

Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

```
Clear["Global`*"]
```

3. Inverse. If $w = f[z]$ is any transformation that has an inverse, prove the fact that f and its inverse have the same fixed points.

```
Clear["Global`*"]
```

This may be a pud proof, but... I assume that $w=f[x]$ has at least one fixed point. However many it has, consider an arbitrary fixed point z_1 in z -plane which gets mapped to w -plane by $w=f[z]$, to a point called w_1 . And the inverse of w , w^{-1} , a mapping in its own right, takes the point w_1 and operates on it to map it to the z -plane, to the exact point where it originated, that being the inverse function's function. Now $w_1 = z_1$ because z_1 is a fixed point for w . By observing w^{-1} as it maps w_1 to z_1 , with which it is equal, I can see that w_1 is a fixed point for w^{-1} . The establishment of the fixed point in w^{-1} is separate and independent of the establishment of z_1 as a fixed point for w , yet, as if I didn't already know, I can look at z_1 and w_1 and see that they are equal. Therefore the mapping functions have a fixed point that is the same, and if they have this arbitrary one, they share all in common.

5. Derive the mapping in Example 2 from numbered line (2) on p. 746.

```
Clear["Global`*"]
```

```
z1 = 0
```

```
0
```

```
z2 = 1
```

```
1
```

Mathematica cannot swallow ∞ assigned to a variable in this context. For now, it needs to be symbolic, as below. (Note: the w - z assignment formula has its own means of dealing with occurrences of ∞ , but with Mathematica I can take care of it a different way.)

```
z3 = a
```

```
a
```

```
w1 = -1
```

```
-1
```

```
w2 = -i
```

```
-i
```

```
w3 = 1
```

```
1
```

$$\text{Solve}\left[\frac{w - w_1}{w - w_3} \left(\frac{w_2 - w_3}{w_2 - w_1}\right) == \frac{z - z_1}{z - z_3} \left(\frac{z_2 - z_3}{z_2 - z_1}\right), \{w, z\}\right]$$

Solve::vars: Equations may not give solution for all "solve" variables >>

$$\left\{\left\{w \rightarrow \frac{-i a - (1 - i) z + a z}{i a - (1 + i) z + a z}\right\}\right\}$$

$$\text{Limit}\left[\frac{-i a - (1 - i) z + a z}{i a - (1 + i) z + a z}, a \rightarrow \infty\right]$$

$$\frac{-i + z}{i + z}$$

The expression in the green cell above matches the answer to example 2 on p. 748.

8 - 16 LFTs from three points and images

Find the LFT that maps the given three points onto the three given points in the respective order.

9. 1, i , -1 onto i , -1 , $-i$

```
Clear["Global`*"]
```

```
z1 = 1
```

```
1
```

```
z2 = i
```

```
i
```

```
z3 = -1
```

```
-1
```

```
w1 = i
```

```
i
```

```
w2 = -1
```

```
-1
```

```
w3 = -i
```

```
-i
```

$$\text{Solve}\left[\frac{w - w_1}{w - w_3} \left(\frac{w_2 - w_3}{w_2 - w_1}\right) == \frac{z - z_1}{z - z_3} \left(\frac{z_2 - z_3}{z_2 - z_1}\right), \{w\}\right]$$

$$\{\{w \rightarrow i z\}\}$$

11. $-1, 0, 1$ onto $-i, -1, i$

```
Clear["Global`*"]
```

```
z1 = -1
```

```
-1
```

```
z2 = 0
```

```
0
```

```
z3 = 1
```

```
1
```

```
w1 = -i
```

```
-i
```

```
w2 = -1
```

```
-1
```

```
w3 = i
```

```
i
```

```
Solve[ $\frac{w - w_1}{w - w_3} \left( \frac{w_2 - w_3}{w_2 - w_1} \right) == \frac{z - z_1}{z - z_3} \left( \frac{z_2 - z_3}{z_2 - z_1} \right), \{w\}]$ 
```

$$\left\{ \left\{ w \rightarrow \frac{i + z}{-i + z} \right\} \right\}$$

13. 0, 1, ∞ onto ∞, 1, 0

```
Clear["Global`*"]
```

```
z1 = 0
```

```
0
```

```
z2 = 1
```

```
1
```

```
z3 = a
```

```
a
```

```
w1 = a
```

```
a
```

```
w2 = 1
```

```
1
```

```
w3 = 0
```

```
0
```

$$\text{Solve}\left[\frac{w - w_1}{w - w_3} \left(\frac{w_2 - w_3}{w_2 - w_1}\right) == \frac{z - z_1}{z - z_3} \left(\frac{z_2 - z_3}{z_2 - z_1}\right), \{w\}\right]$$

$$\left\{\left\{w \rightarrow \frac{a - z}{1 - 2z + az}\right\}\right\}$$

$$\text{Limit}\left[\frac{a - z}{1 - 2z + az}, a \rightarrow \infty\right]$$

$$\frac{1}{z}$$

$$15. \quad 1, \quad i, \quad 2 \text{ onto } 0, \quad -i - 1, \quad -\frac{1}{2}$$

```
Clear["Global`*"]
```

$$z_1 = 1$$

$$1$$

$$z_2 = i$$

$$i$$

$$z_3 = 2$$

$$2$$

$$w_1 = 0$$

$$0$$

$$w_2 = -i - 1$$

$$-1 - i$$

$$w_3 = -\frac{1}{2}$$

$$-\frac{1}{2}$$

$$\text{Solve}\left[\frac{w - w_1}{w - w_3} \left(\frac{w_2 - w_3}{w_2 - w_1}\right) == \frac{z - z_1}{z - z_3} \left(\frac{z_2 - z_3}{z_2 - z_1}\right), \{w\}\right]$$

$$\left\{\left\{w \rightarrow \frac{1 - z}{z}\right\}\right\}$$

17. Find an LFT that maps $|z| \leq 1$ onto $|w| \leq 1$ so that $z = \frac{i}{2}$ is mapped onto $w = 0$.
Sketch the images of the lines $x = \text{const}$ and $y = \text{const}$.

```
Clear["Global`*"]
```

Numbered line (3) on p. 749 is part of an example that works this problem out for me, except it is left in general form, the z -plane point z_0 being mapped to the origin of the w -plane, while the z -unit-circle is mapped to the w -unit-circle.

$$w = \frac{z - z_0}{c \bar{z} - 1}, \quad c = z_0^*, \quad \text{Abs}[z_0] < 1$$

One caveat here is that of the need to tailor c , the conjugate of z_0 , to the specific z_0 chosen.

$$z_0 = \frac{i}{2}$$

$$\frac{i}{2}$$

$$z_{\text{con}} = z_0^*$$

$$-\frac{i}{2}$$

$$w[z_] = \frac{z - \frac{i}{2}}{z_{\text{con}} z - 1}$$

$$\frac{-\frac{i}{2} + z}{-1 - \frac{i z}{2}}$$

Simplify[%]

$$\frac{1 + 2 i z}{-2 i + z}$$

The cell below demonstrates that the green cell agrees with the text answer in content.

$$\text{PossibleZeroQ}\left[\frac{1 + 2 i z}{-2 i + z} - \frac{2 z - i}{-(i z + 2)}\right]$$

True

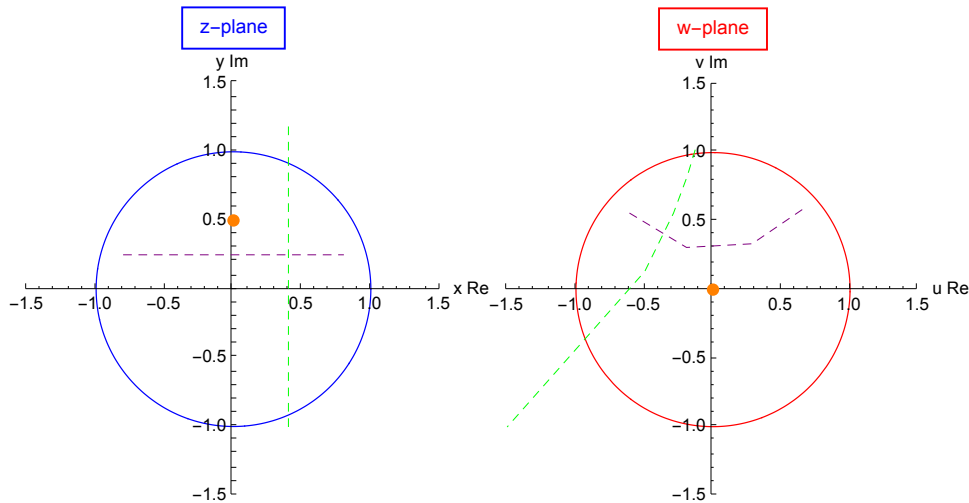
The transformation w maps the point $z=0+\frac{i}{2}$ on the z -plane to $w=0+0i$ on the w -plane. In the z -plane plot its location is consistent with the vertical axis location, and in the w -plane plot it is shown correctly.

As for the original and transformed circles, and constant lines, the mapping w is allowed to do its work directly wherever possible. Two intersecting test lines demonstrate that intersection angles are preserved under conformal mapping.

```

Row[{ParametricPlot[{Re[(e^i t + 0)], Im[(e^i t)]}, {t, 0, 2 π},
  ImageSize → 250, PlotRange → {{-1.5, 1.5}, {-1.5, 1.5}},
  AspectRatio → Automatic, PlotStyle → {Blue, Thickness[0.003]},
  Epilog → {{Orange, PointSize[0.03], Point[{0, 1/2}]},
    {Dashed, Green, Line[{0.4, -1}, {0.4, 1.2}]}},
    {Dashed, Purple, Line[{-0.8, 0.25}, {0.8, 0.25}]}]},
  AxesLabel → {"x Re", "y Im"}, PlotLabel →
  Style[Framed[" z-plane "], 10, Blue, Background → White]},
ParametricPlot[{Re[(1 + 2 i e^i t) / (-2 i + e^i t)], Im[(1 + 2 i e^i t) / (-2 i + e^i t)]}, {t, 0, 2 π},
  ImageSize → 250, PlotRange → {{-1.5, 1.5}, {-1.5, 1.5}},
  AspectRatio → Automatic, PlotStyle → {Red, Thickness[0.003]},
  Epilog → {{Orange, PointSize[0.03], Point[{Re[(1 + 2 i z) / (-2 i + z) /. z → (0 + i/2)],
    Im[(1 + 2 i z) / (-2 i + z) /. z → (0 + i/2)]]}], {Dashed, Green,
  Line[{Re[(1 + 2 i z) / (-2 i + z) /. z → (0.4 - i)], Im[(1 + 2 i z) / (-2 i + z) /. z → (0.4 - i)]},
    {Re[(1 + 2 i z) / (-2 i + z) /. z → (0.4 - 0.5 i)], Im[(1 + 2 i z) / (-2 i + z) /. z → (0.4 - 0.5 i)]},
    {Re[(1 + 2 i z) / (-2 i + z) /. z → (0.4 - 0 i)], Im[(1 + 2 i z) / (-2 i + z) /. z → (0.4 - 0 i)]},
    {Re[(1 + 2 i z) / (-2 i + z) /. z → (0.4 + 0.5 i)], Im[(1 + 2 i z) / (-2 i + z) /. z → (0.4 + 0.5 i)]},
    {Re[(1 + 2 i z) / (-2 i + z) /. z → (0.4 + 1.2 i)],
    Im[(1 + 2 i z) / (-2 i + z) /. z → (0.4 + 1.2 i)]}}],
    {Dashed, Purple, Line[{Re[(1 + 2 i z) / (-2 i + z) /. z → (-0.8 + 0.25 i)],
    Im[(1 + 2 i z) / (-2 i + z) /. z → (-0.8 + 0.25 i)]}, {Re[(1 + 2 i z) / (-2 i + z) /.
    z → (-0.3 + 0.25 i)], Im[(1 + 2 i z) / (-2 i + z) /. z → (-0.3 + 0.25 i)]}, {Re[(1 + 2 i z) / (-2 i + z) /.
    z → (0.2 + 0.25 i)], Im[(1 + 2 i z) / (-2 i + z) /. z → (0.2 + 0.25 i)]},
    {Re[(1 + 2 i z) / (-2 i + z) /. z → 0.8 + 0.25 i], Im[(1 + 2 i z) / (-2 i + z) /. z → 0.8 + 0.25
    i]}]}], AxesLabel → {"u Re", "v Im"}, PlotLabel →
  Style[Framed[" w-plane "], 10, Red, Background → White]}]}

```



19. Find an analytic function $w = f[z]$ that maps the region $0 \leq \text{Arg}[z] \leq \frac{\pi}{4}$ onto the unit disk $|w| \leq 1$.

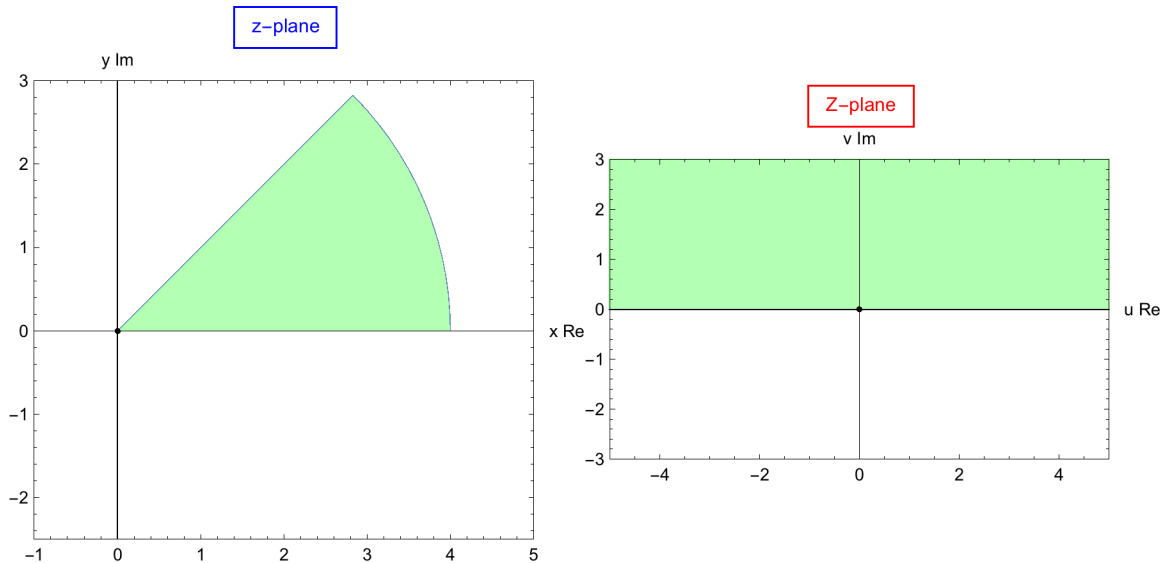
Clear["Global`*"]

A hint from s.m. points me to example 2 on p. 739. This explains how to map a wedge sector onto an upper half plane. The upshot is that a wedge sector $0 \leq \theta \leq \frac{\pi}{n}$ can be mapped to the upper half plane $v \geq 0$ using the transform z^n . In this problem I have a sector from 0 to $\frac{\pi}{4}$, so $n = 4$. As far as the r value goes, it can be anything above about 2, but it can't be ∞ . I assume that setting it at 4 arbitrarily will not undermine the argument that the demo is general.

```

Row[{ParametricPlot[{Re[r (e^i t)], Im[r (e^i t)]], {t, 0,  $\frac{\pi}{4}$ },
  {r, 0, 4}, ImageSize → 300, PlotStyle → {Green, Thickness[0.004]},
  Epilog → {{Point[{0, 0}]}}, Axes → True, AxesLabel → {"x Re", "y Im"},
  PlotRange → {{-1, 5}, {3, -2.5}}, PlotLabel →
    Style[Framed[" z-plane "], 10, Blue, Background → White] ],
ParametricPlot[{Re[(r e^i t)^4], Im[(r e^i t)^4]}, {t, 0,  $\frac{\pi}{4}$ }, {r, 0, 4},
  ImageSize → 300, PlotStyle → {Green, Thickness[0.004]},
  Epilog → {{Point[{Re[(r e^i t)^4 /. {t → 0, r → 0}],
    Im[(r e^i t)^4 /. {t → 0, r → 0}]}]}}, AxesLabel →
    {"u Re", "v Im"}, PlotRange → {{-5, 5}, {3, -3}}, PlotLabel →
    Style[Framed[" Z-plane "], 10, Red, Background → White] ]}]

```



As the plot windows above show, the mapping scheme is successful so far. The next phase is to figure out how to map the upper half plane to a unit circle on the w-plane. The strategy used by the s.m., which I will follow, will be to use the mapping formula advanced in problem 5 and following. To do this I need to pick three points on Z-plane and three others on w-plane and then calculate the expression to do it. As suggested by s.m., I choose -1, 0, and 1 on the u axis of Z-plane, and intend to map them to $(-1, -i, \text{ and } 1)$, on the unit circle, in the w-plane.

$$Z_1 = -1$$

$$-1$$

$$Z_2 = 0$$

$$0$$

$$z_3 = 1$$

$$1$$

$$w_1 = -1$$

$$-1$$

$$w_2 = -i$$

$$-i$$

$$w_3 = 1$$

$$1$$

$$\text{exp1} = \text{Simplify}\left[\text{Solve}\left[\frac{w - w_1}{w - w_3} \left(\frac{w_2 - w_3}{w_2 - w_1}\right) == \frac{z - z_1}{z - z_3} \left(\frac{z_2 - z_3}{z_2 - z_1}\right), \{w\}\right]\right]$$

$$\left\{\left\{w \rightarrow \frac{1 + i z}{i + z}\right\}\right\}$$

To re-wind this mapping back to the original sector, I make a substitution,

$$\text{exp2} = \text{exp1} /. z \rightarrow z^4$$

$$\left\{\left\{w \rightarrow \frac{1 + i z^4}{i + z^4}\right\}\right\}$$

Since this does not look exactly like the text answer, I check its equivalence, then confer the green.

$$\text{PossibleZeroQ}\left[\frac{1 + i z^4}{i + z^4} - \frac{(z^4 - i)}{(-i z^4 + 1)}\right]$$

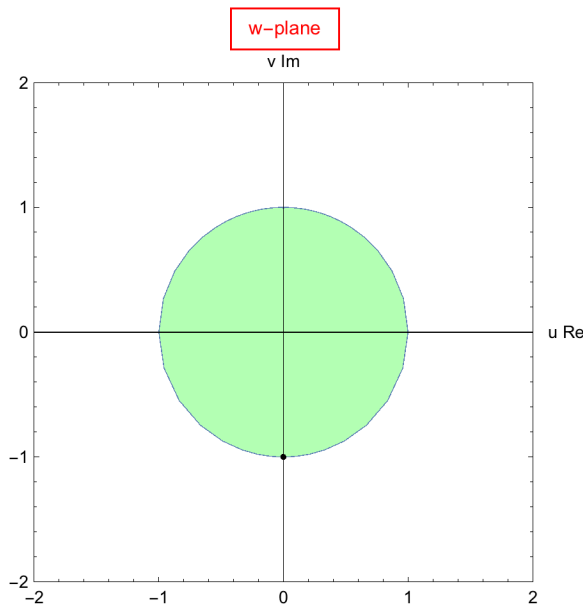
True

Time to set up the final mapping transit, taking points from the z-plane pie sector to the w-plane unit circle.

```

ParametricPlot[ {Re[  $\frac{(r e^{i t})^4 - i}{-i (r e^{i t})^4 + 1}$  ], Im[  $\frac{(r e^{i t})^4 - i}{-i (r e^{i t})^4 + 1}$  ] }, {t, 0,  $\frac{\pi}{4}$ },
{r, 0, 4}, ImageSize -> 300, PlotStyle -> {Green, Thickness[0.004]},
Epilog -> { {Point[ {Re[  $\frac{(r e^{i t})^4 - i}{-i (r e^{i t})^4 + 1}$  ] /. {t -> 0, r -> 0} ],
Im[  $\frac{(r e^{i t})^4 - i}{-i (r e^{i t})^4 + 1}$  ] /. {t -> 0, r -> 0} ] ] } },
AxesLabel -> {"u Re", "v Im"}, PlotRange -> {{-2, 2}, {-2, 2}},
PlotLabel -> Style[Framed[ " w-plane "], 10, Red, Background -> White] ]

```



The origin point of the z-plane ends up at the bottom of the unit circle in the w-plane.