+ω^2y = r(t)
with r(t) as given below.

6. r(t) = sin αt + sin βt, $\omega^2 \neq \alpha^2, \beta^2$

```
Clear["Global`*"]
```

```
r[t_] := Sin[αt] + Sin[βt] /; {{ω² ≠ α²}, {ω² ≠ β²}}
```

```
DSolve[y''[t] + ω² y[t] == r[t], y[t], t]
```

$$\left\{\left\{y[t] \to C[1]\,Cos[t\,\omega] + Cos[t\,\omega]\int_1^t -\frac{r[K[1]]\,Sin[\omega\,K[1]]}{\omega}\,dK[1] + C[2]\,Sin[t\,\omega] + \left(\int_1^t \frac{Cos[\omega\,K[2]]\,r[K[2]]}{\omega}\,dK[2]\right)Sin[t\,\omega]\right\}\right\}$$

An even-numbered problem. Is the answer correct? Can’t check it.

7. r(t) = sin t, ω = 0.5, 0.9, 1.1, 1.5, 10

```
Clear["Global`*"]
```

```
r[t_] := Sin[t]
eq1 = DSolve[y''[t] + ω² y[t] == r[t], y[t], t]
```

$$\left\{\left\{y[t] \to C[1]\,Cos[t\,\omega] + C[2]\,Sin[t\,\omega] + \frac{Cos[t\,\omega]^2\,Sin[t] + Sin[t]\,Sin[t\,\omega]^2}{-1+\omega^2}\right\}\right\}$$

$$eq2 = eq1\ /.\ \frac{1}{-1+\omega^2}\left(Cos[t\,\omega]^2\,Sin[t] + Sin[t]\,Sin[t\,\omega]^2\right) \to \frac{Sin[t]}{-1+\omega^2}$$

$$\left\{\left\{y[t] \to C[1]\,Cos[t\,\omega] + \frac{Sin[t]}{-1+\omega^2} + C[2]\,Sin[t\,\omega]\right\}\right\}$$

above: making a trig identity substitution by hand to conform the green cell to the text answer. The sequence of ω s makes it look like a table could be built, but not of the solution function, because the arbitrary constants blur everything. Instead the text focuses on the particle $\frac{1}{-1+\omega^2}$, listing the calculated values for each ω.

$$ome[\omega\_] = \frac{1}{-1+\omega^2}$$

$$\frac{1}{-1+\omega^2}$$

```
m = Table[ome[ω], {ω, {0.5, 0.9, 1.1, 1.5, 10}}]
```

$\{-1.33333, -5.26316, 4.7619, 0.8, \frac{1}{99}\}$

```
N[TableForm[{{0.5, -1.3333333333333333`}, {0.9, -5.263157894736843`},
   {1.1, 4.761904761904757`}, {1.5, 0.8`}, {10, 1/99}},
  TableHeadings → {{}, {"ω", "m[ω]"}}]]
```

ω	m[ω]
0.5	-1.33333
0.9	-5.26316
1.1	4.7619
1.5	0.8
10.	0.010101

The above matches the text, though the table construction seemed more time-consuming than profitable.

11. $r(t) = \begin{cases} -1 & \text{if } -\pi < t < 0 \\ 1 & \text{if } 0 < t < \pi \end{cases} \quad |\omega| \neq 1, 3, 5, \ldots$

```
Clear["Global`*"]
```

```
r[t_] = Piecewise[{{-1, -π < t < 0}, {1, 0 < t < π}}]
```

$\begin{cases} -1 & -\pi < t < 0 \\ 1 & 0 < t < \pi \\ 0 & \text{True} \end{cases}$

First r[t] is considered by finding its Fourier series.

```
e3 = ExpToTrig[
  FourierSeries[Piecewise[{{-1, -π < t < 0}, {1, 0 < t < π}}], t, 6]]
```

$\frac{4 \text{Sin}[t]}{\pi} + \frac{4 \text{Sin}[3 t]}{3 \pi} + \frac{4 \text{Sin}[5 t]}{5 \pi}$

The above doesn' t look bad at all. The general term is $\frac{4}{n \pi}$ Sin[n t],
with n = 1, 3, 5 ... In the text example,
the general term of the Fourier series is set equal to the ODE without apology,
so I will do it too. At this point in the problem,
I am supposed to switch over to considering the ODE,
including that series general term for r[t].

```
eq1 = FullSimplify[DSolve[y''[t] + ω^2 y[t] == 4/(n π) Sin[n t], y[t], t]]
```

$$\left\{\left\{y[t] \to C[1] \operatorname{Cos}[t\,\omega] - \frac{4\,\operatorname{Sin}[n\,t]}{n^3\,\pi - n\,\pi\,\omega^2} + C[2]\,\operatorname{Sin}[t\,\omega]\right\}\right\}$$

```
eq11 = eq1 /. n → 1
```

$$\left\{\left\{y[t] \to C[1] \operatorname{Cos}[t\,\omega] - \frac{4\,\operatorname{Sin}[t]}{\pi - \pi\,\omega^2} + C[2]\,\operatorname{Sin}[t\,\omega]\right\}\right\}$$

```
eq13 = eq1 /. n → 3
```

$$\left\{\left\{y[t] \to C[1] \operatorname{Cos}[t\,\omega] - \frac{4\,\operatorname{Sin}[3\,t]}{27\,\pi - 3\,\pi\,\omega^2} + C[2]\,\operatorname{Sin}[t\,\omega]\right\}\right\}$$

```
eq15 = eq1 /. n → 5
```

$$\left\{\left\{y[t] \to C[1] \operatorname{Cos}[t\,\omega] - \frac{4\,\operatorname{Sin}[5\,t]}{125\,\pi - 5\,\pi\,\omega^2} + C[2]\,\operatorname{Sin}[t\,\omega]\right\}\right\}$$

This seemed to be going so well. But I could not (quite) get to the text answer. The yellow cells should show the text answer, but the central term of the text answer presents the model $\frac{4}{\pi}\,\frac{\operatorname{Sin}[n\,t]}{\omega^2-(4\,n-1)^2}$, instead of the yellow $\frac{4}{n\,\pi}\,\frac{\operatorname{Sin}[n\,t]}{n^2-\omega^2}$, and I don't understand this result. I checked the integration in Symbolab, and it agreed with *Mathematica* as far as the integration is concerned. Certainly it is possible the text answer is correct.

13 - 16 Steady-State Damped Oscillations

Find the steady-state oscillations of y”+cy’+y = r(t) with c>0 and r(t) as given. Note that the spring constant is k=1. Show the details. In probs. 14 - 16 sketch r(t).

13. $r(t) = \sum_{n=1}^{N} (a_n \cos nt + b_n \sin nt)$

```
Clear["Global`*"]
```

Here r[t] is already a series. `r[t_] = ∑_{n=1}^{N} (a Cos[n t] + b Sin[n t])`. Using a method seen in the solutions manual, I will drop the subscripts of the coefficients a and b. (This problem is being solved after finishing problem 15, for which solutions manual assistance was available.) I will consider r[t] to be a single term of the series.

```
r[t_] = a Cos[n t] + b Sin[n t]
a Cos[n t] + b Sin[n t]
```

```
r'[t]
b n Cos[n t] - a n Sin[n t]
```

```
r''[t]
```

$$-a\,n^2\,\text{Cos}[n\,t]-b\,n^2\,\text{Sin}[n\,t]$$

```
partic = r''[t] + c r'[t] + r[t]
```

$$a\,\text{Cos}[n\,t]-a\,n^2\,\text{Cos}[n\,t]+b\,\text{Sin}[n\,t]-b\,n^2\,\text{Sin}[n\,t]+c\,(b\,n\,\text{Cos}[n\,t]-a\,n\,\text{Sin}[n\,t])$$

```
eq1 = Simplify[partic]
```

$$\left(a+b\,c\,n-a\,n^2\right)\text{Cos}[n\,t]+\left(b-a\,c\,n-b\,n^2\right)\text{Sin}[n\,t]$$

For this problem, evidently the RHS will have both sine and cosine terms. The value of N is unknown, but it could encompass any number of 2π cycles. The coefficients must keep the same ratios at all points of the trig circle, so I take the guess that A_n will be solved when the function is at zero (cosine function is max), and B_n will be solved
when the function is at $\pi/2$ (sine function is max). So eq2 will be for A_n:

```
eq2 = Solve[{a + b*c*n - a*n^2 == 1, b - a*c*n - b*n^2 == 0}, {a, b}]
```

$$\left\{\left\{a\to-\frac{-1+n^2}{1-2\,n^2+c^2\,n^2+n^4},\ b\to\frac{c\,n}{1-2\,n^2+c^2\,n^2+n^4}\right\}\right\}$$

To assemble A_n I suppose that all I need to do is multiply the numerators by the relevant coefficients and add these two together. (I can already check the 'D_n' value, the denominator, with the text and confirm that it agrees.)

$$\text{bigA}=\text{Simplify}\left[-\frac{\left(-1+n^2\right)\text{asubn}}{1-2\,n^2+c^2\,n^2+n^4}+\frac{(c\,n)\,\text{bsubn}}{1-2\,n^2+c^2\,n^2+n^4}\right]$$

$$\frac{\text{asubn}+\text{bsubn}\,c\,n-\text{asubn}\,n^2}{1+\left(-2+c^2\right)n^2+n^4}$$

The method works for A_n above, which agrees with the text. Now to try to figure out B_n, which I predict must come into alignment at trig $\pi/2$:

```
eq3 = Solve[{a + b*c*n - a*n^2 == 0, b - a*c*n - b*n^2 == -1}, {a, b}]
```

$$\left\{\left\{a\to\frac{c\,n}{1-2\,n^2+c^2\,n^2+n^4},\ b\to-\frac{1-n^2}{1-2\,n^2+c^2\,n^2+n^4}\right\}\right\}$$

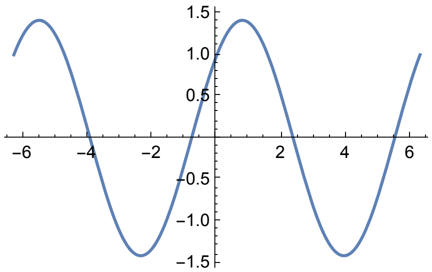
$$\text{BigB}=\text{Simplify}\left[\frac{(c\,n)\,\text{asubn}}{1-2\,n^2+c^2\,n^2+n^4}-\frac{\left(-1+n^2\right)\text{bsubn}}{1-2\,n^2+c^2\,n^2+n^4}\right]$$

$$\frac{\text{bsubn}+\text{asubn}\,c\,n-\text{bsubn}\,n^2}{1+\left(-2+c^2\right)n^2+n^4}$$

The method works for B_n too, *except* that in order to get the sign of a_n to agree with the text, it was necessary to choose -$\pi/2$ as the point of evaluation, so that the a_n part of the B_n

ensemble could be positive in sign. I don't know how to interpret that requirement.

```
Plot[Cos[t] + Sin[t], {t, -2 π, 2 π}]
```



The plot (above) does not look quite as expected. I feel I should emphasize that the described solution method is largely speculation.

15. r(t) = $t(\pi^2 - t^2)$ if -π<t<π , and r(t+2π) = r(t)

This problem is covered in the solutions manual. The observation, made there and visible from problem description, is that the function r[t] is odd and that the function's cycle is 2π. At this point I check the Fourier series.

```
Clear["Global`*"]
```

```
eq1 = FourierSeries[t (π² - t²), t, 1]
```

$6\,i\,e^{-i\,t} - 6\,i\,e^{i\,t}$

```
eq2 = ExpToTrig[6 i e^(-i t) - 6 i e^(i t)]
```

```
12 Sin[t]
```

So at this point I know the form of the output series. No cosine term. I don't take the '12' too seriously, it is still subject to some variation.

The s.m. refers to the method of finding a particular solution in Example 1 on p. 493, and sees it as y"+cy'+y=b_n sin nt. Here the s.m. makes reference to Example 1 on p. 493 of the text, where in a similar situation the y_p is set to y= A cos nt + B sin nt. The motivation for this is an entry in Table 2.1, "*Method of Undetermined Coefficients*, where, upon finding r[t] equal to k sin ωx, a preliminary choice for $y_p(x)$ is taken as K cos ωx +M sin ωx. So at this point I have [1]: y=A cos nt + B sin nt, and I go on to assign [2]: y'=-A sin nt + B cos nt, and also [3]: y"=-A cos nt -B sin nt.

```
partic = (y''[t] + c y'[t] + y[t])
```

```
y[t] + c y′[t] + y″[t]
```

partic is the LHS

```
r[t] = A Cos[n t] + B Sin[n t] +
   c (-n A Sin[n t] + n B Cos[n t]) - n^2 A Cos[n t] - n^2 B Sin[n t]
```

$$A\,\mathrm{Cos}[n\,t] - A\,n^2\,\mathrm{Cos}[n\,t] + B\,\mathrm{Sin}[n\,t] - B\,n^2\,\mathrm{Sin}[n\,t] + c\,(B\,n\,\mathrm{Cos}[n\,t] - A\,n\,\mathrm{Sin}[n\,t])$$

r[t] is the consolidation of plugging values of the 3 equations into LHS and adding them up.

```
Simplify[r[t]]
```

$$\left(A + B\,c\,n - A\,n^2\right)\mathrm{Cos}[n\,t] + \left(B - A\,c\,n - B\,n^2\right)\mathrm{Sin}[n\,t]$$

Now it is time to solve for coefficients of the r[t] complex. Final coefficient of cos must be zero (since it doesn't appear in final r) and final coefficient of sin must be b_n. As for n, it can vary in series fashion. It is necessary to humor Mathematica a bit, as for instance not using variables beginning with captitals, and, for just this once, eschewing subscripts (m is standing in for b_n);

```
eq3 = Solve[{a + b * c * n - a * n^2 == 0, b - a * c * n - b * n^2 == m}, {a, b}]
```

$$\left\{\left\{a \to -\frac{c\,m\,n}{1 - 2\,n^2 + c^2\,n^2 + n^4},\; b \to -\frac{m\left(-1 + n^2\right)}{1 - 2\,n^2 + c^2\,n^2 + n^4}\right\}\right\}$$

Solve does the solve thing, and sets the denominator to the correct value of D_n. In the cell below, it will be done in the determinant way.

$$\mathrm{dee} = \mathrm{Det}\left[\begin{pmatrix} 1 - n^2 & c\,n \\ -c\,n & 1 - n^2 \end{pmatrix}\right]$$

$$1 - 2\,n^2 + c^2\,n^2 + n^4$$

The s.m. now goes on to find A and B, using determinants, but will it thereby find what **Solve** came up with above? The current step is to find b_n, which Mathematica has not yet found, and which it cannot find by modifying eq3 for the search. But the s.m. goes back to a table on page 487, where is says that an odd function with period 2π should follow the formula $b_n = \frac{2}{\pi}\int_0^{2\pi} f(x) \sin nx\, dx$ and $n = 1, 2, \ldots$ Okay, I'll follow.

$$\mathrm{bn} = \frac{2}{\pi}\,\mathrm{Integrate}\left[t\left(\pi^2 - t^2\right)\mathrm{Sin}[n\,t],\ \{t, 0, \pi\}\right]$$

$$\frac{1}{n^4\,\pi}2\left(-6\,n\,\pi\,\mathrm{Cos}[n\,\pi] - 2\left(-3 + n^2\,\pi^2\right)\mathrm{Sin}[n\,\pi]\right)$$

```
int = bn /. Cos[n π] → (-1)^n
```

$$\frac{1}{n^4\,\pi}2\left(-6\,(-1)^n\,n\,\pi - 2\left(-3 + n^2\,\pi^2\right)\mathrm{Sin}[n\,\pi]\right)$$

```
bn = int /. Sin[n π] → 0
```

$$-\frac{12\ (-1)^n}{n^3}$$

With two invaluable trig substitutions provided by s.m., b_n is determined, above, green. I now have the value of ‘m’ in eq3, and I want to use it to find the total A, using the numerator of the ‘a’ part of eq3.

```
aaa = -cn (bn)
```

$$-cn\ b_n$$

$$\text{aaaa} = \text{aaa} \;/.\; b_n \to -\frac{12\ (-1)^n}{n^3}$$

$$\frac{12\ (-1)^n\ cn}{n^3}$$

```
aaaaa = aaaa / dee
```

$$\frac{12\ (-1)^n\ cn}{n^3\ \left(1 - 2\ n^2 + c^2\ n^2 + n^4\right)}$$

Above is the final value of A, which agrees with the text answer.

$$\text{bbb} = -\ \left(-1 + n^2\right)\ b_n$$

$$\left(1 - n^2\right)\ b_n$$

$$\text{bbbb} = \text{bbb} \;/.\; b_n \to -\frac{12\ (-1)^n}{n^3}$$

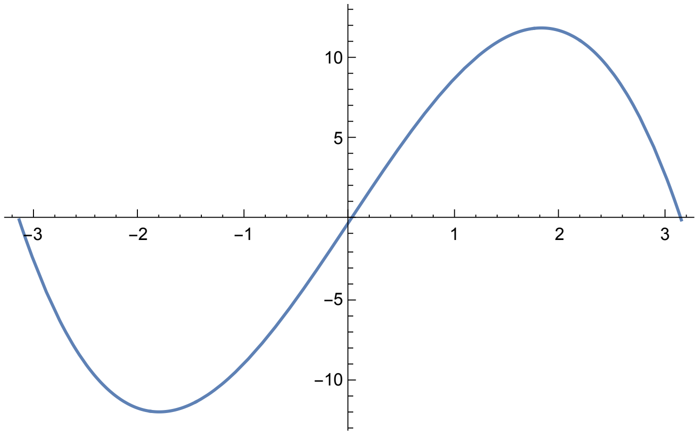
$$-\frac{12\ (-1)^n\ \left(1 - n^2\right)}{n^3}$$

```
bbbbb = bbbb / dee
```

$$-\frac{12\ (-1)^n\ \left(1 - n^2\right)}{n^3\ \left(1 - 2\ n^2 + c^2\ n^2 + n^4\right)}$$

Above is the final answer of B, which agrees with the text answer. (Note that (-1)^n resolves to (-1)$^{n+1}$.) This problem also requires a sketch of r[t].

```
rtplot = Plot[t (π² - t²), {t, -π, π}]
```



17.

19.