1 - 6 Elastic deformations

Given A in a deformation y = A x, find the principal directions and corresponding factors of extension or contraction.

1. $\begin{pmatrix} 3.0 & 1.5 \\ 1.5 & 3.0 \end{pmatrix}$

```
Clear["Global`*"]
```

$$aA = \begin{pmatrix} 3.0 & 1.5 \\ 1.5 & 3.0 \end{pmatrix}$$

```
{{3., 1.5}, {1.5, 3.}}
```

```
e1 = {vals, vecs} = Eigensystem[aA]
```

```
{{4.5, 1.5}, {{0.707107, 0.707107}, {-0.707107, 0.707107}}}
```

Above: The answer agrees with the text. As explained in the s.m., the eigenvalues describe the magnitude of the deformation, and the eigenvectors describe the angle of application.

$$\alpha = \text{ArcCot}\left[\frac{0.7071067811865475`}{0.7071067811865475`}\right]$$

```
0.7853981633974483`
```

$$\frac{\alpha}{\text{Degree}}$$

```
45.
```

$$\beta = \text{ArcCot}\left[\frac{-0.7071067811865475`}{0.7071067811865475`}\right]$$

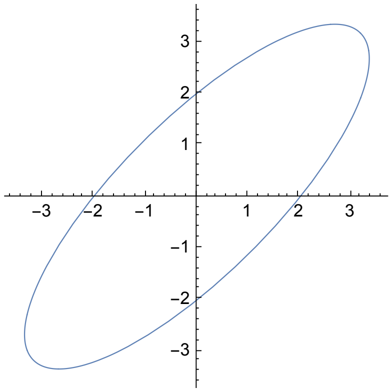
```
-0.785398
```

$$\frac{\beta}{\text{Degree}}$$

```
-45.
```

Above: So the 4.5 magnitude distortion is directed along an axis of 45 degrees (from horizontal), and the 1.5 magnitude distortion is directed along an axis of -45 degrees.

```
ParametricPlot[{4.5` Cos[t] Cos[π/4] - 1.5` Sin[t] Sin[π/4],
  4.5` Cos[t] Sin[π/4] + 1.5` Sin[t] Cos[π/4]},
 {t, 0, 2 π}, ImageSize → 200, PlotStyle → Thickness[0.003]]
```



3. $\begin{pmatrix} 7 & \sqrt{6} \\ \sqrt{6} & 2 \end{pmatrix}$

```
Clear["Global`*"]
```

aA = $\begin{pmatrix} 7 & \sqrt{6} \\ \sqrt{6} & 2 \end{pmatrix}$

$\{\{7, \sqrt{6}\}, \{\sqrt{6}, 2\}\}$

```
e1 = {vals, vecs} = Eigensystem[aA]
```

$\{\{8, 1\}, \{\{\sqrt{6}, 1\}, \{-\frac{1}{\sqrt{6}}, 1\}\}\}$

Above: The eigenvalues and eigenvectors match the answer in the text.

```
e1[[2, 1, 1]]
```

$\sqrt{6}$

e2 = α = ArcCot$\left[\frac{\text{e1[[2, 1, 1]]}}{\text{e1[[2, 1, 2]]}}\right]$ // N

0.387597

e3 = $\frac{\text{e2}}{\text{Degree}}$

22.2077

Above: So the distortion of magnitude 8 is directed along an axis of +22.2 degrees above horizontal.

```
e1[[2, 2]]
```

$\{-\frac{1}{\sqrt{6}}, 1\}$

$$e4 = \beta = \text{ArcCot}\left[\frac{e1[[2, 2, 1]]}{e1[[2, 2, 2]]}\right] // N$$

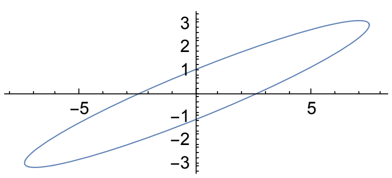
-1.1832

$$e5 = \frac{e4}{\text{Degree}}$$

-67.7923

Above: And the distortion of magnitude 1 is directed along an axis of −67.8 degrees above horizontal.

```
ParametricPlot[{8.` Cos[t] Cos[.387] - 1` Sin[t] Sin[.387],
  8.` Cos[t] Sin[.387] + 1` Sin[t] Cos[.387]},
 {t, 0, 2 π}, ImageSize → 200, PlotStyle → Thickness[0.003]]
```



5. $\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$

```
Clear["Global`*"]
```

$$aA = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$

$\{\{1, \frac{1}{2}\}, \{\frac{1}{2}, 1\}\}$

```
e1 = {vals, vecs} = Eigensystem[aA]
```

$\{\{\frac{3}{2}, \frac{1}{2}\}, \{\{1, 1\}, \{-1, 1\}\}\}$

Above: The eigenvalues and eigenvectors agree with the answer in the text.

```
vecs[[1, 1]]
```

1

```
e2 = ArcCot[vecs[[1, 1]] / vecs[[1, 2]]]
```

$\frac{\pi}{4}$

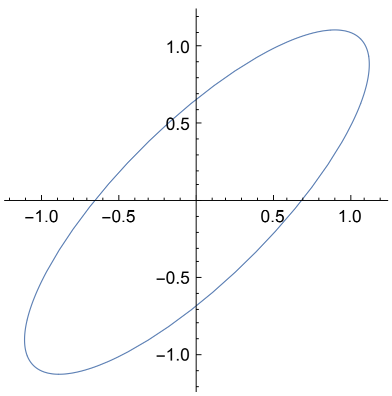
Above: So, the distortion of magnitude $\frac{3}{2}$ is directed along an axis rotated +45 degrees.

```
e3 = ArcCot[vecs[[2, 1]] / vecs[[2, 2]]]
```

$-\frac{\pi}{4}$

And the distortion of magnitude $\frac{1}{2}$ is directed along an axis rotated -45 degrees.

```
ParametricPlot[
 {3/2 Cos[t] Cos[π/4] - 1/2 Sin[t] Sin[π/4], 3/2 Cos[t] Sin[π/4] + 1/2 Sin[t] Cos[π/4]},
 {t, 0, 2 π}, ImageSize → 200, PlotStyle → Thickness[0.003]]
```



7 - 9 Markov processes
Find the limit state of the Markov process modeled by the given matrix.

7. $\begin{pmatrix} 0.2 & 0.5 \\ 0.8 & 0.5 \end{pmatrix}$

Comment: The below closely follows the process and arguments of example 2 on p. 331 of the text.

```
Clear["Global`*"]
```

```
aA = (0.2 0.5
      0.8 0.5)
```

```
{{0.2, 0.5}, {0.8, 0.5}}
```

```
e1 = Transpose[aA]
```

```
{{0.2, 0.8}, {0.5, 0.5}}
```

```
e2 = {1, 1}
{1, 1}
```

```
e3 = e1.e2
{1., 1.}
```

Above: Because the transpose of aA, multiplied against a unity vector, equals the unity vector, it is concluded that the transpose of aA has 1 as an eigenvalue. And because the transpose of aA has 1 as an eigenvalue, then aA also has 1 as an eigenvalue in consequence of theorem 3 in section 8.1, p. 328.

```
e4 = aA - IdentityMatrix[2]
{{-0.8, 0.5}, {0.8, -0.5}}
```

Above: The first step in identifying the eigenvector corresponding to the eigenvalue 1 for aA, the existence of which was established.

```
e5 = RowReduce[e4]
{{1, -0.625}, {0, 0}}
```

Above: The second step.

```
e6 = {x1, x2}
{x1, x2}
```

Above: Bringing the last player onto the stage.

```
e7 = Thread[e4.e6 == 0]
{-0.8 x1 + 0.5 x2 == 0, 0.8 x1 - 0.5 x2 == 0}
```

Above: e5 has an empty row. This empty row means that one coordinate of the eigenvector for eigenvalue 1 can be assigned arbitrarily.

```
e8 = Solve[e7, {x1, x2}]
Solve::svars: Equations may not give solutions for all "solve" variables >>
{{x2 → 0. + 1.6 x1}}
```

Above: The assignment will be for x1=5.

```
e9 = e8 /. x1 → 5
{{x2 → 8.}}
```

Above: The answers for the eigenvector corresponding to eigenvalue 1 for aA agrees with the text. The eigenvector is {5, 8}.

9. $\begin{pmatrix} 0.6 & 0.1 & 0.2 \\ 0.4 & 0.1 & 0.4 \\ 0 & 0.8 & 0.4 \end{pmatrix}$

```
Clear["Global`*"]
```

```
aA = (0.6 0.1 0.2
      0.4 0.1 0.4
      0   0.8 0.4)
```

```
{{0.6, 0.1, 0.2}, {0.4, 0.1, 0.4}, {0, 0.8, 0.4}}
```

```
e1 = Transpose[aA]
```

```
{{0.6, 0.4, 0}, {0.1, 0.1, 0.8}, {0.2, 0.4, 0.4}}
```

```
e2 = e1.{1, 1, 1}
```

```
{1., 1., 1.}
```

```
e3 = aA - IdentityMatrix[3]
```

```
{{-0.4, 0.1, 0.2}, {0.4, -0.9, 0.4}, {0, 0.8, -0.6}}
```

```
e4 = RowReduce[e3]
```

```
{{1, 0., -0.6875}, {0, 1, -0.75}, {0, 0, 0}}
```

Above: There is an empty row in the row echelon version of aA.

```
e5 = {x1, x2, x3}
```

```
{x1, x2, x3}
```

```
e6 = Thread[e4.e5 == 0]
```

```
{0. + x1 - 0.6875 x3 == 0, x2 - 0.75 x3 == 0, True}
```

```
e7 = Solve[e6, e5]
```

```
Solve::svars : Equations may not give solutions for all "solve" variables. >>
```

```
{{x2 → 0. + 1.09091 x1, x3 → 0. + 1.45455 x1}}
```

```
e8 = e7 /. x1 → 11
```

Above: Since I have the answer to refer to, I know that x1 coordinate should be 11.

```
{{x2 → 12., x3 → 16.}}
```

Above: The answer matches the text. The eigenvector sought equals {11, 12, 16}

10 - 12 Age-specific population
Find the growth rate in the Leslie model (see example 3, p. 331) with the matrix as given.

11. $\begin{pmatrix} 0 & 3.45 & 0.6 \\ 0.90 & 0 & 0 \\ 0 & 0.45 & 0 \end{pmatrix}$

```
Clear["Global`*"]
```

$$aA = \begin{pmatrix} 0 & 3.45 & 0.6 \\ 0.90 & 0 & 0 \\ 0 & 0.45 & 0 \end{pmatrix}$$

```
{{0, 3.45, 0.6}, {0.9, 0, 0}, {0, 0.45, 0}}

e1 = {400, 400, 400}
{400, 400, 400}

e2 = aA.e1
{1620., 360., 180.}

e3 = aA.e2
{1350., 1458., 162.}

e4 = aA.e3
{5127.3, 1215., 656.1}
```

$$e6 = \mathrm{Det}\left[\begin{matrix} -LL & 3.45 & 0.6 \\ 0.90 & -LL & 0 \\ 0 & 0.45 & -LL \end{matrix}\right]$$

$$1.\ \left(0.243 + 3.105\, LL - 1.\, LL^3\right)$$

```
e7 = Solve[e6 == 0]
{{LL → -1.72158}, {LL → -0.0784162}, {LL → 1.8}}
```

Above: There is one positive root, LL=1.8.

```
e8 = aA - 1.8 IdentityMatrix[3]
{{-1.8, 3.45, 0.6}, {0.9, -1.8, 0.}, {0., 0.45, -1.8}}

e9 = {x1, x2, x3}
{x1, x2, x3}

e10 = Thread[e8.e9 == 0]
{-1.8 x1 + 3.45 x2 + 0.6 x3 == 0,
 0. + 0.9 x1 - 1.8 x2 == 0, 0. + 0.45 x2 - 1.8 x3 == 0}

e11 = Solve[e10]
{{x2 → 0. + 0.5 x1, x3 → 0. + 0.125 x1}}
```

```
e12 = e11 /. x1 → 1
{{x2 → 0.5, x3 → 0.125}}
```

```
e13 = {1, .5, .125}
{1, 0.5, 0.125}
```

$$e14 = \frac{1}{1.625}$$

```
0.615385
```

```
e15 = 1200 e14
738.462
```

Above: This number has to be multiplied by the first coordinate of the eigenvector: 1.

```
e16 = e15 .5
369.231
```

Above: This number has to be multiplied by the second coordinate of the eigenvector: 0.5.

```
e17 = e15 .125
92.3077
```

Above: This number has to be multiplied by the third coordinate of the eigenvector: .125.

```
e18 = e15 + e16 + e17
1200.
```

Above: The three initial classes are shown to be equal to the original number.

Above: By an odd coincidence, the eigenvalue-derived factor (which is the sum of the eigenvector coordinates) WAS exactly the same for this problem as for the example 3 on p 331. As for the problem answer, the book only gives the eigenvalue, 1.8. This value is shown as e7[[3]]. As for the calculation of initial class sizes for ‘proportional growth’, it took awhile to figure that one out.

```
2.3 / .4
5.75
```

```
3.45 / .6
5.75
```

```
.6 / .3
```

```
2.
```

```
.9 / .45
```

```
2.
```

The ‘odd coincidence’ noted above is explained. The matrices (text example/text problem) have linearly dependent entries for key locations.

13 - 15 Leontief models

13. Leontief input-output model. suppose that three industries are interrelated so that their outputs are used as inputs by themselves, according to the 3 × 3 **consumption matrix**

$$\begin{pmatrix} 0.1 & 0.5 & 0 \\ 0.8 & 0 & 0.4 \\ 0.1 & 0.5 & 0.6 \end{pmatrix}$$

where a_{jk} is the fraction of the output of industry k consumed (purchased) by industry j. Let p_j be the price charged by industry j for its total output. A problem is to find prices so that for each industry, total expenditures equal total income. Show that this leads to **A p = p**, where **p** $= \{\{p_1, p_2, p_3\}\}^{\intercal}$, and find a solution **p** with nonnegative p_1, p_2, p_3.

This problem would consist of finding an eigenvalue equal to 1, like the Markov problem.

```
Clear["Global`*"]
```

$$\texttt{aA} = \begin{pmatrix} 0.1 & 0.5 & 0 \\ 0.8 & 0 & 0.4 \\ 0.1 & 0.5 & 0.6 \end{pmatrix}$$

```
{{0.1, 0.5, 0}, {0.8, 0, 0.4}, {0.1, 0.5, 0.6}}
```

```
e1 = Transpose[aA]
{{0.1, 0.8, 0.1}, {0.5, 0, 0.5}, {0, 0.4, 0.6}}
```

```
e2 = {1, 1, 1}
{1, 1, 1}
```

```
e3 = e1.e2
{1., 1., 1.}
```

Above: So the transpose has an eigenvalue equal to 1, implying that the matrix aA also has one. Now to find it.

```
e4 = aA - IdentityMatrix[3]
{{-0.9, 0.5, 0}, {0.8, -1, 0.4}, {0.1, 0.5, -0.4}}

e5 = RowReduce[e4]
{{1, 0., -0.4}, {0, 1, -0.72}, {0, 0, 0}}

e6 = {x1, x2, x3}
{x1, x2, x3}

e7 = Thread[e5.e6 == 0]
{0. + x1 - 0.4 x3 == 0, x2 - 0.72 x3 == 0, True}

e8 = Solve[e7, e6]
Solve::svars : Equations may not give solutions for all "solve" variables >>
{{x2 → 0. + 1.8 x1, x3 → 0. + 2.5 x1}}

e9 = e8 /. x1 → 10

{{x2 → 18., x3 → 25.}}
```

Above: The answer matches the text. (The 10 of course was taken from the answer.)

```
738 × .125
92.25
```