Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

```
Clear["Global`*"]
```

1 - 6 Vector norms
Compute the norms (5), (6),(7). Compute a corresponding unit vector (vector of norm 1) with respect to the l_∞ – norm.

1. {{1, -3, 8, 0, -6, 0}}

```
Clear["Global`*"]
```

Defining the vector.

```
A = {1, -3, 8, 0, -6, 0}
{1, -3, 8, 0, -6, 0}
```

Finding the three norms asked for.

```
Table[Norm[A, n], {n, {1, 2, ∞}}]
```

$\{18, \sqrt{110}, 8\}$

To get a unit vector from A, I will need to take into account which norm the unit-ness is relative to. The problem says it is the ∞ norm. Now to the ∞ norm, the size of a vector is not the Euclidean length, but rather the size of the largest component making it up. In this case the largest component is 8. In a vector of unit size, as evaluated according to the ∞ norm, the largest component will have size 1. Therefore, all the components of the starting vector need to be divided by 8, thus

```
{0.125, -0.375, 1, 0, -0.75, 0}
```

It is unexpected to me because its l_2-norm is not equal to 1. That is

```
Norm[{0.125, -0.375, 1, 0, -0.75, 0}]
1.31101
```

By not specifying which norm I desired, I get the default norm-2. See problem 7 for a discussion of different types of norms.

3. {{0.2, 0.6, -2.1, 3.0}}

The steps in this problem are the same as in problem 1.

```
Clear["Global`*"]
```

```
a = {0.2, 0.6, -2.1, 3.0}
{0.2, 0.6, -2.1, 3.}
```

```
Table[Norm[a, n], {n, {1, 2, ∞}}]
```

```
{5.9, 3.71618, 3.}
```

$\frac{1}{3}$ `{0.2`, 0.6`, -2.1`, 3.`}`

```
{0.0666667, 0.2, -0.7, 1.}
```

5. {{1, 1, 1, 1, 1}}

```
Clear["Global`*"]
```

```
a = {1, 1, 1, 1, 1}
{1, 1, 1, 1, 1}
```

```
Table[Norm[a, n], {n, {1, 2, ∞}}]
```

$\{5, \sqrt{5}, 1\}$

$\frac{1}{1}$ `{1, 1, 1, 1, 1}`

```
{1, 1, 1, 1, 1}
```

7. For what x ={{a,b,c}} will $\text{Abs}[x]_1 = \text{Abs}[x]_2$?

There is something tricky here in the nomenclature that I didn't pick up on at first. Note that in the problem description, the subscripts are outside of their expressions. $\text{Abs}[x]_1 \neq \text{Abs}[x_1]$. The subscript outside refers to numbered lines (5), (6), (7) on p. 866. The three numbered lines are shown below, and refer to three different types of norms. In Mathematica the $\|x\|_2$ type norm is the default and is the only type I have been used to.

$\|x\|_1 = |x_1| + \cdots + |x_n|$ ("l_1-norm")

$\|x\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}$ ("Euclidean" or "l_2-norm")

$\|x\|_\infty = \max_j |x_j|$ ("l_∞-norm")

So in trying to solve the problem, I would

```
Clear["Global`*"]
```

```
Solve[Norm[{a, b, c}, 1] == Norm[{a, b, c}, 2], {a, b, c}]
```

Solve::ifun:
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

Solve::svars: Equations may not give solutions for all "solve" variables. >>

```
{{a → 0, b → 0}}
```

This may be fairly close to the text answer, which was a*b+b*c+c*a=0. However, it may be that the Mathematica answer is only a subset of the text answer, in which case it is deficient. Let me try a Reduce on the text answer.

```
Reduce[a b + bc + ca == 0, {a, b, c}]
```

$$(bc == -ca \,\&\&\, a == 0) \;||\; \left(a \neq 0 \,\&\&\, b == \frac{-bc - ca}{a}\right)$$

If I'm looking at it right, either of the two possibilities above implies that two of the three factors equal zero. If true, this means that Mathematica is correct in its answer.

9 - 16 Matrix norms, condition numbers
Compute the matrix norm and the condition number corresponding to the l_1 – vector norm.

9. {{2, 1}, {0, 4}}

```
Clear["Global`*"]
```

```
a = {{2, 1}, {0, 4}}
```

```
{{2, 1}, {0, 4}}
```

```
Norm[a, 1]
```

```
5
```

```
LinearAlgebra`MatrixConditionNumber[a]
```

$$\frac{5}{2}$$

11. $\{\{\sqrt{5}, 5\}, \{0, \sqrt{5}\}\}$

```
Clear["Global`*"]
```

$$a = \{\{\sqrt{5}, 5\}, \{0, \sqrt{5}\}\}$$

$$\{\{\sqrt{5}, 5\}, \{0, \sqrt{5}\}\}$$

```
Norm[a, 1]
```

$5 + \sqrt{5}$

```
LinearAlgebra`MatrixConditionNumber[a]
```

$\left(1 + \frac{1}{\sqrt{5}}\right)\left(5 + \sqrt{5}\right)$

```
Simplify[%]
```

$2\left(3 + \sqrt{5}\right)$

From the above, it appears that the matrix norm makes up part of the condition number.

13. {{-2, 4, -1}, {-2, 3, 0}, {7, -12, 2}}

```
Clear["Global`*"]

a = {{-2, 4, -1}, {-2, 3, 0}, {7, -12, 2}}
{{-2, 4, -1}, {-2, 3, 0}, {7, -12, 2}}

Norm[a, 1]
19

LinearAlgebra`MatrixConditionNumber[a]
273

Norm[a, 1] Norm[Inverse[a], 1]
247
```

In this problem I see that the condition number that Mathematica provides can disagree with the one calculated by the definition in the text. According to Wolfram *MathWorld*, inputs such as [a,1], [a,2] or [a,∞] should be acceptable, but Mathematica would not take them. I will now adopt the text procedure for determining the condition number.

15. {{-20, 0, 0}, {0, 0.05, 0}, {0, 0, 20}}

```
Clear["Global`*"]

a = {{-20, 0, 0}, {0, 0.05, 0}, {0, 0, 20}}
{{-20, 0, 0}, {0, 0.05, 0}, {0, 0, 20}}

Norm[a, 1]
20.
```

```
LinearAlgebra`MatrixConditionNumber[a]
400.
```

```
Norm[a, 1] Norm[Inverse[a], 1]
```

```
400.
```

17. Verify (11) for x = {{3,15,-4}} taken with the l_∞-norm and the matrix in problem 13.

```
Clear["Global`*"]
```

Numbered line (11) on p. 867 contains the expression ||**A x**|| ≤ ||**A**|| ||**x**|| . With x identified, A must be

```
A = {{-2, 4, -1}, {-2, 3, 0}, {7, -12, 2}}
{{-2, 4, -1}, {-2, 3, 0}, {7, -12, 2}}
```

The vector x can be vertical or horizontal as far as Mathematica is concerned, but for the product with A, x needs to be vertical.

```
x = {{3}, {15}, {-4}}
{{3}, {15}, {-4}}
```

The double vertical line symbols are meant to signify Norms. So since the problem description mentions the infinity norm, I have to assume that is the desired mode.

```
Norm[A.x, ∞]
```

```
167
```

```
Norm[A, ∞] Norm[x, ∞]
```

```
315
```

Because I got the right answer means, I guess, that I understood the terms of the problem.

19 - 20 Ill-conditioned systems.
Solve A x = b_1, A x = b_2. Compare the solutions and comment. Compute the condition number of A.

```
19. A = {{4.50, 3.55}, {3.55, 2.80}};
 b₁ = {{5.2}, {4.1}} ; b₂ = {{5.2}, 4.0}}
```

```
Clear["Global`*"]
```

```
a = {{4.50, 3.55}, {3.55, 2.80}}
{{4.5, 3.55}, {3.55, 2.8}}
```

```
b1 = {{5.2}, {4.1}}
{{5.2}, {4.1}}

b2 = {{5.2}, {4.0}}
{{5.2}, {4.}}

LinearSolve[a, b1]
{{-2.}, {4.}}

LinearSolve[a, b2]
{{-144.}, {184.}}

LinearAlgebra`MatrixConditionNumber[a]
25921.

Norm[a, 1] Norm[Inverse[a], 1]
25921.
```

From the discussion in the text, a large condition number leaves the system vulnerable to large relative errors in calculation.

21. Residual. For A x = b_1 in problem 19, guess what the residual of $\tilde{x}$ = {{-10.0, 14.1}}^{T}, very poorly approximating {{-2, 4}}^{T}, might be. Then calculate and comment.

According to numbered line (1) on p. 865, $\mathbf{r} = \mathbf{b} - \mathbf{A}\,\tilde{\mathbf{x}}$, where $\tilde{x}$ is an approximate solution. Though I stumbled around, I got there. I found that the example at *https://astro.temple.edu/~d-hill001/course/NUMANAL_SP_2017/Lectures/Section%203_4_Error%20and%20Condition_2017.pdf* helped me.

```
Clear["Global`*"]

x1 = Transpose[{{-2, -4}}]
{{-2}, {-4}}

x2 = Transpose[{{-10.0, 14.1}}]
{{-10.}, {14.1}}
```

I need to pull in the multiplicands from problem 19.

```
a = {{4.50, 3.55}, {3.55, 2.80}}
{{4.5, 3.55}, {3.55, 2.8}}

b1 = {{5.2}, {4.1}}
{{5.2}, {4.1}}
```

```
app = b1 - a.x2
```

```
{{0.145}, {0.12}}
```

23. CAS experiment. Hilbert matrices. The 3×3 Hilbert matrix is

$$\left\{\left\{1, \frac{1}{2}, \frac{1}{3}\right\}, \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right\}, \left\{\frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}\right\}$$

The n×n Hilbert matrix is $H_n = \{\{h_{jk}\}\}$, where $h_{jk} = \frac{1}{(j+k-1)}$. (Similar matrices occur in curve fitting by least squares.) Compute the condition number $\kappa\,(H_n)$ for the matrix norm corresponding to the l_∞- (or l_1-) vector norm, for n = 2,3,...,6 (or further if you wish). Try to find a formula that gives reasonable approximate values of these rapidly growing numbers. Solve a few linear systems of your choice, involving an H_n.

The condition numbers for 1-norm and ∞-norm are the same, which is something I guess the parenthesis in the problem statement is telling me.

```
Grid[
 Table[{n, Norm[HilbertMatrix[n], 1] Norm[Inverse[HilbertMatrix[n]], 1]},
  {n, 2, 6}], Frame → All]
```

2	27
3	748
4	28 375
5	943 656
6	29 070 279

```
Grid[Table[
  {n, Norm[HilbertMatrix[n], ∞] Norm[Inverse[HilbertMatrix[n]], ∞]},
  {n, 2, 6}], Frame → All]
```

2	27
3	748
4	28 375
5	943 656
6	29 070 279

The 2-norm condition numbers are different. Number 5 below is mentioned in Wikipedia.

One of the things the problem asked for was an approximate formula for the Hilbert condition number. At *https://blogs.mathworks.com/cleve/2013/02/02/hilbert-matrices/#73083b00-1b97-4570-a516-31796a031dc4* I found such a formula, but for l_2 only.

$\kappa\,(H_n) \approx 0.01133\, e^{3.49\, n}$

```
Grid[Table[
  {n, N[Norm[HilbertMatrix[n], 2] Norm[Inverse[HilbertMatrix[n]]], 2],
   0.01133 ⅇ^(3.49 n)}, {n, 2, 6}], Frame → All]
```

2	19.	12.1788
3	5.2×10^{2}	399.294
4	1.6×10^{4}	13 091.2
5	4.8×10^{5}	429 209.
6	1.5×10^{7}	1.4072×10^{7}

```
HilbertMatrix[2].b1
```

{{7.25}, {3.96667}}

```
thr = {{-2, 4, -1}, {-2, 3, 0}, {7, -12, 2}}
```

{{-2, 4, -1}, {-2, 3, 0}, {7, -12, 2}}

```
HilbertMatrix[3].thr
```

$$\left\{\left\{-\frac{2}{3}, \frac{3}{2}, -\frac{1}{3}\right\}, \left\{\frac{1}{12}, 0, 0\right\}, \left\{\frac{7}{30}, -\frac{19}{60}, \frac{1}{15}\right\}\right\}$$