Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

```
Clear["Global`*"]
```

3. Inverse. If w = f[z] is any transformation that has an inverse, prove the fact that f and its inverse have the same fixed points.

```
Clear["Global`*"]
```

This may be a pud proof, but... I assume that w=f[x] has at least one fixed point. However many it has, consider an arbitrary fixed point z_1 in z-plane which gets mapped to w-plane by w=f[z], to a point called w_1. And the inverse of w, w^{-1}, a mapping in its own right, takes the point w_1 and operates on it to map it to the z-plane, to the exact point where it originated, that being the inverse function's function. Now $w_1 = z_1$ because z_1 is a fixed point for w. By observing w^{-1} as it maps w_1 to z_1, with which it is equal, I can see that w_1 is a fixed point for w^{-1}. The establishment of the fixed point in w^{-1} is separate and independent of the establishment of z_1 as a fixed point for w, yet, as if I didn't already know, I can look at z_1 and w_1 and see that they are equal. Therefore the mapping functions have a fixed point that is the same, and if they have this arbitrary one, they share all in common.

5. Derive the mapping in Example 2 from numbered line (2) on p. 746.

```
Clear["Global`*"]
```

$z_1 = 0$

0

$z_2 = 1$

1

Mathematica cannot swallow ∞ assigned to a variable in this context. For now, it needs to be symbolic, as below. (Note: the w-z assignment formula has its own means of dealing with occurrences of ∞, but with Mathematica I can take care of it a different way.)

$z_3 = a$

a

$w_1 = -1$

-1

$w_2 = -i$

$-i$

$w_3 = 1$

1

$$\text{Solve}\left[\frac{w - w_1}{w - w_3}\left(\frac{w_2 - w_3}{w_2 - w_1}\right) == \frac{z - z_1}{z - z_3}\left(\frac{z_2 - z_3}{z_2 - z_1}\right), \{w, z\}\right]$$

Solve::svars: Equations may not give solutions for all "solve" variables ≫

$$\left\{\left\{w \to \frac{-i\,a - (1 - i)\,z + a\,z}{i\,a - (1 + i)\,z + a\,z}\right\}\right\}$$

$$\text{Limit}\left[\frac{-i\,a - (1 - i)\,z + a\,z}{i\,a - (1 + i)\,z + a\,z}, a \to \infty\right]$$

$$\frac{-i + z}{i + z}$$

The expression in the green cell above matches the answer to example 2 on p. 748.

8 - 16 LFTs from three points and images

Find the LFT that maps the given three points onto the three given points in the respective order.

9. 1, i, -1 onto i, -1, -i

```
Clear["Global`*"]
```

$z_1 = 1$

1

$z_2 = i$

i

$z_3 = -1$

-1

$w_1 = i$

i

$w_2 = -1$

-1

$w_3 = -i$

$-i$

$$\text{Solve}\left[\frac{w - w_1}{w - w_3}\left(\frac{w_2 - w_3}{w_2 - w_1}\right) == \frac{z - z_1}{z - z_3}\left(\frac{z_2 - z_3}{z_2 - z_1}\right), \{w\}\right]$$

$\{\{w \to i\,z\}\}$

11. - 1, 0, 1 onto - i, -1, i

```
Clear["Global`*"]
```

$z_1 = -1$

-1

$z_2 = 0$

0

$z_3 = 1$

1

$w_1 = -i$

$-i$

$w_2 = -1$

-1

$w_3 = i$

i

$$\text{Solve}\left[\frac{w - w_1}{w - w_3}\left(\frac{w_2 - w_3}{w_2 - w_1}\right) == \frac{z - z_1}{z - z_3}\left(\frac{z_2 - z_3}{z_2 - z_1}\right), \{w\}\right]$$

$$\left\{\left\{w \to \frac{i + z}{-i + z}\right\}\right\}$$

13. 0, 1, ∞ onto ∞, 1, 0

```
Clear["Global`*"]
```

$z_1 = 0$

0

$z_2 = 1$

1

$z_3 = a$

a

$w_1 = a$

a

$w_2 = 1$

1

$w_3 = 0$

0

```
Solve[(w - w1)/(w - w3) ((w2 - w3)/(w2 - w1)) == (z - z1)/(z - z3) ((z2 - z3)/(z2 - z1)), {w}]
```

$$\left\{\left\{w \to \frac{a - z}{1 - 2 z + a z}\right\}\right\}$$

```
Limit[(a - z)/(1 - 2 z + a z), a → ∞]
```

$$\frac{1}{z}$$

15. 1, i, 2 onto 0, $-i - 1$, $-\frac{1}{2}$

```
Clear["Global`*"]
```

$z_1 = 1$

1

$z_2 = i$

i

$z_3 = 2$

2

$w_1 = 0$

0

$w_2 = -i - 1$

$-1 - i$

$w_3 = -\frac{1}{2}$

$-\frac{1}{2}$

```
Solve[(w - w1)/(w - w3) ((w2 - w3)/(w2 - w1)) == (z - z1)/(z - z3) ((z2 - z3)/(z2 - z1)), {w}]
```

$$\left\{\left\{w \to \frac{1 - z}{z}\right\}\right\}$$

17. Find an LFT that maps $|z| \le 1$ onto $|w| \le 1$ so that $z = \frac{i}{2}$ is mapped onto $w = 0$. Sketch the images of the lines x = const and y=const.

```
Clear["Global`*"]
```

Numbered line (3) on p. 749 is part of an example that works this problem out for me, except it is left in general form, the z-plane point z_0 being mapped to the origin of the w-plane, while the z-unit-circle is mapped to the w-unit-circle.

$w = \frac{z - z_0}{c\, z - 1}$, $c = z_0{}^*$, Abs[z_0] < 1

One caveat here is that of the need to tailor c, the conjugate of z_0, to the specific z_0 chosen.

$z_0 = \frac{i}{2}$

$\frac{i}{2}$

zcon = $z_0{}^*$

$-\frac{i}{2}$

w[z_] = $\frac{z - \frac{i}{2}}{\text{zcon}\, z - 1}$

$\frac{-\frac{i}{2} + z}{-1 - \frac{i z}{2}}$

Simplify[%]

$\frac{1 + 2 i z}{-2 i + z}$

The cell below demonstrates that the green cell agrees with the text answer in content.

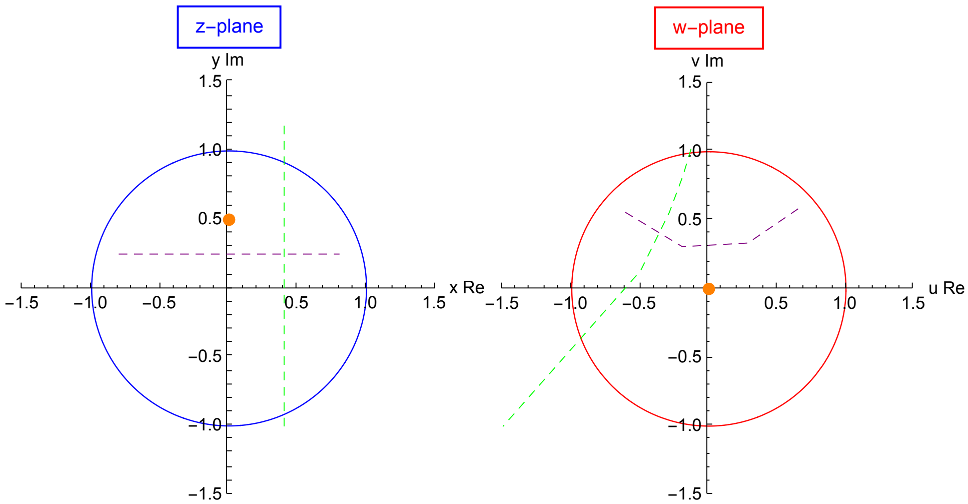
PossibleZeroQ[$\frac{1 + 2 i z}{-2 i + z} - \frac{2 z - i}{-(i z + 2)}$]

True

The transformation w maps the point z=0+$\frac{i}{2}$ on the z-plane to w=0+0i on the w-plane. In the z-plane plot its location is consistent with the vertical axis location, and in the w-plane plot it is shown correctly.

As for the original and transformed circles, and constant lines, the mapping w is allowed to do its work directly wherever possible. Two intersecting test lines demonstrate that intersection angles are preserved under conformal mapping.

```
Row[{ParametricPlot[{Re[(ⅇ^(ⅈ t) + 0)], Im[(ⅇ^(ⅈ t))]}, {t, 0, 2 π},
   ImageSize → 250, PlotRange → {{-1.5, 1.5}, {-1.5, 1.5}},
   AspectRatio → Automatic, PlotStyle → {Blue, Thickness[0.003]},
   Epilog → {{Orange, PointSize[0.03], Point[{0, 1/2}]},
     {Dashed, Green, Line[{{0.4, -1}, {0.4, 1.2}}]},
     {Dashed, Purple, Line[{{-0.8, 0.25}, {0.8, 0.25}}]}},
   AxesLabel → {"x Re", "y Im"}, PlotLabel →
    Style[Framed[ "  z-plane "], 10, Blue, Background → White]],
  ParametricPlot[{Re[((1 + 2 ⅈ ⅇ^(ⅈ t))/(-2 ⅈ + ⅇ^(ⅈ t)))], Im[((1 + 2 ⅈ ⅇ^(ⅈ t))/(-2 ⅈ + ⅇ^(ⅈ t)))]}, {t, 0, 2 π},
   ImageSize → 250, PlotRange → {{-1.5, 1.5}, {-1.5, 1.5}},
   AspectRatio → Automatic, PlotStyle → {Red, Thickness[0.003]},
   Epilog → {{Orange, PointSize[0.03], Point[{Re[(1 + 2 ⅈ z)/(-2 ⅈ + z) /. z → (0 + ⅈ/2)],
        Im[(1 + 2 ⅈ z)/(-2 ⅈ + z) /. z → (0 + ⅈ/2)]}]}, {Dashed, Green,
      Line[{{Re[(1 + 2 ⅈ z)/(-2 ⅈ + z) /. z → (0.4 - ⅈ)], Im[(1 + 2 ⅈ z)/(-2 ⅈ + z) /. z → (0.4 - ⅈ)]},
        {Re[(1 + 2 ⅈ z)/(-2 ⅈ + z) /. z → (0.4 - 0.5 ⅈ)], Im[(1 + 2 ⅈ z)/(-2 ⅈ + z) /. z → (0.4 - 0.5 ⅈ)]},
        {Re[(1 + 2 ⅈ z)/(-2 ⅈ + z) /. z → (0.4 - 0 ⅈ)], Im[(1 + 2 ⅈ z)/(-2 ⅈ + z) /. z → (0.4 - 0 ⅈ)]},
        {Re[(1 + 2 ⅈ z)/(-2 ⅈ + z) /. z → (0.4 + 0.5 ⅈ)], Im[(1 + 2 ⅈ z)/(-2 ⅈ + z) /. z → (0.4 + 0.5 ⅈ)]},
        {Re[(1 + 2 ⅈ z)/(-2 ⅈ + z) /. z → (0.4 + 1.2 ⅈ)],
         Im[(1 + 2 ⅈ z)/(-2 ⅈ + z) /. z → (0.4 + 1.2 ⅈ)]}}]},
     {Dashed, Purple, Line[{{Re[(1 + 2 ⅈ z)/(-2 ⅈ + z) /. z → (-0.8 + 0.25 ⅈ)],
         Im[(1 + 2 ⅈ z)/(-2 ⅈ + z) /. z → (-0.8 + 0.25 ⅈ)]}, {Re[(1 + 2 ⅈ z)/(-2 ⅈ + z) /.
           z → (-0.3 + 0.25 ⅈ)], Im[(1 + 2 ⅈ z)/(-2 ⅈ + z) /. z → (-0.3 + 0.25 ⅈ)]}, {Re[
          (1 + 2 ⅈ z)/(-2 ⅈ + z) /. z → (0.2 + 0.25 ⅈ)], Im[(1 + 2 ⅈ z)/(-2 ⅈ + z) /. z → (0.2 + 0.25 ⅈ)]},
        {Re[(1 + 2 ⅈ z)/(-2 ⅈ + z) /. z → 0.8 + 0.25 ⅈ], Im[(1 + 2 ⅈ z)/(-2 ⅈ + z) /. z → 0.8 + 0.25
              ⅈ]}}]}}, AxesLabel → {"u Re", "v Im"}, PlotLabel →
    Style[Framed[ "  w-plane "], 10, Red, Background → White]]}]
```

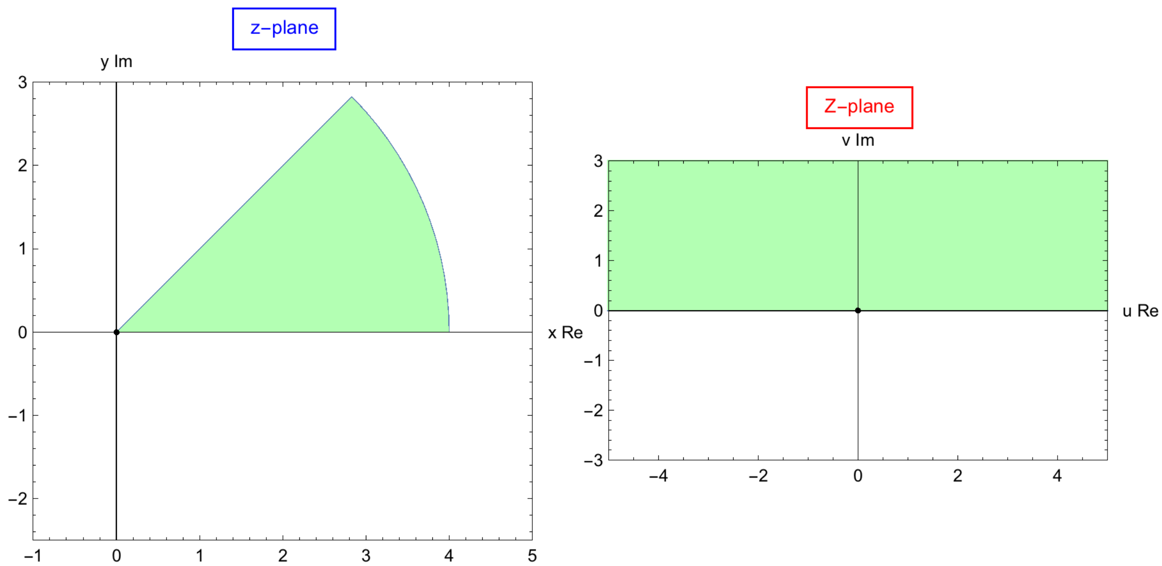


19. Find an analytic function $w = f[z]$ that maps the region $0 \leq \text{Arg}[z] \leq \frac{\pi}{4}$ onto the unit disk $|w| \leq 1$.

```
Clear["Global`*"]
```

A hint from s.m. points me to example 2 on p. 739. This explains how to map a wedge sector onto an upper half plane. The upshot is that a wedge sector $0 \leq \ \leq \frac{\pi}{n}$ can be mapped to the upper half plane v ≥ 0 using the transform z^n. In this problem I have a sector from 0 to $\frac{\pi}{4}$, so n =4. As far as the r value goes, it can be anything above about 2, but it can't be ∞. I assume that setting it at 4 arbitrarily will not undermine the argument that the demo is general.

```
Row[{ParametricPlot[{Re[r (ⅇ^(ⅈ t))], Im[(r ⅇ^(ⅈ t))]}, {t, 0, π/4},
    {r, 0, 4}, ImageSize → 300, PlotStyle → {Green, Thickness[0.004]},
    Epilog → {{Point[{0, 0}]}}, Axes → True, AxesLabel → {"x Re", "y Im"},
    PlotRange → {{-1, 5}, {3, -2.5}}, PlotLabel →
     Style[Framed[ "  z-plane "], 10, Blue, Background → White]] ,
  ParametricPlot[{Re[(r ⅇ^(ⅈ t))^4], Im[(r ⅇ^(ⅈ t))^4]}, {t, 0, π/4}, {r, 0, 4},
    ImageSize → 300, PlotStyle → {Green, Thickness[0.004]},
    Epilog → {{Point[{Re[(r ⅇ^(ⅈ t))^4 /. {t → 0, r → 0}],
          Im[(r ⅇ^(ⅈ t))^4 /. {t → 0, r → 0}]}]}}, AxesLabel →
     {"u Re", "v Im"}, PlotRange → {{-5, 5}, {3, -3}}, PlotLabel →
     Style[Framed[ "  Z-plane "], 10, Red, Background → White]] }]
```



As the plot windows above show, the mapping scheme is successful so far. The next phase is to figure out how to map the upper half plane to a unit circle on the w-plane. The strategy used by the s.m., which I will follow, will be to use the mapping formula advanced in problem 5 and following. To do this I need to pick three points on Z-plane and three others on w-plane and then calculate the expression to do it. As suggested by s.m., I choose -1, 0, and 1 on the u axis of Z-plane, and intend to map them to (-1, $-i$, and 1), on the unit circle, in the w-plane.

```
Z1 = -1
```

```
-1
```

```
Z2 = 0
```

```
0
```

```
Z3 = 1
```

1

```
w1 = -1
```

-1

```
w2 = -i
```

-i

```
w3 = 1
```

1

$$\text{exp1} = \text{Simplify}\left[\text{Solve}\left[\frac{w - w_1}{w - w_3}\left(\frac{w_2 - w_3}{w_2 - w_1}\right) == \frac{Z - Z_1}{Z - Z_3}\left(\frac{Z_2 - Z_3}{Z_2 - Z_1}\right), \{w\}\right]\right]$$

$$\left\{\left\{w \to \frac{1 + i\,Z}{i + Z}\right\}\right\}$$

To re-wind this mapping back to the original sector, I make a substitution,

$$\text{exp2} = \text{exp1} \;/.\; Z \to z^4$$

$$\left\{\left\{w \to \frac{1 + i\,z^4}{i + z^4}\right\}\right\}$$

Since this does not look exactly like the text answer, I check its equivalence, then confer the green.

$$\text{PossibleZeroQ}\left[\frac{1 + i\,z^4}{i + z^4} - \frac{\left(z^4 - i\right)}{\left(-i\,z^4 + 1\right)}\right]$$

True

Time to set up the final mapping transit, taking points from the z-plane pie sector to the w-plane unit circle.

$$\text{ParametricPlot}\left[\left\{\text{Re}\left[\frac{\left(r\, e^{i\,t}\right)^4 - i}{-i\,\left(r\, e^{i\,t}\right)^4 + 1}\right],\ \text{Im}\left[\frac{\left(r\, e^{i\,t}\right)^4 - i}{-i\,\left(r\, e^{i\,t}\right)^4 + 1}\right]\right\},\ \left\{t,\ 0,\ \frac{\pi}{4}\right\},\right.$$

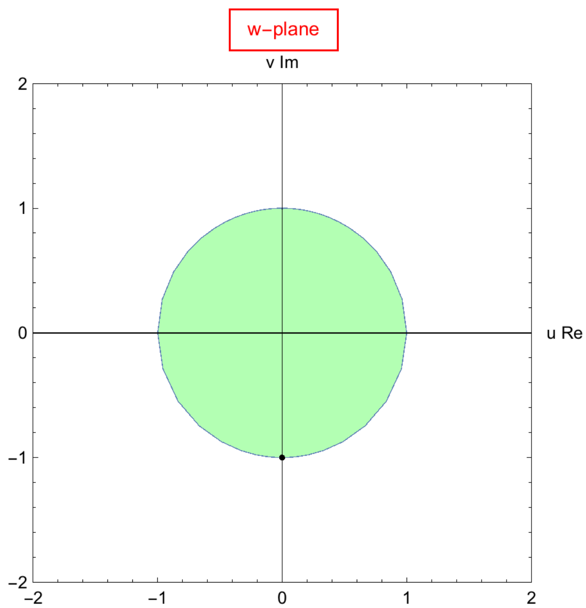
$$\{r,\ 0,\ 4\},\ \text{ImageSize} \to 300,\ \text{PlotStyle} \to \{\text{Green},\ \text{Thickness}[0.004]\},$$

$$\text{Epilog} \to \left\{\left\{\text{Point}\left[\left\{\text{Re}\left[\frac{\left(r\, e^{i\,t}\right)^4 - i}{-i\,\left(r\, e^{i\,t}\right)^4 + 1}\ /.\ \{t \to 0,\ r \to 0\}\right],\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\text{Im}\left[\frac{\left(r\, e^{i\,t}\right)^4 - i}{-i\,\left(r\, e^{i\,t}\right)^4 + 1}\ /.\ \{t \to 0,\ r \to 0\}\right]\right\}\right]\right\}\right\},$$

$$\text{AxesLabel} \to \{\text{"u Re"},\ \text{"v Im"}\},\ \text{PlotRange} \to \{\{-2,\ 2\},\ \{-2,\ 2\}\},$$

$$\left.\text{PlotLabel} \to \text{Style}[\text{Framed}[\text{"   w-plane "}],\ 10,\ \text{Red},\ \text{Background} \to \text{White}]\right]$$



The origin point of the z-plane ends up at the bottom of the unit circle in the w-plane.