Ordinary Differential Equation

Chapter III : Laplace Transformation

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Definition

Let $f : [0, \infty) \to \mathbb{R}$ and S be a subset of $\mathbb{R}$ such that $\displaystyle\int_0^\infty e^{-st} f(t)dt$ converges for all $s \in S$. We define the function $F : S \to \mathbb{R}$ by

$$F(s) = \int_0^\infty e^{-st} f(t)dt, \quad s \in S.$$

We say that $F(s)$ is **Laplace transform** of $f(t)$. Sometimes, we denote by $\mathcal{L}\{f(t)\}(s)$ or $\mathcal{L}\{f(t)\}$

Example

Find $\mathcal{L}\{1\}$.

Example

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Solution

$$\begin{aligned}
\mathcal{L}\{1\} &= \int_0^\infty e^{-st}dt \\
&= \lim_{k\to\infty} \int_0^k e^{-st}dt \\
&= \lim_{k\to\infty} \left(\frac{1}{s} - \frac{e^{-sk}}{s}\right) \\
&= \frac{1}{s}. \quad \square
\end{aligned}$$

Example

Find $\mathcal{L}\{e^{at}\}$, where a is a constant.

Example

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<u>Solution</u>

$$\begin{aligned}
\mathcal{L}\{e^{at}\} &= \int_0^\infty e^{at}e^{-st}dt \\
&= \lim_{k\to\infty}\int_0^k e^{-(s-a)t}dt \\
&= \lim_{k\to\infty}\left(\frac{1}{s-a} - \frac{e^{-(s-a)k}}{s-a}\right) \\
&= \frac{1}{s-a}. \quad \square
\end{aligned}$$

Example

Find $\mathcal{L}\{\cos bt\}$, where b is a constant

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Solution Note that

$$\int e^{at} \cos btdt = \frac{ae^{at} \cos bt + be^{at} \sin bt}{a^2 + b^2} + C.$$

So,

$$\begin{aligned} \mathcal{L}\{\cos bt\} &= \int_0^\infty e^{-st} \cos btdt \\ &= \lim_{k\to\infty} \left(\frac{-se^{-sk} \cos bk + be^{-sk} \sin bt}{s^2 + b^2} + \frac{s}{s^2 + b^2} \right) \\ &= \frac{s}{s^2 + b^2}. \quad \square \end{aligned}$$

Or using the Euler's identity

$$\cos bt = \frac{e^{ib} + e^{-ib}}{2}, \quad \text{where} \quad i^2 = -1.$$

So,

$$\begin{aligned}
\mathcal{L}\{\cos bt\} &= \frac{1}{2}\int_0^\infty e^{-st}(e^{ib} + e^{-ib})dt \\
&= \frac{1}{2}\left(\int_0^\infty e^{ib}e^{-st}dt + \int_0^\infty e^{-ib}e^{-st}dt\right) \\
&= \frac{1}{2}\left(\mathcal{L}\{e^{ib}\} + \mathcal{L}\{e^{-ib}\}\right) \\
&= \frac{1}{2}\left(\frac{1}{s-ib} + \frac{1}{s+ib}\right) \\
&= \frac{s}{s^2+b^2}.
\end{aligned}$$

Example

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Solution Note that

$$\int e^{at} \sin btdt = \frac{ae^{at} \sin bt - be^{at} \cos bt}{a^2 + b^2} + C.$$

So,

$$\begin{aligned}
\mathcal{L}\{\sin bt\} &= \int_0^\infty e^{-st} \sin btdt \\
&= \lim_{k \to \infty} \left(\frac{-se^{-sk} \sin bk - be^{-sk} \sin bt}{s^2 + b^2} + \frac{b}{s^2 + b^2} \right) \\
&= \frac{b}{s^2 + b^2}. \quad \square
\end{aligned}$$

Or using the Euler's identity

$$\cos bt = \frac{e^{ib} - e^{-ib}}{2i}, \quad \text{where} \quad i^2 = -1.$$

So,

$$\begin{aligned}
\mathcal{L}\{\sin bt\} &= \frac{1}{2i}\int_0^{\infty} e^{-st}(e^{ib} - e^{-ib})dt \\
&= \frac{1}{2i}\left(\int_0^{\infty} e^{ib}e^{-st}dt - \int_0^{\infty} e^{-ib}e^{-st}dt\right) \\
&= \frac{1}{2i}\left(\mathcal{L}\{e^{ib}\} - \mathcal{L}\{e^{-ib}\}\right) \\
&= \frac{1}{2i}\left(\frac{1}{s-ib} - \frac{1}{s+ib}\right) \\
&= \frac{b}{s^2+b^2}.
\end{aligned}$$

Example

Find $\mathcal{L}\{t^n\}$, where n is a positive integer.

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<u>Solution</u> Note that

$$\begin{aligned}
\int_0^\infty t^n e^{-st}dt &= \lim_{k\to\infty}\int_0^k t^n e^{-st}dt \\
&= \lim_{k\to\infty}\left(-\frac{t^n e^{-st}}{s}\Bigg]_{t=0}^{t=k} + \frac{n}{s}\int_0^k t^n e^{-st}dt\right) \\
&= \lim_{k\to\infty}\left(-\frac{k^n e^{-sk}}{s} + \frac{n}{s}\int_0^k t^{n-1} e^{-st}dt\right) \\
&= \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt.
\end{aligned}$$

By mathematical induction, we can show that

$$\int_0^\infty t^n e^{-st} dt = \frac{n(n-1)\ldots 3 \cdot 2 \cdot 1}{s^n} \int_0^\infty e^{-st} dt = \frac{n!}{s^n} \int_0^\infty e^{-st} dt.$$

Note that

$$\int_0^\infty e^{-st} dt = \mathcal{L}\{1\} = \frac{1}{s}.$$

So,

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}. \quad \square$$

Example

Find $\mathcal{L}\{f(t)\}$, where $f(t) = \begin{cases} 1 & ; 0 < t < 4 \\ t & ; t \geq 4 \end{cases}$.

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Solution

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^4 e^{-st}dt + \int_4^\infty te^{-st} \\
&= -\frac{e^{-st}}{s}\Big]_{t=0}^{t=4} + \lim_{k\to\infty}\left(-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\right)\Big]_{t=4}^{t=k} \\
&= -\frac{e^{-4s}}{s} + \frac{1}{s} + \frac{4e^{-4s}}{s} + \frac{e^{-4s}}{s^2} \\
&= \frac{1}{s} + \frac{(3s+1)e^{-4s}}{s^2}. \quad \square
\end{aligned}$$

Definition

Let $f : [a, b] \to \mathbb{R}$ be function. We say that f is a **piecewise continuous** on $[a, b]$ if there exists $t_1, \ldots, t_i \in (a, b)$, where $t_1 < \ldots < t_i$ that satisfy the following conditions

1. f is continuous on (t_{k-1}, t_k) for $k = 1, \ldots, i+1$. (We let $t_0 = a$ and $t_{i+1} = b$)
2. For each subinterval $[t_{k-1}, t_k]$, f is discontinuous at the endpoint of subinterval t_{k-1} and t_k, moreover the limits

$$\lim_{t \to t_{k-1}^+} f(t), \quad \text{and} \quad \lim_{t \to t_k^-} f(t)$$

exists.

Definition

We say that $f(t)$ is of **exponential order** if there exist $\alpha, t_0 \in \mathbb{R}$ and $M > 0$ such that

$$|f(t)| \leq Me^{\alpha t}$$

for $t \geq t_0$, denoted by $f(t) = O(e^{\alpha t})$.

Theorem

If $\lim_{t\to\infty} e^{-\alpha t}|f(t)|$ exists for some $\alpha \in \mathbb{R}$, then $f(t) = O(e^{\alpha t})$.

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Theorem

If $\lim\limits_{t\to\infty} e^{-\alpha t}|f(t)| = \infty$ for all $\alpha \in \mathbb{R}$, then $f(t)$ is not of exponential order.

Example

Show that $f(t) = \dfrac{e^{2t}}{t+3}$ is of exponential order.

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Solution It is easy to see that

$$\lim_{t\to\infty} \frac{e^{2t} \cdot e^{-2t}}{t+3} = \lim_{t\to\infty} \frac{1}{t+2} = 0,$$

so $\dfrac{e^{2t}}{t+3} = O(e^{2t})$. $\square$

Example

Show that $f(t) = e^{t^2}$ is not of exponential order.

Example

Show that $f(t) = e^{t^2}$ is not of exponential order.

Solution Let $\alpha \in \mathbb{R}$. Note that $\lim\limits_{t\to\infty}(t^2 - \alpha t) = \infty$, so

$$\lim_{t\to\infty} e^{t^2 - \alpha t} = \infty. \quad \square$$

Thus, e^{t^2} is not of exponential order. $\square$

Example

Show that $f(t) = t^n$ is of exponential order for all $n \in \mathbb{N}$.

Example

Show that $f(t) = t^n$ is of exponential order for all $n \in \mathbb{N}$.

Solution Let $\alpha > 0$. By using L'hóstipal's rule n times, we have

$$\lim_{s\to\infty} t^n e^{-\alpha t} = \lim_{s\to\infty} \frac{t^n}{e^{\alpha t}} = \frac{n!}{\alpha^n e^{\alpha t}} = 0.$$

So, $t^n = O(e^{\alpha t})$ for $\alpha > 0$. $\square$

Theorem : Existence of Laplace Transformation

Let $f(t)$ be function such that

$$\int_0^{t_0} e^{-st} f(t)dt$$

exists for all $t_0 > 0$ and there exists $\alpha > 0$ such that $f(t) = O(e^{\alpha t})$, then the Laplace transform $\mathcal{L}\{f(t)\}$ exists for $s > \alpha$.

Theorem

Let $f : [0, T] \to \mathbb{R}$ be continuous function for all $T > 0$ and $f(t) = O(e^{\alpha t})$ for some $\alpha > 0$, then the Laplace transform $\mathcal{L}\{f(t)\}$ exists for $s > \alpha$.

Theorem

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Theorem

Let $f : [0, T] \to \mathbb{R}$ be continuous function for all $T > 0$ and $f(t) = O(e^{\alpha t})$ for some $\alpha > 0$ and $F(s) = \mathcal{L}\{f(t)\}$, then $\lim_{s\to\infty} F(s) = 0$.

Definition

A function f is a **function of class** A if

1. f is continuous function on $[0, T]$ for all $T > 0$,
2. $f(t) = O(e^{\alpha t})$ for some $\alpha \in \mathbb{R}$.

From the definition of function of class A, we have

Theorem

Let f be a function of class A, then $\mathcal{L}\{f(t)\}$ exists.

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Let f be a function of class A and $F(s) = \mathcal{L}\{f(t)\}$, then $\lim\limits_{s\to\infty} F(s) = 0$.

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Theorem

Let f be a function of class A and $F(s) = \mathcal{L}\{f(t)\}$, then $\lim_{s\to\infty} F(s) = 0$.

Remark

If f is **not** a function of class A, $\mathcal{L}\{f(t)\}$ **might** exists.

Example

Let $f(t) = t^{-\frac{1}{2}}$. Show that $\mathcal{L}\{f(t)\}$ exists and find its Laplace transform.

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Let $f(t) = t^{-\frac{1}{2}}$. Show that $\mathcal{L}\{f(t)\}$ exists and find its Laplace transform.

Solution Note that

$$\lim_{t\to\infty} t^{-\frac{1}{2}} e^{-t} = 0,$$

so $t^{-\frac{1}{2}} = O(e^t)$. By using advanced calculus, we can show that

$$\int_0^{t_0} e^{-st} t^{-\frac{1}{2}} dt$$

for $t_0 > 0$. So Laplace transform of $t^{-\frac{1}{2}}$ exists.

Next, we consider

$$\int_0^\infty e^{-st} t^{-\frac{1}{2}} dt$$

Let $r = st$, then $dr = -sdt$. So $t = 0 \to r = 0$ and if $t \to \infty$, then $r \to \infty$ and

$$\int_0^\infty e^{-st} t^{-\frac{1}{2}} dt = \frac{1}{\sqrt{s}} \int_0^\infty e^{-r} r^{-\frac{1}{2}} dr.$$

By advanced calculus knowledge, we can show that

$$\int_0^\infty e^{-r} r^{-\frac{1}{2}} dr = \sqrt{\pi}.$$

So,

$$\mathcal{L}\{t^{-\frac{1}{2}}\} = \sqrt{\frac{\pi}{s}}. \quad \square$$

Remark

Let $r > -1$. We can show that the Laplace transform of t^r exists and

$$\mathcal{L}\{t^r\} = \frac{\Gamma(r+1)}{s^{r+1}},$$

where $\Gamma(x)$ is **Gamma function**, defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$$

for $x > 0$.

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$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$$

for $x > 0$.

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Corollary

If $\lim_{s\to\infty} F(s) \neq 0$, then there is no function f of class A such that $\mathcal{L}\{f(t)\} = F(s)$.

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Example

Let $F(s) = \dfrac{s(2s+4)}{s^2+1}$, then there is no function f of class A such that $\mathcal{L}\{f(t)\} = F(s)$ since $\lim_{s\to\infty} F(s) = 2 \neq 0$.

Question

Is it true that $f_1(t) = f_2(t)$ if $\mathcal{L}\{f_1(t)\} = \mathcal{L}\{f_2(t)\}$?

Example

Let $f_1, f_2 : [0,\infty) \to \mathbb{R}$ by

$$f_1(t) = \begin{cases} 2 & ; 0 < t < 3 \\ 1 & ; t = 1 \\ t & ; t > 3 \end{cases} \quad \text{and} \quad f_2(t) = \begin{cases} 2 & ; 0 < t < 3 \\ t & ; t \geq 3 \end{cases}$$

for all $t \in [0,\infty)$. Show that
$\mathcal{L}\{f_1(t)\} = \mathcal{L}\{f_2(t)\} = \dfrac{2}{s} + \dfrac{(s+1)e^{-3s}}{s^2}, s > 0.$

From problem, we have

$$\begin{aligned}
\mathcal{L}\{f_1(t)\} &= \int_0^3 2e^{-st}dt + \int_3^\infty te^{-st}dt \\
&= -2\left(\frac{e^{-st}}{s}\right)\Bigg]_{t=0}^{t=3} + \lim_{k\to\infty}\left(-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\right)\Bigg]_{t=3}^{t=k} \\
&= -\frac{2e^{-3t}}{s} + \frac{2}{s} + \frac{3e^{-3t}}{s} + \frac{e^{-3t}}{s^2} \\
&= \frac{2}{s} + \frac{(s+1)e^{-3t}}{s^2}.
\end{aligned}$$

On the other hand, we can show that

$$\mathcal{L}\{f_2(t)\} = \frac{2}{s} + \frac{(s+1)e^{-3t}}{s^2}. \quad \square$$

From the previous example, it is easy to see that

$$\lim_{s\to\infty} F(s) = \lim_{s\to\infty}\left(\frac{2}{s} + \frac{(s+3)e^{-3s}}{s^2}\right) = 0,$$

f_1 and f_2 are discontinuous at $x = 3$ and $f_1(x) = f_2(x)$ for $x \neq 3$.

Theorem

Let $f, g : [0, T] \to \mathbb{R}$ be continuous function for every $T > 0$ and there exists $\alpha \in \mathbb{R}$ such that $\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\}, s > \alpha$, then $f(t) = g(t)$ for every $t \in [0, \infty)$ such that f and g are continuous at t. Moreover, if f and g are continuous on $[0, \infty)$, then $f = g$.

Example

Consider
$$F(s) = \frac{1}{s-a}.$$

Note that $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$ and $f(t) = e^{at}$ is continuous on $[0,\infty)$. So $f(t) = e^{at}$ is the only one continuous function on $[0,\infty)$ such that
$$\mathcal{L}\{f(t)\} = \frac{1}{s-a}. \quad \square$$

Theorem

Let $F : (s_0, \infty) \to \mathbb{R}$ for some $s_0 \in \mathbb{R}$ and $\lim_{s\to\infty} F(s) = 0$, then there exists unique continuous function f on $[0, \infty)$ and f is of exponential order such that

$$\mathcal{L}\{f(t)\} = F(s).$$

We say that f is the **inverse of Laplace transform** of $F(s)$, denoted by $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

From section 1 we have the inverse of Laplace transform of the following functions,

1. $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1.$
2. $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}.$
3. $\mathcal{L}^{-1}\left\{\frac{b}{s^2+b^2}\right\} = \sin bt.$
4. $\mathcal{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\} = \cos bt.$
5. $\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n, n \in \mathbb{N}.$
6. $\mathcal{L}^{-1}\left\{\frac{\Gamma(x+1)}{s^{x+1}}\right\} = t^x, x > -1$

Theorem : Linearity Property

Let $f(t)$ and $g(t)$ be functions such that the Laplace transform $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{g(t)\}$ exist for $s > \alpha$ and $s > \beta$, respectively, and a, b are constants, then

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\},$$

where $s > \max\{\alpha, \beta\}$.

Theorem : First Shifting Theorem

Let $f(t)$ be a function that the Laplace transform $F(s)$ exists for $s > \alpha$ and a is a constant, then

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a),$$

where $s > a + \alpha$.

Theorem : First Shifting Theorem

Let $f(t)$ be a function that the Laplace transform $F(s)$ exists for $s > \alpha$ and a is a constant, then

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a),$$

where $s > a + \alpha$.

We use the symbol for this operation by $\mathcal{L}\{f(t)\}_{s \to s-a}$.

Theorem : Multiplication by t^n Property

Let $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n} F(s),$$

where $n \in \mathbb{N}$.

Example

Find $\mathcal{L}\{t \sin bt\}$, where b is a constant.

Example

Find $\mathcal{L}\{t\sin bt\}$, where b is a constant.

Solution

$$\begin{aligned}
\mathcal{L}\{t\sin bt\} &= -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{\sin bt\} \\
&= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{b}{s^2+b^2}\right) \\
&= \frac{2bs}{(s^2+b^2)^2}. \quad \square
\end{aligned}$$

Example

Find $\mathcal{L}\{t^4 e^{2t}\}$

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Find $\mathcal{L}\{t^4e^{2t}\}$

Solution

$$\begin{aligned}\mathcal{L}\{t^4e^{2t}\} &= \mathcal{L}\{t^4\}_{s\to s-2} \\ &= \left.\frac{4!}{s^5}\right|_{s\to s-2} \\ &= \frac{4!}{(s-2)^5}. \quad \square\end{aligned}$$

Or

$$\begin{aligned}\mathcal{L}\{t^4 e^{2t}\} &= \frac{\mathrm{d}^4}{\mathrm{d}s^4}\mathcal{L}\{e^{2t}\} \\ &= \frac{\mathrm{d}^4}{\mathrm{d}s^4}\left(\frac{1}{s-2}\right) \\ &= \frac{4!}{(s-2)^5}. \quad \square\end{aligned}$$

Theorem

Let $\mathcal{L}\{f(t)\} = F(s), P(t) = a_n t^n + \ldots + a_1 t + a_0$ and $D = \dfrac{d}{ds}$, then

$$\mathcal{L}\{P(t)f(t)\} = P(-D)F(s).$$

Example

Find $\mathcal{L}\{(4+5t-t^2)\cos t\}$

Example

Find $\mathcal{L}\{(4+5t-t^2)\cos t\}$

Solution

$$\begin{aligned}\mathcal{L}\{(4+5t-t^2)\cos t\} &= 4\mathcal{L}\{\cos t\} - 5\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{\cos t\} - \frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{L}\{\cos t\} \\ &= \frac{4s}{s^2+1} - 5\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s}{s^2+1}\right) - \frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(\frac{s}{s^2+1}\right) \\ &= \frac{4s}{s^2+1} - \frac{5-5s^2}{(s^2+1)^2} - \frac{2s^3-6s}{(s^2+1)^3}. \qquad \square\end{aligned}$$

Example

Find $\mathcal{L}^{-1}\left\{\ln\left(1+\frac{c^2}{s^2}\right)\right\}$, where c is a constant.

Example

Find $\mathcal{L}^{-1}\left\{\ln\left(1+\frac{c^2}{s^2}\right)\right\}$, where c is a constant.

Solution Let $f(t) = \mathcal{L}^{-1}\left\{\ln\left(1+\frac{c^2}{s^2}\right)\right\}$, then

$$\mathcal{L}\{f(t)\} = \ln\left(1+\frac{c^2}{s^2}\right).$$

So

$$\mathcal{L}\{tf(t)\} = \frac{\mathrm{d}}{\mathrm{d}s}\left(\ln\left(1+\frac{c^2}{s^2}\right)\right) = \frac{2s}{s^2+c^2} - \frac{2}{s}.$$

Hence,

$$tf(t) = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+c^2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 2\cos ct - 2$$

$$f(t) = \frac{2\cos ct - 1}{t}. \quad \square$$

Example

Find $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\}$.

Example

Find $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\}$.

Solution Note that

$$\frac{s}{(s^2+1)^2} = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s^2+1}\right) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{\sin t\} = \frac{1}{2}\mathcal{L}\{t\sin t\}.$$

So,

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2}t\sin t. \quad \square$$

Theorem : Laplace transform of the *n*th derivative

Let f be n times continuous differentiable function on $[0, \infty)$, then

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1}f(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

Example

Find $\mathcal{L}\{\sin^2 t\}$.

Example

Find $\mathcal{L}\{\sin^2 t\}$.

Solution Let $f(t) = \sin^2 t$, then

$$f(0) = 0 \quad \text{and} \quad f'(t) = 2\sin t\cos t = \sin 2t.$$

So,

$$\begin{aligned}\mathcal{L}\{\sin 2t\} &= s\mathcal{L}\{\sin^2 t\} - f(0)\\ \frac{2}{s^2+4} &= s\mathcal{L}\{\sin^2 t\}.\end{aligned}$$

Thus,

$$\mathcal{L}\{\sin^2 t\} = \frac{2}{s(s^2+4)}. \quad \square$$

Example

Find the particular solution of IVP

$$y'' + 2y' + 5y = 0, \quad y(0) = 2, \quad y'(0) = -4.$$

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$$y'' + 2y' + 5y = 0, \quad y(0) = 2, \quad y'(0) = -4.$$

Solution Let $Y(s) = \mathcal{L}\{y\}$, then

$$\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) - 2,$$

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2s + 4.$$

So,

$$\begin{aligned}(s^2Y(s) - 2s + 4) + 2(sY(s) - 2) + 5Y(s) &= 0\\ (s^2 + 2s + 5)Y(s) &= 2s + 4\\ Y(s) &= \frac{2s+4}{s^2+2s+5}.\end{aligned}$$

Hence,

$$\begin{aligned}y &= \mathcal{L}^{-1}\left\{\frac{2s+2}{(s+1)^2+1} + \frac{2}{(s+1)^2+1}\right\}\\ &= 2\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}\\ &= 2\mathcal{L}^{-1}\left\{\left.\frac{s}{s^2+1}\right|_{s+1\to s}\right\} + 2\mathcal{L}^{-1}\left\{\left.\frac{1}{s^2+1}\right|_{s+1\to s}\right\}\\ &= e^{-t}\cos t + e^{-t}\sin t. \quad \square\end{aligned}$$

Example

Find the particular solution of IVP

$$y'' + y = 2t, \quad y\left(\frac{\pi}{4}\right) = \frac{\pi}{2}, \quad y'\left(\frac{\pi}{4}\right) = 2 - \sqrt{2}.$$

Example

Find the particular solution of IVP

$$y'' + y = 2t, \quad y\left(\frac{\pi}{4}\right) = \frac{\pi}{2}, \quad y'\left(\frac{\pi}{4}\right) = 2 - \sqrt{2}.$$

Solution Let $Y(s) = \mathcal{L}\{y\}, y(0) = A$ and $y'(0) = B$. Then

$$\mathcal{L}\{y''\} = s^2Y(s) - As - B.$$

So,

$$(s^2Y(s) - As - B) + Y(s) = \frac{2}{s^2}$$

$$Y(s) = \frac{2}{s^2(s^2+1)} + \frac{As}{s^2+1} + \frac{B}{s^2+1}.$$

Note that

$$\frac{2}{s^2(s^2+1)} = \frac{2}{s^2} - \frac{2}{s^2+1}.$$

Then,

$$Y(s) = \frac{2}{s^2} - \frac{2}{s^2+1} + \frac{As}{s^2+1} + \frac{B}{s^2+1}.$$

So,

$$\begin{aligned} y &= 2\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + A\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + B\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\ &= 2t - 2\sin t + A\cos t + B\sin t. \end{aligned}$$

Hence,

$$y' = 2 - 2\cos t - A\sin t + B\cos t.$$

From $y\left(\dfrac{\pi}{4}\right) = \dfrac{\pi}{2}$ and $y'\left(\dfrac{\pi}{4}\right) = 2 - \sqrt{2}$, we have

$$\frac{\pi}{2} = \frac{\pi}{2} - \sqrt{2} + \frac{A}{\sqrt{2}} + \frac{B}{\sqrt{2}} \quad \rightarrow \quad A + B = 2$$

$$2 - \sqrt{2} = 2 - \sqrt{2} - \frac{A}{\sqrt{2}} + \frac{B}{\sqrt{2}} \quad \rightarrow \quad A = B.$$

This implies $A = B = 1$. Thus, the particular solution is

$$y = 2 - 2\cos t - \sin t + \cos t = 2 - \cos t - \sin t. \quad \square$$

Theorem : Laplace transform of an integral

Let $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\left\{ \int_a^t f(x)dx \right\} = \frac{1}{s}F(s) - \frac{1}{s}\int_0^a f(x)dx.$$

Remark

If $a = 0$, then

$$\mathcal{L}\left\{ \int_0^t f(x)dx \right\} = \frac{1}{s}F(s).$$

Example

Find $\mathcal{L}\left\{ \int_0^t x\sin(80x)dx \right\}$.

Example

Find $\mathcal{L}\left\{\int_0^t x\sin(80x)dx\right\}$.

Solution

$$\begin{aligned}\mathcal{L}\left\{\int_0^t x\sin(80x)dx\right\} &= \frac{1}{s}\mathcal{L}\{t\sin 80t\}\\ &= -\frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{\sin 80t\}\\ &= -\frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{80}{s^2+80^2}\right)\\ &= \frac{160}{(s^2+80^2)^2}. \qquad \square\end{aligned}$$

Example

Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s(s^2+1)}\right\}$.

Example

Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s(s^2+1)}\right\}$.

Solution Note that $\mathcal{L}\{\sin t\} = \dfrac{1}{s^2+1}$. So

$$\begin{aligned}\frac{1}{s(s^2+1)} &= \frac{1}{s}\mathcal{L}\{\sin t\} \\ &= \mathcal{L}\left\{\int_0^t \sin x dx\right\} \\ &= \mathcal{L}\{1-\cos t\}.\end{aligned}$$

Thus,

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1-\cos t. \quad \square$$

Theorem

Let $\mathcal{L}\{f(t)\} = F(s)$ and $\displaystyle\lim_{t \to 0^+} \frac{f(t)}{t}$ exists, then

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(r)dr.$$

Example

1. Find $\mathcal{L}\left\{\dfrac{\sin t}{t}\right\}$.
2. Evaluate $\displaystyle\int_0^\infty \frac{\sin t}{t}dt$ and $\displaystyle\int_0^\infty e^{-t}\frac{\sin t}{t}dt$.

Example

1. Find $\mathcal{L}\left\{\dfrac{\sin t}{t}\right\}$.
2. Evaluate $\displaystyle\int_0^\infty \frac{\sin t}{t}dt$ and $\displaystyle\int_0^\infty e^{-t}\frac{\sin t}{t}dt$.

<u>Solution</u>

1. Note that $\displaystyle\lim_{t\to 0^+}\frac{\sin t}{t} = 1$ and $\mathcal{L}\{\sin t\} = \dfrac{1}{s^2+1}$, so

$$\begin{aligned}\mathcal{L}\left\{\frac{\sin t}{t}\right\} &= \int_s^\infty \frac{1}{r^2+1}dr = \lim_{k\to\infty} \arctan r\Big]_{r=s}^{r=k} \\ &= \lim_{k\to\infty}(\arctan k - \arctan s) = \frac{\pi}{2} - \arctan s. \qquad \square\end{aligned}$$

2. From 1, we have

$$\begin{aligned}\int_0^\infty \frac{\sin t}{t}dt = \int_0^\infty e^{-0t}\frac{\sin t}{t}dt &= \mathcal{L}\left\{\frac{\sin t}{t}\right\}(0)\\ &= \frac{\pi}{2} - \arctan 0\\ &= \frac{\pi}{2}\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty e^{-t}\frac{\sin t}{t}dt &= \mathcal{L}\left\{\frac{\sin t}{t}\right\}(1)\\ &= \frac{\pi}{2} - \arctan 1\\ &= \frac{\pi}{4}. \quad \square\end{aligned}$$

Example

Find $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\}$.

Example

Find $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\}$.

Solution Let $\mathcal{L}\{f(t)\} = \dfrac{s}{(s^2+1)^2}$ for some function f. Then

$$\begin{aligned}\mathcal{L}\left\{\frac{f(t)}{t}\right\} &= \int_s^\infty \frac{r}{(r^2+1)^2}dr \\ &= -\frac{1}{2}\lim_{k\to\infty}\frac{1}{r^2+1}\Bigg]_{r=s}^{r=k} \\ &= -\frac{1}{2}\lim_{k\to\infty}\left(\frac{1}{k^2+1}-\frac{1}{s^2+1}\right) \\ &= \frac{1}{2(s^2+1)}.\end{aligned}$$

Hence,

$$\frac{f(t)}{t} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \frac{1}{2}\sin t.$$

That is

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = f(t) = \frac{1}{2}t\sin t. \quad \square$$

Definition

Heaviside step function (or **unit step function**) is a function $H : \mathbb{R} \to \mathbb{R}$ defined by

$$H(t) = \begin{cases} 0 & ; t < 0 \\ 1 & ; t > 0 \end{cases}$$

for $x \in \mathbb{R}$.

Remark

Let a and b be nonnegative real numbers, then

$$H(t-a)=\begin{cases}0 & ;t<a\\ 1 & ;t>a\end{cases},\quad H(a-t)=\begin{cases}1 & ;t<a\\ 0 & ;t>a\end{cases}$$

$$H(b(t-a))=\begin{cases}0 & ;t<a\\ 1 & ;t>a\end{cases},\quad H(b(a-t))=\begin{cases}1 & ;t<a\\ 0 & ;t>a\end{cases}.$$

Remark

Let a and b be nonnegative real numbers, then

$$H(t-a) = \begin{cases} 0 & ; t < a \\ 1 & ; t > a \end{cases}, \quad H(a-t) = \begin{cases} 1 & ; t < a \\ 0 & ; t > a \end{cases}$$

$$H(b(t-a)) = \begin{cases} 0 & ; t < a \\ 1 & ; t > a \end{cases}, \quad H(b(a-t)) = \begin{cases} 1 & ; t < a \\ 0 & ; t > a \end{cases}.$$

It is easy to check that

$$H(b(t-a)) = H(t-a) \quad \text{and} \quad H(b(a-t)) = H(a-t)$$

for every nonnegative real numbers a and b.

Example

Write a piecewise function

$$f(t) = \begin{cases} t^2 & ; 0 < t < 1 \\ 2 & ; t > 1 \end{cases}$$

in terms of Heaviside step functions.

Theorem

Let $a \geq 0$, then $\mathcal{L}\{H(t-a)\} = \frac{1}{s}e^{-as}$.

Example

Write a piecewise function

$$f(t) = \begin{cases} -1 & ; 0 < t < 1 \\ 2 & ; 1 < t < 3 \\ 1 & ; t > 3 \end{cases}$$

in terms of Heaviside step functions.

Example

Write a piecewise function

$$f(t) = \begin{cases} \sin x & ; 0 < t < 2 \\ e^t & ; 2 < t < 5 \\ t^2 & ; t > 5 \end{cases}$$

in terms of Heaviside step functions.

Theorem : Second Shifting Theorem

Let $\mathcal{L}\{f(t)\} = F(s)$ and $a \in \mathbb{R}$, then $\mathcal{L}\{H(t-a)f(t-a)\} = e^{-as}F(s)$.

Theorem

Let $a \in \mathbb{R}$, then $\mathcal{L}\{H(t-a)f(t)\} = e^{-as}\mathcal{L}\{f(t+a)\}$.

Example

Find $\mathcal{L}\{f(t)\}$, where $f(t) = \begin{cases} t & ; 0 < t < 1 \\ 1 & ; t > 1 \end{cases}$

Example

Find $\mathcal{L}\{f(t)\}$, where $f(t) = \begin{cases} \cos t & ; 0 < t < 3 \\ 4 & ; 3 < t < 5 \\ t^3 & ; t > 5 \end{cases}$.

Example

Evaluate $\mathcal{L}^{-1}\left\{\frac{1}{s}(e^{-2s} - 3e^{-5s})\right\}$.

Example

Find $\mathcal{L}^{-1}\left\{\dfrac{e^{-3s}}{s-1}\right\}$.

Example

Evaluate $\int_{\pi}^{\infty} e^{-4t} \sin t dt$.

Definition

A function f is said to be **periodic** if there exists $T > 0$ such that $f(t+T) = f(t)$ for all $t \in \mathbb{R}$. A number T for which this is the case is called a **period** of f.

Theorem

If f is a periodic function with period T and f is continuous on every interval with length T, then

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt,$$

where $s > 0$.

Example

Find $\mathcal{L}\{\cos bt\}$.

Example

Fixed $T, k > 0$. Let $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} k & ; 0 < t < T \\ -k & ; T < t < 2T \end{cases}$$

and $f(t+2T) = f(t)$ for all $t \in \mathbb{R}$. Find $\mathcal{L}\{f(t)\}$.

Example

Let $\omega > 0$ and $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} \sin \omega t & ; 0 < t < \frac{\pi}{\omega} \\ 0 & ; \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$$

and $f\left(t + \dfrac{2\pi}{\omega}\right) = f(t)$ for all $t \in \mathbb{R}$. Find $\mathcal{L}\{f(t)\}$.

Example

Let $k \in \mathbb{R}$. Find $\mathcal{L}^{-1}\left\{\dfrac{k}{s} - \dfrac{ke^{-s}}{s(1-e^{-s})}\right\}$.

Definition

Let $\epsilon \neq 0$ and $t_0 \in \mathbb{R}$. Define $\delta_\epsilon : \mathbb{R} \to \mathbb{R}$ by

$$\delta_\epsilon(t - t_0) = \begin{cases} \dfrac{1}{\epsilon} & ; t \in [t_0, t_0 + \epsilon] \\ 0 & ; \text{otherwise} \end{cases}.$$

Definition

Let $\epsilon \neq 0$ and $t_0 \in \mathbb{R}$. Define $\delta_\epsilon : \mathbb{R} \to \mathbb{R}$ by

$$\delta_\epsilon(t - t_0) = \begin{cases} \dfrac{1}{\epsilon} & ; t \in [t_0, t_0 + \epsilon] \\ 0 & ; \text{otherwise} \end{cases}.$$

Remark

It is easy to see show that

$$\int_{-\infty}^{\infty} \delta_\epsilon(t - t_0)dt = 1.$$

Definition

Unit impulse function or **Direc delta function** is a function $\delta(t - t_0)$ defined by

$$\delta(t - t_0) = \lim_{\epsilon \to 0} \delta_\epsilon(t - t_0).$$

Theorem

Let $f : \mathbb{R} \to \mathbb{R}$ and $t_0 \in \mathbb{R}$, then

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0).$$

Theorem : Laplace Transform of Direc Delta Function

Let $t_0 \in \mathbb{R}$, then $\mathcal{L}\{\delta(t-t_0)\} = e^{-t_0 s}$

Example

Let $\alpha > 0$ and $\beta \in \mathbb{R}$. Find solution of BVP

$$y'' + \alpha^2 y = \delta(t - \beta), \quad y(0) = y(1) = 0.$$

Definition

Let $f(t)$ and $g(t)$ be continuous functions on interval $[0, T]$. The **convolution** of f and g is written $f(t) * g(t)$ or $(f * g)(t)$, defined by

$$f(t) * g(t) = \int_0^t f(x)g(t-x)dx.$$

Definition

Let $f(t)$ and $g(t)$ be continuous functions on interval $[0,T]$. The **convolution** of f and g is written $f(t)*g(t)$ or $(f*g)(t)$, defined by

$$f(t)*g(t)=\int_0^t f(x)g(t-x)dx.$$

Remark

$$f*g=g*f.$$

Theorem : Convolution Theorem

Let $f(t)$ and $g(t)$ be continuous functions on interval $[0, T]$. If $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{g(t)\}$ exist, then

$$\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}.$$

Corollary

Let $f(t)$ and $g(t)$ be continuous functions on interval $[0,T]$ and $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$, then

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \mathcal{L}^{-1}\{F(s)\} * \mathcal{L}^{-1}\{G(s)\} = (f * g)(t).$$

Example

Find $e^t * \sin t$.

Example

Let $a \in \mathbb{R}$. Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2(s-a)}\right\}$.

Example

Find $\mathcal{L}^{-1}\left\{\dfrac{2s}{(s^2+16)^2}\right\}$.