Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

1 - 10 Line integrals: evaluation by Green's theorem

Evaluate $\int_C F(r).dr$ counterclockwise around the boundary C of the region R by Green's theorem, where

1. $F = \{y, -x\}$, C the circle $x^2 + y^2 = \frac{1}{4}$

Note: Rogawski has an example which I followed in form.

```
P[x_, y_] = y
Q[x_, y_] = -x
```

```
y
```

```
-x
```

Inspect the derivative set to judge continuity

```
D[P[x, y], x]
D[P[x, y], y]
D[Q[x, y], x]
D[Q[x, y], y]
```

```
0
```

```
1
```

```
-1
```

```
0
```

All of the above derivatives are definitely continuous inside the path, so Green's should apply.

$$\int_0^{2\pi}\int_0^{1/8}(D[Q[x, y], x] - D[P[x, y], y])\, dy\, dx$$

$$-\frac{\pi}{2}$$

The above answer matches the text. I had trouble with the limits of the integrals. Paul's notes (http://tutorial.math.lamar.edu/Classes/CalcIII/GreensTheorem.aspx) solved a Green's problem with
circular path, explaining that it was done in polar coordinates. That sounded good and I copied the method.

3. $F = \{x^2 e^y, y^2 e^x\}$, R the rectangle with vertices $\{0, 0\}$, $\{2, 0\}$, $\{2, 3\}$, $\{0, 3\}$

```
Clear["Global`*"]
```

```
P[x_, y_] = x^2 ⅇ^y
Q[x_, y_] = y^2 ⅇ^x
```

$e^y x^2$

$e^x y^2$

```
D[P[x, y], x]
D[P[x, y], y]
D[Q[x, y], x]
D[Q[x, y], y]
```

$2 e^y x$

$e^y x^2$

$e^x y^2$

$2 e^x y$

I believe all of the above derivatives are continuous everywhere.

$$\int_0^2 \int_0^3 (D[Q[x, y], x] - D[P[x, y], y])\, dy\, dx$$

$$-\frac{19}{3} + 9 e^2 - \frac{8 e^3}{3}$$

The above answer matches the text. Limits of integration were not a problem.

5. $F = \{x^2 + y^2, x^2 - y^2\}$, R : $1 \le y \le 2 - x^2$

```
Clear["Global`*"]
```

```
P[x_, y_] = x^2 + y^2
Q[x_, y_] = x^2 - y^2
```

$x^2 + y^2$

$x^2 - y^2$

```
D[P[x, y], x]
D[P[x, y], y]
D[Q[x, y], x]
D[Q[x, y], y]
```

$2 x$

$2 y$

$2 x$

$-2 y$

The above derivatives are continuous.

$$\int_{-1}^{1}\int_{1}^{2-x^2} (D[Q[x, y], x] - D[P[x, y], y])\, dy\, dx$$

$$-\frac{56}{15}$$

The above answer matches the second part of the problem's answer.

$$\int\int (D[Q[x, y], x] - D[P[x, y], y])\, dy\, dx$$

x (x - y) y

The first part of the problem is not solved here. I can't see a limit to the boundary on x, but using infinity does
not work either.

7. $F = grad[x^3 Cos[x y]^2,\ R$ as in problem 5

```
Clear["Global`*"]
```

```
f[x_, y_] = x^3 Cos[x y]^2
whatisit = Grad[f[x, y], {x, y}]
```

x^3 Cos[x y]^2

{3 x^2 Cos[x y]^2 - 2 x^3 y Cos[x y] Sin[x y], -2 x^4 Cos[x y] Sin[x y]}

```
P[x_, y_]  =  3 x^2 Cos[x y]^2 - 2 x^3 y Cos[x y] Sin[x y]
Q[x_, y_]  =  -2 x^4 Cos[x y] Sin[x y]
```

3 x^2 Cos[x y]^2 - 2 x^3 y Cos[x y] Sin[x y]

-2 x^4 Cos[x y] Sin[x y]

```
D[P[x, y], x]
D[P[x, y], y]
D[Q[x, y], x]
D[Q[x, y], y]
```

6 x Cos[x y]^2 - 2 x^3 y^2 Cos[x y]^2 - 12 x^2 y Cos[x y] Sin[x y] + 2 x^3 y^2 Sin[x y]^2

-2 x^4 y Cos[x y]^2 - 8 x^3 Cos[x y] Sin[x y] + 2 x^4 y Sin[x y]^2

-2 x^4 y Cos[x y]^2 - 8 x^3 Cos[x y] Sin[x y] + 2 x^4 y Sin[x y]^2

-2 x^5 Cos[x y]^2 + 2 x^5 Sin[x y]^2

As for the continuity of the four lines of expressions above, I think the polys have to be continuous.
As for the trig expressions, I know of no reason why they should not be continuous, so I assume that

they are.

$$\int_{-1}^{1}\int_{1}^{2-x^2} (\text{D[Q[x, y], x] - D[P[x, y], y]})\, dy\, dx$$

```
0
```

The expression from the previous problem is brought down, since the y domain is the same for this problem as for the last. The zero answer matches the text for the part where x is given values between -1 and 1. Then the answer section asks 'Why'? Good question. I copied the format for doing Green's Function problems, that's why.

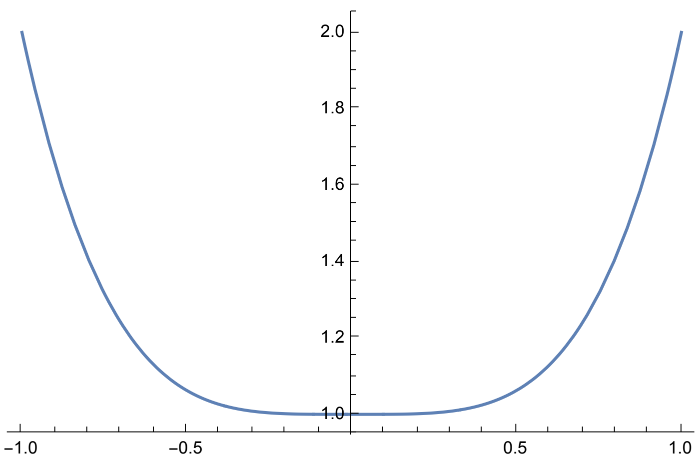
9. $F = \left\{e^{y/x},\ e^{y}\,\text{Log}[x] + 2\,x\right\}$, R : $1 + x^4 \le y \le 2$

This problem is part of the set 1 - 10, so it is asking for the same thing as the others, do Green's theorem on a
counterclockwise path. But it seems harder than the rest.

```
Clear["Global`*"]
```

```
quizz = 1 + x^4
Plot[quizz, {x, -1, 1}]
```

```
1 + x^4
```




```
P[x_, y_] = e^(y/x)
Q[x_, y_] = e^y Log[x] + 2 x
```

$e^{\frac{y}{x}}$

```
2 x + e^y Log[x]
```

```
Reduce[1 + x^4 ≤ 2, x]
```

```
-1 ≤ x ≤ 1
```

$-1 \le x \le 1$

$-1 \le x \le 1$

```
D[P[x, y], x]
D[P[x, y], y]
D[Q[x, y], x]
D[Q[x, y], y]
```

$$-\frac{e^{\frac{y}{x}}\, y}{x^2}$$

$$\frac{e^{\frac{y}{x}}}{x}$$

$$2 + \frac{e^{y}}{x}$$

$$e^{y}\,\mathrm{Log}[x]$$

For the first three of the above, x must not be zero in order for the expressions to be continuous inside the path. Outside of that, continuity does not seem to be an issue.

```
N[16 / 5]
```

3.2

$$\text{stet} = \int_{-1}^{-0.001}\int_{1+x^4}^{1} (D[Q[x, y], x] - D[P[x, y], y])\, dy\, dx$$

$$\int_{-1}^{-0.001}\left(e^{\frac{1}{x}}\left(-1 + e^{x^3}\right) - \frac{e\left(-1 + e^{x^4}\right)}{x} - 2\,x^4\right) dx$$

$$\text{stetN2} = N\left[\int_{-1.293196}^{-0.001}\int_{1+x^4}^{1} (D[Q[x, y], x] - D[P[x, y], y])\, dy\, dx\right]$$

3.20008

There is a problem with finding the limits of integration for x. The y limits are not a problem. But x cannot be zero, though it can be anything above zero. Trying a few limit values, it seems possible to get close to the answer (yellows).

13 - 17 Integral of the normal derivative

Using (9), p. 437, find the value of $\int \frac{\partial w}{\partial n}\, ds$ taken counterclockwise over the boundary C of the region R.

13. w = Cosh[x], R the triangle with vertices {0, 0}, {4, 2}, {0, 2}

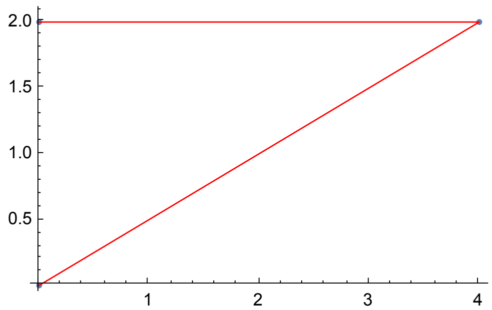
```
Clear["Global`*"]
```

This problem is included in the s.m., p. 181. There it is represented that transforming the normal derivative of a Laplacian of the cited function into a double integral is what needs to be done.

```
Laplacian[Cosh[x], {x}]
Cosh[x]
```

```
mypoints = {{0, 0}, {4, 2}, {0, 2}}
{{0, 0}, {4, 2}, {0, 2}}
```

```
a = ListPlot[mypoints, ImageSize → 250];
b = ListLinePlot[mypoints, PlotStyle → {Red, Thickness[0.003]}];
Show[a, b]
```



By inspection it is seen that, for the hypotheneuse, $y=\frac{x}{2}$, or $x = 2y$. So the s.m. says that what is
needed is a double integral with x going from 0 to 2 y and y going from 0 to 2.

blaso = $\int_0^2 \int_0^{2y}$ Cosh[x] dx dy

$\frac{1}{2}$ (-1 + Cosh[4])

The above answer agrees with the text's.

15. $w = e^x \operatorname{Cos}[y] + x y^3$, R : $1 \le y \le 10 - x^2$, $x \ge 0$

```
Clear["Global`*"]
```

here = Laplacian[e^x Cos[y] + x y^3, {x, y}]

6 x y

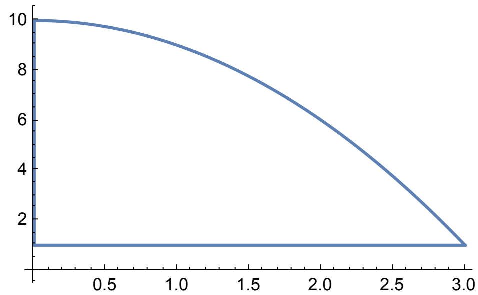
The above agrees with the text's calculation of the Laplacian.

Reduce[$1 \le y \le 10 - x^2$ && $x \ge 0$]

($0 \le x < 3$ && $1 \le y \le 10 - x^2$) || (x == 3 && y == 1)

```
p1 = Plot[10 - x^2, {x, 0, 3}];
plist = {{0, 10}, {0, 1}, {3, 1}}
p2 = ListLinePlot[plist];
Show[p1, p2]
```

```
{{0, 10}, {0, 1}, {3, 1}}
```



Above is the path. Now to write an integral with limits that walk around it ccw.

$$\text{blastiddo} = \int_0^3 \int_1^{10-x^2} 6\,x\,y\,dy\,dx$$

```
486
```

The above answer matches the text’s. It seemed appropriate to make *d*y the inner integral.

17. $w = x^3 - y^3$, $0 \le y \le x^2$, $|x| \le 2$

```
Clear["Global`*"]
```

$$\text{Lap} = \text{Laplacian}[x^3 - y^3, \{x, y\}]$$

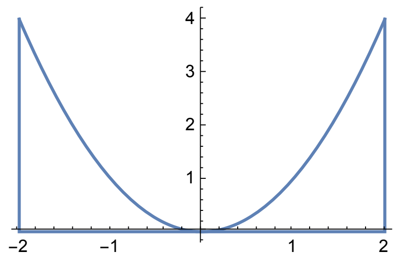
```
6 x - 6 y
```

The above agrees with the text’s calculation of the Laplacian.

```
p1 = Plot[x², {x, -2, 2}, ImageSize → 200];
plist = {{0, 0}, {2, 0}, {2, 4}}
p2 = ListLinePlot[plist, ImageSize → 200];
p2list = {{-2, 4}, {-2, 0}, {0, 0}}
p3 = ListLinePlot[p2list, ImageSize → 200];
Show[p1, p2, p3]
```

```
{{0, 0}, {2, 0}, {2, 4}}
```

```
{{-2, 4}, {-2, 0}, {0, 0}}
```



Above is the path. Now to write an integral with appropriate limits of integration.

blastiddo = $\int_{-2}^{2}\int_{0}^{x^2} (6x - 6y)\, dy\, dx$

$-\frac{192}{5}$

$\frac{-192.}{5}$

-38.4

The above answer agrees with the text's. Again I elected to put *d*y on the inside.

19. Show that w = e^x Sin[y] satisfies Laplace's equation ∇w = 0 and, using numbered line (12), p. 438, integrate w($\frac{dw}{dn}$) counterclockwise around the boundary curve C of the rectangle 0≤x≤2, 0≤y≤5

```
Clear["Global`*"]
```

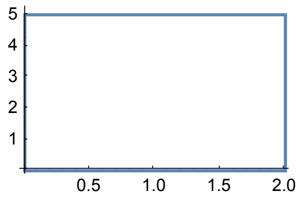
```
eq[x_, y_] = ⅇ^x Sin[y]
ⅇ^x Sin[y]
```

```
Lap = Laplacian[ⅇ^x Sin[y], {x, y}]
0
```

```
ppoints = {{0, 0}, {2, 0}, {2, 5}, {0, 5}, {0.001, 0}}
```

```
{{0, 0}, {2, 0}, {2, 5}, {0, 5}, {0.001, 0}}
```

```
ListLinePlot[ppoints, ImageSize → 150]
```



The problem instructions refer to numbered line (12), an equation contained in the problems, and shown below.

$$(12)\quad \int\!\!\int_R \left(\frac{dw}{dx}\right)^2 + \left(\frac{dw}{dy}\right)^2 dx\, dy = \oint_C w \frac{dw}{dn} ds$$

The partial derivatives in the top line, for the present problem, are the raps:

```
rap1 = D[ⅇ^x Sin[y], x]
ⅇ^x Sin[y]

rap2  = D[ⅇ^x Sin[y], y]
ⅇ^x Cos[y]

sq = rap1^2 + rap2^2
ⅇ^(2 x) Cos[y]^2 + ⅇ^(2 x) Sin[y]^2

sq1  = TrigReduce[sq]
ⅇ^(2 x)
```

And the top line filled in and executed:

$$\text{outsq} = \int_0^5 \int_0^2 (\text{sq1})\, dx\, dy$$

$$\frac{5}{2}\left(-1 + e^4\right)$$

The line above agrees with the text’s answer.