Green cells below match corresponding answers in the text.

1 - 5 Legendre polynomials and functions

5. Obtain P_6 and P_7.

```
Clear["Global`*"]
```

It turns out that Legendre polynomials are available from a built-in command.

```
LegendreP[6, x]
```

$$\frac{1}{16}\left(-5+105\,x^2-315\,x^4+231\,x^6\right)$$

```
LegendreP[7, x]
```

$$\frac{1}{16}\left(-35\,x+315\,x^3-693\,x^5+429\,x^7\right)$$

11 - 15 Further formulas

11. ODE. Find a solution of $\left(a^2-x^2\right) y''-2\,x\,y'+n\,(n+1)\,y=0,\ a\neq 0$, by reduction to the Legendre equation.

```
Clear["Global`*"]
```

```
eqn = (a^2 - x^2) y''[x] - 2 x y'[x] + n (n + 1) y[x] == 0
```

$$n\,(1+n)\,y[x]-2\,x\,y'[x]+\left(a^2-x^2\right) y''[x]==0$$

```
sol = DSolve[eqn, y, x, Assumptions → a ≠ 0]
```

$$\left\{\left\{y\to \text{Function}\left[\{x\},\ C[1]\,\text{LegendreP}\left[n,\ \frac{x}{a}\right]+C[2]\,\text{LegendreQ}\left[n,\ \frac{x}{a}\right]\right]\right\}\right\}$$

```
sol1 = sol /. {C[1] → 1, C[2] → 1, n → 1, a → 1}
```

$$\left\{\left\{y\to \text{Function}\left[\{x\},\ 1\,\text{LegendreP}\left[1,\ \frac{1}{1}\,x\right]+1\,\text{LegendreQ}\left[1,\ \frac{1}{1}\,x\right]\right]\right\}\right\}$$

$$\text{LegendreP}\left[1,\ \frac{1}{1}\,x\right]+1\,\text{LegendreQ}\left[1,\ \frac{1}{1}\,x\right]$$

$$-1+x+x\left(-\frac{1}{2}\,\text{Log}[1-x]+\frac{1}{2}\,\text{Log}[1+x]\right)$$

15. Associated Legendre functions $P_n^k[x]$ are needed, e.g. in quantum physics. They are defined by $P_n^k[x]=\left(1-x^2\right)^{k/2}\frac{d^k p_n[x]}{d\,x^k}$ and are solutions of the ODE $\left(1-x^2\right) y''-2\,x\,y'+q[x]\,y=0$ where $q[x]=n(n+1)-k^2/\left(1-x^2\right)$. Find $P_1^1[x]$, $P_2^1[x]$, $P_2^2[x]$, and $P_4^2[x]$ and verify that they satisfy numbered line (16) in yellow above.

$$P_1^1[x] = \left(1 - x^2\right)^{1/2} \frac{d^1 p_1[x]}{d x^1}$$

```
LegendreP[1, x]
```

x

$$P_1^1[x] = \left(1 - x^2\right)^{1/2}$$

$$P_2^1[x] = \left(1 - x^2\right)^{1/2} \frac{d^1 p_2[x]}{d x^1}$$

```
LegendreP[2, x]
```

$$\frac{1}{2} \left(-1 + 3 x^2\right)$$

```
D[%, x]
```

3 x

$$\left(1 - x^2\right)^{1/2} * \%$$

$$3 x \sqrt{1 - x^2}$$

$$P_2^2[x] = \left(1 - x^2\right)^{2/2} \frac{d^2 p_2[x]}{d x^2}$$

```
LegendreP[2, x]
```

$$\frac{1}{2} \left(-1 + 3 x^2\right)$$

```
D[%, {x, 2}]
```

3

$$3 \left(1 - x^2\right)$$

$$P_4^2[x] = \left(1 - x^2\right)^{2/2} \frac{d^2 p_4[x]}{d x^2}$$

```
LegendreP[4, x]
```

$$\frac{1}{8} \left(3 - 30 x^2 + 35 x^4\right)$$

```
D[%, {x, 2}]
```

$$\frac{1}{8} \left(-60 + 420 x^2\right)$$

```
FullSimplify[% * (1 - x^2)]
```

$$-\frac{15}{2}\left(1-8x^2+7x^4\right)$$

```
PossibleZeroQ[-15/2 (1 - 8 x^2 + 7 x^4) - ((1 - x^2) (105 x^2 - 15) / 2)]
```

```
True
```

The green cells above match the answers in the text. (The last, though slightly different in format, is verified by the cyan cell.)