Ordinary Differential Equation

Chapter I : Basic Knowledge

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Table of Indefinite Integrals

1. $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx.$
2. $\int cf(x)dx = c\int f(x)dx.$
3. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, where $n \neq -1$.
4. $\int \frac{1}{x}dx = \ln|x| + C.$

Table of Indefinite Integral : Exponential Functions

1. $\int e^x dx = e^x + C.$
2. $\int a^x dx = \frac{a^x}{\ln a} + C$, where $a > 0$ and $a \neq 1$.

Table of Indefinite Integral : Trigonometric Functions

1. $$\int \sin x dx = -\cos x + C.$$
2. $$\int \cos x dx = \sin x + C.$$
3. $$\int \sec^2 x dx = \tan x + C.$$
4. $$\int \csc^2 x dx = -\cot x + C.$$
5. $$\int \sec x \tan x dx = \sec x + C.$$
6. $$\int \csc x \cot x dx = -\csc x + C.$$

Table of Indefinite Integral : Trigonometric Functions

7. $$\int \tan x dx = \ln|\sec x| + C.$$
8. $$\int \sec x dx = \ln|\sec x + \tan x| + C.$$
9. $$\int \cot x dx = \ln|\sin x| + C.$$
10. $$\int \csc x dx = \ln|\csc x - \cot x| + C.$$

Table of Indefinite Integral : Some Fraction Functions

1. $\displaystyle\int \frac{1}{x^2+1}dx = \arctan x + C.$
2. $\displaystyle\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C.$
3. $\displaystyle\int \frac{1}{|x|\sqrt{x^2-1}}dx = \operatorname{arcsec} x + C.$

Theorem : The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

Example

Evaluate $\int x^3 \sin(x^4 + 9)dx$.

Example

Evaluate $\int x^3 \sin(x^4+9)dx$.

Solution Let $u = x^4 + 9$, then $du = 4x^3 dx$. So

$$\begin{aligned}\int x^3 \sin(x^4+9)dx &= \frac{1}{4}\int \sin u du \\ &= -\frac{1}{4}\cos u + C \\ &= -\frac{1}{4}\cos(x^4+9) + C. \quad \square\end{aligned}$$

Example

Find $\displaystyle\int \sqrt[3]{4x+1}\,dx.$

Example

Find $\int \sqrt[3]{4x+1}dx$.

Solution Let $u = 4x+1$, then $du = 4dx$. So

$$\begin{aligned}
\int \sqrt[3]{4x+1}dx &= \frac{1}{4}\int \sqrt[3]{u}du \\
&= \frac{1}{4}\int u^{\frac{1}{3}}du \\
&= \frac{1}{4}\left(\frac{u^{\frac{4}{3}}}{\frac{4}{3}}\right) + C \\
&= \frac{3}{16}(4x+1)^{\frac{4}{3}} + C. \quad \square
\end{aligned}$$

Example

Evaluate $\displaystyle\int \tan x dx$.

Example

Evaluate $\displaystyle\int \tan x dx$.

<u>Solution</u> Note that

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx.$$

Let $u = \cos x$, then $du = -\sin x dx$. So

$$\begin{aligned}\int \tan x dx &= -\int \frac{1}{u} du \\ &= -\ln|u| + C \\ &= -\ln|\cos x| + C \\ &= \ln|\sec x| + C. \quad \square\end{aligned}$$

Example

Find $\int \sec^4 3\theta d\theta$.

Example

Find $\int \sec^4 3\theta d\theta$.

Solution Let $u = 3\theta$, then $du = 3d\theta$. So

$$\int \sec^4 3\theta d\theta = \frac{1}{3}\int \sec^4 u du.$$

Note that

$$\int \sec^4 u du = \int \sec^2 u(1+\tan^2 u)du = \int \sec^2 u du + \int \tan^2 u \sec^2 u du.$$

Consider $\tan^2 u \sec^2 u du$, let $w = \tan u$, then $dw = \sec^2 u du$. So

$$\int \tan^2 u \sec^2 u du = \int w^2 dw = \frac{w^3}{3} + C = \frac{\tan^3 u}{3} + C.$$

So,

$$\int \sec^4 u du = \tan u + \frac{\tan^3 u}{3} + C.$$

That is

$$\int \sec^4 3\theta d\theta = \frac{\tan 3\theta}{3} + \frac{\tan^3 3\theta}{9} + C. \quad \square$$

Find $\int e^x dx$.

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It is easy to see that

$$\int e^x dx = e^x + C,$$

where C is a constant.

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Try to find $\int xe^x dx$.

Recall

Let u and v be differentiable functions on the same interval. Then

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

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Definition

The **differential** dx represents an infinitely small change in variable x.

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Let u and v be differentiable functions on the same interval. Then

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Definition

The **differential** dx represents an infinitely small change in variable x. Let $y = f(x)$, the differential df of f is related to dx by

$$df = f'(x)dx.$$

We integrate $\dfrac{d(uv)}{dx}$ respect to x on both side, so we have

$$\int \frac{d(uv)}{dx}dx = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx.$$

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Hence,

$$uv = \int udv + \int vdu.$$

That is

$$\int udv = uv - \int vdu.$$

Definition

The formula

$$\int u dv = uv - \int v du$$

is called the **formula of integration by parts**.

Example

Find $\displaystyle\int x\cos x dx$.

Example

Find $\displaystyle\int x\cos x dx$.

Solution Let $u = x$ and $dv = \cos x dx$.
Then $du = dx$ and $v = \sin x + C_1$. Hence

$$\begin{aligned}\int x\cos x dx &= x(\sin x + C_1) - \int (\sin x + C_1) dx \\ &= x\sin x + C_1 x + \cos x - C_1 x + C_2 \\ &= x\sin x + \cos x + C_2. \quad \square\end{aligned}$$

Example

Find $\displaystyle\int xe^x dx$.

Example

Find $\int xe^x dx$.

Solution Let $u = x$ and $dv = e^x dx$.
Then $du = dx$ and $v = e^x + C_1$. Hence

$$\begin{aligned}\int xe^x dx &= x(e^x + C_1) - \int (e^x + C_1) dx \\ &= xe^x + C_1 x - e^x - C_1 x + C_2 \\ &= xe^x - e^x + C_2. \quad \square\end{aligned}$$

Example

Find $\int \ln x dx$.

Example

Find $\int \ln x dx$.

Solution Let $u = \ln x$ and $dv = dx$.
Then $du = \dfrac{dx}{x}$ and $v = x + C_1$. So

$$\begin{aligned}\int \ln x dx &= (x + C_1)\ln x - \int (x + C_1)\frac{dx}{x} \\ &= x\ln x + C_1 \ln x - x - C_1 \ln x + C_2 \\ &= x\ln x - x + C_2. \quad \square\end{aligned}$$

Example

Find $\displaystyle\int \sin\sqrt{x}dx$.

Example

Find $\int \sin\sqrt{x}dx$.

Solution First, we let $u = \sqrt{x}$, then $du = \dfrac{dx}{2\sqrt{x}}$. That is $dx = 2\sqrt{x}du = 2udu$. So

$$\int \sin\sqrt{x}dx = 2\int u\sin u du.$$

Next, we consider $\int u \sin u du$. Let $v = u$ and $dw = \sin u du$. Then $dv = du$ and $w = -\cos u$. So

$$\begin{aligned}\int u \sin u du &= -u \cos u + \int \cos u du \\ &= -u \cos u + \sin u + C \\ &= -\sqrt{x} \cos \sqrt{x} + \sin \sqrt{x} + C.\end{aligned}$$

Thus,

$$\int \sin \sqrt{x} dx = -2\sqrt{x} \cos \sqrt{x} + 2 \sin \sqrt{x} + C. \quad \square$$

Example

Find $\int \cos(\ln x)dx$.

Example

Find $\int \cos(\ln x)dx$.

Solution First, we let $u = \ln x$, then $du = \frac{dx}{x}$. That is

$$dx = xdu = e^u du.$$

So

$$\int \cos(\ln x)dx = \int e^u \cos u du.$$

Next, let $v = e^u$ and $dw = \cos u du$.
Then $dv = e^u du$ and $w = \sin u$. So

$$\int e^u \cos u du = e^u \sin u - \int e^u \sin u du.$$

Next, we consider $\int e^u \sin u du$.
Let $p = e^u$ and $dq = \sin u du$.
Then $dp = e^u du$ and $q = -\cos u$. Hence

$$\int e^u \sin u du = -e^u \cos u + \int e^u \cos u du.$$

Therefore,

$$\int e^u \sin u du = e^u \sin u + e^u \cos u - \int e^u \cos u du$$
$$2\int e^u \sin u du = e^u \sin u + e^u \cos u + C$$
$$\int e^u \sin u du = \frac{1}{2}(e^u \sin u + e^u \cos u) + C.$$

Thus

$$\begin{aligned}\int \cos(\ln x)dx &= \frac{1}{2}(e^{\ln x}\sin\ln x + e^{\ln x}\cos\ln x) + C \\ &= \frac{1}{2}(x\sin\ln x + x\cos\ln x) + C. \quad \square\end{aligned}$$

Example

Find $\int e^{ax} \cos bx dx$, where $a, b \in \mathbb{R}$ such that $a^2 + b^2 > 0$.

Example

Find $\int e^{ax} \cos bx dx$, where $a, b \in \mathbb{R}$ such that $a^2 + b^2 > 0$.

Solution Let $u = e^{ax}$ and $dv = \cos bx dx$.
Then $du = ae^{ax} dx$ and $v = \frac{1}{b} \sin bx$. So

$$\int e^{ax} \cos bx dx = \frac{a}{b} e^{ax} \sin bx - \frac{a}{b} \int e^{ax} \sin bx dx.$$

Next, we consider $\int e^{ax} \sin bx dx$.
Let $p = e^{ax}$ and $dq = \sin bx dx$.
Then $dp = ae^{ax} dx$ and $q = -\frac{1}{b} \cos bx$. Hence

$$\int e^{ax} \sin bx dx = -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b} \int e^{ax} \cos bx dx.$$

Therefore,

$$\int e^{ax} \cos bx dx = \frac{a}{b} e^{ax} \sin bx + \frac{a}{b^2} e^{ax} \cos bx - \frac{a^2}{b^2} \int e^{ax} \cos bx dx$$

$$\frac{a^2 + b^2}{b^2} \int e^{ax} \cos bx dx = \frac{a}{b} e^{ax} \sin bx + \frac{a}{b^2} e^{ax} \cos bx + C$$

$$\int e^{ax} \cos bx dx = \frac{1}{a^2 + b^2} (be^{ax} \sin bx + ae^{ax} \cos bx) + C. \quad \square$$

Try to find $\int \sin^3 x dx$.

Try to find $\int \sin^3 x dx$.

Try to find $\int \tan^3 x \sec^5 x dx$.

Try to find $\displaystyle\int \sin^3 x dx$.

Try to find $\displaystyle\int \tan^3 x \sec^5 x dx$.

How to find

$$\int \sin^m x \cos^n x dx, \int \tan^m x \sec^n x dx,$$

where $m, n \in \mathbb{N} \cup \{0\}$.

Trigonometry Identities

1. $\cos^2 x + \sin^2 x = 1.$
2. $1 + \tan^2 x = \sec^2 x.$
3. $\cot^2 x + 1 = \csc^2 x.$
4. $\sin 2x = 2 \sin x \cos x.$
5. $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x.$

Theorem : Binomial Theorem

Let $k \in \mathbb{N} \cup \{0\}$, then

$$(x+y)^k = \binom{k}{0}x^k + \binom{k}{1}x^{k-1}y + \ldots + \binom{k}{n-1}xy^{k-1} + \binom{k}{k}y^k,$$

where $\binom{n}{r} = \dfrac{n!}{(n-r)!r!}$.

Strategy for Evaluating $\int \sin^m x \cos^n xdx$

1. If n is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of $\sin x$:

$$\begin{aligned}\int \sin^m x \cos^{2k+1} xdx &= \int \sin^m x(\cos^2 x)^k \cos xdx \\ &= \int \sin^m x(1 - \sin^2 x)^k \cos xdx.\end{aligned}$$

   Then substitute $u = \sin x$.

Strategy for Evaluating $\int \sin^m x \cos^n x dx$

2. If m is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of $\cos x$:

$$\begin{aligned}\int \sin^{2k+1} x \cos^n x dx &= \int (\sin^2 x)^k \cos^n x \sin x dx \\ &= \int (1 - \cos^2 x)^k \cos^n x \sin x dx.\end{aligned}$$

Then substitute $u = \cos x$.

Strategy for Evaluating $\int \sin^m x \cos^n x dx$

2. If m is odd ($m = 2k+1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of $\cos x$:

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Then substitute $u = \cos x$.

Remark

If m and n are odd, either 1) or 2) can be used.

Strategy for Evaluating $\int \sin^m x \cos^n x dx$

3. If m and n are even ($m = 2k, n = 2l$), use the half-angle identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x), \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

to express factor in term of $\cos 2x$:

$$\begin{aligned}\int \sin^{2k} x \cos^{2l} x dx &= \int (\sin^2 x)^k (\cos^2 x)^l dx \\ &= \frac{1}{2^{k+l}} \int (1 - \cos 2x)^k (1 + \cos 2x)^l dx.\end{aligned}$$

Then use binomial theorem to expand a binomial expression and integrate.

Example

Evaluate $\int \sin^2 x \cos^3 x dx$.

Example

Evaluate $\displaystyle\int \sin^2 x \cos^3 x dx$.

Solution Let $u = \sin x$, then $du = \cos x dx$. So

$$\begin{aligned}\int \sin^2 x \cos^3 x dx &= \int \sin^2 x(1 - \sin^2 x) \cos x dx \\ &= \int u^2(1 - u^2) du \\ &= \int (u^2 - u^4) du \\ &= \frac{u^3}{3} - \frac{u^5}{5} + C \\ &= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C. \quad \square\end{aligned}$$

Example

Evaluate $\displaystyle\int \sin^3 x dx$.

Example

Evaluate $\displaystyle\int \sin^3 x dx$.

Solution Let $u = \cos x$, then $du = -\sin x dx$. So

$$\begin{aligned}\int \sin^3 x dx &= \int (1-\cos^2 x)\sin x dx \\ &= -\int (1-u^2)du \\ &= -u + \frac{u^3}{3} + C \\ &= -\cos x + \frac{\cos^3 x}{3} + C. \quad \square\end{aligned}$$

Example

Find $\displaystyle\int \sin^2 x \cos^2 x dx$.

Example

Find $\displaystyle\int \sin^2 x \cos^2 x dx$.

Solution

$$\begin{aligned}
\int \sin^2 x \cos^2 x dx &= \frac{1}{4}\int (1-\cos 2x)(1+\cos 2x)dx \\
&= \frac{1}{4}\int (1-\cos^2 2x)dx \\
&= \frac{1}{4}\int \left(1-\frac{1}{2}(1+\cos 4x)\right)dx \\
&= \frac{1}{8}\int (1-\cos 4x)dx \\
&= \frac{1}{8}\left(x-\frac{\sin 4x}{4}\right)+C. \quad \square
\end{aligned}$$

Or Solution

$$\begin{aligned}\int \sin^2 x \cos^2 x dx &= \frac{1}{4}\int (2\sin x \cos x)^2 dx\\ &= \frac{1}{4}\int \sin^2 2x dx\\ &= \frac{1}{8}\int (1-\cos 4x) dx\\ &= \frac{1}{8}\left(x - \frac{\sin 4x}{4}\right) + C. \quad \square\end{aligned}$$

Strategy for Evaluating $\int \tan^m x \sec^n x dx$

1. If n is even ($n = 2k, k \geq 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\begin{aligned}\int \tan^m x \sec^{2k} x dx &= \int \tan^m x (\sec^2 x)^{k-1} \sec^2 x dx \\ &= \int \tan^m x (1 + \tan^2 x)^{k-1} \sec^2 x dx.\end{aligned}$$

   Then subsitute $u = \tan x$.

Strategy for Evaluating $\int \tan^m x \sec^n x dx$

2. If m is odd ($m = 2k+1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\begin{aligned}\int \tan^{2k+1} x \sec^n x dx &= \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x dx \\ &= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x dx.\end{aligned}$$

Then substitute $u = \sec x$.

Strategy for Evaluating $\int \tan^m x \sec^n x dx$

3. If m is even and n is odd ($m = 2k, n = 2l + 1$), use $\tan^2 x = \sec^2 x - 1$ to express the factor in terms of $\sec x$:

$$\begin{aligned}\int \tan^{2k} x \sec^{2l+1} x dx &= \int (\tan^2 x)^k \sec^{2l+1} x dx \\ &= \int (\sec^2 x - 1)^k \sec^{2l+1} x dx \\ &= \sum_{r=0}^{k} \left[(-1)^r \binom{k}{r} \int \sec^{2r+2l+1} x dx \right].\end{aligned}$$

Next, we consider $\int \sec^q x dx$, where q is odd. Note that

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Let $u = \sec^{q-2} x$ and $dv = \sec^2 xdx$, then
$du = (q-2)\sec^{q-2} x \tan xdx$ and $v = \tan x$.

Next, we consider $\displaystyle\int \sec^q x dx$, where q is odd. Note that

$$\int \sec^q x dx = \int \sec^{q-2} x \sec^2 x dx.$$

Let $u = \sec^{q-2} x$ and $dv = \sec^2 x dx$, then
$du = (q-2)\sec^{q-2} x \tan x dx$ and $v = \tan x$.
By integral by parts formula, we get

$$\int \sec^q x dx = \sec^{q-2} x \tan x - (q-2) \int \sec^{q-2} x \tan^2 x dx$$

Next, we consider $\displaystyle\int \sec^q x dx$, where q is odd. Note that

$$\int \sec^q x dx = \int \sec^{q-2} x \sec^2 x dx.$$

Let $u = \sec^{q-2} x$ and $dv = \sec^2 x dx$, then
$du = (q-2)\sec^{q-2} x \tan x dx$ and $v = \tan x$.
By integral by parts formula, we get

$$\int \sec^q x dx = \sec^{q-2} x \tan x - (q-2)\int \sec^{q-2} x \tan^2 x dx$$

$$= \sec^{q-2} x \tan x - (q-2)\int \sec^q x dx + (q-2)\int \sec^{q-2} x dx$$

Hence,

$$(q-1)\int \sec^q x dx = \sec^{q-2} x \tan x + (q-2)\int \sec^{q-2} x dx$$
$$\int \sec^q x dx = \frac{\sec^{q-2} x \tan x}{q-1} + \frac{q-2}{q-1}\int \sec^{q-2} x dx. \quad (*)$$

Next, we find $\int \sec^{q-2} x dx$ by using the same argument.
We continue this process until the power of secant is 1.

The formula $(*)$ is called a **reduction formula** for $\sec x$.

Example

Find $\displaystyle\int \sec^4 x \tan^2 x dx$.

Example

Find $\displaystyle\int \sec^4 x \tan^2 x dx$.

Solution Let $u = \tan x$, then $du = \sec^2 x dx$. So

$$\begin{aligned}
\int \sec^4 x \tan^2 x dx &= \int (1 + \tan^2 x) \tan^2 x \sec^2 x dx \\
&= \int (1 + u^2) u^2 du \\
&= \int (u^2 + u^4) du \\
&= \frac{u^3}{3} + \frac{u^5}{5} + C \\
&= \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C. \quad \square
\end{aligned}$$

Example

Evaluate $\int \sec^5 x \tan^3 x dx$.

Example

Evaluate $\displaystyle\int \sec^5 x \tan^3 xdx$.

Solution Let $u = \sec x$, then $du = \sec x \tan xdx$. So

$$\begin{aligned}
\int \sec^5 x \tan^3 xdx &= \int \sec^4 x(\sec^2 x - 1) \sec x \tan xdx \\
&= \int u^4(u^2 - 1)du \\
&= \int (u^6 - u^4)du \\
&= \frac{u^7}{7} - \frac{u^5}{5} + C \\
&= \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C. \quad \square
\end{aligned}$$

Example

Find $\displaystyle\int \sec^3 x dx$.

Example

Find $\displaystyle\int \sec^3 x dx$.

Solution Let $u = \sec x$ and $dv = \sec^2 x dx$.
Then $du = \sec x \tan x dx$ and $v = \tan x$. So

$$\begin{aligned}\int \sec^3 x dx &= \sec x \tan x - \int \tan^2 x \sec x dx \\ &= \sec x \tan x - \int (\sec^2 x - 1) \sec x dx \\ &= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \\ &= \sec x \tan x - \int \sec^3 x dx + \ln|\sec x + \tan x| + C.\end{aligned}$$

Hence

$$2\int \sec^3 dx = \sec x \tan x + \ln|\sec x + \tan x| + C$$

$$\int \sec^3 x dx = \frac{1}{2}\Big(\sec x \tan x + \ln|\sec x + \tan x|\Big) + C. \quad \square$$

Example

Evaluate $\int \tan^2 x \sec x dx$.

Example

Evaluate $\displaystyle\int \tan^2 x \sec x dx$.

Solution Note that

$$\begin{aligned}\int \tan^2 x \sec x dx &= \int (\sec^2 x - 1) \sec x dx \\ &= \int \sec^3 x dx - \int \sec x dx.\end{aligned}$$

By the previous example, we have

$$\begin{aligned}\int \tan^2 x \sec x dx &= \frac{1}{2}\Big(\sec x \tan x + \ln|\sec x + \tan x|\Big) \\ &\quad - \ln|\sec x + \tan x| + C. \\ &= \frac{1}{2}\Big(\sec x \tan x - \ln|\sec x + \tan x|\Big) + C. \quad \square\end{aligned}$$

Example

Find $\int \cot^2 x \csc^2 x dx$.

Example

Find $\displaystyle\int \cot^2 x \csc^2 x dx$.

Solution First, we let $u = \dfrac{\pi}{2} - x$, then $du = -dx$. So

$$\begin{aligned}\int \cot^2 x \csc^2 x dx &= -\int \cot^2\left(\frac{\pi}{2} - u\right)\csc^2\left(\frac{\pi}{2} - u\right)du \\ &= -\int \tan^2 u \sec^2 u du.\end{aligned}$$

Next, we consider $\int \tan^2 u \sec^2 u du$. Let $w = \tan u$, then $dw = \sec^2 u du$. So

$$\begin{aligned}\int \tan^2 u \sec^2 u du &= \int w^2 dw \\ &= \frac{w^3}{3} + C \\ &= \frac{\tan^3 u}{3} + C.\end{aligned}$$

Hence

$$\int \cot^2 x \csc^2 x dx = -\frac{1}{3}\tan^3\left(\frac{\pi}{2} - x\right) + C = -\frac{\cot^3 x}{3} + C. \quad \square$$

Strategy for Evaluating $\int \sin mx \cos nx dx$, $\int \sin mx \sin nx dx$ or $\int \cos mx \cos nx dx$

Use the corresponding identity:

1. $2 \sin A \cos B = \sin(A - B) + \sin(A + B)$.
2. $2 \cos A \cos B = \cos(A - B) + \cos(A + B)$.
3. $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$.

Example

Find $\displaystyle\int \sin 4x \cos 5x dx$.

Example

Find $\displaystyle\int \sin 4x \cos 5x dx$.

<u>Solution</u>

$$\begin{aligned}
\int \sin 4x \cos 5x dx &= \frac{1}{2}\int (\sin 9x + \sin(-x))dx \\
&= \frac{1}{2}\int (\sin 9x - \sin x)dx \\
&= \frac{1}{2}\left(-\frac{\cos 9x}{9} + \cos x\right) + C. \quad \square
\end{aligned}$$

Example

Find $\displaystyle\int \sin x \cos 2x \sin 3x dx$.

Example

Find $\displaystyle\int \sin x \cos 2x \sin 3x dx.$

<u>Solution</u>

$$\begin{aligned}
\int \sin x \cos 2x \sin 3x dx &= \frac{1}{2}\int (\sin 3x + \sin(-x)) \sin 3x dx \\
&= \frac{1}{2}\int (\sin^2 3x - \sin x \sin 3x) dx \\
&= \frac{1}{2}\int \Big(\frac{1}{2}(1 - \cos 6x) - \frac{1}{2}(\cos(-2x) - \cos 4x)\Big) dx \\
&= \frac{1}{4}\int (1 - \cos 6x - \cos 2x + \cos 4x) dx \\
&= \frac{1}{4}\left(x - \frac{\sin 6x}{6} - \frac{\sin 2x}{2} + \frac{\sin 4x}{4}\right) + C. \quad \square
\end{aligned}$$

Example

Evaluate $\displaystyle\int e^x \cos(3e^x)\cos(4e^x)dx$.

Example

Evaluate $\int e^x \cos(3e^x)\cos(4e^x)dx$.

Solution Let $u = e^x$, then $du = e^x dx$. So

$$\begin{aligned}\int e^x \cos(3e^x)\cos(4e^x)dx &= \int \cos 3u \cos 4u du \\ &= \frac{1}{2}\int(\cos 7u + \cos u)du \\ &= \frac{1}{2}\left(\frac{\sin 7u}{7} + \sin u\right) + C \\ &= \frac{\sin(7e^x)}{14} + \frac{\sin e^x}{2} + C. \quad \square\end{aligned}$$

Find $\displaystyle\int x\sqrt{x^2+16}\,dx$.

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It is easy to see that

$$\int x\sqrt{x^2+16}dx = \frac{1}{3}(x^2+16)^{\frac{3}{2}} + C.$$

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Try to find $\int \sqrt{x^2+16}dx$.

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Try to find $\int \sqrt{x^2+16}dx$.

Try to evaluate $\int \sqrt{r^2-x^2}dx$, where $r>0$.

There are three kinds of trigonometric substitutions. We can use them when we see one of the expressions

There are three kinds of trigonometric substitutions. We can use them when we see one of the expressions

$$\sqrt{a^2 - x^2}, \quad \sqrt{a^2 + x^2}, \quad \text{or} \quad \sqrt{x^2 - a^2},$$

where $a > 0$.

Expression $\sqrt{a^2 - x^2}$

$$\sqrt{a^2 - x^2}$$

1. Let $x = a\sin\theta$, where $\theta \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.
2. $dx = a\cos\theta d\theta$.
3. $\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2\cos^2\theta} = a|\cos\theta| = a\cos\theta$.

Expression $\sqrt{x^2+a^2}$

$$\sqrt{x^2+a^2}$$

1. Let $x = a\tan\theta$, where $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
2. $dx = a\sec^2\theta d\theta$.
3. $\sqrt{x^2+a^2} = \sqrt{a^2\tan^2\theta + a^2} = \sqrt{a^2\sec^2\theta} = a|\sec\theta| = a\sec\theta$.

Expression $\sqrt{x^2 - a^2}$

$$\sqrt{x^2 - a^2}$$

1. Let $x = a\sec\theta$, where $\theta \in \left[0, \dfrac{\pi}{2}\right) \cup \left[\pi, \dfrac{3\pi}{2}\right)$.
2. $dx = a\sec\theta\tan\theta d\theta$.
3. $\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2\tan^2\theta} = a|\tan\theta| = a\tan\theta$

Example

Find $\displaystyle\int \frac{\sqrt{100 - x^2}}{x^2}\,dx.$

Example

Find $\displaystyle\int \frac{\sqrt{100-x^2}}{x^2}dx.$

<u>Solution</u> Let $x = 10\sin\theta$, then $dx = 10\cos\theta d\theta$. So

$$\begin{aligned}
\int \frac{\sqrt{100-x^2}}{x^2}dx &= \int \frac{\sqrt{100-100\sin^2\theta}}{100\sin^2\theta}10\cos\theta d\theta \\
&= \int \frac{100\cos^2\theta}{100\sin^2\theta}d\theta \\
&= \int \cot^2\theta d\theta \\
&= \int (\csc^2\theta - 1)d\theta \\
&= -\cot\theta - \theta + C.
\end{aligned}$$

From $\sin\theta = \dfrac{x}{10}$, we get

$$\theta = \arcsin\left(\frac{x}{10}\right), \quad \text{and} \quad \cot\theta = \frac{\sqrt{100-x^2}}{x}.$$

Thus

$$\int \frac{\sqrt{100-x^2}}{x^2}dx = -\frac{\sqrt{100-x^2}}{x} - \arcsin\left(\frac{x}{10}\right) + C. \quad \square$$

Example

Find $\displaystyle\int x^2\sqrt{x^2+16}\,dx$.

Example

Find $\displaystyle\int x^2\sqrt{x^2+16}dx$.

Solution Let $x = 4\tan\theta$, then $dx = 4\sec^2\theta d\theta$. So

$$\begin{aligned}\int x^2\sqrt{x^2+16}dx &= 256\int \tan^2\theta\sec^3\theta d\theta\\ &= 256\left(\int\sec^5\theta d\theta - \int\sec^3\theta d\theta\right).\end{aligned}$$

We consider a term $\int \sec^5\theta d\theta$.
Let $u = \sec^3\theta$ and $dv = \sec^2\theta d\theta$.
Then $du = 3\sec^3\theta\tan\theta d\theta$ and $v = \tan\theta$. So

$$\begin{aligned}\int \sec^5\theta d\theta &= \sec^3\theta\tan\theta - 3\int \sec^3\theta\tan^2\theta d\theta \\ &= \sec^3\theta\tan\theta - 3\int \sec^5\theta d\theta + 3\int \sec^3\theta d\theta.\end{aligned}$$

Hence

$$\int \sec^5\theta d\theta = \frac{1}{4}\sec^3\theta\tan\theta + \frac{3}{4}\int \sec^3\theta d\theta.$$

That is

$$\int x^2\sqrt{x^2+16}dx = 256\left(\frac{1}{4}\sec^3\theta\tan\theta - \frac{1}{4}\int\sec^3\theta d\theta\right).$$

Note that

$$\int\sec^3\theta d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C.$$

Thus

$$\int x^2\sqrt{x^2+16}dx = 256\left(\frac{1}{4}\sec^3\theta\tan\theta - \frac{1}{8}\sec\theta\tan\theta - \frac{1}{8}\ln|\sec\theta+\tan\theta|\right) + C.$$

From $\tan\theta = \dfrac{x}{4}$, we obtain $\sec\theta = \dfrac{\sqrt{x^2+16}}{4}$. Therefore,

$$\begin{aligned}\int x^2\sqrt{x^2+16}\,dx &= 256\left(\frac{1}{1024}x(x^2+16)^{\frac{3}{2}} - \frac{1}{128}x\sqrt{x^2+16}\right.\\ &\quad\left. - \frac{1}{8}\ln\left|\frac{\sqrt{x^2+16}+x}{4}\right|\right) + C\\ &= \frac{1}{4}x(x^2+16)^{\frac{3}{2}} - 2x\sqrt{x^2+16}\\ &\quad - 32\ln\left|\frac{\sqrt{x^2+16}+x}{4}\right|\Bigg) + C. \quad \square\end{aligned}$$

Example

Find $\int e^x \sqrt{25e^{2x} - 16}dx$.

Example

Find $\int e^x\sqrt{25e^{2x}-16}dx$.

Solution Let $5e^x = 4\sec\theta$, then $5e^x dx = 4\sec\theta\tan\theta d\theta$. So

$$\int e^x\sqrt{25e^x-16}dx = \frac{16}{5}\int \tan^2\theta\sec\theta d\theta$$

Note that

$$\int \tan^2\theta\sec\theta d\theta = \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta+\tan\theta| + C$$

and from $\sec\theta = \frac{5e^x}{4}$, we get $\tan\theta = \frac{\sqrt{25e^{2x}-16}}{4}$. Hence

$$\int e^x\sqrt{25e^x-16}dx = \frac{1}{2}e^x\sqrt{5e^x-16} - \frac{8}{5}\ln\left|\frac{5e^x+\sqrt{25e^{2x}-16}}{4}\right| + C. \quad \square$$

Example

Find the area of enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

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$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

<u>Solution</u> Note that the area of enclosed by the ellipse is equal to

$$\frac{2b}{a}\int_{-a}^{a}\sqrt{a^2 - x^2}dx.$$

Let $x = a\sin\theta$, then $dx = a\cos\theta d\theta$.
If $x = a \to \sin\theta = 1 \to \theta = \dfrac{\pi}{2}$.
If $x = -a \to \sin\theta = -1 \to \theta = -\dfrac{\pi}{2}$. Hence

$$\begin{aligned}
\int_{-a}^{a} \sqrt{a^2 - x^2}dx &= a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta d\theta \\
&= \frac{a^2}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \cos 2\theta) d\theta \\
&= \frac{a^2}{2} \left(\theta + \frac{\sin 2\theta}{2} \Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \right) \\
&= \frac{\pi a^2}{2}.
\end{aligned}$$

Thus, the area of enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is

$$\left(\frac{2b}{a}\right)\left(\frac{\pi a^2}{2}\right) = \pi ab. \quad \square$$

Example

Evaluate $\displaystyle\int_{\frac{\sqrt{2}}{3}}^{\frac{2}{3}} \frac{dx}{x^5\sqrt{9x^2-1}}$.

Example

Evaluate $\displaystyle\int_{\frac{\sqrt{2}}{3}}^{\frac{2}{3}} \frac{dx}{x^5\sqrt{9x^2-1}}$.

Solution Let $3x = \sec\theta$, then $3dx = \sec\theta\tan\theta d\theta$.
If $x = \dfrac{2}{3} \to \sec\theta = 2 \to \theta = \dfrac{\pi}{3}$.
If $x = \dfrac{\sqrt{2}}{3} \to \sec\theta = \sqrt{2} \to \theta = \dfrac{\pi}{4}$.

So

$$\begin{aligned}
\int_{\frac{\sqrt{2}}{3}}^{\frac{2}{3}} \frac{dx}{x^5\sqrt{9x^2-1}} &= 81\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec\theta\tan\theta d\theta}{\sec^5\theta\tan\theta} \\
&= 81\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^4\theta d\theta \\
&= \frac{81}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (1+\cos 2\theta)^2\theta d\theta \\
&= \frac{81}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (1+2\cos 2\theta+\cos^2 2\theta)d\theta \\
&= \frac{81}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\frac{3}{2}+2\cos 2\theta+\frac{1}{2}\cos 4\theta)d\theta
\end{aligned}$$

$$\int_{\frac{\sqrt{2}}{3}}^{\frac{2}{3}} \frac{dx}{x^5\sqrt{9x^2-1}} = \frac{81}{4}\left(\frac{3\theta}{2} + \sin 2\theta + \frac{1}{8}\sin 4\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}\Bigg)$$

$$= \frac{81}{4}\left(\frac{\pi}{8} + \frac{7\sqrt{3}}{16} - 1\right). \quad \square$$

Find $\displaystyle\int \frac{dx}{x+1}$.

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It is easy to verify that

$$\int \frac{dx}{x+1} = \ln|x+1| + C.$$

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Evaluate $\displaystyle\int \frac{dx}{(x^2+1)(x-1)^2}$.

Find $\int \frac{dx}{x+1}$.

It is easy to verify that

$$\int \frac{dx}{x+1} = \ln|x+1| + C.$$

Evaluate $\int \frac{dx}{(x^2+1)(x-1)^2}$.

How to find

$$\int \frac{P(x)}{Q(x)} dx,$$

where $P(x)$ and $Q(x)$ are polynomials and $Q(x) \not\equiv 0$.

Definition

The **partial fraction decomposition** or **partial fraction expression** of rational function is the operation that consist in expressing the fraction as the sum of a polynomial and one or several fractions with a simpler denominator.

Definition

The rational function

$$\frac{P(x)}{Q(x)}$$

is called **proper** if $\deg P < \deg Q$. A rational function which is not proper is **improper**.

Theorem : Division Algorithm for Polynomial

Let $P(x)$ and $Q(x)$ be polynomials such that $Q(x) \not\equiv 0$, then there exists the unique pair of polynomial $S(x)$ and $R(x)$ that

$$P(x) = Q(x)S(x) + R(x),$$

where $R(x) \equiv 0$ or $\deg R < \deg P$.

Theorem : Division Algorithm for Polynomial

Let $P(x)$ and $Q(x)$ be polynomials such that $Q(x) \not\equiv 0$, then there exists the unique pair of polynomial $S(x)$ and $R(x)$ that

$$P(x) = Q(x)S(x) + R(x),$$

where $R(x) \equiv 0$ or $\deg R < \deg P$.

So, if $\dfrac{P(x)}{Q(x)}$ is improper, by using division algorithm for polynomial, we get that there exists the polynomials $S(x)$ and $R(x)$ such that

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)},$$

where $\deg R < \deg Q$.

Partial Fraction for Proper Rational Function

1. $Q(x)$ is a product of distinct linear factors. That is

$$Q(x) = (a_1x + b_1)(a_2x + b_2) \dots (a_kx + b_k),$$

where no factor repeated (and no factor is a constant multiple of another).
So there exists constants $A_1, A_2, \dots, A_k$ such that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{ax_1 + b_1} + \frac{A_2}{a_2x + b_2} + \dots + \frac{A_k}{a_kx + b_k}.$$

Partial Fraction for Proper Rational Function

2. $Q(x)$ is a product of distinct linear factors, some of which are repeated.
   If $(a_1x + b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $\dfrac{A_1}{a_1x + b}$, we would use

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \ldots + \frac{A_r}{(a_1x + b_1)^r}.$$

Partial Fraction for Proper Rational Function

3. $Q(x)$ contains irreducible quadratic factor, none of which is repeated.
   If $Q(x)$ has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then the expression for rational function will have a term of the form

$$\frac{Ax + B}{ax^2 + bx + c},$$

   where A and B are constant.

Partial Fraction for Proper Rational Function

4. $Q(x)$ contains a repeated irreducible quadratic factor.
If $(ax^2+bx+c)^r$, where $b^2-4ac<0$, occurs in the factorization of $Q(x)$, then instead of the single term $\dfrac{Ax+B}{ax^2+bx+c}$, we would use

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\ldots+\frac{A_rx+B_r}{(ax^2+bx+c)^r}.$$

Recall

1. $\displaystyle\int \frac{dx}{x} = \ln|x| + C.$
2. $\displaystyle\int \frac{dx}{1+x^2} = \arctan x + C.$

We have 2 ways to find coefficients of partial fractions :

1. Plug in the x-values.
2. Comparing coefficients.

Example

Find $\displaystyle\int \frac{dx}{x^2 - \pi^2}$.

Example

Find $\displaystyle\int \frac{dx}{x^2-\pi^2}$.

Solution Note that $\displaystyle\frac{1}{x^2-\pi^2} = \frac{1}{(x-\pi)(x+\pi)} = \frac{A}{x-\pi} + \frac{B}{x+\pi}$,

where $A, B \in \mathbb{R}$.

So,

$$1 = A(x+\pi) + B(x-\pi).$$

If $x = \pi \to 2\pi A = 1 \to A = \dfrac{1}{2\pi}$.

If $x = -\pi \to -2\pi B = 1 \to B = -\dfrac{1}{2\pi}$.

Hence,

$$\begin{aligned}\int \frac{dx}{x^2-\pi^2} &= \int \left(\frac{1}{2\pi(x-\pi)} - \frac{1}{2\pi(x+\pi)}\right)dx \\ &= \frac{1}{2\pi}\ln|x-\pi| - \frac{1}{2\pi}\ln|x+\pi| + C. \quad \square\end{aligned}$$

Or, from

$$1 = A(x+\pi) + B(x-\pi) = (A+B)x + (\pi A - \pi B).$$

By comparing coefficients, we obtain

$$\begin{cases} A + B = 0 \\ \pi A - \pi B = 1 \end{cases}.$$

We are solving the linear equation system, we get

$$A = \frac{1}{2\pi} \quad \text{and} \quad B = -\frac{1}{2\pi}.$$

That is

$$\frac{1}{x^2-\pi^2} = \frac{1}{2\pi(x-\pi)} - \frac{1}{2\pi(x+\pi)}.$$

Example

Evaluate $\displaystyle\int \frac{3x^2 - 8x + 13}{x^3 + x^2 - 5x + 3}dx.$

Example

Evaluate $\int \frac{3x^2 - 8x + 13}{x^3 + x^2 - 5x + 3} dx$.

Solution Note that

$$\frac{3x^2 - 8x + 13}{x^3 + x^2 - 5x + 3} = \frac{3x^2 - 8x + 13}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3},$$

where $A, B, C \in \mathbb{R}$. So,

$$3x^2 - 8x + 13 = A(x-1)(x+3) + B(x+3) + C(x-1)^2.$$

If $x = 1 \rightarrow 4B = 8 \rightarrow B = 2.$
If $x = -3 \rightarrow 16C = 64 \rightarrow C = 4.$
If $x = 0 \rightarrow -3A + 3B + C = 13 \rightarrow -3A + 6 + 4 = 13 \rightarrow A = -1.$
That is

$$\int \frac{3x^2 - 8x + 13}{x^3 + x^2 - 5x + 3} dx = \int \left(-\frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{4}{x+3} \right) dx$$
$$= -\ln|x-1| - \frac{2}{x-1} + 4\ln|x+3| + C. \quad \square$$

Example

Find $\displaystyle\int \frac{dx}{x^3 - 1}$.

Example

Find $\displaystyle\int \frac{dx}{x^3 - 1}$.

Solution Note that

$$\frac{1}{x^3-1} = \frac{1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1},$$

where $A, B, C \in \mathbb{R}$. Then

$$1 = A(x^2+x+1)+(Bx+C)(x-1) = (A+B)x^2+(A-B+C)x+(A-C).$$

By comparing coefficients, we have

$$\begin{cases} A + B \qquad\quad = 0 \\ A - B + C = 0 \\ A \qquad\quad - C = 1 \end{cases}$$

If $x = 1 \to 3A = 1 \to A = \dfrac{1}{3}$.
So, $B = -\dfrac{1}{3}$ and $C = -\dfrac{2}{3}$. Hence

$$\begin{aligned} \int \frac{dx}{x^3 - 1} &= \int \left(\frac{1}{3(x-1)} - \frac{1}{3} \left(\frac{x+2}{x^2+x+1} \right) \right) dx \\ &= \frac{1}{3} \ln|x-1| - \frac{1}{3} \int \frac{x+2}{x^2+x+1} dx. \end{aligned}$$

We consider $\int \frac{x+2}{x^2+x+1}dx$. Note that

$$\begin{aligned}
\int \frac{x+2}{x^2+x+1}dx &= \frac{1}{2}\left(\int \frac{2x+1}{x^2+x+1}dx + \int \frac{3}{x^2+x+1}dx\right) \\
&= \frac{1}{2}\ln|x^2+x+1| + \frac{3}{2}\int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}dx \\
&= \frac{1}{2}\ln|x^2+x+1| + 2\int \frac{1}{\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)^2+1}dx \\
&= \frac{1}{2}\ln|x^2+x+1| + \sqrt{3}\arctan\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)+C.
\end{aligned}$$

Thus

$$\int \frac{dx}{x^3-1} = \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln|x^2+x+1| - \frac{1}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)+C. \quad \square$$

Example

Find $\displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$.

Example

Find $\displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$.

Solution By long division algorithm for polynomial, we obtain

$$\begin{aligned}\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx &= \int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 + \frac{4}{x^2+1}\right)dx \\ &= \frac{x^7}{7} - \frac{2x^6}{3} + x^5 - \frac{4x^3}{3} + 4x + 4\arctan x\Big]_0^1 \\ &= \frac{22}{7} - \pi. \quad \square\end{aligned}$$

Example

Evaluate $\displaystyle\int \frac{dx}{(x+1)\sqrt{x+6}}$.

Example

Evaluate $\displaystyle\int \frac{dx}{(x+1)\sqrt{x+6}}$.

Solution Let $u = \sqrt{x+6}$, then $x = u^2 - 6$ and $dx = 2udu$. So

$$\int \frac{dx}{(x+1)\sqrt{x+6}} = 2\int \frac{udu}{(u^2-5)u} = 2\int \frac{du}{u^2-5}.$$

Note that

$$\frac{1}{u^2-5} = \frac{1}{(u-\sqrt{5})(u+\sqrt{5})} = \frac{A}{u-\sqrt{5}} + \frac{B}{u+\sqrt{5}},$$

where $A, B \in \mathbb{R}$. Hence

$$1 = A(u+\sqrt{5}) + B(u-\sqrt{5}).$$

If $u = \sqrt{5} \to 2\sqrt{5}A = 1 \to A = \dfrac{1}{2\sqrt{5}}$.

If $u = -\sqrt{5} \to -2\sqrt{5}B = 1 \to B = -\dfrac{1}{2\sqrt{5}}$. Thus,

$$\begin{aligned}
\int \frac{dx}{(x+1)\sqrt{x+6}} &= \frac{1}{\sqrt{5}} \int \left(\frac{1}{u-\sqrt{5}} - \frac{1}{u+\sqrt{5}} \right) du \\
&= \frac{1}{\sqrt{5}} (\ln|u-\sqrt{5}| - \ln|u+\sqrt{5}|) + C \\
&= \frac{1}{\sqrt{5}} (\ln|\sqrt{x+6}-\sqrt{5}| - \ln|\sqrt{x+6}+\sqrt{5}|) + C. \quad \square
\end{aligned}$$