2 - 5 Review: radius of convergence

3. $\sum_{m=0}^{\infty}\left(\frac{-1}{k}\right)^m x^{2m}$

```
Clear["Global`*"]
```

$$\text{Sum}\left[\frac{(-1)^m}{k^m}x^{2m},\ \{m,0,\infty\},\ \text{GenerateConditions}\to\text{True}\right]$$

$$\text{ConditionalExpression}\left[\frac{k}{k+x^2},\ \text{Abs}[k]>\text{Abs}[x]^2\ \&\&\ k\neq 0\ \&\&\ k+x^2\neq 0\ \&\&\ 1+\frac{x^2}{k}\neq 0\right]$$

$$\text{SumConvergence}\left[\frac{(-1)^m}{k^m}x^{2m},\ m\right]$$

$$\text{Abs}[k]>\text{Abs}[x]^2$$

1. Above: This does not look exactly like the answer, but I believe it says the same thing.

5. $\sum_{m=0}^{\infty}\left(\frac{2}{3}\right)^m x^{2m}$

```
Clear["Global`*"]
```

$$\text{SumConvergence}\left[\left(\frac{2}{3}\right)^m x^{2m},\ m\right]$$

$$\text{Abs}[x]<\sqrt{\frac{3}{2}}$$

The answer in the green cells above match the answers in the text.

6 - 9 Series solutions by hand

Apply the power series method. Do this by hand, not by a CAS, to get a feel for the method, e.g. why a series may terminate, or has even powers only, etc.

7. y' = -2 x y

```
Clear["Global`*"]
```

```
e1 = DSolve[y'[x] == -2 x y[x], y[x], x]
```

$$\{\{y[x]\to e^{-x^2}\,C[1]\}\}$$

```
e2 = e1 /. C[1] → a0
```
$\{\{y[x] \to e^{-x^2} a_0\}\}$

```
e3 =
 Series[a0 e^(-x^2), {x, 0, 8}]
```
$$a_0 - a_0 x^2 + \frac{a_0 x^4}{2} - \frac{a_0 x^6}{6} + \frac{a_0 x^8}{24} + O[x]^9$$

```
e4 = Collect[e3, a0]
```
$$\left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24}\right) a_0$$

The answer in the green cells above match the answers in the text.

9. y'' + y = 0

```
Clear["Global`*"]
```

```
e1 = DSolve[y''[x] + y[x] == 0, y[x], x]
```
{{y[x] → C[1] Cos[x] + C[2] Sin[x]}}

```
e2 = e1 /. {C[1] → a0, C[2] → a1}
```
$\{\{y[x] \to \text{Cos}[x]\, a_0 + \text{Sin}[x]\, a_1\}\}$

```
e3 = e2[[1, 1, 2]]
```
$\text{Cos}[x]\, a_0 + \text{Sin}[x]\, a_1$

```
e4 = Series[e3, {x, 0, 8}]
```
$$a_0 + a_1 x - \frac{a_0 x^2}{2} - \frac{a_1 x^3}{6} + \frac{a_0 x^4}{24} + \frac{a_1 x^5}{120} - \frac{a_0 x^6}{720} - \frac{a_1 x^7}{5040} + \frac{a_0 x^8}{40320} + O[x]^9$$

The answer in the green cells above match the answers in the text.

10 - 14 Series solutions

Find a power series solution in powers of x.

11. $y'' - y' + x^2 y = 0$

```
Clear["Global`*"]
```

```
e1 = y[x_] = Sum[am x^m, {m, 0, 6}]
```
$a_0 + x a_1 + x^2 a_2 + x^3 a_3 + x^4 a_4 + x^5 a_5 + x^6 a_6$

```
e2 = y'[x]
```
$a_1 + 2 x a_2 + 3 x^2 a_3 + 4 x^3 a_4 + 5 x^4 a_5 + 6 x^5 a_6$

```
e3 = y''[x]
```

$2 a_2 + 6 x a_3 + 12 x^2 a_4 + 20 x^3 a_5 + 30 x^4 a_6$

```
e6 = y''[x] - y'[x] + x^2 y[x] == 0
```

$-a_1 + 2 a_2 - 2 x a_2 + 6 x a_3 - 3 x^2 a_3 + 12 x^2 a_4 - 4 x^3 a_4 + 20 x^3 a_5 - 5 x^4 a_5 + 30 x^4 a_6 - 6 x^5 a_6 + x^2 \left(a_0 + x a_1 + x^2 a_2 + x^3 a_3 + x^4 a_4 + x^5 a_5 + x^6 a_6\right) == 0$

```
e7 = Expand[e6]
```

$x^2 a_0 - a_1 + x^3 a_1 + 2 a_2 - 2 x a_2 + x^4 a_2 + 6 x a_3 - 3 x^2 a_3 + x^5 a_3 + 12 x^2 a_4 - 4 x^3 a_4 + x^6 a_4 + 20 x^3 a_5 - 5 x^4 a_5 + x^7 a_5 + 30 x^4 a_6 - 6 x^5 a_6 + x^8 a_6 == 0$

```
e8 = Collect[e7, x]
```

$-a_1 + 2 a_2 + x (-2 a_2 + 6 a_3) + x^6 a_4 + x^2 (a_0 - 3 a_3 + 12 a_4) + x^7 a_5 + x^3 (a_1 - 4 a_4 + 20 a_5) + x^5 (a_3 - 6 a_6) + x^8 a_6 + x^4 (a_2 - 5 a_5 + 30 a_6) == 0$

$e9 = \text{Solve}[2 a_2 == a_1, a_2]$

$\left\{\left\{a_2 \to \frac{a_1}{2}\right\}\right\}$

Above: x^0

$e11 = \text{Solve}[2 a_2 == 6 a_3, a_3] \text{ /. } a_2 \text{ -> } \frac{a_1}{2}$

$\left\{\left\{a_3 \to \frac{a_1}{6}\right\}\right\}$

Above: x^1

$e13 = \text{Expand}\left[\text{Solve}[a_0 - 3 a_3 + 12 a_4 == 0, a_4] \text{ /. } a_3 \to \frac{a_1}{6}\right]$

$\left\{\left\{a_4 \to -\frac{a_0}{12} + \frac{a_1}{24}\right\}\right\}$

Above: x^2

$e14 = \text{Simplify}\left[\text{Solve}[a_1 - 4 a_4 + 20 a_5 == 0, a_5] \text{ /. } a_4 \text{ -> } \frac{1}{12}\left(-a_0 + \frac{a_1}{2}\right)\right]$

$\left\{\left\{a_5 \to \frac{1}{120} (-2 a_0 - 5 a_1)\right\}\right\}$

Above: x^3

$e15 = \text{Simplify}\left[\text{Solve}[a_2 - 5 a_5 + 30 a_6 == 0, a_6] \text{ /. } \left\{a_5 \text{ -> } \frac{1}{120} (-2 a_0 - 5 a_1), a_2 \to \frac{a_1}{2}\right\}\right]$

$\left\{\left\{a_6 \to \frac{1}{720} (-2 a_0 - 17 a_1)\right\}\right\}$

$$e16 = y[x] \mathbin{/.} \left\{a_2 \to \frac{a_1}{2},\ a_3 \to \frac{a_1}{6},\ a_4 \to -\frac{a_0}{12} + \frac{a_1}{24},\ a_5 \to \frac{1}{120}(-2\,a_0 - 5\,a_1),\ a_6 \to \frac{1}{720}(-2\,a_0 - 17\,a_1)\right\}$$

$$a_0 + \frac{1}{720} x^6 (-2\,a_0 - 17\,a_1) + \frac{1}{120} x^5 (-2\,a_0 - 5\,a_1) + x^4 \left(-\frac{a_0}{12} + \frac{a_1}{24}\right) + x\,a_1 + \frac{x^2\,a_1}{2} + \frac{x^3\,a_1}{6}$$

$$e17 = \text{Collect}[e16, \{a_0, a_1\}]$$

$$\left(1 - \frac{x^4}{12} - \frac{x^5}{60} - \frac{x^6}{360}\right) a_0 + \left(x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{24} - \frac{17\,x^6}{720}\right) a_1$$

Above: The answer in the green cell matches the text answer.

13. $y'' + (1 + x^2)\, y = 0$

```
Clear["Global`*"]
```

$$e1 = y[x\_] = \text{Sum}[a_m\, x^m, \{m, 0, 7\}]$$

$$a_0 + x\,a_1 + x^2\,a_2 + x^3\,a_3 + x^4\,a_4 + x^5\,a_5 + x^6\,a_6 + x^7\,a_7$$

$$e2 = y'[x]$$

$$a_1 + 2\,x\,a_2 + 3\,x^2\,a_3 + 4\,x^3\,a_4 + 5\,x^4\,a_5 + 6\,x^5\,a_6 + 7\,x^6\,a_7$$

$$e3 = y''[x]$$

$$2\,a_2 + 6\,x\,a_3 + 12\,x^2\,a_4 + 20\,x^3\,a_5 + 30\,x^4\,a_6 + 42\,x^5\,a_7$$

$$e4 = y''[x] + (1 + x^2)\, y[x] == 0$$

$$2\,a_2 + 6\,x\,a_3 + 12\,x^2\,a_4 + 20\,x^3\,a_5 + 30\,x^4\,a_6 + 42\,x^5\,a_7 + (1 + x^2)\,(a_0 + x\,a_1 + x^2\,a_2 + x^3\,a_3 + x^4\,a_4 + x^5\,a_5 + x^6\,a_6 + x^7\,a_7) == 0$$

$$e5 = \text{Expand}[e4]$$

$$a_0 + x^2\,a_0 + x\,a_1 + x^3\,a_1 + 2\,a_2 + x^2\,a_2 + x^4\,a_2 + 6\,x\,a_3 + x^3\,a_3 + x^5\,a_3 + 12\,x^2\,a_4 + x^4\,a_4 + x^6\,a_4 + 20\,x^3\,a_5 + x^5\,a_5 + x^7\,a_5 + 30\,x^4\,a_6 + x^6\,a_6 + x^8\,a_6 + 42\,x^5\,a_7 + x^7\,a_7 + x^9\,a_7 == 0$$

$$e6 = \text{Collect}[e5, x]$$

$$a_0 + 2\,a_2 + x\,(a_1 + 6\,a_3) + x^2\,(a_0 + a_2 + 12\,a_4) + x^3\,(a_1 + a_3 + 20\,a_5) + x^8\,a_6 + x^6\,(a_4 + a_6) + x^4\,(a_2 + a_4 + 30\,a_6) + x^9\,a_7 + x^7\,(a_5 + a_7) + x^5\,(a_3 + a_5 + 42\,a_7) == 0$$

$$e7 = \text{Solve}[a_0 + 2\,a_2 == 0, a_2]$$

$$\left\{\left\{a_2 \to -\frac{a_0}{2}\right\}\right\}$$

$$e8 = \text{Solve}[a_1 + 6\, a_3 == 0,\ a_3]$$

$$\left\{\left\{a_3 \to -\frac{a_1}{6}\right\}\right\}$$

$$e9 = \text{Solve}[a_0 + a_2 + 12\, a_4 == 0,\ a_4]\ /.\ a_2 \to -\frac{a_0}{2}$$

$$\left\{\left\{a_4 \to -\frac{a_0}{24}\right\}\right\}$$

Above: x^2

$$e10 = \text{Solve}[a_1 + a_3 + 20\, a_5 == 0,\ a_5]\ /.\ a_3 \to -\frac{a_1}{6}$$

$$\left\{\left\{a_5 \to -\frac{a_1}{24}\right\}\right\}$$

Above: x^3

$$e11 = \text{Solve}[a_2 + a_4 + 30\, a_6 == 0,\ a_6]\ /.\ \left\{a_2 \to -\frac{a_0}{2},\ a_4 \to -\frac{a_0}{24}\right\}$$

$$\left\{\left\{a_6 \to \frac{13\, a_0}{720}\right\}\right\}$$

Above: x^4

$$e12 = \text{Solve}[a_3 + a_5 + 42\, a_7 == 0,\ a_7]\ /.\ \left\{a_3 \to -\frac{a_1}{6},\ a_5 \to -\frac{a_1}{24}\right\}$$

$$\left\{\left\{a_7 \to \frac{5\, a_1}{1008}\right\}\right\}$$

Above: x^5

$$e12 = y[x]\ /.\ \left\{a_2 \to -\frac{a_0}{2},\ a_3 \to -\frac{a_1}{6},\ a_4 \to -\frac{a_0}{24},\ a_5 \to -\frac{a_1}{24},\ a_6 \to \frac{13\, a_0}{720},\ a_7 \to \frac{5\, a_1}{1008}\right\}$$

$$a_0 - \frac{x^2\, a_0}{2} - \frac{x^4\, a_0}{24} + \frac{13\, x^6\, a_0}{720} + x\, a_1 - \frac{x^3\, a_1}{6} - \frac{x^5\, a_1}{24} + \frac{5\, x^7\, a_1}{1008}$$

$$e13 = \text{Collect}[e12,\ \{a_0,\ a_1\}]$$

$$\left(1 - \frac{x^2}{2} - \frac{x^4}{24} + \frac{13\, x^6}{720}\right) a_0 + \left(x - \frac{x^3}{6} - \frac{x^5}{24} + \frac{5\, x^7}{1008}\right) a_1$$

$$e14 = \text{Normal}\left[\left(1 - \frac{x^2}{2} - \frac{x^4}{24} + \frac{13\, x^6}{720}\right) a_0 + \left(x - \frac{x^3}{6} - \frac{x^5}{24} + \frac{5\, x^7}{1008}\right) a_1\right]\ /.\ x \to 1$$

$$\frac{343\, a_0}{720} + \frac{803\, a_1}{1008}$$

Above: The answer in the green cell matches the text answer. The cell below the answer is an experiment for doing IVP.

16 - 19 CAS problems. IVPs

Solve the initial value problem by a power series. Graph the partial sums of the powers up to and including x^5. Find the value of the sum s (5 digits) at x_1.

17. y'' + 3 x y' + 2 y = 0, y[0] = 1, y'[0] = 1, x = 0.5

```
Clear["Global`*"]
```

```
e1 = y[x_] = Sum[a_m x^m, {m, 0, 5}]
```

$a_0 + x\, a_1 + x^2\, a_2 + x^3\, a_3 + x^4\, a_4 + x^5\, a_5$

```
e4 = y''[x] + 3 x y'[x] + 2 y[x] == 0
```

$2\, a_2 + 6\, x\, a_3 + 12\, x^2\, a_4 + 20\, x^3\, a_5 + 3\, x\, \left(a_1 + 2\, x\, a_2 + 3\, x^2\, a_3 + 4\, x^3\, a_4 + 5\, x^4\, a_5\right) + 2\, \left(a_0 + x\, a_1 + x^2\, a_2 + x^3\, a_3 + x^4\, a_4 + x^5\, a_5\right) == 0$

```
e5 = Expand[e4]
```

$2\, a_0 + 5\, x\, a_1 + 2\, a_2 + 8\, x^2\, a_2 + 6\, x\, a_3 + 11\, x^3\, a_3 + 12\, x^2\, a_4 + 14\, x^4\, a_4 + 20\, x^3\, a_5 + 17\, x^5\, a_5 == 0$

```
e6 = Collect[e5, x]
```

$2\, a_0 + 2\, a_2 + x\, (5\, a_1 + 6\, a_3) + 14\, x^4\, a_4 + x^2\, (8\, a_2 + 12\, a_4) + 17\, x^5\, a_5 + x^3\, (11\, a_3 + 20\, a_5) == 0$

```
e7 = Solve[2 a_0 + 2 a_2 == 0, a_2]
```

$\{\{a_2 \to -a_0\}\}$

```
e8 = Solve[5 a_1 + 6 a_3 == 0, a_3]
```

$\left\{\left\{a_3 \to -\frac{5\, a_1}{6}\right\}\right\}$

Above: x^1

```
e9 = Solve[8 a_2 + 12 a_4 == 0, a_4] /. a_2 → -a_0
```

$\left\{\left\{a_4 \to \frac{2\, a_0}{3}\right\}\right\}$

Above: x^2

```
e10 = Solve[11 a_3 + 20 a_5 == 0, a_5] /. a_3 → -5 a_1/6
```

$\left\{\left\{a_5 \to \frac{11\, a_1}{24}\right\}\right\}$

Above: x^3

Above: With discovery of a_5, all the coefficient values for calculation of s have been found.

```
e19 = y[x] /. {a2 → -a0, a3 → -5 a1/6, a4 → 2 a0/3, a5 → 11 a1/24}
```

$$a_0 - x^2 a_0 + \frac{2 x^4 a_0}{3} + x a_1 - \frac{5 x^3 a_1}{6} + \frac{11 x^5 a_1}{24}$$

Above. This is the general solution. The initial value condition of $y(0) = 1$ will make $a_0 = 1$, and the other initial value condition of $y'(0) = 1$ will make $a_1 = 1$.

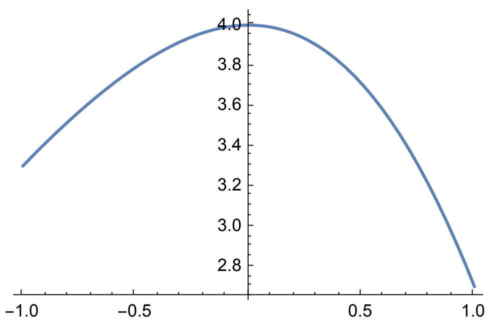
```
e20 = s[x_] = e19 /. {a0 → 1, a1 → 1}
```

$$1 + x - x^2 - \frac{5 x^3}{6} + \frac{2 x^4}{3} + \frac{11 x^5}{24}$$

```
s[1 / 2]
```

$$\frac{923}{768}$$

```
Plot[s[x], {x, -1, 1}, PlotRange → Automatic, ImageSize → 250]
```



The answers in the green cells above match the answers in the text.

19. $(x - 2)\, y' = x\, y$, $y[0] = 4$, $x_1 = 2$

```
Clear["Global`*"]
```

```
e1 = y[x_] = Sum[a_m x^m, {m, 0, 5}]
```

$$a_0 + x a_1 + x^2 a_2 + x^3 a_3 + x^4 a_4 + x^5 a_5$$

```
e2 = (x - 2) y'[x] - x y[x] == 0
```

$$(-2 + x)\left(a_1 + 2 x a_2 + 3 x^2 a_3 + 4 x^3 a_4 + 5 x^4 a_5\right) - x\left(a_0 + x a_1 + x^2 a_2 + x^3 a_3 + x^4 a_4 + x^5 a_5\right) == 0$$

```
e3 = Expand[e2]
```

$$-x a_0 - 2 a_1 + x a_1 - x^2 a_1 - 4 x a_2 + 2 x^2 a_2 - x^3 a_2 - 6 x^2 a_3 + 3 x^3 a_3 - x^4 a_3 - 8 x^3 a_4 + 4 x^4 a_4 - x^5 a_4 - 10 x^4 a_5 + 5 x^5 a_5 - x^6 a_5 == 0$$

```
e4 = Collect[e3, x]
```

$$-2 a_1 + x (-a_0 + a_1 - 4 a_2) + x^2 (-a_1 + 2 a_2 - 6 a_3) + x^3 (-a_2 + 3 a_3 - 8 a_4) + x^4 (-a_3 + 4 a_4 - 10 a_5) - x^6 a_5 + x^5 (-a_4 + 5 a_5) == 0$$

Below: a_1 , which will be the coefficient of x in the final equation, has no business sticking out by itself.

$e5 = \text{Solve}[-2 a_1 == 0, a_1]$

$\{\{a_1 \to 0\}\}$

Below: This value of a_0 was set with the belief that it is necessary for the initial condition, $y(0) = 4$.

$e6 = \text{Solve}[-a_0 + a_1 - 4 a_2 == 0, a_2] \text{ /. } \{a_0 \to 4, a_1 \to 0\}$

$\{\{a_2 \to -1\}\}$

$e7 = \text{Simplify}[\text{Solve}[-a_1 + 2 a_2 - 6 a_3 == 0, a_3] \text{ /. } \{a_2 \to -1, a_1 \to 0\}]$

$\left\{\left\{a_3 \to -\frac{1}{3}\right\}\right\}$

$e8 = \text{Simplify}\left[\text{Solve}[-a_2 + 3 a_3 - 8 a_4 == 0, a_4] \text{ /. } \left\{a_2 \to -1, a_3 \to -\frac{1}{3}\right\}\right]$

$\{\{a_4 \to 0\}\}$

$e9 = \text{Simplify}\left[\text{Solve}[-a_3 + 4 a_4 - 10 a_5 == 0, a_5] \text{ /. } \left\{a_3 \to -\frac{1}{3}, a_4 \to 0\right\}\right]$

$\left\{\left\{a_5 \to \frac{1}{30}\right\}\right\}$

Above: Discovery of a_5 gives all the coefficients necessary to express s up to fifth power of x.

$e10 = y[x] \text{ /. } \left\{a_0 \to 4, a_1 \to 0, a_2 \to -1, a_3 \to -\frac{1}{3}, a_4 \to 0, a_5 \to \frac{1}{30}\right\}$

$$4 - x^2 - \frac{x^3}{3} + \frac{x^5}{30}$$

```
e11 = s[x_] = e10
```

$$4 - x^2 - \frac{x^3}{3} + \frac{x^5}{30}$$

```
s[0]
```

4

```
s[2]
```

$$-\frac{8}{5}$$

```
Plot[s[x], {x, -1, 1}, PlotRange → Automatic, ImageSize → 250]
```