

2 - 15 Systems of ODEs

Using the Laplace transform and showing the details of your work, solve the IVP:

$$3. \quad y_1' = -y_1 + 4 y_2, \quad y_2' = 3 y_1 - 2 y_2, \quad y_1[0] = 3, \quad y_2[0] = 4$$

```
Clear["Global`*"]
```

```
e1 = DSolve[{y1'[t] == -y1[t] + 4 y2[t],  
            y2'[t] == 3 y1[t] - 2 y2[t], y1[0] == 3, y2[0] == 4}, {y1, y2}, t]
```

```
{ {y1 -> Function[{t}, e-5 t (-1 + 4 e7 t)],  
  y2 -> Function[{t}, e-5 t (1 + 3 e7 t)] } }
```

Above: The answer matches the text's. It seems that DSolve is able to do this without any reference to Laplace.

$$5. \quad y_1' = y_2 + 1 - u(t - 1), \quad y_2' = -y_1 + 1 - u(t - 1), \quad y_1[0] = 0, \quad y_2[0] = 0$$

```
Clear["Global`*"]
```

```
e1 = DSolve[{y1'[t] == y2[t] + 1 - UnitStep[t - 1],  
            y2'[t] == -y1[t] + 1 - UnitStep[t - 1], y1[0] == 0, y2[0] == 0}, {y1, y2}, t]
```

```
{ {y1 -> Function[{t}, -Cos[t] + Cos[t]2 + Sin[t] + Sin[t]2 +  
  Cos[1] Cos[t] UnitStep[-1 + t] - Cos[t]2 UnitStep[-1 + t] +  
  Cos[t] Sin[1] UnitStep[-1 + t] - Cos[1] Sin[t] UnitStep[-1 + t] +  
  Sin[1] Sin[t] UnitStep[-1 + t] - Sin[t]2 UnitStep[-1 + t]],  
  y2 -> Function[{t}, Cos[t] - Cos[t]2 + Sin[t] - Sin[t]2 -  
  Cos[1] Cos[t] UnitStep[-1 + t] + Cos[t]2 UnitStep[-1 + t] +  
  Cos[t] Sin[1] UnitStep[-1 + t] - Cos[1] Sin[t] UnitStep[-1 + t] -  
  Sin[1] Sin[t] UnitStep[-1 + t] + Sin[t]2 UnitStep[-1 + t]] } }
```

```
e2 = e1[[1, 1, 2, 2]]
```

```
-Cos[t] + Cos[t]2 + Sin[t] + Sin[t]2 +  
Cos[1] Cos[t] UnitStep[-1 + t] - Cos[t]2 UnitStep[-1 + t] +  
Cos[t] Sin[1] UnitStep[-1 + t] - Cos[1] Sin[t] UnitStep[-1 + t] +  
Sin[1] Sin[t] UnitStep[-1 + t] - Sin[t]2 UnitStep[-1 + t]
```

```
e3 = Collect[e2, UnitStep[-1 + t]]
```

```
-Cos[t] + Cos[t]2 + Sin[t] + Sin[t]2 +  
(Cos[1] Cos[t] - Cos[t]2 + Cos[t] Sin[1] -  
Cos[1] Sin[t] + Sin[1] Sin[t] - Sin[t]2) UnitStep[-1 + t]
```

```

e4 = e3 /. Cos[t]^2 + Sin[t] + Sin[t]^2 -> 1 + Sin[t]
1 - Cos[t] + Sin[t] + (Cos[1] Cos[t] - Cos[t]^2 + Cos[t] Sin[1] -
  Cos[1] Sin[t] + Sin[1] Sin[t] - Sin[t]^2) UnitStep[-1 + t]

e5 = e4 /. -Cos[t]^2 + Cos[t] Sin[1] - Cos[1] Sin[t] + Sin[1] Sin[t] - Sin[t]^2 ->
  -1 + Cos[t] Sin[1] - Cos[1] Sin[t] + Sin[1] Sin[t]
1 - Cos[t] + Sin[t] +
  (-1 + Cos[1] Cos[t] + Cos[t] Sin[1] - Cos[1] Sin[t] + Sin[1] Sin[t])
  UnitStep[-1 + t]

```

Working in a couple of trig identities.

```
FullSimplify[Sin[1 - t] == Sin[1] Cos[t] - Cos[1] Sin[t]]
```

True

```

e6 = e5 /. Cos[t] Sin[1] - Cos[1] Sin[t] -> Sin[1 - t]
1 - Cos[t] + Sin[t] +
  (-1 + Cos[1] Cos[t] + Sin[1 - t] + Sin[1] Sin[t]) UnitStep[-1 + t]

```

```
FullSimplify[Cos[t - 1] == Cos[t] Cos[1] + Sin[t] Sin[1]]
```

True

```

e7 =
e6 /. (Cos[1] Cos[t] + Sin[1 - t] + Sin[1] Sin[t]) -> (Cos[t - 1] + Sin[1 - t])
1 - Cos[t] + Sin[t] + (-1 + Cos[1 - t] + Sin[1 - t]) UnitStep[-1 + t]

```

```

PossibleZeroQ[
  (1 - Cos[t] + Sin[t] + (-1 + Cos[1 - t] + Sin[1 - t]) UnitStep[-1 + t]) -
  (-Cos[t] + Sin[t] + 1 + UnitStep[-1 + t] (-1 + Cos[t - 1] - Sin[t - 1]))]

```

True

Above: The answer matches the text answer (for y1) in content, as shown by the PZQ above. And a couple more trig identities.

```
Sin[1 - x] == -Sin[x - 1]
```

True

```
Cos[1 - x] == Cos[x - 1]
```

True

```

e8 = e1[[1, 2, 2, 2]]
Cos[t] - Cos[t]^2 + Sin[t] - Sin[t]^2 -
  Cos[1] Cos[t] UnitStep[-1 + t] + Cos[t]^2 UnitStep[-1 + t] +
  Cos[t] Sin[1] UnitStep[-1 + t] - Cos[1] Sin[t] UnitStep[-1 + t] -
  Sin[1] Sin[t] UnitStep[-1 + t] + Sin[t]^2 UnitStep[-1 + t]

e9 = Collect[e8, UnitStep[-1 + t]]
Cos[t] - Cos[t]^2 + Sin[t] - Sin[t]^2 +
  (-Cos[1] Cos[t] + Cos[t]^2 + Cos[t] Sin[1] -
    Cos[1] Sin[t] - Sin[1] Sin[t] + Sin[t]^2) UnitStep[-1 + t]

e10 =
  e9 /. (Cos[t]^2 + Cos[t] Sin[1] - Cos[1] Sin[t] - Sin[1] Sin[t] + Sin[t]^2) ->
    (1 + Cos[t] Sin[1] - Cos[1] Sin[t] - Sin[1] Sin[t])
Cos[t] - Cos[t]^2 + Sin[t] - Sin[t]^2 +
  (1 - Cos[1] Cos[t] + Cos[t] Sin[1] - Cos[1] Sin[t] - Sin[1] Sin[t])
  UnitStep[-1 + t]

e11 = e10 /. -Cos[t]^2 + Sin[t] - Sin[t]^2 -> -1 + Sin[t]
-1 + Cos[t] + Sin[t] +
  (1 - Cos[1] Cos[t] + Cos[t] Sin[1] - Cos[1] Sin[t] - Sin[1] Sin[t])
  UnitStep[-1 + t]

e12 = e11 /. -Cos[1] Cos[t] + Cos[t] Sin[1] - Cos[1] Sin[t] - Sin[1] Sin[t] ->
  -Cos[1 - t] + Cos[t] Sin[1] - Cos[1] Sin[t]
-1 + Cos[t] + Sin[t] +
  (1 - Cos[1 - t] + Cos[t] Sin[1] - Cos[1] Sin[t]) UnitStep[-1 + t]

e13 = e12 /. Cos[t] Sin[1] - Cos[1] Sin[t] -> Sin[1 - t]
-1 + Cos[t] + Sin[t] + (1 - Cos[1 - t] + Sin[1 - t]) UnitStep[-1 + t]

PossibleZeroQ[
  (-1 + Cos[t] + Sin[t] + (1 - Cos[1 - t] + Sin[1 - t]) UnitStep[-1 + t]) -
  (Cos[t] + Sin[t] - 1 + UnitStep[-1 + t] (1 - Cos[t - 1] - Sin[t - 1]))]
True

```

Above: The answer in green matches the text answer (for y_2) in content, as shown by the PZQ above.

$$\begin{aligned}
 7. \quad y_1' &= 2y_1 - 4y_2 + u(t-1)e^t, \\
 y_2' &= y_1 - 3y_2 + u(t-1)e^t, \quad y_1[0] = 3, \quad y_2[0] = 0
 \end{aligned}$$

```
Clear["Global`*"]
```

```
e1 = DSolve[{y1'[t] == 2 y1[t] - 4 y2[t] + UnitStep[t - 1] e^t, y2'[t] ==
  y1[t] - 3 y2[t] + UnitStep[t - 1] e^t, y1[0] == 3, y2[0] == 0}, {y1, y2}, t]
{{y1 -> Function[{t},
   $\frac{1}{3} e^{-2t} (-3 + 12 e^{3t} - e^3 \text{UnitStep}[-1 + t] + e^{3t} \text{UnitStep}[-1 + t])$ ],
  y2 -> Function[{t},  $\frac{1}{3} e^{-2t} (-3 + 3 e^{3t} - e^3 \text{UnitStep}[-1 + t] + e^{3t} \text{UnitStep}[-1 + t])$ ]}}
```

```
e2 = e1[[1, 1, 2, 2]]
```

```
 $\frac{1}{3} e^{-2t} (-3 + 12 e^{3t} - e^3 \text{UnitStep}[-1 + t] + e^{3t} \text{UnitStep}[-1 + t])$ 
```

```
e3 = Collect[e2, UnitStep[-1 + t]]
```

```
 $\frac{1}{3} e^{-2t} (-3 + 12 e^{3t}) + \frac{1}{3} e^{-2t} (-e^3 + e^{3t}) \text{UnitStep}[-1 + t]$ 
```

```
PossibleZeroQ[ $\left(\frac{1}{3} e^{-2t} (-3 + 12 e^{3t}) + \frac{1}{3} e^{-2t} (-e^3 + e^{3t}) \text{UnitStep}[-1 + t]\right) -$ 
 $\left(-e^{-2t} + 4 e^t + \frac{1}{3} \text{UnitStep}[-1 + t] (-e^{3-2t} + e^t)\right)$ ]
```

```
True
```

Above: This answer matches the text in content (y1), as shown by the PZQ above.

```
e4 = e1[[1, 2, 2, 2]]
```

```
 $\frac{1}{3} e^{-2t} (-3 + 3 e^{3t} - e^3 \text{UnitStep}[-1 + t] + e^{3t} \text{UnitStep}[-1 + t])$ 
```

```
e5 = Collect[e4, UnitStep[-1 + t]]
```

```
 $\frac{1}{3} e^{-2t} (-3 + 3 e^{3t}) + \frac{1}{3} e^{-2t} (-e^3 + e^{3t}) \text{UnitStep}[-1 + t]$ 
```

```
PossibleZeroQ[ $\left(\frac{1}{3} e^{-2t} (-3 + 3 e^{3t}) + \frac{1}{3} e^{-2t} (-e^3 + e^{3t}) \text{UnitStep}[-1 + t]\right) -$ 
 $\left(-e^{-2t} + e^t + \frac{1}{3} \text{UnitStep}[-1 + t] (-e^{3-2t} + e^t)\right)$ ]
```

```
True
```

Above: This answer matches the text in content (y2), as shown by the PZQ above.

```
9. y1' = 4 y1 + y2, y2' = -y1 + 2 y2, y1[0] = 3, y2[0] = 1
```

```
Clear["Global`*"]
```

```
e1 = DSolve[{y1'[t] == 4 y1[t] + y2[t],
            y2'[t] == -y1[t] + 2 y2[t], y1[0] == 3, y2[0] == 1}, {y1, y2}, t]
{ {y1 -> Function[{t}, e^{3 t} (3 + 4 t)], y2 -> Function[{t}, -e^{3 t} (-1 + 4 t)] }}
```

Above: The answer matches the text answer.

11. $y_1'' = y_1 + 3 y_2$, $y_2'' = 4 y_1 - 4 e^t$, $y_1[0] = 2$,
 $y_1'[0] = 3$, $y_2[0] = 1$, $y_2'[0] = 2$

```
Clear["Global`*"]
e1 = DSolve[{y1''[t] == y1[t] + 3 y2[t], y2''[t] == 4 y1[t] - 4 e^t,
            y1[0] == 2, y1'[0] == 3, y2[0] == 1, y2'[0] == 2}, {y1, y2}, t]
{ {y1 -> Function[{t}, (1/7) e^t (4 + 7 e^t + 3 Cos[sqrt(3) t]^2 + 3 Sin[sqrt(3) t]^2)],
  y2 -> Function[{t}, (1/7) e^t (4 + 7 e^t - 4 Cos[sqrt(3) t]^2 - 4 Sin[sqrt(3) t]^2)] }}
```

```
e2 = e1[[1, 1, 2, 2]]
```

$$\frac{1}{7} e^t \left(4 + 7 e^t + 3 \cos[\sqrt{3} t]^2 + 3 \sin[\sqrt{3} t]^2 \right)$$

```
e3 = FullSimplify[e2]
```

$$e^t (1 + e^t)$$

Above: The answer matches the text answer (y1).

```
e4 = e1[[1, 2, 2, 2]]
```

$$\frac{1}{7} e^t \left(4 + 7 e^t - 4 \cos[\sqrt{3} t]^2 - 4 \sin[\sqrt{3} t]^2 \right)$$

```
e5 = FullSimplify[e4]
```

$$e^{2 t}$$

Above: The answer matches the text answer (y2).

13. $y_1'' + y_2 = -101 \sin[10 t]$, $y_2'' + y_1 = 101 \sin[10 t]$,
 $y_1[0] = 0$, $y_1'[0] = 6$, $y_2[0] = 8$, $y_2'[0] = -6$

```
Clear["Global`*"]
```

```
e1 =
DSolve[{y1''[t] + y2[t] == -101 Sin[10 t], y2''[t] + y1[t] == 101 Sin[10 t],
y1[0] == 0, y1'[0] == 6, y2[0] == 8, y2'[0] == -6}, {y1, y2}, t]

{{y1 -> Function[{t}, -4 e^t + 4 Cos[t] + Sin[10 t]],
y2 -> Function[{t}, 4 e^t + 4 Cos[t] - Sin[10 t] ]}}
```

Above: The answers match the text answers (y1 & y2).

```
15. y1' + y2' = 2 Sinh[t], y2' + y3' = e^t,
y3' + y1' = 2 e^t + e^-t, y1[0] = 1, y2[0] = 1, y3[0] = 0
```

```
Clear["Global`*"]

e1 = DSolve[{y1'[t] + y2'[t] == 2 Sinh[t], y2'[t] + y3'[t] == e^t,
y3'[t] + y1'[t] == 2 e^t + e^-t, y1[0] == 1, y3[0] == 0}, {y1, y2, y3}, t]

{{y1 -> Function[{t}, e^t], y2 -> Function[{t}, e^-t (1 + e^t C[2])],
y3 -> Function[{t}, e^-t (-1 + e^2 t)]}}
```

Above: The answers match those of the text for y1, y2, and y3 (with choice of constant = 0 for C[2]).