Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

2 - 11 Cauchy-Riemann equations

Are the following functions analytic? Use (1) on p. 625 or (7) on p. 628.

3. f[z] = e^{-2x} (Cos[2 y] − i Sin[2 y])

```
Clear["Global`*"]
```

f[x_, y_] = e^{-2x} (Cos[2 y] − i Sin[2 y])

e^{-2x} (Cos[2 y] − i Sin[2 y])

The test for analyticity comes from Weisstein's Wolfram Mathworld.

u[x_, y_] = e^{-2x} Cos[2 y]

e^{-2x} Cos[2 y]

v[x_, y_] = −e^{-2x} Sin[2 y]

−e^{-2x} Sin[2 y]

```
D[u[x, y], x]
```

−2 e^{-2x} Cos[2 y]

```
D[v[x, y], y]
```

−2 e^{-2x} Cos[2 y]

```
-D[u[x, y], y]
```

2 e^{-2x} Sin[2 y]

```
D[v[x, y], x]
```

2 e^{-2x} Log[e] Sin[2 y]

The function f passes the test described in Wolfram Mathworld (cyan cells equal and pink cells equal) and is therefore analytic, yes.

5. f[z] = Re[z^2] − i Im[z^2]

```
Clear["Global`*"]
```

z = x + i y

x + i y

```
f[x_, y_] = Re[z^2] - ⅈ Im[z^2]
```

$-i\,\mathrm{Im}[(x + i y)^2] + \mathrm{Re}[(x + i y)^2]$

```
ComplexExpand[f[x, y]]
```

$x^2 - 2 i x y - y^2$

```
u[x_, y_] = x^2 - y^2
```

$x^2 - y^2$

```
v[x_, y_] = -2 x y
```

$-2 x y$

```
D[u[x, y], x]
```

$2 x$

```
D[v[x, y], y]
```

$-2 x$

```
-D[u[x, y], y]
```

$2 y$

```
D[v[x, y], x]
```

$-2 y$

Cyan cells and pink cells are not equal in this case, therefore f is not analytic, no.

7. $f[z] = \frac{i}{z^8}$

```
Clear["Global`*"]
```

```
z = x + ⅈ y
```

$x + i y$

```
f[x_, y_] = ⅈ / z^8
```

$\frac{i}{(x + i y)^8}$

```
ComplexExpand[f[x, y]]
```

$$\frac{8 x^7 y}{(x^2+y^2)^8} - \frac{56 x^5 y^3}{(x^2+y^2)^8} + \frac{56 x^3 y^5}{(x^2+y^2)^8} - \frac{8 x y^7}{(x^2+y^2)^8} + i \left(\frac{x^8}{(x^2+y^2)^8} - \frac{28 x^6 y^2}{(x^2+y^2)^8} + \frac{70 x^4 y^4}{(x^2+y^2)^8} - \frac{28 x^2 y^6}{(x^2+y^2)^8} + \frac{y^8}{(x^2+y^2)^8}\right)$$

$$u[x\_, y\_] = \frac{8 x^7 y}{(x^2+y^2)^8} - \frac{56 x^5 y^3}{(x^2+y^2)^8} + \frac{56 x^3 y^5}{(x^2+y^2)^8} - \frac{8 x y^7}{(x^2+y^2)^8};$$

$$v[x\_, y\_] = \frac{x^8}{(x^2+y^2)^8} - \frac{28 x^6 y^2}{(x^2+y^2)^8} + \frac{70 x^4 y^4}{(x^2+y^2)^8} - \frac{28 x^2 y^6}{(x^2+y^2)^8} + \frac{y^8}{(x^2+y^2)^8};$$

```
D[u[x, y], x];
FullSimplify[%]
```

$$-\left(\left(8 \left(9 x^8 y - 84 x^6 y^3 + 126 x^4 y^5 - 36 x^2 y^7 + y^9\right)\right) / \left(x^2+y^2\right)^9\right)$$

```
D[v[x, y], y];
FullSimplify[%]
```

$$-\left(\left(8 \left(9 x^8 y - 84 x^6 y^3 + 126 x^4 y^5 - 36 x^2 y^7 + y^9\right)\right) / \left(x^2+y^2\right)^9\right)$$

```
-D[u[x, y], y];
FullSimplify[%]
```

$$-\left(\left(8 \left(x^9 - 36 x^7 y^2 + 126 x^5 y^4 - 84 x^3 y^6 + 9 x y^8\right)\right) / \left(x^2+y^2\right)^9\right)$$

```
D[v[x, y], x];
FullSimplify[%]
```

$$-\left(\left(8 \left(x^9 - 36 x^7 y^2 + 126 x^5 y^4 - 84 x^3 y^6 + 9 x y^8\right)\right) / \left(x^2+y^2\right)^9\right)$$

From the problem description in can be seen that z cannot be 0; otherwise, since cyan and pink cells are equal to each other, the expression f[z] is analytic, yes.

9. $f[z] = \frac{3 \pi^2}{z^3 + 4 \pi^2 z}$

```
Clear["Global`*"]
```

$$\texttt{f[z]} = \frac{3\,\pi^2}{z^3 + 4\,\pi^2\,z}$$

$$\frac{3\,\pi^2}{4\,\pi^2\,z + z^3}$$

```
ff[x_, y_] = f[z] /. z → x + ⅈ y
```

$$\frac{3\,\pi^2}{4\,\pi^2\,(x + i\,y) + (x + i\,y)^3}$$

```
dr = ComplexExpand[Re[ff[x, y]]];
u[x_, y_] = FullSimplify[dr]
```

$$\left(3\,\pi^2\,x\,\left(4\,\pi^2 + x^2 - 3\,y^2\right)\right) \Big/ \left(\left(x^2 + y^2\right)\,\left(16\,\pi^4 + 8\,\pi^2\,(x - y)\,(x + y) + \left(x^2 + y^2\right)^2\right)\right)$$

```
di = ComplexExpand[Im[ff[x, y]]];
v[x_, y_] = FullSimplify[di]
```

$$\left(3\,\pi^2\,y\,\left(-4\,\pi^2 - 3\,x^2 + y^2\right)\right) \Big/ \left(\left(x^2 + y^2\right)\,\left(16\,\pi^4 + 8\,\pi^2\,(x - y)\,(x + y) + \left(x^2 + y^2\right)^2\right)\right)$$

```
D[u[x, y], x];
FullSimplify[%]
```

$$\frac{3}{8}\left(\frac{2}{x^2 + y^2} - \frac{1}{x^2 + (-2\,\pi + y)^2} - \frac{1}{x^2 + (2\,\pi + y)^2} + 2\,x^2\left(-\frac{2}{\left(x^2 + y^2\right)^2} + \frac{1}{\left(x^2 + (-2\,\pi + y)^2\right)^2} + \frac{1}{\left(x^2 + (2\,\pi + y)^2\right)^2}\right)\right)$$

```
D[v[x, y], y];
FullSimplify[%]
```

$$\frac{3}{8}\left(\frac{2}{x^2 + y^2} - \frac{1}{x^2 + (-2\,\pi + y)^2} - \frac{1}{x^2 + (2\,\pi + y)^2} + 2\,x^2\left(-\frac{2}{\left(x^2 + y^2\right)^2} + \frac{1}{\left(x^2 + (-2\,\pi + y)^2\right)^2} + \frac{1}{\left(x^2 + (2\,\pi + y)^2\right)^2}\right)\right)$$

```
-D[u[x, y], y];
FullSimplify[%]
```

$$\frac{3}{4}\,x\left(\frac{2\,y}{\left(x^2 + y^2\right)^2} + \frac{2\,\pi - y}{\left(x^2 + (-2\,\pi + y)^2\right)^2} - \frac{2\,\pi + y}{\left(x^2 + (2\,\pi + y)^2\right)^2}\right)$$

```
D[v[x, y], x];
FullSimplify[%]
```

$$\frac{3}{4}\,x\left(\frac{2\,y}{\left(x^2+y^2\right)^2}+\frac{2\,\pi-y}{\left(x^2+(-2\,\pi+y)^2\right)^2}-\frac{2\,\pi+y}{\left(x^2+(2\,\pi+y)^2\right)^2}\right)$$

Here is another case where z is not allowed to equal zero; with that exception, cyans and pinks match, so the function is judged analytic, yes.

11. f[z] = Cos[x] Cosh[y] - I Sin[x] Sinh[y]

```
Clear["Global`*"]
f[x_, y_] = Cos[x] Cosh[y] - I Sin[x] Sinh[y]
Cos[x] Cosh[y] - ⅈ Sin[x] Sinh[y]
f[z] = Cos[x] Cosh[y] - I Sin[x] Sinh[y]
Cos[x] Cosh[y] - ⅈ Sin[x] Sinh[y]
u[x_, y_] = Cos[x] Cosh[y]
Cos[x] Cosh[y]
v[x_, y_] = -Sin[x] Sinh[y]
-Sin[x] Sinh[y]
D[u[x, y], x]
-Cosh[y] Sin[x]
D[v[x, y], y]
-Cosh[y] Sin[x]
-D[u[x, y], y]
-Cos[x] Sinh[y]
D[v[x, y], x]
-Cos[x] Sinh[y]
```

In this case there are no domain restrictions, and the Cauchy-Riemann test is passed by u and v, yes.

Are the following functions harmonic? If your answer is yes, find a corresponding analytic function **f[z]=u[x,y]+ⅈv[x,y].**

13. $u = x\,y$

```
Clear["Global`*"]
```

```
u[x_, y_] = x y
```

```
x y
```

```
Simplify[Laplacian[u[x, y], {x, y}]]
```

```
0
```

The function u passes the test for harmonic function. Now to look for a corresponding analytic function.

```
D[u[x, y], x]
```

```
y
```

$v_y = u_x = y$ and $v_x = -u_y = -x$

according to the Cauchy-Riemann criteria, which I must follow. Integrating the first equation with respect to y and differentiating the result with respect to x, I get

$v = \frac{1}{2} y^2 + h[x]$ and $v_x = \frac{dh}{dx}$

A comparison with the last v_x with the expressions in the yellow cell shows that $\frac{dh}{dx} =$

$-x$ or $h[x] = -\frac{1}{2} x^2$

Thus the following results:

$$f[z] = u + i\,v = x\,y + i\left(\frac{1}{2} y^2 + -\frac{1}{2} x^2 + C\right)$$

$$\text{out} = \text{Simplify}\left[x\,y + i\left(\frac{1}{2} y^2 + -\frac{1}{2} x^2 + C\right)\right]$$

$$i\,C - \frac{1}{2} i\,(x + i\,y)^2$$

```
out1 = out /. (x + ⅈ y) → z
```

$$i\,C - \frac{i\,z^2}{2}$$

$$\text{Solve}\left[-\frac{1}{2}\, i \left(z^2 + c\right) == i\, C - \frac{i\, z^2}{2},\ C\right]$$

$$\left\{\left\{C \to -\frac{c}{2}\right\}\right\}$$

The green cell above matches the text answer, modified by the value of C (real) shown in the purple cell.

15. $u = \dfrac{x}{x^2 + y^2}$

```
Clear["Global`*"]
```

$$u[x\_,\ y\_] = \frac{x}{x^2 + y^2}$$

$$\frac{x}{x^2 + y^2}$$

```
Simplify[Laplacian[u[x, y], {x, y}]]
```

0

The function u passes the test for harmonic function. Now to look for a corresponding analytic function.

```
D[u[x, y], x]
```

$$-\frac{2\, x^2}{\left(x^2 + y^2\right)^2} + \frac{1}{x^2 + y^2}$$

```
-D[u[x, y], y]
```

$$\frac{2\, x\, y}{\left(x^2 + y^2\right)^2}$$

$$v_y = u_x = -\frac{2\, x^2}{\left(x^2 + y^2\right)^2} + \frac{1}{x^2 + y^2} \quad \text{and} \quad v_x = -u_y = \frac{2\, x\, y}{\left(x^2 + y^2\right)^2}$$

Integrating the first equation with respect to y and differentiating the result with respect to x, I get

$$\text{vup} = \int \left(-\frac{2\, x^2}{\left(x^2 + y^2\right)^2} + \frac{1}{x^2 + y^2}\right) dy$$

$$-\frac{y}{x^2 + y^2}$$

vup2 = vup + h[x] + C

$$C - \frac{y}{x^2 + y^2} + h[x]$$

v_x = D[vup2, x]

$$\frac{2\,x\,y}{\left(x^2 + y^2\right)^2} + h'[x]$$

A comparison with the expressions in the yellow cell shows that $\frac{dh}{dx} = 0$ or h[x] = C1.
Thus f[z] = u[x, y] + i v[x, y] and (making C2=C+C1)

$$f[z] = \frac{x}{x^2 + y^2} + i\left(-\frac{y}{x^2 + y^2} + C2\right)$$

$$\frac{x}{x^2 + y^2} + i\left(C2 - \frac{y}{x^2 + y^2}\right)$$

f1[z] = Simplify[f[z]]

$$\frac{1 + i\,C2\,x - C2\,y}{x + i\,y}$$

f2[z] = f1[z] /. (x + i y) → z

$$\frac{1 + i\,C2\,x - C2\,y}{z}$$

$$f3[z] = \frac{1 + i\,C2\,(x + i\,y)}{z} == \frac{1 + i\,C2\,z}{z} == \frac{1}{z} + i\,C2;$$

This answer does not match the text because a real constant C is left sitting next to an imaginary unit.

17. v = (2 x + 1) y

This one has the twist of looking for u instead of the usual v.

```
Clear["Global`*"]
```

```
v[x_, y_] = (2 x + 1) y
```

(1 + 2 x) y

```
Simplify[Laplacian[v[x, y], {x, y}]]
```

0

The function v passes the test for harmonic function. Now to look for a corresponding analytic function.

```
-D[v[x, y], x]
```

```
-2 y
```

```
D[v[x, y], y]
```

```
1 + 2 x
```

$u_x = v_y = 1 + 2\,x$ and $u_y = -v_x = -2\,y$

Integrating the first equation with respect to x and differentiating the result with respect to y, I get

$$up = \int (1 + 2\,x)\,dx$$

$$x + x^2$$

```
up2 = up + h[y] + c
```

$$c + x + x^2 + h[y]$$

```
uy = D[up2, y]
```

```
h′[y]
```

A comparison with the expressions in the yellow cell shows that $\frac{dh}{dy} = -2\,y$ or $h[y] = -y^2$. Thus

f[z] = u[x, y] + i v[x, y] and

$$f[z] = x + x^2 + c - y^2 + i\,((2\,x + 1)\,y)$$

$$c + x + x^2 + i\,(1 + 2\,x)\,y - y^2$$

```
f1[z] = FullSimplify[f[z]]
```

$$c + (x + i\,y)\,(1 + x + i\,y)$$

```
f2[z] = f1[z] /. (x + i y) → z
```

```
c + z (1 + z)
```

19. $v = e^x\,\mathrm{Sin}[2\,y]$

Again my quarry is the u function instead of the v function.

```
Clear["Global`*"]
```

$$v[x\_, y\_] = e^x\,\mathrm{Sin}[2\,y]$$

$$e^x\,\mathrm{Sin}[2\,y]$$

```
Simplify[Laplacian[v[x, y], {x, y}]]
```

$-3 e^{x} \text{Sin}[2 y]$

The green cell above is not 0; therefore the function is not harmonic.

21 - 24 Determine a and b so that the given function is harmonic and find a harmonic conjugate.

21. $u = e^{\pi x} \text{Cos}[a v]$

This looks pretty intimidating as written; I'm going to start by assuming a typo, and insert y for v.

```
Clear["Global`*"]
```

$u[x\_, y\_] = e^{\pi x} \text{Cos}[\pi y]$

$e^{\pi x} \text{Cos}[\pi y]$

```
Simplify[Laplacian[u[x, y], {x, y}]]
```

0

It looks like **a** needs to equal π in order to have a harmonic function.

```
D[u[x, y], x]
```

$e^{\pi x} \pi \text{Cos}[\pi y]$

```
-D[u[x, y], y]
```

$e^{\pi x} \pi \text{Sin}[\pi y]$

$v_y = u_x = e^{\pi x} \pi \text{Cos}[\pi y]$ and $v_x = -u_y = e^{\pi x} \pi \text{Sin}[\pi y]$

Integrating the first equation with respect to y and differentiating the result with respect to x, I get

$\text{vup} = \int (e^{\pi x} \pi \text{Cos}[\pi y]) \, dy$

$e^{\pi x} \text{Sin}[\pi y]$

As usual Mathematica neglects to insert a constant of integration. However, in this case the omission lands on the text answer.

```
vup2 = vup + h[x]
```

$h[x] + e^{\pi x} \text{Sin}[\pi y]$

```
vx = D[vup2, x]
```

$e^{\pi x} \pi \, \text{Sin}[\pi y] + h'[x]$

A comparison of v_x with the expressions in the yellow cell shows that $\frac{dh}{dx} = 0$ or $h[x] = C$. Thus $f[z] = u[x, y] + i\, v[x, y]$ and

```
f[z] = e^(π x) Cos[π y] + i (e^(π x) Sin[π y] + c)
```

$e^{\pi x} \text{Cos}[\pi y] + i \left(c + e^{\pi x} \text{Sin}[\pi y]\right)$

The green cell above agrees with the text answer for v[x,y]. However, for f[z], I believe a constant has to come in there. Unless I was wrong about the typo, and (due to principles not understood by me) that accounts for the text dispensing with the constant.

23. $u = a\, x^3 + b\, x\, y$

```
Clear["Global`*"]
```

```
u[x_, y_] = a x^3 + b x y
```

$a\, x^3 + b\, x\, y$

```
Simplify[Laplacian[u[x, y], {x, y}]]
```

$6\, a\, x$

It appears that **a** must equal zero for the function to be harmonic.

```
u[x_, y_] = b x y
```

$b\, x\, y$

```
Simplify[Laplacian[u[x, y], {x, y}]]
```

0

```
D[u[x, y], x]
```

$b\, y$

```
-D[u[x, y], y]
```

$-b\, x$

$v_y = u_x = b\, y$ and $v_x = -u_y = -b\, x$

Integrating the first equation with respect to y and differentiating the result with respect to x, I get

```
vup = ∫b y dy
```

$$\frac{b\,y^2}{2}$$

```
vup2 = vup + h[x] + c
```

$$c + \frac{b\,y^2}{2} + h[x]$$

```
vx = D[vup2, x]
```

$$h'[x]$$

A comparison with the expressions in the yellow cell shows that $\frac{dh}{dx} =.$

$-b\,x$ or $h[x] = -\frac{b}{2}x^2$

Thus $f[z] = u[x, y] + i\, v[x, y]$ and

$$f[z] = b\,x\,y + i\left[\frac{b\,y^2}{2} - \frac{b\,x^2}{2} + c\right]$$

$$b\,x\,y + i\left[c - \frac{b\,x^2}{2} + \frac{b\,y^2}{2}\right]$$

```
f1[z] = Simplify[f[z]]
```

$$b\,x\,y + i\left[c + \frac{1}{2}b\left(-x^2 + y^2\right)\right]$$

The green cell matches the text answer. I noticed this when I saw that only v is covered in the answer, not f[z].

25. CAS Project. Equipotential Lines. Write a program for graphing equipotential lines u=const of a harmonic function u and of its conjugate v on the same axes. Apply the program to (a) $u = x^2 - y^2$, $v = 2\,x\,y$, (*b*) $u = x^3 - 3\,x^2\,y - y^3$, $v = 3\,x^2\,y - y^3$.

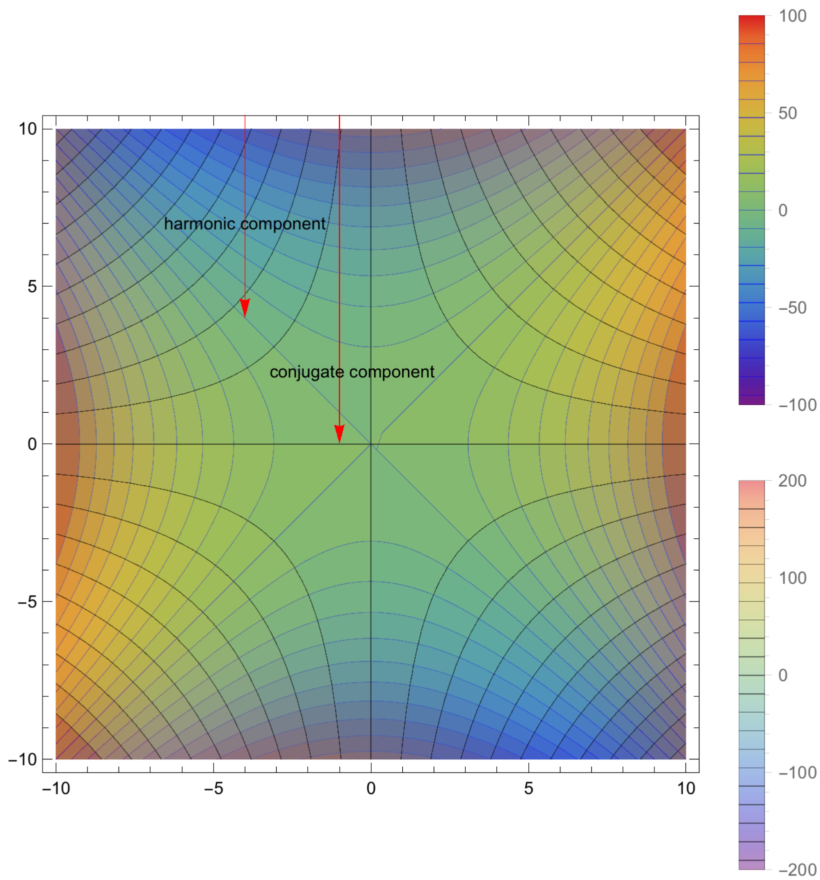
Part (a)

In a spooky coincidence, the exact same equations for part (a) were the subject of discussion in Mathematica StackExchange question #153214.

```
cp1 = ContourPlot[x^2 - y^2, {x, -10, 10},
   {y, -10, 10}, Contours → 20, PlotLegends → Automatic,
   ColorFunction → "Rainbow", ContourStyle → Blue,
   Epilog → {{Red, Arrowheads[.03], Arrow[{{-4, 11}, {-4, 4}}]},
     {Text["harmonic component", {-4, 7}]},
     {Red, Arrowheads[.03], Arrow[{{-1, 11}, {-1, 0}}]},
     {Text["conjugate component", {-0.6, 2.3}]}}];
```

```
cp2 = ContourPlot[2 x y, {x, -10, 10},
   {y, -10, 10}, Mesh → None, (*ContourShading→None,*)
   ColorFunction → Function[f, Opacity[.5, ColorData["Rainbow"][f]]],
   Contours → 20, PlotLegends → Automatic];

Show[cp1, cp2 /. EdgeForm[] → EdgeForm[Opacity[0]]]
```



Part (b)

```
Clear["Global`*"]

cp1 = ContourPlot[x³ - 3 x² y - y³, {x, -10, 10},
   {y, -10, 10}, Contours → 20, PlotLegends → Automatic,
   ColorFunction → "Rainbow", ContourStyle → Blue,
   Epilog → {{Red, Arrowheads[.03], Arrow[{{-4, 11}, {-4, 5.3}}]},
     {Text["harmonic component", {-4, 7}]},
     {Red, Arrowheads[.03], Arrow[{{0, 11}, {0, 5.75}}]},
     {Text["conjugate component", {0, 9}]}}];
```

```
cp2 = ContourPlot[3 x^2 y - y^3, {x, -10, 10},
   {y, -10, 10}, Mesh → None, (*ContourShading→None,*)
   ColorFunction → Function[f, Opacity[.5, ColorData["Rainbow"][f]]],
   Contours → 20, PlotLegends → Automatic];

Show[cp1, cp2 /. EdgeForm[] → EdgeForm[Opacity[0]]]
```