Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

1 - 10 Find the path and sketch it.

1. $z[t\_] = \left(1 + \frac{i}{2}\right) t \;,\; 2 \le t \le 5$

For the first plot at least, this can be handled by treating i as equal to 1, and labeling the axes appropriately.

```
Clear["Global`*"]
```

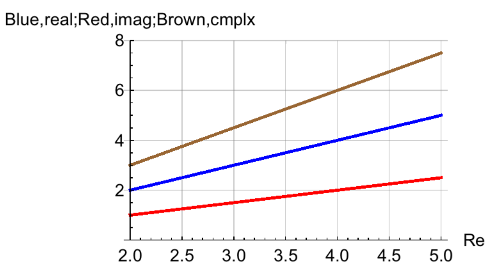
```
z[t_] = (1 + i/2) t
```

$\left(1 + \frac{i}{2}\right) t$

```
p1 = Plot[{Re[z[t]], Im[z[t]]}, {t, 2, 5}, PlotStyle → {Blue, Red},
   AxesLabel → {"Re", "Blue,real;Red,imag;Brown,cmplx"},
   ImageSize → 250, PlotRange → {0, 8}, GridLines → Automatic];
```

```
p2 = Plot[t + t/2, {t, 2, 5}, PlotStyle → {Brown},
   AxesLabel → Automatic, ImageSize → 250];
```

```
Show[p1, p2]
```



3. $z[t\_] = t + 2\, i\, t^2 \;,\; 1 \le t \le 2$

The **Plot** function is very limited, but seems to handle the first two problems.

```
Clear["Global`*"]
```

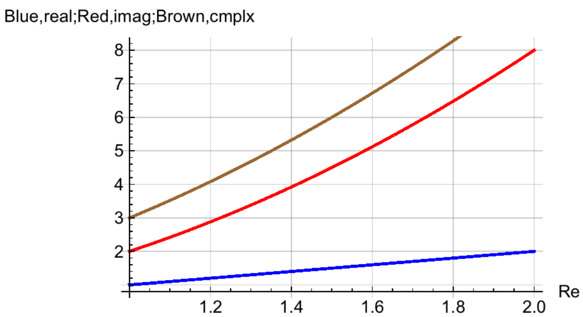
```
z[t_] = t + 2 i t^2
```

$t + 2\, i\, t^2$

```
p1 = Plot[{Re[z[t]], Im[z[t]]}, {t, 1, 2}, PlotStyle → {Blue, Red},
    AxesLabel → {"Re", "Blue,real;Red,imag;Brown,cmplx"},
    ImageSize → 300, PlotRange → Full, GridLines → Automatic];
```

```
p2 = Plot[t + 2 t^2, {t, 1, 2},
    PlotStyle → {Brown}, AxesLabel → Automatic, ImageSize → 300];
```

```
Show[p1, p2]
```



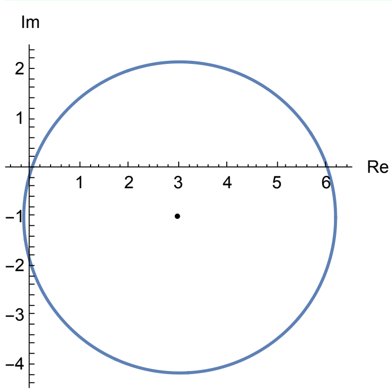
5. $z[t\_] = 3 - i + \sqrt{10}\, e^{-it}$, $0 \le t \le 2\pi$

```
Clear["Global`*"]
```

```
z[t_] = 3 - i + √10 e^(-i t)
```

$(3 - i) + \sqrt{10}\, e^{-i t}$

```
ParametricPlot[{Re[z[t]], Im[z[t]]}, {t, 0, 2 π}, ImageSize → 200,
 Epilog -> {PointSize[0.014], Point[{3, -1}]}, AxesLabel → {"Re", "Im"}]
```



The radius is $\sqrt{10}$, and $\sqrt{3^2 + 1^2}$ =10, so the circle goes through the origin. Watching the result of increasing t from $\frac{\pi}{2}$to π reveals in which direction the function develops. The function is oriented clockwise.

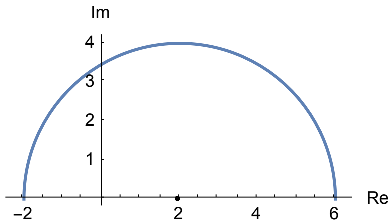
7. $z[t\_] = 2 + 4\, e^{\pi i t/2}$, $0 \le t \le 2$

```
Clear["Global`*"]
```

```
z[t_] = 2 + 4 e^(π i t/2)
```

$2 + 4\, e^{\frac{i \pi t}{2}}$

```
ParametricPlot[{Re[z[t]], Im[z[t]]}, {t, 0, 2}, ImageSize → 200,
 Epilog -> {PointSize[0.014], Point[{2, 0}]}, AxesLabel → {"Re", "Im"}]
```



Increasing t from 1 to 2 shows that the function is oriented counterclockwise.

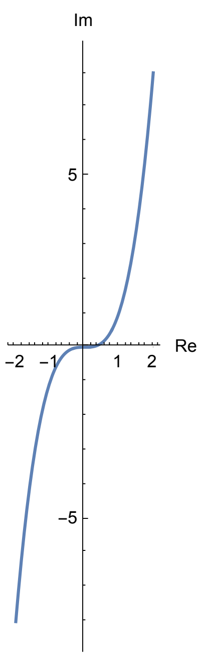
9. $z[t\_] = t + i\, t^3$, $-2 \leq t \leq 2$

```
Clear["Global`*"]
```

```
z[t_] = t + i t^3
```

$t + i\, t^3$

```
ParametricPlot[{Re[z[t]], Im[z[t]]}, {t, -2, 2}, ImageSize → 100,
 Epilog -> {PointSize[0.014], Point[{2, 0}]}, AxesLabel → {"Re", "Im"}]
```



Increasing t-max from 0 to 2 shows that the function is oriented from lower left to upper right.

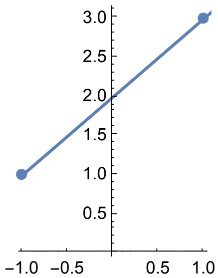
11 - 20 Find a parametric representation

11. Segment from (-1, 1) to (1, 3)

This one turns out to be just finding the equation of a line.

```
Clear["Global`*"]
```

```
p1 = ListPlot[{{-1, 1}, {1, 3}}, ImageSize → 110,
   AspectRatio → 1.33, PlotMarkers → {Automatic, Small}];
p2 = Plot[x + 2, {x, -1, 3}, ImageSize → 110];
Show[p1, p2]
```



The nuts and bolts of the equation calculation.

```
m = (3 - 1) / (1 - (-1))
```

1

```
as = y - y1 == m (x - x1)
```

$y - y_1 == x - x_1$

```
as1 = as /. {x1 → 1, y1 → 3}
```

$-3 + y == -1 + x$

```
Solve[as1, {y}]
```

{{y → 2 + x}}

```
ap = x + y ⅈ /. y → 2 + x
```

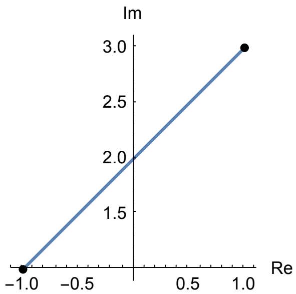
$x + i\,(2 + x)$

```
ap1 = ap /. x → t
```

$t + i\,(2 + t)$

This expresses the equation of the line in parametric terms.

```
ParametricPlot[{Re[t + ⅈ (2 + t)], Im[t + ⅈ (2 + t)]}, {t, -1, 1},
 ImageSize → 150, Epilog -> {PointSize[0.035], Point[{{-1, 1}, {1, 3}}]},
 AxesLabel → {"Re", "Im"}]
```



13. Upper half of Abs[z - 2 + I] == 2 from (4, -1) to (0, -1)

The first thing I have to understand are the points given, (4,-1), (0,-1). From the other given expressions I have to think these are points in the complex plane. When it says “upper half” it makes me think it is a circle.

On the site at *https://mathhelpboards.com/analysis-50/equation-circle-complex-plane-6771.html* I found the following

Let $z = x + i\, y$, $a = s + i\, t$

Then the equation of the circle can be written as

$|z - a| = r$ (a = center, r = radius)

$(x - s)^2 + (y - t)^2 = r^2$

$x^2 + y^2 - 2\, x\, s - 2\, y\, t + t^2 + s^2 = r^2$

Although the gist of the above lines seems to be to get back to ℝ completely, some of the expressions can be used in the present ℂ case.

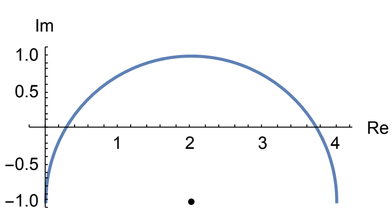
```
Clear["Global`*"]
```

```
z[t_] = Abs[z - (2 - ⅈ)] == 2
```

```
Abs[(-2 + ⅈ) + z] == 2
```

The center is already recognizable from the above, as is the radius, from the problem description.

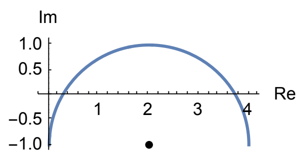
```
ParametricPlot[{2 Cos[t] + 2, 2 Sin[t] - 1}, {t, 0, π}, ImageSize → 200,
 Epilog -> {PointSize[0.02], Point[{{2, -1}, {1, 3}}]},
 AxesLabel → {"Re", "Im"}]
```



Above: By trying the upper t boundary of $\frac{\pi}{2}$ before using π, I can see that the curve develops to the left and is therefore oriented counter-clockwise, as the problem description requires. Knowing the center and radius, it is only necessary to figure out how to express these in parametric plot parlance, which is easy. So there is a plot. However, the text answer is not yet achieved.

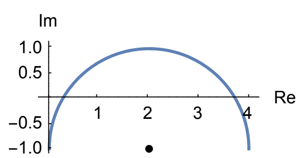
Below is shown another style of expressing the function z parametrically, closer to the text answer style.

```
ParametricPlot[{Re[2 - ⅈ + 2 (Cos[t] + ⅈ Sin[t])],
  Im[2 - ⅈ + 2 (Cos[t] + ⅈ Sin[t])]}, {t, 0, π}, ImageSize → 150,
 Epilog -> {PointSize[0.035], Point[{{2, -1}, {1, 3}}]},
 AxesLabel → {"Re", "Im"}]
```



And below yet a further progression in style, this one close enough to get the green for text answer match.

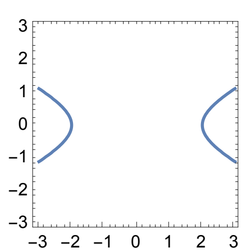
```
ParametricPlot[{Re[2 - ⅈ + 2 ⅇ^(ⅈ t)], Im[2 - ⅈ + 2 ⅇ^(ⅈ t)]}, {t, 0, π},
 ImageSize → 150, Epilog -> {PointSize[0.035], Point[{{2, -1}, {1, 3}}]},
 AxesLabel → {"Re", "Im"}]
```



15. $x^2 - 4y^2 == 4$, the branch through (2, 0)

From the form of the equation, it can be seen that a hyperbolic is being described.

```
ContourPlot[x^2/4 - y^2 == 1, {x, -3, 3}, {y, -3, 3}, ImageSize → 120]
```

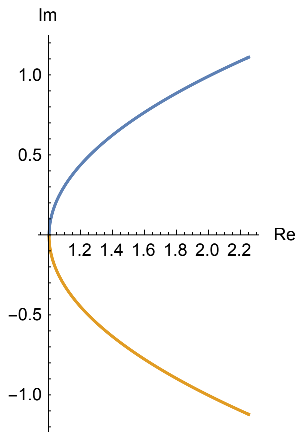


```
Solve[x^2/4 - y^2 == 1, y]
```

$$\left\{\left\{y \to -\frac{1}{2}\sqrt{-4 + x^2}\right\}, \left\{y \to \frac{1}{2}\sqrt{-4 + x^2}\right\}\right\}$$

Using blind parameterization with x -> t

```
pduo = ParametricPlot[{{Re[t^2/4 + 1/2 Sqrt[-4 + t^2] ⅈ], Im[t^2/4 + 1/2 Sqrt[-4 + t^2] ⅈ]},
   {Re[t^2/4 - 1/2 Sqrt[-4 + t^2] ⅈ], Im[t^2/4 - 1/2 Sqrt[-4 + t^2] ⅈ]}},
  {t, 2, 3}, ImageSize → 150, AxesLabel → {"Re", "Im"}]
```



The domain of t was determined by trial and error. It can’t be less than 2, and must extend somewhat to the right of 2.

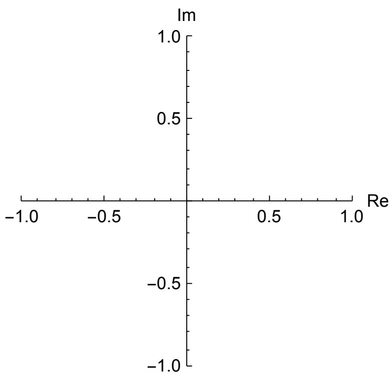
17. Abs[z + a + I b] == r, clockwise

This is a circle, expressible as

```
HoldForm[Abs[z - (-a - ⅈ b)] == r]
Abs[z - (-a - ⅈ b)] == r
```

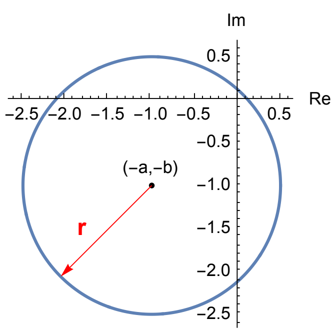
The radius is r, and the center is the point (-a,-b). To make a plot,

```
deco = ParametricPlot[{r Cos[t] + a, r Sin[t] - b}, {t, 0, 2 π},
  ImageSize → 200, Epilog -> {PointSize[0.02], Point[{{2, -1}, {1, 3}}]},
  AxesLabel → {"Re", "Im"}]
```



Taking a green ribbon for an empty plot! To get the plot to show something, make substitutions, as using {-1,-1} for {-a,-b} and 1.5 for r.

```
deco1 = ParametricPlot[{1.5 Cos[t] - 1, 1.5 Sin[t] - 1},
  {t, 0, 2 π}, ImageSize → 170, Epilog ->
   {{PointSize[0.02], Point[{-1, -1}]}, {Text["(-a,-b)", {-1, -0.8}]},
    {Text[Style["r", Bold, Red, Medium], {-1.8, -1.5}]},
    {Red, Arrowheads[.05], Arrow[{{-1, -1},
        {-1 - 1.5 Cos[π/4], -1 - 1.5 Cos[π/4]}}]}}, AxesLabel → {"Re", "Im"}]
```



19. Parabola $y = 1 - \frac{1}{4} x^2 \quad (-2 \le x \le 2)$

```
Clear["Global`*"]
```

In the first step of parameterization, let x = t

$$y == 1 - \frac{1}{4} x^2$$

$$y == 1 - \frac{x^2}{4}$$

$$y == 1 - \frac{t^2}{4}$$

$$y == 1 - \frac{t^2}{4}$$

Now check the boundary values of y,

$$y == 1 - \frac{t^2}{4} \text{ /. } t \to -2$$

$$y == 0$$

$$y == 1 - \frac{t^2}{4} \text{ /. } t \to 2$$

$$y == 0$$

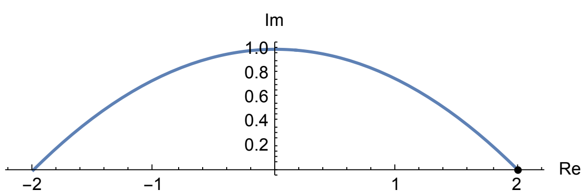
The boundary values have no effect on the value of y, so

```
z[t_] = x[t] + ⅈ y[t]
x[t] + ⅈ y[t]
```

$$z[t\_] = t + i \left(1 - \frac{1}{4} t^2\right)$$

$$t + i \left(1 - \frac{t^2}{4}\right)$$

```
ParametricPlot[{Re[z[t]], Im[z[t]]}, {t, -2, 2}, ImageSize → 300,
 Epilog -> {PointSize[0.014], Point[{2, 0}]}, AxesLabel → {"Re", "Im"}]
```



21 - 30 Integration

Integrate by the first method or state why it does not apply and use the second method.

21. $\int_C \text{Re}[z] \, dz$, the shortest path from $1 + i$ to $3 + i$

I guess I should find the shortest distance between these two points. For points (a,b) and (s,t) the distance is

$$d = \sqrt{(s - a)^2 + (t - b)^2}$$

$$d1 = \sqrt{(1 - 3)^2 + (1 - 1)^2}$$

2

or

```
d2 = √((3 - 1)² + (1 - 1)²)
```

2

This distance calculation is just an aside, I believe.

```
Clear["Global`*"]
```

$$\int_{1+i}^{3+i} z \, dz$$

4 + 2 i

The answer in yellow does not match the text answer. In the general case of such complex integrations, it appears that Mathematica prefers not to see the region of application. However, since the answer is questionable, I can try separation:

$$\int_{1+i}^{3+i} \mathrm{Re}[z] \, dz$$

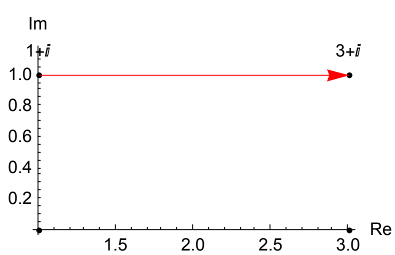
4

$$\int_{1+i}^{3+i} \mathrm{Im}[z] \, dz$$

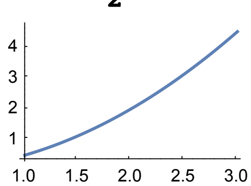
2

Done separately, the Mathematica answer is found. However, see below, as numbered line (9) does not say to do the parts separately. First, a couple of plots can't hurt.

```
Graphics[{{Point[{1, 0}]}, {Point[{3, 0}]}, {Point[{1, 1}]},
  {Point[{3, 1}]}, {Red, Arrowheads[.07], Arrow[{{1, 1}, {3, 1}}]},
  {Text["3+i", {3.0, 1.15}]}, {Text["1+i", {1, 1.15}]}},
 Axes → True, ImageSize → 200, AxesLabel → {"Re", "Im"}]
```




```
Plot[x^2/2, {x, 1, 3}, ImageSize → 125]
```



Discussion of the discrepancy. Numbered line (9) on p. 647, I think, tends to support Mathematica. The burden of numbered line (9) seems to be that, for analytic functions, the z form can be retained during the integration. This would therefore seem to evaluate as

```
(3 + i)^2 (*top evaluation*)
```

$8 + 6\,i$

```
(1 + i)^2 (*bottom evaluation*)
```

$2\,i$

$\frac{1}{2}(8 + 4\,i)$

```
(*applying integration factor to difference between top and bottom*)
```

$4 + 2\,i$

in agreement with Mathematica's answer (but not that of the text).

23. $\int_C e^z\,dz$, C, the shortest path from πi to $2\pi i$

```
Clear["Global`*"]
```

$\int_{\pi i}^{2\pi i} e^z\,dz$

2

25. $\int_C z\,\mathrm{Exp}[z^2]\,dz$, C from 1 along the axes to i

```
Clear["Global`*"]
```

$\int_1^i z\,e^{z^2}\,dz$

$-\frac{-1 + e^2}{2\,e}$

```
TrigToExp[-Sinh[1]]
```

$\frac{1}{2\,e} - \frac{e}{2}$

```
Together[1/(2 ⅇ) - ⅇ/2]
```

$$\frac{1-e^2}{2\,e}$$

The green cell above is equivalent to the text answer, as shown by the yellow cells above.

27. $\int_C \text{Sec}[z]^2 \, dz$, any path from $\frac{\pi}{4}$ to $\frac{\pi\, i}{4}$

```
Clear["Global`*"]
```

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{4} i} \text{Sec}[z]^2 \, dz$$

$$-1 + i\, \text{Tanh}\left[\frac{\pi}{4}\right]$$

The green cell above matches the text answer.

29. $\int_C \text{Im}\left[z^2\right] dz$,
counterclockwise around the triangle with vertices 0, 1, i

```
Clear["Global`*"]
```

$$\text{eq1} = \int_0^1 \text{Im}\left[z^2\right] dz$$

0

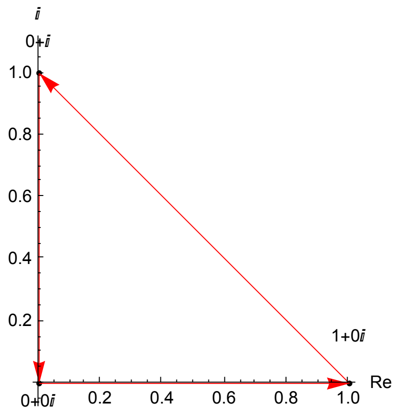
$$\text{eq2} = \int_1^{i} z^2 \, dz$$

$$-\frac{1}{3} - \frac{i}{3}$$

```
eq3 = Together[eq1 + eq2]
```

$$-\frac{1}{3} - \frac{i}{3}$$

```
Graphics[{{Point[{1, 0}]}, {Point[{0, 0}]},
  {Point[{0, 1}]}, {Red, Arrowheads[.07], Arrow[{{1, 0}, {0, 1}}]},
  {Red, Arrowheads[.07], Arrow[{{0, 1}, {0, 0}}]},
  {Text["1+0ⅈ", {1.0, 0.15}]}, {Text["0+0ⅈ", {0, -0.06}]},
  {Red, Arrowheads[.07], Arrow[{{0, 0}, {1, 0}}]},
  {Text["0+ⅈ", {0, 1.1}]}}, Axes → True,
 ImageSize → 200, AxesLabel → {"Re", "ⅈ"}]
```



The green cell above matches the text answer, except I cannot by any means get the terms to merge. The arrows show the counter-clockwise sense.

The fact that Mathematica came through with the text answer on 23, 25, 27, and 29 helps build confidence about no. 21.