Lots of buzz and energy about Frobenius and his method. Probably more than the average person can put up with. Problems 5, 9, 11, 15, 17 and 19 were worked straightforwardly without resorting to Frobenius.

The following example is found at *https : // wps.prenhall.com/wps/media/objects/884/905578/Assignment8_ 2 _V5.pdf.* It seems to work for certain equations, and I keep it here for general interest. It does not work on problem 9, however. Reading the included comments may give insight on where it can be expected to work.

```
Clear["Global`*"]
```

```
eqn = 2 x^2 (x + 1) y''[x] + 3 x (x + 1)^3 y'[x] - (1 - x^2) y[x] == 0;
```

The first step is to verify that x=0 is in fact a regular singular point of this equation. This is done by dividing through by the leading coefficient, which is $2\,x^2\,(x+1)$. This will put the equation in standard form.

$$y''[x] + \frac{3\,x\,(x+1)^3}{2\,x^2\,(x+1)}\,y'[x] - \frac{\left(1-x^2\right)}{2\,x^2\,(x+1)}\,y[x] == 0;$$

The standard form, shown in numbered line (1) on p. 180, includes a $\frac{1}{x}$sub-term for the y'[x] term and a $\frac{1}{x^2}$sub-term for the y[x] term. Thus of the remainder of the terms, called p[x] and q[x], the p[x] will be the coefficient for the y' and the q[x] will be the coefficient for the y.

$$q[x\_] = \text{Simplify}\left[-\frac{\left(1-x^2\right)}{2\,(x+1)}\right]$$

$$\frac{1}{2}\,(-1+x)$$

$$p[x\_] = \text{Simplify}\left[\frac{3\,(1+x)^2}{2}\right]$$

$$\frac{3}{2}\,(1+x)^2$$

Taking a good look at the two declarations above, I see that neither one has any x^{-m} factor in it. This may be necessary for the present example to work. Now to set the number of terms in the series.

```
n = 7;
```

Expand p[x] in a Maclaurin series

$$\text{pseries} = \text{Series}\left[\frac{p[x]}{x},\ \{x,\ 0,\ n\}\right]\ //\ \text{Normal}$$

$$3 + \frac{3}{2\,x} + \frac{3\,x}{2}$$

and also q[x]

```
qseries = Series[q[x]/x^2, {x, 0, n}] // Normal
```

$$-\frac{1}{2x^2}+\frac{1}{2x}$$

The first step in making a generic Frobenius series is the following

```
coeffs = Array[c, n, 0];
```

Another component is determined by

```
lotsofxpowers = Table[x^(r+j), {j, 0, 2 n}];
```

And a third necessary component comes from

```
xpowers = Table[x^(r+j), {j, 0, n - 1}];
```

Stirring and gelling the series is accomplished by

```
y = coeffs.xpowers;
```

The first derivative of the series will be needed

```
yprime = D[y, x]
```

$$r\,x^{-1+r}\,c[0]+(1+r)\,x^{r}\,c[1]+(2+r)\,x^{1+r}\,c[2]+(3+r)\,x^{2+r}\,c[3]+(4+r)\,x^{3+r}\,c[4]+(5+r)\,x^{4+r}\,c[5]+(6+r)\,x^{5+r}\,c[6]$$

Note above that the largest minus exponent in yprime is x^{-1}. Now to find the second derivative also

```
y2prime = D[y, {x, 2}]
```

$$(-1+r)\,r\,x^{-2+r}\,c[0]+r\,(1+r)\,x^{-1+r}\,c[1]+(1+r)\,(2+r)\,x^{r}\,c[2]+(2+r)\,(3+r)\,x^{1+r}\,c[3]+(3+r)\,(4+r)\,x^{2+r}\,c[4]+(4+r)\,(5+r)\,x^{3+r}\,c[5]+(5+r)\,(6+r)\,x^{4+r}\,c[6]$$

Note above that the largest minus exponent in y2prime is x^{-2}. The differential equation itself is the next to be created. Because of the limited presence of minus exponents on x, multiplying through by x^2 here normalizes the series so that the lowest order term appearing will be x^r.

```
x^2 * (y2prime + pseries * yprime + qseries * y) // Expand;
```

The series will adopt an appearance of order after executing the following

```
lhs = Collect[%, lotsofxpowers]
```

$$x^r\left(-\frac{c[0]}{2}+\frac{1}{2}r\,c[0]+r^2\,c[0]\right)+$$
$$x^{1+r}\left(\frac{c[0]}{2}+3\,r\,c[0]+c[1]+\frac{5}{2}r\,c[1]+r^2\,c[1]\right)+$$
$$x^{2+r}\left(\frac{3}{2}r\,c[0]+\frac{7\,c[1]}{2}+3\,r\,c[1]+\frac{9\,c[2]}{2}+\frac{9}{2}r\,c[2]+r^2\,c[2]\right)+$$
$$x^{3+r}\left(\frac{3\,c[1]}{2}+\frac{3}{2}r\,c[1]+\frac{13\,c[2]}{2}+3\,r\,c[2]+10\,c[3]+\frac{13}{2}r\,c[3]+r^2\,c[3]\right)+$$
$$x^{4+r}\left(3\,c[2]+\frac{3}{2}r\,c[2]+\frac{19\,c[3]}{2}+3\,r\,c[3]+\frac{35\,c[4]}{2}+\frac{17}{2}r\,c[4]+r^2\,c[4]\right)+$$
$$x^{5+r}\left(\frac{9\,c[3]}{2}+\frac{3}{2}r\,c[3]+\frac{25\,c[4]}{2}+3\,r\,c[4]+27\,c[5]+\frac{21}{2}r\,c[5]+r^2\,c[5]\right)+$$
$$x^{8+r}\left(9\,c[6]+\frac{3}{2}r\,c[6]\right)+x^{7+r}\left(\frac{15\,c[5]}{2}+\frac{3}{2}r\,c[5]+\frac{37\,c[6]}{2}+3\,r\,c[6]\right)+$$
$$x^{6+r}\left(6\,c[4]+\frac{3}{2}r\,c[4]+\frac{31\,c[5]}{2}+3\,r\,c[5]+\frac{77\,c[6]}{2}+\frac{25}{2}r\,c[6]+r^2\,c[6]\right)$$

Note above that **`Collect`** has ordered the terms so that the x^r term comes first. This may not always be the case, and if it is not, it will affect the following steps. A necessary preliminary step to finding the indicial roots is

```
firstcoeff = lhs[[1, 2]]
```

$$-\frac{c[0]}{2}+\frac{1}{2}r\,c[0]+r^2\,c[0]$$

Note two things about firstcoeff above. It contains elements of r, and it contains a single constant parameter, here c[0]. Both of these characteristics are needed to make the example work. To reveal the roots, set the above expression to zero and solve as follows.

```
indroots = Solve[firstcoeff == 0, r];
```

Order the roots by size

```
maxroot = Max[r /. indroots]
```

$$\frac{1}{2}$$

```
minroot = Min[r /. indroots]
```

-1

The next step is the first in a series of six steps to find the first solution, y_1.

```
lhsy1 = lhs /. r → maxroot;
```

To determine the values of the arbitrary coefficients, a preliminary step is

$$\texttt{y1series} = \frac{\texttt{lhsy1}}{x^{\texttt{maxroot}}}\ \texttt{// Distribute;}$$

followed by the next step, which makes a list of coefficients.

```
y1seriescoeffs = Take[CoefficientList[y1series, x], n - 1];
```

The higher-degree coefficients in terms of C[0] will be found by

```
y1coeffs = Solve[{y1seriescoeffs == 0, c[0] == 1}][[1]];
```

At this point the first solution, $y_{1,}$ can be created by

```
y = Drop[coeffs, -1].Drop[xpowers, -1]
```

$$x^r c[0] + x^{1+r} c[1] + x^{2+r} c[2] + x^{3+r} c[3] + x^{4+r} c[4] + x^{5+r} c[5]$$

followed by the final necessary step

```
y1 = Collect[y /. r → maxroot /. y1coeffs, x]
```

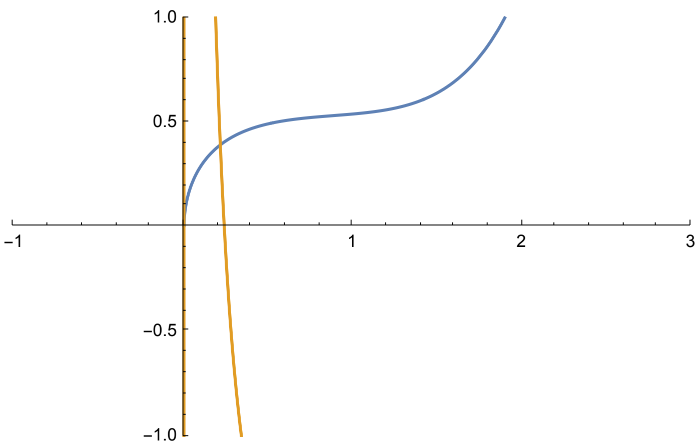
$$\sqrt{x} - \frac{4 x^{3/2}}{5} + \frac{13 x^{5/2}}{28} - \frac{134 x^{7/2}}{945} - \frac{2741 x^{9/2}}{332640} + \frac{20429 x^{11/2}}{772200}$$

To find the second solution, y_2, requires a similar set of six steps.

```
lhsy2 = lhs /. r → minroot;
y2series = lhsy2 / x^minroot // Distribute;
y2seriescoeffs = Take[CoefficientList[y2series, x], n - 1];
y2coeffs = Solve[{y2seriescoeffs == 0, c[0] == 1}][[1]];
y = Drop[coeffs, -1].Drop[xpowers, -1];
y2 = Collect[y /. r → minroot /. y2coeffs, x];
```

And now the two solutions, y_1 and y_2, are plotted. Maybe this doesn't work after all. The roots don't seem to check out.

```
Plot[{y1, y2}, {x, -1, 3}, PlotRange → {{-1, 3}, {-1, 1}}]
```



2 - 13 Frobenius method

Find a basis of solutions by the Frobenius method. Try to identify the series as expansions of known functions.

3. x y'' + 2 y' + x y = 0

```
Clear["Global`*"]
```

$1. Starting the problem over, adopting new symbols. The **Hold** command does not seem to work as I would like, it gets in the way. I don't have nomenclature to designate a Mathematica pseudosum.

e7 = f[x_] = a_m x^{m+r}

$x^{m+r}\, a_m$

$2. What is meant by above is the **Sum[a_m x^{m+r}, {m, 0, ∞}]**

e8 = f'[x]

$(m + r)\, x^{-1+m+r}\, a_m$

e9 = f''[x]

$(-1 + m + r)\,(m + r)\, x^{-2+m+r}\, a_m$

e10 = x^2 f''[x] + 2 x f'[x] + x^2 f[x] == 0

$2\,(m + r)\, x^{m+r}\, a_m + (-1 + m + r)\,(m + r)\, x^{m+r}\, a_m + x^{2+m+r}\, a_m == 0$

$3. Above: The exponential powers of the first two 'x' factors are equal, but the third is two higher. Since these represent infinite sums, it would not affect their individual (or collective) summations if the indices were adjusted to all match. As the s.m. suggested, this can be accomplished by effectively subtracting 2 from the power of x in the third occurence of that variable. This would make the third pseudosum start at 2 (instead of 0) to compensate. As for a_m, that element would be changed to a_{m-2} in order to have it start at the same place as before the change.

e11 = e10 /. $\left(x^{2+m+r}\, a_m\right) \to \left(x^{m+r}\, a_{m-2}\right)$

$x^{m+r}\, a_{-2+m} + 2\,(m + r)\, x^{m+r}\, a_m + (-1 + m + r)\,(m + r)\, x^{m+r}\, a_m == 0$

$4. Above: Last reminder of the form change. Having all the factors mix together should not invalidate anything. However, if I find myself in a position where I want to assign 0 or 1 to the index m, the coefficient m_{-2+m} will drop out.

e12 = Expand[e11]

$x^{m+r}\, a_{-2+m} + m\, x^{m+r}\, a_m + m^2\, x^{m+r}\, a_m + r\, x^{m+r}\, a_m + 2\, m\, r\, x^{m+r}\, a_m + r^2\, x^{m+r}\, a_m == 0$

e13 = Simplify[e12]

$x^{m+r}\left(a_{-2+m} + \left(m + m^2 + r + 2\, m\, r + r^2\right) a_m\right) == 0$

$5. Below: The indicial equation. (What makes it the indicial equation is setting $m = 0$.) In transferring from above, the factor **a_{-2+m}** was ignored.

e14 = Solve$\left[m + m^2 + r + 2\, m\, r + r^2 == 0,\ r\right]$ /. m → 0

{{r → -1}, {r → 0}}

$6. Above: The sol'n of the indicial equation. The s.m. wants to look at the larger root first, $r = 0$. Where am I? After I equalized powers of x, expanded, and simplified, I got e13. Now I will look at e13 again, but with r evaluated.

```
e15 = e13 /. {r → 0}
```

$x^m \left(a_{-2+m} + \left(m + m^2\right) a_m\right) == 0$

```
e16 = Solve[(a_{-2+m} + (m + m^2) a_m) == 0, a_m] /. {m → 1, a_{-2+m} → 0}
```

$\{\{a_1 \to 0\}\}$

$7. Making $m = 1$ in the above allows the finding of a_1. Since $m < 2$, the coefficient a_{-2+m} will be zero.

```
e17 = e13 /. {a_{-2+m} → 0, r → 0}
```

$\left(m + m^2\right) x^m a_m == 0$

$8. Above: Getting a look at the heart of the equation updated.

```
A = {};
Do[A = Union[A, Solve[(a_{-2+m} + (m + m^2) a_m) == 0, a_m]], {m, 2, 7}];
A
```

$\left\{\left\{a_2 \to -\frac{a_0}{6}\right\}, \left\{a_3 \to -\frac{a_1}{12}\right\}, \left\{a_4 \to -\frac{a_2}{20}\right\}, \left\{a_5 \to -\frac{a_3}{30}\right\}, \left\{a_6 \to -\frac{a_4}{42}\right\}, \left\{a_7 \to -\frac{a_5}{56}\right\}\right\}$

$9. What the above set A does not include is the factor $a_0 x^0$. The s.m recommends assigning the value 1 to a_0, and this will be added when the opportunity presents. The starting value of m is 2, because that is the lowest value for which a_{-2+m} has meaning. There is no assigned value for a_1 yet. But now everything has a value based on either a_0 or a_1.

```
B = {};
```

$\text{Eliminate}\left[\left\{a_4 == -\frac{a_2}{20},\ a_2 == -\frac{a_0}{6}\right\},\ a_2\right]$

$120\, a_4 == a_0$

$\text{Eliminate}\left[\left\{a_6 == -\frac{a_4}{42},\ 120\, a_4 == a_0\right\},\ a_4\right]$

$-5040\, a_6 == a_0$

$\text{Eliminate}\left[\left\{a_5 == -\frac{a_3}{30},\ a_3 == -\frac{a_1}{12}\right\},\ a_3\right]$

$360\, a_5 == a_1$

$\text{Eliminate}\left[\left\{a_7 == -\frac{a_5}{56},\ 360\, a_5 == a_1\right\},\ a_5\right]$

$-20\,160\, a_7 == a_1$

$$B = \left\{\{a_0 \to 1\}, \left\{a_2 \to -\frac{a_0}{6}\right\}, \left\{a_3 \to -\frac{a_1}{12}\right\}, \left\{a_4 \to \frac{a_0}{120}\right\}, \left\{a_5 \to \frac{a_1}{360}\right\}, \left\{a_6 \to -\frac{a_0}{5040}\right\}, \left\{a_7 \to -\frac{a_1}{20160}\right\}\right\}$$

$$\left\{\{a_0 \to 1\}, \left\{a_2 \to -\frac{a_0}{6}\right\}, \left\{a_3 \to -\frac{a_1}{12}\right\}, \left\{a_4 \to \frac{a_0}{120}\right\}, \left\{a_5 \to \frac{a_1}{360}\right\}, \left\{a_6 \to -\frac{a_0}{5040}\right\}, \left\{a_7 \to -\frac{a_1}{20160}\right\}\right\}$$

```
cs = {2!, 3!, 4!, 5!, 6!, 7!, 8!}
```

{2, 6, 24, 120, 720, 5040, 40 320}

$$B[[3]] = B[[3]] \;/.\; \frac{a_2}{120} \to \frac{a_1}{24}$$

$$\left\{a_3 \to -\frac{a_1}{12}\right\}$$

$$B[[4]] = B[[4]] \;/.\; a_2 \to a_0$$

$$\left\{a_4 \to \frac{a_0}{120}\right\}$$

$$B[[5]] = B[[5]] \;/.\; a_3 \to a_1$$

$$\left\{a_5 \to \frac{a_1}{360}\right\}$$

$$B[[6]] = B[[6]] \;/.\; a_4 \to a_0$$

$$\left\{a_6 \to -\frac{a_0}{5040}\right\}$$

$$B[[7]] = B[[7]] \;/.\; a_5 \to a_1$$

$$\left\{a_7 \to -\frac{a_1}{20160}\right\}$$

$$\text{e19} = \text{TableForm}[\text{Table}[\{m, a_m, B[[m]]\}, \{m, 2, 7\}], \text{TableHeadings} \to \{\{\}, \{\text{"m"}, \text{"}a_m\text{ "}, \text{"B[m] "}, \text{"}a_1\text{"}\}\}]$$

m	a_m	B[m]	a_1
2	a_2	$a_2 \to -\frac{a_0}{6}$	
3	a_3	$a_3 \to -\frac{a_1}{12}$	
4	a_4	$a_4 \to \frac{a_0}{120}$	
5	a_5	$a_5 \to \frac{a_1}{360}$	
6	a_6	$a_6 \to -\frac{a_0}{5040}$	
7	a_7	$a_7 \to -\frac{a_1}{20160}$	

$10. There are two series. Let me see if I can separate them:

```
B1 = {B[[3, 1, 2]], B[[5, 1, 2]], B[[7, 1, 2]]}
```

$$\left\{-\frac{a_1}{12}, \frac{a_1}{360}, -\frac{a_1}{20160}\right\}$$

$11. There is a problem with B1. However, since it will not be used, I won't investigate it now.

```
B2 = {B[[1, 1, 2]], B[[2, 1, 2]], B[[4, 1, 2]], B[[6, 1, 2]]}
```

$$\left\{1, -\frac{a_0}{6}, \frac{a_0}{120}, -\frac{a_0}{5040}\right\}$$

```
e20 = TableForm[Table[{m, B2[[m]]}, {m, 1, 4}],
  TableHeadings → {{}, {"term", "B2[m] "}}]
```

term	B2[m]
1	1
2	$-\frac{a_0}{6}$
3	$\frac{a_0}{120}$
4	$-\frac{a_0}{5040}$

```
y1 = Sum[B2[[s]] x^(2 (s-1)), {s, 1, 4}]
```

$$1 - \frac{x^2 a_0}{6} + \frac{x^4 a_0}{120} - \frac{x^6 a_0}{5040}$$

$12. To do as s.m., a_0 was assigned a value of 1. Then y1 = $1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!}$

```
(*y2=Sum[B1[[s]] x^s,{s,1,4}]*)
```

This series is y2 = $-\frac{x^0}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!}$

$13. The above series will not be used. Below is shown the definition version of what will now be called y1. It agrees with the text answer for y_1.

```
Series[Sin[x]/x, {x, 0, 4}]
```

$$1 - \frac{x^2}{6} + \frac{x^4}{120} + O[x]^5$$

$14. To get the second sol'n in the basis, y2, it is recommended by the s.m. to march off and do reduction of order, covered in Sec 2.1 of the text. From that perspective it is deemed important to put the original equation into standard form,

```
e21 = {y''[x] + (2/x) y'[x] + y[x] == 0}
```

$$\left\{y[x] + \frac{2 y'[x]}{x} + y''[x] == 0\right\}$$

```
p[x_] = 2/x
```

$\frac{2}{x}$

```
-Integrate[p[x], x, GenerateConditions → True]
```

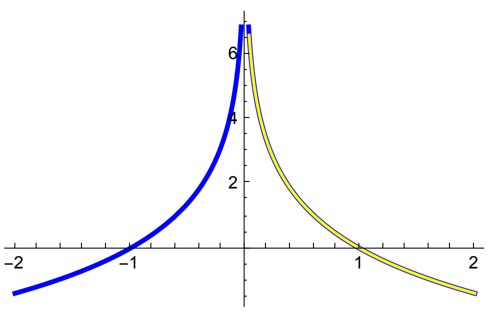
```
-2 Log[x]
```

$15. Above: Mathematica does not bother to show that the correct answer involves Abs[x]. The plot below shows the difference. This doesn't seem to affect this particular answer, but I don't like the omission.

```
plot1 = Plot[-2 Log[x], {x, -2, 2}, PlotStyle → Yellow, ImageSize → 250];
```

```
plot2 =
  Plot[-2 Log[Abs[x]], {x, -2, 2}, PlotStyle → {Blue, Thickness[0.01]}];
```

```
Show[plot2, plot1]
```



$16. Putting the integral into another form.

```
Exp[-Integrate[p[x], x]]
```

$\frac{1}{x^2}$

$17. The expression: $U = \frac{1}{y_1^2} e^{-\int p \, dx}$ is from section 2.1, p.52, where one sol'n to a homogeneous linear ODE with constant coefficents is already known and you are tracking down the other part of the basis. Putting it to use,

```
capU = Exp[-Integrate[p[x], x]] / (Sin[x]/x)^2
```

```
Csc[x]^2
```

```
smallu = Integrate[capU, x]
```

```
-Cot[x]
```

```
y2 = smallu Sin[x]/x
```

$-\frac{\text{Cos}[x]}{x}$

$18. The s.m. points out that since it is a combo of y1 and y2 and there are arbitrary coefficients involved, the minus sign on y2 is not necessary, and the version shown in the answer section (no minus sign) can be claimed. I will still claim green here.

5. x y'' + (2 x + 1) y' + (x + 1) y = 0

```
Clear["Global`*"]
e1 = {x y''[x] + (2 x + 1) y'[x] + (x + 1) y[x] == 0}
{(1 + x) y[x] + (1 + 2 x) y′[x] + x y″[x] == 0}
sol = DSolve[e1, y, x]
```

{{y → Function[{x}, e^{-x} C[1] + e^{-x} C[2] Log[x]]}}

```
e1 /. sol // Simplify
{{True}}
```

Mathematica can solve this one without all the Frobenius gingerbread. It should be noted that the two parts (terms) of the sol'n shown in green above are considered two separate sol'ns, y1 and y2. This was the same case with the first problem, no. 3. In the text answer C[1] and C[2] are equal to 1. The answer agrees with that of the text.

7. y'' + (x - 1) y = 0

```
Clear["Global`*"]
eqn = y''[x] + (x - 1) y[x] == 0
(-1 + x) y[x] + y″[x] == 0
sol = DSolve[eqn, y, x]
```

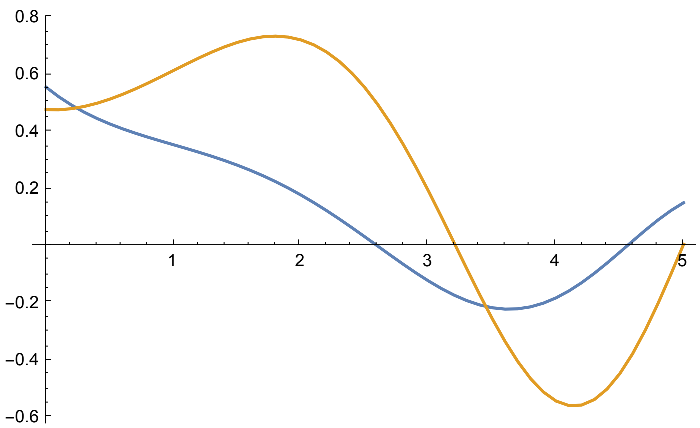
{{y → Function[{x}, AiryAi[$-(-1)^{1/3}$ (1 - x)] C[1] + AiryBi[$-(-1)^{1/3}$ (1 - x)] C[2]]}}

```
eqn /. sol // Simplify
{True}
```

Though Mathematica finds an apparently viable solution, it is not user friendly. Getting a plot is possible, though not all that easy.

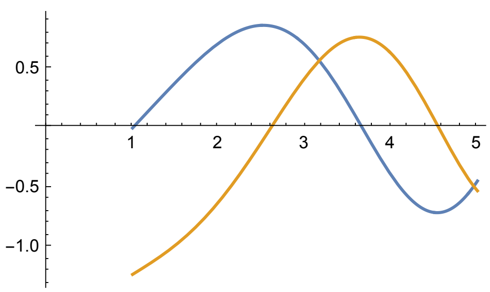
```
frt1 = ComplexExpand[
   Re[Evaluate[Table[{x, N[AiryAi[-(-1)^(1/3) (1 - x)]]}, {x, 0, 5, 0.1}]]]];
frt2 = ComplexExpand[Re[Evaluate[
      Table[{x, N[AiryBi[-(-1)^(1/3) (1 - x)]]}, {x, 0, 5, 0.1}]]]];
```

```
ListLinePlot[{frt1, frt2}]
```



As a matter of interest, I experimented with the Cocalc site, (https://cocalc.com), where Sage leverages Maxima to get a friendlier Bessel solution with the command: **sage: desolve(diff(y,x,2)+(x-1)*y==0,y,contrib_ode=True,show_method=True)**

$$\text{Plot}\left[\left\{\sqrt{x-1}\ \text{BesselJ}\left[\frac{1}{3}, \frac{2}{3}(x-1)^{3/2}\right], \sqrt{x-1}\ \text{BesselY}\left[\frac{1}{3}, \frac{2}{3}(x-1)^{3/2}\right]\right\}, \{x, 0, 5\}, \text{ImageSize} \to 250\right]$$



However, looking at the plot, this does not seem to be the same set of functions as the ones found by Mathematica.

Finally, there is a rather large section of dead pipe attached to this problem which I am leaving in for the time being, which approaches this problem in the usual Frobenius way, but without complete resolution. Maybe I will come back to it later. Finally finally, I list the text answer for reference:

$$y_1 = 1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{30}x^5 + \frac{1}{144}x^6 - \ \cdots$$
$$y_2 = x + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{120}x^5 - \frac{1}{120}x^6 + \ \cdots$$

I think the sequence of coefficients in y_2 looks suspicious.

```
y[x_] = a_m x^(m+r)
```
$x^{m+r} a_m$

```
e1 = y''[x] + (x - 1) y[x] == 0
```
$(-1+m+r)(m+r)x^{-2+m+r}a_m + (-1+x)x^{m+r}a_m == 0$

Powers do not match. Adjusting.

```
e2 = e1 /. ((-1 + x) x^(m+r) a_m) → ((-1 + x) x^(-2+m+r) a_(m-2))
```
$(-1+x)x^{-2+m+r}a_{-2+m} + (-1+m+r)(m+r)x^{-2+m+r}a_m == 0$

```
e3 = Expand[e2]
```
$-x^{-2+m+r}a_{-2+m} + x^{-1+m+r}a_{-2+m} - m\,x^{-2+m+r}a_m + m^2x^{-2+m+r}a_m - r\,x^{-2+m+r}a_m + 2\,m\,r\,x^{-2+m+r}a_m + r^2x^{-2+m+r}a_m == 0$

```
e4 = Simplify[e3]
```
$x^{-1+m+r}\left((-1+x)a_{-2+m} + \left(m^2+(-1+r)r+m(-1+2r)\right)a_m\right) == 0$

This is where it is assumed $m = 0$, so the indexes less than zero can be ignored. Above is the indicial equation.

```
e5 = Solve[(m^2 + (-1 + r) r + m (-1 + 2 r)) a_m == 0, r] /. m → 0
```
{{r → 1}, {r → 0}}

Above: The sol'n of the indicial equation. Choosing $r = 0$ makes a simpler path, no?

```
e6 = e4 /. r → 0
```
$x^{-1+m}\left((-1+x)a_{-2+m} + \left(-m+m^2\right)a_m\right) == 0$

```
e7 = Solve[(-1 + x) a_(-2+m) + (-m + m^2) a_m == 0, a_m] /. a_(-2+m) → 0
```
{{a_m → 0}}

```
e8 = e4 /. {r → 0}
```
$x^{-1+m}\left((-1+x)a_{-2+m} + \left(-m+m^2\right)a_m\right) == 0$

```
A = {}; Do[
 A = Union[A, Solve[x^(-1+m) ((-1 + x) a_(-2+m) + (-m + m^2) a_m) == 0, a_m]], {m, 0, 9}]; A
```
$\left\{\left\{a_2 \to -\frac{1}{2}(-1+x)a_0\right\}, \left\{a_3 \to -\frac{1}{6}(-1+x)a_1\right\}, \left\{a_4 \to -\frac{1}{12}(-1+x)a_2\right\}, \left\{a_5 \to -\frac{1}{20}(-1+x)a_3\right\}, \left\{a_6 \to -\frac{1}{30}(-1+x)a_4\right\}, \left\{a_7 \to -\frac{1}{42}(-1+x)a_5\right\}, \left\{a_8 \to -\frac{1}{56}(-1+x)a_6\right\}, \left\{a_9 \to -\frac{1}{72}(-1+x)a_7\right\}\right\}$

The challenge at this point is to get everything expressed as functions of a_0 and a_1.

```
AA = {}
```

{}

$$\text{Simplify}\left[\text{Eliminate}\left[\left\{a_2 == -\frac{1}{2}(-1+x)\,a_0,\ a_4 == -\frac{1}{12}(-1+x)\,a_2\right\},\ a_2\right]\right]$$

$$(-1+x)^2\,a_0 == 24\,a_4$$

$$\text{Solve}\left[(-1+x)^2\,a_0 == 24\,a_4,\ a_4\right]$$

$$\left\{\left\{a_4 \to \frac{1}{24}(-1+x)^2\,a_0\right\}\right\}$$

$$\text{Simplify}\left[\text{Eliminate}\left[\left\{a_4 == \frac{1}{24}(-1+x)^2\,a_0,\ a_6 == -\frac{1}{30}(-1+x)\,a_4\right\},\ a_4\right]\right]$$

$$(-1+x)^3\,a_0 + 720\,a_6 == 0$$

$$\text{Solve}\left[(-1+x)^3\,a_0 + 720\,a_6 == 0,\ a_6\right]$$

$$\left\{\left\{a_6 \to -\frac{1}{720}(-1+x)^3\,a_0\right\}\right\}$$

$$\text{Simplify}\left[\text{Eliminate}\left[\left\{a_6 == -\frac{1}{720}(-1+x)^3\,a_0,\ a_8 == -\frac{1}{56}(-1+x)\,a_6\right\},\ a_6\right]\right]$$

$$(-1+x)^4\,a_0 == 40\,320\,a_8$$

$$\text{Solve}\left[(-1+x)^4\,a_0 == 40\,320\,a_8,\ a_8\right]$$

$$\left\{\left\{a_8 \to \frac{(-1+x)^4\,a_0}{40\,320}\right\}\right\}$$

The above takes care of the even coefficients. Now for the odd.

$$\text{Simplify}\left[\text{Eliminate}\left[\left\{a_3 == -\frac{1}{6}(-1+x)\,a_1,\ a_5 == -\frac{1}{20}(-1+x)\,a_3\right\},\ a_3\right]\right]$$

$$(-1+x)^2\,a_1 == 120\,a_5$$

$$\text{Solve}\left[(-1+x)^2\,a_1 == 120\,a_5,\ a_5\right]$$

$$\left\{\left\{a_5 \to \frac{1}{120}(-1+x)^2\,a_1\right\}\right\}$$

$$\text{Simplify}\left[\text{Eliminate}\left[\left\{a_5 == \frac{1}{120}(-1+x)^2\,a_1,\ a_7 == -\frac{1}{42}(-1+x)\,a_5\right\},\ a_5\right]\right]$$

$$(-1+x)^3\,a_1 + 5040\,a_7 == 0$$

$$\text{Solve}\left[(-1+x)^3\,a_1 + 5040\,a_7 == 0,\ a_7\right]$$

$$\left\{\left\{a_7 \to -\frac{(-1+x)^3\,a_1}{5040}\right\}\right\}$$

```
FindSequenceFunction[{-1/2, -1/6, -1/12, -1/20, -1/30, -1/42}, x]
```

$$-\frac{1}{x\,(1+x)}$$

```
Together[-1/120 + 1/144]
```

$$-\frac{1}{720}$$

9. 2 x (x - 1) y'' - (x + 1) y' + y = 0

```
Clear["Global`*"]
eqn = 2 x (x - 1) y''[x] - (x + 1) y'[x] + y[x] == 0
y[x] - (1 + x) y′[x] + 2 (-1 + x) x y″[x] == 0
sol = DSolve[eqn, y, x]
```

$$\{\{y \to \text{Function}[\{x\}, \sqrt{x}\, C[1] - 2\,(1+x)\, C[2]]\}\}$$

```
eqn /. sol // Simplify
{True}
```

It appears that Frobenius's method is not necessary with this problem. In order to make the green cell match the text answer, I choose C[1]=1 and C[2]=$-\frac{1}{2}$.

11. x y'' + (2 - 2 x) y' + (x - 2) y = 0

```
Clear["Global`*"]
eqn = x y''[x] + (2 - 2 x) y'[x] + (x - 2) y[x] == 0
(-2 + x) y[x] + (2 - 2 x) y′[x] + x y″[x] == 0
sol = DSolve[eqn, y, x]
```

$$\{\{y \to \text{Function}[\{x\}, \frac{e^x\, C[1]}{x} + e^x\, C[2]]\}\}$$

```
eqn /. sol // Simplify
{True}
```

Again the Frobenius method is not necessary. To make the green cell match the text answer, I choose C[1] = C[2]=1.

13. x y'' + (1 - 2 x) y' + (x - 1) y = 0

```
Clear["Global`*"]
```

```
e1 = x y''[x] + (1 - 2 x) y'[x] + (x - 1) y[x] == 0
(-1 + x) y[x] + (1 - 2 x) y′[x] + x y″[x] == 0
```

```
e2 = x e1
x ((-1 + x) y[x] + (1 - 2 x) y′[x] + x y″[x] == 0)
```

```
e3 = x ((-1 + x) y[x] + (1 - 2 x) y′[x] + x y″[x]) == 0
x ((-1 + x) y[x] + (1 - 2 x) y′[x] + x y″[x]) == 0
```

```
e4 = Expand[e3]
```

$-x\,y[x] + x^2\,y[x] + x\,y'[x] - 2\,x^2\,y'[x] + x^2\,y''[x] == 0$

```
e5 = Collect[e4, {y''[x], y'[x], y[x]}]
```

$(-x + x^2)\,y[x] + (x - 2\,x^2)\,y'[x] + x^2\,y''[x] == 0$

$x\,b(x)$ must equal $(x - 2\,x^2)$ and $c(x)\,y$ must equal $(-x + x^2)$. So $b(x)$ equals $(1 - 2\,x)$. To find out b_0 and c_0, it is necessary to expand them.

```
e6 = Series[1 - 2 x, {x, 0, 2}]
```

$1 - 2\,x + O[x]^3$

So $b_0 = 1$.

```
e7 = Series[-x + x^2, {x, 0, 2}]
```

$-x + x^2 + O[x]^3$

Because there is no constant term in the expansion of $c(x)$, the s.m. tells me that $c_0 = 0$.

The sol'n series will look like:

$y1[x\_] = a_m\,x^{m+r}$

$x^{m+r}\,a_m$

```
y1'[x]
```

$(m + r)\,x^{-1+m+r}\,a_m$

```
y1''[x]
```

$(-1 + m + r)\,(m + r)\,x^{-2+m+r}\,a_m$

Since I know $b(x)$ and $c(x)$ and b_0 and c_0, I can write the indicial equation.

```
e8 = Solve[r (r - 1) + 1 r + 0 == 0, r]
{{r → 0}, {r → 0}}
```

Mathematica is telling me it is a **double root**. Now I can write the original equation,

```
e9 = x y1''[x] + (1 - 2 x) y1'[x] + (x - 1) y1[x] == 0
```

$(-1 + m + r)\,(m + r)\,x^{-1+m+r}\,a_m + (m + r)\,(1 - 2\,x)\,x^{-1+m+r}\,a_m + (-1 + x)\,x^{m+r}\,a_m == 0$

```
e10 = e9 /. {r → 0}
```

$$(-1+m)\, m\, x^{-1+m} a_m + m\,(1-2x)\, x^{-1+m} a_m + (-1+x)\, x^m a_m == 0$$

```
e11 = Collect[e10, (-1 + m)]
```

$$m\, x^{-1+m} a_m + (-1+m)\, m\, x^{-1+m} a_m - x^m a_m - 2\, m\, x^m a_m + x^{1+m} a_m == 0$$

The five factors above are the same factors as in the s.m.

```
e12 = Collect[e11, {x^(-1+m), x^m, x^(1+m)}]
```

$$m^2 x^{-1+m} a_m + x^{1+m} a_m + x^m (-a_m - 2\, m\, a_m) == 0$$

This also matches, at the place where the powers are adjusted.

```
e13 = e12 /. {(m^2 x^(-1+m) a_m) → ((s + 1)^2 a_(s+1) x^s),
   (x^(1+m) a_m) → (a_(s-1) x^s), (x^m (-a_m - 2 m a_m)) → (-(2 s + 1) a_s x^s)}
```

$$x^s a_{-1+s} + (-1-2s)\, x^s a_s + (1+s)^2 x^s a_{1+s} == 0$$

The powers are adjusted. If shown as real sums, the third factor (sum) would be $\{s, -1, \infty\}$, the second $\{s, 0, \infty\}$, and the first $\{s, 1, \infty\}$.

For the case of $s = -1$, only the third factor would work, fitting into that index range.

```
e14 = Solve[(1 + s)^2 x^s a_(1+s) == 0, a_(s+1)] /. s → -1
```

$$\{\{a_0 \to 0\}\}$$

In the case of $s = 0$, both second and third factors can accommodate.

```
e15 = Solve[(-1 - 2 s) x^s a_s + (1 + s)^2 x^s a_(1+s) == 0, a_s] /. s → 0
```

$$\{\{a_0 \to a_1\}\}$$

In the case of $s > 0$, all three factors can accommodate.

```
e16 = Solve[x^s a_(-1+s) + (-1 - 2 s) x^s a_s + (1 + s)^2 x^s a_(1+s) == 0, a_s] /. s → 1
```

$$\left\{\left\{a_1 \to \frac{1}{3}(a_0 + 4\, a_2)\right\}\right\}$$

```
e17 = Solve[a_1 == 1/3 (a_0 + 4 a_2), a_2] /. a_1 → a_0
```

$$\left\{\left\{a_2 \to \frac{a_0}{2}\right\}\right\}$$

```
e18 = Solve[x^s a_(-1+s) + (-1 - 2 s) x^s a_s + (1 + s)^2 x^s a_(1+s) == 0, a_(s+1)]
```

$$\left\{\left\{a_{1+s} \to \frac{-a_{-1+s} + a_s + 2\, s\, a_s}{(1+s)^2}\right\}\right\}$$

```
e19 = e18 /. {s → 1, a_0 → 1}
```

$$\left\{\left\{a_2 \to \frac{1}{4}(-a_0 + 3\, a_1)\right\}\right\}$$

```
loc = {};
e20 =
 Do[loc = Union[loc, Solve[x^s a_{-1+s} + (-1 - 2 s) x^s a_s + (1 + s)^2 x^s a_{1+s} == 0,
      a_{s+1}] /. {a_0 → 1, a_1 → 1}], {s, 1, 4}];
loc
```

This marks the point where values for both a_0 and a_1 are shown as assigned.

$$\left\{\left\{a_2 \to \frac{1}{2}\right\}, \left\{a_3 \to \frac{1}{9}(-1+5 a_2)\right\}, \left\{a_4 \to \frac{1}{16}(-a_2+7 a_3)\right\}, \left\{a_5 \to \frac{1}{25}(-a_3+9 a_4)\right\}\right\}$$

$$\text{e21} = \text{Solve}\left[a_3 == \frac{1}{9}(-1+5 a_2), a_3\right] /. a_2 \to \frac{1}{2}$$

$$\left\{\left\{a_3 \to \frac{1}{6}\right\}\right\}$$

$$\text{e22} = \text{Solve}\left[a_4 == \frac{1}{16}(-a_2+7 a_3), a_4\right] /. \left\{a_2 \to \frac{1}{2}, a_3 \to \frac{1}{6}\right\}$$

$$\left\{\left\{a_4 \to \frac{1}{24}\right\}\right\}$$

$$\text{e23} = \text{Solve}\left[a_5 == \frac{1}{25}(-a_3+9 a_4), a_5\right] /. \left\{a_2 \to \frac{1}{2}, a_3 \to \frac{1}{6}, a_4 \to \frac{1}{24}\right\}$$

$$\left\{\left\{a_5 \to \frac{1}{120}\right\}\right\}$$

```
e24 = cs = {2!, 3!, 4!, 5!, 6!, 7!, 8!}
{2, 6, 24, 120, 720, 5040, 40320}
```

$$\text{e25} = \text{y1}[\text{x\_}] = \text{Sum}[a_m x^m, \{m, 0, 4\}]$$

$$a_0 + x a_1 + x^2 a_2 + x^3 a_3 + x^4 a_4$$

In e20 I said that $a_0 = a_1 = 1$. Thus y1 $= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdot\cdot\cdot$ Looks like e^x.

$$y_1 = \text{Series}[e^x, \{x, 0, 4\}]$$

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + O[x]^5$$

The green cell above matches the text answer for y_1. It is still necessary to get a second solution. Again, the method is reduction of order. The first step is to put the equation into standard form.

$$\text{e26} = \frac{1}{x}(-y[x] + x\,y[x] + y'[x] - 2 x\,y'[x] + x\,y''[x]) == 0$$

$$\frac{-y[x] + x\,y[x] + y'[x] - 2 x\,y'[x] + x\,y''[x]}{x} == 0$$

```
e28 = Collect[e26, {y''[x], y'[x], y[x]}]
```

$$\frac{(-1+x)\,y[x]}{x}+\frac{(1-2x)\,y'[x]}{x}+y''[x]==0$$

$$e29 = p[x\_] = \frac{(1-2x)}{x}$$

$$\frac{1-2x}{x}$$

Following the procedure for reduction of order,

```
e30 = -Integrate[p[x], x]
2 x - Log[x]
```

Using log identity, this is

$$e31 = e30\ /.\ -\text{Log}[x] \to \text{Log}\left[\frac{1}{x}\right]$$

$$2x + \text{Log}\left[\frac{1}{x}\right]$$

Again, Mathematica forgot to use the **Abs** function when integrating a fraction.

```
e32 = Exp[e31]
```

$$\frac{e^{2x}}{x}$$

Continuing to follow the reduction recipe, we have our big U and little u. As shown in s.m.,

$$e33 = \text{bigU}[x\_] = \frac{1}{(e^{x})^{2}}\left(\frac{e^{2x}}{x}\right)$$

$$\frac{1}{x}$$

And as for little u,

```
e34 = u[x_] = Integrate[bigU[x], x]
Log[x]
```

Again, should be **Abs**

$$y2 = u[x]\, e^{x}$$

$$e^{x}\,\text{Log}[x]$$

The green cell above matches the text answer for y_2.

15 - 20 Hypergeometric ODE

Find a general solution in terms of hypergeometric functions.

15. 2 x (1 - x) y" - (1 + 6 x) y' - 2 y = 0

```
Clear["Global`*"]
```

```
eqn = 2 x (1 - x) y''[x] - (1 + 6 x) y'[x] - 2 y[x] == 0
```

$$-2\,y[x] - (1 + 6\,x)\,y'[x] + 2\,(1 - x)\,x\,y''[x] == 0$$

```
sol = DSolve[eqn, y, x]
```

$$\left\{\left\{y \to \text{Function}\left[\{x\},\ \frac{x^{3/2}\,C[1]}{(1-x)^{5/2}} + \frac{2\left(-\sqrt{1-x} + 4\sqrt{1-x}\;x + 3\,x^{3/2}\,\text{ArcSin}\left[\sqrt{x}\right]\right) C[2]}{3\sqrt{1-x}\;(-1+x)^2}\right]\right\}\right\}$$

```
eqn /. sol // Simplify
```

{True}

$$x^{3/2}\,\text{Hypergeometric2F1}\left[\frac{5}{2}, \frac{5}{2}, \frac{5}{2}, x\right]$$

$$\frac{x^{3/2}}{(1-x)^{5/2}}$$

$$\text{Hypergeometric2F1}\left[1, 1, -\frac{1}{2}, x\right]$$

$$\text{Hypergeometric2F1}\left[1, 1, -\frac{1}{2}, x\right]$$

Mathematica returns what looks like a valid solution. The top pink cell contents, contained in the text answer, can be found in the solution, but the bottom pink expression, also from the text solution, is returned unevaluated.

17. $4\,x\,(1 - x)\,y'' + y' + 8\,y = 0$

```
Clear["Global`*"]
```

```
eqn = 4 x (1 - x) y''[x] + y'[x] + 8 y[x] == 0
```

$$8\,y[x] + y'[x] + 4\,(1 - x)\,x\,y''[x] == 0$$

```
sol = DSolve[eqn, y, x]
```

$$\left\{\left\{y \to \text{Function}\left[\{x\},\ (1-x)^{5/4}\,x^{3/4}\,C[1] - \frac{4}{15}\left(5 - 40\,x + 32\,x^2\right) C[2]\right]\right\}\right\}$$

```
eqn /. sol // Simplify
```

{True}

$x^{3/4}$ Hypergeometric2F1$\left[\frac{7}{4}, -\frac{5}{4}, \frac{7}{4}, x\right]$

$(1-x)^{5/4}\, x^{3/4}$

The green cell above matches the text answer, with C[1]=1 and C[2]=-$\frac{3}{4}$, (assuming the text constants A and B both equal 1).

19. $2\left(t^2-5t+6\right)\ddot{y}+(2t-3)\dot{y}-8y=0$

```
Clear["Global`*"]
```

```
eqn = 2 (t^2 - 5 t + 6) y''[t] + (2 t - 3) y'[t] - 8 y[t] == 0
```

$-8\,y[t] + (-3+2t)\,y'[t] + 2\left(6-5t+t^2\right) y''[t] == 0$

```
sol = DSolve[eqn, y, t]
```

$\left\{\left\{y \to \text{Function}\left[\{t\}, \frac{(2-t)^{1/4}(-3+t)^{1/4}(-2+t)^{5/4}(-17+6t)\,C[1]}{6\,(3-t)^{3/4}} + \frac{4\,(2-t)^{1/4}(-3+t)^{3/4}\left(111-104t+24t^2\right) C[2]}{5\,(3-t)^{3/4}(-2+t)^{1/4}}\right]\right\}\right\}$

```
eqn /. sol // Simplify
```

{True}

$(t-2)^{3/2}$ Hypergeometric2F1$\left[\frac{7}{2}, -\frac{1}{2}, \frac{5}{2}, t-2\right]$

$\frac{(17-6t)\,(-2+t)^{3/2}}{5\sqrt{3-t}}$

Hypergeometric2F1$\left[2, -2, -\frac{1}{2}, t-2\right]$

$-111+104t-24t^2$

$$\text{Solve}\Big[\frac{(2-t)^{1/4}\,(-3+t)^{1/4}\,(-2+t)^{5/4}\,(-17+6\,t)\,C[1]}{6\,(3-t)^{3/4}} + \frac{4\,(2-t)^{1/4}\,(-3+t)^{3/4}\,\left(111-104\,t+24\,t^2\right)\,C[2]}{5\,(3-t)^{3/4}\,(-2+t)^{1/4}} == \frac{(17-6\,t)\,(-2+t)^{3/2}}{5\,\sqrt{3-t}} - 111 + 104\,t - 24\,t^2,\ \{C[1],\ C[2]\}\Big]$$

Solve::svars: Equations may not give solutions for all "solve" variables ≫

$$\left\{\left\{C[2] \to -\frac{5\,(3-t)^{3/4}\,(-2+t)^{1/4}\,\left(111-\frac{(17-6\,t)\,(-2+t)^{3/2}}{5\,\sqrt{3-t}}-104\,t+24\,t^2\right)}{4\,(2-t)^{1/4}\,(-3+t)^{3/4}\,\left(111-104\,t+24\,t^2\right)} - \frac{5\,(-2+t)^{3/2}\,(-17+6\,t)\,C[1]}{24\,\sqrt{-3+t}\,\left(111-104\,t+24\,t^2\right)}\right\}\right\}$$

Though I see some vague similarities, there is nothing to definitely link the yellow with the text answer, which, if the hypergeometric function correspondence is valid, as it seems to be, is sum of pinks.