Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

```
Clear["Global`*"]
```

1 - 10 Zeros

Determine the location and order of the zeros.

1. $\text{Sin}\left[\frac{1}{2} z\right]^4$

```
Clear["Global`*"]
```

$\text{f1}[z\_] = \text{Sin}\left[\frac{1}{2} z\right]^4$

$\text{Sin}\left[\frac{z}{2}\right]^4$

I make a table to look at. The first non-zero derivative position is at 4th order, that is in agreement with text answer. (There is no need to look at odd z-coefficients.) The location consists of even multiples of 2 π.

```
TableForm[Table[
  {i, ± (i π), D[f1[z], {z, i}], D[f1[z], {z, i}] /. z → i π}, {i, 0, 6, 2}],
 TableHeadings → {{}, {"D-Num.", "z", "Derivative", "Value"}}]
```

D-Num.	z	Derivative	Value
0	± 0	$\text{Sin}\left[\frac{z}{2}\right]^4$	0
2	$\pm(2\pi)$	$3\,\text{Cos}\left[\frac{z}{2}\right]^2 \text{Sin}\left[\frac{z}{2}\right]^2 - \text{Sin}\left[\frac{z}{2}\right]^4$	0
4	$\pm(4\pi)$	$\frac{3}{2}\,\text{Cos}\left[\frac{z}{2}\right]^4 - 12\,\text{Cos}\left[\frac{z}{2}\right]^2 \text{Sin}\left[\frac{z}{2}\right]^2 + \frac{5}{2}\,\text{Sin}\left[\frac{z}{2}\right]^4$	$\frac{3}{2}$
6	$\pm(6\pi)$	$-\frac{15}{2}\,\text{Cos}\left[\frac{z}{2}\right]^4 + 48\,\text{Cos}\left[\frac{z}{2}\right]^2 \text{Sin}\left[\frac{z}{2}\right]^2 - \frac{17}{2}\,\text{Sin}\left[\frac{z}{2}\right]^4$	$-\frac{15}{2}$

3. $(z + 81\, i)^4$

```
Clear["Global`*"]
```

$\text{f2}[z\_] = (z + 81\, i)^4$

$(81\, i + z)^4$

Obviously f2 has a simple zero, and there is evidently a theorem that states that if a function has a simple zero, it has a zero at a raised integer power. So to investigate

```
TableForm[
 Table[{i, D[f2[z], {z, i}], D[f2[z], {z, i}] /. z → -81 ⅈ}, {i, 6}]]
```

1	$4\,(81\,i + z)^3$	0
2	$12\,(81\,i + z)^2$	0
3	$24\,(81\,i + z)$	0
4	24	24
5	0	0
6	0	0

Since the 4th order is the first occasion when the evaluated derivative does not equal zero, then f2 has a 4th order zero. The location is -81 i.

5. $z^{-2}\,\mathrm{Sin}[\pi z]^2$

```
Clear["Global`*"]
```

```
f3[z_] = z^-2 Sin[π z]^2
```

$$\frac{\mathrm{Sin}[\pi z]^2}{z^2}$$

I split the results into two tables because of line length. When z=2, the derivative is non-zero, because the cosine part of the third term survives. This is second order, and the location is z=±2.

```
TableForm[Table[{i, ± (i ), D[f3[z], {z, i}]}, {i, 0, 3}],
 TableHeadings → {{}, {"D-Num.", "z", "Derivative"}}]
```

D-Num.	z	Derivative
0	±0	$\frac{\mathrm{Sin}[\pi z]^2}{z^2}$
1	±1	$\frac{2\pi\,\mathrm{Cos}[\pi z]\,\mathrm{Sin}[\pi z]}{z^2} - \frac{2\,\mathrm{Sin}[\pi z]^2}{z^3}$
2	±2	$-\frac{8\pi\,\mathrm{Cos}[\pi z]\,\mathrm{Sin}[\pi z]}{z^3} + \frac{6\,\mathrm{Sin}[\pi z]^2}{z^4} + \frac{2\pi^2\,\mathrm{Cos}[\pi z]^2 - 2\pi^2\,\mathrm{Sin}[\pi z]^2}{z^2}$
3	±3	$\frac{36\pi\,\mathrm{Cos}[\pi z]\,\mathrm{Sin}[\pi z]}{z^4} - \frac{8\pi^3\,\mathrm{Cos}[\pi z]\,\mathrm{Sin}[\pi z]}{z^2} - \frac{24\,\mathrm{Sin}[\pi z]^2}{z^5} - \frac{6\left(2\pi^2\,\mathrm{Cos}[\pi z]^2 -\right.}{z^3}$

```
TableForm[Table[{i, ± (i ), D[f3[z], {z, i}] /. z → i }, {i, 0, 3}],
 TableHeadings → {{}, {"D-Num.", "z", "Derivative", "Value"}}]
```

Power::infy : Infinite expression $\frac{1}{0^2}$ encountered. ≫

Infinity::indet : Indeterminate expression 0 ComplexInfinity encountered. ≫

D-Num.	z	Derivative
0	±0	Indeterminate
1	±1	0
2	±2	$\frac{\pi^2}{2}$
3	±3	$-\frac{4\pi^2}{9}$

7. $z^4 + (1 - 8\,i)\, z^2 - 8\,i$

```
Clear["Global`*"]
```

```
f4[z_] = z^4 + (1 - 8 i) z^2 - 8 i
```

$-8\,i + (1 - 8\,i)\, z^2 + z^4$

Solving the function shows that solutions exist at the zeroth or simple order, and gives the locations.

```
Solve[f4[z] == 0, z]
```

$\{\{z \to -2 - 2\,i\}, \{z \to -i\}, \{z \to i\}, \{z \to 2 + 2\,i\}\}$

9. Sin[2 z] Cos[2 z]

```
Clear["Global`*"]
```

```
f5[z_] = Sin[2 z] Cos[2 z]
```

Cos[2 z] Sin[2 z]

If I suspect a simple solution, I should try to solve. I find that simple solutions exist, with locations shown below.

```
Solve[f5[z] == 0, z]
```

$$\{\{z \to \text{ConditionalExpression}[\pi\, C[1],\ C[1] \in \text{Integers}]\},$$
$$\{z \to \text{ConditionalExpression}[\tfrac{1}{2}(-\tfrac{\pi}{2} + 2\pi\, C[1]),\ C[1] \in \text{Integers}]\},$$
$$\{z \to \text{ConditionalExpression}[\tfrac{1}{2}(\tfrac{\pi}{2} + 2\pi\, C[1]),\ C[1] \in \text{Integers}]\},$$
$$\{z \to \text{ConditionalExpression}[\tfrac{1}{2}(\pi + 2\pi\, C[1]),\ C[1] \in \text{Integers}]\}\}$$

13 - 22 Singularities

Determine the location of the singularities, including at infinity. For poles also state the order.

13. $\dfrac{1}{(z + 2\,i)^2} - \dfrac{z}{z - i} + \dfrac{z + 1}{(z - i)^2}$

```
Clear["Global`*"]
```

By inspection and without further ado, I see the problem function has two second order poles, one at i, another at $-2i$.

15. $z\,\text{Exp}\left[\frac{1}{(z-1-i)^2}\right]$

```
Clear["Global`*"]
```

Theorem 4 on p. 717 has implications. Looking at the inverse of the problem function, since

$$\left(z\,\text{Exp}\left[\frac{1}{(z-1-i)^2}\right]\right)^{-1}$$

$$\frac{e^{-\frac{1}{((-1-i)+z)^2}}}{z}$$

$$\text{Limit}\left[\frac{e^{-\frac{1}{((-1-i)+z)^2}}}{z},\ z\to\infty\right]$$

0

has a zero at ∞ (first order), it means that its inverse, the problem function, has a (simple) pole at infinity.

$$\text{f6}[z\_] = z\,\text{Exp}\left[\frac{1}{(z-1-i)^2}\right]$$

$$e^{\frac{1}{((-1-i)+z)^2}}\,z$$

If I try writing

$$\text{Series}\left[z\,\text{Exp}\left[\frac{1}{(z-(1+i))^2}\right],\ \{z,\ 1+i,\ 5\}\right]$$

$$e^{\frac{1}{((-1-i)+z)^2}}\left((1+i)+(z-(1+i))+O[z-(1+i)]^6\right)$$

what is returned by Mathematica is not useful, apparently because Mathematica does not wish to sacrifice what it considers the most logical form of the expression. What I can do is write

$$\text{ser} = \text{Normal@Series}[z\,\text{Exp}[z],\ \{z,\ 0,\ 6\}]\ /.\ z\to\frac{1}{(z-(1+i))^2}$$

$$\frac{1}{120\,((-1-i)+z)^{12}}+\frac{1}{24\,((-1-i)+z)^{10}}+\frac{1}{6\,((-1-i)+z)^{8}}+\frac{1}{2\,((-1-i)+z)^{6}}+\frac{1}{((-1-i)+z)^{4}}+\frac{1}{((-1-i)+z)^{2}}$$

(after MMAstackexchange question 44203, answered by poweierstrass) and there is more to look at. Although the terms in the request are limited in number to 6, it is clear that the **principal part** of the series potentially includes an infinite number of terms of the form $z-(1+i)$. According to the text this is the requirement for declaring the point $1+i$

as an **essential singularity**.

17. Cot[z]^4

```
Clear["Global`*"]
```

First the basics

$$\text{Cot}[z] == \frac{1}{\text{Tan}[z]} == \frac{\text{Cos}[z]}{\text{Sin}[z]}$$

```
line1 = Solve[Sin[z] == 0, z]
```

{{z → ConditionalExpression[2 π C[1], C[1] ∈ Integers]},
 {z → ConditionalExpression[π + 2 π C[1], C[1] ∈ Integers]}}

```
Series[Sin[z], {z, 2 π, 5}]
```

$$(z - 2\pi) - \frac{1}{6}(z - 2\pi)^3 + \frac{1}{120}(z - 2\pi)^5 + O[z - 2\pi]^6$$

$$\text{Series}\left[\frac{\text{Cos}[z]}{\text{Sin}[z]}, \{z, 2\pi, 5\}\right]$$

$$\frac{1}{z - 2\pi} - \frac{1}{3}(z - 2\pi) - \frac{1}{45}(z - 2\pi)^3 - \frac{2}{945}(z - 2\pi)^5 + O[z - 2\pi]^6$$

I can conclude that Cot[z] is singular where Sin[z] is 0. The above cell shows 2π to be a simple pole of Cot[z].

Looking next at

$$\text{Cot}[z]^4 == \frac{\text{Cos}[z]^4}{\text{Sin}[z]^4}$$

True

$$\text{Series}\left[\frac{\text{Cos}[z]^4}{\text{Sin}[z]^4}, \{z, 2\pi, 5\}\right]$$

$$\frac{1}{(z - 2\pi)^4} - \frac{4}{3(z - 2\pi)^2} + \frac{26}{45} - \frac{64}{945}(z - 2\pi)^2 - \frac{19(z - 2\pi)^4}{2835} + O[z - 2\pi]^6$$

Since

$$\text{Sin}[0]^4 == \text{Sin}[0]$$

True

and likewise for other integer multiples of 2π, Cot[z]^4 has the same poles as Cot[z] (as shown in line1), except now the poles are 4th order.

Using the trick used in the last problem for getting Mathematica to overcome its shyness in providing series details, I can try

```
ser = Series[Cos[z]^4, {z, 0, 8}] /. z → 1/z
```

SeriesData::sdatv: First argument $\frac{1}{z}$ is not a valid variable ≫

$$1 - 2\left(\frac{1}{z}\right)^2 + \frac{5}{3}\left(\frac{1}{z}\right)^4 - \frac{34}{45}\left(\frac{1}{z}\right)^6 + \frac{13}{63}\left(\frac{1}{z}\right)^8 + O\left[\frac{1}{z}\right]^9$$

The version supplied above shows a developed series with a principal part which is infinite in extent. If z is replaced by ∞, the base function has an essential singularity at ∞. And by theorem 4 on p. 717, this essential singularity is also shared by the $\text{Cot}[z]^4$ function.

19. $\frac{1}{(e^z - e^{2z})}$

```
Clear["Global`*"]
```

```
Solve[(1/(e^z - e^(2 z)))^-1 == 0, z]
```

{{z → ConditionalExpression[2 i π C[1], C[1] ∈ Integers]}}

By theorem 4, p. 717, the zeros revealed above imply poles at the same locations in the inverse, i.e. the problem version of the function. The zeros are first order, so the poles are simple.

Looking further,

```
Series[1/(e^z - e^(2 z)), {z, 0, 5}]
```

$$-\frac{1}{z} + \frac{3}{2} - \frac{13 z}{12} + \frac{z^2}{2} - \frac{119 z^3}{720} + \frac{z^4}{24} - \frac{253 z^5}{30240} + O[z]^6$$

According to example 5 on p. 718, the function e^z has an essential singularity at ∞. The problem function, which incorporates this function as an element, must share the characteristic.

21. $\frac{e^{1/(z-1)}}{e^z - 1}$

```
Clear["Global`*"]
```

```
def = Series[e^(1/(z-1))/(e^z - 1), {z, 0, 5}]
```

$$\frac{1}{e\, z}-\frac{3}{2\, e}+\frac{z}{12\, e}+\frac{59\, z^3}{720\, e}+\frac{z^4}{8\, e}+\frac{815\, z^5}{6048\, e}+O[z]^6$$

```
tyu = ExpToTrig[e^(1/(z-1))/(e^z - 1)]
```

$$\frac{\text{Cosh}\left[\frac{1}{-1+z}\right]}{-1+\text{Cosh}[z]+\text{Sinh}[z]}+\frac{\text{Sinh}\left[\frac{1}{-1+z}\right]}{-1+\text{Cosh}[z]+\text{Sinh}[z]}$$

The following diagnoses a singularity.

```
Solve[-1 + Cosh[z] + Sinh[z] == 0, z]
```

```
{{z → ConditionalExpression[2 ⅈ π C[1], C[1] ∈ Integers]}}
```

And the following tests it.

```
tyu1 = N[tyu /. z → 2 π ⅈ]
```

Power::infy : Infinite expression $\frac{1}{0}$ encountered. >>

Power::infy : Infinite expression $\frac{1}{0}$ encountered. >>

Infinity::indet : Indeterminate expression ComplexInfinity + ComplexInfinity encountered. >>

```
Indeterminate
```

I still need to do an inventory of essential singularities. I see that e^z is lurking in the denominator. I re-use the argument that, as the possessor of an essential singularity at ∞ (by example 5, p.718), its incorporation into the problem function transfers the trait to the problem function. The same statement holds for the numerator of the problem function, where the assumption of the value of 1 by z shows the essential singularity at that point, also as mentioned in example 5.