Green cells below match corresponding answers in the text.

1 - 5 Legendre polynomials and functions

5. Obtain P_6 and P_7.

```
ClearAll["Global`*"]
```

It turns out that Legendre polynomials are available from a built-in command.

```
LegendreP[6, x]
```

$$\frac{1}{16} \left(-5 + 105 \, x^2 - 315 \, x^4 + 231 \, x^6\right)$$

```
LegendreP[7, x]
```

$$\frac{1}{16} \left(-35 \, x + 315 \, x^3 - 693 \, x^5 + 429 \, x^7\right)$$

11 - 15 Further formulas

11. ODE. Find a solution of $\left(a^2 - x^2\right) y'' - 2 \, x \, y' + n \, (n + 1) \, y = 0$, $a \neq 0$, by reduction to the Legendre equation.

```
ClearAll["Global`*"]
```

```
eqn = (a² - x²) y''[x] - 2 x y'[x] + n (n + 1) y[x] == 0
```

$$n \, (1 + n) \, y[x] - 2 \, x \, y'[x] + \left(a^2 - x^2\right) y''[x] == 0$$

```
sol = DSolve[eqn, y, x, Assumptions → a ≠ 0]
```

$$\left\{\left\{y \to \text{Function}\left[\{x\}, \, C[1] \, \text{LegendreP}\left[n, \, \frac{x}{a}\right] + C[2] \, \text{LegendreQ}\left[n, \, \frac{x}{a}\right]\right]\right\}\right\}$$

```
sol1 = sol /. {C[1] → 1, C[2] → 1, n → 1, a → 1}
```

$$\left\{\left\{y \to \text{Function}\left[\{x\}, \, 1 \, \text{LegendreP}\left[1, \, \frac{1}{1} x\right] + 1 \, \text{LegendreQ}\left[1, \, \frac{1}{1} x\right]\right]\right\}\right\}$$

$$\text{LegendreP}\left[1, \, \frac{1}{1} x\right] + 1 \, \text{LegendreQ}\left[1, \, \frac{1}{1} x\right]$$

$$-1 + x + x \left(-\frac{1}{2} \, \text{Log}[1 - x] + \frac{1}{2} \, \text{Log}[1 + x]\right)$$

15. Associated Legendre functions $P_n^k[x]$ are needed, e.g. in quantum physics. They are defined by $P_n^k[x] = \left(1 - x^2\right)^{k/2} \frac{d^k \, p_n[x]}{dx^k}$ and are solutions of the ODE

$\left(1 - x^2\right) y'' - 2 x y' + q[x] y = 0$ where $q[x] = n(n+1) - k^2 / \left(1 - x^2\right)$. Find $P_1^1[x]$, $P_2^1[x]$, $P_2^2[x]$, and $P_4^2[x]$ and verify that they satisfy numbered line (16) in yellow above.

$P_1^1[x] = \left(1 - x^2\right)^{1/2} \dfrac{d^1 p_1[x]}{d x^1}$

LegendreP[1, x]

x

$P_1^1[x] = \left(1 - x^2\right)^{1/2}$

$P_2^1[x] = \left(1 - x^2\right)^{1/2} \dfrac{d^1 p_2[x]}{d x^1}$

LegendreP[2, x]

$\dfrac{1}{2} \left(-1 + 3 x^2\right)$

D[%, x]

3 x

$\left(1 - x^2\right)^{1/2} * \%$

$3 x \sqrt{1 - x^2}$

$P_2^2[x] = \left(1 - x^2\right)^{2/2} \dfrac{d^2 p_2[x]}{d x^2}$

LegendreP[2, x]

$\dfrac{1}{2} \left(-1 + 3 x^2\right)$

D[%, {x, 2}]

3

$3 \left(1 - x^2\right)$

$P_4^2[x] = \left(1 - x^2\right)^{2/2} \dfrac{d^2 p_4[x]}{d x^2}$

LegendreP[4, x]

$\dfrac{1}{8} \left(3 - 30 x^2 + 35 x^4\right)$

D[%, {x, 2}]

$$\frac{1}{8}\left(-60 + 420\, x^2\right)$$

FullSimplify$\left[\% * \left(1 - x^2\right)\right]$

$$-\frac{15}{2}\left(1 - 8\, x^2 + 7\, x^4\right)$$

PossibleZeroQ$\left[-\frac{15}{2}\left(1 - 8\, x^2 + 7\, x^4\right) - \left(\left(1 - x^2\right)\left(105\, x^2 - 15\right)\big/2\right)\right]$

True

The green cells above match the answers in the text. (The last, though slightly different in format, is verified by the cyan cell.)