

Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

### 1 - 8 Parametric surface representation

Familiarize yourself with parametric representations of important surfaces by deriving a representation (1), by finding the **parameter curves** (curves  $u = \text{const}$  and  $v = \text{const}$ ) of the surface and a normal vector  $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$  of the surface.

1. xy-plane  $\mathbf{r}(u, v) = (u, v)$  (thus  $u \mathbf{i} + v \mathbf{j}$ ; similarly in problems 2 - 8).

3. Cone  $\mathbf{r}(u, v) = \{u \cos[v], u \sin[v], c u\}$

```
Clear["Global`*"]
```

I am given that

$$\mathbf{x} = u \cos[v];$$

$$\mathbf{y} = u \sin[v];$$

$$\mathbf{z} = c u$$

$$c u$$

If I happen to have the brainwave to look at  $x^2 + y^2$ ,

$$\mathbf{x}^2 + \mathbf{y}^2$$

$$u^2 \cos^2[v] + u^2 \sin^2[v]$$

I see that it is equal to  $u^2$ . At this point I can find one sort of parameter curve, by setting  $u = \text{const}$ . If  $u = \text{const}$ .

then  $u^2 = \text{const}$ . and  $x^2 + y^2 = \text{const}$ ., that is, What about when  $v = \text{const}$ ?

equals a circle. So circles are one of the parameter curves.

Then by the definition of the coordinates, the quantity  $y/x = \text{some constant}$ , and  $y = kx$ . That describes a straight line. So straight lines are the other parameter curve.

$$\mathbf{yip} = \text{Simplify}\left[\text{Solve}\left[\mathbf{x}^2 + \mathbf{y}^2 = \frac{\mathbf{z}^2}{c^2}, \{\mathbf{z}\}\right]\right]$$

$$\left\{\left\{\mathbf{z} \rightarrow -\sqrt{c^2 \left(\mathbf{x}^2 + \mathbf{y}^2\right)}\right\}, \left\{\mathbf{z} \rightarrow \sqrt{c^2 \left(\mathbf{x}^2 + \mathbf{y}^2\right)}\right\}\right\}$$

Or,

$$\mathbf{z} = c \sqrt{\mathbf{x}^2 + \mathbf{y}^2}$$

$z = f(x, y)$  is the formula for a cone, if you can recognize it, according to the s.m.. Now I need to find a

surface normal on this surface. To do this I will probably need both partial derivatives.

```

ru = D[{u Cos[v], u Sin[v], c u}, u]
{Cos[v], Sin[v], c}

rv = D[{u Cos[v], u Sin[v], c u}, v]
{-u Sin[v], u Cos[v], 0}

cprod = Cross[ru, rv]
{-c u Cos[v], -c u Sin[v], u Cos[v]^2 + u Sin[v]^2}

TrigReduce[u Cos[v]^2 + u Sin[v]^2]
u

cprodf = cprod /. u Cos[v]^2 + u Sin[v]^2 -> u
{-c u Cos[v], -c u Sin[v], u}

```

The above line holds the normal vector the problem was looking for.

#### 5. Paraboloid of revolution $\mathbf{r}(u, v) = \{u \cos[v], u \sin[v], u^2\}$

```

Clear["Global`*"]

x = u Cos[v]
u Cos[v]

y = u Sin[v]
u Sin[v]

z = u^2
u^2

TrigReduce[x^2 + y^2]
u^2

```

So  $x^2 + y^2 = z$ . What about the parameter curves? If  $u = \text{const.}$ , then  $z$  is const., and  $x^2 + y^2 = \text{const.}$ , in other words, circles again. If  $v$  is const., then the coordinates of the parameter curve look like  $(c_1 u, c_2 u, u^2)$ , which will describe a parabola. Now I am ready to go on to the normal vector calculation.

```

ru = D[{u Cos[v], u Sin[v], u^2}, u]
{Cos[v], Sin[v], 2 u}

rv = D[{u Cos[v], u Sin[v], u^2}, v]
{-u Sin[v], u Cos[v], 0}

```

```

cprod = Cross[ru, rv]
{-2 u^2 Cos[v], -2 u^2 Sin[v], u Cos[v]^2 + u Sin[v]^2}

cprodf = cprod /. u Cos[v]^2 + u Sin[v]^2 -> u
{-2 u^2 Cos[v], -2 u^2 Sin[v], u}

```

The above line contains the normal vector expression, which agrees with the text's answer.

### 7. Ellipsoid $r(u, v) = \{a \cos[v] \cos[u], b \cos[v] \sin[u], c \sin[v]\}$

```

Clear["Global`*"]

x = a Cos[v] Cos[u]
y = b Cos[v] Sin[u]
z = c Sin[v]

a Cos[u] Cos[v]
b Cos[v] Sin[u]

c Sin[v]

firs = Cos[v] Cos[u];
seco = Cos[v] Sin[u];
thir = Sin[v];
FullSimplify[firs^2 + seco^2 + thir^2]

1

```

The somewhat amazing result stated on the above line allows the ellipsoid equation to become  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ .

The ratio of the coefficients  $a$  and  $b$  (and  $c$ ) describe the eccentricity of the ellipsoid. If  $u = \text{const.}$ , then a surface incorporating the still-active  $v$  is created, an ellipse. If  $v = \text{const.}$ , then a different kind of ellipse is expressed, still dependent on the ratio between  $a$  and  $b$ . I can try to find the normal vector.

```

ru = D[{a Cos[v] Cos[u], b Cos[v] Sin[u], c Sin[v]}, u]
{-a Cos[v] Sin[u], b Cos[u] Cos[v], 0}

rv = D[{a Cos[v] Cos[u], b Cos[v] Sin[u], c Sin[v]}, v]
{-a Cos[u] Sin[v], -b Sin[u] Sin[v], c Cos[v]}

cprod = Cross[ru, rv]
{b c Cos[u] Cos[v]^2, a c Cos[v]^2 Sin[u],
 a b Cos[u]^2 Cos[v] Sin[v] + a b Cos[v] Sin[u]^2 Sin[v]}

```

```
In[5]:= PossibleZeroQ[
  (a b Cos[u]^2 Cos[v] Sin[v] + a b Cos[v] Sin[u]^2 Sin[v]) - (a b Cos[v] Sin[v]) ]
Out[5]:= True
```

The PZQ above verifies the equality.

```
cprodf = cprod /.
  a b Cos[u]^2 Cos[v] Sin[v] + a b Cos[v] Sin[u]^2 Sin[v] -> a b Cos[v] Sin[v]
{b c Cos[u] Cos[v]^2, a c Cos[v]^2 Sin[u], a b Cos[v] Sin[v]}
```

The above line contains the text answer for the normal vector.

11. Satisfying numbered line (4), p. 441. Represent the paraboloid in problem 5 so that  $\tilde{\mathbf{N}}(0, 0) \neq 0$  and show  $\tilde{\mathbf{N}}$ .

13. Representation  $z = f(x, y)$ . Show that  $z = f(x, y)$  or  $g = z - f(x, y) = 0$  can be written  $\left(f_u = \frac{\partial f}{\partial u}, \text{ etc.}\right)$

Even with treatment in the s.m., I don't understand what is supposed to happen with the above problem.

#### 14 - 19 Derive a parametric representation

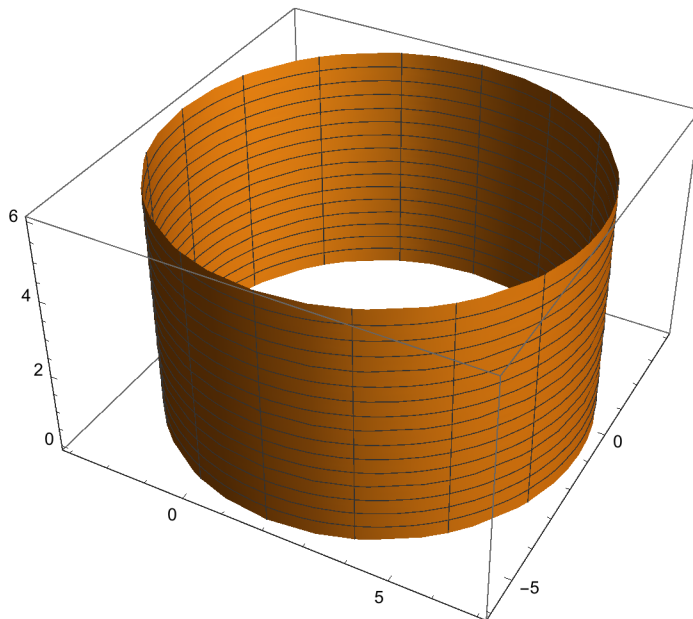
Find a normal vector. The answer gives one representation; there are many. Sketch the surface and parameter curves.

15. Cylinder of revolution  $(x - 2)^2 + (y + 1)^2 = 25$

```
Clear["Global`*"]
```

The constants just specify the location of the center of the cylinder.

```
ParametricPlot3D[{2 + 5 Cos[u], -1 + 5 Sin[u], v}, {u, 0, 2 π}, {v, 0, 6}]
```



The parametric curves are shown in the figure. In the constant- $u$  plane they are circles. In the constant- $v$  plane they are straight lines.

```
par[u_, v_] = {2 + 5 Cos[u], -1 + 5 Sin[u], v}
{2 + 5 Cos[u], -1 + 5 Sin[u], v}
```

Above is the parametric equation for the cylinder. Below I will take the partial derivatives so I can find a normal vector by the cross product of them.

```
fir = D[{2 + 5 Cos[u], -1 + 5 Sin[u], v}, {u}]
{-5 Sin[u], 5 Cos[u], 0}
```

```
sec = D[{2 + 5 Cos[u], -1 + 5 Sin[u], v}, {v}]
{0, 0, 1}
```

```
norm = Cross[fir, sec]
```

```
{5 Cos[u], 5 Sin[u], 0}
```

The vector shown in the line above is a normal vector, and agrees with the text's answer.

17. Sphere  $x^2 + (y + 2.8)^2 + (z - 3.2)^2 = 2.25$

```
Clear["Global`*"]
```

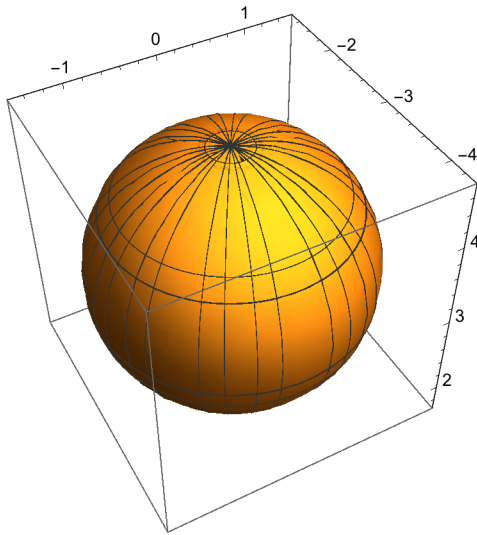
Again, the constants 2.8 and -3.2 merely locate the center of the sphere. In the cartesian formula for the sphere, the quantity 2.25 above is the square of the radius.

```
radd = (2.25)^.5
```

```
1.5
```

And so my parametric version will look like

```
parsph[u_, v_] =
  {1.5 Sin[u] Cos[v], 1.5 Sin[u] Sin[v] - 2.8, 1.5 Cos[u] + 3.2}
{1.5` Cos[v] Sin[u], -2.8` + 1.5` Sin[u] Sin[v], 3.2` + 1.5` Cos[u]}
ParametricPlot3D[{1.5 Sin[u] Cos[v],
  1.5 Sin[u] Sin[v] - 2.8, 1.5 Cos[u] + 3.2}, {u, -7, 7}, {v, -7, 7}]
```



As for parametric curves, they are circles in both the constant- $u$  and constant- $v$  planes, and are shown in the figure. And the center of the sphere seems to be in the same location as in the cartesian one.

```
fir = D[{1.5 Sin[u] Cos[v], 1.5 Sin[u] Sin[v] - 2.8, 1.5 Cos[u] + 3.2}, {u}]
{1.5 Cos[u] Cos[v], 1.5 Cos[u] Sin[v], -1.5 Sin[u]}

sec = D[{1.5 Sin[u] Cos[v], 1.5 Sin[u] Sin[v] - 2.8, 1.5 Cos[u] + 3.2}, {v}]
{-1.5 Sin[u] Sin[v], 1.5 Cos[v] Sin[u], 0}

norm = Simplify[Cross[fir, sec]]
{2.25 Cos[v] Sin[u]^2, 2.25 Sin[u]^2 Sin[v], 2.25 Cos[u] Sin[u]}
```

The above answer does not match the text's, but maybe it is equivalent. My  $\text{Sin}[u]$  terms match the  $\text{Cos}[v]$  terms in the answer, and my  $\text{Cos}[u]$  matches the  $\text{Sin}[v]$  terms. Anyway, the structure of the normal vector is consistent with that in the answer.

19. Hyperbolic cylinder  $x^2 - y^2 = 1$

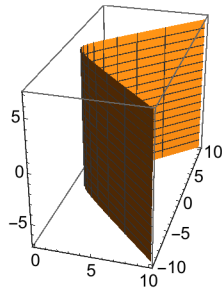
```
Clear["Global`*"]
```

I found a reference to a hyperbolic identity which is analogous to  $\sin^2 x + \cos^2 x = 1$ . It is  $\cosh^2 x - \sinh^2 x = 1$ . Since the squared exponent is already in the problem expression, there is no need to repeat it.

```
parhyp[u_, v_] = {Cosh[u], Sinh[u], v}

{Cosh[u], Sinh[u], v}

ParametricPlot3D[{Cosh[u], Sinh[u], v}, {u, -3, 3}, {v, -7, 7}]
```



As for parameter curves, I think the ones in constant- $u$  planes are hyperbolas (or at least half hyperbolas). The ones in constant- $v$  planes are straight lines.

```
fir = D[{Cosh[u], Sinh[u], v}, {u}]
{Sinh[u], Cosh[u], 0}

sec = D[{Cosh[u], Sinh[u], v}, {v}]
{0, 0, 1}

norm = Cross[fir, sec]

{Cosh[u], -Sinh[u], 0}
```

The above line agrees with the normal vector contained in the text's answer.