

Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

1 - 6 Calculation of the divergence

Find $\text{div } \mathbf{v}$ and its value at P.

$$1. \mathbf{v} = \{x^2, 4y^2, 9z^2\}, \quad P : \left(-1, 0, \frac{1}{2}\right)$$

```
Clear["Global`*"]
```

```
vv[x_, y_, z_] = Div[{x^2, 4 y^2, 9 z^2}, {x, y, z}]
```

$$2x + 8y + 18z$$

$$\text{vv}\left[-1, 0, \frac{1}{2}\right]$$

$$7$$

$$3. \mathbf{v} = (x^2 + y^2)^{-1} [x, y]$$

```
Clear["Global`*"]
```

```
vv[x_, y_] = Div[{x/(x^2 + y^2), y/(x^2 + y^2)}, {x, y}]
```

$$-\frac{2x^2}{(x^2 + y^2)^2} - \frac{2y^2}{(x^2 + y^2)^2} + \frac{2}{x^2 + y^2}$$

```
Simplify[%]
```

$$0$$

$$5. \mathbf{v} = x^2 y^2 z^2 [x, y, z], \quad P : (3, -1, 4)$$

```
Clear["Global`*"]
```

```
vv[x_, y_, z_] = Div[{x^3 y^2 z^2, x^2 y^3 z^2, x^2 y^2 z^3}, {x, y, z}]
```

$$9x^2 y^2 z^2$$

$$\text{vv}[3, -1, 4]$$

$$1296$$

$$7. \text{ For what } v_3 \text{ is } \mathbf{v} = [e^x \cos[y], e^x \sin[y], v_3] \text{ solenoidal?}$$

```
Clear["Global`*"]
```

Set up a function for the div

```
vv[x_, y_, z_] = Div[{e^x Cos[y], e^x Sin[y], v3}, {x, y, z}]
2 e^x Cos[y]
```

The output of the function has no v_3 factor. This suggests a table to experiment

```
Table[{n, Div[{e^x Cos[y], e^x Sin[y], n z}, {x, y, z}]}, {n, 0, 4}]
{{0, 2 e^x Cos[y]}, {1, 1 + 2 e^x Cos[y]},
 {2, 2 + 2 e^x Cos[y]}, {3, 3 + 2 e^x Cos[y]}, {4, 4 + 2 e^x Cos[y]}}
```

From the table it is seen that whatever the coefficient of z is, that will be reflected against $2 e^x \cos[y]$ in the calculation of div. So if I want a zero outcome (solenoidal), I had better make the third place factor equal $-2 e^x \cos[y] z$. Trying

```
Div[{e^x Cos[y], e^x Sin[y], -2 e^x Cos[y] z}, {x, y, z}]
0
```

Success. So $v_3 = -2 e^x \cos[y] z$ is the answer.

11. Incompressible flow. Show that the flow with velocity vector $\mathbf{v} = y \mathbf{i}$ is incompressible. Show that the particles that at time $t = 0$ are in the cube whose faces are portions of the planes $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$ occupy at $t = 1$ the volume 1.

15 - 20 Laplacian

Calculate $\nabla^2 f$ by numbered line (3) on p. 404. Check by direct differentiation. Indicate when (3) is simpler.

15. $f = \cos[x]^2 + \sin[y]^2$

```
Clear["Global`*"]
```

```
e1 = Div[Grad[Cos[x]^2 + Sin[y]^2, {x, y}], {x, y}]
-2 Cos[x]^2 + 2 Cos[y]^2 + 2 Sin[x]^2 - 2 Sin[y]^2
```

```
e2 = FullSimplify[e1]
```

$-2 \cos[2x] + 2 \cos[2y]$

17. $f = \log[x^2 + y^2]$

```
Clear["Global`*"]
```

$$\mathbf{e1} = \text{Div}[\text{Grad}[\text{Log}[\mathbf{x}^2 + \mathbf{y}^2], \{\mathbf{x}, \mathbf{y}\}], \{\mathbf{x}, \mathbf{y}\}]$$

$$- \frac{4 \mathbf{x}^2}{(\mathbf{x}^2 + \mathbf{y}^2)^2} - \frac{4 \mathbf{y}^2}{(\mathbf{x}^2 + \mathbf{y}^2)^2} + \frac{4}{\mathbf{x}^2 + \mathbf{y}^2}$$

$\mathbf{e2} = \text{FullSimplify}[\mathbf{e1}]$

0

19. $f = \frac{1}{(\mathbf{x}^2 + \mathbf{y}^2 + \mathbf{z}^2)}$

$\text{Clear}["\text{Global`*}"]$

$$\mathbf{e1} = \text{Div}[\text{Grad}[\frac{1}{\mathbf{x}^2 + \mathbf{y}^2 + \mathbf{z}^2}, \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}], \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}]$$

$$\frac{8 \mathbf{x}^2}{(\mathbf{x}^2 + \mathbf{y}^2 + \mathbf{z}^2)^3} + \frac{8 \mathbf{y}^2}{(\mathbf{x}^2 + \mathbf{y}^2 + \mathbf{z}^2)^3} + \frac{8 \mathbf{z}^2}{(\mathbf{x}^2 + \mathbf{y}^2 + \mathbf{z}^2)^3} - \frac{6}{(\mathbf{x}^2 + \mathbf{y}^2 + \mathbf{z}^2)^2}$$

$\mathbf{e2} = \text{FullSimplify}[\mathbf{e1}]$

$$\frac{2}{(\mathbf{x}^2 + \mathbf{y}^2 + \mathbf{z}^2)^2}$$