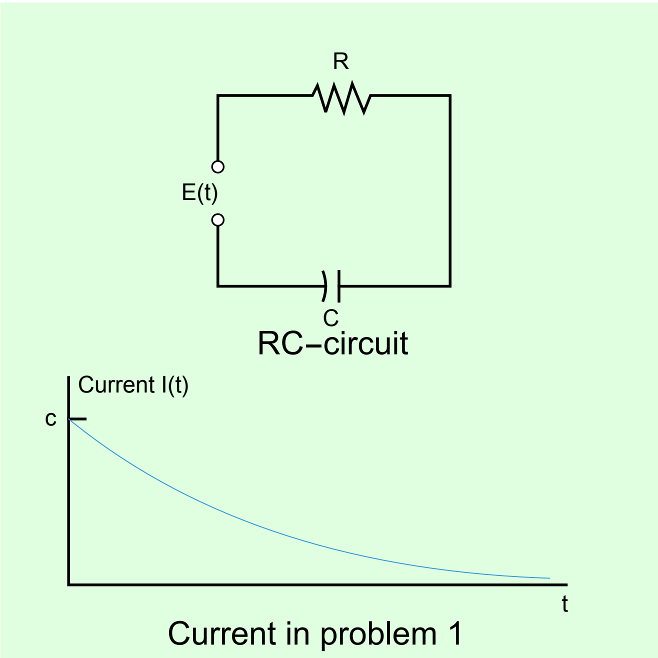
1 - 6 RLC-Circuits: special cases

1. RC-Circuit. Model the RC-Circuit in the figure below. Find the current due to a constant E.



In[9]:=
```
Clear["Global`*"]
```

The problem is asking for a look at RC circuit, not RLC.

The site *https : // www.intmath.com/differential - equations/6 - rc - circuits.php* assumes a constant voltage source, just what the problem specifies. Below: There is no inductance here, only R and C.

In[99]:=
```
eqnw = rR (D[eye[t], t]) + eye[t] / cC == 0
```

Out[99]= $\frac{\text{eye}[t]}{\text{cC}} + \text{rR}\,\text{eye}'[t] == 0$

Within a certain range of capacitance and resistance, the plot resembles the one in the problem description, and can be manipulated to imitate changing parameters, with the voltage remaining constant.

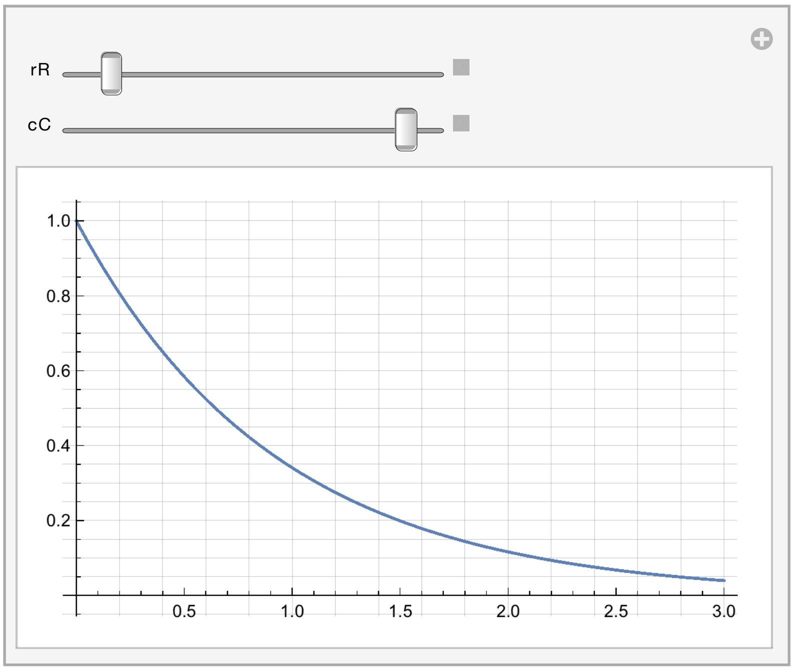
In[100]:=
```
sol2 = DSolve[eqnw, eye, t]
```

Out[100]= $\{\{\text{eye} \to \text{Function}[\{t\}, e^{-\frac{t}{\text{cC rR}}} C[1]]\}\}$

It looks like the current is normalized to 1 at t=0, and the fraction of its max value at a given time needs to be estimated from the underlying grid.

In[101]:=
```
Manipulate[Plot[ⅇ^(-t/(cC rR)), {t, 0, 3}, PlotRange → All, GridLines → All
 ], {rR, 0.2, 10}, {cC, 0.01, 1} ]
```

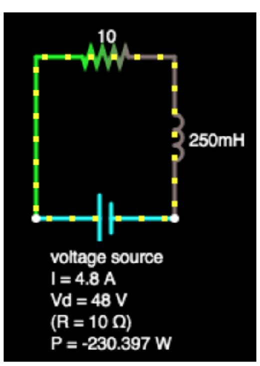
Out[101]=



A random scrap from a different perspective, kept as interesing junk.

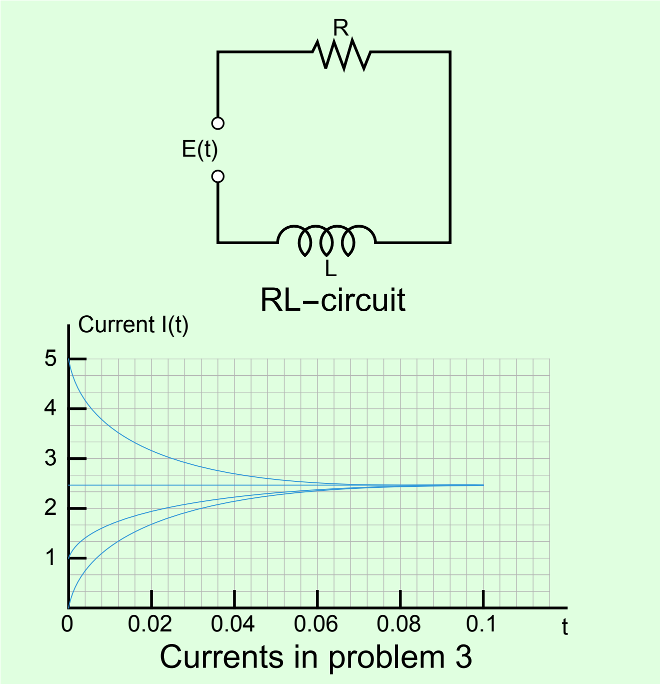
```
{ind, cap, res} = {l i'[t] == v_l[t], v_c'[t] == 1 / c i[t], r i[t] == v_r[t]};
kirchhoff = v_l[t] + v_c[t] + v_r[t] == v_s[t];
```

3. RL-Circuit. Model the RL-circuit in the figure below. Find a general solution when R, L, E are any constants. Graph or sketch solutions when L = 0.25 H, R = 10 Ω, and E = 48 V.



The above screenshot came from the online app at *https://falstad.com/circuit/*. The current it shows agrees with the old formula for current, I=E/R, and was captured after the resistance had plenty of time to decay. And that's all it is, except that there is a time constant to apply. The time constant becomes ever smaller as the operation time increases. Since the problem description talks in terms of a constant state, it seems the time constant would become vanishingly small, leaving merely I=E/R=4.8 amps.

In[205]:=
```
Clear["Global`*"]
```



When there are a lot of variables to watch, the Manipulate command is the only way I know to get an overview. The box below is based on the material at *https://www.electronics-tutorials.ws/inductor/lr-circuits.html* and may not agree with the text in detail.

In[206]:= $\text{eye}[\text{vee\_}, \text{are\_}, \text{ell\_}, \text{tee\_}] = \frac{\text{vee}}{\text{are}}\left(1 - e^{-\frac{\text{are tee}}{\text{ell}}}\right)$

Out[206]= $$\frac{\left(1 - e^{-\frac{\text{are tee}}{\text{ell}}}\right)\text{vee}}{\text{are}}$$

It takes some time for the current to reach its max value. From t=0.4 on in the green grid below, the circuit current is nominal.

In[207]:=
```
Grid[Table[{tee, eye[48, 10, 0.25, tee]}, {tee, 0, 0.6, 0.1}], Frame → All]
```

Out[207]=

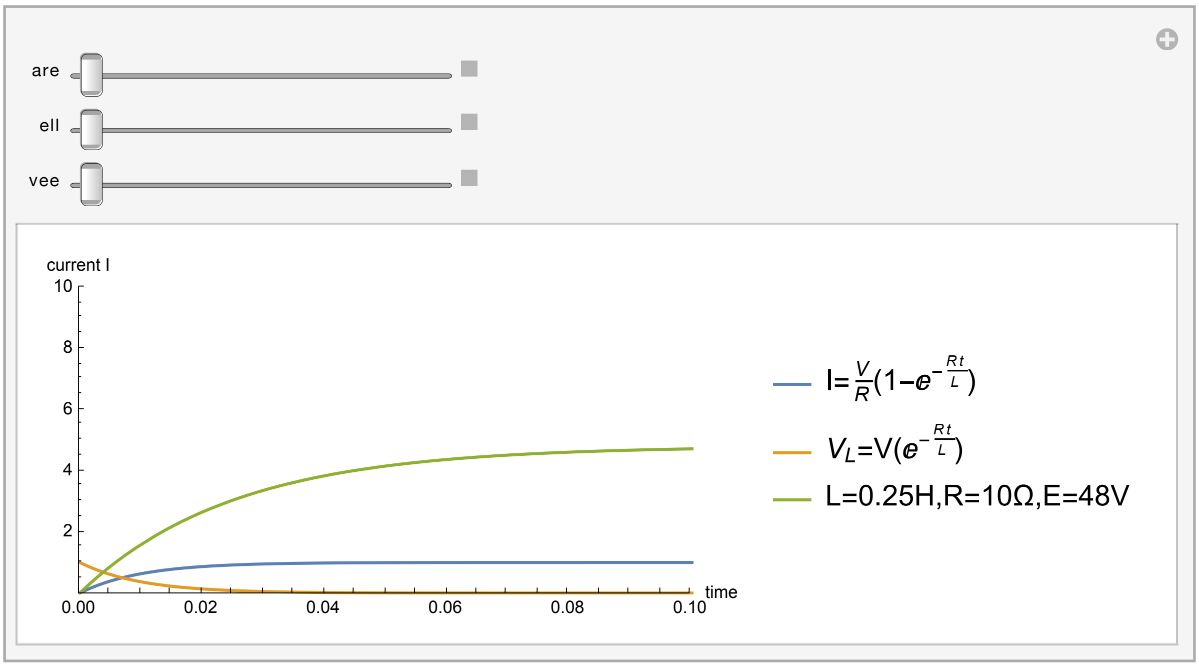
0.	0.
0.1	4.71208
0.2	4.79839
0.3	4.79997
0.4	4.8
0.5	4.8
0.6	4.8

In[208]:=
```
veel[vee_, are_, ell_, tee_] = vee (ⅇ^(-(are tee)/ell))
```

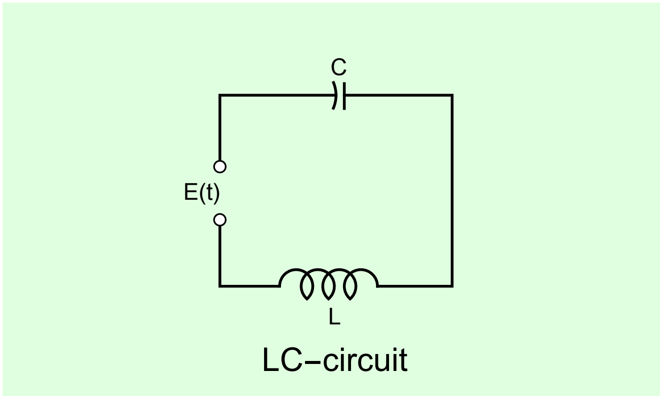
Out[208]= $e^{-\frac{\text{are tee}}{\text{ell}}}$ vee

In[209]:=
```
Manipulate[
 Plot[{Abs[eye[vee, are, ell, tee]], Abs[veel[vee, are, ell, tee]],
   Abs[eye[48, 10, 0.25, tee]]}, {tee, 0, 5},
  PlotLegends → {"I=V/R(1-ⅇ^(-Rt/L))", "V_L=V(ⅇ^(-Rt/L))", "L=0.25H,R=10Ω,E=48V"},
  PlotRange → {{0, 0.1}, {0, 10}}, AxesLabel → {"time", "current I"},
  AspectRatio → 0.5], {are, 1, 200}, {ell, 0.01, 10} , {vee, 1, 50}]
```

Out[209]=



5. LC-Circuit. This is an RLC-circuit with negligibly small R (analog of an undamped mass-spring system). Find the current when L=0.5 H, C = 0.005 F, and E = Sin[t V], assuming zero initial current and charge.



I ran across a couple of snippets, including one from the *Mathematica* documentation, suggesting that state space modeling would be a good way to look at circuits in Mathematica. I use it here.

In[102]:=
```
Clear["Global`*"]
```

In[103]:=
```
eqns = {eL q''[t] + aR q'[t] + 1/cC q[t] == Vee[t]};
```

In[104]:=
```
m1 = StateSpaceModel[eqns,
  {{q[t], 0}, {q'[t], 0}}, {{Vee[t], 0}}, {q'[t]}, t]
```

Out[104]=

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -\frac{1}{cC\ eL} & -\frac{aR}{eL} & \frac{1}{eL} \\ \hline 0 & 1 & 0 \end{array}\right)^{\mathcal{S}}$$

Here I put in the given parameters, taking the opportunity to equate the resistance with zero.

In[115]:=
```
ms = m1 /. {cC → 0.005, eL → 0.5, aR → 0}
```

Out[115]=

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -400. & 0. & 2. \\ \hline 0 & 1 & 0 \end{array}\right)^{\mathcal{S}}$$

The way to get output from a state space model is to use the command **`OutputResponse`**. Since the voltage depends on a periodic function, I drop the V for the input field, the voltage, following the lead of the s.m.. (Although I'm starting to think that the "V" in the problem description may have just been a label.)

In[116]:=
```
outz = OutputResponse[{ms}, Sin[t ], t]
```

Out[116]=
$$\{(1.46082\times 10^{-17} + 0.0526316\, i)\, ((0. + 0.0952381\, i)\, \text{Cos}[20.\, t] - (0. + 1.\, i)\, \text{Cos}[19.\, t]\, \text{Cos}[20.\, t] + (0. + 0.904762\, i)\, \text{Cos}[20.\, t]\, \text{Cos}[21.\, t] - (1.66533\times 10^{-16} - 7.21645\times 10^{-17}\, i)\, \text{Cos}[20.\, t]\, \text{Sin}[19.\, t] - (2.24688\times 10^{-17} + 6.60847\times 10^{-19}\, i)\, \text{Sin}[20.\, t] + (2.35922\times 10^{-16} + 6.93889\times 10^{-18}\, i)\, \text{Cos}[19.\, t]\, \text{Sin}[20.\, t] - (2.13454\times 10^{-16} + 6.27805\times 10^{-18}\, i)\, \text{Cos}[21.\, t]\, \text{Sin}[20.\, t] + (5.96745\times 10^{-17} - 1.\, i)\, \text{Sin}[19.\, t]\, \text{Sin}[20.\, t] + (1.50673\times 10^{-16} - 6.52917\times 10^{-17}\, i)\, \text{Cos}[20.\, t]\, \text{Sin}[21.\, t] - (5.39912\times 10^{-17} - 0.904762\, i)\, \text{Sin}[20.\, t]\, \text{Sin}[21.\, t])\}$$

It is necessary to clean up the result with a small **Chop**.

In[140]:=
$$\text{outt} = \text{Chop}[\text{ComplexExpand}[\text{Re}[\text{outz}]],\, 10^{-16}] \;//\; \text{FullSimplify}$$

Out[140]=
$$\{0.00501253\, \text{Cos}[1.\, t] - 0.00501253\, \text{Cos}[20.\, t] + 3.46945\times 10^{-18}\, \text{Cos}[39.\, t]\}$$

Recognizing the periodic value of cosine, I can get the expression ready for a second chop by doing

In[127]:=
```
outtf = outt /. Cos[39. t] → 1
```

Out[127]=
$$\{3.46945\times 10^{-18} + 0.00501253\, \text{Cos}[1.\, t] - 0.00501253\, \text{Cos}[20.\, t]\}$$

And then the **Chop**.

In[129]:=
$$\text{outtff} = \text{Chop}[\%,\, 10^{-17}]$$

Out[129]=
```
{0.00501253 Cos[1. t] - 0.00501253 Cos[20. t]}
```

Testing the identity of those coefficients

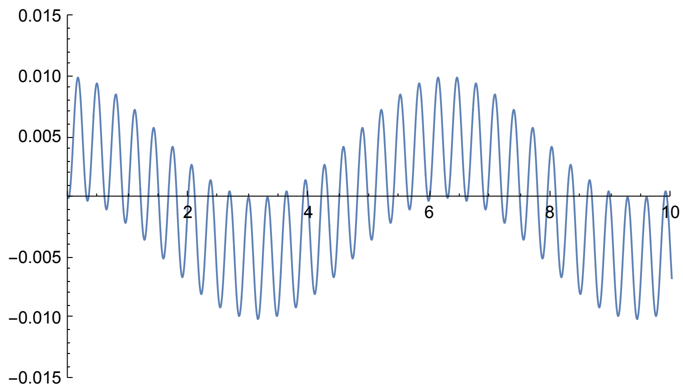
In[118]:=
```
1 / 0.005012531328320802`
```

Out[118]=
```
199.5
```

I find that the answer matches the text answer, justifying the green coloration above.

The plot is interesting.

In[144]:=
```
Plot[outtff, {t, 0, 10}, ImageSize → 350, AspectRatio → 0.6,
 PlotRange → {{-0.01, 10}, {-0.015, 0.015}}, PlotStyle → Thickness[0.003]]
```

Out[144]=


7 - 18 General RLC-circuits

7. Tuning. In tuning a sterio system to a radio station, we adjust the tuning control (turn a knob) that changes C (or perhaps L) in an RLC-circuit so that the amplitude of the steady-state current, numbered line (5), p. 95 becomes maximum. For what C will this happen?

It is where the particular solution of the homogeneous equation is maximized. Numbered line (5) looks like

$I_p\ (t) = I_0 \, \mathrm{Sin}[\omega\, t - \Theta]$

The quantity θ is known as the phase lag, and, I suppose, the signal is best, I_p maximized, when θ equals zero.

8 - 14 Find the steady-state current in the RLC-circuit in the figure below for the given data.

9. R = 4 Ω, L = 0.1 H, C = 0.05 F, E = 110 V

$L\, D[q[t], \{t, 2\}] + R\, D[q[t], t] - \frac{1}{C} q[t] = v[t]$

```
eqn = 0.1 q''[t] + 4 q'[t] - 1/0.05 q[t] == 110
```

```
-20. q[t] + 4 q′[t] + 0.1 q″[t] == 110
```

```
sol = DSolve[eqn, q, t]
```

$\{\{q \to \mathrm{Function}[\{t\}, -5.5 + e^{-44.4949\, t}\, C[1] + e^{4.4949\, t}\, C[2]]\}\}$

If C[1]=C[2]=0, then the green cell above matches the text answer.

11. R = 12 Ω, L = 0.4 H, C = $\frac{1}{80}$F, E = 220 Sin[10 t] V

The state space method has been working where former methods I tried did not, so it makes sense to stick with it.

In[102]:=
```
Clear["Global`*"]
```

In[103]:=
```
eqns = {eL q''[t] + aR q'[t] + 1/cC q[t] == Vee[t]};
```

In[104]:=
```
m1 = StateSpaceModel[eqns,
  {{q[t], 0}, {q'[t], 0}}, {{Vee[t], 0}}, {q'[t]}, t]
```

Out[104]=
$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -\frac{1}{cC\ eL} & -\frac{aR}{eL} & \frac{1}{eL} \\ \hline 0 & 1 & 0 \end{array}\right)^{\mathcal{S}}$$

Here I put in the given parameters.

In[145]:=
```
ms = m1 /. {cC → 1/80, eL → 0.4, aR → 12}
```

Out[145]=
$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -200. & -30. & 2.5 \\ \hline 0 & 1 & 0 \end{array}\right)^{\mathcal{S}}$$

The way to get output from a state space model is to use the command **OutputResponse**.

In[146]:=
```
outz = OutputResponse[{ms}, 220 Sin[10 t ], t]
```

Out[146]=
$$\{0. + e^{-30.\,t}\,(22.\,e^{10.\,t} - 27.5\,e^{20.\,t} - 7.10543\times10^{-15}\,e^{20.\,t}\,\mathrm{Cos}[10.\,t] + 5.5\,e^{30.\,t}\,\mathrm{Cos}[10.\,t] + 7.10543\times10^{-15}\,e^{20.\,t}\,\mathrm{Sin}[10.\,t] + 16.5\,e^{30.\,t}\,\mathrm{Sin}[10.\,t] + 7.10543\times10^{-15}\,e^{40.\,t}\,\mathrm{Sin}[10.\,t])\}$$

It is necessary to clean up the result with a **Chop**.

In[153]:=
```
outt = Chop[outz, 10^-14] // FullSimplify
```

Out[153]=
$$\{22.\,e^{-20.\,t} - 27.5\,e^{-10.\,t} + 5.5\,\mathrm{Cos}[10.\,t] + 16.5\,\mathrm{Sin}[10.\,t]\}$$

I guess the e factors can be dropped if they are small enough, say, at 3 seconds.

In[150]:=
```
N[-27.500000000000007` e^-10.` t] /. t → 3
```

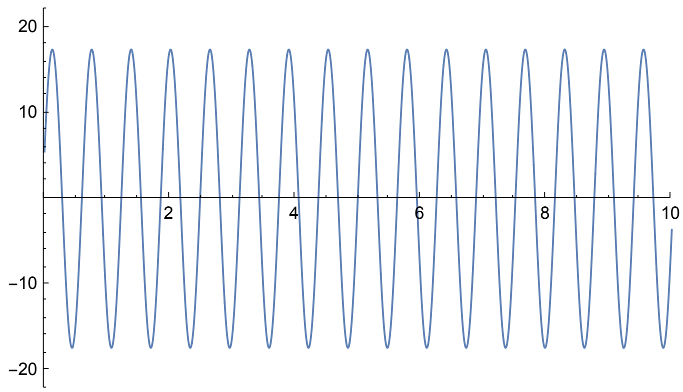
Out[150]=
$$-2.57335\times10^{-12}$$

Evidently the text considers that size to be negligible, leaving

```
5.5 Cos[10. t] + 16.5 Sin[10. t]
```

as the answer. The plot looks routine.

In[157]:=
```
Plot[5.5 Cos[10. t] + 16.5 Sin[10. t],
 {t, 0, 10}, ImageSize → 350, AspectRatio → 0.6,
 PlotRange → {{-0.01, 10}, {-22, 22}}, PlotStyle → Thickness[0.003]]
```

Out[157]=


13. R = 12, L = 1.2 H, C = $\frac{20}{3}*10^{-3}$ F, E = 12,000 Sin[25 t] V

$C = \frac{20}{3} * \frac{1}{1000} = \frac{20}{3000} = \frac{2}{300}$

In[185]:=
```
Clear["Global`*"]
```

In[186]:=
```
eqns = {eL q''[t] + aR q'[t] + 1/cC q[t] == Vee[t]};
```

In[187]:=
```
m1 = StateSpaceModel[eqns,
  {{q[t], 0}, {q'[t], 0}}, {{Vee[t], 0}}, {q'[t]}, t]
```

Out[187]=
$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -\frac{1}{cC\ eL} & -\frac{aR}{eL} & \frac{1}{eL} \\ \hline 0 & 1 & 0 \end{array}\right)^{\mathcal{S}}$$

Here I put in the given parameters.

In[188]:=
```
ms = m1 /. {cC → 20/3 * 10^-3, eL → 1.2, aR → 12}
```

Out[188]=
$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -125. & -10. & 0.833333 \\ \hline 0 & 1 & 0 \end{array}\right)^{\mathcal{S}}$$

The way to get output from a state space model is to use the command **OutputResponse**.

In[189]:=
```
outz = OutputResponse[{ms}, 12 000 Sin[25 t ], t]
```

Out[189]=

$$\{(0. + 0.\,i) - (400. + 1.56319\times10^{-14}\,i)\,e^{-5.\,t}\,((-1. + 0.\,i)\,\text{Cos}[10.\,t] + (1. + 0.\,i)\,e^{5.\,t}\,\text{Cos}[10.\,t]^2\,\text{Cos}[25.\,t] + (0.75 - 4.80505\times10^{-16}\,i)\,\text{Sin}[10.\,t] - (3.19744\times10^{-16} - 3.21521\times10^{-16}\,i)\,e^{5.\,t}\,\text{Cos}[10.\,t]\,\text{Cos}[25.\,t]\,\text{Sin}[10.\,t] + (1. + 2.45581\times10^{-16}\,i)\,e^{5.\,t}\,\text{Cos}[25.\,t]\,\text{Sin}[10.\,t]^2 - (0.5 - 7.49623\times10^{-17}\,i)\,e^{5.\,t}\,\text{Cos}[10.\,t]^2\,\text{Sin}[25.\,t] + (1.42109\times10^{-16} - 9.97247\times10^{-17}\,i)\,e^{5.\,t}\,\text{Cos}[10.\,t]\,\text{Sin}[10.\,t]\,\text{Sin}[25.\,t] - (0.5 - 1.39035\times10^{-16}\,i)\,e^{5.\,t}\,\text{Sin}[10.\,t]^2\,\text{Sin}[25.\,t])\}$$

It is necessary to clean up the result with a **Chop**.

In[192]:=
```
outt = Chop[ComplexExpand[Re[outz]], 10^-15] // Simplify
```

Out[192]=

$$\{-300.\,e^{-5.\,t}\,\text{Sin}[10.\,t] + \text{Cos}[10.\,t]\,(400.\,e^{-5.\,t} + 1.27898\times10^{-13}\,\text{Cos}[25.\,t]\,\text{Sin}[10.\,t]) - 2.84217\times10^{-14}\,\text{Sin}[20.\,t]\,\text{Sin}[25.\,t] + \text{Cos}[10.\,t]^2\,(-400.\,\text{Cos}[25.\,t] + 200.\,\text{Sin}[25.\,t]) + \text{Sin}[10.\,t]^2\,(-400.\,\text{Cos}[25.\,t] + 200.\,\text{Sin}[25.\,t])\}$$

There is a $\sin^2+\cos^2$ trig identity in the above, but I'm going to have to pull it out by hand.

In[194]:=

$$\text{outhnd} = -300.\,e^{-5.\,t}\,\text{Sin}[10.\,t] + \text{Cos}[10.\,t]\,(400.\,e^{-5.\,t}) + (-400.\,\text{Cos}[25.\,t] + 200.\,\text{Sin}[25.\,t])$$

Out[194]=

$$400.\,e^{-5.\,t}\,\text{Cos}[10.\,t] - 400.\,\text{Cos}[25.\,t] - 300.\,e^{-5.\,t}\,\text{Sin}[10.\,t] + 200.\,\text{Sin}[25.\,t]$$

In[195]:=

$$\text{outhnd2} = \text{Collect}[\text{outhnd},\ e^{-5.\,t}]$$

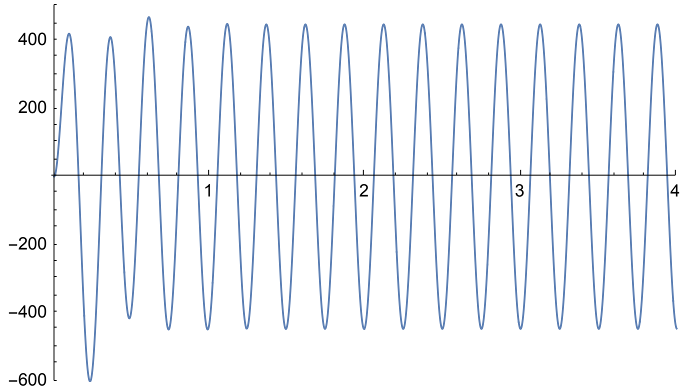
Out[195]=

$$-400.\,\text{Cos}[25.\,t] + e^{-5.\,t}\,(400.\,\text{Cos}[10.\,t] - 300.\,\text{Sin}[10.\,t]) + 200.\,\text{Sin}[25.\,t]$$

While I was pulling things out by hand, I pulled out a choppable term. The text constant B is equal to -300. The text constant A is equal to 1 in one position and 400 in another position. That makes my answer wrong, technically. I guess I should make it yellow, though I don't feel it is a just action to do so. I feel like it is correct.

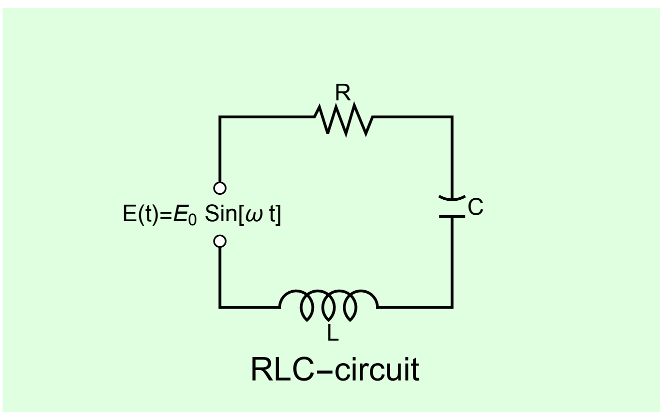
In[204]:=
```
Plot[-400.` Cos[25.` t] + e^(-5.` t) (400.` Cos[10.` t] - 300.` Sin[10.` t]) +
  200.` Sin[25.` t], {t, 0, 4}, ImageSize → 350, AspectRatio → 0.6,
 PlotRange → {{-0.01, 4}, {-600, 500}}, PlotStyle → Thickness[0.003]]
```

Out[204]=



15. Cases of damping. What are the conditions for an RLC-circuit to be (I) overdamped, (II) critically damped, (III) underdamped? What is the critical resistance R_{crit} (the analog of the critical damping constant $2\sqrt{m k}$?

16 - 18 Solve the initial value problem for the RLC-circuit shown below, with the given data, assuming zero initial current and charge. Graph or sketch the solution.



17. R = 6 Ω, L = 1 H, C = 0.04 F, E = 600(Cos[t] + 4 Sin[t])V

In[158]:=
```
Clear["Global`*"]
```

In[159]:=
```
eqns = {eL q''[t] + aR q'[t] + 1/cC q[t] == Vee[t]};
```

In[160]:=
```
m1 = StateSpaceModel[eqns,
  {{q[t], 0}, {q'[t], 0}}, {{Vee[t], 0}}, {q'[t]}, t]
```

Out[160]=
$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -\dfrac{1}{cC\ eL} & -\dfrac{aR}{eL} & \dfrac{1}{eL} \\ \hline 0 & 1 & 0 \end{array}\right)^{\mathcal{S}}$$

Here I put in the given parameters.

In[174]:=
```
ms = m1 /. {cC → 0.04, eL → 1, aR → 6}
```

Out[174]=
$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -25. & -6 & 1 \\ \hline 0 & 1 & 0 \end{array}\right)^{\mathcal{S}}$$

The way to get output from a state space model is to use the command **OutputResponse**.

In[175]:=
```
outz = OutputResponse[{ms}, 600 (Cos[t] + 4 Sin[t]), t]
```

Out[175]=
$$\{(0. + 0.\,i) + e^{-3.\,t}\big((-100. - 1.11022\times10^{-14}\,i)\,\mathrm{Cos}[4.\,t] + (100. + 1.11022\times10^{-14}\,i)\,e^{3.\,t}\,\mathrm{Cos}[t]\,\mathrm{Cos}[4.\,t]^2 - (1.87214\times10^{-14} - 1.65445\times10^{-14}\,i)\,e^{3.\,t}\,\mathrm{Cos}[4.\,t]^2\,\mathrm{Sin}[t] + (75. + 1.52656\times10^{-14}\,i)\,\mathrm{Sin}[4.\,t] - (8.65974\times10^{-15} + 1.80411\times10^{-14}\,i)\,e^{3.\,t}\,\mathrm{Cos}[t]\,\mathrm{Cos}[4.\,t]\,\mathrm{Sin}[4.\,t] + (2.27374\times10^{-13} + 2.91161\times10^{-14}\,i)\,e^{3.\,t}\,\mathrm{Cos}[4.\,t]\,\mathrm{Sin}[t]\,\mathrm{Sin}[4.\,t] + (100. - 1.14492\times10^{-14}\,i)\,e^{3.\,t}\,\mathrm{Cos}[t]\,\mathrm{Sin}[4.\,t]^2 - (0. + 7.91555\times10^{-14}\,i)\,e^{3.\,t}\,\mathrm{Sin}[t]\,\mathrm{Sin}[4.\,t]^2\big)\}$$

In[176]:=
```
outt = Chop[ComplexExpand[Re[outz]]] // Simplify
```

Out[176]=
$$\{-100.\,e^{-3.\,t}\,\mathrm{Cos}[4.\,t] + 100.\,\mathrm{Cos}[t]\,\mathrm{Cos}[4.\,t]^2 + \mathrm{Sin}[4.\,t]\,(75.\,e^{-3.\,t} + 100.\,\mathrm{Cos}[t]\,\mathrm{Sin}[4.\,t])\}$$

In[177]:=
$$\text{outtf} = \text{Collect}[\text{outt}, e^{-3.\,t}]$$

Out[177]=
$$\{100.\,\mathrm{Cos}[t]\,\mathrm{Cos}[4.\,t]^2 + 100.\,\mathrm{Cos}[t]\,\mathrm{Sin}[4.\,t]^2 + e^{-3.\,t}\,(-100.\,\mathrm{Cos}[4.\,t] + 75.\,\mathrm{Sin}[4.\,t])\}$$

I can see the $\sin^2+\cos^2$ identity in the above, but will have to take it out by hand.

$$100.\,\mathrm{Cos}[t] + e^{-3.\,t}\,(-100.\,\mathrm{Cos}[4.\,t] + 75.\,\mathrm{Sin}[4.\,t])$$

And with that, the above cell matches the text answer.

In[184]:=
```
Plot[100. Cos[t] + e^(-3. t) (-100. Cos[4. t] + 75. Sin[4. t]),
 {t, 0, 10}, ImageSize → 350, AspectRatio → 0.6,
 PlotRange → {{-0.01, 10}, {-125, 125}}, PlotStyle → Thickness[0.003]]
```

Out[184]=