Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

4 - 8 Double Fourier Series
Represent f(x,y) by a series (15), where

$$u[x, y, 0] = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\,\text{Sin}\left[\frac{m\,\pi\,x}{a}\right]\,\text{Sin}\left[\frac{n\,\pi\,y}{b}\right] = f[x, y]$$

and

5. f(x,y)=y, a=b=1

```
Clear["Global`*"]
```

For this type of problem, numbered line (15) is shown above. After a little development, the text presents numbered line (18), p.582, which is the **generalized Euler formula**:

$$B_{mn} = \frac{4}{a\,b}\int_0^b\int_0^a f[x, y]\,\text{Sin}\left[\frac{m\,\pi\,x}{a}\right]\,\text{Sin}\left[\frac{n\,\pi\,y}{b}\right]\,dx\,dy \qquad n, m \to 1, 2, \ldots$$

in the case of this problem,

$$\textbf{Bmn} = 4\int_0^1\int_0^1 y\,\text{Sin}[m\,\pi\,x]\,\text{Sin}[n\,\pi\,y]\,dx\,dy$$

$$\frac{4\,(-1+\text{Cos}[m\,\pi])\,(n\,\pi\,\text{Cos}[n\,\pi]-\text{Sin}[n\,\pi])}{m\,n^2\,\pi^3}$$

If m is even then Bmn is zero (because **Cos[m π]** would then equal 1), else if m is odd, then

$$\textbf{Bmno} = \textbf{Bmn} \;/.\; \left\{(-1+\text{Cos}[m\,\pi]) \to -2,\ \text{Cos}[n\,\pi] \to (-1)^{n+1},\ \text{Sin}[n\,\pi] \to 0\right\}$$

$$-\frac{8\,(-1)^{1+n}}{m\,n\,\pi^2}$$

The green cell above matches the text answer for B_{mn}. There is no text answer for u(x,y,0), but it would be the pattern shown above, in cyan, with the restriction that m be odd. The general token $B_{1\,n}$ does not have a negative sign, which I guess is why (-1)$^{n+1}$ was chosen as the formula for the sign of Cos[n π].

7. f(x,y)=x y, a and b arbitrary

$$\textbf{Bmn} = \frac{4}{a\,b}\int_0^b\int_0^a x\,y\,\text{Sin}\left[\frac{m\,\pi\,x}{a}\right]\,\text{Sin}\left[\frac{n\,\pi\,y}{b}\right]\,dx\,dy$$

$$\frac{4\,a\,b\,(m\,\pi\,\text{Cos}[m\,\pi]-\text{Sin}[m\,\pi])\,(n\,\pi\,\text{Cos}[n\,\pi]-\text{Sin}[n\,\pi])}{m^2\,n^2\,\pi^4}$$

The circumstances are not the same as in the last problem. No pattern, even or odd, makes B_{mn} equal zero.

```
Bmno = Bmn /. {Sin[n π] → 0, Sin[m π] → 0}
```

$$\frac{4\,a\,b\,\mathrm{Cos}[m\,\pi]\,\mathrm{Cos}[n\,\pi]}{m\,n\,\pi^2}$$

I hope the above cell would do if required, because I had to cheat by looking at the answer to see the clever device for getting the sign:

```
Bmnf = Bmno /. Cos[m π] Cos[n π] → (-1)^(m+n)
```

$$\frac{4\,(-1)^{m+n}\,a\,b}{m\,n\,\pi^2}$$

The above green cell matches the text answer.

11 - 13 Square Membrane
Find the deflection u(x,y,t) of the square membrane of side π and $c^2 = 1$ for initial velocity 0 and initial deflection

11. 0.1 Sin[2 x] Sin[4 y]

To do this problem I need numbered line (9) on p. 580:

$$\lambda = \lambda_{mn} = c\,\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \qquad m,\ n \to 1,\ 2,\ 3,\ \ldots$$

and numbered line (14) on p. 582 :

$$u[x,\ y,\ t] = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\mathrm{Bmn}\,\mathrm{Cos}[\lambda_{mn}\,t] + \mathrm{Bast}_{mn}\,\mathrm{Sin}[\lambda_{mn}\,t]\right)\mathrm{Sin}\left[\frac{m\,\pi\,x}{a}\right]\mathrm{Sin}\left[\frac{n\,\pi\,y}{b}\right]$$

and numbered line (18) on p. 582 :

$$B_{mn} = \frac{4}{a\,b}\int_0^b\int_0^a f[x,\ y]\,\mathrm{Sin}\left[\frac{m\,\pi\,x}{a}\right]\mathrm{Sin}\left[\frac{n\,\pi\,y}{b}\right]dx\,dy \qquad m,\ n \to 1,\ 2,\ 3,\ \ldots$$

as well as numbered line (19) on p. 583 :

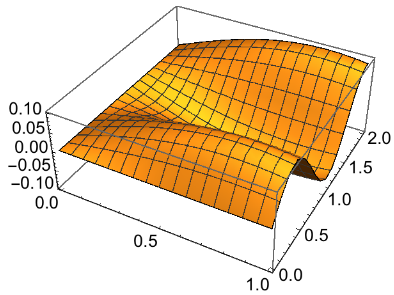
$$\mathrm{Bast}_{mn} = \frac{4}{a\,b\,\lambda_{mn}}\int_0^b\int_0^a g[x,\ y]\,\mathrm{Sin}\left[\frac{m\,\pi\,x}{a}\right]\mathrm{Sin}\left[\frac{n\,\pi\,y}{a}\right]dx\,dy \qquad m,\ n \to 1,\ 2,\ 3\ \ldots$$

```
Clear["Global`*"]
```

The initial displacement is defined.

```
f[x_, y_] = 0.1 Sin[2 x] Sin[4 y]
0.1 Sin[2 x] Sin[4 y]
```

```
Plot3D[f[x, y], {x, 0, 1}, {y, 0, 2}, ImageSize → 200]
```



The Fourier coefficients are computed:

```
a[n_, m_] = Integrate[
   Integrate[f[x, y] Sin[m π y / 3.14], {y, 0, 3.14}] Sin[n π x / 3.14], {x, 0, 3.14}];
Grid[Table[a[n, m], {n, 1, 10}, {m, 1, 10}]]
```

4.51578×10^{-8} + 0. i	-1.12925×10^{-7} + 0. i	2.9066×10^{-7} + 0. i	0.00016688 + 0. i	-3.75232×10^{-7} + 0. i	2.02826×10^{-7} + 0. i	-1.43458×10^{-7} + 0. i	1.12734×10^{-7} + 0. i	-9.36646×10^{-8} + 0. i	8. … 0
0.0000667338 + 0. i	-0.000016688 + 0. i	0.000429534 + 0. i	0.246613 + 0. i	-0.000554514 + 0. i	0.000299734 + 0. i	-0.000212001 + 0. i	0.000166598 + 0. i	-0.000138417 + 0. i	0. … 0
-8.11085×10^{-8} + 0. i	2.02826×10^{-7} + 0. i	-5.22058×10^{-7} + 0. i	-0.000299734 + 0. i	6.73959×10^{-7} + 0. i	-3.64298×10^{-7} + 0. i	2.57667×10^{-7} + 0. i	-2.02484×10^{-7} + 0. i	1.68232×10^{-7} + 0. i	-1 … 0
4.50816×10^{-8} + 0. i	-1.12734×10^{-7} + 0. i	2.90169×10^{-7} + 0. i	0.000166598 + 0. i	-3.74599×10^{-7} + 0. i	2.02484×10^{-7} + 0. i	-1.43216×10^{-7} + 0. i	1.12544×10^{-7} + 0. i	-9.35064×10^{-8} + 0. i	8. … 0

-3.22⋱058 × 10^{-8} + 0. i	8.053⋱63 × 10^{-8} + 0. i	-2.07⋱29⋱4 × 10^{-7} + 0. i	-0.000⋱11⋱90⋱16 + 0. i	2.676⋱09 × 10^{-7} + 0. i	-1.44⋱65⋱2 × 10^{-7} + 0. i	1.023⋱12 × 10^{-7} + 0. i	-8.04⋱00⋱3 × 10^{-8} + 0. i	6.68 × 10^{-8} + 0. i	-5 0
2.536⋱37 × 10^{-8} + 0. i	-6.3⋱42⋱65 × 10^{-8} + 0. i	1.632⋱55 × 10^{-7} + 0. i	0.0000⋱93731 + 0. i	-2.1⋱07⋱56 × 10^{-7} + 0. i	1.139⋱21 × 10^{-7} + 0. i	-8.0⋱57⋱6 × 10^{-8} + 0. i	6.331⋱94 × 10^{-8} + 0. i	-5.2⋱60⋱85 × 10^{-8} + 0. i	4. 0
-2.10⋱433 × 10^{-8} + 0. i	5.262⋱24 × 10^{-8} + 0. i	-1.35⋱44⋱6 × 10^{-7} + 0. i	-0.0000⋱7776⋱49 + 0. i	1.748⋱56 × 10^{-7} + 0. i	-9.45⋱15⋱7 × 10^{-8} + 0. i	6.685⋱06 × 10^{-8} + 0. i	-5.25⋱33⋱6 × 10^{-8} + 0. i	4.364⋱71 × 10^{-8} + 0. i	-3 0
1.803⋱75 × 10^{-8} + 0. i	-4.5⋱10⋱6 × 10^{-8} + 0. i	1.160⋱99 × 10^{-7} + 0. i	0.00006⋱66571 + 0. i	-1.4⋱98⋱8 × 10^{-7} + 0. i	8.101⋱53 × 10^{-8} + 0. i	-5.7⋱30⋱18 × 10^{-8} + 0. i	4.502⋱98 × 10^{-8} + 0. i	-3.7⋱41⋱27 × 10^{-8} + 0. i	3. 0
-1.58⋱123 × 10^{-8} + 0. i	3.954⋱15 × 10^{-8} + 0. i	-1.01⋱77⋱7 × 10^{-7} + 0. i	-0.0000⋱5843⋱41 + 0. i	1.3139 × 10^{-7} + 0. i	-7.10⋱21 × 10^{-8} + 0. i	5.023⋱29 × 10^{-8} + 0. i	-3.94⋱74⋱8 × 10^{-8} + 0. i	3.279⋱73 × 10^{-8} + 0. i	-2 0
1.409⋱22 × 10^{-8} + 0. i	-3.5⋱23⋱99 × 10^{-8} + 0. i	9.070⋱47 × 10^{-8} + 0. i	0.00005⋱20772 + 0. i	-1.1⋱70⋱97 × 10^{-7} + 0. i	6.329⋱48 × 10^{-8} + 0. i	-4.4⋱76⋱82 × 10^{-8} + 0. i	3.518⋱04 × 10^{-8} + 0. i	-2.9⋱22⋱94 × 10^{-8} + 0. i	2. 0

```
(*Lambda[n_,m_]=(n π)^2 + (m π/2)^2;*)
```

The eigenvalues are computed:

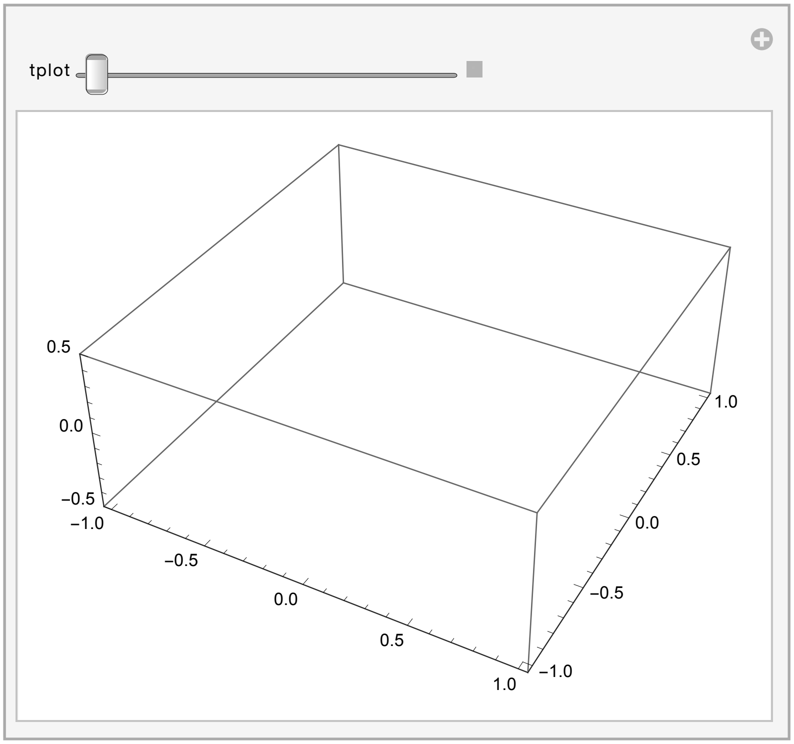
$$\mathbf{Lambda[m\_,\ n\_]} = \pi \sqrt{\frac{m^2}{(3.14)^2} + \frac{n^2}{(3.14)^2}}$$

$$\sqrt{0.101424\, m^2 + 0.101424\, n^2}\ \pi$$

The solution, truncated at N and M, respectively, is given by

```
u[x_, y_, t_, N_, M_] := Sum[Sum[
    a[n, m] Cos[Lambda[n, m] t] Sin[(n π x)/3.14] Sin[(m π y)/3.14], {n, 1, N}], {m, 1, M}]
uplot = u[x, y, t, 10, 20];
```

```
Manipulate[Plot3D[uplot /. t → tplot, {x, 0, 1}, {y, 0, 2},
  PlotRange → {All, All, {-1/2, 1/2}}], {tplot, 0, 5}]
```




```
u[x, y, t, 2, 4]
```

$(4.51578\times10^{-8} + 0.\, i)$ `Cos[1.41493 t] Sin[1.00051 x] Sin[1.00051 y] +`
$(0.0000667338 + 0.\, i)$ `Cos[2.2372 t] Sin[2.00101 x] Sin[1.00051 y] -`
$(1.12925\times10^{-7} + 0.\, i)$ `Cos[2.2372 t] Sin[1.00051 x] Sin[2.00101 y] -`
$(0.00016688 + 0.\, i)$ `Cos[2.82986 t] Sin[2.00101 x] Sin[2.00101 y] +`
$(2.9066\times10^{-7} + 0.\, i)$ `Cos[3.16388 t] Sin[1.00051 x] Sin[3.00152 y] +`
$(0.000429534 + 0.\, i)$ `Cos[3.60738 t] Sin[2.00101 x] Sin[3.00152 y] +`
$(0.00016688 + 0.\, i)$ `Cos[4.1252 t] Sin[1.00051 x] Sin[4.00203 y] +`
$(0.246613 + 0.\, i)$ `Cos[4.4744 t] Sin[2.00101 x] Sin[4.00203 y]`

$(4.4744)^2$

20.0203

Although the above yellow cell is not exactly the text answer, it has the same form. There appears to be something wrong with this problem. Maybe a typo? (Except for the leading coefficient, the last yellow cell line is very close to the text answer.)

****
****

Following example 2 on p. 582,

c = 1

1

To test Mathematica's abilities, B_{mn} is calculted two ways:

$$B_{mn} = \frac{4}{\pi^2} \int_0^\pi \int_0^\pi .1\, \text{Sin}[2\, x]\, \text{Sin}[4\, y]\, \text{Sin}[m\, x]\, \text{Sin}[n\, y]\, dy\, dx$$

$$\frac{0.324228\, \text{Sin}[m\, \pi]\, \text{Sin}[n\, \pi]}{(-4. + m^2)\,(-16. + n^2)}$$

$$\text{Bmny} = \frac{.4}{\pi^2} \int_0^\pi \frac{1}{2}\, (\text{Cos}[(n-4)\, y] - \text{Cos}[(n+4)\, y])\, dy$$

$$\frac{0.162114\, \text{Sin}[n\, \pi]}{-16 + n^2}$$

$$\text{Bmnx} = \int_0^\pi \frac{1}{2}\, (\text{Cos}[(2-m)\, x] - \text{Cos}[(2+m)\, x])\, dx$$

$$\frac{2\, \text{Sin}[m\, \pi]}{-4 + m^2}$$

bmn = Bmny Bmnx

$$\frac{0.324228\, \text{Sin}[m\, \pi]\, \text{Sin}[n\, \pi]}{(-4 + m^2)\,(-16 + n^2)}$$

$$\text{bmnc} = \frac{0.3242277876554809`\, (\text{Cos}[\pi\, (m-n)] - \text{Cos}[\pi\, (m+n)])\, \frac{1}{2}}{(-4 + m^2)\,(-16 + n^2)}$$

$$\frac{0.162114\, (\text{Cos}[(m-n)\, \pi] - \text{Cos}[(m+n)\, \pi])}{(-4 + m^2)\,(-16 + n^2)}$$

bmnc makes use of a trig identity that I hoped would help out with the usability of bm.

$$\lambda_{mn} = \text{Simplify}\left[c\, \pi \sqrt{\frac{m^2}{\pi^2} + \frac{n^2}{\pi^2}}\,\right]$$

$$\sqrt{m^2 + n^2}$$

```
uxyt2 =
 Sum[Sum[bmnc (1/2 (Cos[m x - n y] - Cos[m x + n y])) Cos[λmn t], {m, 1, ∞, 2}],
  {n, 1, ∞, 2}]
```

$$\text{Sum}\left[\text{Sum}\left[\left(0.0810569\,(\text{Cos}[(m-n)\,\pi] - \text{Cos}[(m+n)\,\pi])\,\text{Cos}\left[\sqrt{m^2+n^2}\;t\right]\,(\text{Cos}[m\,x - n\,y] - \text{Cos}[m\,x + n\,y])\right)\Big/\left(\left(-4+m^2\right)\left(-16+n^2\right)\right),\ \{m,1,\infty,2\}\right],\ \{n,1,\infty,2\}\right]$$

The trig identity is used again. But there are still problems.

I was not able to get the text answer, which looks very nice. The yellow cell above, which was assembled using the rules as best as I was able to apply them, looks very messy.

$$u[x\_,\,y\_,\,t\_,\,m\_,\,n\_,\,j\_,\,k\_] := \text{Sum}\left[\text{Sum}\left[\frac{1}{\left(-4+m^2\right)\left(-16+n^2\right)}\,0.08105694691387022\grave{}\;(\text{Cos}[(m-n)\,\pi] - \text{Cos}[(m+n)\,\pi])\,\text{Cos}\left[\sqrt{m^2+n^2}\;t\right]\,(\text{Cos}[m\,x - n\,y] - \text{Cos}[m\,x + n\,y]),\ \{m,1,\infty,2\}\right],\ \{n,1,\infty,2\}\right]$$

It's not blowing up at the moment, but I have little hope of verifying it.

```
u[x, y, t, m, n, 1, 1]
```

$$\text{Sum}\left[\text{Sum}\left[\left(0.0810569\,(\text{Cos}[(m-n)\,\pi] - \text{Cos}[(m+n)\,\pi])\,\text{Cos}\left[\sqrt{m^2+n^2}\;t\right]\,(\text{Cos}[m\,x - n\,y] - \text{Cos}[m\,x + n\,y])\right)\Big/\left(\left(-4+m^2\right)\left(-16+n^2\right)\right),\ \{m,1,\infty,2\}\right],\ \{n,1,\infty,2\}\right]$$

****
****

$$\text{partialuo}[j\_,\,k\_,\,x\_,\,y\_,\,t\_] := \text{Sum}\left[0.1\ \text{Sin}[m\,x]\ \text{Sin}[n\,y]\ \text{Cos}\left[\sqrt{m^2+n^2}\;t\right],\ \{m,j,j,2\},\ \{n,k,k,2\}\right]$$

```
part1 = partialuo[2, 4, x, y, t]
```

$$0.1\,\text{Cos}\left[2\sqrt{5}\;t\right]\ \text{Sin}[2\,x]\ \text{Sin}[4\,y]$$

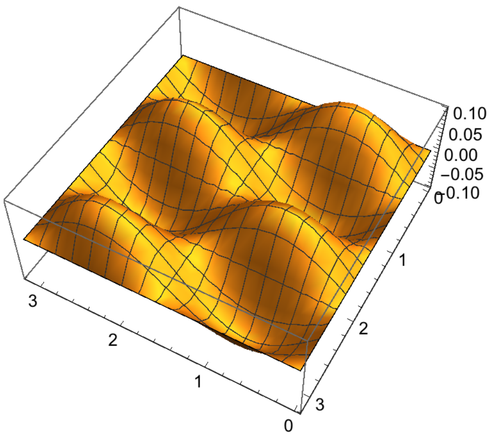
Just playing around, trying to get the text answer, I came up with the above yellow cell. The below yellow cell also gets there. No justification for this entertainment unfortunately.

```
f[x_, y_] = 0.1 Sin[2 x] Sin[4 y]
0.1 Sin[2 x] Sin[4 y]
```

```
partialuo2[j_, k_, x_, y_, t_] :=
 Sum[f[x, y] Cos[Sqrt[m^2 + n^2] t], {m, j, j, 2}, {n, k, k, 2}]
part2 = partialuo[2, 4, x, y, t]
```

$0.1 \cos[2\sqrt{5}\, t] \sin[2 x] \sin[4 y]$

```
uti = 0.1 Sin[2 x] Sin[4 y];
Plot3D[Evaluate[uti], {x, 0, Pi}, {y, 0, Pi},
 PlotPoints → {20, 20}, ViewPoint → {-1.5, 3, 0.5}]
```



13. 0.1 x y(π - x)(π - y)

```
Clear["Global`*"]
```

$c = 1;\ \lambda_{mn} = \text{Simplify}\left[c\,\pi\sqrt{\frac{m^2}{\pi^2} + \frac{n^2}{\pi^2}}\,\right]$

$\sqrt{m^2 + n^2}$

$f[x\_, y\_] = 0.1\, x\, y\, (\pi - x)\, (\pi - y)$

$0.1\, (\pi - x)\, x\, (\pi - y)\, y$

The expression in the cell below is not complete, because I have pulled out a factor of $.4/\pi^2$ to save for later.

$\text{Bmn1} = \int_0^\pi \int_0^\pi x\, y\, (\pi - x)\, (\pi - y) \sin[m\, x] \sin[n\, y]\, dx\, dy$

$$\frac{(-2 + 2\cos[m\,\pi] + m\,\pi \sin[m\,\pi])\ (-2 + 2\cos[n\,\pi] + n\,\pi \sin[n\,\pi])}{m^3\, n^3}$$

Now as for the cell above: I want to save the denominator, and to evaluate the numerator:

```
Sum[(-2 + 2 Cos[m π] + m π Sin[m π]) (-2 + 2 Cos[n π] + n π Sin[n π]),
 {m, 1, 1, 2}, {n, 1, 1, 2}]
16

6.4 / .4
16.
```

Combining the operations above means that I have a total leading factor now of $\frac{6.4}{\pi^2}$, and all that is left of Bmnl is $\frac{1}{m^3 n^3}$.

$$\text{outeq} = \text{Simplify}\left[\text{partialu}[j\_, k\_, x\_, y\_, t\_] := \sum_{n=1}^{k}\sum_{m=1}^{j} \text{Bmn}\,\text{Sin}[m\,x]\,\text{Sin}[n\,y]\,\text{Cos}\left[\sqrt{m^2+n^2}\;t\right],\ \text{Assumptions} \to \{m, n \in \text{OddQ}\}\right]$$

$$\text{outeq1} = \text{Simplify}\left[\text{partialu}[j\_, k\_, x\_, y\_, t\_] := \frac{6.4}{\pi^2}\sum_{n=1}^{k}\sum_{m=1}^{j} \frac{1}{m^3\,n^3}\,\text{Sin}[m\,x]\,\text{Sin}[n\,y]\,\text{Cos}\left[\sqrt{m^2+n^2}\;t\right],\ \text{Assumptions} \to \{m, n \in \text{OddQ}\}\right]$$

The above green cell matches the answer of the text.

14 - 19 Rectangular Membrane

17. Find eigenvalues of the rectangular membrane of sides a = 2 and b = 1 to which there correspond two or more different (independent) eigenfunctions.

```
Clear["Global`*"]
```

$$\lambda_{mn} = c\,\pi\sqrt{\frac{m^2}{4}+\frac{n^2}{1}}$$

$$c\sqrt{\frac{m^2}{4}+n^2}\;\pi$$

$$\text{eig}[m\_, n\_] = \frac{m^2}{4}+n^2$$

$$\frac{m^2}{4}+n^2$$

```
Solve[m^2/4 == n^2, {m, n}]
```

Solve::svars: Equations may not give solutions for all "solve" variables ≫

$\{\{n \to -\frac{m}{2}\}, \{n \to \frac{m}{2}\}\}$

```
Table[eig[m, n], {m, {4, 16}}, {n, {16, 14}}]
{{260, 200}, {320, 260}}

eig[10, 5]
50

Table[eig[m, n], {m, {2, 4, 6, 8}}, {n, {1, 2, 3, 4}}]
{{2, 5, 10, 17}, {5, 8, 13, 20}, {10, 13, 18, 25}, {17, 20, 25, 32}}
```

So for example $\lambda_{8,3} = \lambda_{6,4} = c\sqrt{25}\,\pi$. These are much smaller m,n than the text uses in its answer, but if I understand correctly, they are okay.

19. Deflection. Find the deflection of the membrane of sides a and b with $c^2 = 1$ for the initial deflection
$f(x, y) = \text{Sin}\left[\frac{6\pi x}{a}\right] \text{Sin}\left[\frac{2\pi y}{b}\right]$ and initial velocity 0.

```
Clear["Global`*"]
```

$$c = 1;\ \lambda_{mn} = \text{Simplify}\left[c\,\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}\right]$$

$$\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}\,\pi$$

$$f[x\_, y\_] = \text{Sin}\left[\frac{6\pi x}{a}\right] \text{Sin}\left[\frac{2\pi y}{b}\right]$$

$$\text{Sin}\left[\frac{6\pi x}{a}\right] \text{Sin}\left[\frac{2\pi y}{b}\right]$$

$$B_{mn} = \frac{4}{a\,b}\int_0^b\int_0^a f[x, y]\,\text{Sin}\left[\frac{m\pi x}{a}\right] \text{Sin}\left[\frac{n\pi y}{b}\right] dx\,dy$$

$$\frac{48\,\text{Sin}[m\pi]\,\text{Sin}[n\pi]}{(-36 + m^2)(-4 + n^2)\,\pi^2}$$

$$u[x\_, y\_, t\_, j\_, k\_] = \sum_{m=1}^{j}\sum_{n=1}^{k}(B_{mn}\,\text{Cos}[\lambda_{mn}\,t])\,\text{Sin}\left[\frac{m\pi x}{a}\right] \text{Sin}\left[\frac{n\pi y}{b}\right]$$

0

This problem has exactly the same problem as problem 11, and I have not yet figured out how to avoid it.