```
Clear["Global`*"]
```

1 - 6 Mixing problems.

1. Find out, without calculation, whether doubling the flow rate in example 1 has the same effect as halfing the tank sizes. (Give a reason.)

I see the answer to this problem is yes, which surprised me.

3. Derive the eigenvectors in example 1 without consulting this book.

$$A = \begin{pmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{pmatrix}$$

```
{{-0.02, 0.02}, {0.02, -0.02}}
```

```
Eigensystem[A]
{{-0.04, 0.}, {{0.707107, -0.707107}, {0.707107, 0.707107}}}
```

As there is no text answer to this problem, I can't determine whether my guess is right or wrong.

5. If you extend example 1, p. 130 by a tank T_3 of the same size as the others, and connected to T_2 by two tubes with flow rates as between T_1 and T_2, what system of ODEs will you get?

The example in the text is basically the diagram below, except only the first two tanks. Working first with the example conditions,

```
ClearAll["Global`*"]
```

```
eqn1 = y1'[x] == -0.02 y1[x] + 0.02 y2[x];
eqn2 = y2'[x] == 0.02 y1[x] - 0.02 y2[x];
ics = {y1[0] == 0, y2[0] == 150};
```

The first **DSolve** will be to get a general solution of the system.

```
sol = DSolve[{eqn1, eqn2}, {y1, y2}, x]
```

$$\{\{y_1 \to \text{Function}[\{x\},\ 0.5\, e^{-0.04 x} (1. + 1.\, e^{0.04 x})\, C[1] + 0.5\, e^{-0.04 x} (-1. + 1.\, e^{0.04 x})\, C[2]],$$
$$y_2 \to \text{Function}[\{x\},\ 0.5\, e^{-0.04 x} (-1. + 1.\, e^{0.04 x})\, C[1] + 0.5\, e^{-0.04 x} (1. + 1.\, e^{0.04 x})\, C[2]]\}\}$$

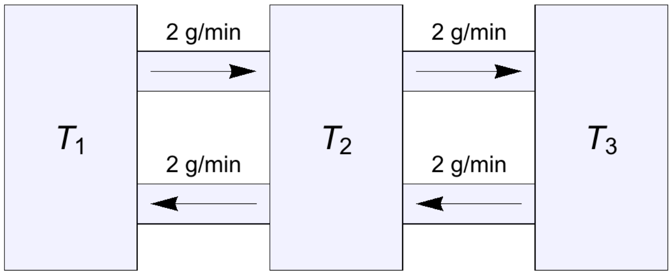
The solution checks.

$$\text{Chop}[\text{Simplify}[\text{eqn1} /. \text{sol}],\ 10^{-17}]$$

```
{True}
```

```
Chop[Simplify[eqn2 /. sol], 10^-17]
```

```
{True}
```



Still working with the text example, in which there are two tanks, I can solve for the initial conditions, in which all 150 pounds of fertilizer starts out in tank T_2.

```
sol2 = DSolve[{eqn1, eqn2, ics}, {y1, y2}, x]
```

$$\{\{y_1 \to \text{Function}[\{x\}, 75.\, e^{-0.04\, x} (-1. + 1.\, e^{0.04\, x})],\ y_2 \to \text{Function}[\{x\}, 75.\, e^{-0.04\, x} (1. + 1.\, e^{0.04\, x})]\}\}$$

The question posed by the example is the time required for the first tank, T_1, to accumulate at least half the fertilizer that is in tank T_2. That will happen when T_1 has 50 pounds and T_2 has 100 pounds.

$$\text{Solve}[74.99999999999999\grave{}\ e^{-0.04\grave{}\ x} (-1.\grave{} + 1.\grave{}\ e^{0.04\grave{}\ x}) == 50, x]$$

Solve::ifun:
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

$$\{\{x \to 27.4653\}\}$$

$$\text{Solve}[75.\grave{}\ e^{-0.04\grave{}\ x} (1.\grave{} + 1.\grave{}\ e^{0.04\grave{}\ x}) == 100, x]$$

Solve::ifun:
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

$$\{\{x \to 27.4653\}\}$$

The above answers match the text example pretty well. (The example gives 27.5 minutes as the time, and displays it on a graph, figure 78, p. 131.) Now, what will the system of ODEs look like with the addition of tank T_3? It is still just the circulation in and out, for each tank. Tank T_1 remains unchanged, its circulation limited to T_2. The circulation in tank T_2 will double, since it will have 4 gpm in and 4 gpm out. The outflow can be described as $2\, y_2$. And there will be 2 gpm going to T_1, as well as 2 gpm going to T_3.

So altogether the equation for T_2 will be $y_2' = 0.02\,(y_1 - 2\, y_2 + y_3)$. As for T_3, it will be just like T_1, except on the other side of T_2, thus $y_3' = 0.02\,(y_2 - y_3)$. This identification of the system of equations is all the problem description asks for.

But let me work it out. Suppose the 150 lbs of fertilzer starts out in T_2 as before, and it is desired to know when T_1 and T_2 have accumulated 25 pounds of fertilizer (which I think should be at the same time.)

```
eqn3 = y1'[x] == -0.02 y1[x] + 0.02 y2[x];
eqn4 = y2'[x] == 0.02 y1[x] - 2 (0.02 y2[x]) + 0.02 y3[x];
eqn5 = y3'[x] == -0.02 y3[x] + 0.02 y2[x];
```

Mathematica is capable of solving the 3-equation problem, and the answer checks.

```
sol3 = DSolve[{eqn3, eqn4, eqn5}, {y1, y2, y3}, x];

Chop[Simplify[eqn3 /. sol3], 10^-17]
{True}

Chop[Simplify[eqn4 /. sol3], 10^-17]
{True}

Chop[Simplify[eqn5 /. sol3], 10^-17]
{True}
```

In the revised set of initial conditions, the 150 pounds of fertilizer is still deposited in T_2.

```
ics2 = {y1[0] == 0, y2[0] == 150, y3[0] == 0};

sol4 = DSolve[{eqn3, eqn4, eqn5, ics2}, {y1, y2, y3}, x]
```

$$\{\{y_1 \to \text{Function}[\{x\}, 50.\, e^{-0.08 x} (-1.\, e^{0.02 x} + 6.73463 \times 10^{-18} e^{0.06 x} + 1.\, e^{0.08 x})],$$
$$y_2 \to \text{Function}[\{x\}, 50.\, e^{-0.08 x} (2.\, e^{0.02 x} - 7.47694 \times 10^{-34} e^{0.06 x} + 1.\, e^{0.08 x})],$$
$$y_3 \to \text{Function}[\{x\}, 50.\, e^{-0.08 x} (-1.\, e^{0.02 x} - 6.73463 \times 10^{-18} e^{0.06 x} + 1.\, e^{0.08 x})]\}\}$$

And the time in minutes to get half of the fertilizer into the two auxillary tanks is sought.

```
Solve[50.00000000000001` e^(-0.08000000000000002` x)
   (-1.` e^(0.020000000000000004` x) + 6.734631938767573 6`*^-18 e^(0.06000000000000001` x) +
     1.` e^(0.08000000000000002` x)) == 25, x]
```

Solve::ifun:
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

```
{{x → 11.5525}}
```

```
Solve[50.00000000000002` e^(-0.08000000000000002` x)
   (1.999999999999999` e^(0.020000000000000004` x) - 7.476943440795785`*^-34
     e^(0.06000000000000001` x) + 1.` e^(0.08000000000000002` x)) == 100, x]
```

Solve::ifun:
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

```
{{x → 11.5525}}
```

```
Solve[50.00000000000002` e^(-0.08000000000000002` x)
   (-1.` e^(0.020000000000000004` x) - 6.734631938767571`*^-18 e^(0.06000000000000001` x) +
     1.` e^(0.08000000000000002` x)) == 25, x]
```

Solve::ifun:
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

```
{{x → 11.5525}}
```

The above cells show that with the circulation doubled, the time to distribute one third of the fertilizer out of tank T_2 is much reduced, in fact by

```
1 - 11.552453009332412` / 27.465307216702744`
```

```
0.57938
```

more than 50 percent.

7 - 9 Electrical network
In example 2, find the currents:

7. If the initial currents are 0 A and -3 A (minus meaning the $I_2(0)$ flows against the direction of the arrow).

```
ClearAll["Global`*"]
```

In example 2 the applicable matrix is found as

$$\begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix}$$

```
{{-4, 4}, {-1.6, 1.2}}
```

Mathematica , in calculating eigenvectors, always normalizes any which have any entries, in the parent matrix, which are floats. In this case I can pull the following into agreement with the text (which does not normalize the eigenvectors here) by rationalizing.

```
Rationalize[-1.6]
```

$$-\frac{8}{5}$$

```
Rationalize[1.2]
```

$$\frac{6}{5}$$

$$A = \begin{pmatrix} -4 & 4 \\ -\frac{8}{5} & \frac{6}{5} \end{pmatrix}$$

$$\left\{\{-4, 4\}, \left\{-\frac{8}{5}, \frac{6}{5}\right\}\right\}$$

For which the applicable eigenvalues and eigenvectors can be found as

```
{vals, vecs} = Eigensystem[A]
```

$$\left\{\left\{-2, -\frac{4}{5}\right\}, \left\{\{2, 1\}, \left\{\frac{5}{4}, 1\right\}\right\}\right\}$$

which I can then decimalize

```
NumberForm[N[{vals, vecs}], 3]
{{-2., -0.8}, {{2., 1.}, {1.25, 1.}}}
```

Scooping up at a later stage in the example, there will be two equations for the two circuit loops.

$I_1 = 2\, c_1\, e^{-2\,t} + c_2\, e^{-0.8\,t} + 3$ **and** $I_2 = c_1\, e^{-2\,t} + 0.8\, c_2\, e^{-0.8\,t}$

For the case where t=0, the example, at top of p. 134, states these as

$I_1[0] = 2\, c_1 + c_2 + 3 = 0$ **and** $I_2[0] = c_1 + 0.8\, c_2 = -3$

The alteration, from example 2, for this problem is that at t=0 the two current values are 0 and -3 Amp respectively, so the above equations can be solved by

$\text{Solve}[2\, c_1 + c_2 + 3 == 0\ \&\&\ c_1 + 0.8\, c_2 == -3,\ \{c_1, c_2\}]$

$\{\{c_1 \to 1.,\ c_2 \to -5.\}\}$

Then I will have

$I_1[t] = \left(2\, c_1\, e^{-2\,t} + c_2\, e^{-0.8\,t} + 3\right)$ /.
$\{c_1 \to 0.9999999999999997\grave{},\ c_2 \to -4.999999999999999\grave{}\}$

$3 + 2.\ e^{-2\,t} - 5.\ e^{-0.8\,t}$

and

$I_2[t] = c_1\, e^{-2\,t} + 0.8\, c_2\, e^{-0.8\,t}$ /.
$\{c_1 \to 0.9999999999999997\grave{},\ c_2 \to -4.999999999999999\grave{}\}$

$1.\ e^{-2\,t} - 4.\ e^{-0.8\,t}$

The text answer only encompasses the constant values in green above, not the actual resulting current equations.

9. If the initial currents in example 2 are 28 A and 14 A.

The use of example 2 on p. 132 is not finished, there is this additional problem concerning it. Using the last problem, and jumping down to the pertinent expressions

$\text{Solve}[2\, c_1 + c_2 + 3 == 28\ \&\&\ c_1 + 0.8\, c_2 == 14,\ \{c_1, c_2\}]$

$\{\{c_1 \to 10.,\ c_2 \to 5.\}\}$

The above green cell matches the text answer. The text answer skips the final equations, so I will also.

10 - 13 Conversion to systems
Find a general solution of the given ODE (a) by first converting it to a system, (b), as given.

11. 4 y'' - 15 y' - 4 y = 0

(a) Convert to a system. Conversion to a system seems like it would be useful in some cases. However, as long as DSolve can get it done with such conversion, it is a little difficult to get motivated about it.

(b) As given

```
eqn = 4 y''[x] - 15 y'[x] - 4 y[x] == 0
-4 y[x] - 15 y′[x] + 4 y′′[x] == 0
```

```
sol = DSolve[eqn, y, x]
```

$\{\{y \to \text{Function}[\{x\}, e^{-x/4}\, C[1] + e^{4x}\, C[2]]\}\}$

```
eqn /. sol // Simplify
{True}
```

The answer in yellow above is correct, but not listed in the text answer. Instead, the text answer includes a vector of constants, which I think are ultimately absorbed by the constants shown above.

13. y'' + 2 y' - 24 y = 0

```
ClearAll["Global`*"]
```

(b) As given

```
eqn = y''[x] + 2 y'[x] - 24 y[x] == 0
-24 y[x] + 2 y′[x] + y′′[x] == 0
```

```
sol = DSolve[eqn, y, x]
```

$\{\{y \to \text{Function}[\{x\}, e^{-6x}\, C[1] + e^{4x}\, C[2]]\}\}$

```
eqn /. sol // Simplify
{True}
```

The answer in green above matches the answer in the text.

15. CAS experiment. Electrical network.
(a) In Example 2, p. 132, choose a sequence of values of *C* that increases beyond bound,

and compare the corresponding sequences of eigenvalues of **A**. What limits of these sequences do your numeric values (approximately) suggest?

(b) Find these limits analytically.

(c) Explain your result physically.

(d) Below what value (approximately) must you decrease C to get vibrations?