

### 1 - 6 Elastic deformations

Given  $A$  in a deformation  $y = A x$ , find the principal directions and corresponding factors of extension or contraction.

$$1. \begin{pmatrix} 3.0 & 1.5 \\ 1.5 & 3.0 \end{pmatrix}$$

```
Clear["Global`*"]
```

$$aA = \begin{pmatrix} 3.0 & 1.5 \\ 1.5 & 3.0 \end{pmatrix}$$

```
{{3., 1.5}, {1.5, 3.}}
```

```
e1 = {vals, vecs} = Eigensystem[aA]
```

```
{{{4.5, 1.5}, {{0.707107, 0.707107}, {-0.707107, 0.707107}}}}
```

Above: The answer agrees with the text. As explained in the s.m., the eigenvalues describe the magnitude of the deformation, and the eigenvectors describe the angle of application.

$$\alpha = \text{ArcCot} \left[ \frac{0.7071067811865475}{0.7071067811865475} \right]$$

```
0.7853981633974483`
```

$\alpha$   
Degree

45.

$$\beta = \text{ArcCot} \left[ \frac{-0.7071067811865475}{0.7071067811865475} \right]$$

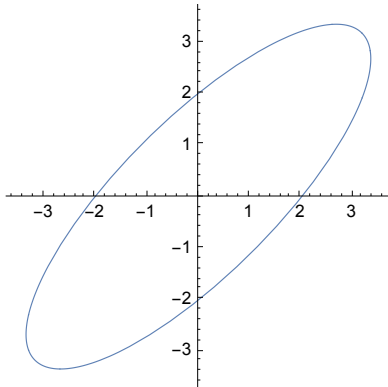
```
-0.785398
```

$\beta$   
Degree

-45.

Above: So the 4.5 magnitude distortion is directed along an axis of 45 degrees (from horizontal), and the 1.5 magnitude distortion is directed along an axis of -45 degrees.

```
ParametricPlot[{4.5` Cos[t] Cos[ $\frac{\pi}{4}$ ] - 1.5` Sin[t] Sin[ $\frac{\pi}{4}$ ],
  4.5` Cos[t] Sin[ $\frac{\pi}{4}$ ] + 1.5` Sin[t] Cos[ $\frac{\pi}{4}$ ]},
{t, 0, 2  $\pi$ }, ImageSize -> 200, PlotStyle -> Thickness[0.003]]
```



$$3. \begin{pmatrix} 7 & \sqrt{6} \\ \sqrt{6} & 2 \end{pmatrix}$$

```
Clear["Global`*"]
```

$$\mathbf{aA} = \begin{pmatrix} 7 & \sqrt{6} \\ \sqrt{6} & 2 \end{pmatrix}$$

$$\{\{7, \sqrt{6}\}, \{\sqrt{6}, 2\}\}$$

```
e1 = {vals, vecs} = Eigensystem[aA]
```

$$\{\{8, 1\}, \{\{\sqrt{6}, 1\}, \{-\frac{1}{\sqrt{6}}, 1\}\}\}$$

Above: The eigenvalues and eigenvectors match the answer in the text.

```
e1[[2, 1, 1]]
```

$$\sqrt{6}$$

$$\mathbf{e2} = \alpha = \text{ArcCot}\left[\frac{\mathbf{e1}[[2, 1, 1]]}{\mathbf{e1}[[2, 1, 2]]}\right] // \mathbf{N}$$

```
0.387597
```

$$\mathbf{e3} = \frac{\mathbf{e2}}{\text{Degree}}$$

```
22.2077
```

Above: So the distortion of magnitude 8 is directed along an axis of +22.2 degrees above

horizontal.

```
e1[[2, 2]]
```

$$\left\{-\frac{1}{\sqrt{6}}, 1\right\}$$

```
e4 = β = ArcCot[ $\frac{e1[[2, 2, 1]]}{e1[[2, 2, 2]]}$ ] // N
```

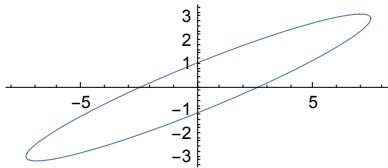
```
-1.1832
```

```
e5 =  $\frac{e4}{\text{Degree}}$ 
```

```
-67.7923
```

Above: And the distortion of magnitude 1 is directed along an axis of  $-67.8$  degrees above horizontal.

```
ParametricPlot[{8.` Cos[t] Cos[.387] - 1` Sin[t] Sin[.387],  
8.` Cos[t] Sin[.387] + 1` Sin[t] Cos[.387]},  
{t, 0, 2 π}, ImageSize → 200, PlotStyle → Thickness[0.003]]
```



$$5. \quad \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$

```
Clear["Global`*"]
```

$$\mathbf{aA} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$

$$\left\{\left\{1, \frac{1}{2}\right\}, \left\{\frac{1}{2}, 1\right\}\right\}$$

```
e1 = {vals, vecs} = Eigensystem[aA]
```

$$\left\{\left\{\frac{3}{2}, \frac{1}{2}\right\}, \left\{\{1, 1\}, \{-1, 1\}\right\}\right\}$$

Above: The eigenvalues and eigenvectors agree with the answer in the text.

```
vecs[[1, 1]]
```

```
1
```

$$e2 = \text{ArcCot} \left[ \frac{\text{vecs}[[1, 1]]}{\text{vecs}[[1, 2]]} \right]$$

$$\frac{\pi}{4}$$

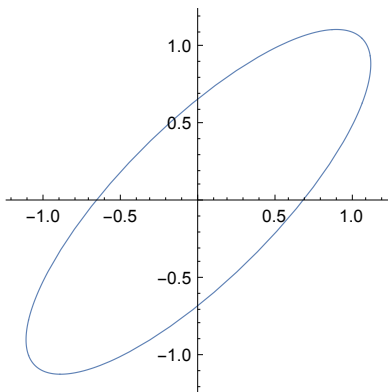
Above: So, the distortion of magnitude  $\frac{3}{2}$  is directed along an axis rotated +45 degrees.

$$e3 = \text{ArcCot} \left[ \frac{\text{vecs}[[2, 1]]}{\text{vecs}[[2, 2]]} \right]$$

$$-\frac{\pi}{4}$$

And the distortion of magnitude  $\frac{1}{2}$  is directed along an axis rotated -45 degrees.

```
ParametricPlot[
  {  $\frac{3}{2} \cos[t] \cos\left[\frac{\pi}{4}\right] - \frac{1}{2} \sin[t] \sin\left[\frac{\pi}{4}\right], \frac{3}{2} \cos[t] \sin\left[\frac{\pi}{4}\right] + \frac{1}{2} \sin[t] \cos\left[\frac{\pi}{4}\right] \},
  \{t, 0, 2\pi\}, \text{ImageSize} \rightarrow 200, \text{PlotStyle} \rightarrow \text{Thickness}[0.003]$ 
```



### 7 - 9 Markov processes

Find the limit state of the Markov process modeled by the given matrix.

$$7. \begin{pmatrix} 0.2 & 0.5 \\ 0.8 & 0.5 \end{pmatrix}$$

Comment: The below closely follows the process and arguments of example 2 on p. 331 of the text.

```
Clear["Global`*"]
```

$$aA = \begin{pmatrix} 0.2 & 0.5 \\ 0.8 & 0.5 \end{pmatrix}$$

```
{{0.2, 0.5}, {0.8, 0.5}}
```

```
e1 = Transpose[aA]
```

```
{{0.2, 0.8}, {0.5, 0.5}}
```

```
e2 = {1, 1}
{1, 1}
```

```
e3 = e1.e2
{1., 1.}
```

Above: Because the transpose of  $aA$ , multiplied against a unity vector, equals the unity vector, it is concluded that the transpose of  $aA$  has 1 as an eigenvalue. And because the transpose of  $aA$  has 1 as an eigenvalue, then  $aA$  also has 1 as an eigenvalue in consequence of theorem 3 in section 8.1, p. 328.

```
e4 = aA - IdentityMatrix[2]
{{-0.8, 0.5}, {0.8, -0.5}}
```

Above: The first step in identifying the eigenvector corresponding to the eigenvalue 1 for  $aA$ , the existence of which was established.

```
e5 = RowReduce[e4]
{{1, -0.625}, {0, 0}}
```

Above: The second step.

```
e6 = {x1, x2}
{x1, x2}
```

Above: Bringing the last player onto the stage.

```
e7 = Thread[e4.e6 == 0]
{-0.8 x1 + 0.5 x2 == 0, 0.8 x1 - 0.5 x2 == 0}
```

Above:  $e5$  has an empty row. This empty row means that one coordinate of the eigenvector for eigenvalue 1 can be assigned arbitrarily.

```
e8 = Solve[e7, {x1, x2}]
```

Solve::vars: Equations may not give solution for all "solve" variables>>

```
{{x2 → 0. + 1.6 x1}}
```

Above: The assignment will be for  $x_1=5$ .

```
e9 = e8 /. x1 → 5
{{x2 → 8.}}
```

Above: The answers for the eigenvector corresponding to eigenvalue 1 for  $aA$  agrees with the text. The eigenvector is  $\{5, 8\}$ .

$$9. \begin{pmatrix} 0.6 & 0.1 & 0.2 \\ 0.4 & 0.1 & 0.4 \\ 0 & 0.8 & 0.4 \end{pmatrix}$$

```

Clear["Global`*"]

aA =  $\begin{pmatrix} 0.6 & 0.1 & 0.2 \\ 0.4 & 0.1 & 0.4 \\ 0 & 0.8 & 0.4 \end{pmatrix}$ 
{{0.6, 0.1, 0.2}, {0.4, 0.1, 0.4}, {0, 0.8, 0.4}}

e1 = Transpose[aA]
{{0.6, 0.4, 0}, {0.1, 0.1, 0.8}, {0.2, 0.4, 0.4}}

e2 = e1.{1, 1, 1}
{1., 1., 1.}

e3 = aA - IdentityMatrix[3]
{{-0.4, 0.1, 0.2}, {0.4, -0.9, 0.4}, {0, 0.8, -0.6}}

e4 = RowReduce[e3]
{{1, 0., -0.6875}, {0, 1, -0.75}, {0, 0, 0}}

Above: There is an empty row in the row echelon version of aA.

e5 = {x1, x2, x3}
{x1, x2, x3}

e6 = Thread[e4.e5 == 0]
{0. + x1 - 0.6875 x3 == 0, x2 - 0.75 x3 == 0, True}

e7 = Solve[e6, e5]
Solve::vars: Equations may not give solution for all "solve" variables>>
{{x2 -> 0. + 1.09091 x1, x3 -> 0. + 1.45455 x1}}

```

```
e8 = e7 /. x1 -> 11
```

Above: Since I have the answer to refer to, I know that x1 coordinate should be 11.

```
{{x2 -> 12., x3 -> 16.}}
```

Above: The answer matches the text. The eigenvector sought equals {11, 12, 16}

### 10 - 12 Age-specific population

Find the growth rate in the Leslie model (see example 3, p. 331) with the matrix as given.

$$11. \begin{pmatrix} 0 & 3.45 & 0.6 \\ 0.90 & 0 & 0 \\ 0 & 0.45 & 0 \end{pmatrix}$$

```
Clear["Global`*"]
```

$$aA = \begin{pmatrix} 0 & 3.45 & 0.6 \\ 0.90 & 0 & 0 \\ 0 & 0.45 & 0 \end{pmatrix}$$

```
{{0, 3.45, 0.6}, {0.9, 0, 0}, {0, 0.45, 0}}
```

```
e1 = {400, 400, 400}
```

```
{400, 400, 400}
```

```
e2 = aA.e1
```

```
{1620., 360., 180.}
```

```
e3 = aA.e2
```

```
{1350., 1458., 162.}
```

```
e4 = aA.e3
```

```
{5127.3, 1215., 656.1}
```

$$e6 = \text{Det} \begin{bmatrix} -LL & 3.45 & 0.6 \\ 0.90 & -LL & 0 \\ 0 & 0.45 & -LL \end{bmatrix}$$

$$1. (0.243 + 3.105 LL - 1. LL^3)$$

```
e7 = Solve[e6 == 0]
```

```
{{LL → -1.72158}, {LL → -0.0784162}, {LL → 1.8}}
```

Above: There is one positive root,  $LL=1.8$ .

```
e8 = aA - 1.8 IdentityMatrix[3]
```

```
{{-1.8, 3.45, 0.6}, {0.9, -1.8, 0.}, {0., 0.45, -1.8}}
```

```
e9 = {x1, x2, x3}
```

```
{x1, x2, x3}
```

```
e10 = Thread[e8.e9 == 0]
```

```
{-1.8 x1 + 3.45 x2 + 0.6 x3 == 0,  
0. + 0.9 x1 - 1.8 x2 == 0, 0. + 0.45 x2 - 1.8 x3 == 0}
```

```
e11 = Solve[e10]
```

```
{{x2 → 0. + 0.5 x1, x3 → 0. + 0.125 x1}}
```

$$e_{12} = e_{11} / .x1 \rightarrow 1$$

$$\{\{x2 \rightarrow 0.5, x3 \rightarrow 0.125\}\}$$

$$e_{13} = \{1, .5, .125\}$$

$$\{1, 0.5, 0.125\}$$

$$e_{14} = \frac{1}{1.625}$$

$$0.615385$$

$$e_{15} = 1200 e_{14}$$

$$738.462$$

Above: This number has to be multiplied by the first coordinate of the eigenvector: 1.

$$e_{16} = e_{15} .5$$

$$369.231$$

Above: This number has to be multiplied by the second coordinate of the eigenvector: 0.5.

$$e_{17} = e_{15} .125$$

$$92.3077$$

Above: This number has to be multiplied by the third coordinate of the eigenvector: .125.

$$e_{18} = e_{15} + e_{16} + e_{17}$$

$$1200.$$

Above: The three initial classes are shown to be equal to the original number.

Above: By an odd coincidence, the eigenvalue-derived factor (which is the sum of the eigenvector coordinates) WAS exactly the same for this problem as for the example 3 on p 331. As for the problem answer, the book only gives the eigenvalue, 1.8. This value is shown as  $e7[[3]]$ . As for the calculation of initial class sizes for 'proportional growth', it took awhile to figure that one out.

$$2.3 / .4$$

$$5.75$$

$$3.45 / .6$$

$$5.75$$



.6 / .3

2.

.9 / .45

2.

The 'odd coincidence' noted above is explained. The matrices (text example/text problem) have linearly dependent entries for key locations.

### 13 - 15 Leontief models

13. Leontief input-output model. suppose that three industries are interrelated so that their outputs are used as inputs by themselves, according to the  $3 \times 3$  **consumption matrix**

$$\begin{pmatrix} 0.1 & 0.5 & 0 \\ 0.8 & 0 & 0.4 \\ 0.1 & 0.5 & 0.6 \end{pmatrix}$$

where  $a_{jk}$  is the fraction of the output of industry  $k$  consumed (purchased) by industry  $j$ .

Let  $p_j$  be the price charged by industry  $j$  for its total output. A problem is to find prices so that for each industry, total expenditures equal total income. Show that this leads to  $\mathbf{A} \mathbf{p} = \mathbf{p}$ , where  $\mathbf{p} = \{p_1, p_2, p_3\}^+$ , and find a solution  $\mathbf{p}$  with nonnegative  $p_1, p_2, p_3$ .

This problem would consist of finding an eigenvalue equal to 1, like the Markov problem.

```
Clear["Global`*"]
```

```
aA =  $\begin{pmatrix} 0.1 & 0.5 & 0 \\ 0.8 & 0 & 0.4 \\ 0.1 & 0.5 & 0.6 \end{pmatrix}$ 
```

```
{{0.1, 0.5, 0}, {0.8, 0, 0.4}, {0.1, 0.5, 0.6}}
```

```
e1 = Transpose[aA]
```

```
{{0.1, 0.8, 0.1}, {0.5, 0, 0.5}, {0, 0.4, 0.6}}
```

```
e2 = {1, 1, 1}
```

```
{1, 1, 1}
```

```
e3 = e1.e2
```

```
{1., 1., 1.}
```

Above: So the transpose has an eigenvalue equal to 1, implying that the matrix  $\mathbf{aA}$  also has one. Now to find it.

```

e4 = aA - IdentityMatrix[3]
{{-0.9, 0.5, 0}, {0.8, -1, 0.4}, {0.1, 0.5, -0.4}}

e5 = RowReduce[e4]
{{1, 0., -0.4}, {0, 1, -0.72}, {0, 0, 0}}

e6 = {x1, x2, x3}
{x1, x2, x3}

e7 = Thread[e5.e6 == 0]
{0. + x1 - 0.4 x3 == 0, x2 - 0.72 x3 == 0, True}

e8 = Solve[e7, e6]
Solve::vars: Equations may not give solution for all "solve" variables >>
{{x2 -> 0. + 1.8 x1, x3 -> 0. + 2.5 x1}}

```

```
e9 = e8 /. x1 -> 10
```

```
{{x2 -> 18., x3 -> 25.}}
```

Above: The answer matches the text. (The 10 of course was taken from the answer.)

738 × .125

92.25