3 - 10 Reduction of order

Reduce to first order and solve, showing each step in detail.

3. $y'' + y' = 0$

Reduction of order is something that Mathematica does not generally need to do.

```
eqn = y''[x] + y'[x] == 0
y′[x] + y′′[x] == 0
```

```
sol = DSolve[eqn, y, x]
```

```
{{y → Function[{x}, -e^-x C[1] + C[2]]}}
```

```
eqn /. sol // Simplify
{True}
```

5. $y\, y'' = 3\,(y')^2$

```
eqn = y[x] y''[x] == 3 y'[x]^2
y[x] y′′[x] == 3 y′[x]^2
```

```
sol = DSolve[eqn, y, x]
```

$$\left\{\left\{y \to \text{Function}\left[\{x\}, \frac{C[2]}{\sqrt{2\,x + C[1]}}\right]\right\}\right\}$$

```
eqn /. sol // Simplify
{True}
```

The text answer is $1 \Big/ \sqrt{c_1\, x + c_2}$. So Mathematica and the text answer each have assigned a value to one of their three constants. This leaves leeway for the remaining assignments to be made in such a way that the two solutions become equivalent.

7. $y'' + y'^{\,3}\text{Sin}[y] = 0$

```
Clear["Global`*"]
```

This problem is a topsy-turvy little trip with an inverted domain. The substitution $z = y'[x]$ is made. Afterwards there is the form

```
eqn2 = z'[y] z[y] == -z[y]^3 Sin[y]
z[y] z′[y] == -Sin[y] z[y]^3
```

Which can be processed by DSolve into the solution

```
sol2 = DSolve[eqn2, z, y]
```

$$\left\{\{z \to \text{Function}[\{y\}, 0]\}, \left\{z \to \text{Function}\left[\{y\}, \frac{1}{-C[1] - \text{Cos}[y]}\right]\right\}\right\}$$

The above green cell agrees with the text, though the text uses the inverted form of the fractional expression, calling it $\frac{dx}{dy}$. Using the terms of the substitution, the solution checks out.

```
eqn2 /. sol2 // Simplify
{True, True}
```

The next step is to reverse the substitution level by solving again.

```
eqn3 = -x'[y] == C[1] + Cos[y]
-x′[y] == C[1] + Cos[y]
```

```
sol3 = DSolve[eqn3, x, y]
```

```
{{x → Function[{y}, -y C[1] + C[2] - Sin[y]]}}
```

The green cell above matches the final answer in the text, with the provision that the sign on the constant -C[1] is opposite to the constant c_1 in the text. The second use of DSolve also checks out true.

```
eqn3 /. sol3
{True}
```

9. x^2 y'' -5x y' +9 y = 0, $y_1 = x^3$

```
Clear["Global`*"]
```

The substitution $y_1 = x^3$ works as advertised as a singular solution. If it is ignored,

```
eqn = x^2 y''[x] - 5 x y'[x] + 9 y[x] == 0
9 y[x] - 5 x y′[x] + x^2 y″[x] == 0
```

then Mathematica comes up with an equivalent solution, so long as C[1] is assigned the value 0 and C[2] is assigned the value $\frac{1}{3}$.

```
sol = DSolve[eqn, y, x]
```

$$\{\{y \to \text{Function}[\{x\}, x^3 C[1] + 3 x^3 C[2] \text{Log}[x]]\}\}$$

The Mathematica solution, neither more nor less general than the text, checks out.

```
eqn /. sol // Simplify
{True}
```

11 - 14 Applications of reducible ODEs

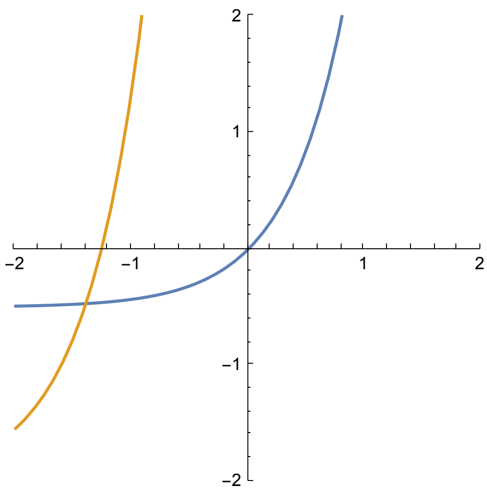
11. Curve. Find the curve through the origin in the xy-plane which satisfies $y'' = 2\, y'$ and whose tangent at the origin has slope 1.

```
Clear["Global`*"]
eqn = y''[x] == 2 y'[x]
y''[x] == 2 y'[x]
sol = DSolve[{eqn, y'[0] == 1, y[0] == 0}, y, x]
```

$\{\{y \to \text{Function}[\{x\}, \frac{1}{2}(-1 + e^{2x})]\}\}$

The plot below shows that the text answer meets neither of the two requirements stated for the solution. The function in the yellow cell above meets both.

Plot[$\{\frac{1}{2}(-1 + e^{2x}), -2 + 25\, e^{2x}\}$, {x, -2, 2}, AspectRatio → Automatic, PlotRange → {{-2, 2}, {-2, 2}}, ImageSize → 250]



13. Motion. If, in the motion of a small body on a straight line, the sum of the velocity and acceleration equals a positive constant, how will the distance y[t] depend on the initial velocity and position?

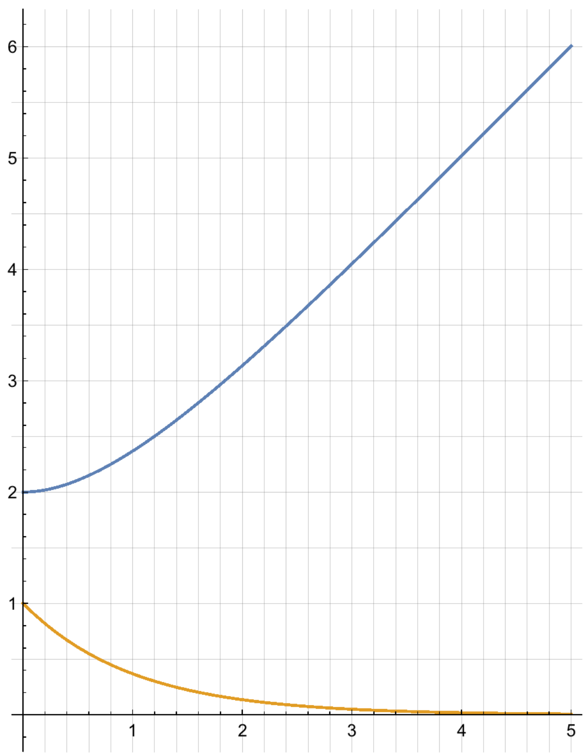
```
Clear["Global`*"]
```

First, there is an objection against the statement that the sum of velocity and acceleration equals a constant. The two quantities have different units, so they can't be added. The problem must mean to stipulate that the sum of the coefficients of acceleration and velocity add to a constant. To try to understand this a little bit, I will plot the text answer.

```
y[t_] = c1 e^-t + k t + c2
k t + e^-t c1 + c2
```

The grid squares do not appear as squares, but the axes's major ticks seem to be about equal. The problem is supposed to be about travel along a straight line; here the straight line must be the y-axis. With my choice of c_1, c_2, and $k = 1$, the starting point must be y = 2, and sum of acceleration and velocity must be 1, and the starting velocity must be 1.

```
Plot[{e^-t + t + 1, e^-t}, {t, 0, 5},
 AspectRatio → 1.3, ImageSize → 300, GridLines → All]
```




```
tid = N[Table[{t, e^-t + t + 1}, {t, 0, 15}]]
{{0., 2.}, {1., 2.36788}, {2., 3.13534},
 {3., 4.04979}, {4., 5.01832}, {5., 6.00674}, {6., 7.00248},
 {7., 8.00091}, {8., 9.00034}, {9., 10.0001}, {10., 11.},
 {11., 12.}, {12., 13.}, {13., 14.}, {14., 15.}, {15., 16.}}
```

What can be seen from the two cells below is that by the time t=14, acceleration has nearly disappeared, which means that added velocity is also nearly gone, and the travel velocity is at the rate of the starting velocity.

```
tir = Table[tid[[n]][[2]] - tid[[n]][[1]], {n, 15}]
{2., 1.36788, 1.13534, 1.04979, 1.01832, 1.00674, 1.00248,
 1.00091, 1.00034, 1.00012, 1.00005, 1.00002, 1.00001, 1., 1.}
```

$N[e^{-14}]$

8.31529×10^{-7}

```
y'[t]
```

$k - e^{-t} c_1$

15 - 19 General solution. Initial value problem (IVP)
(More in the next set.) (a) Verify that the given functions are linearly independent and form a basis of solutions of the given ODE. (b) Solve the IVP. Graph or sketch the solution.

15. 4 y'' + 25 y = 0, y[0] = 3.0, y'[0] = -2.5, Cos[2.5 x], Sin[2.5 x]

```
Clear["Global`*"]
```

By inspection, the two trig expressions are independent. To test whether they are solutions,

```
eqn = 4 y''[x] + 25 y[x] == 0
25 y[x] + 4 y''[x] == 0
```

```
sol = DSolve[{eqn, y[0] == 3.0, y'[0] == -2.5}, y, x]
```

$\left\{\left\{y \to \text{Function}\left[\{x\}, 3.\ \text{Cos}\left[\frac{5x}{2}\right] - 1.\ \text{Sin}\left[\frac{5x}{2}\right]\right]\right\}\right\}$

The solution checks.

```
eqn /. sol // Simplify
{True}
```

The two proposed solutions check.

```
eqn /. Cos[2.5 x] // Simplify
```

ReplaceAll::reps: {Cos[2.5x]} is neither a list of replacement rules nor a valid dispatch table, and so cannot be used for replacing. >>

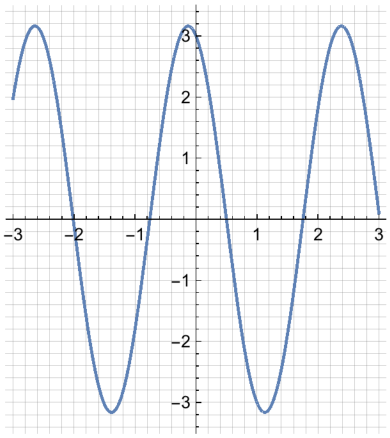
```
True /. Cos[2.5 x]
```

```
eqn /. Sin[2.5 x] // Simplify
```

ReplaceAll::reps: {Sin[2.5x]} is neither a list of replacement rules nor a valid dispatch table, and so cannot be used for replacing. >>

```
True /. Sin[2.5 x]
```

```
Plot[3.` Cos[5 x/2] - 1.` Sin[5 x/2], {x, -3, 3},
 AspectRatio → Automatic, ImageSize → 200, GridLines → All]
```



17. $4 x^2 y'' - 3 y = 0$, $y(1) = -3$, $y'(1) = 0$, $x^{3/2}$, $x^{-1/2}$

```
Clear["Global`*"]
```

```
eqn = 4 x^2 y''[x] - 3 y[x] == 0
```

$-3\,y[x] + 4\,x^2\,y''[x] == 0$

```
sol = DSolve[{eqn, y[1] == -3, y'[1] == 0}, y, x]
```

$\left\{\left\{y \to \text{Function}\left[\{x\}, -\frac{3\,(3 + x^2)}{4\,\sqrt{x}}\right]\right\}\right\}$

```
eqn /. sol
```

{True}

Although they look a little different due to their format, the green cell above and the text answer are equivalent.

$\text{PossibleZeroQ}\left[-\frac{3\,(3 + x^2)}{4\,\sqrt{x}} - \left(-0.75\,x^{3/2} - 2.25\,x^{-1/2}\right)\right]$

True

Checking the proposed solutions is a little more complicated than usual.

$\text{d2} = \text{D}\left[x^{3/2}, \{x, 2\}\right]$

$$\frac{3}{4\,\sqrt{x}}$$

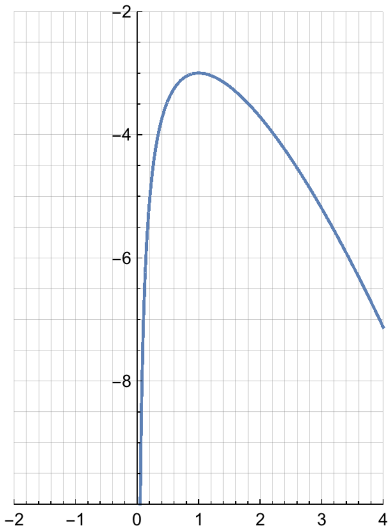
```
eqn /. {y[x] -> x^(3/2), y''[x] → d2}
True

d22 = D[x^(-1/2), {x, 2}]
```

$$\frac{3}{4\,x^{5/2}}$$

```
eqn /. {y[x] -> x^(-1/2), y''[x] → d22}
True
```

```
Plot[- 3 (3 + x^2)/(4 Sqrt[x]), {x, -2, 4}, AspectRatio → Automatic,
 ImageSize → 200, GridLines → All, PlotRange → {{-2, 4}, {-10, -2}}]
```



19. $y'' + 2\,y' + 2\,y = 0$, $y(0) = 0$, $y'(0) = 15$, e^{-x} Cos[x], e^{-x} Sin[x]

```
Clear[eqn, sol]
eqn = y''[x] + 2 y'[x] + 2 y[x] == 0;
sol = DSolve[{eqn, y[0] == 0, y'[0] == 15}, y, x]
{{y → Function[{x}, 15 e^-x Sin[x]]}}

eqn /. sol // Simplify
{True}

f1[x_] = e^-x Cos[x]
e^-x Cos[x]
```

```
d1 = D[f1[x], x]
-e^-x Cos[x] - e^-x Sin[x]

d2 = D[f1[x], {x, 2}]
2 e^-x Sin[x]

eqn /. {y[x] → f1[x], y'[x] → d1, y''[x] → d2} // Simplify
True

f2[x_] = e^-x Sin[x]
e^-x Sin[x]

d11 = D[f2[x], x]
e^-x Cos[x] - e^-x Sin[x]

d22 = D[f2[x], {x, 2}]
-2 e^-x Cos[x]

eqn /. {y[x] → f2[x], y'[x] → d11, y''[x] → d22} // Simplify
True

Plot[15 e^-x Sin[x], {x, -1, 2}, AspectRatio → Automatic,
 ImageSize → 150, GridLines → All, PlotRange → {{-1, 2}, {-1, 5}}]
```