Ordinary Differential Equation

Chapter II : Differential Equation

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Definition

The **general linear equation** of order n is an equation in a form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = f(x), \tag{1}$$

where $a_n(x) \neq 0$ for some interval I. $a_0(x), \ldots, a_n(x)$ are **coefficient functions** of equation and $a_n(x)$ is a **leading coefficient function**.

If $f(x) = 0$ for all $x \in I$, the equation (1) is called **homogeneous equation**, otherwise (1) is called **non-homogeneous equation** and $f(x)$ is **non-homogeneous term** of (1) and

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = 0$$

is an **related homogeneous equation** of (1).

Definition

We say that the general ODE

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = f(x)$$

is **normal** on an interval I if $a_0(x), \ldots, a_n(x)$ and $f(x)$ are continuous on I and $a_n(x) \neq 0$ for all $x \in I$.

Example

Consider

$$y'' - \frac{1}{x}y + 2y = \sin x.$$

Note that $a_2(x) = 1, a_1(x) = -\frac{1}{x}, a_0(x) = 2$ and $f(x) = \sin x$.
It is easy to see that $a_0(x), a_2(x)$ and $f(x)$ are continuous for every $x \in \mathbb{R}$ but $a_1(x)$ is discontinuous at $x = 0$ and $a_2(x) \neq 0$ for every $x \in \mathbb{R}$. So the following equation is normal on every interval which not contain 0.
For example $(0, \infty)$ and $[11, 80)$.

Theorem : Existence and Uniqueness Theorem

Let

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = f(x)$$

is normal on $I, x_0 \in I$ and $y_0, \ldots, y_{n-1} \in \mathbb{R}$. Then there exists the unique solution $y = y(x)$ of (1) that satisfies conditions

$$y(x_0) = y_0, y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}.$$

Example

Show that $y = e^{2x}$ is the only solution of

$$y'' - 3y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = 2.$$

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Solution From $y = e^{2x}$, then $y' = 2e^2x$ and $y'' = 4e^{2x}$, so

$$y'' - 3y' + 2y = 4e^{2x} - 6e^{2x} + 2e^{2x} = 0.$$

Note that $y(0) = e^0 = 1$ and $y'(0) = 2e^0 = 2$. By theorem, $y = e^{2x}$ is the only solution of the following ODE. $\square$

Corollary

Let

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = 0$$

is normal on I. Then $y(x) = 0$ is the only solution that satisfies the following equation and conditions

$$y(x_0) = y'(x_0) = \ldots = y^{(n-1)}(x_0) = 0.$$

Definition

A solution $y(x) = 0$ is called **trivial solution**, otherwise $y(x) \neq 0$ is called **nontrivial solution**.

Theorem : Superposition Principle

Let $y_1, \ldots, y_m$ be solutions of an linear ODE

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = 0 \qquad (*)$$

and $c_1, \ldots, c_m$ are constants, then

$$y = c_1y_1 + \ldots + c_my_m$$

is a solution of (*).

Definition

We say that

$$y = c_1 y_1 + \ldots + c_m y_m$$

is a **linear combination** of functions $y_1, \ldots, y_m$.

Example

Let $y_1 = e^{-x}$ and $y_2 = e^{2x}$. It is easy to show that y_1 and y_2 are solutions of

$$y'' - y' - 2y = 0 \tag{**}$$

on $\mathbb{R}$. By superposition principle,

$$y = c_1e^{-x} + c_2e^{2x}$$

are solutions of (**) for all $c_1, c_2 \in \mathbb{R}$.

Example

We can show that

$$y_1 = e^{-x} \quad \text{and} \quad y_2 = e^{2x}$$

are solutions of nonlinear homogeneous ODE

$$yy'' - (y')^2 = 0$$

but $y = c_1e^{-x} + c_2e^{2x}$ is a solution if $c_1 = 0$ or $c_2 = 0$. So, the superposition principle is not valid for nonlinear ODE.

Definition

Let $f_1, \ldots, f_n$ be functions on I. We say that $f_1, \ldots, f_n$ are **linearly independent** if for every $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$c_1 f_1(x) + \ldots + c_n f_n(x) = 0$$

for all $x \in I$, then $c_1 = \ldots = c_n = 0$.

Example

Show that $\sin x$ and $\cos x$ are linearly independent on $\mathbb{R}$.

Example

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Solution Let $c_1, c_2 \in \mathbb{R}$ such that

$$c_1 \sin x + c_2 \cos x = 0$$

for all $x \in \mathbb{R}$.
If $x = 0$, then $c_2 = c_1 \sin 0 + c_2 \cos 0 = 0$.
If $x = \frac{\pi}{2}$, then $c_1 = c_1 \sin\left(\frac{\pi}{2}\right) = 0$.
Thus, $\sin x$ and $\cos x$ are linearly independent on $\mathbb{R}$. $\square$

Example

Show that $f_1(x) = |x|$ and $f_2(x) = x$ are linearly independent on $\mathbb{R}$.

Example

Show that $f_1(x) = |x|$ and $f_2(x) = x$ are linearly independent on $\mathbb{R}$.

<u>Solution</u> Let $c_1, c_2 \in \mathbb{R}$ such that

$$c_1|x| + c_2x = 0$$

for all $x \in \mathbb{R}$.
If $x = 1$, then $c_1 + c_2 = 0$.
If $x = -1$, then $c_1 - c_2 = 0$.
So, $c_1 = c_2 = 0$. Thus, $|x|$ and x are linearly independent on $\mathbb{R}$. $\square$

Theorem

For every linear ODE of order n

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = 0 \qquad (*)$$

that normal on an interval I, there exists solutions $y_1, \ldots, y_n$ which are linearly independent and every particular solution of (*) can be written as a linear combination of $y_1, \ldots, y_n$.

Definition

Let $f_1, f_2, \ldots, f_n$ be $n-1$ times differentiable functions on $\mathbb{R}$, then the determinant

$$W(x) =: W(f_1, \ldots, f_n : x) := \begin{vmatrix} f_1(x) & f_2(x) & \ldots & f_n(x) \\ f_1'(x) & f_2'(x) & \ldots & f_n'(x) \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \ldots & f_n^{(n-1)}(x) \end{vmatrix}$$

is called the **Wronskian** of $f_1, f_2, \ldots, f_n$.

Theorem

Let $f_1, \ldots, f_n$ be $n-1$ differentiable functions on an interval I. If $W(f_1, \ldots, f_n : x) \neq 0$ for some $x \in I$, then $f_1, \ldots, f_n$ are linearly independent.

Theorem

Let $f_1, \ldots, f_n$ be $n-1$ differentiable functions on an interval I. If $W(f_1, \ldots, f_n : x) \neq 0$ for some $x \in I$, then $f_1, \ldots, f_n$ are linearly independent.

Corollary

Let $f_1, \ldots, f_n$ be $n-1$ differentiable functions on an interval I. If $f_1, \ldots, f_n$ are linearly dependent, then $W(f_1, \ldots, f_n : x) = 0$ for all $x \in I$.

Theorem

Let $f_1, \ldots, f_n$ be $n-1$ differentiable functions on an interval I. If $W(f_1, \ldots, f_n : x) \neq 0$ for some $x \in I$, then $f_1, \ldots, f_n$ are linearly independent.

Corollary

Let $f_1, \ldots, f_n$ be $n-1$ differentiable functions on an interval I. If $f_1, \ldots, f_n$ are linearly dependent, then $W(f_1, \ldots, f_n : x) = 0$ for all $x \in I$.

Caution

Converse of the theorem is not true.

Example

Use Wronskian to show that x, xe^x and x^2e^x are linearly independent on $\mathbb{R}$.

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<u>Solution</u> Note that

$$W(x) = \begin{vmatrix} x & xe^x & x^2e^x \\ 1 & (x+1)e^x & (x^2+2x)e^x \\ 0 & (x+2)e^x & (x^2+4x+2)e^x \end{vmatrix} = e^{2x}x^3 \neq 0$$

for some $x \in \mathbb{R}$. So, x, xe^x and x^2e^x are linearly independent on $\mathbb{R}$. $\square$

Example

Let f_1 and f_2 be real-valued functions on the set of real numbers by

$$f_1(x) = \begin{cases} x^3 & ; x \geq 0 \\ 0 & ; x < 0, \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} 0 & ; x \geq 0 \\ x^3 & ; x < 0 \end{cases}$$

for all $x \in \mathbb{R}$.

1. Show that f_1 and f_2 are differentiable on $\mathbb{R}$.
2. Show that the Wronskian of f_1 and f_2 is identically zero.
3. Show that f_1 and f_2 are linearly independent.

1. First, we find the derivatives of f_1 and f_2 Since

$$\lim_{x\to 0^+} \frac{f_1(x)-f_1(0)}{x} = \lim_{x\to 0^+} x^2 = 0 = \lim_{x\to 0^+} \frac{f_1(x)-f_1(0)}{x}$$

and

$$\lim_{x\to 0^+} \frac{f_2(x)-f_2(0)}{x} = 0 = \lim_{x\to 0^-} x^2 = \lim_{x\to 0^-} \frac{f_2(x)-f_2(0)}{x}.$$

Hence, $f_1'(0)$ and $f_2'(0)$ exist and $f_1'(0) = 0 = f_2'(0)$. So,

$$f_1'(x) = \begin{cases} 3x^2 & ; x > 0 \\ 0 & ; x \leq 0 \end{cases}, \quad \text{and} \quad f_2'(x) = \begin{cases} 0 & ; x \geq 0 \\ 3x^2 & ; x < 0 \end{cases}. \quad \square$$

2. Let $x \in \mathbb{R}$.
   If $x < 0$, then
   $$W(x) = \begin{vmatrix} 0 & x^3 \\ 0 & 3x^2 \end{vmatrix} = 0.$$
   If $x > 0$, then
   $$W(x) = \begin{vmatrix} x^3 & 0 \\ 3x^2 & 0 \end{vmatrix} = 0.$$
   If $x = 0$, then
   $$W(x) = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}.$$
   So, $W(x) = 0$ for all $x \in \mathbb{R}$. $\square$

3. Let $k_1, k_2 \in \mathbb{R}$ such that

$$k_1 f_1(x) + k_2 f_2(x) = 0$$

for all $x \in \mathbb{R}$.
If $x = 1$, then

$$f_1(x) = 1, \quad \text{and} \quad f_2(x) = 0.$$

So, $k_1 = 0$.
If $x < 1$, then

$$f_1(x) = 0, \quad \text{and} \quad f_2(x) = 1.$$

So, $k_2 = 0$. This implies that f_1 and f_2 are form a linearly independent set. □

Theorem

Let $y_1, \ldots, y_n$ be particular solutions of an linear ODE of order n

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = 0$$

that normal on an interval I, then either $W(y_1, \ldots, y_n : x)$ is identically zero or $W(y_1, \ldots, y_n : x) \neq 0$ for all $x \in I$.

Theorem

Let $y_1, \ldots, y_n$ be particular solutions of an linear ODE of order n

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = 0$$

that normal on an interval I, then $y_1, \ldots, y_n$ are linearly independent if and only if $W(y_1, \ldots, y_n : x) \neq 0$ for all $x \in I$.

Example

We can show that $y_1 = e^{-x}$ and $y_2 = e^{2x}$ are solutions of

$$y'' - y' - 2y = 0$$

and this equation is normal on $\mathbb{R}$. Note that

$$W(y_1, y_2 : x) = \begin{vmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{vmatrix} = 3e^{x} \neq 0$$

for all $x \in \mathbb{R}$. So y_1 and y_2 are linearly independent.

Example

Let $\omega > 0$. We can show that $y_1 = \sin \omega x$ and $y_2 = 4 \sin \omega x$ are solutions of

$$y'' + \omega^2 y = 0.$$

Note that

$$W(y_1, y_2 : x) = \begin{vmatrix} \sin \omega x & 4 \sin \omega x \\ \omega \cos \omega x & 4\omega \cos \omega x \end{vmatrix} = 0$$

for all $x \in \mathbb{R}$. So y_1 and y_2 are linearly dependent.

Theorem

Let $y_1, \ldots, y_n$ be linearly independent particular solutions of a linear ODE

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = 0 \qquad (*)$$

that normal on an interval I, then every solution of (*) on I is a linear combination of $y_1, \ldots, y_n$.

Definition

Functions $y_1, \ldots, y_n$ those linearly independent particular solutions of a linear ODE order n on an interval I are called **fundamental solutions** or **basis** of all solutions on I and a linear combination

$$y = c_1 y_1 + \ldots + c_n y_n,$$

where $c_1, \ldots, c_n$ are constants, is **complete solution** or **general solution** of equation.

Example

Use functions x^{-2} and $x^{-2}\ln x$ to find a complete solution (or general solution) of

$$x^2y'' + 5xy' + 4y = 0$$

on $[0, \infty)$.

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Solution First, we check that $y_1 = x^{-2}$ and $y_2 = x^{-2}\ln x$ are solutions of this ODE. Note that

$$y_1' = -2x^{-3}, \quad y_1'' = 6x^{-4}$$

and

$$y_2' = x^{-3} - 2x^{-3}\ln x, \quad y_2'' = -3x^{-3} - 2x^{-4} + 6x^{-4}\ln x.$$

Then,

$$x^2y_i'' + 5xy_i' + 4y_i = 0$$

for $i = 1, 2$. Since

$$W(y_1, y_2 : x) = \begin{vmatrix} x^{-2} & x^{-2}\ln x \\ -2x^{-3} & x^{-3} - 2x^{-3}\ln x \end{vmatrix} = x^{-5} \neq 0$$

for $x \in (0, \infty)$. So, y_1 and y_2 are linearly independent on $(0, \infty)$. Thus, a general solution of ODE is

$$y = c_1x^{-2} + c_2x^{-2}\ln x,$$

where c_1, c_2 are constants. $\square$

In this section, we will find an another solution y_2 if we know that y_1 is one of solution of

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0,$$

where $a_2(x), a_1(x)$ and $a_0(x)$ are functions and y_1, y_2 are linearly independent.

Note that y_1 and y_2 are solutions, so

$$a_2(x)y_i'' + a_1(x)y_i' + a_0(x)y_i = 0,$$

for $i = 1, 2$. Since y_1 and y_2 are linearly independent, there exists a nonconstant function $u(x)$ such that

$$y_2 = uy_1.$$

Then

$$y_2' = uy_1' + u'y_1 \quad \text{and} \quad y_2'' = uy_1'' + 2u'y_1' + u''y_1.$$

So,

$$a_2(x)(uy_1'' + 2u'y_1' + u''y_1) + a_1(x)(uy_1' + u'y_1) + a_0(x)(uy_1) = 0.$$

Then

$$u''a_2(x)y_1 + u'(2a_2(x)y_1' + a_1(x)y_1) + u(a_2(x)y_1'' + a_1(x)y_1' + a_0(x)y_1) = 0$$

$$u''a_2(x)y_1 + u'(2a_2(x)y_1' + a_1(x)y_1) = 0$$

$$\frac{u''}{u'} = -2\frac{y_1'}{y_1} - \frac{a_1(x)}{a_2(x)}.$$

Integrate respect to x on both sides, hence

$$\ln u' = -2\ln y_1 - \int \frac{a_1(x)}{a_2(x)}dx = \ln y_1^{-2} - \int \frac{a_1(x)}{a_2(x)}dx + C.$$

So

$$u' = Ay_1^{-2}e^{-\int \frac{a_1(x)}{a_2(x)}dx} \quad \to \quad u = A\int y_1^{-2}e^{-\int \frac{a_1(x)}{a_2(x)}dx}dx + C_2.$$

Hence

$$y_2 = Ay_1 \int y_1^{-2} e^{-\int \frac{a_1(x)}{a_2(x)}dx} dx + C_2 y_1.$$

Let $\tilde{y}_2 = y_1 \int y_1^{-2} e^{-\int \frac{a_1(x)}{a_2(x)}dx} dx$. Note that

$$W(y_1, \tilde{y}_2 : x) = \begin{vmatrix} y_1 & y_1 \int y_1^{-2} e^{-\int \frac{a_1(x)}{a_2(x)}dx} dx \\ y_1' & y_1^{-1} e^{-\int \frac{a_1(x)}{a_2(x)}dx} + y_1' \int y_1^{-2} e^{-\int \frac{a_1(x)}{a_2(x)}dx} dx \end{vmatrix}$$
$$= e^{-\int \frac{a_1(x)}{a_2(x)}dx} \neq 0,$$

for all $x \in \mathbb{R}$. So, y_1 and $\tilde{y}_2$ are linearly independent. That is another solution is

$$y_1 \int y_1^{-2} e^{-\int \frac{a_1(x)}{a_2(x)}dx} dx.$$

Theorem

Let y_1 be one of solution of

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0, \quad a_2(x) \neq 0$$

then a complete solution (or general solution) is

$$y = c_1y_1 + c_2y_2,$$

where

$$y_2 = y_1 \int y_1^{-2} e^{-\int \frac{a_1(x)}{a_2(x)}dx} dx$$

and c_1, c_2 are constants.
(We already shown that y_1 and y_2 are fundamental solutions of this ODE.)

Example

Find a general solution of

$$x^2y'' - 3xy' + 3y = 0,$$

where $y = x$ is one of its fundamental solution.

Example

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$$x^2y'' - 3xy' + 3y = 0,$$

where $y = x$ is one of its fundamental solution.

Solution Note that $a_2(x) = x^2, a_1(x) = -3x$. So

$$y_2 = x\int x^{-2}e^{3\int \frac{1}{x}dx}dx = x\int x^{-2}e^{3\ln|x|}dx$$
$$= x\int x^{-2}x^3dx = x\int xdx = \frac{x^3}{2} + Cx.$$

Hence, a general solution is

$$y = c_1x + c_2x^3,$$

where c_1, c_2 are constants. $\square$

Example

Find a general solution of

$$4x^2y'' - 8xy' + 9y = 0,$$

where $y = x^{\frac{3}{2}}$ is one of its fundamental solution.

Example

Find a general solution of

$$4x^2y'' - 8xy' + 9y = 0,$$

where $y = x^{\frac{3}{2}}$ is one of its fundamental solution.

<u>Solution</u> Note that $a_2(x) = 4x^2, a_1(x) = -8x$. So

$$\begin{aligned} y_2 &= x^{\frac{3}{2}} \int x^{-3} e^{2\int \frac{1}{x}dx} dx = x^{\frac{3}{2}} \int x^{-3} e^{2\ln|x|} dx \\ &= x^{\frac{3}{2}} \int x^{-3} x^2 dx = x^{\frac{3}{2}} \int x^{-1} dx = x^{\frac{3}{2}} \ln|x| + Cx^{\frac{3}{2}}. \end{aligned}$$

Hence, a general solution is

$$y = c_1 x^{\frac{3}{2}} \ln|x| + c_2 x^{\frac{3}{2}},$$

where c_1, c_2 are constants.

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Find a general solution of

$$y'' - 4xy' + 2(2x^2 - 1)y = 0,$$

where $y = e^{x^2}$ is one of its fundamental solution.

Example : KVIS ODE Final Examination 2560

Find a general solution of

$$y'' - 4xy' + 2(2x^2 - 1)y = 0,$$

where $y = e^{x^2}$ is one of its fundamental solution.

Solution Note that $a_2(x) = 1, a_1(x) = -4x$ and $a_0(x) = 2(2x^2 - 1)$. So

$$y_2 = e^{x^2} \int e^{-2x^2} e^{\int 4x dx} dx = e^{x^2} \int dx = xe^{x^2}.$$

So, a general solution is

$$y = c_1 e^{x^2} + c_2 x e^{x^2},$$

where c_1, c_2 are constants. $\square$

Example

Find a general solution of

$$(2x+1)y'' - (4x+4)y' + 4y = 0,$$

where $y = e^{2x}$ is one of its fundamental solution.

Example

Find a general solution of

$$(2x+1)y'' - (4x+4)y' + 4y = 0,$$

where $y = e^{2x}$ is one of its fundamental solution.

<u>Solution</u> Note that $a_2(x) = 2x+1, a_1(x) = -(4x+4)$. So

$$y_2 = e^{2x}\int e^{-4x}e^{4\int \frac{x+1}{2x+1}dx}dx = e^{2x}\int e^{-4x}e^{2x+\ln|2x+1|}dx$$

$$= e^{2x}\int e^{-2x}(2x+1)dx = -(x+1) + Ce^{2x}.$$

Hence, a general solution is

$$y = c_1(x+1) + c_2e^{2x},$$

where c_1, c_2 are constants.

In this section, we consider the second order ODE

$$ay'' + by' + cy = 0, \tag{2}$$

where a, b, c are constants and $a \neq 0$.
First, we try to find solutions by let $y = e^{rx}$, where r is a constant, then

$$y' = re^{rx} \quad \text{and} \quad y'' = r^2 e^{rx}.$$

So

$$e^{rx}(ar^2 + br + c) = 0. \tag{3}$$

Note that $e^{rx} \neq 0$ for all $x \in \mathbb{R}$, so

$$ar^2 + br + c = 0.$$

This equation is called a **auxiliary equation** or **characteristic equation** of equation (2).

Then, we have

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We separate into 3 cases.

1. $b^2 - 4ac > 0$ ((3) has two distinct real roots.)
2. $b^2 - 4ac = 0$ ((3) has two repeated real roots.)
3. $b^2 - 4ac < 0$ ((3) has two complex conjugate roots.)

Case I : $b^2 - 4ac > 0$.

Then, (3) has two distinct real roots, say r_1 and r_2. Hence

$$y_1 = e^{r_1 x} \quad \text{and} \quad y_2 = e^{r_2 x}$$

are solutions of (2). Note that

$$W(y_1, y_2 : x) = \begin{vmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{vmatrix} = (r_2 - r_1) e^{(r_1 + r_2)x} \neq 0$$

for all $x \in \mathbb{R}$. That is y_1 and y_2 are linearly independent, so a general solution of equation (2) is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x},$$

where c_1 and c_2 are constants.

Case II : $b^2 - 4ac = 0$.

Then, (3) has two repeated real roots, say $r = -\dfrac{b}{2a}$. So

$$y_1 = e^{rx}$$

is one of solution of equation (2). Next, we will find another solution y_2 that linearly independent with y_1 by

$$y_2 = y_1 \int \frac{1}{y_1^2} e^{-\int \frac{b}{a} dx} dx = e^{rx} \int \frac{e^{-\frac{b}{a}x}}{e^{2r}} dx = e^{rx} \int dx = xe^{rx}.$$

So, a general solution of (2) is

$$y = c_1 e^{rx} + c_2 x e^{rx} = e^{rx}(c_1 + c_2 x),$$

where c_1 and c_2 are constants.

Case III : $b^2 - 4ac < 0$.
Then, (3) has two complex conjugate roots, say

$$r_1 = \alpha + i\beta \quad \text{and} \quad r_2 = \bar{r}_1 = \alpha - i\beta \qquad ; \alpha, \beta \in \mathbb{R}, \quad i^2 = -1.$$

So,

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} = e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x})$$

is a general solution of (2). Next, we will simplify y as a real-valued function by using Euler's identity

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad \theta \in \mathbb{R}.$$

Then,

$$\begin{aligned} y &= e^{\alpha x}(c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)) \\ &= e^{\alpha x}((c_1+c_2)\cos\beta x + (ic_1 - ic_2)\sin\beta x) \\ &= C_1 e^{\alpha x}\cos\beta x + C_2 e^{\alpha x}\sin\beta x, \end{aligned}$$

where C_1, C_2 are constants. Let $y_1 = e^{\alpha x}\cos\beta x$ and $y_2 = e^{\alpha x}\sin\beta x$. Note that

$$\begin{aligned} W(y_1, y_2 : x) &= \begin{vmatrix} e^{\alpha x}\cos\beta x & e^{\alpha x}\sin\beta x \\ e^{\alpha x}(-\beta\sin\beta x + \alpha\cos\beta x) & e^{\alpha x}(\beta\cos\beta x + \alpha\sin\beta x) \end{vmatrix} \\ &= e^{2\alpha x}(\beta\cos^2\beta x + \beta\sin^2\beta x) = \beta e^{2\alpha x} \neq 0 \end{aligned}$$

for all $x \in \mathbb{R}$. So a general solution of (2) is

$$y = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x),$$

where c_1 and c_2 are constants.

Example

Find a general solution of $y'' - 2y' = 0$.

Example

Find a general solution of $y'' - 2y' = 0$.

Solution Note that the characteristic equation is

$$r^2 - 2r = 0 \quad \rightarrow \quad r(r-2) = 0,$$

so $r = 0, 2$. Thus, a general solution of this ODE is

$$y = c_1 + c_2 e^{2x},$$

where c_1, c_2 are constants. $\square$

Example

Find a general solution of $y'' - 3y' - y = 0$.

Example

Find a general solution of $y'' - 3y' - y = 0$.

Solution The characteristic equation is

$$r^2 - 3r - 1 = 0 \quad \rightarrow \quad r = \frac{3 \pm \sqrt{13}}{2}.$$

Thus, a general solution of this ODE is

$$y = c_1 e^{\left(\frac{3+\sqrt{13}}{2}\right)x} + c_2 e^{\left(\frac{3-\sqrt{13}}{2}\right)x},$$

where c_1, c_2 are constants. $\square$

Example

Find a general solution of $y'' - 4y' + 4y = 0$.

Example

Find a general solution of $y'' - 4y' + 4y = 0$.

Solution The characteristic equation is

$$r^2 - 4r + 4 = 0 \quad \rightarrow \quad r = 2, 2.$$

Thus, a general solution of this ODE is

$$y = c_1 e^{2x} + c_2 x e^{2x},$$

where c_1, c_2 are constants. □

Example

Find the particular solution of

$$y'' - 10y' + 25y = 0, \quad y(0) = 3, \quad y'(0) = 4.$$

Example

Find the particular solution of

$$y'' - 10y' + 25y = 0, \quad y(0) = 3, \quad y'(0) = 4.$$

Solution The characteristic equation is

$$r^2 - 10r + 25 = 0 \quad \rightarrow \quad r = 5, 5.$$

Thus, a general solution of this ODE is

$$y = c_1 e^{5x} + c_2 x e^{5x},$$

where c_1, c_2 are constants. So,

$$y' = 5c_1 e^{5x} + 5c_2 x e^{5x} + c_2 e^{5x}.$$

Note that $y(0) = 3$ and $y'(0) = 4$, we have $c_1 = 3$ and $c_2 = -11$. Thus, the particular solution is

$$y = 3e^{5x} - 11xe^{5x}. \quad \square$$

Example

Find a general solution of $y'' + 5y = 0$.

Example

Find a general solution of $y'' + 5y = 0$.

Solution The characteristic equation is

$$r^2 + 5 = 0 \quad \rightarrow \quad r = i\sqrt{5}, -i\sqrt{5}.$$

Thus, a general solution of this ODE is

$$y = c_1 \cos \sqrt{5}x + c_2 \sin \sqrt{5}x,$$

where c_1, c_2 are constants. $\square$

Example

Find the particular solution of

$$4y'' + 16y' + 17y = 0, \quad y(0) = 1, \quad y(\pi) = 0.$$

Example

Find the particular solution of

$$4y'' + 16y' + 17y = 0, \quad y(0) = 1, \quad y(\pi) = 0.$$

Solution The characteristic equation is

$$4r^2 + 16r + 17 = 0 \quad \rightarrow \quad r = -2 \pm \frac{1}{2}i.$$

Thus, a general solution of this ODE is

$$y = e^{-2x}\left(c_1 \cos\frac{x}{2} + c_2 \sin\frac{x}{2}\right),$$

where c_1, c_2 are constants.

From $y(0) = 1$ and $y(\pi) = 0$, we have

$$c_1 = 1 \quad \text{and} \quad c_2 = 0.$$

Thus, the particular solutioni is

$$y = e^{-2x} \cos \frac{x}{2}. \quad \square$$

Next, we will find a solution of a linear homogeneous equation ODE with with constant coefficients order n,

$$a_n y^{(n)} + \ldots + a_1 y' + a_0 y = 0, \tag{4}$$

where $a_n, \ldots, a_1, a_0$ are constants.
We can show that the auxiliary equation is

$$a_n r^n + \ldots + a_1 r + a_0 = 0. \tag{5}$$

Theorem

If y_i is a particular solution of (4) that corresponding to a root of (5) r_i and r_i is repeated k times, where $k \geq 2$, then

$$xy_i, x^2y_i, \ldots, x^{k-1}y_i$$

are particular solutions of (4) and a linear combination of all particular solutions of (4) is a general solution of (4).

Example

Find a general solution of

$$y''' - 5y'' + 6y' = 0.$$

Example

Find a general solution of

$$y''' - 5y'' + 6y' = 0.$$

Solution The characteristic equation is

$$r^3 - 5r^2 + 6r = 0 \quad \rightarrow \quad r = 0, -2, -3.$$

So, a general solution is

$$y = c_1 + c_2 e^{-2x} + c_3 e^{-3x},$$

where c_1, c_2, c_3 are constants. $\square$

Example

Find the particular solution of

$$y''' + 3y'' + 3y' + y = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 1.$$

Example

Find the particular solution of

$$y''' + 3y'' + 3y' + y = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 1.$$

Solution The characteristic equation is

$$r^3 + 3r^2 + 3r + 1 = 0 \quad \rightarrow \quad r = -1, -1, -1.$$

Thus, a general solution of this ODE is

$$y = (c_1 + c_2 x + c_3 x^2)e^{-x},$$

where c_1, c_2, c_3 are constants.

From $y(0) = 0$, we have $c_1 = 0$. Since $y'(0) = 1$ and

$y' = -(c_2x + c_3x^2)e^{-x} + (c_2 + 2c_3x)e^{-x} = (c_2 + (2c_3 - c_2)x + c_3x^2)e^{-x}$,

so, we obtain $c_2 = 1$. From

$$y'' = ((2c_3 - 2) + x - c_3x^2)e^{-x},$$

and $y''(0) = 1$, so $c_3 = \dfrac{3}{2}$. Thus, the particular solution is

$$y = \left(x + \frac{3}{2}x^2\right)e^{-x}. \quad \square$$

Example

Find a general solution of

$$y^{(4)} - 2y''' + 6y'' - 8y' + 8y = 0.$$

Example

Find a general solution of

$$y^{(4)} - 2y''' + 6y'' - 8y' + 8y = 0.$$

Solution The characteristic equation

$$r^4 - 2r^3 + 6r^2 - 8r + 8 = 0 \quad \rightarrow \quad r = \pm i\sqrt{2}, 1 \pm i.$$

So, a general solution is

$$y = c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x + e^x(c_3 \cos\sqrt{x} + c_3 \sin\sqrt{x}),$$

where c_1, c_2, c_3, c_4 are constants. $\square$

Example

Find a general solution of

$$y''' + 4y'' - 3y' - 18y = 0.$$

Example

Find a general solution of

$$y''' + 4y'' - 3y' - 18y = 0.$$

Solution The characteristic equation is

$$r^3 + 4r^2 - 3r - 18 = 0 \quad \rightarrow \quad (r-2)(r+3)^2 = 0.$$

So, $r = 2, -3, -3$. Thus a general solution is

$$y = c_1 e^{2x} + c_2 e^{-3x} + c_3 x e^{-3x},$$

where c_1, c_2, c_3 are constants. □

Example

Find a general solution of

$$y^{(4)} - 8y''' + 26y'' - 40y' + 25y = 0.$$

Example

Find a general solution of

$$y^{(4)} - 8y''' + 26y'' - 40y' + 25y = 0.$$

Solution The characteristic equation is

$$r^4 - 8r^3 + 26r^2 - 40r + 25 = 0 \quad \rightarrow \quad (r^2 - 4r + 5)^2 = 0.$$

So, $r = 2+i, 2+i, 2-i, 2-i$. Thus a general solution is

$$y = e^{2x}(c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x),$$

where c_1, c_2, c_3, c_4 are constants. $\square$

Example

Find a general solution of

$$y^{(8)} + 6y^{(6)} - 32y'' = 0.$$

Example

Find a general solution of

$$y^{(8)} + 6y^{(6)} - 32y'' = 0.$$

Solution The characteristic equation is

$$r^8 + 6r^6 - 32r^2 = 0 \quad \rightarrow \quad r^2(r^2 - 2)(r^2 + 4)^2 = 0.$$

So, $r = 0, 0, \pm\sqrt{2}, 2i, 2i, -2i, -2i$. Thus a general solution is

$$y = c_1 + c_2 x + c_3 e^{\sqrt{2}x} + c_4 e^{-\sqrt{2}x} + c_5 \cos 2x + c_6 \sin 2x + c_7 x \cos 2x + c_8 x \sin 2x,$$

where $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8$ are constants. $\square$