MEC302: Embedded Computer Systems

Theme I: Modeling Dynamic Behaviors

Lecture 2 – Continuous Dynamics

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Lecture outline

- Mathematical models and their types;
- Continuous dynamics modelling:
  - Newtonian (classical) mechanics;
  - Model order reduction;
  - Actor models;
- Model properties;
- Feedback control.

Mathematical models

What is a mathematical model and why do we need them?

- Mathematical models represent real-life problems in math terms and expressions (e.g., a system of equations, state machines);
- Mathematical models help us understand the real-world processes (i.e., physical systems, computer systems and both at the same time);
- Design of ECS require clear understanding of the interaction between the dynamics of ECS and its environment, which can be obtained from detailed modelling.

“All models are wrong, but some are useful.”

by George E. P. Box (British statistician)

Types of mathematical models

Mathematical models can be:

- Deductive, inductive or floating – derivation method (e.g., based on theory, from empirical observations or arbitrary (expected) structure).
- Static or dynamic – time dependency (time-invariant or time-dependent);
- Linear or nonlinear – dependency to varying parameters or variables;
- Explicit or implicit – output parameters can/cannot be directly derived from the input;
- Deterministic or probabilistic – if randomness cannot/can determine the model behavior; 确定性或概率性，是否随机性 能/不能 确定模型的行为
- Discrete or continuous – operate with discrete or continuous quantities;

Continuous dynamics

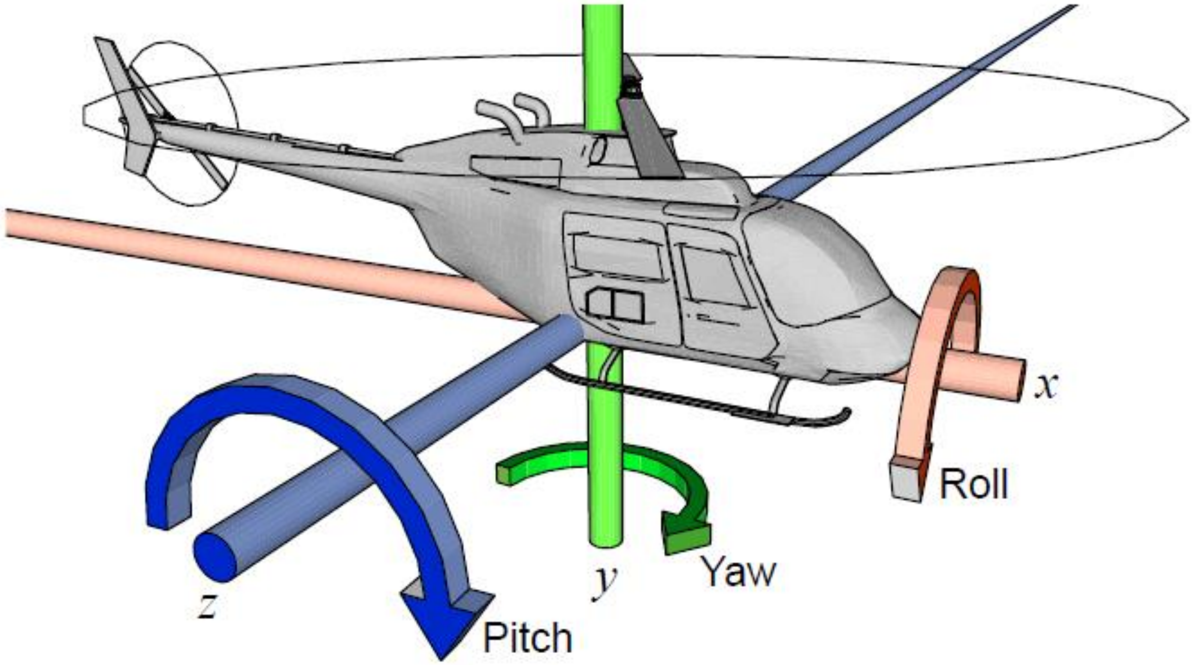
Continuous dynamics modelling:

- Studies dynamics of physical systems, which may include mechanical objects, electric circuits, chemical and biological processes, etc.;
- Represents evolution of an object or system in time according to a certain rule (i.e., physics laws);
- Modeled using differential or integral equations – accurate for continuous and “smooth” processes, where:
  - Differential equation: $$F\left(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \dots, \frac{d^nx}{dt^n}\right) = 0;$$
  - Solution of a diff.eq.: $$x = f(t).$$

Newtonian (classical) mechanics

An object can represented with six degrees of freedom:

- Three to define a position:
  - Coordinated on axes: x, y, z;
- Three to define its orientation:
  - Rotation angles: $\theta_x, \theta_y, \theta_z$.



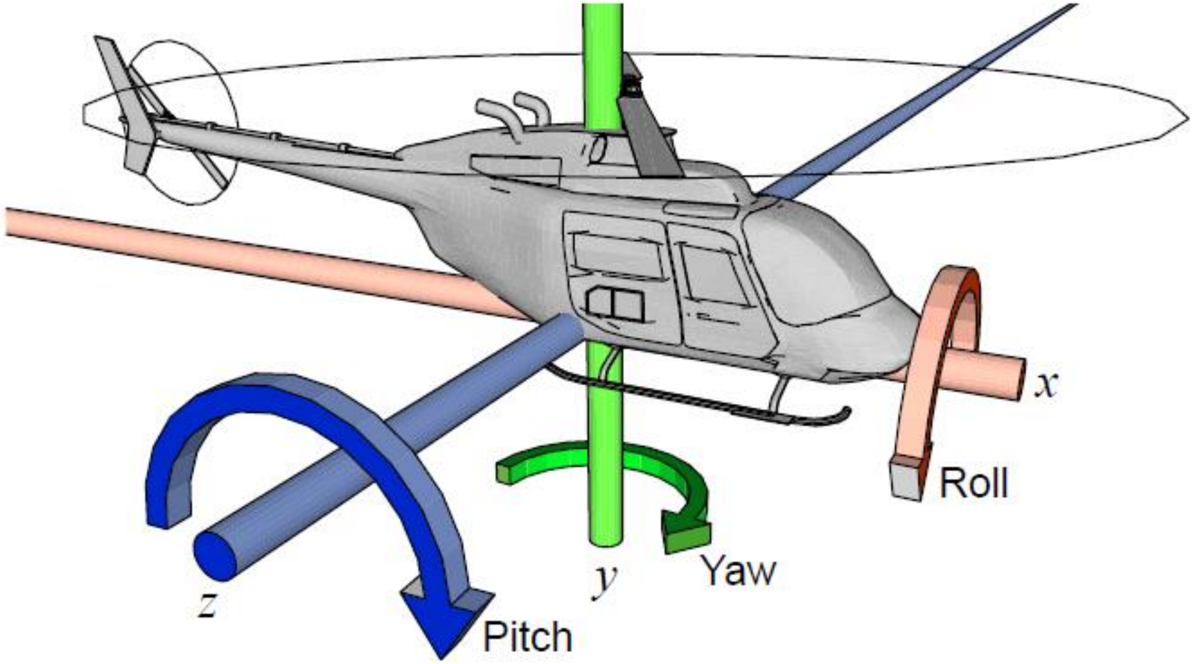
Effectively these can be represented with two vectors:

$$\boldsymbol{x}: R \to R^3 \text{ and } \boldsymbol{\theta}: R \to R^3$$

Newtonian (classical) mechanics

To track its position (along $\boldsymbol{x}: R \to R^3$) in time:

- Newton's second law: $\boldsymbol{F}(t) = M\ddot{\boldsymbol{x}}(t)$



Object's velocity:

$$\dot{\boldsymbol{x}}(t) = \dot{\boldsymbol{x}}(0) + \int_0^t \ddot{\boldsymbol{x}}(\tau)d\tau$$

$$\dot{\boldsymbol{x}}(t) = \dot{\boldsymbol{x}}(0) + \frac{1}{M}\int_0^t \boldsymbol{F}(\tau)d\tau \ \forall\, t > 0$$

Object's position:

$$\boldsymbol{x}(t) = \boldsymbol{x}(0) + \int_0^t \dot{\boldsymbol{x}}(\tau)d\tau$$

$$\boldsymbol{x}(t) = \boldsymbol{x}(0) + t\,\dot{\boldsymbol{x}}\,(0) + \frac{1}{M}\int_0^t \int_0^\tau \boldsymbol{F}(\alpha)\,d\alpha\,d\tau \ \forall\, t > 0$$

Newtonian (classical) mechanics

To track its orientation (along $\boldsymbol{\theta}: R \to R^3$) in time:

- Newton's second law (for rotation):

$$\boldsymbol{T}(t) = \frac{d}{dt}\left(\boldsymbol{I}(t)\,\dot{\boldsymbol{\theta}}(\boldsymbol{t})\right)$$

For a spherical object:

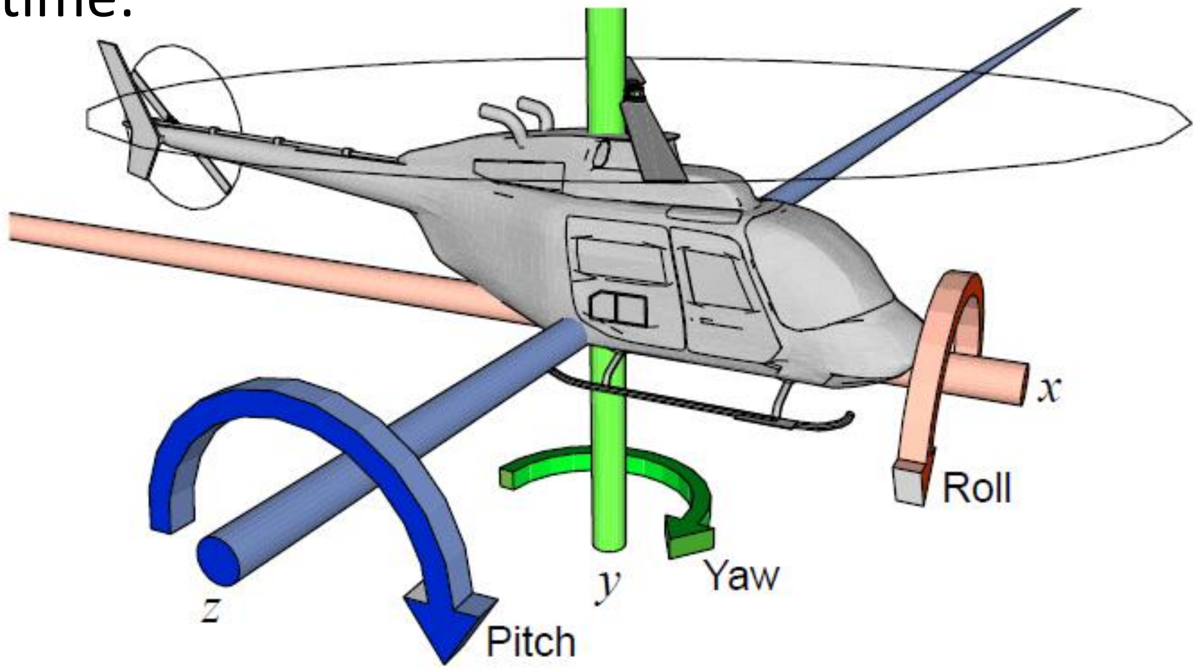
$$\boldsymbol{T}(t) = I\ddot{\boldsymbol{\theta}}(\boldsymbol{t})$$

Object's angular velocity:

$$\dot{\boldsymbol{\theta}}(t) = \dot{\boldsymbol{\theta}}(0) + \frac{1}{I}\int_0^t \boldsymbol{T}(\tau)d\tau \; \forall\, t > 0$$

Object's orientation:

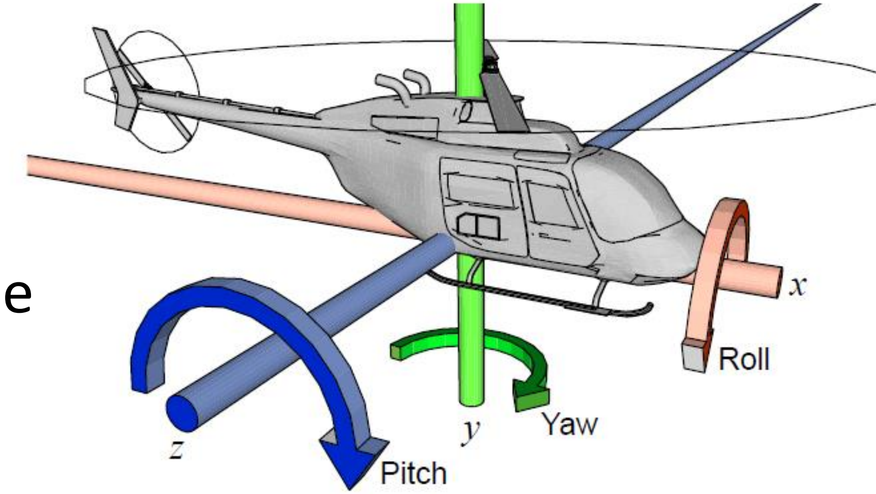
$$\boldsymbol{\theta}(t) = \boldsymbol{\theta}(0) + t\,\dot{\boldsymbol{\theta}}(0) + \frac{1}{I}\int_0^t\int_0^\tau \boldsymbol{T}(\alpha)\; d\alpha\; d\tau \; \forall\, t > 0$$



Moment of inertia tensor for an arbitrary object:

$$\boldsymbol{I}(t) = \begin{bmatrix} I_{xx}(t) & I_{xy}(t) & I_{xz}(t) \\ I_{yx}(t) & I_{yy}(t) & I_{yz}(t) \\ I_{zx}(t) & I_{zy}(t) & I_{zz}(t) \end{bmatrix}$$

Model order reduction



Model order reduction is a formal procedure to reduce the number of model's degrees of freedom.

模型降阶：减少模型自由度

To diversify control channels (and of course for simplicity):

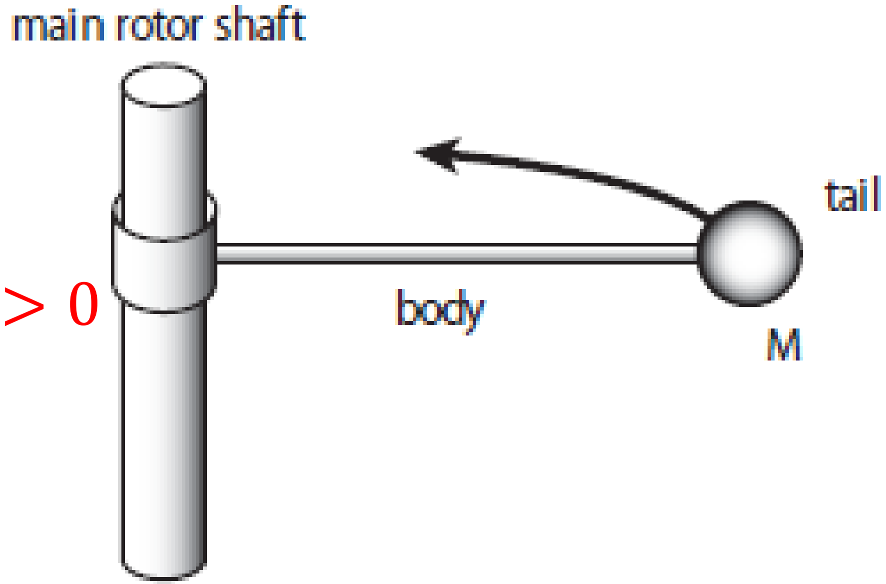
多样化控制渠道

- For angular velocity around y-axis:

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t T_y(\tau) d\tau \; \forall \; t > 0$$

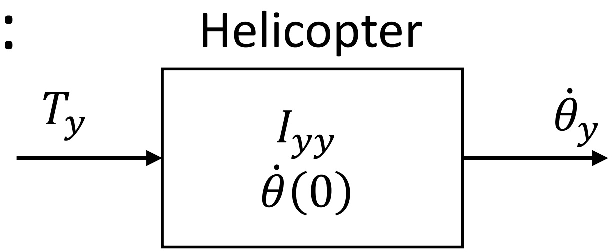
- For orientation around y-axis:

$$\theta_y(t) = \theta_y(0) + t\,\dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t \int_0^\tau T_y(\alpha)\, d\alpha\, d\tau \; \forall \; t > 0$$



Actor model

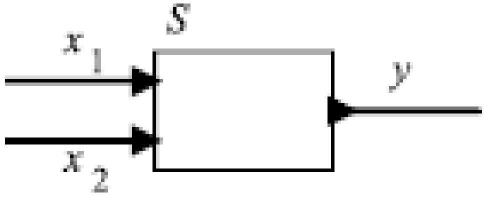
Helicopter's actor model:



$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t T_y(\tau) d\tau$$

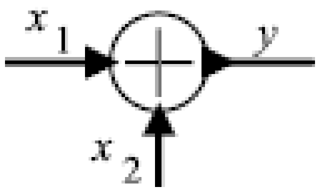
From control theory, actor models:

1. May have multiple inputs/outputs;



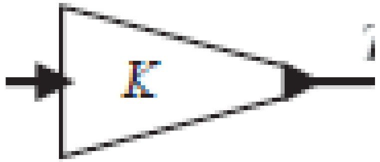
$$y = f(x_1, x_2)$$

2. Signal adder (particular example of 1.);



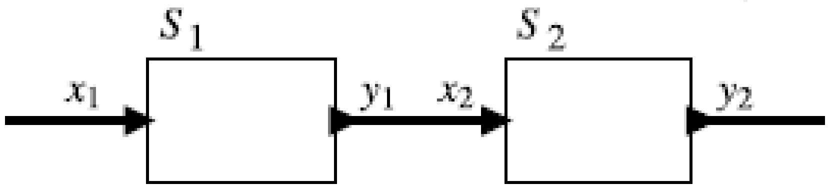
$$y = x_1 + x_2$$

3. Gain function (multiplication by *K*);



$$y = Kx$$

4. (De)composable



$$y_2 = f_2(f_1(x_1))$$

Model properties

Main model properties:

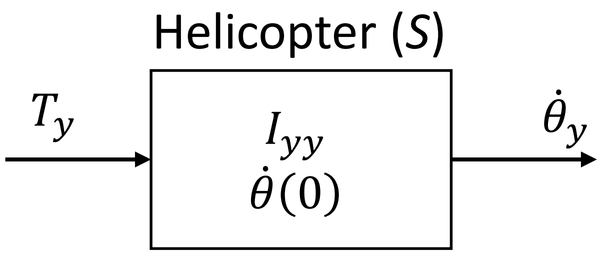
- **Causality** – output depends only on current and past inputs;
- **Memory** – output depends not only on the current inputs, but also on past inputs;
- **Linearity** – the outputs are proportional to the inputs;
- **Time Invariance** – model behavior does not depend on time;
- **Stability** – the output is bounded for all bounded input signals.

As a rule:

- To design an effective control, a model/object/system need to be Causal, Not Memoryless, Linear, Time Invariant and Stable

Model properties

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t T_y(\tau)d\tau$$



Causality – the output depends only on current and past inputs: 因果性

Causal model is defined as follows:

$$x_1 \Big|_{t\le\tau} = x_2 \Big|_{t\le\tau} \Rightarrow S(x_1)\Big|_{t\le\tau} = S(x_2)\Big|_{t\le\tau} \quad \forall x_1, x_2 \in X$$

– depends on past and present inputs!

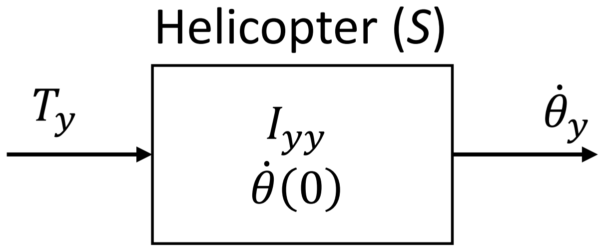
Strictly causal model is defined:

$$x_1 \Big|_{t<\tau} = x_2 \Big|_{t<\tau} \Rightarrow S(x_1)\Big|_{t\le\tau} = S(x_2)\Big|_{t\le\tau} \quad \forall x_1, x_2 \in X$$

– depends only on past inputs!

Is the helicopter model causal, strictly causal or non-causal?

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}}\int_0^t T_y(\tau)d\tau$$



Model properties

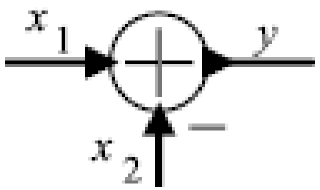
Memory – the output depends not only on the current input, but also on the past inputs.

Memoryless model is defined as follows:

$$\exists\ f: A \to B \mid f\big(x(t)\big) = \big(S(x)\big)(t)\ \forall\ t \in R$$

– model output depends only on the current input

For example, Adder ($y = x_1 + x_2$) is memoryless.

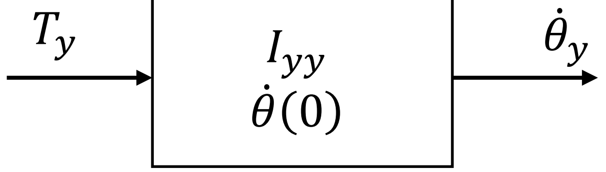


Is the helicopter model memoryless or not?

Model properties

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t T_y(\tau) d\tau$$

Helicopter (S)



Linearity – the output(s) is proportional to the input(s):

Linear model is defined as follows:

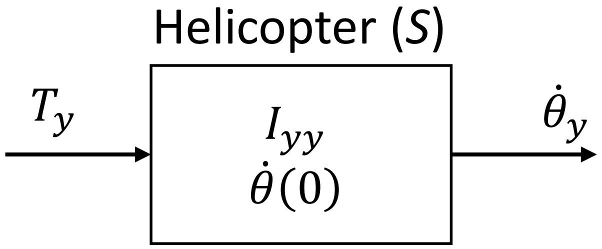
$$S(ax) = a\, S(x)\ \forall\ x \in X, a \in R$$

For example, Adder($y = x_1 + x_2$) and Gain($y = Kx$) are linear.

Is the helicopter model linear or not?

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}}\int_0^t T_y(\tau)d\tau$$

Model properties



Time Invariance – model behavior does not depend on time.

Time invariant model is defined as follows:

$$S\big(D_\tau(x)\big) = D_\tau\big(S(x)\big)\ \forall\ x \in X, \tau \in R,$$

where $\big(D_\tau(x)\big)(t) = x(t-\tau)\ \forall\ x \in X, t \in R$.

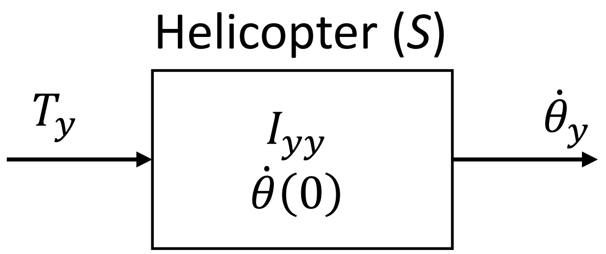
– the model is linear to delay function.

Obviously, Adder($y = x_1 + x_2$) and Gain($y = Kx$) are time invariant.

Is the helicopter model time invariant or not?

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t T_y(\tau) d\tau$$

Model properties



Stability – the output is bounded for all bounded input signals.

A model is bounded-input bounded-output (BIBO) stable if:

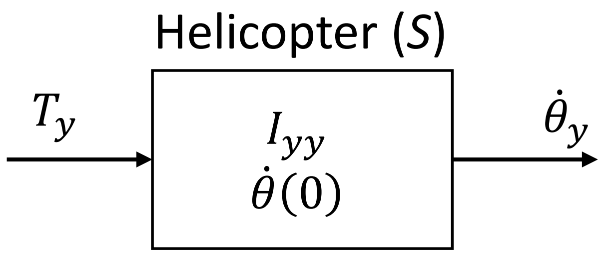
$$\exists |A|, |B| < \infty : \left| \big(S(x)\big)(t) \right| \leq |A| \; \forall \; x(t) \leq |B|, t \in R$$

Clearly, Adder($y = x_1 + x_2$) and Gain($y = Kx$) are BIBO stable.

Is the helicopter model BIBO stable?

Helicopter model properties

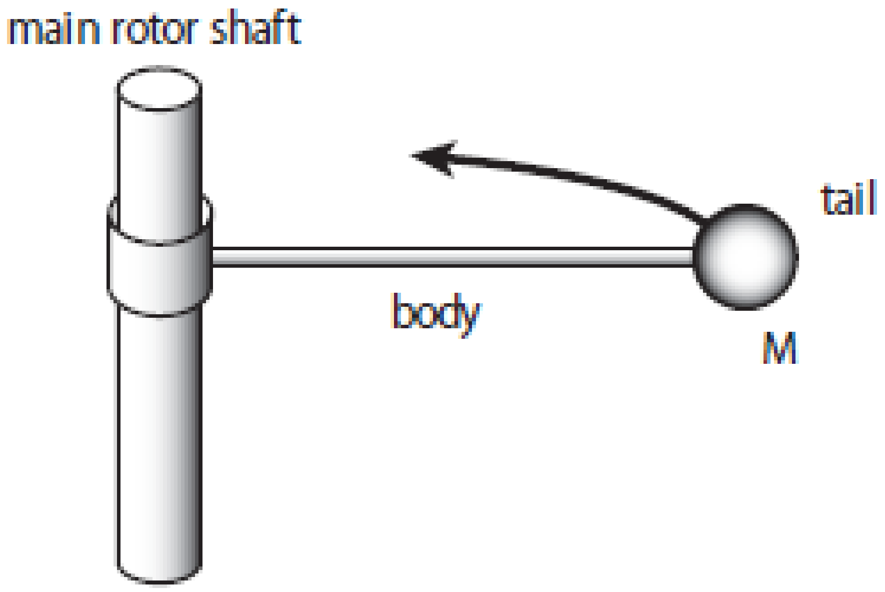
$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t T_y(\tau) d\tau$$



Helicopter model is:

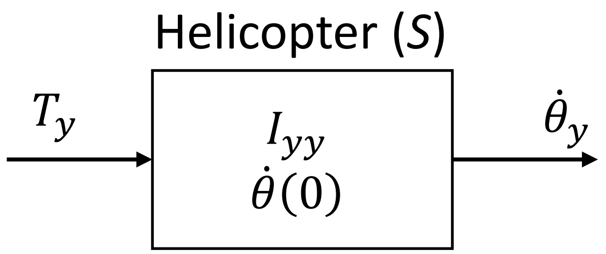
- Strictly causal;
- Not memoryless;
- Linear*
- Time invariant*
- BIBO unstable

- Can we control such system?



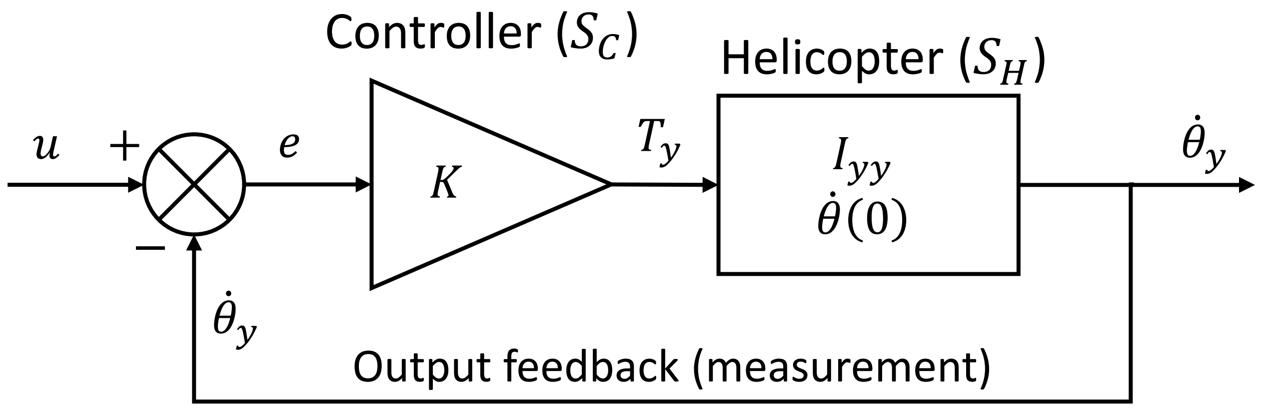
* – if $\dot{\theta}_y(0) = 0$

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t T_y(\tau) d\tau$$



Feedback Control

Let us introduce a feedback control loop:



Effective Transfer Function (TF) becomes:

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{K}{I_{yy}} \int_0^t \left(u(\tau) - \dot{\theta}_y(\tau) \right) d\tau$$

Discrete difference equation (for $\Delta t \to 0$):

$$\dot{\theta}_y(s+1) = \dot{\theta}_y(s) + \frac{K}{I_{yy}} \sum_{s=1}^{S} \left(u(s) - \dot{\theta}_y(s) \right) \Delta t$$

To check if it is stable, we can:

- Model it using LabView or Simulink;
- Brute force – reformulate TF into a difference equation and solve it iteratively;
- Solve the differential equation and check if $\dot{\theta}_y(t)$ is bounded.

Feedback Control

Feedback control model TF:

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{K}{I_{yy}} \int_0^t \left(u(\tau) - \dot{\theta}_y(\tau)\right) d\tau$$

To find if the system is stable, let's solve its differential equation ($\dot{\theta}_y(t) = f(t)$):

1. Consider $u(t) = U$, then helicopter ODE is as follows:

$$\frac{d\dot{\theta}_y(t)}{dt} = \frac{K}{I_{yy}}\left(U - \dot{\theta}_y(t)\right)$$

2. Let's rearrange/separate ODE variables:

$$\frac{1}{U - \dot{\theta}_y(t)} d\dot{\theta}_y(t) = \frac{K}{I_{yy}} dt$$

3. Now we can integrate both parts separately:

$$-\ln\left(U - \dot{\theta}_y(t)\right) - \ln C = \frac{K}{I_{yy}} t$$

4. Manipulate with it (get rid of logarithm):

$$\left(U - \dot{\theta}_y(t)\right) C = e^{\frac{-K}{I_{yy}}t}$$

5. Express $\dot{\theta}_y(t)$:

$$\dot{\theta}_y(t) = U - \frac{1}{C} e^{\frac{-K}{I_{yy}}t}$$

6. For initial conditions ($\dot{\theta}_y(0) = 0$), $C = \frac{1}{U}$:

$$\dot{\theta}_y(t) = U\left(1 - e^{\frac{-K}{I_{yy}}t}\right)$$

7. Thus, $\lim_{t\to\infty} \dot{\theta}_y(t) = U \; \forall \; K > 0$

To sum up

- Mathematical models help us understand the real-world processes (e.g., physical systems, computer systems and both at the same time);
- Continuous dynamics modelling uses differential or integral equations to model continuous and “smooth” processes (e.g., Newtonian mechanics can be used to model mechanical systems);
- Actor model provides a graphical representation of the mathematical model (e.g., physical system), which can be effective for the analysis and system design;
- Model properties (i.e., Causality, Memorylessness, Linearity, Time Invariance, and Stability) can show if a system is suitable for control:
  - I.e., Causal, Not Memoryless, Linear, Time Invariant and Stable system is deemed to be suitable for control.
- Feedback control is effective to improve system properties (e.g., Stability).

The End

See you next time (March 6)