1 - 6 Eigenvalues and vectors

Is the given matrix Hermitian? Skew-Hermitian? Unitary?

1. $\begin{pmatrix} 6 & i \\ -i & 6 \end{pmatrix}$

```
Clear["Global`*"]
```

e1 = $\begin{pmatrix} 6 & i \\ -i & 6 \end{pmatrix}$

{{6, i}, {-i, 6}}

```
e2 = Transpose[e1] // MatrixForm
```

$\begin{pmatrix} 6 & -i \\ i & 6 \end{pmatrix}$

```
e3 = Inverse[e1] // MatrixForm
```

$\begin{pmatrix} \frac{6}{35} & -\frac{i}{35} \\ \frac{i}{35} & \frac{6}{35} \end{pmatrix}$

Above: It is Hermitian. It is not skew-Hermitian. It is not Unitary.

```
e4 = {vals, vecs} = Eigensystem[e1]
```

{{7, 5}, {{i, 1}, {-i, 1}}}

The green cells above, containing eigenvalues, eigenvectors, and matrix ID, agree with the text answer.

3. $\begin{pmatrix} \frac{1}{2} & i\sqrt{\frac{3}{4}} \\ i\sqrt{\frac{3}{4}} & \frac{1}{2} \end{pmatrix}$

```
Clear["Global`*"]
```

e1 = $\begin{pmatrix} \frac{1}{2} & i\sqrt{\frac{3}{4}} \\ i\sqrt{\frac{3}{4}} & \frac{1}{2} \end{pmatrix}$

$\{\{\frac{1}{2}, \frac{i\sqrt{3}}{2}\}, \{\frac{i\sqrt{3}}{2}, \frac{1}{2}\}\}$

```
e2 = Transpose[e1] // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} & \frac{i\sqrt{3}}{2} \\ \frac{i\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

```
e3 = Inverse[e1] // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} & -\frac{i\sqrt{3}}{2} \\ -\frac{i\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

It is not Hermitian. It is not skew-Hermitian. It is Unitary.

```
e4 = {vals, vecs} = Eigensystem[e1]
```

$$\left\{\left\{\frac{1}{2}\left(1 + i\sqrt{3}\right), \frac{1}{2}\left(1 - i\sqrt{3}\right)\right\}, \{\{1, 1\}, \{-1, 1\}\}\right\}$$

The green cells above, containing eigenvalues, eigenvectors, and matrix ID, agree with the text answer.

5. $\begin{pmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}$

```
Clear["Global`*"]
```

$$e1 = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}$$

$\{\{i, 0, 0\}, \{0, 0, i\}, \{0, i, 0\}\}$

```
e2 = Transpose[e1] // MatrixForm
```

$$\begin{pmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}$$

```
e3 = Inverse[e1] // MatrixForm
```

$$\begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{pmatrix}$$

Above: the matrix is both Unitary and Skew-Hermitian.

```
e4 = {vals, vecs} = Eigensystem[e1]
```

$\{\{i, i, -i\}, \{\{0, 1, 1\}, \{1, 0, 0\}, \{0, -1, 1\}\}\}$

The green cells above, containing eigenvalues, eigenvectors, and matrix ID, agree with the text answer.

7. Pauli spin matrices. Find the eigenvalues and eigenvectors of the so-called Pauli spin matrices and show that

$S_x S_y = i S_z$, $S_y S_x = -i S_z$, $S_x^2 = S_y^2 = S_z^2 = I$, where

$S_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $S_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $S_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

```
Clear["Global`*"]
```

$sx = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

```
{{0, 1}, {1, 0}}
```

$sy = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

```
{{0, -i}, {i, 0}}
```

$sz = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

```
{{1, 0}, {0, -1}}

e1 = FullSimplify[sx.sy == i sz]
True

e2 = FullSimplify[sy.sx == -i sz]
True

e3 = FullSimplify[sx.sx == sy.sy == sz.sz == IdentityMatrix[2]]
True
```

9 - 12 Complex forms

Is the matrix A Hermitian or skew-Hermitian? Find $\bar{x}^T A x$

9. $\begin{pmatrix} 4 & 3-2i \\ 3+2i & -4 \end{pmatrix}$

```
Clear["Global`*"]
```

$e1 = \begin{pmatrix} 4 & 3-2i \\ 3+2i & -4 \end{pmatrix}$

```
{{4, 3 - 2 i}, {3 + 2 i, -4}}
```

```
e2 = xx = {-4 i, 2 + 2 i}
{-4 i, 2 + 2 i}
```

```
e3 = Transpose[e1] // MatrixForm
```

$$\begin{pmatrix} 4 & 3+2i \\ 3-2i & -4 \end{pmatrix}$$

```
e4 = Inverse[e1] // MatrixForm
```

$$\begin{pmatrix} \frac{4}{29} & \frac{3}{29}-\frac{2i}{29} \\ \frac{3}{29}+\frac{2i}{29} & -\frac{4}{29} \end{pmatrix}$$

The matrix is Hermitian.

```
e5 = xbar = {4 i, 2 - 2 i}
{4 i, 2 - 2 i}
```

```
e6 = xbar.e1.xx
```

16

The two green cells above match the text answer for matrix ID and the value of $\bar{\mathbf{x}}^{T} \mathbf{A}\, \mathbf{x}$.

11. $A = \begin{pmatrix} i & 1 & 2+i \\ -1 & 0 & 3i \\ -2+i & 3i & i \end{pmatrix}$, $x = \begin{pmatrix} 1 \\ i \\ -i \end{pmatrix}$

```
Clear["Global`*"]
```

$$e1 = \begin{pmatrix} i & 1 & 2+i \\ -1 & 0 & 3i \\ -2+i & 3i & i \end{pmatrix}$$

```
{{i, 1, 2 + i}, {-1, 0, 3 i}, {-2 + i, 3 i, i}}
```

```
xx = {1, i, -i}
{1, i, -i}
```

```
xbar = {1, -i, i}
{1, -i, i}
```

```
e2 = Transpose[e1] // MatrixForm
```

$$\begin{pmatrix} i & -1 & -2+i \\ 1 & 0 & 3i \\ 2+i & 3i & i \end{pmatrix}$$

```
e3 = Inverse[e1] // MatrixForm
```

$$\begin{pmatrix} \frac{9 i}{2} & -\frac{5}{2} - \frac{3 i}{2} & -\frac{3}{2} \\ \frac{5}{2} - \frac{3 i}{2} & 2 i & \frac{1}{2} + \frac{i}{2} \\ \frac{3}{2} & -\frac{1}{2} + \frac{i}{2} & \frac{i}{2} \end{pmatrix}$$

The matrix is skew-Hermitian.

```
e4 = xbar.e1.xx
```

$-6 i$

The two green cells above match the text answer for matrix ID and the value of $\bar{\mathbf{x}}^T \mathbf{A}\, \mathbf{x}$.