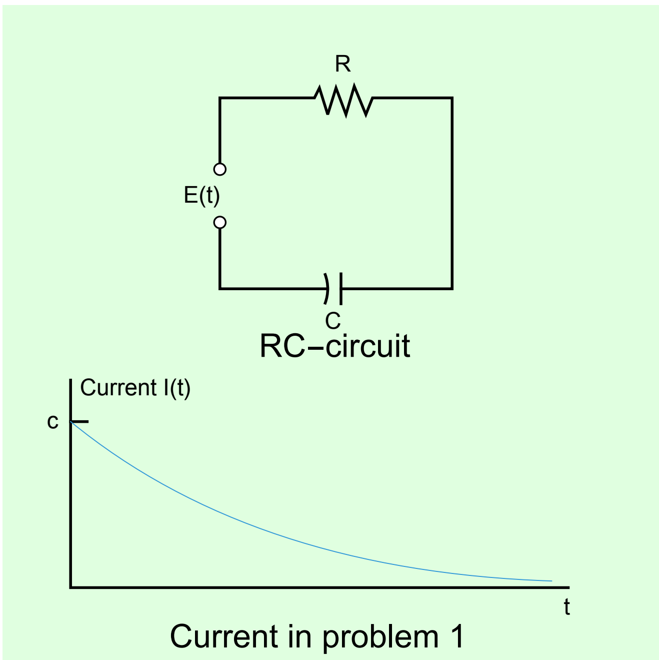
1 - 6 RLC-Circuits: special cases

1. RC-Circuit. Model the RC-Circuit in the figure below. Find the current due to a constant E.

```
Clear["Global`*"]
```



I ran across a couple of snippets suggesting that state space modeling would be a good way to look at circuits in Mathematica. However, I didn't find a cookbook recipe laid out, and didn't try to invest the time to get results.

```
eqns = {L q''[t] + R q'[t] + 1/C q[t] == V[t]};
```

$$\mathrm{eqns} = \left\{L\, q''[t] + R\, q'[t] + \frac{1}{C} q[t] == V[t]\right\};$$

```
m1 = StateSpaceModel[eqns,
  {{q[t], 0}, {q'[t], 0}}, {{V[t], 0}}, {q'[t]}, t]
```

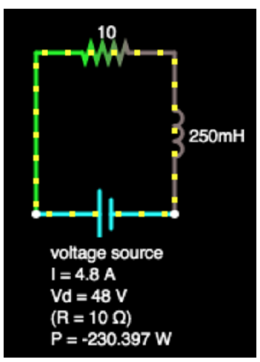
$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -\frac{1}{C\,L} & -\frac{R}{L} & \frac{1}{L} \\ \hline 0 & 1 & 0 \end{array}\right)^{\mathcal{S}}$$

$$\text{Simplify}\left[\text{OutputResponse}\left[\left\{\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -\frac{1}{C\,L} & -\frac{R}{L} & \frac{1}{L} \\ \hline 0 & 1 & 0 \end{array}\right)^{\mathcal{S}}, \{1, 0.1`\}\right\}, 0, t\right]\right]$$

$$\left\{\frac{1}{\sqrt{C}\,\sqrt{-4\,L + C\,R^2}}\left(e^{-\frac{\left(R-\frac{\sqrt{-4\,L+C\,R^2}}{\sqrt{C}}\right)t}{2\,L}}\left(-1. - 0.05\,C\,R + 0.05\,\sqrt{C}\,\sqrt{-4\,L + C\,R^2}\right) + e^{-\frac{\left(R+\frac{\sqrt{-4\,L+C\,R^2}}{\sqrt{C}}\right)t}{2\,L}}\left(1. + 0.05\,C\,R + 0.05\,\sqrt{C}\,\sqrt{-4\,L + C\,R^2}\right)\right)\right\}$$

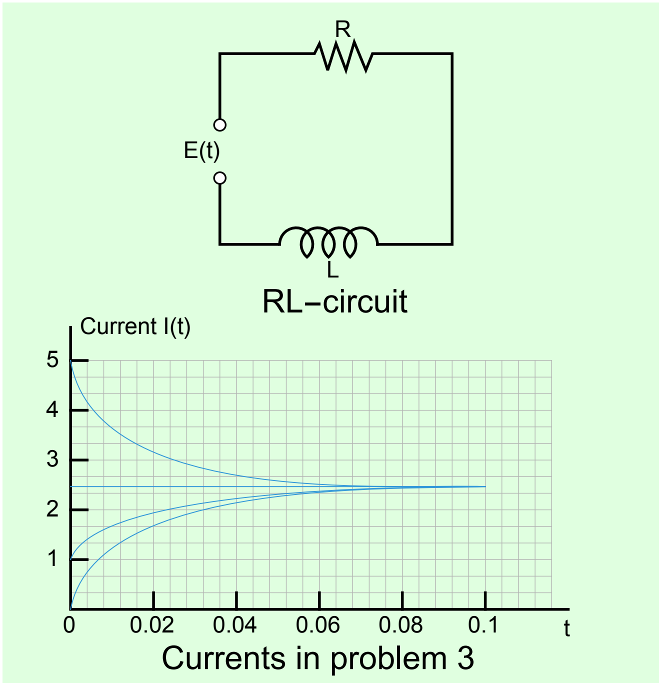
```
{ind, cap, res} = {l i'[t] == v_l[t], v_c'[t] == 1/c i[t], r i[t] == v_r[t]};
kirchhoff = v_l[t] + v_c[t] + v_r[t] == v_s[t];
```

3. RL-Circuit. Model the RL-circuit in the figure below. Find a general solution when R, L, E are any constants. Graph or sketch solutions when L = 0.25 H, R = 10 Ω, and E = 48 V.



The above screenshot came from the online app at *https://falstad.com/circuit/*. The current it shows agrees with the old formula for current, I=E/R, and was captured after the resistance had plenty of time to decay. And that's all it is, except that there is a time constant to apply. The time constant becomes ever smaller as the operation time increases. Since the problem description talks in terms of a constant state, it seems the time constant would become vanishingly small, leaving merely I=E/R=4.8 amps.

```
Clear["Global`*"]
```



When there are a lot of variables to watch, the Manipulate command is the only way I know to get an overview. The box below is based on the material at *https://www.electronics-tutorials.ws/inductor/lr-circuits.html* and may not agree with the text in detail.

eye[vee_, are_, ell_, tee_] = $\frac{\text{vee}}{\text{are}}\left(1 - e^{-\frac{\text{are tee}}{\text{ell}}}\right)$

$$\frac{\left(1 - e^{-\frac{\text{are tee}}{\text{ell}}}\right)\text{vee}}{\text{are}}$$

It takes some time for the current to reach its max value. From t=0.4 on in the green grid below, the circuit current is nominal.

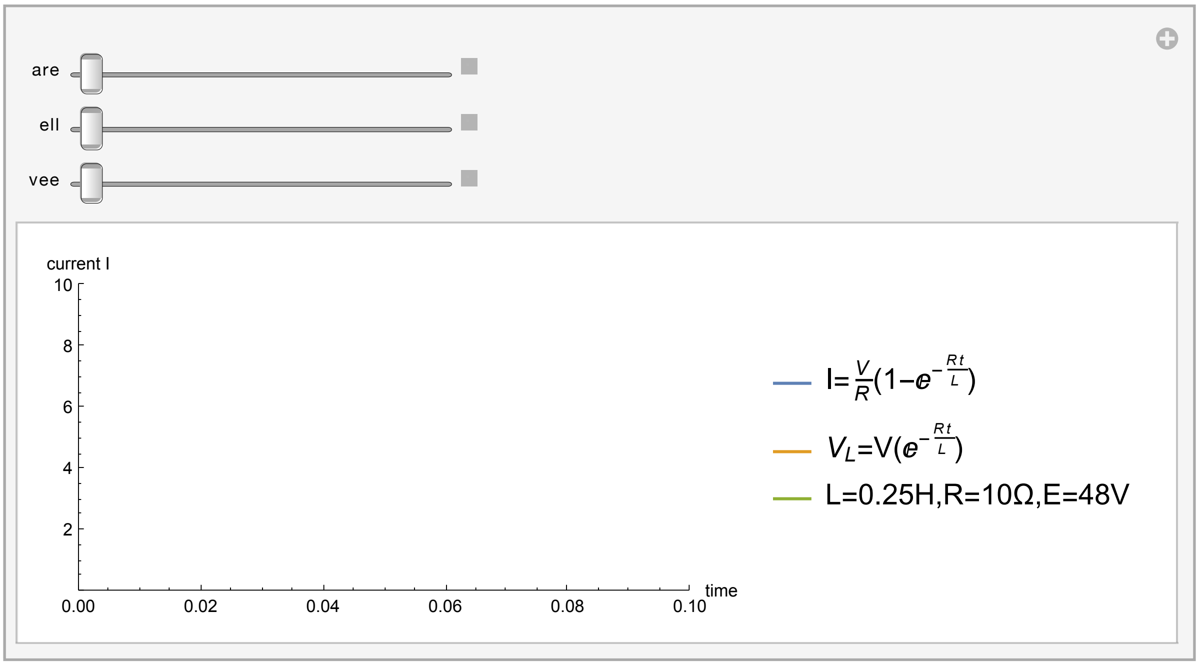
```
Grid[Table[{tee, eye[48, 10, 0.25, tee]}, {tee, 0, 0.6, 0.1}], Frame → All]
```

0.	0.
0.1	4.71208
0.2	4.79839
0.3	4.79997
0.4	4.8
0.5	4.8
0.6	4.8

veel[vee_, are_, ell_, tee_] = vee $\left(e^{-\frac{\text{are tee}}{\text{ell}}}\right)$

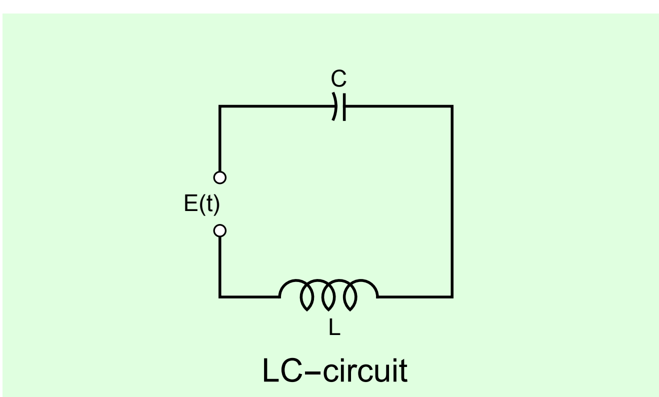
$$e^{-\frac{\text{are tee}}{\text{ell}}}\,\text{vee}$$

```
Manipulate[
 Plot[{Abs[eye[vee, are, ell, tee]], Abs[veel[vee, are, ell, tee]],
   Abs[eye[48, 10, 0.25, tee]]}, {tee, 0, 5},
  PlotLegends → {"I=V/R(1-ⅇ^(-Rt/L))", "V_L=V(ⅇ^(-Rt/L))", "L=0.25H,R=10Ω,E=48V"},
  PlotRange → {{0, 0.1}, {0, 10}}, AxesLabel → {"time", "current I"},
  AspectRatio → 0.5], {are, 1, 200}, {ell, 0.01, 10} , {vee, 1, 50}]
```



5. LC-Circuit. This is an RLC-circuit with negligibly small R (analog of an undamped mass-spring system). Find the current when L=0.5 H, C = 0.005 F, and E = Sin[t V], assuming zero initial current and charge.

```
Clear["Global`*"]
```



I found the site *https://en.wikiversity.org/wiki/RLC_circuit*, which has a formula which is shared

by the text, i.e.
L D[q[t], {t, 2}] + R D[q[t], t] - $\frac{1}{C}$ q[t] = v[t]

R being negligibly small will make the first derivative term disappear, leaving

$$\text{eqn} = 0.5\, q''[t] - \frac{1}{0.005} q[t] == \text{Sin}[t\, V]$$

$$-200.\, q[t] + 0.5\, q''[t] == \text{Sin}[t\, V]$$

I will see what DSolve can do with this problem as it stands.

```
sol = DSolve[{eqn}, q, t]
```

$$\left\{\left\{q \to \text{Function}\left[\{t\}, e^{20.\, t} C[1] + e^{-20.\, t} C[2] - \frac{2.\,(0. + 1.\, \text{Sin}[t\, V])}{400. + 1.\, V^2}\right]\right\}\right\}$$

DSolve pulls out a fully real solution which checks.

```
eqn /. sol // FullSimplify
{True}
```

$$\text{ExpToTrig}\left[e^{20.`\, t} + e^{-20.`\, t}\right]$$

$$2\, \text{Cosh}[20.\, t]$$

So to clean it up,

$$2\, \text{Cosh}[20\, t] - \frac{2\, \text{Sin}[t\, V]}{400 + V^2}$$

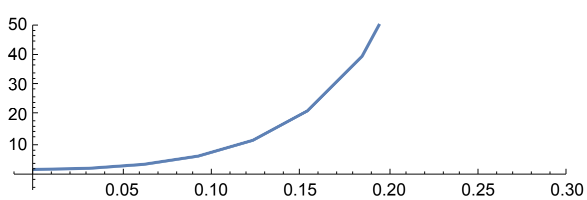
Which doesn't look too hairy, though not the same as the text answer.

Example 1 on p. 96 of the text could repay close study on this problem. Roots of the characteristic equation, along with reactance, can yield the coefficients of the particular equation after some work. One thing that bothers me is the vagueness of the voltage function. Maybe something cancels out to make the answer come out so neat. The text answer, by the way, is $\frac{I = 2(\text{Cos}[t] - \text{Cos}[20\, t])}{399}$.

Freezing some parameter values in order to take a look at the thing.

The plot does not look like figure 63 on p. 97.

$$\text{Plot}\left[2\, \text{Cosh}[20\, t] - \frac{2\, \text{Sin}[t]}{401}, \{t, 0, 100\}, \text{ImageSize} \to 300, \text{AspectRatio} \to .3, \text{PlotRange} \to \{\{-0.01, 0.3\}, \{-5, 50\}\}\right]$$



7 - 18 General RLC-circuits

7. Tuning. In tuning a sterio system to a radio station, we adjust the tuning control (turn a knob) that changes C (or perhaps L) in an RLC-circuit so that the amplitude of the

steady-state current, numbered line (5), p. 95 becomes maximum. For what C will this happen?

It is where the particular solution of the homogeneous equation is maximized. Numbered line (5) looks like

$I_p(t) = I_0 \text{Sin}[\omega t - \theta]$

The quantity θ is known as the phase lag, and, I suppose, the signal is best, I_p maximized, when θ equals zero.

8 - 14 Find the steady-state current in the RLC-circuit in the figure below for the given data.

9. R = 4 Ω, L = 0.1 H, C = 0.05 F, E = 110 V

L D[q[t], {t, 2}] + R D[q[t], t] - $\frac{1}{C}$ q[t] = v[t]

eqn = 0.1 q''[t] + 4 q'[t] - $\frac{1}{0.05}$ q[t] == 110

-20. q[t] + 4 q′[t] + 0.1 q′′[t] == 110

sol = DSolve[eqn, q, t]

{{q → Function[{t}, -5.5 + $e^{-44.4949\,t}$ C[1] + $e^{4.4949\,t}$ C[2]]}}

If C[1]=C[2]=0, then the green cell above matches the text answer.

11. R = 12 Ω, L = 0.4 H, C = $\frac{1}{80}$F, E = 220 Sin[10 t] V

eqn = 0.4 q''[t] + 12 q'[t] - 80 q[t] == 220 Sin[10 t]

-80 q[t] + 12 q′[t] + 0.4 q′′[t] == 220 Sin[10 t]

sol = DSolve[eqn, q, t]

{{q → Function[{t}, $e^{-35.6155\,t}$ C[1] + $e^{5.61553\,t}$ C[2] + 0.0974775 $e^{-8.88178\times10^{-16}\,t}$ (-10.4039 Cos[10. t] + 1. $e^{8.88178\times10^{-16}\,t}$ Cos[10. t] - 5.84233 Sin[10. t] - (3.56155 + 1.18695 × 10^{-16} i) $e^{8.88178\times10^{-16}\,t}$ Sin[10. t])]}}

```
FullSimplify[
 ⅇ^(-35.6155281280883` t) C[1] + ⅇ^(5.615528128088303` t) C[2] + 0.09747747404159644`
   ⅇ^(-8.881784197001252`*^-16 t) (-10.403882032022072` Cos[10.` t] +
     1.` ⅇ^(8.881784197001252`*^-16 t) Cos[10.` t] - 5.842329219213245`
      Sin[10.` t] - (3.56155281280883` + 1.186954912693127`*^-16 ⅈ)
      ⅇ^(8.881784197001252`*^-16 t) Sin[10.` t]) /. {C[1] → 0, C[2] → 0}]
```

$e^{-8.88178\times10^{-16}\,t}\left(\left(-1.01414+0.0974775\,e^{8.88178\times10^{-16}\,t}\right)\mathrm{Cos}[10.\,t]+\left(-0.569495-\left(0.347171+1.15701\times10^{-17}\,i\right)e^{8.88178\times10^{-16}\,t}\right)\mathrm{Sin}[10.\,t]\right)$

```
FullSimplify[ⅇ^(-8.881784197001252`*^-16 t)
  ((-1.014144140708263` + 0.09747747404159644` ⅇ^(8.881784197001252`*^-16 t))
    Cos[10.` t] + (-0.5694954948083195` - (0.3471711718583475` +
          1.1570136669058967`*^-17 ⅈ) ⅇ^(8.881784197001252`*^-16 t)) Sin[10.` t])]
```

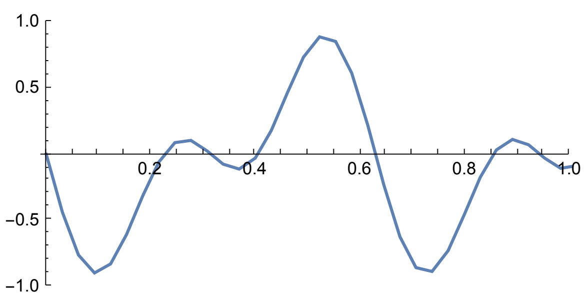
$e^{-8.88178\times10^{-16}\,t}\left(\left(-1.01414+0.0974775\,e^{8.88178\times10^{-16}\,t}\right)\mathrm{Cos}[10.\,t]+\left(-0.569495-\left(0.347171+1.15701\times10^{-17}\,i\right)e^{8.88178\times10^{-16}\,t}\right)\mathrm{Sin}[10.\,t]\right)$

Dump the imaginary.

```
ⅇ^(-8.881784197001252`*^-16 t)
  ((-1.014144140708263` + 0.09747747404159644` ⅇ^(8.881784197001252`*^-16 t))
    Cos[10.` t] + (-0.5694954948083195`)
    ⅇ^(8.881784197001252`*^-16 t)) Sin[10.` t])
```

Try to plot it.

```
Plot[ⅇ^(-8.881784197001252`*^-16 t)
  ((-1.014144140708263` + 0.09747747404159644` ⅇ^(8.881784197001252`*^-16 t))
    Cos[10.` t] + (-0.5694954948083195`) ⅇ^(8.881784197001252`*^-16 t))
  Sin[10.` t], {t, 0, 100}, ImageSize → 300, AspectRatio → .5,
 PlotRange → {{-0.01, 1}, {-1, 1}}]
```



By the way, the trig elements in the text answer are present in the yellow. The text answer is I=5.5Cos[10 t]+16.5 Sin[10 t] amperes.

13. R = 12, L = 1.2 H, C = $\frac{20}{3}$*10^{-3} F, E = 12,000 Sin[25 t] V

$C = \frac{20}{3} * \frac{1}{1000} = \frac{20}{3000} = \frac{2}{300}$

```
eqn = 1.2 q''[t] + 12 q'[t] - 300/2 q[t] == 12 000 Sin[25 t]
-150 q[t] + 12 q′[t] + 1.2 q″[t] == 12 000 Sin[25 t]

sol = DSolve[eqn, q, t]
{{q → Function[{t}, e^(-17.2474 t) C[1] + e^(7.24745 t) C[2] - (4. - 7.08198 × 10^-16 i)
    ((1. + 0. i) Cos[25. t] + (3. + 5.31148 × 10^-16 i) Sin[25. t])]}}

(e^(-17.24744871391589` t) + e^(7.24744871391589` t) - (4.) Cos[25.` t] + (3.) Sin[25.` t]) /.
 t → 0.5
33.2869
```

The next few cells try out the Assini method of undetermined coefficients.

```
hAndp[odeH_, rhs_, y_, x_] :=
  Module[{wronskian, u1, u2, solH, y1, y2, leadingC},
   leadingC = Cases[odeH, c_ y''[x] :> c];
   leadingC = If[leadingC === {}, 1, First@leadingC];
   solH = (y[x] /. First@DSolve[odeH == 0, y[x], x]);
   {y1, y2} = solH /. C[1] y1_ + C[2] y2_ :> {y1, y2};
   (*basis solutions*)
   wronskian = Det[{{y1, y2}, {D[y1, x], D[y2, x]}}];
   u1 = -Integrate[y2 rhs / (leadingC * wronskian), x];
   u2 = Integrate[y1 rhs / (leadingC * wronskian), x];
   {solH, Simplify[y1 u1 + y2 u2]}];

odeH = -150 q[t] + 12 q′[t] + 1.2` q″[t];
rhs = 12 000 Sin[25 t];

{yh, yp} = hAndp[odeH, rhs, q, t]
{e^(-17.2474 t) C[1] + e^(7.24745 t) C[2],
 (-4. - 7.08198 × 10^-16 i) Cos[25. t] - 12. Sin[25. t]}

fullSolution = yh + yp
e^(-17.2474 t) C[1] + e^(7.24745 t) C[2] -
 (4. + 7.08198 × 10^-16 i) Cos[25. t] - 12. Sin[25. t]
```

Trying to clean and standardize a little.

```
mfs = e^(-17.24744871391589` t) + e^(7.24744871391589` t) - (4.) Cos[25.` t] - 12.` Sin[25.` t]
e^(-17.2474 t) + e^(7.24745 t) - 4. Cos[25. t] - 12. Sin[25. t]

mfs1 = mfs /. t → 0.5
34.2817
```

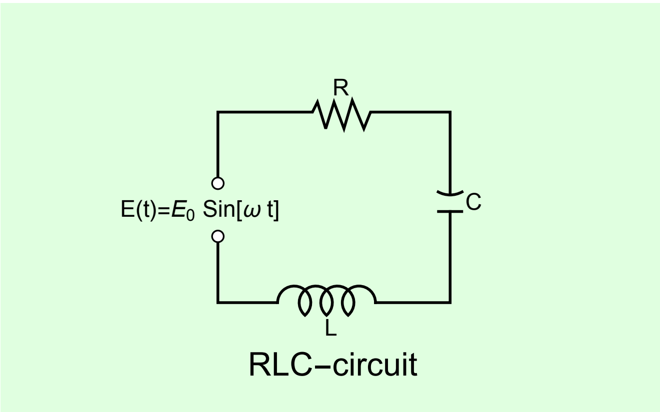
```
(ⅇ^(-5 t) (Cos[10 t] + Sin[10 t]) - 400 Cos[25 t] + 200 Sin[25 t]) /. t → 0.5
```

```
-412.439
```

As can be seen above, the text answer (pink) is not achieved.

15. Cases of damping. What are the conditions for an RLC-circuit to be (I) overdamped, (II) critically damped, (III) underdamped? What is the critical resistance R_{crit} (the analog of the critical damping constant $2\sqrt{m\,k}$?

16 - 18 Solve the initial value problem for the RLC-circuit shown below, with the given data, assuming zero initial current and charge. Graph or sketch the solution.



17. R = 6 Ω, L = 1 H, C = 0.04 F, E = 600(Cos[t] + 4 Sin[t])V

$L\,D[q[t], \{t, 2\}] + R\,D[q[t], t] - \frac{1}{C}\,q[t] = v[t]$

```
Clear["Global`*"]
```

```
eqn = 1 q''[t] + 6 q'[t] - 25 q[t] == 600 (Cos[t] + 4 Sin[t])
```

```
-25 q[t] + 6 q′[t] + q′′[t] == 600 (Cos[t] + 4 Sin[t])
```

```
sol = DSolve[eqn, q, t]
```

$$\Big\{\Big\{q \to \text{Function}\Big[\{t\},\; e^{\left(-3-\sqrt{34}\right) t}\, C[1] + e^{\left(-3+\sqrt{34}\right) t}\, C[2] + \frac{300\,(25\,\text{Cos}[t] + 49\,\text{Sin}[t])}{\left(-22 + 3\sqrt{34}\right)\left(22 + 3\sqrt{34}\right)}\Big]\Big\}\Big\}$$

The solution found by Mathematica checks.

```
eqn /. sol // Simplify
```

```
{True}
```

$$\frac{300\,(25\,\mathrm{Cos}[t] + 49\,\mathrm{Sin}[t])}{\left(-22 + 3\sqrt{34}\right)\left(22 + 3\sqrt{34}\right)}\ /.\ t \to 0.5$$

-76.5698

$$\left(e^{-3\,t}\,(-100\,\mathrm{Cos}[4\,t] + 75\,\mathrm{Sin}[4\,t]) + 100\,\mathrm{Cos}[t]\right)\ /.\ t \to 0.5$$

112.261

The text answer (pink) was not found.

$$\mathrm{Plot}\left[\frac{300\,(25\,\mathrm{Cos}[t] + 49\,\mathrm{Sin}[t])}{\left(-22 + 3\sqrt{34}\right)\left(22 + 3\sqrt{34}\right)},\ \{t, -4, 15\},\ \mathrm{ImageSize} \to 300,\ \mathrm{AspectRatio} \to .5,\ \mathrm{PlotRange} \to \{\{-4, 15\}, \{-100, 100\}\}\right]$$