Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

```
Clear["Global`*"]
```

1 - 8 Comments on text and examples

1. Verify theorem 1 for the integral of z^2 over the boundary of the square with vertices ±1 $\pm i$. Hint. Use deformation.

3. Deformation principle. Can we exclude from Example 4 that the integral is also zero over the contour in Prob. 1?

5. Connectedness. What is the connectedness of the domain in which $\frac{\text{Cos}[z^2]}{z^4+1}$ is analytic?

7. Deformation. Can we conclude in Example 2 that the integral of $\frac{1}{z^2+4}$ over **(a)** $|z-2|=2$ and **(b)** $|z-2|=3$ is zero?

9 - 19 Cauchy's Theorem applicable?
Integrate f(z) counterclockwise around the unit circle. Indicate whether Cauchy's integral theorem applies.

9. $f[z\_] = \text{Exp}[-z^2]$

The s.m. covers this problem and is helpful.

```
Clear["Global`*"]
```

```
h[z_] = Exp[-z^2]
```

e^{-z^2}

```
f[x_, y_] = h[z] /. z → (x + ⅈ y)
```

$e^{-(x+i\,y)^2}$

```
ComplexExpand[%]
```

$e^{-x^2+y^2}\,\text{Cos}[2\,x\,y] - i\,e^{-x^2+y^2}\,\text{Sin}[2\,x\,y]$

First, it is true that the unit circle is a simple closed path enclosed in a simply connected domain D. It remains necessary to test analyticity to secure Cauchy theorem coverage.

```
u[x_, y_] = e^(-x^2+y^2) Cos[2 x y]
```

$e^{-x^2+y^2}\,\text{Cos}[2\,x\,y]$

```
v[x_, y_] = - e^(-x^2+y^2) Sin[2 x y]
```

$-e^{-x^2+y^2}\,\mathrm{Sin}[2\,x\,y]$

```
D[u[x, y], x]
```

$-2\,e^{-x^2+y^2}\,x\,\mathrm{Cos}[2\,x\,y]-2\,e^{-x^2+y^2}\,y\,\mathrm{Sin}[2\,x\,y]$

```
D[v[x, y], y]
```

$-2\,e^{-x^2+y^2}\,x\,\mathrm{Cos}[2\,x\,y]-2\,e^{-x^2+y^2}\,y\,\mathrm{Sin}[2\,x\,y]$

```
-D[u[x, y], y]
```

$-2\,e^{-x^2+y^2}\,y\,\mathrm{Cos}[2\,x\,y]+2\,e^{-x^2+y^2}\,x\,\mathrm{Sin}[2\,x\,y]$

```
D[v[x, y], x]
```

$-2\,e^{-x^2+y^2}\,y\,\mathrm{Cos}[2\,x\,y]+2\,e^{-x^2+y^2}\,x\,\mathrm{Sin}[2\,x\,y]$

Cyan and pink cells match. The function proves to be analytic for all real x and y. It meets the requirements of Cauchy's theorem, and therefore the contour integral equals zero.

11. $f[z\_]=\dfrac{1}{2\,z-1}$

```
Clear["Global`*"]
```

$h[z\_]=\dfrac{1}{2\,z-1}$

$\dfrac{1}{-1+2\,z}$

$f[x\_,\,y\_]=h[z]\,/.\,z\to(x+i\,y)$

$\dfrac{1}{-1+2\,(x+i\,y)}$

```
ComplexExpand[%]
```

$-\dfrac{1}{(-1+2\,x)^2+4\,y^2}+\dfrac{2\,x}{(-1+2\,x)^2+4\,y^2}-\dfrac{2\,i\,y}{(-1+2\,x)^2+4\,y^2}$

As with problem 9, it is true that the unit circle is a simple closed path enclosed in a simply connected domain D. It remains necessary to test analyticity to secure Cauchy theorem coverage.

$$u[x\_, y\_] = -\frac{1}{(-1+2x)^2+4y^2}+\frac{2x}{(-1+2x)^2+4y^2}$$

$$-\frac{1}{(-1+2x)^2+4y^2}+\frac{2x}{(-1+2x)^2+4y^2}$$

$$v[x\_, y\_] = -\frac{2y}{(-1+2x)^2+4y^2}$$

$$-\frac{2y}{(-1+2x)^2+4y^2}$$

```
D[u[x, y], x]
```

$$\frac{4(-1+2x)}{\left((-1+2x)^2+4y^2\right)^2}-\frac{8x(-1+2x)}{\left((-1+2x)^2+4y^2\right)^2}+\frac{2}{(-1+2x)^2+4y^2}$$

```
D[v[x, y], y]
```

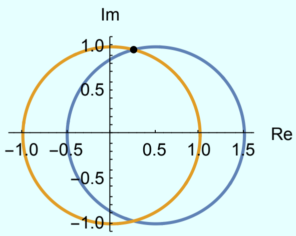
$$\frac{16y^2}{\left((-1+2x)^2+4y^2\right)^2}-\frac{2}{(-1+2x)^2+4y^2}$$

The cyan cells are not equal, therefore the function is not analytic, and therefore Cauchy's theorem cannot apply.
The s.m. has another way to deduce this which is easier: the function becomes discontinuous at $z=\frac{1}{2}$, and this is reached inside the unit circle, therefore Cauchy's theorem is unavailable. However, the s.m. goes on to do the contour integral.

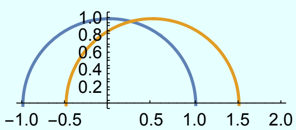
The s.m. gets around the problem of discontinuity by using the method of deformation of path, an allowable method described on p. 656. Instead of allowing z to take on the value of $\frac{1}{2}$, a new function, $z(t)=\frac{1}{2}+e^{i t}$ is substituted, moving the unit circle away from the discontinuous point.

Note: The cells below in cyan do not apply directly to the problem, being only feel-good extra steps.

```
ParametricPlot[{{Re[1/2 + ⅇ^(ⅈ t)], Im[ⅇ^(ⅈ t)]}, {Re[ⅇ^(ⅈ t)], Im[ⅇ^(ⅈ t)]}},
 {t, 0, 2 π}, ImageSize → 150, AxesLabel → {"Re", "Im"},
 Epilog -> {PointSize[0.03], Point[{0.25, 0.968246}]}]
```




```
Plot[{√(1 - x²), √(1 - (x - 1/2)²)}, {x, -1, 2},
 AspectRatio → Automatic, ImageSize → 150]
```




```
Solve[√(1 - x²) - √(1 - (x - 1/2)²) == 0, x]
```

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

$$\left\{\left\{x \to \frac{1}{4}\right\}\right\}$$

```
N[√(1 - (1/4)²)]
```

0.968246

With the new z, the s.m. achieves the following

```
f[z_] = h[z] /. z → (1/2 + ⅇ^(ⅈ t))
```

$$\frac{1}{-1 + 2\left(\frac{1}{2} + e^{i t}\right)}$$

Now comes an important addition to the procedure. This is use of numbered line (10) on page 647:

$$\int_C f[z]\, dz = \int_a^b f[z[t]]\, \dot{z}[t]\, dt \text{ where } \dot{z}[t] = D[z[t], t]$$

Incorporating the derivative of z into the contour integral. This derivative,

$$D\left[\frac{1}{2} + e^{i t}, t\right]$$

$$i\, e^{i t}$$

turns the expression into

$$\text{Integrate}\left[\frac{i\, e^{i t}}{-1 + 2\left(\frac{1}{2} + e^{i t}\right)}, \{t, 0, 2\pi\}\right]$$

$$i\,\pi$$

Matching the answer in the text.

Integrating around the unit circle, it doesn’t matter where to start and finish, so the above green answer is valid. However, it feels a little better to make the shared point between original and new unit circles the start/end point.

```
z = 0.25 + i 0.9682458365518543`
```

```
0.25 + 0.968246 i
```

```
ArcTan[0.9682458365518543` / 0.25]
```

```
1.31812
```

$$\text{Integrate}\left[\frac{i\, e^{i t}}{-1 + 2\left(\frac{1}{2} + e^{i t}\right)}, \{t, 0, 1.318116071652818`\}\right]$$

```
0. + 0.659058 i
```

$$\text{Integrate}\left[\frac{i\, e^{i t}}{-1 + 2\left(\frac{1}{2} + e^{i t}\right)}, \{t, 1.318116071652818`, 2\pi\}\right]$$

```
0. + 2.48253 i
```

```
0.659058035826409` ⅈ + 2.4825346177633842` ⅈ
```

```
0. + 3.14159 ⅈ
```

Repeating the answer obtained when integrating from 0 to 2 π.

13. $f[z\_] = \dfrac{1}{z^4 - 1.1}$

```
Clear["Global`*"]
```

$$h[z\_] = \frac{1}{z^4 - 1.1}$$

$$\frac{1}{-1.1 + z^4}$$

```
f[x_, y_] = h[z] /. z → (x + ⅈ y)
```

$$\frac{1}{-1.1 + (x + i\, y)^4}$$

```
ComplexExpand[%]
```

$$-\frac{1.1}{\left(4 x^3 y - 4 x y^3\right)^2 + \left(-1.1 + x^4 - 6 x^2 y^2 + y^4\right)^2} + \frac{x^4}{\left(4 x^3 y - 4 x y^3\right)^2 + \left(-1.1 + x^4 - 6 x^2 y^2 + y^4\right)^2} - \frac{6 x^2 y^2}{\left(4 x^3 y - 4 x y^3\right)^2 + \left(-1.1 + x^4 - 6 x^2 y^2 + y^4\right)^2} + \frac{y^4}{\left(4 x^3 y - 4 x y^3\right)^2 + \left(-1.1 + x^4 - 6 x^2 y^2 + y^4\right)^2} + i \left(-\frac{4 x^3 y}{\left(4 x^3 y - 4 x y^3\right)^2 + \left(-1.1 + x^4 - 6 x^2 y^2 + y^4\right)^2} + \frac{4 x y^3}{\left(4 x^3 y - 4 x y^3\right)^2 + \left(-1.1 + x^4 - 6 x^2 y^2 + y^4\right)^2}\right)$$

Working here with a somewhat unruly looking thing. It remains necessary to test analyticity to secure Cauchy theorem coverage.

$$u[x\_, y\_] = -\frac{1.1`}{(4x^3 y - 4 x y^3)^2 + (-1.1` + x^4 - 6 x^2 y^2 + y^4)^2} + \frac{x^4}{(4x^3 y - 4 x y^3)^2 + (-1.1` + x^4 - 6 x^2 y^2 + y^4)^2} - \frac{6 x^2 y^2}{(4x^3 y - 4 x y^3)^2 + (-1.1` + x^4 - 6 x^2 y^2 + y^4)^2} + \frac{y^4}{(4x^3 y - 4 x y^3)^2 + (-1.1` + x^4 - 6 x^2 y^2 + y^4)^2}$$

$$-\frac{1.1}{(4x^3 y - 4 x y^3)^2 + (-1.1 + x^4 - 6 x^2 y^2 + y^4)^2} + \frac{x^4}{(4x^3 y - 4 x y^3)^2 + (-1.1 + x^4 - 6 x^2 y^2 + y^4)^2} - \frac{6 x^2 y^2}{(4x^3 y - 4 x y^3)^2 + (-1.1 + x^4 - 6 x^2 y^2 + y^4)^2} + \frac{y^4}{(4x^3 y - 4 x y^3)^2 + (-1.1 + x^4 - 6 x^2 y^2 + y^4)^2}$$

$$v[x\_, y\_] = -\frac{4 x^3 y}{(4x^3 y - 4 x y^3)^2 + (-1.1` + x^4 - 6 x^2 y^2 + y^4)^2} + \frac{4 x y^3}{(4x^3 y - 4 x y^3)^2 + (-1.1` + x^4 - 6 x^2 y^2 + y^4)^2}$$

$$-\frac{4 x^3 y}{(4x^3 y - 4 x y^3)^2 + (-1.1 + x^4 - 6 x^2 y^2 + y^4)^2} + \frac{4 x y^3}{(4x^3 y - 4 x y^3)^2 + (-1.1 + x^4 - 6 x^2 y^2 + y^4)^2}$$

```
D[u[x, y], x];
FullSimplify[%]
```

$$\left(x \left(-4.\, x^{10} + 28.\, x^8 y^2 + x^4 y^2 \left(26.4 + 56.\, y^4\right) + x^6 \left(8.8 + 88.\, y^4\right) + y^2 \left(14.52 + 8.8\, y^4 - 20.\, y^8\right) + x^2 \left(-4.84 + 26.4\, y^4 - 20.\, y^8\right)\right)\right) \Big/ \left(1.21 + x^8 + 4.\, x^6 y^2 - 2.2\, y^4 + y^8 + x^2 y^2 \left(13.2 + 4.\, y^4\right) + x^4 \left(-2.2 + 6.\, y^4\right)\right)^2$$

```
D[v[x, y], y];
```

```
FullSimplify[%]
```

$$\left(x\left(-4.\,x^{10}+28.\,x^8y^2+x^4y^2\left(26.4+56.\,y^4\right)+x^6\left(8.8+88.\,y^4\right)+y^2\left(14.52+8.8\,y^4-20.\,y^8\right)+x^2\left(-4.84+26.4\,y^4-20.\,y^8\right)\right)\right)\Big/\left(1.21+x^8+4.\,x^6y^2-2.2\,y^4+y^8+x^2y^2\left(13.2+4.\,y^4\right)+x^4\left(-2.2+6.\,y^4\right)\right)^2$$

```
-D[u[x, y], y];
FullSimplify[%]
```

$$\left(y\left(20.\,x^{10}+20.\,x^8y^2+x^4y^2\left(-26.4-88.\,y^4\right)+x^6\left(-8.8-56.\,y^4\right)+x^2\left(-14.52-26.4\,y^4-28.\,y^8\right)+y^2\left(4.84-8.8\,y^4+4.\,y^8\right)\right)\right)\Big/\left(1.21+x^8+4.\,x^6y^2-2.2\,y^4+y^8+x^2y^2\left(13.2+4.\,y^4\right)+x^4\left(-2.2+6.\,y^4\right)\right)^2$$

```
D[v[x, y], x];
FullSimplify[%]
```

$$\left(y\left(20.\,x^{10}+20.\,x^8y^2+x^4y^2\left(-26.4-88.\,y^4\right)+x^6\left(-8.8-56.\,y^4\right)+x^2\left(-14.52-26.4\,y^4-28.\,y^8\right)+y^2\left(4.84-8.8\,y^4+4.\,y^8\right)\right)\right)\Big/\left(1.21+x^8+4.\,x^6y^2-2.2\,y^4+y^8+x^2y^2\left(13.2+4.\,y^4\right)+x^4\left(-2.2+6.\,y^4\right)\right)^2$$

Cyan and pink cells match. The function proves to be analytic for all real x and y. It meets the requirements of Cauchy's theorem, and therefore the contour integral equals zero.

The s.m. took a different approach to this problem, and analyzed it logically. The problem instructions are to integrate on the unit circle. As it turns out, the closest points of discontinuity are exterior to the unit circle, and noting this fact avoids further calculations.

15. f[z_] = Im[z]

```
Clear["Global`*"]
```

I'll try this problem, which is not in the s.m., by following the '**steps in applying theorem 2**' on p. 648:

(A) Represent the path C in the form z(t), (a ≤ t ≤ b).

(B) Calculate the derivative $\dot{z}(t)$ = dz/dt.

(C) Substitute z(t) for every z in f(z) (hence x(t) for x and y(t) for y).

(D) Integrate f[z(t)]$\dot{z}(t)$ over t from a to b.

Step (A). Looking at example 5 on p. 648, the path of the unit circle is represented as

```
z[t_] = Cos[t] + i Sin[t]
Cos[t] + i Sin[t]
```

Step (B). Calculate the derivative $\dot{z}(t)$

```
devz = D[z[t], t]
ⅈ Cos[t] - Sin[t]
```

Step (C). Substitute z(t) for every z in f(z)

```
f[z_] = Im[z] /. z → (Cos[t] + ⅈ Sin[t])
Im[Cos[t]] + Re[Sin[t]]
```

Step (D). Integrate $f[z(t)]\dot{z}(t)$ over t from a to b.

```
Integrate[ f[z] devz, {t, 0, 2 π}]
```

```
-π
```

The text answer is the same as the green cell above.

Belatedly, I check the analyticity,

```
u[x_, y_] = 0
0
```

```
v[x_, y_] = y
y
```

```
D[u[x, y], x]
```

```
0
```

```
D[v[x, y], y]
```

```
1
```

The cyan cells are not equal, therefore the function is not analytic, and therefore Cauchy' s theorem cannot apply.

17. $f[z\_] = \frac{1}{|z|^2}$

```
Clear["Global`*"]
```

$$f[x\_, y\_] = \frac{1}{\text{Abs}[x + y\,ⅈ]^2}$$

$$\frac{1}{\text{Abs}[x + ⅈ\,y]^2}$$

```
ComplexExpand[%]
```

$$\frac{1}{x^2 + y^2}$$

Since there is no imaginary component, it looks like D[v[x,y],y] will equal zero, whereas D[u[x,y],x] will not. So the function is not analytic. Skipping, then, the Cauchy Theorem option, I will go on to an attempted solution.

Having had luck the last problem, I will again try the '**steps in applying theorem 2**' on p. 648:

(A) Represent the path C in the form z(t), (a ≤ t ≤ b).

(B) Calculate the derivative $\dot{z}(t) = dz/dt$.

(C) Substitute z(t) for every z in f(z) (hence x(t) for x and y(t) for y).

(D) Integrate $f[z(t)]\dot{z}(t)$ over t from a to b.

Step (A). Looking at example 5 on p. 648, the path of the unit circle is represented as

```
z[t_] = e^(i t)
```

$e^{i t}$

(B) Calculate the derivative $\dot{z}(t) = dz/dt$.

```
devz = D[z[t], t]
```

$i\, e^{i t}$

(C) Substitute z(t) for every z in f(z) (hence x(t) for x and y(t) for y).

$$f[z[t]] = \frac{1}{2\, e^{2 i t}}$$

$$\frac{1}{2} e^{-2 i t}$$

(D) Integrate $f[z(t)]\dot{z}(t)$ over t from a to b.

$$\text{Integrate}\left[\frac{1}{2} e^{-2 i t}\, \text{devz}, \{t, 0, 2\pi\}\right]$$

0

The green cell matches the text answer. In this case the **Exp** version of f[z] worked, but the **Trig** version did not. In the last problem it was the reverse. In this problem a zero integral is encountered which, however, does not qualify for its zip status because of the Cauchy Theorem, but rather because of its innate zeroness.

19. $f[z\_] = z^3 \, \text{Cot}[z]$

20 - 30 Further Contour Integrals

Evaluate the integral. Does Cauchy's theorem apply?

```
Clear["Global`*"]
```

```
h[z_] = z^3 Cot[z]
```

z^3 Cot[z]

```
f[x_, y_] = h[z] /. z → (x + y ⅈ)
```

$(x + i\, y)^3$ Cot[x + ⅈ y]

```
ComplexExpand[%]
```

$$-\frac{x^3 \operatorname{Sin}[2 x]}{\operatorname{Cos}[2 x]-\operatorname{Cosh}[2 y]}+\frac{3 x y^2 \operatorname{Sin}[2 x]}{\operatorname{Cos}[2 x]-\operatorname{Cosh}[2 y]}-\frac{3 x^2 y \operatorname{Sinh}[2 y]}{\operatorname{Cos}[2 x]-\operatorname{Cosh}[2 y]}+\frac{y^3 \operatorname{Sinh}[2 y]}{\operatorname{Cos}[2 x]-\operatorname{Cosh}[2 y]}+i\left(-\frac{3 x^2 y \operatorname{Sin}[2 x]}{\operatorname{Cos}[2 x]-\operatorname{Cosh}[2 y]}+\frac{y^3 \operatorname{Sin}[2 x]}{\operatorname{Cos}[2 x]-\operatorname{Cosh}[2 y]}+\frac{x^3 \operatorname{Sinh}[2 y]}{\operatorname{Cos}[2 x]-\operatorname{Cosh}[2 y]}-\frac{3 x y^2 \operatorname{Sinh}[2 y]}{\operatorname{Cos}[2 x]-\operatorname{Cosh}[2 y]}\right)$$

$$u[x\_, y\_] = -\frac{x^3 \operatorname{Sin}[2 x]}{\operatorname{Cos}[2 x]-\operatorname{Cosh}[2 y]}+\frac{3 x y^2 \operatorname{Sin}[2 x]}{\operatorname{Cos}[2 x]-\operatorname{Cosh}[2 y]}-\frac{3 x^2 y \operatorname{Sinh}[2 y]}{\operatorname{Cos}[2 x]-\operatorname{Cosh}[2 y]}+\frac{y^3 \operatorname{Sinh}[2 y]}{\operatorname{Cos}[2 x]-\operatorname{Cosh}[2 y]}$$

$$-\frac{x^3 \operatorname{Sin}[2 x]}{\operatorname{Cos}[2 x]-\operatorname{Cosh}[2 y]}+\frac{3 x y^2 \operatorname{Sin}[2 x]}{\operatorname{Cos}[2 x]-\operatorname{Cosh}[2 y]}-\frac{3 x^2 y \operatorname{Sinh}[2 y]}{\operatorname{Cos}[2 x]-\operatorname{Cosh}[2 y]}+\frac{y^3 \operatorname{Sinh}[2 y]}{\operatorname{Cos}[2 x]-\operatorname{Cosh}[2 y]}$$

$$v[x\_, y\_] = -\frac{3 x^2 y \operatorname{Sin}[2 x]}{\operatorname{Cos}[2 x]-\operatorname{Cosh}[2 y]}+\frac{y^3 \operatorname{Sin}[2 x]}{\operatorname{Cos}[2 x]-\operatorname{Cosh}[2 y]}+\frac{x^3 \operatorname{Sinh}[2 y]}{\operatorname{Cos}[2 x]-\operatorname{Cosh}[2 y]}-\frac{3 x y^2 \operatorname{Sinh}[2 y]}{\operatorname{Cos}[2 x]-\operatorname{Cosh}[2 y]}$$

$$-\frac{3 x^2 y \operatorname{Sin}[2 x]}{\operatorname{Cos}[2 x]-\operatorname{Cosh}[2 y]}+\frac{y^3 \operatorname{Sin}[2 x]}{\operatorname{Cos}[2 x]-\operatorname{Cosh}[2 y]}+\frac{x^3 \operatorname{Sinh}[2 y]}{\operatorname{Cos}[2 x]-\operatorname{Cosh}[2 y]}-\frac{3 x y^2 \operatorname{Sinh}[2 y]}{\operatorname{Cos}[2 x]-\operatorname{Cosh}[2 y]}$$

```
D[u[x, y], x];
```

```
FullSimplify[%]
```

```
(-4 x (x^2 - 3 y^2) +
    2 Cosh[2 y] (2 x (x^2 - 3 y^2) Cos[2 x] + 3 (x - y) (x + y) Sin[2 x]) +
    3 (-x^2 + y^2) Sin[4 x] +
    4 y (-3 x Cos[2 x] + 3 x Cosh[2 y] + (-3 x^2 + y^2) Sin[2 x]) Sinh[2 y]) /
  (2 (Cos[2 x] - Cosh[2 y])^2)
```

```
D[v[x, y], y];
FullSimplify[%]
```

```
(-4 x (x^2 - 3 y^2) +
    2 Cosh[2 y] (2 x (x^2 - 3 y^2) Cos[2 x] + 3 (x - y) (x + y) Sin[2 x]) +
    3 (-x^2 + y^2) Sin[4 x] +
    4 y (-3 x Cos[2 x] + 3 x Cosh[2 y] + (-3 x^2 + y^2) Sin[2 x]) Sinh[2 y]) /
  (2 (Cos[2 x] - Cosh[2 y])^2)
```

```
-D[u[x, y], y];
FullSimplify[%]
```

```
(4 y (-3 x^2 + y^2) - 6 x y Sin[4 x] +
    2 Cos[2 x] ((6 x^2 y - 2 y^3) Cosh[2 y] + 3 (x - y) (x + y) Sinh[2 y]) +
    4 x Sin[2 x] (3 y Cosh[2 y] + (x^2 - 3 y^2) Sinh[2 y]) -
    3 (x - y) (x + y) Sinh[4 y]) / (2 (Cos[2 x] - Cosh[2 y])^2)
```

```
D[v[x, y], x];
FullSimplify[%]
```

```
(4 y (-3 x^2 + y^2) - 6 x y Sin[4 x] +
    2 Cos[2 x] ((6 x^2 y - 2 y^3) Cosh[2 y] + 3 (x - y) (x + y) Sinh[2 y]) +
    4 x Sin[2 x] (3 y Cosh[2 y] + (x^2 - 3 y^2) Sinh[2 y]) -
    3 (x - y) (x + y) Sinh[4 y]) / (2 (Cos[2 x] - Cosh[2 y])^2)
```

Cyan and pink cells match. The function proves to be analytic for all real x and y. It meets the requirements of Cauchy's theorem, and therefore the contour integral equals zero.

21. $\oint \frac{1}{z - 3 i} \, dz$, the circle $|z| = \pi$ counterclockwise

```
Clear["Global`*"]
```

$$\oint \frac{1}{z - 3\,i}\,dz$$

I will first try the example 5 steps.

(A) Represent the path C in the form z(t), (a ≤ t ≤ b).

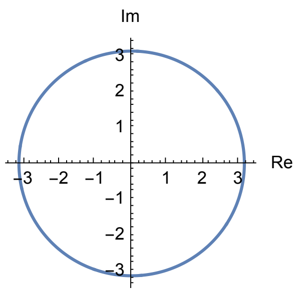
(B) Calculate the derivative $\dot{z}(t) = dz/dt$.

(C) Substitute z(t) for every z in f(z) (hence x(t) for x and y(t) for y).

(D) Integrate $f[z(t)]\dot{z}(t)$ over t from a to b.

Step (A). Since z[t_] = $e^{i\,t}$ is a unit circle, I suppose that $\pi\, e^{i\,t}$ is a circle of radius π.

```
ParametricPlot[{Re[π e^(i t)], Im[π e^(i t)]},
 {t, 0, 2 π}, ImageSize → 150, AxesLabel → {"Re", "Im"}]
```



The plot looks okay as far as radius, I think.

```
z[t_] = π e^(i t)
```

$e^{i\,t}\,\pi$

Step (B).

```
devz = D[z[t], t]
```

$i\, e^{i\,t}\,\pi$

Step(C).

```
Integrate[1/(π e^(i t) - 3 i) devz, {t, 0, 2 π}]
```

$2\,i\,\pi$

The green cell answer matches that of the text.

23. $\oint \frac{2\,z - 1}{z^2 - z}\,dz$,
counterclockwise around an ellipse with foci at origin and (2, 0). Use partial fractions.

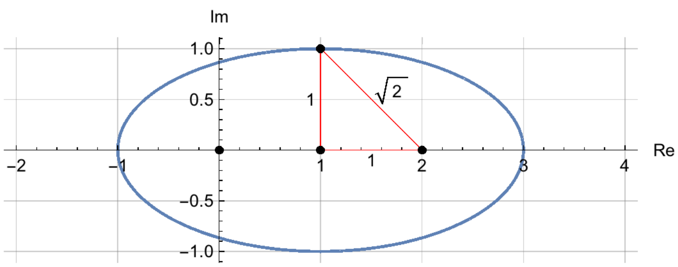
This problem is worked in the s.m.

```
Clear["Global`*"]
```

$$\oint \frac{2z-1}{z^2-z}\, dz$$

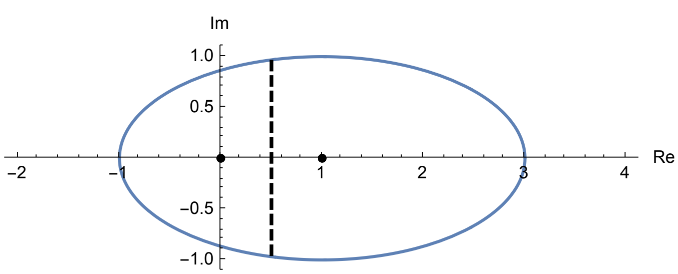
```
ContourIntegral[(-1 + 2 z)/(-z + z^2), z]
```

```
Plot[y /. Solve[((x - 1)/2)^2 + (y/1)^2 == 1],
 {x, -2, 4}, AspectRatio → Automatic, ImageSize → 350,
 AxesLabel → {"Re", "Im"}, GridLines → Automatic,
 Epilog → {{Red, Line[{{1, 0}, {2, 0}}]}, {Text["1", {1.5, -0.1}]},
   {Red, Line[{{1, 0}, {1, 1}}]}, {Text["1", {0.9, 0.5}]},
   {Red, Line[{{1, 1}, {2, 0}}]}, {Text["√2", {1.7, 0.6}]},
   {PointSize[0.014], Point[{1, 1}]}, {PointSize[0.014], Point[{1, 0}]},
   {PointSize[0.014], Point[{2, 0}]}, {PointSize[0.014], Point[{0, 0}]}}]
```



The above plot displays an ellipse which satisfies the problem description. The problem is actually simple, provided example 6 on p. 656 is used, as pointed out by the s.m. The problem's problem is that there are two points of discontinuity: 0 and 1. The hint to use partial fractions solves the issue by splitting the domain into two parts as shown schematically below. This schematic sketch represents two paths, each incorporating the dashed line, which will be added together.

```
Plot[y /. Solve[((x - 1)/2)^2 + (y/1)^2 == 1], {x, -2, 4},
 AspectRatio → Automatic, ImageSize → 350, AxesLabel → {"Re", "Im"},
 Epilog → {{Thick, Dashed, Line[{{0.5, 0.95}, {0.5, -0.95}}]},
   {PointSize[0.014], Point[{1, 0}]}, {PointSize[0.014], Point[{0, 0}]}}]
```



The principle of deformation of path, illustrated in example 6 on p. 656, covers this situation nicely. It works because the problem function can be factored.

```
Factor[(2 z - 1)/(z^2 - z)]
```

$$\frac{-1+2z}{(-1+z)\,z}$$

```
Apart[%]
```

$$\frac{1}{-1+z}+\frac{1}{z}$$

```
egr1 = (-1 + z)^-1 (*note exponent of -1  *)
```

$$\frac{1}{-1+z}$$

```
egr2 = z^-1 (*note exponent of -1  *)
```

$$\frac{1}{z}$$

Taking a look at numbered line (3) on p. 656,

$$\oint (z - z_0)^m \, dz = \begin{cases} 2\pi i & m = -1 \\ 0 & m \neq -1 \text{ and integer} \end{cases}$$

The point z_0 is understood to be the point of discontinuity, and in the first factor is equal to 1.

$$\texttt{ci1} = \oint \texttt{egr1}\, dz == 2\pi i$$

$$\texttt{ContourIntegral}\left[\frac{1}{-1+z}, z\right] == 2 i \pi$$

For this other factor of the function, the point z_0 is equal to 0.

$$\texttt{ci2} = \oint \texttt{egr2}\, dz == 2\pi i$$

$$\texttt{ContourIntegral}\left[\frac{1}{z}, z\right] == 2 i \pi$$

$$\texttt{ci3} == \texttt{ci1} + \texttt{ci2} == 4\pi i$$

$$\texttt{ci3} == \left(\texttt{ContourIntegral}\left[\frac{1}{-1+z}, z\right] == 2 i \pi\right) + \left(\texttt{ContourIntegral}\left[\frac{1}{z}, z\right] == 2 i \pi\right) == 4 i \pi$$

Instead of taking a shortcut and strong-arming Mathematica into submission, there is a procedure on *MathWorld* under the heading Contour Integral, which could possibly be used to derive the answer more legitimately.

25. $\oint \frac{e^z}{z} \, dz$,

C consists of $|z| = 2$ counterclockwise and $|z| = 1$ clockwise.

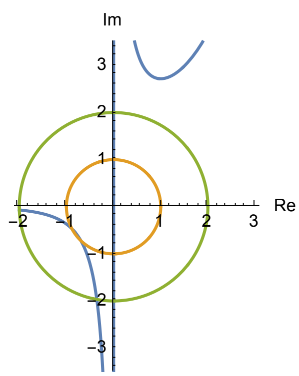
```
Clear["Global`*"]
```

$$\left(\frac{e^z}{z}\right)^{-1}$$

$$e^{-z}\, z$$

In the above form there are no values of z_0 which cause a discontinuity. The form that allows a discontinuity at $z_0=0$ is the form $\left(\frac{e^z}{z}\right)^1$, and then $m=1$, which means the integral equals 0. This answer agrees with the text.

```
Plot[{e^x/x, y /. Solve[(x^2 + y^2 == 1)], y /. Solve[(x^2 + y^2 == 4)]},
 {x, -2, 3}, AspectRatio → Automatic, ImageSize → 150,
 AxesLabel → {"Re", "Im"}, PlotRange → {-3.5, 3.5}]
```



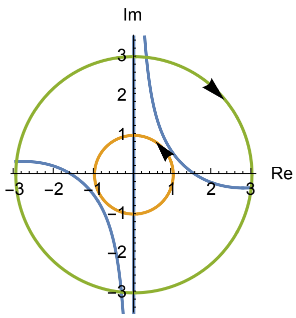
A plot of the annular domain and the candidate contour. The contour is not a closed path.

27. $\oint \frac{\text{Cos}[z]}{z}\, dz$,
C consists of $|z| = 1$ counterclockwise and $|z| = 3$ clockwise.

```
Clear["Global`*"]
```

This time I'll make the plot first.

```
Plot[{Cos[x]/x, y /. Solve[(x^2 + y^2 == 1)], y /. Solve[(x^2 + y^2 == 9)]},
 {x, -3, 3}, AspectRatio → Automatic, ImageSize → 150,
 AxesLabel → {"Re", "Im"}, PlotRange → {-3.5, 3.5},
 Epilog → {{Arrowheads[.1], Arrow[{{2.1, 2.1}, {2.3, 1.9}}]},
   {Arrowheads[.09], Arrow[{{0.7, 0.67}, {0.55, 0.85}}]}}]
```



The above plot of the annular domain superimposed with candidate contour.

$$\left(\frac{\text{Cos}[z]}{z}\right)^{-1} == \frac{z}{\text{Cos}[z]}$$

True

So an alternate form of the current function is

$$\left(\frac{z}{\text{Cos}[z]}\right)^{-1}$$

In the form shown above, the expression does admit of a discontinuity at the point $z_0 = \frac{\pi}{2}$. And the value $\frac{\pi}{2}$ is contained in the domain. However, I believe that because of the opposite orientation of the directions of the annuli, as shown by arrowheads, the integral will equal zero. A site with some treatment of this is: *http://www.math.unm.edu/~nitsche/courses/313/s16/lec19_int5.pdf*. Also, the text, on p. 658, seems to reinforce this idea when talking about branch cuts. So this problem could end up equaling zero not through deformation of path and numbered line (3), but through calculation by means of branch cuts, one cut cutting out the point of discontinuity. I will skip the green cell coloring, though.

29. $\oint \frac{\text{Sin}[z]}{z + 2 i z} dz$, C : $|z - 4 - 2 i| = 5.5$ clockwise.