1 - 5 Similar matrices have equal eigenvalues

Verify this for **A** and $\mathbf{A} = \boldsymbol{P}^{-1}\mathbf{A}\,\mathbf{P}$. If y is an eigenvector of **P**, show that $\mathbf{x} = \mathbf{P}\,\mathbf{y}$ are eigenvectors of **A**. (The object $\boldsymbol{P}^{-1}\mathbf{A}\,\mathbf{P}$ will be frequently employed as one gooey mass, so I'll refer to it as the melange.)

1. $A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$, $P = \begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix}$

```
Clear["Global`*"]
```

$\mathbf{AA} = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$

```
{{3, 4}, {4, -3}}
```

$\mathbf{PP} = \begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix}$

```
{{-4, 2}, {3, -1}}
```

```
e1 = Inverse[PP]
```

$\left\{\left\{\frac{1}{2}, 1\right\}, \left\{\frac{3}{2}, 2\right\}\right\}$

```
e2 = e1.AA.PP
{{-25, 12}, {-50, 25}}
```

```
e3 = {vals, vecs} = Eigensystem[e2]
{{-5, 5}, {{3, 5}, {2, 5}}}
```

```
e4 = {vals, vecs} = Eigensystem[PP]
```

$\left\{\left\{\frac{1}{2}\left(-5-\sqrt{33}\right), \frac{1}{2}\left(-5+\sqrt{33}\right)\right\}, \left\{\left\{\frac{1}{6}\left(-3-\sqrt{33}\right), 1\right\}, \left\{\frac{1}{6}\left(-3+\sqrt{33}\right), 1\right\}\right\}\right\}$

```
e5 = {vals, vecs} = Eigensystem[AA]
{{-5, 5}, {{-1, 2}, {2, 1}}}
```

Above: Some basic declarations and calculations to set up the problem. The spectrum matches the text answer, i.e. I have that the eigenvalues of the melange equal the eigenvalues of **AA**.

```
e14 = e1.{-1, 2}
```

$\left\{\frac{3}{2}, \frac{5}{2}\right\}$

```
e15 = e2.{3/2, 5/2} == -5 {3/2, 5/2}
```

True

```
e16 = e1.{2, 1}
```

{2, 5}

```
e17 = e2.{2, 5} == 5 {2, 5}
```

True

Above. This confirms the second half of Theorem 3, sec 8.4, p. 340. I know the theorem states that eigenvectors of **aA** are good with the melange (after being dotted with Inverse[**pP**]), but not the other way round. The other way round is what this problem instruction is trying to establish. Is it enforceable?

```
e19 = PP.{3, 5}
```

{-2, 4}

```
e20 = AA.e19 == -5 e19
```

True

```
e21 = PP.{2, 5}
```

{2, 1}

```
e22 = AA.e21 == 5 e21
```

True

Above: Okay, it works. That explains where those **x** vectors in the answer came from. The eigenvectors from the melange work when dotted with **pP**.

3. $A = \begin{pmatrix} 8 & -4 \\ 2 & 2 \end{pmatrix}$, $P = \begin{pmatrix} 0.28 & 0.96 \\ -0.96 & 0.28 \end{pmatrix}$

```
Clear["Global`*"]
```

```
aA = (8  -4
      2   2)
```

{{8, -4}, {2, 2}}

```
pP = ( 0.28  0.96
      -0.96  0.28)
```

{{0.28, 0.96}, {-0.96, 0.28}}

```
e1 = Inverse[pP]
```

{{0.28, -0.96}, {0.96, 0.28}}

```
e2 = e1.aA.pP
```

```
{{3.008, -0.544}, {5.456, 6.992}}
```

```
e3 = {vals, vecs} = Eigensystem[aA]
```

```
{{6, 4}, {{2, 1}, {1, 1}}}
```

```
e4 = {vals, vecs} = Eigensystem[e2]
```

```
{{6., 4.}, {{0.178885, -0.98387}, {-0.480833, 0.876812}}}
```

Above: **aA** and the melange share eigenvalues.

```
e5 = {vals, vecs} = Eigensystem[pP]
```

```
{{0.28 + 0.96 i, 0.28 - 0.96 i},
 {{0.707107 + 0. i, 0. + 0.707107 i}, {0.707107 + 0. i, 0. - 0.707107 i}}}
```

```
e6 = e1.{2, 1}
```

```
{-0.4, 2.2}
```

```
e7 = e2.e6 == 6 e6
```

```
True
```

```
e8 = e1.{1, 1}
```

```
{-0.68, 1.24}
```

```
e9 = e2.e8 == 4 e8
```

```
True
```

Above, e6 through e9: eigenvectors of **aA,** after being dotted with Inverse[**pP**], work as eigenvectors of the melange.

```
e10 = pP.{0.17888543819998307`, -0.9838699100999075`}
```

```
{-0.894427, -0.447214}
```

```
e11 = aA.e10 == 6 e10
```

```
True
```

```
e12 = pP.{-0.48083261120685233`, 0.8768124086713189`}
```

```
{0.707107, 0.707107}
```

```
e13 = aA.e12 == 4 e12
```

```
True
```

Above, e10 through e13: eigenvectors of the melange, after being dotted with **pP**, work as eigenvectors of **aA**.

5. $A = \begin{pmatrix} -5 & 0 & 15 \\ 3 & 4 & -9 \\ -5 & 0 & 15 \end{pmatrix}$, $P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

```
Clear["Global`*"]
```

$$aA = \begin{pmatrix} -5 & 0 & 15 \\ 3 & 4 & -9 \\ -5 & 0 & 15 \end{pmatrix}$$

```
{{-5, 0, 15}, {3, 4, -9}, {-5, 0, 15}}
```

$$pP = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
{{0, 1, 0}, {1, 0, 0}, {0, 0, 1}}

e1 = Inverse[pP]

{{0, 1, 0}, {1, 0, 0}, {0, 0, 1}}

e2 = e1.aA.pP

{{4, 3, -9}, {0, -5, 15}, {0, -5, 15}}

e3 = {vals, vecs} = Eigensystem[aA]

{{10, 4, 0}, {{1, -1, 1}, {0, 1, 0}, {3, 0, 1}}}

e4 = {vals, vecs} = Eigensystem[pP]

{{-1, 1, 1}, {{-1, 1, 0}, {0, 0, 1}, {1, 1, 0}}}

e5 = {vals, vecs} = Eigensystem[e2]

{{10, 4, 0}, {{-1, 1, 1}, {1, 0, 0}, {0, 3, 1}}}
```

Above: it is seen that **aA** and the melange share the same eigenvalues, {10, 4, 0}.

```
e6 = e1.{1, -1, 1}

{-1, 1, 1}

e7 = e2.e6 == 10 e6

True

e8 = e1.{0, 1, 0}

{1, 0, 0}

e9 = e2.e8 == 4 e8

True
```

```
e10 = e1.{3, 0, 1}
{0, 3, 1}

e11 = e2.e10 == 0 e10
True
```

Above, e6 through e11: it is seen that the eigenvectors of **aA** work as eigenvectors for the melange if first dotted with Inverse[**pP**].

```
e12 = pP.{-1, 1, 1}
{1, -1, 1}

e13 = aA.e12 == 10 e12
True

e14 = pP.{1, 0, 0}
{0, 1, 0}

e15 = aA.e14 == 4 e14
True

e16 = pP.{0, 3, 1}
{3, 0, 1}

e17 = aA.e16 == 0 e16
True
```

Above, e12 through e17: it is seen that the eigenvectors of the melange work as eigenvectors for **aA** if first dotted with **pP**.

9 - 16 Diagonalization of matrices

Find an eigenbasis (a basis of eigenvectors) and diagonalize.

9. $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

Example 4 on p. 342 shows how to get this done.

```
Clear["Global`*"]
```

$$\texttt{aA} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

```
{{1, 2}, {2, 4}}
```

Find the eigenvectors

```
e1 = {vals, vecs} = Eigensystem[aA]
{{5, 0}, {{1, 2}, {-2, 1}}}
```

Make a matrix of a couple of the eigenvectors, think of this as P from above

```
e2 = (1  -2
      2   1)
{{1, -2}, {2, 1}}
```

Get the inverse of the eigenvector matrix, this will be P^T

```
e4 = Inverse[e2]
```

$\{\{\frac{1}{5}, \frac{2}{5}\}, \{-\frac{2}{5}, \frac{1}{5}\}\}$

Then muliply to make a conglomerate.

```
e4.aA.e2
{{5, 0}, {0, 0}}
```

Above: This is what I need. It is already diagonalized. The answer agrees with the text.

11. $\begin{pmatrix} -19 & 7 \\ -42 & 16 \end{pmatrix}$

```
Clear["Global`*"]

e1 = (-19  7
      -42  16)
{{-19, 7}, {-42, 16}}

e2 = {vals, vecs} = Eigensystem[e1]
{{-5, 2}, {{1, 2}, {1, 3}}}

e3 = (1  1
      2  3)
{{1, 1}, {2, 3}}

e4 = Inverse[e3]
{{3, -1}, {-2, 1}}

e5 = e4.e1.e3
{{-5, 0}, {0, 2}}

Det[e5]
-10

e6 = {{2, 0}, {0, -5}}
{{2, 0}, {0, -5}}
```

```
Det[e6]
-10

Eigenvalues[e5]
{-5, 2}

Eigenvalues[e6]
{-5, 2}
```

Above: Mathematica has found e5 as the answer, but the text shows e6. e5 and e6 have the same determinant, trace, and eigenvalues, and I suspect they are similar. However, I do not know how to find P such that e6=P^{-1}.e5.P, so I can't take a green.

13. $\begin{pmatrix} 4 & 0 & 0 \\ 12 & -2 & 0 \\ 21 & -6 & 1 \end{pmatrix}$

```
Clear["Global`*"]
```

e1 = $\begin{pmatrix} 4 & 0 & 0 \\ 12 & -2 & 0 \\ 21 & -6 & 1 \end{pmatrix}$

```
{{4, 0, 0}, {12, -2, 0}, {21, -6, 1}}

e2 = {vals, vecs} = Eigensystem[e1]
{{4, -2, 1}, {{1, 2, 3}, {0, 1, 2}, {0, 0, 1}}}
```

e3 = $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$

```
{{1, 0, 0}, {2, 1, 0}, {3, 2, 1}}

e4 = Inverse[e3]
{{1, 0, 0}, {-2, 1, 0}, {1, -2, 1}}

e5 = e4.e1.e3
{{4, 0, 0}, {0, -2, 0}, {0, 0, 1}}
```

Above: The looked-for diagonalized matrix. The text answer agrees.

15. $\begin{pmatrix} 4 & 3 & 3 \\ 3 & 6 & 1 \\ 3 & 1 & 6 \end{pmatrix}$

```
Clear["Global`*"]
```

```
e1 = (4 3 3
      3 6 1
      3 1 6)
{{4, 3, 3}, {3, 6, 1}, {3, 1, 6}}

e2 = {vals, vecs} = Eigensystem[e1]
{{10, 5, 1}, {{1, 1, 1}, {0, -1, 1}, {-2, 1, 1}}}

e3 = (1  0 -2
      1 -1  1
      1  1  1)
{{1, 0, -2}, {1, -1, 1}, {1, 1, 1}}

e4 = Inverse[e3]
```

$$\left\{\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}, \left\{0, -\frac{1}{2}, \frac{1}{2}\right\}, \left\{-\frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right\}\right\}$$

```
e5 = e4.e1.e3
{{10, 0, 0}, {0, 5, 0}, {0, 0, 1}}
```

Above: another similar-looking lookalike. The text gives the answer {{10,0,0},{0,1,0},{0,0,5}}.

```
e6 = {{10, 0, 0}, {0, 1, 0}, {0, 0, 5}}
{{10, 0, 0}, {0, 1, 0}, {0, 0, 5}}

Det[e5]
50

Det[e6]
50

Eigenvalues[e5]
{10, 5, 1}

Eigenvalues[e6]
{10, 5, 1}

Tr[e5]
16

Tr[e6]
16
```

Again, I can't prove they are similar, but with three common qualities, it's suggestive.

17 - 23 Principal axes. Conic sections

What kind of conic section (or pair of straight lines) is given by the quadratic form? Transform it to principal axes. Express $\mathbf{x}^T = \{x_1, x_2\}$ in terms of the new coordinate vector $\mathbf{y}^T = \{y_1, y_2\}$, as in example 6, p. 344.

17. $7\, x_1{}^2 + 6\, x_1\, x_2 + 7\, x_2{}^2 = 200$

```
Clear["Global`*"]
```

```
e1 = 7 x1^2 + 6 x1 x2 + 7 x2^2 == 200
```

$7\, x1^2 + 6\, x1\, x2 + 7\, x2^2 == 200$

$$e2 = \begin{pmatrix} 7 & 3 \\ 3 & 7 \end{pmatrix}$$

```
{{7, 3}, {3, 7}}
```

Above: The text answer identifies the matrix as C.

```
e3 = {x1, x2}
```

```
{x1, x2}
```

```
e4 = e3.e2.e3
```

```
x1 (7 x1 + 3 x2) + x2 (3 x1 + 7 x2)
```

```
e5 = Expand[e4]
```

$7\, x1^2 + 6\, x1\, x2 + 7\, x2^2$

```
e6 = {vals, vecs} = Eigensystem[e2]
```

```
{{10, 4}, {{1, 1}, {-1, 1}}}
```

Above: according to example 6, p. 344, numbered line (10) on p 343 becomes:

```
e7 = 10 y1^2 + 4 y2^2
```

$10\, y1^2 + 4\, y2^2$

And further,

```
e8 = e7 == 200
```

$10\, y1^2 + 4\, y2^2 == 200$

Above: the equation is the same as in the text, except numbering of constants is reversed.

$$e9 = e8 \;/.\; \left\{\left(10\, y1^2 + 4\, y2^2\right) \to \left(\frac{10\, y1^2}{200} + \frac{4\, y2^2}{200}\right),\ 200 \to 1\right\}$$

$$\frac{y1^2}{20} + \frac{y2^2}{50} == 1$$

Above: this is the line where the type of conic section is identified. Here there are different coefficients to the squared terms, and the signs are the same. This identifies an ellipse.

```
e10 = e2 - 10 IdentityMatrix[2]
{{-3, 3}, {3, -3}}
```

Above: the part of example 6, p. 344 where normalized eigenvectors are calculated.

```
e11 = Thread[e10.e3 == 0]
{-3 x1 + 3 x2 == 0, 3 x1 - 3 x2 == 0}

e12 = Solve[e11]
{{x2 → x1}}

e16 = {1, 1}
{1, 1}
```

Above: the components of the first eigenvector have been calculated.

```
e13 = e2 - 4 IdentityMatrix[2]
{{3, 3}, {3, 3}}

e14 = Thread[e13.e3 == 0]
{3 x1 + 3 x2 == 0, 3 x1 + 3 x2 == 0}

e15 = Solve[e14]
{{x2 → -x1}}
```

Above: This looks good. However, the vectors are supposed to be orthonormal. They need to be normalized.

```
e16n = Normalize[e16]
```

$$\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$$

```
e17 = {1, -1}
{1, -1}

e17n = Normalize[e17]
```

$$\left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}$$

$$\text{e18} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\left\{\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, \left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}\right\}$$

Above: the matrix of eigenvectors is put together by hand.

```
e19 = {y1, y2}
```

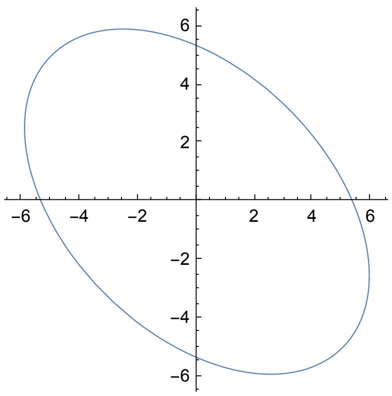
```
{y1, y2}
```

```
e20 = Thread[e18.e19 == e3]
```

$$\left\{\frac{y1}{\sqrt{2}} + \frac{y2}{\sqrt{2}} == x1,\ \frac{y1}{\sqrt{2}} - \frac{y2}{\sqrt{2}} == x2\right\}$$

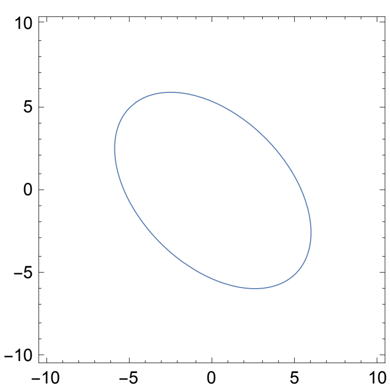
This mostly matches the text. Perhaps the rotation would be opposite to the text. In Wikipedia is an article on the Principal axis theorem, which was of use.

```
ParametricPlot[{√20 Cos[t] Cos[π/4] - √50 Sin[t] Sin[π/4],
  √20 Cos[t] Sin[π/4] + √50 Sin[t] Cos[π/4]},
 {t, 0, 2 π}, ImageSize → 200, PlotStyle → Thickness[0.003]]
```



Above: Would this be it, or would it go the other way? If I had to label x1 and x2 axes, which would be which?

```
ContourPlot[7 x1^2 + 6 x1 x2 + 7 x2^2 == 200, {x1, -10, 10},
 {x2, -10, 10}, ImageSize → 200, ContourStyle → Thickness[0.003]]
```



Above: Bravo home team.

19. $3\,x_1^{\,2} + 22\,x_1\,x_2 + 3\,x_2^{\,2} = 0$

```
Clear["Global`*"]
```

$$e1 = \begin{pmatrix} 3 & 11 \\ 11 & 3 \end{pmatrix}$$

```
{{3, 11}, {11, 3}}
```

Above: The matrix identified in the text answer.

```
e2 = {x1, x2}
{x1, x2}
```

```
e3 = e2.e1.e2
x2 (11 x1 + 3 x2) + x1 (3 x1 + 11 x2)
```

```
e4 = Expand[e3]
```

$3\,x1^2 + 22\,x1\,x2 + 3\,x2^2$

```
e5 = {vals, vecs} = Eigensystem[e1]
{{14, -8}, {{1, 1}, {-1, 1}}}
```

Below: According to example6, p. 344, expression (10) on p 343 becomes

$e6 = 14\,y1^2 - 8\,y2^2$

$14\,y1^2 - 8\,y2^2$

Below: and further,

$e7 = 14\,y1^2 - 8\,y2^2 == 0$

$14\,y1^2 - 8\,y2^2 == 0$

Above: this is the line where the type of conic section is identified. The signs on the squared terms are opposite, which would normally indicate a hyperbola. However, the equation is equal to zero, and I guess that’s what makes it a pair of lines. The equation appears in the text answer.

```
e8 = e1 - 14 IdentityMatrix[2]
{{-11, 11}, {11, -11}}
```

Above: the part of example 6, p. 344 where normalized eigenvectors are calculated.

```
e9 = Thread[e8.e2 == 0]
{-11 x1 + 11 x2 == 0, 11 x1 - 11 x2 == 0}
```

```
e10 = Solve[e9]
{{x2 → x1}}
```

```
e11 = {1, 1}
```
```
{1, 1}
```

```
e12 = Normalize[e11]
```
$$\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$$

Above: one normalized vector down, one to go.

```
e13 = e1 + 8 IdentityMatrix[2]
```
```
{{11, 11}, {11, 11}}
```

```
e14 = Thread[e13.e2 == 0]
```
```
{11 x1 + 11 x2 == 0, 11 x1 + 11 x2 == 0}
```

```
e15 = Solve[e14]
```
```
{{x2 → -x1}}
```

```
e16 = {1, -1}
```
```
{1, -1}
```

```
e17 = Normalize[e16]
```
$$\left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}$$

Above: both normalized eigenvectors have been found.

$$\mathbf{e18} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\left\{\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, \left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}\right\}$$

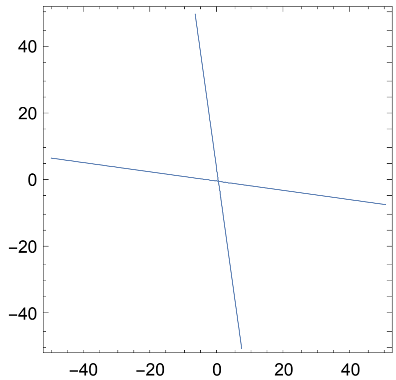
Above: the matrix of eigenvectors is put together by hand. The expression appears in the text.

```
e19 = {y1, y2}
```
```
{y1, y2}
```

```
e20 = Thread[e18.e19 == e2]
```
$$\left\{\frac{y1}{\sqrt{2}} + \frac{y2}{\sqrt{2}} == x1, \frac{y1}{\sqrt{2}} - \frac{y2}{\sqrt{2}} == x2\right\}$$

Above: it looks like something has a 45 degree rotation. But it is not an ellipse. This time the answers match the text well. The text says pair of straight lines, which is obvious, since all factors are linear.

```
ContourPlot[3 x1^2 + 22 x1 x2 + 3 x2^2 == 0, {x1, -50, 50},
 {x2, -50, 50}, ImageSize → 200, ContourStyle → Thickness[0.003]]
```



Above: with new tool, **ContourPlot**, the plot is easily executed.

21. $x_1^2 - 12\, x_1\, x_2 + x_2^2 = 70$

```
Clear["Global`*"]
```

I am at example 6, p. 344. The coefficients for the squared terms are equal here as in example 6, and they make up the diagonal. The $x_1\, x_2$ term is divided in half and makes up the off-diagonal 'corners' of the matrix.

$$e1 = \begin{pmatrix} 1 & -6 \\ -6 & 1 \end{pmatrix}$$

```
{{1, -6}, {-6, 1}}
```

Above: The matrix C is identified as per the text.

```
e2 = {x1, x2}
{x1, x2}
```

Below: The polynomial equation's coefficients are made from the above matrix, the 1 and the 6.

```
Solve[(1 - λ)^2 - (6)^2 == 0, λ]
{{λ → -5}, {λ → 7}}
```

And the polynomial is readily available as

```
e7 = 7 y1^2 - 5 y2^2 == 70
```

$$7\, y_1^2 - 5\, y_2^2 == 70$$

Above: this is the line where the type of conic section is identified. The squared factors differ in sign, and the sum is greater than zero. Hence it is a hyperbola. It agrees with text.

Below: beginning the part of example6, p. 344 where normalized eigenvectors are calculated based on
(A-λ I)x=0, and for this case λ_1 = 7 and λ_2 = -5.

So,

```
e8 = e1 - 7 IdentityMatrix[2]
{{-6, -6}, {-6, -6}}

e9 = Thread[e8.e2 == 0]
{-6 x1 - 6 x2 == 0, -6 x1 - 6 x2 == 0}

e10 = Solve[e9]
{{x2 → -x1}}

e11 = {1, -1}
{1, -1}

e12 = Normalize[e11]
```

$$\left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}$$

Above: one normalized eigenvector down, one to go.

```
e13 = e1 + 5 IdentityMatrix[2]
{{6, -6}, {-6, 6}}

e14 = Thread[e13.e2 == 0]
{6 x1 - 6 x2 == 0, -6 x1 + 6 x2 == 0}

e15 = Solve[e14]
{{x2 → x1}}

e16 = {1, 1}
{1, 1}

e17 = Normalize[e16]
```

$$\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$$

Above: both normalized eigenvectors have been found.

```
e18 = {{1/Sqrt[2], 1/Sqrt[2]}, {-1/Sqrt[2], 1/Sqrt[2]}}
```

$$e18 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\left\{\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, \left\{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}\right\}$$

Above: the matrix of eigenvectors is put together by hand.

```
e19 = {y1, y2}
```

```
{y1, y2}
```

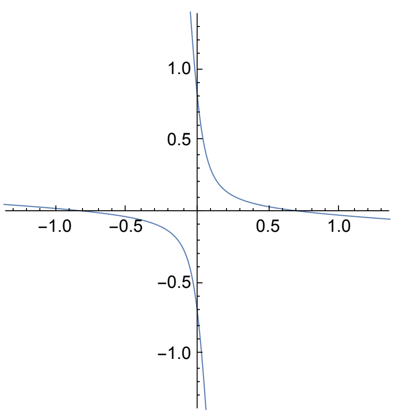
Following the wind-up section of example 6, and fetching the appropriate vectors connoting x_1, x_2, y_1, y_2 ,

```
e20 = Thread[e18.e19 == e2]
```

$$\left\{\frac{y1}{\sqrt{2}} + \frac{y2}{\sqrt{2}} == x1, -\frac{y1}{\sqrt{2}} + \frac{y2}{\sqrt{2}} == x2\right\}$$

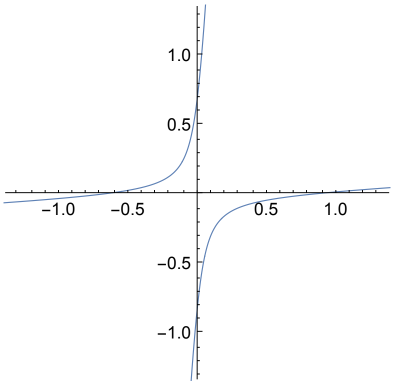
The parametric version of the hyperbola in e7 would be $x = \frac{2}{\sqrt{70}}$ secant t, $y = \frac{\sqrt{5}}{\sqrt{70}}$ tangent t.

$$\text{ParametricPlot}\left[\left\{\frac{2}{\sqrt{70}}\text{Sec}[t]\,\text{Cos}\left[\frac{\pi}{4}\right] - \frac{\sqrt{5}}{\sqrt{70}}\text{Tan}[t]\,\text{Sin}\left[\frac{\pi}{4}\right], \frac{2}{\sqrt{70}}\text{Sec}[t]\,\text{Sin}\left[\frac{\pi}{4}\right] + \frac{\sqrt{5}}{\sqrt{70}}\text{Tan}[t]\,\text{Cos}\left[\frac{\pi}{4}\right]\right\}, \{t, 0, 2\pi\}, \text{ImageSize} \to 200, \text{PlotStyle} \to \text{Thickness}[0.003]\right]$$



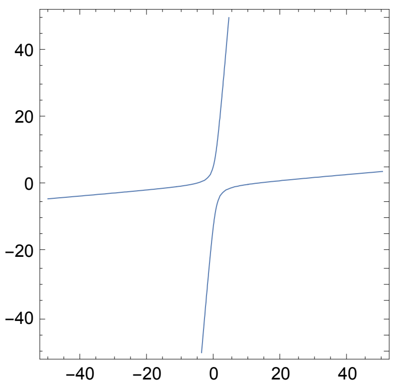
Above: the plot shows the sign differences for multiple angle formula, I think. But does not take into account the sign differences for the problem equation. Those are tried out below:

```
ParametricPlot[{2/Sqrt[70] Sec[t] Cos[-π/4] - Sqrt[5]/Sqrt[70] Tan[t] Sin[-π/4],
  2/Sqrt[70] Sec[t] Sin[-π/4] + Sqrt[5]/Sqrt[70] Tan[t] Cos[-π/4]},
 {t, 0, 2 π}, ImageSize → 200, PlotStyle → Thickness[0.003]]
```



Above: playing with the signs makes it come out looking more like it should.

```
ContourPlot[x1^2 - 12 x1 x2 + x2^2 == 70, {x1, -50, 50},
 {x2, -50, 50}, ImageSize → 200, ContourStyle → Thickness[0.003]]
```



23. $-11\, x_1^2 + 84\, x_1\, x_2 + 24\, x_2^2 = 156$

```
Clear["Global`*"]
```

$$e1 = \begin{pmatrix} -11 & 42 \\ 42 & 24 \end{pmatrix}$$

```
{{-11, 42}, {42, 24}}
```

Above: the matrix C is identified and agrees with the text. Here I see the important difference made when the coefficients of the two squared terms are not equal. They both occupy the diagonal in their own identity.

```
e2 = {x1, x2}
```

{x1, x2}

From this point down to the next green cell, the ground covered is not exactly by shortcut by way of example 6, as it was in the procedure in problem 21.

```
e3 = e2.e1.e2
```

x2 (42 x1 + 24 x2) + x1 (-11 x1 + 42 x2)

```
e4 = Expand[e3]
```

$-11\, x1^2 + 84\, x1\, x2 + 24\, x2^2$

```
e5 = {vals, vecs} = Eigensystem[e1]
```

{{52, -39}, {{2, 3}, {-3, 2}}}

Below: According to example 6 on p. 344, expression (10), p. 343 becomes

```
e6 = 52 y1^2 - 39 y2^2
```

$52\, y1^2 - 39\, y2^2$

Below: and further,

```
e7 = e6 == 156
```

$52\, y1^2 - 39\, y2^2 == 156$

Above: this is the line where the type of conic section is identified. The squared factors differ in sign, making it a hyperbola. It matches the equation in the text answer.

```
e8 = e1 - 52 IdentityMatrix[2]
```

{{-63, 42}, {42, -28}}

```
e9 = Thread[e8.e2 == 0]
```

{-63 x1 + 42 x2 == 0, 42 x1 - 28 x2 == 0}

```
e10 = Solve[e9]
```

$\left\{\left\{x2 \to \frac{3\, x1}{2}\right\}\right\}$

```
e11 = {1, 3 / 2}
```

$\left\{1, \frac{3}{2}\right\}$

```
e12 = Normalize[e11]
```

$\left\{\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right\}$

Above: one normalized eigenvector down, one to go.

```
e13 = e1 + 39 IdentityMatrix[2]
{{28, 42}, {42, 63}}
```

```
e14 = Thread[e13.e2 == 0]
{28 x1 + 42 x2 == 0, 42 x1 + 63 x2 == 0}
```

```
e15 = Solve[e14]
```

$$\left\{\left\{x2 \to -\frac{2\, x1}{3}\right\}\right\}$$

$$e16 = \left\{1, -\frac{2}{3}\right\}$$

$$\left\{1, -\frac{2}{3}\right\}$$

```
e17 = Normalize[e16]
```

$$\left\{\frac{3}{\sqrt{13}}, -\frac{2}{\sqrt{13}}\right\}$$

Above: both normalized eigenvectors have been found.

$$e18 = \begin{pmatrix} \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{13}} \end{pmatrix}$$

$$\left\{\left\{\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right\}, \left\{\frac{3}{\sqrt{13}}, -\frac{2}{\sqrt{13}}\right\}\right\}$$

Above: the matrix of eigenvectors is put together by hand. It matches the matrix shown in the text answer.

```
e19 = {y1, y2}
{y1, y2}
```

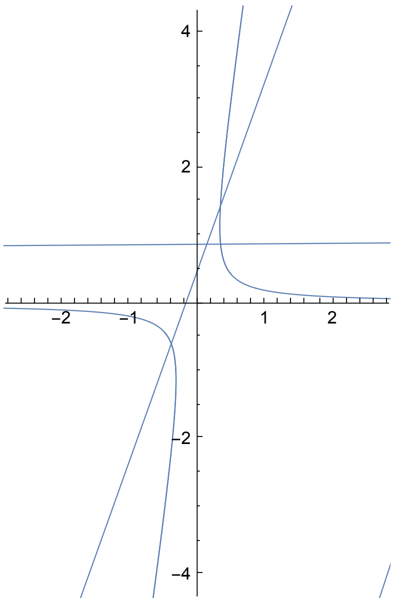
```
e20 = Thread[e18.e19 == e2]
```

$$\left\{\frac{2\, y1}{\sqrt{13}} + \frac{3\, y2}{\sqrt{13}} == x1, \frac{3\, y1}{\sqrt{13}} - \frac{2\, y2}{\sqrt{13}} == x2\right\}$$

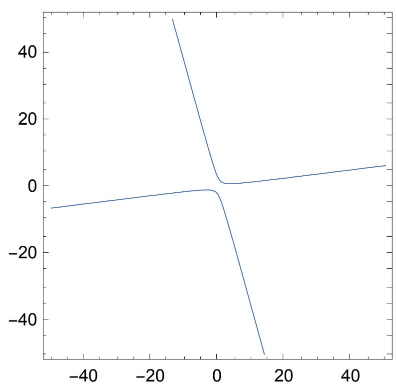
The parametric version of the hyperbola in e7 would be

$x = \sqrt{\frac{52}{156}}$ secant t, $y = \sqrt{\frac{39}{156}}$ tangent t.

$$\text{ParametricPlot}\Big[\Big\{\sqrt{\tfrac{52}{156}}\,\text{Sec}[t]\,\text{Cos}\Big[\tfrac{2}{\sqrt{13}}\Big]-\sqrt{\tfrac{39}{156}}\,\text{Tan}[t]\,\text{Sin}\Big[\tfrac{3}{\sqrt{13}}\Big],\ \sqrt{\tfrac{52}{156}}\,\text{Sec}[t]\,\text{Sin}\Big[\tfrac{3}{\sqrt{13}}\Big]+\sqrt{\tfrac{39}{156}}\,\text{Tan}[t]\,\text{Cos}\Big[-\tfrac{2}{\sqrt{13}}\Big]\Big\},\ \{t, 0, 3\pi\},\ \text{ImageSize}\to 200,\ \text{PlotStyle}\to\text{Thickness}[0.003]\Big]$$



$$\text{ContourPlot}\big[-11\,x1^2+84\,x1\,x2+24\,x2^2==156,\ \{x1,-50,50\},\ \{x2,-50,50\},\ \text{ImageSize}\to 200,\ \text{ContourStyle}\to\text{Thickness}[0.003]\big]$$



Above: evidently there is quite a way to go to understanding the principal axis theorem correctly.