Week 2

October 24, 2018

Table of Indefinite Integral

Some common integrals

$\int \frac{1}{x^2+1} dx = arctanx + C$

$\int \frac{1}{\sqrt{1-x^2}} dx = arcsinx + C$

Substitution Rule

Theorem: The Substitution Rule

> $\int f(g(x))g'(x)dx = \int f(u)du$, where $u = g(x)$.

Examples

> Evaluate $\int x^3 sin(x^4 + 9)dx$

> Evaluate $\int \sqrt[3]{4x+1}dx$

> Evaluate $\int tanxdx$

$-ln\ cosx + C$

> Evaluate $\int sec^4 3\theta d\theta$

$\frac{1}{3}(tan3\theta + \frac{tan^3 3\theta}{3}) + C$

Integration by Parts

Let u and v be differentiable functions on the same interval. Then

$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

Integrate both sides with respect to dx and we obtain

$\int udv = uv - \int vdu$

Examples

> Evaluate $\int xcosxdx$

Let $u = x$ and $du = cosxdx$. Then $du = dx$ and $u = sinx + C_1$.

$\int xcosx = x(sinx + C_1) - \int (sinx + C_1)dx$

$= xsinx + C_1x + cosx - C_1x + C_2$

$= xsinx + cosx + C$

> Evaluate $\int xe^x dx$

> Evaluate $\int lnxdx$

> Evaluate $\int cos(lnx)dx$

$\frac{1}{2}(xsinlnx + xcoslnx) + C$

> Evaluate $\int e^{ax} cosbxdx$, where $a, b \in \mathbb{R}$ such that $a^2 + b^2 > 0$

$\frac{1}{a^2+b^2}(be^{ax}sinbx+ae^{ax}cosbx)+C$

> Evaluate $\int sin\sqrt{x}dx$

Trigonometric Integrals

An important trigonometric identity is

$cos2x = cos^2 x - sin^2 x = 2cos^2 x - 1 = 1 - 2sin^2 x$

Theorem The Binomial Theorem

> $(x + y)^k = \dots$

Strategies for evaluating $\int sin^m xcos^n xdx$

1. If n is odd, save one cosine factor and use $cos^2 x = 1 - sin^2 x$ to express the remaining factors in terms of $sinx$, then substitute $u = sinx$

   $sin^m xcos^{2k+1} x = sin^m x(1 - sin^2 x)^k cosxdx$
2. If m is odd, save one sine factor and use $sin^2 x = 1 - cos^2 x$ to express the remaining factors in terms of $cosx$, then substitute $u = cosx$

   $sin^{2k+1} xcos^n x = (1 - cos^2 x)^k cos^n xsinx$
3. If m and n are even, use the half-angle identities $sin^2 x = \frac{1}{2}(1 - cos2x)$, $cos^2 x = \frac{1}{2}(1 + cos2x)$, then expand and solve.

   $sin^{2m} cos^{2n} x = \frac{1}{4}(1 - cos2x)^m (1 + cos2x)^n$

Examples

> Evaluate $\int sin^2xcos^2xdx$

Strategies for evaluating $\int tan^m xsec^n xdx$

1. If n is even, save a factor $sec^2 x$ and use $sec^2 x = 1 + tan^2 x$ to express the remaining factors in terms of $tanx$, then substitute $u = tanx$.

   $tan^m xsec^{2k} x = tan^m x(1 + tan^2 x)^{k-1} sec^2 x$
2. If m is odd, save a factor of $secxtanx$ and use $tan^2 x = sec^2 x - 1$ to express the remaining factors in terms of $secx$.
3. If m is even and n is odd, use $tan^2 x = sec^2 x - 1$ to express the factor in terms of $secx$.

   $tan^{2k} sec^{2l+1} xdx = (sec^2 x - 1)^k sec^{2l+1}$

   and use the binomial theorem to expand the term. Note that the powers of all the terms are odd.

   Next, we consider $\int sec^q xdx$, where q is odd.

Take u = $sec^{q-2}xdx$ and $dv = sec^2xdx$.

Then $\int sec^{q}xdx = sec^{q-2}xtanx - (q-2)\int sec^{q-2}xtan^2xdx$

$= sec^{q-2}xtanx - (q-2)\int sec^{q}xdx + (q-2)\int sec^{q-2}xdx$

$(q-1)\int sec^{q}xdx = sec^{q-2}xtanx + (q-2)\int sec^{q-2}xdx$

Finally, we have $\int sec^{q}xdx = \frac{sec^{q-2}xtanx}{q-1} + \frac{q-2}{q-1}\int sec^{q-2}xdx$.

We continue this process until the power of secant is 1.

This is called a *reduction formula* as the problem is reduced to lower indices of $sec^{q}x$ and the parity stays the same.

Examples

> Evaluate $\int sec^5xtan^3xdx$

> Evaluate $\int sec^3xdx$