Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

```
Clear["Global`*"]
```

Since Mathematica documentation has a section that seems tailormade, I might as well start off with that:

Study the vibrations of a stretched string using the wave equation.

```
weqn = D[u[x, t], {t, 2}] == D[u[x, t], {x, 2}]
```

$$u^{(0,2)}[x, t] == u^{(2,0)}[x, t]$$

Specify that the ends of the string remain fixed during the vibrations.

```
bc = {u[0, t] == 0, u[π, t] == 0};
```

Give initial values at different points on the string.

```
ic = {u[x, 0] == x^2 (π - x), u^(0,1)[x, 0] == 0};
```

Solve the initial-boundary value problem. Note: in the boundary conditions this example sets the length of the string at π units, and the string length reappears in the initial conditions equation (x^2(π-x)). This format seemed essential in getting Mathematica to solve the diff eq. But it worked as desired after some experimentation.

```
dsol = DSolve[{weqn, bc, ic}, u, {x, t}] /. {K[1] → m}
```

$$\left\{\left\{u \to \text{Function}\left[\{x, t\}, \sum_{m=1}^{\infty} -\frac{4\,(1+2\,(-1)^m)\,\text{Cos}[t\,m]\,\text{Sin}[x\,m]}{m^3}\right]\right\}\right\}$$

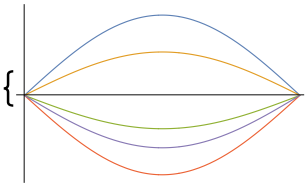
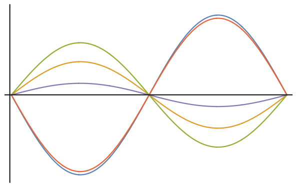
Extract four terms from the inactive sum.

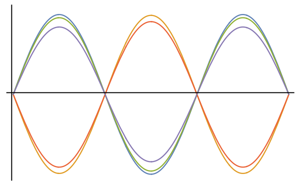
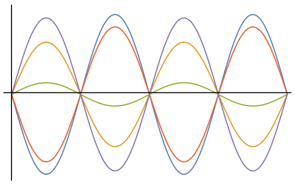
```
asol[x_, t_] = u[x, t] /. dsol[[1]] /. {∞ → 4} // Activate
```

$$4\,\text{Cos}[t]\,\text{Sin}[x] - \frac{3}{2}\,\text{Cos}[2\,t]\,\text{Sin}[2\,x] + \frac{4}{27}\,\text{Cos}[3\,t]\,\text{Sin}[3\,x] - \frac{3}{16}\,\text{Cos}[4\,t]\,\text{Sin}[4\,x]$$

Each term in the sum represents a standing wave.

```
Table[Show[Plot[
   Table[asol[x, t][[m]], {t, 0, 4}] // Evaluate, {x, 0, Pi}, Ticks → False,
   PlotStyle → {Thickness[0.004]}, ImageSize → 150]], {m, 4}]
```

{, ,

, }

5 - 13 Deflection of the String

Find u(x,t) for the string of length L=1 and c^2 = 1 when the initial velocity is zero and the initial deflection with small k (say, 0.01) is as follows. Sketch or graph u(x,t) as in Fig. 291 in the text.

5. k sin 3πx

```
Clear["Global`*"]

weqn = D[u[x, t], {t, 2}] == D[u[x, t], {x, 2}]
u^(0,2)[x, t] == u^(2,0)[x, t]

bc = {u[0, t] == 0, u[1, t] == 0}
{u[0, t] == 0, u[1, t] == 0}

ic = {u[x, 0] == (k Sin[3 π x]) , u^(0,1)[x, 0] == 0}
{u[x, 0] == k Sin[3 π x], u^(0,1)[x, 0] == 0}

dsol = FullSimplify[DSolve[{weqn, bc, ic}, u, {x, t}]]
{{u → Function[{x, t}, k Cos[3 π t] Sin[3 π x]]}}
```

After some slight tinkering, Mathematica came through with text answer. I think the **FullSimplify** definitely helped.

7. kx(1-x)

```
Clear["Global`*"]

weqn = D[u[x, t], {t, 2}] == D[u[x, t], {x, 2}]
u^(0,2)[x, t] == u^(2,0)[x, t]
```

```
bc = {u[0, t] == 0, u[1, t] == 0}
{u[0, t] == 0, u[1, t] == 0}
```

```
ic = {u[x, 0] == (k x) (1 - x) , u(0,1)[x, 0] == 0}
{u[x, 0] == k (1 - x) x, u(0,1)[x, 0] == 0}
```

```
dsol = FullSimplify[DSolve[{weqn, bc, ic}, u, {x, t}]]
```

$$\left\{\left\{u \to \text{Function}\left[\{x, t\}, \sum_{K[1]=1}^{\infty} -\left(\left(4\left(-1+(-1)^{K[1]}\right) k \operatorname{Cos}[\pi\, t\, K[1]] \operatorname{Sin}[\pi\, x\, K[1]]\right) \Big/ \left(\pi^3 K[1]^3\right)\right)\right]\right\}\right\}$$

```
dsol2 = Simplify[dsol /. K[1] → m]
```

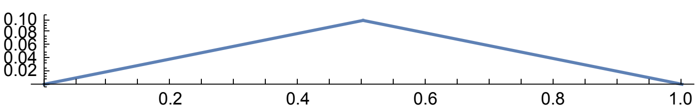
$$\left\{\left\{u \to \text{Function}\left[\{x, t\}, \sum_{m=1}^{\infty} -\frac{1}{\pi^3 m^3} 4\left(-1+(-1)^m\right) k \operatorname{Cos}[\pi\, t\, m] \operatorname{Sin}[\pi\, x\, m]\right]\right\}\right\}$$

The green cell above matches the text’s answer.

9. $\begin{cases} \frac{x}{5} & 0 < x < 1/2 \\ \frac{2}{10} - \frac{x}{5} & 1/2 < x < 1 \end{cases}$

$$\text{rat} = \begin{cases} \frac{x}{5} & 0 < x < 1/2 \\ \frac{2}{10} - \frac{x}{5} & 1/2 < x < 1 \end{cases};$$

```
Plot[rat, {x, 0, 1}, AspectRatio → Automatic]
```



I believe the below series of cells is set up correctly; however, in the brown cell below, Mathematica declines to calculate the answer. (From hints in StackExchange, I put all derivative forms on one side of equals sign in weqn.)

```
Clear["Global`*"]
```

```
weqn = D[u[x, t], {t, 2}] - D[u[x, t], {x, 2}] == 0
u(0,2)[x, t] - u(2,0)[x, t] == 0
```

```
bc = {u[0, t] == 0, u[el, t] == 0}
{u[0, t] == 0, u[el, t] == 0}
```

```
(*ic = u[x, 0] ==
  { Piecewise[{{x/5,0<x<1/2},{1/5-x/5,1/2<x<1}} ],u(0,1)[x, 0] == 0}*)
```

```
ic = u[x, 0] ==
  {Piecewise[{{(2 k/eL) x, 0 < x < eL / 2}, {(2 k/eL) (eL - x), eL / 2 < x < eL}}],
   u^(0,1)[x, 0] == 0}
```

$$u[x, 0] == \left\{ \begin{cases} \frac{2\,k\,x}{eL} & 0 < x < \frac{eL}{2} \\ \frac{2\,k\,(eL-x)}{eL} & \frac{eL}{2} < x < eL \\ 0 & \text{True} \end{cases}, u^{(0,1)}[x, 0] == 0 \right\}$$

```
dsol = FullSimplify[DSolve[{weqn, bc, ic}, u, {x, t}]]
```

$$\text{DSolve}\left[\left\{u^{(0,2)}[x, t] == u^{(2,0)}[x, t], \{u[0, t] == 0, u[1, t] == 0\}, u[x, 0] == \left\{ \begin{cases} \frac{2\,k\,x}{eL} & 0 < x < \frac{eL}{2} \\ \frac{2\,k\,(eL-x)}{eL} & \frac{eL}{2} < x < eL \\ 0 & \text{True} \end{cases}, u^{(0,1)}[x, 0] == 0 \right\}\right\}, u, \{x, t\}\right]$$

So I have to conclude that this approach only works with relatively simple deflection equations. For a successful alternate approach, see problem 11.

Example 1 on p. 550 gives an exact template of the answer and its necessary expression:
$u(x,t) = \frac{8k}{\pi^2}\left[\frac{1}{1^2}\sin\left(\frac{\pi}{L}x\right)\cos\left(\frac{\pi c}{L}\right)t - \frac{1}{3^2}\sin\left(\frac{3\pi}{L}x\right)\cos\left(\frac{3\pi c}{L}\right)t + \frac{1}{5^2}\sin\left(\frac{5\pi}{L}x\right)\cos\left(\frac{5\pi c}{L}\right)t - \cdots\right]$
, or, in this case,

$$\frac{0.8}{\pi^2}\left(\cos \pi t \sin \pi x - \frac{1}{9}\cos 3\pi t \sin 3\pi x + \frac{1}{25}\cos 5\pi t \sin 5\pi x - \cdots\right)$$

11. $\begin{cases} 0 & 0 < x < 1/4 \\ x - 1/4 & 1/4 < x < 1/2 \\ 3/4 - x & 1/2 < x < 3/4 \\ 0 & 3/4 < x < 1 \end{cases}$

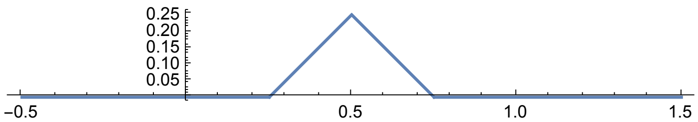
This problem has a more challenging form. I found a very effective solution procedure at *http://math.iit.edu/~fass/461_handouts.html*

```
Clear["Global`*"]
```

Solve the wave equation with the following parameters and initial displacement:

```
c = 1; L = 1; h = 0.25; f[x_] := Piecewise[{{0, 0 < x < L/4},
   {-L/4 + x, L/4 < x < L/2}, {3 L/4 - x, L/2 < x < 3 L/4}, {0, 3 L/4 < x < L}}];
```

```
Plot[f[x], {x, -L/2, 3L/2}, AspectRatio → Automatic, PlotRange → Full]
```



Compute the Fourier coefficients. Since the initial velocity g(x)=0, $B_n = 0$ and

```
A[n_] = (2 / L) Integrate[f[x] Sin[n Pi x / L], {x, 0, L}]
```

$$\frac{1}{n^2 \pi^2} 2 \left(-\mathrm{Sin}\left[\frac{n \pi}{4}\right] + 2\,\mathrm{Sin}\left[\frac{n \pi}{2}\right] - \mathrm{Sin}\left[\frac{3 n \pi}{4}\right]\right)$$

with eigenvalues

```
Lambda[n_] = (c n Pi / L)^2
```

$$n^2 \pi^2$$

The n-th partial sum of the Fourier series solution of the wave equation is

```
u[x_, t_, N_] := Sum[A[n] Cos[Sqrt[Lambda[n]] t] Sin[n Pi x / L], {n, 1, N}]
```

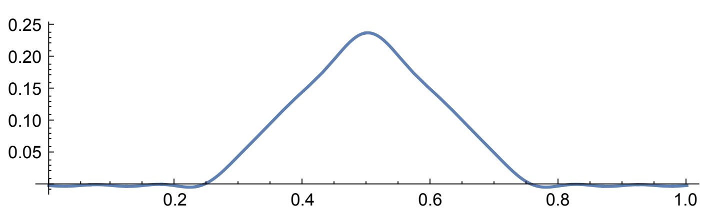
Give the partial sum approximation in a general form.

```
u[x, t, 6]
```

$$\frac{2\left(2-\sqrt{2}\right) \mathrm{Cos}[\pi t]\, \mathrm{Sin}[\pi x]}{\pi^2} + \frac{1}{9 \pi^2} 2 \left(-2-\sqrt{2}\right) \mathrm{Cos}[3 \pi t]\, \mathrm{Sin}[3 \pi x] + \frac{1}{25 \pi^2} 2 \left(2+\sqrt{2}\right) \mathrm{Cos}[5 \pi t]\, \mathrm{Sin}[5 \pi x]$$

The green cell above matches the answer in the text.

```
Plot[u[x, 0, 20], {x, 0, L}, AspectRatio → Automatic]
```



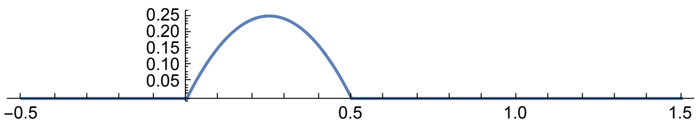
13. $\begin{cases} 2x - 4x^2 & 0 < x < \frac{1}{2} \\ 0 & \frac{1}{2} < x < 1 \end{cases}$

Repeating the procedure used in problem 11,

```
Clear["Global`*"]
```

Solve the wave equation with the following parameters and initial displacement:

```
c = 1; L = 1; h = 0.25;
f[x_] := Piecewise[{{2 x - 4 x^2, 0 < x < L/2}, {0, L/2 < x < L}}];
Plot[f[x], {x, -L/2, 3 L/2}, AspectRatio → Automatic, PlotRange → Full]
```



Compute the Fourier coefficients. Since the initial velocity g(x)=0, $B_n = 0$ and

```
A[n_] = (2 / L) Integrate[f[x] Sin[n Pi x / L], {x, 0, L}]
```

$$-\frac{4\left(-4 + 4\,\text{Cos}\left[\frac{n\pi}{2}\right] + n\,\pi\,\text{Sin}\left[\frac{n\pi}{2}\right]\right)}{n^3\,\pi^3}$$

with eigenvalues

```
Lambda[n_] = (c n Pi / L)^2
```

$$n^2\,\pi^2$$

The n-th partial sum of the Fourier series solution of the wave equation is

```
u[x_, t_, N_] := Sum[A[n] Cos[Sqrt[Lambda[n]] t] Sin[n Pi x / L], {n, 1, N}]
```

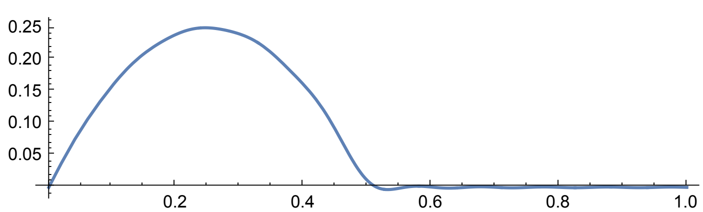
Give the partial sum approximation in a general form.

```
u[x, t, 6]
```

$$-\frac{4\,(-4+\pi)\,\text{Cos}[\pi\,t]\,\text{Sin}[\pi\,x]}{\pi^3} + \frac{4\,\text{Cos}[2\,\pi\,t]\,\text{Sin}[2\,\pi\,x]}{\pi^3} - \frac{1}{27\,\pi^3}4\,(-4-3\,\pi)\,\text{Cos}[3\,\pi\,t]\,\text{Sin}[3\,\pi\,x] - \frac{1}{125\,\pi^3}4\,(-4+5\,\pi)\,\text{Cos}[5\,\pi\,t]\,\text{Sin}[5\,\pi\,x] + \frac{4\,\text{Cos}[6\,\pi\,t]\,\text{Sin}[6\,\pi\,x]}{27\,\pi^3}$$

The green cell above matches the answer in the text.

```
Plot[u[x, 0, 20], {x, 0, L}, AspectRatio → Automatic]
```



15 - 20 Separation of a Fourth-Order PDE. Vibrating Beam

By the principles used in modeling the string it can be shown that small free vertical vibrations of a uniform elastic beam (Fig. 292) are modeled by the fourth-order PDE

(21) $\frac{\partial_2 u}{\partial_2 t} = -c^2 \; \frac{\partial_4 u}{\partial_4 x}$

where c^2 = EIρA (E=Young's modulus of elasticity, I=moment of inertia of the cross section with repsect to the y-axis in the figure, ρ=density, A=cross-sectional area).

Judging by the performance in the last section, I'm not even going to try to get Mathematica to solve equations of

15. Substituting u=F(x)G(t) into (21), show that $\frac{F^{(4)}}{F} = -\frac{\ddot{G}}{c^2 G}$ = const.