Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

1 - 6 Calculation of the divergence

Find div **v** and its value at P.

1. $v = \{x^2, 4y^2, 9z^2\}$, $P : \left(-1, 0, \frac{1}{2}\right)$

```
Clear["Global`*"]
```

$vv[x\_, y\_, z\_] = Div[\{x^2, 4y^2, 9z^2\}, \{x, y, z\}]$

$2x + 8y + 18z$

$vv[-1, 0, \frac{1}{2}]$

7

3. $v = (x^2 + y^2)^{-1}[x, y]$

```
Clear["Global`*"]
```

$vv[x\_, y\_] = Div[\{\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\}, \{x, y\}]$

$$-\frac{2x^2}{(x^2+y^2)^2} - \frac{2y^2}{(x^2+y^2)^2} + \frac{2}{x^2+y^2}$$

```
Simplify[%]
```

0

5. $v = x^2 y^2 z^2 [x, y, z]$, $P : (3, -1, 4)$

```
Clear["Global`*"]
```

$vv[x\_, y\_, z\_] = Div[\{x^3 y^2 z^2, x^2 y^3 z^2, x^2 y^2 z^3\}, \{x, y, z\}]$

$9x^2 y^2 z^2$

```
vv[3, -1, 4]
```

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7. For what v_3 is **v** = $[e^x \mathrm{Cos}[y], e^x \mathrm{Sin}[y], v_3]$ solenoidal?

```
Clear["Global`*"]
```

Set up a function for the div

```
vv[x_, y_, z_] = Div[{ⅇ^x Cos[y], ⅇ^x Sin[y], v3 }, {x, y, z}]
2 ⅇ^x Cos[y]
```

The output of the function has no v3 factor. This suggests a table to experiment

```
Table[{n, Div[{ⅇ^x Cos[y], ⅇ^x Sin[y], n z }, {x, y, z}]}, {n, 0, 4}]
{{0, 2 ⅇ^x Cos[y]}, {1, 1 + 2 ⅇ^x Cos[y]},
 {2, 2 + 2 ⅇ^x Cos[y]}, {3, 3 + 2 ⅇ^x Cos[y]}, {4, 4 + 2 ⅇ^x Cos[y]}}
```

From the table it is seen that whatever the coefficient of z is, that will be reflected against $2\, e^x \mathrm{Cos}[y]$ in the calculation of div. So if I want a zero outcome (solenoidal), I had better make the third place factor equal $-2\, e^x \mathrm{Cos}[y]\, z$. Trying

```
Div[{ⅇ^x Cos[y], ⅇ^x Sin[y], -2 ⅇ^x Cos[y] z }, {x, y, z}]
0
```

Success. So $v_3 = -2\, e^x \mathrm{Cos}[y]\, z$ is the answer.

11. Incompressible flow. Show that the flow with velocity vector v = y i is incompressible. Show that the particles that at time t = 0 are in the cube whose faces are portions of the planes x = 0, x = 1, y = 0, y = 1, z = 0, z = 1 occupy at t = 1 the volume 1.

15 - 20 Laplacian

Calculate $\nabla^2 f$ by numbered line (3) on p. 404. Check by direct differentiation. Indicate when (3) is simpler.

15. $f = \mathrm{Cos}[x]^2 + \mathrm{Sin}[y]^2$

```
Clear["Global`*"]

e1 = Div[Grad[Cos[x]^2 + Sin[y]^2, {x, y}], {x, y}]
-2 Cos[x]^2 + 2 Cos[y]^2 + 2 Sin[x]^2 - 2 Sin[y]^2

e2 = FullSimplify[e1]
-2 Cos[2 x] + 2 Cos[2 y]
```

17. $f = \mathrm{Log}[x^2 + y^2]$

```
Clear["Global`*"]
```

```
e1 = Div[Grad[Log[x^2 + y^2], {x, y}], {x, y}]
```

$$-\frac{4 x^2}{\left(x^2+y^2\right)^2}-\frac{4 y^2}{\left(x^2+y^2\right)^2}+\frac{4}{x^2+y^2}$$

```
e2 = FullSimplify[e1]
```

0

19. $f = \frac{1}{\left(x^2+y^2+z^2\right)}$

```
Clear["Global`*"]
```

```
e1 = Div[Grad[1/(x^2 + y^2 + z^2), {x, y, z}], {x, y, z}]
```

$$\frac{8 x^2}{\left(x^2+y^2+z^2\right)^3}+\frac{8 y^2}{\left(x^2+y^2+z^2\right)^3}+\frac{8 z^2}{\left(x^2+y^2+z^2\right)^3}-\frac{6}{\left(x^2+y^2+z^2\right)^2}$$

```
e2 = FullSimplify[e1]
```

$$\frac{2}{\left(x^2+y^2+z^2\right)^2}$$