I'm not that impressed with the selection of problems in this set. As a supplement, I will start with an example from Paul's online notes, *http://tutorial.math.lamar.edu/Classes/CalcIII/StokesTheorem.aspx*. This is example 1.

```
Clear["Global`*"]
```

```
eF[x_, y_, z_] = {z^2, -3 x y, x^3 y^3}
```

$\{z^2, -3\,x\,y, x^3\,y^3\}$

```
zee[x_, y_] = 5 - x^2 - y^2
```

$5 - x^2 - y^2$

zee will be restricted to a domain which produces a range (vector field) above the plane z=1.

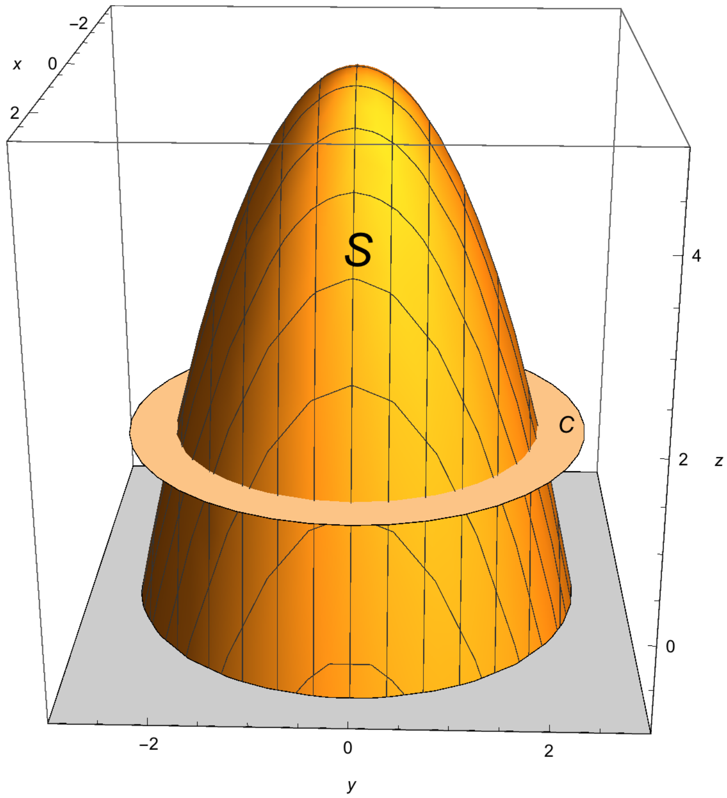
Use Stoke ' s theorem to evaluate $\int_S\int$ Curl[eF, {x, y, z}].dS

I can make a plot of the general surroundings of the example, using a flattened cylinder to imitate the cutoff plane of z.

```
p1 = Plot3D[5 - x^2 - y^2, {x, -3, 3}, {y, -3, 3},
   BoxRatios → {1, 1, 1}, PlotRange → {{-3, 3}, {-3, 3}, {-1, 5}},
   Axes → True, AxesLabel → {x, y, z}];
```

```
p2 = Graphics3D[{{Cylinder[{{0, 0, 1}, {0, 0, 1.001}}, 2.5]},
    {Text[Style[C, Medium], {0, 2.3, 1.001}]},
    {Text[Style[S, Large], {0, 0, 3}]}}, Axes → True];
```

```
Show[p1, p2]
```



In this case the boundary curve C will be where the surface intersects the plane z=1 and so will be the curve

$1 = 5 - x^2 - y^2 \Rightarrow x^2 + y^2 = 4$ at $z = 1$

The parameterization of this curve is

```
curvc[t_] = {2 Cos[t], 2 Sin[t], 1}
{2 Cos[t], 2 Sin[t], 1}
```

with the understanding that $0 \le t \le 2\pi$.

Using Stoke's theorem we can write the surface integral as the following line integral

$$\int_S\int \text{Curl[eF, \{x, y, z\}]} . d\vec{S} = \int_0^{2\pi} \vec{F}\text{[curvc[t]]} . \text{curvc}'\text{[t]}\, dt$$

The above cell uses some notation which is not approved for Mathematica. However, the action which is represented is the integration of a dot product of a vector field, eF, (not illustrated) with the derivative of another curve and its truncating plane, curvc, (illustrated). The right-hand form is made up of two parts, and those parts can be expressed as

```
comp[t_] = eF @@ curvc[t]
```

$\{1, -12\,\text{Cos[t] Sin[t]}, 64\,\text{Cos[t]}^3\,\text{Sin[t]}^3\}$

the above being the composition of the boundary curve function within the vector field function, and

```
curvc'[t]
{-2 Sin[t], 2 Cos[t], 0}
```

the derivative of the boundary curve with respect to t. So let me try out the integration with

$$\int_0^{2\pi} \{1, -12\,\mathrm{Cos}[t]\,\mathrm{Sin}[t], 64\,\mathrm{Cos}[t]^3\,\mathrm{Sin}[t]^3\}.\{-2\,\mathrm{Sin}[t], 2\,\mathrm{Cos}[t], 0\}\,dt$$

0

The answer above in green is the answer which Paul's arrives at.

****

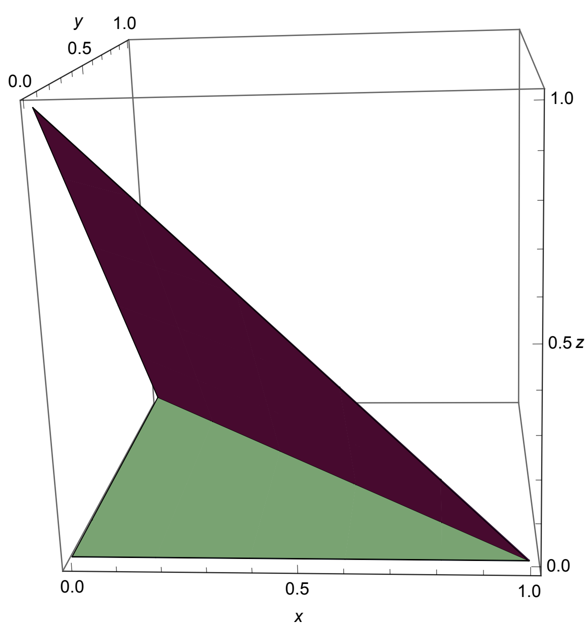
I'm going to do Paul's example 2 also, since it may help out with problem 1.

```
Clear["Global`*"]
```

The function eF will be a dot-ee in the integral, but will not actually appear on stage. The only parts visible in the example are two triangles, one of which will be projected onto the other.

$$\mathrm{eF}[x\_, y\_, z\_] = \{z^2, y^2, x\}$$

$$\{z^2, y^2, x\}$$



It appears that the function **Curl** will not work on the symbolic form of a vector field.

```
aa = Curl[eF, {x, y, z}]
```

```
StructuredArray[ Type: SymmetrizedArray
                 Dimensions: {3, 3}
                 Symmetry: Antisymmetric[{1, 2}] ]
```

```
CurleF = Normal[aa]
```

```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

But when the raw field is substituted as an argument, Mathematica comes up with the correct answer.

```
ab = Curl[{z^2, y^2, x}, {x, y, z}]
{0, -1 + 2 z, 0}
```

I see that **Grad** will not work on the symbolic form of the function expression either, so I have to use the specific descriptive form as an argument.

```
Grad[z - 1 + x + y, {x, y, z}]
{1, 1, 1}
```

The primary curve S in this example (purple) is the triangle with vertices {1,0,0}, {0,1,0}, and {0,0,1}, and assuming the rotation or direction of travel is counterclockwise. The equation of the S plane is

$$x + y + z = 1 \Rightarrow z = g[x, y] = 1 - x - y$$

Catching up with a few equivalences from the example, not all of which contain Mathematica-compatible nomenclature,

$$\int_C \mathrm{eF}.d\vec{S} = \int_S\int \mathrm{Curl}[\mathrm{eF}, \{x, y, z\}].d\vec{S} =$$

$$\int_S\int (2\,z - 1)\,\vec{j}.d\vec{S} = \int_D\int (2\,z - 1)\,\vec{j}.\mathrm{Grad}[z - 1 + x + y, \{x, y, z\}]\,dA$$

I can change the format on one value and plug in another one to prepare the internal environment of the integral. The dot product dots the curl of the external function with the grad of the primary curve.

```
{0, 2 z - 1, 0}.{1, 1, 1} (* curl F dot grad S *)
-1 + 2 z
```

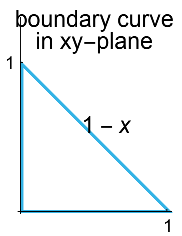
Then I change form of curl-F-dot-grad-S by substitution to reduce variables to the space dimension of C

```
(-1 + 2 z) /. z → (1 - x - y)
-1 + 2 (1 - x - y)
```

The remaining work is in the following form.

$$\int_D\int -1 + 2\,(1 - x - y)\; dA$$

The curve C in example 2 is a very simplistic one, with the projection of the primary curve onto the xy-plane acting as a cut-off boundary. The region D must be described correctly in order for the integral to work.



$$\int_0^1\int_0^{1-x} (-1 + 2\,(1 - x - y))\; dy\, dx$$

$$-\frac{1}{6}$$

The above green cell shows the correct answer to Paul's example 2.

1 - 10 Direct integration of surface integrals

Evaluate the surface integral $\int_S\int$ (curl F) .n dA directly for the given F and S.

1. $F = \{z^2, -x^2, 0\}$, S the rectangle with vertices {0, 0, 0}, {1, 0, 0}, {0, 4, 4}, {1, 4, 4}

```
Clear["Global`*"]
```

The function eF is one which we assume we need to incorporate into an integral, but which does not actually appear physically.

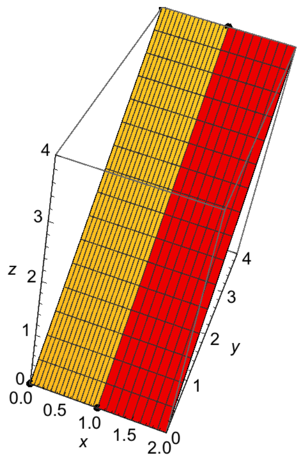
```
eF[x_, y_, z_] = {z^2, -x^2, 0}
{z^2, -x^2, 0}
```

After some experimentation, I came up with a parametric expression for the rectangle defined by the four points in the problem description. The following plots show that the points, the parametric plane equation, and the cartesian system equation are understood.

```
r2 = Graphics3D[{{PointSize[0.03], Point[{0, 0, 0}]},
    {PointSize[0.03], Point[{1, 0, 0}]}, {PointSize[0.03],
     Point[{0, 4, 4}]}, {PointSize[0.03], Point[{1, 4, 4}]}}];
p3 = ParametricPlot3D[{u + 1, 4 v + 4, 4 v + 4}, {u, -1, 0}, {v, -1, 0}];
```

```
p4 = ContourPlot3D[-4 y + 4 z == 0,
   {x, 0, 2}, {y, 0, 4}, {z, 0, 4}, ContourStyle → Red];
```

```
Show[r2, p3, p4, BoxRatios → Automatic, Axes → True, AxesLabel → {x, y, z},
 ImageSize → 150, PlotRange → {{0, 2}, {0, 4}, {0, 4}}]
```



I need the curl of the vector field, F.

```
eFc = Curl[{z², -x², 0}, {x, y, z}]
```

```
{0, 2 z, -2 x}
```

Above: this value does not agree with the text answer, which is {0, 2z, -2z}. I believe the text answer has a typo however, as it produces a zero integral later on.

Regarding the equation of the plane containing the four points.

```
vec1 = {0, 0, 0}
```

```
{0, 0, 0}
```

```
vec2 = {1, 0, 0}
```

```
{1, 0, 0}
```

```
vec3 = {0, 4, 4}
```

```
{0, 4, 4}
```

```
nor1 = Cross[vec1 - vec2, vec1 - vec3]
```

```
{0, -4, 4}
```

The plane's equation, as shown on the above plot, would then be

```
0 (x - 0) - 4 (y - 0) + 4 (z - 0) = 0
```

```
-4 y + 4 z = 0
```

I get the grad of the plane, supposing that this grad is essentially the same as grad S, despite the difference in the number of given points (3 versus 4).

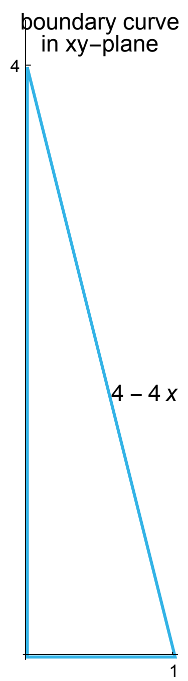
```
gp = Grad[-4 y + 4 z, {x, y, z}]
{0, -4, 4}
```

And dotting curl F with grad S.

```
{0, 2 z, -2 x}.{0, -4, 4}
-8 x - 8 z
```

From the equation of S, I see that z=y.

It remains to decide on a curve C for cut-off. With similar thinking to Paul’s example 2, I choose the projection of S on the xy-plane.



It is necessary to substitute the value of z which I found before, making

$$\int_0^1 \int_0^{4\,(1-x)} (-8\,x - 8\,y)\, dy\, dx$$

$$-\frac{80}{3}$$

The above answer does not match the text. The interior dot product which the text answer gives appears to be based on

```
{0, -2 z, -2 z}.{0, -1, 1}
0
```

And an integral constructed from the above dot product, no matter the limits, would equal zero. For information, the text answer lists ±20 as what appears to be the integral value.

Just for kicks, I played around to come up with the meaningless integral

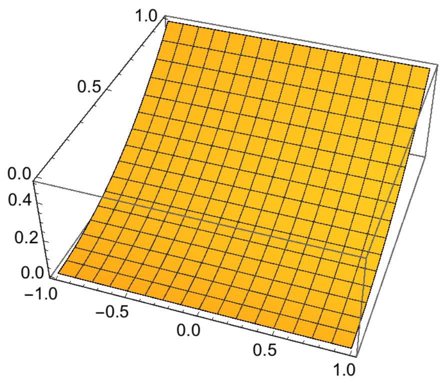
$$\int_0^1 \int_0^{3.405129\,(1-x)} (-8\,x - 8\,y)\, dy\, dx$$

-20.

3. $F = \{e^{-z},\ e^{-z}\,\text{Cos}[y],\ e^{-z}\,\text{Sin}[y]\},\ S:\ z = \frac{y^2}{2},\ -1 \le x \le 1,\ 0 \le y \le 1$

```
Clear["Global`*"]
```

$\texttt{surf} = \texttt{Plot3D}\left[\frac{y^2}{2}, \{x, -1, 1\}, \{y, 0, 1\}\right]$



The s.m. calls attention to the surface as a parabolic cylinder.

$\texttt{fF} = \{e^{-z},\ e^{-z}\,\text{Cos}[y],\ e^{-z}\,\text{Sin}[y]\}$

$\{e^{-z},\ e^{-z}\,\text{Cos}[y],\ e^{-z}\,\text{Sin}[y]\}$

Finding the Curl is not difficult.

$\texttt{curly} = \texttt{Curl}[\{e^{-z},\ e^{-z}\,\text{Cos}[y],\ e^{-z}\,\text{Sin}[y]\},\ \{x, y, z\}]$

$\{e^{-z}\,\text{Cos}[y] + e^{-z}\,\text{Cos}[y],\ -e^{-z},\ 0\}$

Below, eliminating the z expression.

$\texttt{curlyz} = \texttt{curly}\ /.\ z \to \frac{y^2}{2}$

$\left\{e^{-\frac{y^2}{2}}\,\text{Cos}[y] + e^{-\frac{y^2}{2}}\,\text{Cos}[y],\ -e^{-\frac{y^2}{2}},\ 0\right\}$

Below, *Mathematica* was hesitant
 to combine the two instances of $e^{-\frac{y^2}{2}}\,\text{Cos}[y]$.

$\texttt{curlyzz} = \texttt{curlyz}\ /.\ e^{-\frac{y^2}{2}}\,\text{Cos}[y] + e^{-\frac{y^2}{2}}\,\text{Cos}[y] \to 2\,e^{-\frac{y^2}{2}}\,\text{Cos}[y]$

$\left\{2\,e^{-\frac{y^2}{2}}\,\text{Cos}[y],\ -e^{-\frac{y^2}{2}},\ 0\right\}$

Below, writing the surface equation as the s.m. recommended.

```
surf = {x, y, y^2/2}
```

$\left\{x, y, \frac{y^2}{2}\right\}$

Below, finding the partials in preparation for crossing.

```
surfofx = D[surf, {x}]
```

{1, 0, 0}

```
surfofy = D[surf, {y}]
```

{0, 1, y}

Below, crossing gives the normal vector needed.

```
norm = Cross[surfofx, surfofy]
```

{0, -y, 1}

Noting that a quicker way than the pink cells above is to take the **Grad**,

```
Grad[z - y^2/2, {x, y, z}]
```

{0, -y, 1}

Below, the dot product will be the core of the integrand.

```
integr = curlyzz.norm
```

$e^{-\frac{y^2}{2}} y$

Below, the limits are given explicitly.

$$\int_0^1 \int_{-1}^1 \left(e^{-\frac{y^2}{2}} y\right) dx\, dy$$

$$2 - \frac{2}{\sqrt{e}}$$

The above line matches the text's answer, except that the text has +/- on it.

5. $F = \left\{z^2, \frac{3}{2} x, 0\right\}$, S : $0 \le x \le a$, $0 \le y \le a$, $z = 1$

```
Clear["Global`*"]
```

Regarding the vector field F,

$$eF[z\_, y\_, z\_] = \left\{z^2, \frac{3}{2} x, 0\right\}$$

$$\left\{z^2, \frac{3x}{2}, 0\right\}$$

And getting the curl for that one is necessary.

$$Curl\left[\left\{z^2, \frac{3}{2} x, 0\right\}, \{x, y, z\}\right]$$

$$\left\{0, 2z, \frac{3}{2}\right\}$$

The curve S this time happens to be a square in a plane which is parallel to the xy-plane. Three points on this plane are {0,0,1}, {$\frac{a}{4}$, $\frac{a}{2}$,1}, and {a,a,1}. I could (and did) calculate the equation of a plane, but that doesn't get me to the text answer. So I will adopt a craven imitative stance and write S as 2 a+z, whereupon

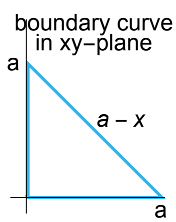
```
Grad[{2 a + z}, {x, y, z}]
{{0, 0, 1}}
```

So there is a dot product between curl of eF and grad of S.

$$\left\{0, 2z, \frac{3}{2}\right\}.\{0, 0, 1\}$$

$$\frac{3}{2}$$

Taking C equal to be the projection of S onto the xy-plane is not now necessary, since the limits on both trees will be constants



So that

$$\int_0^a \int_0^a \frac{3}{2} \, dy \, dx$$

$$\frac{3a^2}{2}$$

The answer in the green cell above matches that in the text.

7. F = {e^y, e^z, e^x}, S : $z = x^2$ ($0 \le x \le 2$, $0 \le y \le 1$)

```
Clear["Global`*"]
```

The curl from the vector field

```
eF[z_, y_, z_] = {ⅇ^y, ⅇ^z, ⅇ^x}
{ⅇ^y, ⅇ^z, ⅇ^x}
```

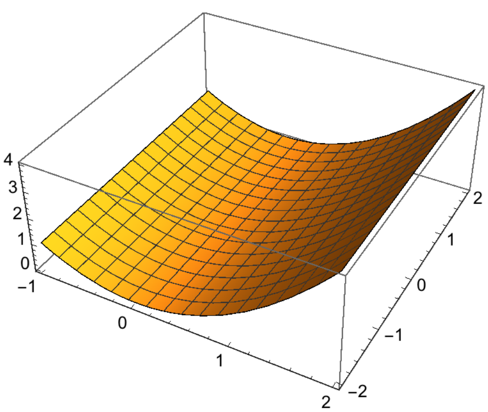
```
Curl[{ⅇ^y, ⅇ^z, ⅇ^x}, {x, y, z}]
```

$\{-e^z, -e^x, -e^y\}$

This time the curve S is a curvy function for a change.

```
eS[x_, y_, z_] = {0, 0, x^2}
{0, 0, x^2}
```

```
Plot3D[x^2, {x, -1, 2}, {y, -2, 2}, ImageSize → 250]
```



I could look at the grad of S.

```
Grad[z - x^2, {x, y, z}]
```

$\{-2x, 0, 1\}$

```
{-ⅇ^z, -ⅇ^x, -ⅇ^y}.{-2 x, 0, 1}
```

$-e^y + 2 e^z x$

$$\int_0^2 \int_0^1 \left(-e^y + 2 e^{x^2} x\right) dy\, dx$$

$1 - 2e + e^4$

The answer above agrees with the text.

9. Verify Stoke's theorem for **F** and *S* in problem 5.

11. Stoke's theorem not applicable. Evaluate
$\oint \mathbf{F.r'}\,ds$, $F = (x^2+y^2)^{-1}\{-y, x\}$, $C: x^2+y^2 = 1$, $z = 0$, oriented clockwise.
Why can Stoke's theorem not be applied? What (false) result would it give?

13 - 20 Evaluation of $\oint_C \mathbf{F.r'}\,ds$

Calculate this line integral by Stoke's theorem for the given F and C. Assume the Cartesian coordinates to be right-handed and the z-component of the surface normal to be nonnegative.

13. $F = \{-5y, 4x, z\}$, C the circle $x^2+y^2 = 16$, $z = 4$

```
Clear["Global`*"]
```

F is the external vector field.

```
eF[x_, y_, z_] = {-5 y, 4 x, z}
{-5 y, 4 x, z}

Curl[{-5 y, 4 x, z}, {x, y, z}]
{0, 0, 9}
```

This seems a little odd, because I am given C but not S. In order to function in this environment I need to follow example 3 on p. 467, which is analogous. As a surface S bounded by C I can take the circular disk $x^2+y^2 \le 16$ in the plane z = 4. Then **n** in Stoke's theorem points in the positive z-direction; thus, **n** = **k**. Hence Curl[F].**n** is simply the component of Curl[F] in the positive z-direction (z being the **k** component). Since F with z = 4 has the constituents $F_1 = -20\,y$, $F_2 = 16\,x$, $F_3 = 4$, I thus obtain

$$\text{Curl[F].n} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 16 - (-20) = 36$$

Hence the integral over S in Stokes's theorem equals 36 times the area 4π of the disk S. This yields the answer

```
36 * 4 π
```

144 π

The above answer does not match that of the text, which is 80π, but see the following note.

$$\int_S\int (9)\,dA = 9\,(\pi)\,(4^2) = 144\,\pi$$

As an interesting take on this problem, the s.m. gives the result shown in the cyan cell above, which is seemingly at odds with the text answer, but in perfect agreement with example 3 (and the yellow cell).

15. F = $\{y^2, x^2, z + x\}$ around the triangle with vertices {0, 0, 0}, {1, 0, 0}, {1, 1, 0}

```
Clear["Global`*"]
```

This is similar to Paul's example 2. I have the vector field

```
eF[x_, y_, z_] = {y^2, x^2, z + x}
{y^2, x^2, x + z}
```

and I get its curl (which matches the answer in the text).

```
Curl[{y^2, x^2, x + z}, {x, y, z}]
```

```
{0, -1, 2 x - 2 y}
```

The three given vectors are set down.

```
vec1 = {0, 0, 0}
{0, 0, 0}

vec2 = {1, 0, 0}
{1, 0, 0}

vec3 = {1, 1, 0}
{1, 1, 0}

Cross[vec2 - vec3, vec2 - vec1]
{0, 0, 1}
```

So an equation of the plane is, unsurprisingly,

```
0 (x - 1) + 0 (y - 0) + 1 (z - 0) == 0
z == 0
```

which will not be used. The triangle that was given is C. Suppose I make S equal to the same triangle, except using {0,0,1}, that is, the vectors

```
vec4 = {0, 0, 1}
{0, 0, 1}

vec5 = {1, 0, 0}
{1, 0, 0}
```

```
vec6 = {1, 1, 0}
{1, 1, 0}
```

Then

```
Cross[vec5 - vec6, vec5 - vec4]
{1, 0, 1}
```

And an equation for S would be

```
(x - 1) + 0 (y - 0) + (z - 0) == 0
-1 + x + z == 0
```

and its grad would be

```
Grad[{-1 + x + z}, {x, y, z}]
{{1, 0, 1}}
```

and the dot product with the curl of F would be

```
{0, -1, 2 x - 2 y}.{1, 0, 1}
2 x - 2 y
```

Now I have it down to an integral over C

$$\int_D\int (2x - 2y)\, dA$$

but I have to watch out about the integration limits, taking a cue from Paul, because if the limits on *d*y are put as 0 to 1, it will not work.

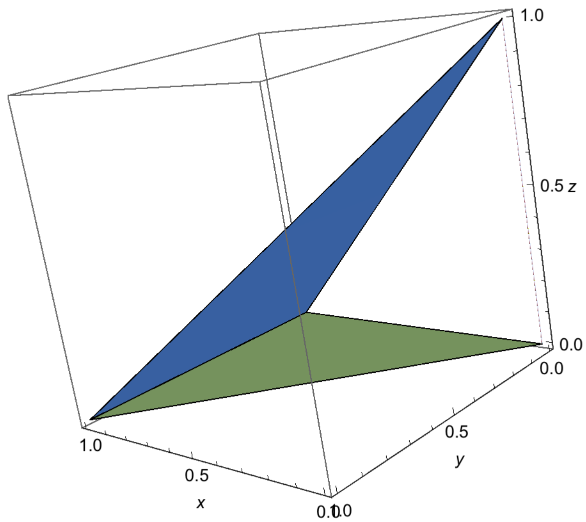
$$\int_0^1\int_0^x (2x - 2y)\, dy\, dx$$

$$\frac{1}{3}$$

The above answer agrees with that in the text.

I can make a plot of the setup showing S and C.

```
Graphics3D[{{RGBColor[0.5, 0.9, 1], Prism[{{0, 0, 0}, {1, 0, 0},
      {1, 1, 0}, {0, 0, 0.001}, {1, 0, 0.001}, {1, 1, 0.001}}]},
  {RGBColor[0.5, 0.9, 1], Prism[{{0, 0, 1}, {1, 0, 0}, {1, 1, 0},
      {0, 0, 0.999}, {1, 0, 0.001}, {1, 1, 0.001}}]},
  {Cylinder[{{0, 0, 0}, {0, 0, 1}}, 0.001]}}, ImageSize → 300,
 Axes → True, AxesLabel → {x, y, z}, BoxRatios → {1, 1, 1}]
```



17. $F = \{0, z^3, 0\}$, C the boundary curve of the cylinder $x^2 + y^2 = 1$, $x \geq 0$, $y \geq 0$, $0 \leq z \leq 1$

```
Clear["Global`*"]
```

There is the vector field

```
eF[x_, y_, z_] = {0, z^3, 0}
```

$\{0, z^3, 0\}$

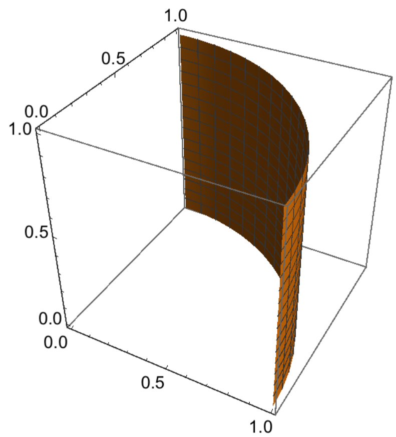
and its curl.

```
Curl[{0, z^3, 0}, {x, y, z}]
```

$\{-3 z^2, 0, 0\}$

And from the text answer it is evident that a parametric version of C (and S) is chosen, which from the constraints on domain reveals itself as a quarter cylinder.

```
ParametricPlot3D[{Cos[u], Sin[u], v},
 {u, 0, π/2}, {v, 0, 1}, ImageSize → 200]
```



Using the parametric notation, the curl could be cast as

$\{-3\, v^2, 0, 0\}$

This problem is very similar in appearance to Paul's example 1. The curve C is at some specific height of the cylinder, and I will assume for a start that z=v=1. Then considering C as a circle at that height,

```
curvc[u_] = {Cos[u], Sin[u], 1}
{Cos[u], Sin[u], 1}
```

If I do the grad of curvc by hand, I would get

```
{-Sin[u], Cos[u], 0}
```

However, the above is not what the text answer gets. I don't have a good rationale, but I could speculate that because the z dimension is what creates C from S, then only the derivatives involving v are taken, and thus the gradient is just

```
{Cos[u], Sin[u], 0}
```

As on a previously used path, I could dot the curl and grad separately

$\{-3\, v^2, 0, 0\}.\{\mathrm{Cos}[u], \mathrm{Sin}[u], 0\}$

$-3\, v^2\, \mathrm{Cos}[u]$

And then do the integral.

$$\int_0^1 \int_0^{\frac{\pi}{2}} \left(-3\, v^2\, \mathrm{Cos}[u]\right) \, du \, dv$$

-1

The answer in yellow above agrees with the text answer.

19. F = {z, e^z, 0},

C the boundary curve of the portion of the cone z = $\sqrt{x^2 + y^2}$,
x ≥ 0, y ≥ 0, 0 ≤ z ≤ 1

In[1]:=
```
Clear["Global`*"]
```

The vector field is

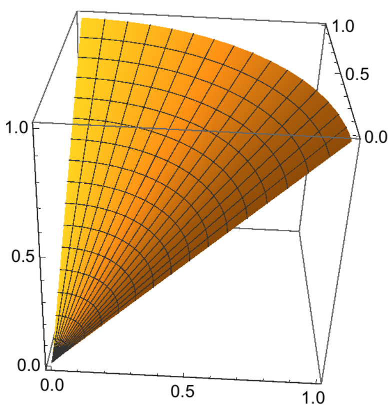
In[2]:=
```
eF[x_, y_, z_] = {z, ⅇ^z, 0}
```

Out[2]= {z, e^z, 0}

In[3]:=
```
Curl[{z, ⅇ^z, 0}, {x, y, z}]
```

Out[3]= {−e^z, 1, 0}

```
ParametricPlot3D[{u Cos[v], u Sin[v], u},
 {u, 0, 1}, {v, 0, π/2}, ImageSize → 200]
```



The following is the dot product which the text answer shows, and which is presumably the core of the integral.

In[4]:=
```
{-ⅇ^z, 1, 0}.{-u Cos[v], -u Sin[v], u}
```

Out[4]= e^z u Cos[v] − u Sin[v]

In[5]:= $\int_0^1 \int_0^{\frac{\pi}{2}} (e^z \, u \operatorname{Cos}[v] - u \operatorname{Sin}[v]) \, dv \, du$

Out[5]= $\frac{1}{2}(-1 + e^z)$

However, this route does not get to the text answer, which is $\frac{1}{2}$. I should also note that the integration limits appear in this case to be listed in the text answer, and are the ones used above.

My preferred route would be

In[7]:= $\{-e^z, 1, 0\}.\{-u \,\text{Sin}[v], u \,\text{Cos}[v], u\}$

Out[7]= $u \,\text{Cos}[v] + e^z u \,\text{Sin}[v]$

because, while differentiating v, it only requires a reason unknown to refrain from differentiating u. This would produce

In[8]:= $\int_0^1 \int_0^{\frac{\pi}{2}} (u \,\text{Cos}[v] + e^z u \,\text{Sin}[v]) \, dv \, du$

Out[8]= $\frac{1}{2} (1 + e^z)$

which appears just as good as the other way, though neither match the text answer.