Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

1 - 8 Parametric surface representation

Familiarize yourself with parametric representations of important surfaces by deriving a representation (1), by finding the **parameter curves** (curves u = const and v = const) of the surface and a normal vector $\mathbf{N} = \boldsymbol{r}_u \times \boldsymbol{r}_v$ of the surface.

1. *xy*-plane **r**(u, v) = (u, v) (thus u **i** + v **j**; similarly in problems 2 - 8).

3. Cone **r**(u, v) = {u Cos[v], u Sin[v], c u}

```
Clear["Global`*"]
```

I am given that

```
x = u Cos[v];
y = u Sin[v];
z = u c
```

```
c u
```

If I happen to have the brainwave to look at $x^2 + y^2$,

```
x^2 + y^2
```

```
u^2 Cos[v]^2 + u^2 Sin[v]^2
```

I see that it is equal to u^2. At this point I can find one sort of parameter curve, by setting u =const. If u=const.

then u^2 = const. and $x^2 + y^2$ = const., that is, What about when v=const.?

equals *a* circle. So circles are one of the parameter curves.

Then by the definition of the coordinates, the quantity y/x=some constant, and y=kx. That describes a straight line. So straight lines are the other parameter curve.

```
yip = Simplify[Solve[x^2 + y^2 == z^2/c^2, {z}]]
```

$$\left\{\left\{z \to -\sqrt{c^2 \left(x^2 + y^2\right)}\right\}, \left\{z \to \sqrt{c^2 \left(x^2 + y^2\right)}\right\}\right\}$$

Or,

$$z = c\sqrt{x^2 + y^2}$$

z=f(x,y) is the formula for a cone, if you can recognize it, according to the s.m.. Now I need to find a

surface normal on this surface. To do this I will probably need both partial derivatives.

```
ru = D[{u Cos[v], u Sin[v], c u}, u]
{Cos[v], Sin[v], c}
```

```
rv = D[{u Cos[v], u Sin[v], c u}, v]
{-u Sin[v], u Cos[v], 0}
```

```
cprod = Cross[ru, rv]
{-c u Cos[v], -c u Sin[v], u Cos[v]^2 + u Sin[v]^2}
```

```
TrigReduce[u Cos[v]^2 + u Sin[v]^2]
u
```

```
cprodf = cprod /. u Cos[v]^2 + u Sin[v]^2 → u
{-c u Cos[v], -c u Sin[v], u}
```

The above line holds the normal vector the problem was looking for.

5. Paraboloid of revolution $\mathbf{r}(\mathrm{u}, \mathrm{v}) = \{\mathrm{u}\ \mathrm{Cos}[\mathrm{v}], \mathrm{u}\ \mathrm{Sin}[\mathrm{v}], u^2\}$

```
Clear["Global`*"]
```

```
x = u Cos[v]
u Cos[v]
```

```
y = u Sin[v]
u Sin[v]
```

```
z = u^2
u^2
```

```
TrigReduce[x^2 + y^2]
u^2
```

So $x^2 + y^2 = z$. What about the parameter curves? If u = const., then z is const., and $x^2 + y^2 =$const., in other words, circles again. If v is const., then the coordinates of the parameter curve look like (c1 u, c2 u, u^2), which will describe a parabola. Now I am ready to go on to the normal vector calculation.

```
ru = D[{u Cos[v], u Sin[v], u^2}, u]
{Cos[v], Sin[v], 2 u}
```

```
rv = D[{u Cos[v], u Sin[v], u^2}, v]
{-u Sin[v], u Cos[v], 0}
```

```
cprod = Cross[ru, rv]
```

$\{-2\,u^2 \text{Cos}[v], -2\,u^2 \text{Sin}[v], u\,\text{Cos}[v]^2 + u\,\text{Sin}[v]^2\}$

```
cprodf = cprod /. u Cos[v]^2 + u Sin[v]^2 → u
```

$\{-2\,u^2 \text{Cos}[v], -2\,u^2 \text{Sin}[v], u\}$

The above line contains the normal vector expression, which agrees with the text's answer.

7. Ellipsoid r(u, v) = {a Cos[v] Cos[u], b Cos[v] Sin[u], c Sin[v]}

```
Clear["Global`*"]
```

```
x = a Cos[v] Cos[u]
y = b Cos[v] Sin[u]
z = c Sin[v]
```

```
a Cos[u] Cos[v]
```

```
b Cos[v] Sin[u]
```

```
c Sin[v]
```

```
firs = Cos[v] Cos[u];
seco = Cos[v] Sin[u];
thir = Sin[v];
FullSimplify[firs^2 + seco^2 + thir^2]
```

```
1
```

The somewhat amazing result stated on the above line allows the ellipsoid equation to become $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

The ratio of the coefficients *a* and *b* (and *c*) describe the eccentricity of the ellipsoid. If u = const., then a surface incorporating the still-active v is created, an ellipse. If v = const., then a different kind of ellipse is expressed, still dependent on the ratio between a and b. I can try to find the normal vector.

```
ru = D[{a Cos[v] Cos[u], b Cos[v] Sin[u], c Sin[v]}, u]
```

```
{-a Cos[v] Sin[u], b Cos[u] Cos[v], 0}
```

```
rv = D[{a Cos[v] Cos[u], b Cos[v] Sin[u], c Sin[v]}, v]
```

```
{-a Cos[u] Sin[v], -b Sin[u] Sin[v], c Cos[v]}
```

```
cprod = Cross[ru, rv]
```

$\{b\,c\,\text{Cos}[u]\,\text{Cos}[v]^2, a\,c\,\text{Cos}[v]^2\,\text{Sin}[u],$
$a\,b\,\text{Cos}[u]^2\,\text{Cos}[v]\,\text{Sin}[v] + a\,b\,\text{Cos}[v]\,\text{Sin}[u]^2\,\text{Sin}[v]\}$

```
EqualTo[a b Cos[u]^2 Cos[v] Sin[v] + a b Cos[v] Sin[u]^2 Sin[v]][
 a b Cos[v] Sin[v]]
```

```
a b Cos[v] Sin[v] == a b Cos[u]^2 Cos[v] Sin[v] + a b Cos[v] Sin[u]^2 Sin[v]
```

MMAflips the compared expressions around, its indication that it concurs in the equality.

```
cprodf = cprod /.
  a b Cos[u]^2 Cos[v] Sin[v] + a b Cos[v] Sin[u]^2 Sin[v] -> a b Cos[v] Sin[v]
```

```
{b c Cos[u] Cos[v]^2, a c Cos[v]^2 Sin[u], a b Cos[v] Sin[v]}
```

The above line contains the text answer for the normal vector.

11. Satisfying numbered line (4), p. 441. Represent the paraboloid in problem 5 so that $\tilde{N}(0, 0) \neq 0$ and show $\tilde{N}$.

13. Representation $z = f(x, y)$. Show that $z = f(x, y)$ or $g = z - f(x, y) = 0$ can be written $\left(f_u = \frac{\partial f}{\partial u}, \text{ etc.}\right)$

Even with treatment in the s.m., I don't understand what is supposed to happen with the above problem.

14 - 19 Derive a parametric representation

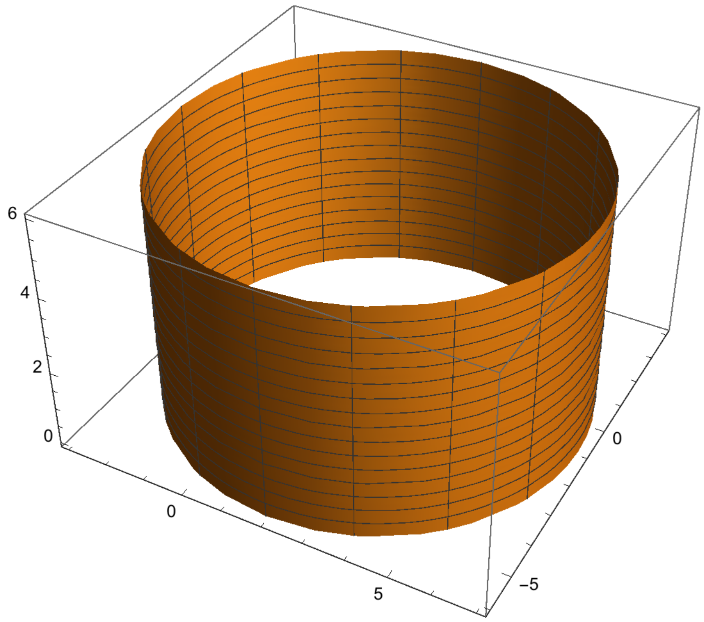
Find a normal vector. The answer gives one representation; there are many. Sketch the surface and parameter curves.

15. Cylinder of revolution $(x - 2)^2 + (y + 1)^2 = 25$

```
Clear["Global`*"]
```

The constants just specify the location of the center of the cylinder.

```
ParametricPlot3D[{2 + 5 Cos[u], -1 + 5 Sin[u], v}, {u, 0, 2 π}, {v, 0, 6}]
```



The parametric curves are shown in the figure. In the constant-u plane they are circles. In the constant-v plane they are straight lines.

```
par[u_, v_] = {2 + 5 Cos[u], -1 + 5 Sin[u], v}
{2 + 5 Cos[u], -1 + 5 Sin[u], v}
```

Above is the parametric equation for the cylinder. Below I will take the partial derivatives so I can find a normal vector by the cross product of them.

```
fir = D[{2 + 5 Cos[u], -1 + 5 Sin[u], v}, {u}]
{-5 Sin[u], 5 Cos[u], 0}
```

```
sec = D[{2 + 5 Cos[u], -1 + 5 Sin[u], v}, {v}]
{0, 0, 1}
```

```
norm = Cross[fir, sec]
```

```
{5 Cos[u], 5 Sin[u], 0}
```

The vector shown in the line above is a normal vector, and agrees with the text’s answer.

17. Sphere $x^2 + (y + 2.8)^2 + (z - 3.2)^2 = 2.25$

```
Clear["Global`*"]
```

Again, the constants 2.8 and -3.2 merely locate the center of the sphere. In the cartesian formula for the sphere, the quantity 2.25 above is the square of the radius.

```
radd = (2.25)^.5
```

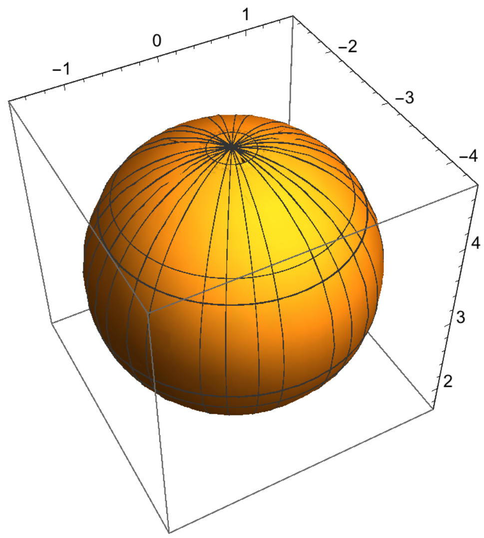
```
1.5
```

And so my parametric version will look like

```
parsph[u_, v_] =
 {1.5 Sin[u] Cos[v], 1.5 Sin[u] Sin[v] - 2.8, 1.5 Cos[u] + 3.2}
```

```
{1.5` Cos[v] Sin[u], -2.8` + 1.5` Sin[u] Sin[v], 3.2` + 1.5` Cos[u]}
```

```
ParametricPlot3D[{1.5 Sin[u] Cos[v],
  1.5 Sin[u] Sin[v] - 2.8, 1.5 Cos[u] + 3.2}, {u, -7, 7}, {v, -7, 7}]
```



As for parametric curves, they are circles in both the constant-u and constant-v planes, and are shown in the figure. And the center of the sphere seems to be in the same location as in the cartesian one.

```
fir = D[{1.5 Sin[u] Cos[v], 1.5 Sin[u] Sin[v] - 2.8, 1.5 Cos[u] + 3.2}, {u}]
{1.5 Cos[u] Cos[v], 1.5 Cos[u] Sin[v], -1.5 Sin[u]}
```

```
sec = D[{1.5 Sin[u] Cos[v], 1.5 Sin[u] Sin[v] - 2.8, 1.5 Cos[u] + 3.2}, {v}]
{-1.5 Sin[u] Sin[v], 1.5 Cos[v] Sin[u], 0}
```

```
norm = Simplify[Cross[fir, sec]]
{2.25 Cos[v] Sin[u]^2, 2.25 Sin[u]^2 Sin[v], 2.25 Cos[u] Sin[u]}
```

The above answer does not match the text’s, but maybe it is equivalent. My Sin[u] terms match the Cos[v] terms in the answer, and my Cos[u] matches the Sin[v] terms. Anyway, the structure of the normal vector is consistent with that in the answer.

19. Hyperbolic cylinder $x^2 - y^2 = 1$

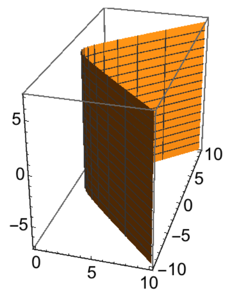
```
Clear["Global`*"]
```

I found a reference to a hyperbolic identity which is analogous to $\sin^2 x + \cos^2 x = 1$. It is $cosh^2 x - \sinh^2 x = 1$. Since the squared exponent is already in the problem expression, there is no need to repeat it.

```
parhyp[u_, v_] = {Cosh[u], Sinh[u], v}
```

```
{Cosh[u], Sinh[u], v}
```

```
ParametricPlot3D[{Cosh[u], Sinh[u], v}, {u, -3, 3}, {v, -7, 7}]
```



As for parameter curves, I think the ones in constant-u planes are hyperbolas (or at least half hyperbolas). The ones in constant-v planes are straight lines.

```
fir = D[{Cosh[u], Sinh[u], v}, {u}]
```

```
{Sinh[u], Cosh[u], 0}
```

```
sec = D[{Cosh[u], Sinh[u], v}, {v}]
```

```
{0, 0, 1}
```

```
norm = Cross[fir, sec]
```

```
{Cosh[u], -Sinh[u], 0}
```

The above line agrees with the normal vector contained in the text's answer.