Ordinary Differential Equation

Chapter II : Differential Equation

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Definition

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An **ordinary differential equation** (ODE) is a differential equation containing one or more functions of one independent variable and its derivatives.

Definition

An **partial differential equation** (PDE) is a differential equation containing one or more functions of two or more independent variables and their partial derivatives.

Example of Ordinary Differential Equation

1. $\dfrac{d^5y}{dx^5} - 2x\dfrac{d^4y}{dx^4} + \sin y\dfrac{dy}{dx} = x^2.$
2. $x^3\dfrac{d^4y}{dx^3} - x^3\dfrac{dy}{dx} + x\dfrac{dy}{dx} = \dfrac{1}{1+y^2}.$
3. $x\left(\dfrac{d^2y}{dx^2}\right)^4 + y\dfrac{dy}{dx} = y^2.$
4. $y^{(7)} - \sqrt{y}y^{(3)} + x\arctan y = 80\sin x.$

Example of Partial Differential Equation

1. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 2u + x^4 - xy.$
2. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 2\dfrac{\partial u}{\partial t} - u^2.$
3. $\dfrac{\partial^3 u}{\partial x^3} + \left(\dfrac{\partial u}{\partial x}\right)^3 = 5\dfrac{\partial^2 u}{\partial t^2}.$

Definition

The **order** of a differential equation is the order of the highest order derivative in the equation.

Definition

The **degree** of a differential equation is the power of the highest order derivative in the equation.

Definition

A **linear differential equation** is a differential equation that satisfy the following conditions

1. Every dependent variables and derivatives of dependent variables has the power of 1.
2. No term of product of dependent variables or/and derivatives of dependent variables.
3. No term of transcendental functions of dependent variables or derivative of dependent variables.

A differential equation is said to be **nonlinear equation** if it is not a linear differential equation.

Example of Linear Equation

1. $x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\frac{\mathrm{d}y}{\mathrm{d}x}=x^3y+\sin x.$
2. $\frac{\mathrm{d}y}{\mathrm{d}x}=xy.$
3. $\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}=u.$
4. $\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}=2\frac{\partial^2u}{\partial t^2}.$

Example of Nonlinear Equation

1. $x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + y\dfrac{\mathrm{d}y}{\mathrm{d}x} = x^3y + \cos x.$
2. $\sqrt{\dfrac{\mathrm{d}y}{\mathrm{d}x}} = xy.$
3. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = e^u.$
4. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y} + \dfrac{\partial^2 u}{\partial y^2} = 2\dfrac{\partial^2 u}{\partial t^2}.$

Every ODE linear equation of order n can be written in form

$$F(x, y, y', \ldots, y^{(n)}) = 0, \tag{1}$$

where F is a function of $n + 2$ values $x, y, y', \ldots, y^{(n)}$.

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$$F(x, y, y', \ldots, y^{(n)}) = 0, \tag{1}$$

where F is a function of $n+2$ values $x, y, y', \ldots, y^{(n)}$.

If (1) is a linear, we can write the (1) in the form

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \ldots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = G(x), \tag{2}$$

where $a_0(x), \ldots, a_n(x)$ and $G(x)$ are function of x on some interval $I \subseteq \mathbb{R}$ and $a_0(x) \neq 0$ on I.

Definition

The equation (2) is a **linear ordinary differential equation of order n with constant coefficients** if $a_0(x), \ldots, a_n(x)$ are all constant functions.

Definition

A **solution** of a differential equation is any function y, which satisfies that differential equation.

Example

Let $a, b \in \mathbb{R}$. Show that $y = ax + be^x$ is a solution of

$$(1-x)y'' + xy' - y = 0.$$

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Solution Note that

$$y' = a + be^x \quad \text{and} \quad y'' = be^x.$$

So,

$$(1-x)y'' + xy' - y = (1-x)(be^x) + x(a + be^x) - (ax + be^x) = 0. \quad \square$$

Definition

A **general solution** of DE of order n is a solution that involves exactly n arbitrary constants.

Definition

A **particular solution** of DE of order n is a solution that obtained by assigning particular values to the arbitrary constants in the general solution.

Example

$y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{3}e^x$ is a general of solution of DE $y'' - 4y = e^x$, while $y = 2e^{2x} - 5e^{-2x} - \dfrac{1}{3}e^x$ is a particular solution.

Definition

A **initial-value problem** (IVP) is a DE of order n $F(x, y, y', \ldots, y^{(n)} = 0)$ with n initial conditions at $x = x_0$;

$$y(x_0) = d_0, y'(x_0) = d_1, \ldots, y^{n-1}(x_0) = d_{n-1},$$

where $d_0, d_1, \ldots, d_{n-1}$ are constants and $y(x)$ is a solution of DE when $x \geq x_0$.

Example

$$y'' + y' - x^3y = \cos x; \quad y(2) = 3, y'(2) = -1$$

is a initial-value problem.

Definition

A **boundary-value problem** (BVP) is a system of DE of order n $F(x, y, y', \ldots, y^{(n)} = 0)$ with n boundary conditions specified at more than one point.

Definition

A **2-point boundary-value problem** is a system of DE of order n $F(x, y, y', \ldots, y^{(n)} = 0)$ with n boundary conditions specified at $x = a$ and $x = b$ $(a < b)$ and $y(x)$ is a solution of DE when $a \leq x \leq b$.

Example

$$y'' + xy' - x^3y = \cos x; y(2) = 3, y'(5) = 0$$

is a 2-point boundary value problem.

Now, we will find a solution of DE of order 1 with degree 1

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) \quad \text{or} \quad M(x,y)dx + N(x,y)dy = 0.$$

Definition

A DE of order 1 with degree 1

$$M(x,y)dx + N(x,y)dy = 0$$

is **seperable equation** if

$$M(x,y) = M_1(x)M_2(y) \quad \text{and} \quad N(x,y) = N_1(x)N_2(y)$$

for some functions M_1, N_1 of x and M_2, N_2 of y.

Strategy to solve separable equation

From

$$M_1(x)M_2(y)dx + N_1(x)N_2(y)dy = 0,$$

we get

$$\frac{M_1(x)}{N_1(x)}dx + \frac{N_2(y)}{M_2(y)}dy = 0.$$

We integrate on both sides, so

$$\int \frac{M_1(x)}{N_1(x)}dx + \int \frac{N_2(y)}{M_2(y)}dy = C,$$

where C is a constant.

Example

Find a general solution of

$$y' - 8xy = 3y.$$

Example

Find a general solution of

$$y' - 8xy = 3y.$$

<u>Solution</u> From problem, we obtain

$$\begin{aligned}(3y + 8xy)dx - dy &= 0\\ y(3 + 8x)dx - dy &= 0\\ (3 + 8x)dx - \frac{dy}{y} &= 0.\end{aligned}$$

We integrate on both sides to obtain a general solution

$$3x + 4x^2 - \ln|y| = C,$$

where C is a constant.

Example

Find a general solution of

$$dx + xydy = y^2dx + ydy.$$

Example

Find a general solution of

$$dx + xydy = y^2dx + ydy.$$

<u>Solution</u> From problem, we have

$$(1-y^2)dx + (xy-y)dy = 0$$
$$\frac{dx}{x-1} + \frac{ydy}{1-y^2} = 0.$$

We integrate on both sides to obtain a general solution

$$\ln|x-1| - \frac{1}{2}\ln|1-y^2| = C,$$

where C is a constant.

Example

Find a general solution of

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos^2 x \cos^2 2y.$$

Example

Find a general solution of

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos^2 x \cos^2 2y.$$

Solution We simplify a problem to get

$$\cos^2 x dx - \sec^2 2y dy = 0$$

$$\frac{1}{2}(1 + \cos 2x)dx - \sec^2 2y dy = 0.$$

We integrate on both sides to obtain

$$\frac{x}{2} + \frac{\sin 2x}{4} - \tan y = C,$$

where C is a constant.

Example

Find the particular solution of IVP

$$xdx + ye^{-x}dy = 0; \quad y(0) = 1.$$

Example

Find the particular solution of IVP

$$xdx + ye^{-x}dy = 0; \quad y(0) = 1.$$

<u>Solution</u> From problem, we have

$$xe^{x}dx + ydy = 0 \quad \rightarrow \quad xe^{x} - e^{x} + \frac{y^2}{2} = C,$$

where C is a constant.
From $y(0) = 1$, we obtain $C = -1 + \frac{1}{2} = -\frac{1}{2}$.
Hence, the particular solution of DE is

$$xe^{x} - e^{x} + \frac{y^2}{2} = -\frac{1}{2}. \quad \square$$

Definition

Let $n \in \mathbb{Z}$ and $F : D \to \mathbb{R}$, where D is a domain on $\mathbb{R}^2$. A function $F(x, y)$ is **homogeneous function of degree** n if

$$F(\lambda x, \lambda y) = \lambda^n F(x, y)$$

for all $\lambda > 0$ and $(x, y) \in D$.

Example

Determine whether or not each of the following functions is homogeneous, and if so of what degree.

1. $F(x, y) = \sqrt{xy} - y$.
2. $F(x, y) = \dfrac{y^3 - xy^2}{x^3 - x^2 y}$.
3. $F(x, y) = x(\ln \sqrt{x^2 + y^2} - \ln y) + ye^{\frac{x}{y}}$.

1. $F(x, y) = \sqrt{xy} - y.$

<u>Solution</u> Let $\lambda > 0$ and $(x, y) \in D_F$, then

$$\begin{aligned} F(\lambda x, \lambda y) &= \sqrt{(\lambda x)(\lambda y)} - (\lambda y) \\ &= \lambda\sqrt{xy} - \lambda y \\ &= \lambda(\sqrt{xy} - y) \\ &= \lambda F(x, y). \end{aligned}$$

Hence F is a homogeneous function with degree 1. $\square$

2. $F(x,y) = \dfrac{y^3 - xy^2}{x^3 - x^2y}$.

Solution Let $\lambda > 0$ and $(x,y) \in D_F$, then

$$\begin{aligned} F(\lambda x, \lambda y) &= \frac{(\lambda y)^3 - (\lambda x)(\lambda y)^2}{(\lambda x)^3 - (\lambda x)^2(\lambda y)} \\ &= \frac{\lambda^3(y^3 - xy^2)}{\lambda^3(x^3 - x^2y)} \\ &= \frac{y^3 - xy^2}{x^3 - x^2y} \\ &= F(x,y). \end{aligned}$$

That is F is a homogeneous function with degree 0. $\square$

3. $F(x,y) = x(\ln\sqrt{x^2+y^2} - \ln y) + ye^{\frac{x}{y}}$.

Solution Let $\lambda > 0$ and $(x,y) \in D_F$, then

$$\begin{aligned}
F(\lambda x, \lambda y) &= (\lambda x)(\ln\left(\sqrt{(\lambda x)^2 + (\lambda y)^2} - \ln(\lambda y)\right)) + (\lambda y)e^{\frac{\lambda x}{\lambda y}} \\
&= (\lambda x)(\ln\left((\lambda\sqrt{x^2+y^2}) - \ln(\lambda y)\right)) + (\lambda y)e^{\frac{x}{y}} \\
&= \lambda\left(x\ln\left(\sqrt{x^2+y^2} - \ln y\right) + ye^{\frac{x}{y}}\right) \\
&= \lambda F(x,y).
\end{aligned}$$

Hence F is a homogeneous function with degree 1. $\square$

Definition

A DE of order 1 with degree 1

$$M(x,y)dx + N(x,y)dy = 0$$

is **homogeneous differential equation** if $M(x,y)$ and $N(x,y)$ are homogeneous functions with same degree.

Strategy to solve homogeneous differential equation

Assume that $M(x,y)$ and $N(x,y)$ are homogeneous functions with degree k for some $k \in \mathbb{Z}$. Then

$$M(x,y) = x^k M\left(1, \frac{y}{x}\right) \quad \text{and} \quad N(x,y) = x^k N\left(1, \frac{y}{x}\right).$$

So,

$$x^k M\left(1, \frac{y}{x}\right) dx + x^k N\left(1, \frac{y}{x}\right) dy = 0.$$

That is

$$M\left(1, \frac{y}{x}\right) dx + N\left(1, \frac{y}{x}\right) dy = 0.$$

Strategy to solve homogeneous differential equation (cont.)

Next, let $u = \dfrac{y}{x}$ or $y = ux$, then $dy = udx + xdu$. So

$$M(1,u)dx + N(1,u)(udx + xdu) = 0$$

and we get

$$(M(1,u) + uN(1,u))dx + xN(1,u)du = 0.$$

That is homogeneous differential equation becomes to separable equation.

Example

Find a general solution of

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x-y}{x+y}.$$

Example

Find a general solution of

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x-y}{x+y}.$$

Solution From problem, we have

$$(x-y)dx - (x+y)dy = 0.$$

That is $M(x,y) = x - y$ and $N(x,y) = -(x+y)$. Note that

$$M(\lambda x, \lambda y) = \lambda x - \lambda y = \lambda(x-y) = \lambda M(x,y) \quad \text{and}$$

$$N(\lambda x, \lambda y) = -(\lambda x + \lambda y) = \lambda(-(x+y)) = \lambda N(x,y)$$

for all $\lambda > 0$ and $(x,y) \in D_M \cap D_N$.

That is $M(x,y)$ and $N(x,y)$ are homogeneous functions with same degree. So, let $y = ux$ and $dy = udx + xdu$. Then

$$\begin{aligned}(x-ux)dx-(x+ux)(udx+xdu) &= 0\\(1-2u-u^2)dx-x(1+u)du &= 0\\\frac{dx}{x}+\frac{(1+u)du}{u^2+2u-1} &= 0.\end{aligned}$$

So,

$$\ln|x|+\frac{1}{2}\ln|u^2+2u-1| = C.$$

Substitutes $u = \dfrac{y}{x}$, so a general solution is

$$\ln|x|+\frac{1}{2}\ln\left|\left(\frac{y}{x}\right)^2+2\left(\frac{y}{x}\right)-1\right| = C,$$

where C is a constant.

Example

Find a general solution of

$$\sqrt{x^2+y^2}dx = xdy - ydx.$$

Example

Find a general solution of

$$\sqrt{x^2+y^2}dx = xdy - ydx.$$

<u>Solution</u> From problem, we obtain

$$(\sqrt{x^2+y^2}+y)dx - xdy = 0.$$

That is $M(x,y) = \sqrt{x^2+y^2}+y$ and $M(x,y) = -x$. Note that

$$M(\lambda x, \lambda y) = \sqrt{(\lambda x)^2+(\lambda y)^2} = \lambda\sqrt{x^2+y^2} = \lambda M(x,y) \quad \text{and}$$

$$N(\lambda x, \lambda y) = -\lambda x = \lambda N(x,y)$$

for all $\lambda > 0$ and $(x,y) \in D_M \cap D_N$.

That is $M(x,y)$ and $N(x,y)$ are homogeneous functions with same degree. So, let $y = ux$ and $dy = udx + xdu$. Then

$$\begin{aligned}(\sqrt{x^2+u^2x^2}+ux)dx - x(udx+xdu) &= 0\\ \sqrt{1+u^2}dx - xdu &= 0\\ \frac{dx}{x} - \frac{du}{\sqrt{1+u^2}} &= 0.\end{aligned}$$

Hence,

$$\ln|x| - \ln|\sqrt{1+u^2}+u| = C.$$

Substitutes $u = \dfrac{y}{x}$, so a general solution is

$$\ln|x| - \ln\left|\sqrt{1+\left(\frac{y}{x}\right)^2}+\frac{y}{x}\right| = C,$$

where C is a constant.

Example

Find a general solution of

$$(x^2y + 2xy^2 - y^3)dx - (2y^3 - xy^2 + x^3)dy = 0.$$

Example

Find a general solution of

$$(x^2y + 2xy^2 - y^3)dx - (2y^3 - xy^2 + x^3)dy = 0.$$

<u>Solution</u> From problem, we get

$$M(x,y) = x^2y + 2xy^2 - y^3, \quad N(x,y) = 2y^3 - xy^2 + x^3.$$

It is easy to check that

$$M(\lambda x, \lambda y) = \lambda^3 M(x,y) \quad \text{and} \quad N(\lambda x, \lambda y) = \lambda^3 N(x,y)$$

for all $\lambda > 0$ and $(x,y) \in D_M \cap D_N$.

Hence $M(x,y)$ and $N(x,y)$ are homogeneous functions with same degree. So, let $y = ux$ and $dy = udx + xdu$, then

$$(2u^2 - 2u^4)dx - x(2u^3 - u^2 + 1)du = 0$$
$$\frac{dx}{x} + \frac{2u^3 - u^2 + 1}{2u^4 - 2u^2}du = 0$$
$$\frac{dx}{x} + \frac{1}{2}\left(-\frac{1}{u^2} + \frac{1}{u-1} + \frac{1}{u+1}\right)du = 0.$$

So,

$$\ln|x| + \frac{1}{2}\left(\frac{1}{u} + \ln|u-1| + \ln|u+1|\right) = C.$$

Substitutes $u = \dfrac{y}{x}$, so a general solution is

$$\ln|x| + \frac{1}{2}\left(\frac{x}{y} + \ln\left|\left(\frac{y}{x}\right) - 1\right| + \ln\left|\left(\frac{y}{x}\right) + 1\right|\right) = C. \quad \square$$

Example

Find a general solution of

$$\left(x^2\sin\left(\frac{y^2}{x^2}\right)-2y^2\cos\left(\frac{y^2}{x^2}\right)\right)dx+2xy\cos\left(\frac{y^2}{x^2}\right)dy=0.$$

Example

Find a general solution of

$$\left(x^2\sin\left(\frac{y^2}{x^2}\right)-2y^2\cos\left(\frac{y^2}{x^2}\right)\right)dx+2xy\cos\left(\frac{y^2}{x^2}\right)dy=0.$$

Solution From problem, we have

$$M(x,y)=x^2\sin\left(\frac{y^2}{x^2}\right)-2y^2\cos\left(\frac{y^2}{x^2}\right),$$

$$N(x,y)=2xy\cos\left(\frac{y^2}{x^2}\right).$$

It is easy to check that $M(x,y)$ and $N(x,y)$ are homogeneous functions of degree 2.

So, let $y = ux$ and $dy = udx + xdu$, then

$$\begin{aligned} \sin\left(u^2\right)dx + 2ux\cos\left(u^2\right)du &= 0 \\ \frac{dx}{x} + 2\cot\left(u^2\right)du &= 0. \end{aligned}$$

So,

$$\ln|x| + \ln\left|\sin\left(u^2\right)\right| = C.$$

Substitutes $u = \dfrac{y}{x}$, so a general solution is

$$\ln|x| + \ln\left|\sin\left(\frac{y^2}{x^2}\right)\right| = C,$$

where C is a constant. $\square$

Definition

Let f be function of n free variables $x_1, \ldots, x_n$. The **total differential of f** is defined by

$$df(x_1, \ldots, x_n) = \frac{\partial f}{\partial x_1} dx_1 + \ldots + \frac{\partial f}{\partial x_n} dx_n.$$

Definition

Let R be a rectangle region in $\mathbb{R}^2$. An equation

$$M(x,y)dx + N(x,y)dy = 0$$

is **exact** if and only if there exists a function $f(x,y)$ such that $M(x,y) = \dfrac{\partial f}{\partial x}$ and $N(x,y) = \dfrac{\partial f}{\partial y}$ for all (x,y) in the region R.

Strategy to solve Exact equation

From the previous definition, we have

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0.$$

It means

$$df(x, y) = 0$$

for all $(x, y) \in R$. Hence $f(x, y) = C$ for some constant C.

Remark

In general, it is very hard to see that what function f that makes $\dfrac{\partial f}{\partial x} = M(x, y)$ and $\dfrac{\partial f}{\partial y} = N(x, y)$.

Example

Find a general solution of

$$(y^3 - 2x)dx + (3xy^2 - 1)dy = 0.$$

Theorem

Let R be rectangle region in $\mathbb{R}^2$. If $\dfrac{\partial M}{\partial y}$ and $\dfrac{\partial N}{\partial x}$ are continuous on R, then

$$M(x,y)dx + N(x,y)dy = 0$$

is exact if and only if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ on R.

Strategy to solve Exact equation (Easy version)

First, we check that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

So, by the definition, there exists the function f such that

$$\frac{\partial f}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial f}{\partial y} = N(x,y).$$

Since $\dfrac{\partial f}{\partial x} = M(x,y)$, we integrate respect to x on both sides, so

$$f(x,y) = \int M(x,y)dx + g(y)$$

for some function g of y.

Strategy to solve Exact equation (Easy version) (Cont.)

Next, we differentiate respect to y on both sides, so

$$N(x,y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)dx + g'(y).$$

We have

$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx.$$

This implies

$$g(y) = \int \left[N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx\right]dy.$$

Strategy to solve Exact equation (Easy version) (Cont.)

Hence,

$$f(x,y) = \int M(x,y)dx + \int \left[N(x,y) - \frac{\partial}{\partial y} \int M(x,y)dx \right] dy.$$

Note that $f(x,y) = C$ for some constant C. So, a general solution of DE is

$$\int M(x,y)dx + \int \left[N(x,y) - \frac{\partial}{\partial y} \int M(x,y)dx \right] dy = C.$$

Remark

If we start with $\frac{\partial f}{\partial y} = N(x,y)$, we integrate respect to y on both sides, so

$$f(x,y) = \int N(x,y)dy + h(x)$$

for some function h of x.
Next, we differentiate respect to x on both sides, so

$$M(x,y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\int N(x,y)dy + h'(x).$$

We have

$$h'(x) = M(x,y) - \frac{\partial}{\partial x}\int N(x,y)dy.$$

Remark (Cont.)

This implies

$$h(x) = \int \left[M(x,y) - \frac{\partial}{\partial x} \int N(x,y)dy \right] dx.$$

Hence,

$$C = f(x,y) = \int N(x,y)dy + \int \left[M(x,y) - \frac{\partial}{\partial x} \int N(x,y)dy \right] dx$$

for some constant C.

Example

Find a general solution of

$$(y^3 - 2x)dx + (3xy^2 - 1)dy = 0.$$

Example

Find a general solution of

$$(y^3 - 2x)dx + (3xy^2 - 1)dy = 0.$$

Solution Note that

$$M(x,y) = y^3 - 2x \quad \text{and} \quad N(x,y) = 3xy^2 - 1.$$

Since

$$\frac{\partial M}{\partial y} = 3y^2 = \frac{\partial N}{\partial x},$$

this equation is exact.

So, there exists function f such that $\dfrac{\partial f}{\partial x} = M(x, y)$ and $\frac{\partial f}{\partial y} = N(x, y)$.
So

$$\begin{aligned} f(x, y) &= \int M(x, y)dx + g(y) \\ &= \int (y^3 - 2x)dx + g(y) \\ &= xy^3 - x^2 + g(y) \end{aligned}$$

for some function g of y.

Then

$$3xy^2 + g'(y) = \frac{\partial f}{\partial y} = 3xy^2 - 1.$$

So,

$$g'(y) = -1 \quad \rightarrow \quad g(y) = -y + C$$

for some constant C. Thus a general solution of this DE is

$$xy^3 - x^2 - y = C. \quad \square$$

Or

$$\begin{aligned}(y^3-2x)dx+(3xy^2-1)dy&=0\\ y^3dx-2xdx+3xy^2dy-dy&=0\\ y^3dx+xd(y^3)-d(x^2)-dy&=0\\ d(xy^3)-d(x^2)-dy&=0\\ d(xy^3-x^2-y)&=0.\end{aligned}$$

So, a general solution of this DE is

$$xy^3-x^2-y=C,$$

where C is a constant. $\square$

Example

Find a general solution of

$$(2x + \cosh xy)dx + \left(\frac{xy\cosh xy - \sinh xy}{y^2}\right)dy = 0,$$

where $\cosh x = \dfrac{e^x + e^{-x}}{2}$ and $\sinh x = \dfrac{e^x - e^{-x}}{2}$.

Example

Find a general solution of

$$(2x + \cosh xy)dx + \left(\frac{xy\cosh xy - \sinh xy}{y^2}\right)dy = 0,$$

where $\cosh x = \dfrac{e^x + e^{-x}}{2}$ and $\sinh x = \dfrac{e^x - e^{-x}}{2}$.

<u>Solution</u> It is easy to show that

$$\frac{\mathrm{d}\cosh x}{\mathrm{d}x} = \frac{e^x - e^{-x}}{2} = \sinh x \quad \text{and} \quad \frac{\mathrm{d}\sinh x}{\mathrm{d}x} = \frac{e^x + e^{-x}}{2} = \cosh x.$$

Note that

$$M(x,y) = 2x + \cosh xy, \quad \text{and} \quad N(x,y) = \frac{xy\cosh xy - \sinh xy}{y^2}.$$

Since,

$$\frac{\partial M}{\partial y} = x\sinh xy = \frac{\partial N}{\partial x}.$$

So, this equation is exact.

Then, there exists a function f such that $\dfrac{\partial f}{\partial x} = M(x,y)$ and $\dfrac{\partial f}{\partial y} = N(x,y)$. So,

$$\begin{aligned} f(x,y) &= \int M(x,y)dx + g(y) \\ &= \int (2x + \cosh xy)dx + g(y) \\ &= x^2 + \frac{\sinh xy}{y} + g(y) \end{aligned}$$

for some function g of y.

Hence,

$$\frac{xy\cosh xy - \sinh xy}{y^2} + g'(y) = \frac{xy\cosh xy - \sinh xy}{y^2}.$$

This implies

$$g'(y) = 0 \quad \rightarrow \quad g(y) = C$$

for some constant C. Thus a general solution of this DE is

$$x^2 + \frac{\sinh xy}{y} = C. \quad \square$$

Or

$$(2x + \cosh xy)dx + \left(\frac{xy\cosh xy - \sinh xy}{y^2}\right)dy = 0$$

$$2xdx + \frac{1}{y^2}(y^2\cosh xydx + xy\cosh xydy - \sinh xydy) = 0$$

$$d(x^2) + \frac{y\cosh xy(ydx + xdy) - \sinh xydy}{y^2} = 0$$

$$d(x^2) + \frac{y\cosh xyd(xy) - \sinh xydy}{y^2} = 0$$

$$d(x^2) + \frac{yd(\sinh xy) - \sinh xydy}{y^2} = 0$$

$$d(x^2) + d\left(\frac{\sinh xy}{y}\right) = 0$$

$$d\left(x^2 + \frac{\sinh xy}{y}\right) = 0.$$

So, a general solution of this DE is

$$x^2 + \frac{\sinh xy}{y} = C,$$

for some constant C. $\square$

Example

Find the particular solution of

$$(ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x)dx + (xe^{xy}\cos 2x - 3)dy = 0, \quad y(3) = 7.$$

Example

Find the particular solution of

$$(ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x)dx + (xe^{xy}\cos 2x - 3)dy = 0, \quad y(3) = 7.$$

Solution Note that

$$M(x,y) = ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x \quad \text{and} \quad N(x,y) = xe^{xy}\cos 2x - 3.$$

Since

$$\frac{\partial M}{\partial y} = xye^{xy}\cos 2x + e^{xy}\cos 2x - 2xe^{xy}\sin 2x = \frac{\partial N}{\partial y},$$

this equation is exact.

So, there exists function f such that $\dfrac{\partial f}{\partial x} = M(x, y)$ and $\dfrac{\partial f}{\partial y} = N(x, y)$.

So

$$\begin{aligned} f(x, y) &= \int N(x, y)dy + h(x) \\ &= \int (xe^{xy}\cos 2x - 3)dy + h(x) \\ &= e^{xy}\cos 2x - 3y + h(x) \end{aligned}$$

for some function h of x.

Then

$$ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x = \frac{\partial f}{\partial x} = ye^{xy}\cos 2x - 2e^{xy}\sin 2x + h'(x).$$

So,

$$h'(x) = 2x \quad \rightarrow \quad h(x) = x^2 + C$$

for some constant C. Thus, a general solution of this DE is

$$e^{xy}\cos 2x - 3y + x^2 = C.$$

From $y(3) = 7$, we obtain

$$C = e^{21}\cos 6 - 21 + 9 = -12 + e^{21}\cos 6.$$

Thus, the particular solution is
$e^{xy}\cos 2x - 3y + x^2 = -12 + e^{21}\cos 6.$ $\square$

If

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

We will find a function $\mu = \mu(x, y)$ that change the first order DE

$$M(x, y)dx + N(x, y)dy = 0$$

to an exact equation. μ is called a **integration factor**.

Method to find μ

First, we multiply μ on both sides,

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0.$$

So

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

and this implies

$$\mu\frac{\partial M}{\partial y} + M\frac{\partial \mu}{\partial y} = \mu\frac{\partial N}{\partial x} + N\frac{\partial \mu}{\partial x}.$$

Method to find μ (Cont.)

Case I $\mu = \mu(x)$, so $\dfrac{\partial \mu}{\partial y} = 0$ and

$$\mu \frac{\partial M}{\partial y} = \mu \frac{\partial N}{\partial x} + N \frac{\mathrm{d}\mu}{\mathrm{d}x}.$$

Hence,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}x}$$

and we integrate respect to x on both sides, so

$$\int \left[\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\right] dx = \int \frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}x} dx.$$

Method to find μ (Cont.)

$$\ln|\mu| = \int \left[\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\right] dx$$

$$\mu(x) = e^{\int \left[\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\right] dx}.$$

Method to find μ (Cont.)

Case II $\mu = \mu(y)$, so $\dfrac{\partial \mu}{\partial x} = 0$ and

$$\mu \frac{\partial M}{\partial y} + M \frac{\mathrm{d}\mu}{\mathrm{d}y} = \mu \frac{\partial N}{\partial x}.$$

Hence,

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}y}$$

and we integrate respect to y on both sides, so

$$\int \left[\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\right] dy = \int \frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}y} dy.$$

Method to find μ (Cont.)

$$\ln|\mu| = \int \left[\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\right] dy$$

$$\mu(y) = e^{\int \left[\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\right] dy}.$$

Example

Find a general solution of

$$y^2 \cos x dx + (4 + 5y \sin x)dy = 0.$$

Solution Note that

$$M(x,y) = y^2 \cos x, \qquad N(x,y) = 4 + 5y \sin x$$

and

$$\frac{\partial M}{\partial y} = 2y \cos 2x, \quad \frac{\partial N}{\partial x} = 5y \cos x.$$

So, this equation is nonexact.

Since

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{3}{y}$$

is a function of y. So, the integration factor is

$$\mu(y) = e^{\int \frac{3}{y} dy} = e^{3\ln|y|} = y^3.$$

Next, we multiply with y^3 on both sides of DE, we get

$$y^5 \cos x dx + (4y^3 + 5y^4 \sin x)dy = 0.$$

So

$$\begin{aligned}
y^5 \cos x dx + (4y^3 + 5y^4 \sin x)dy &= 0 \\
y^5 \cos x dx + 4y^3 dy + 5y^4 \sin x dy &= 0 \\
y^5 d(\sin x) + \sin x d(y^5) + d(y^4) &= 0 \\
d(y^5 \sin x) + d(y^4) &= 0 \\
d(y^5 \sin x + y^4) &= 0.
\end{aligned}$$

Hence, a general solution of this DE is

$$y^5 \sin x + y^4 = C,$$

where C is a constant. □

Example

Find a general solution of

$$y + 2y^3 = (y^3 + 6x)y'.$$

Example

Find a general solution of

$$y + 2y^3 = (y^3 + 6x)y'.$$

Solution Note that $y' = \dfrac{\mathrm{d}y}{\mathrm{d}x}$. So

$$(y + 2y^3)dx - (y^3 + 6x)dy = 0$$

Hence

$$M(x,y) = y + 2y^3, \quad N(x,y) = -(y^3 + 6x)$$

and

$$\frac{\partial M}{\partial y} = 1 + 6y^2, \quad \frac{\partial N}{\partial x} = -6.$$

So, this equation is nonexact.

Since,

$$\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{-7-6y^2}{y+2y^3}$$

is a function of y and

$$\int \frac{-7-6y^2}{y+2y^3}dy=-\int \frac{7}{y}dy+\int \frac{8y}{2y^2+1}dy=-7\ln|y|+2\ln\left(2y^2+1\right).$$

So, the integration factor is

$$\mu(y)=e^{-7\ln|y|+2\ln\left(2y^2+1\right)}=y^{-7}(2y^2+1)^2.$$

Next, we multiply $y^{-7}(2y^2+1)^2$ on both sides of DE, we get

$$(y+2y^3)y^{-7}(2y^2+1)^2dx-(y^3+6x)y^{-7}(2y^2+1)^2dy=0.$$

So, there exists a function f such that

$$\frac{\partial f}{\partial x}=(y+2y^3)y^{-7}(2y^2+1)^2 \quad \text{and} \quad \frac{\partial f}{\partial y}=-(y^3+6x)y^{-7}(2y^2+1)^2.$$

Hence,

$$\begin{aligned} f(x,y) &= \int (y+2y^3)y^{-7}(2y^2+1)^2dx+g(y) \\ &= xy^{-6}(2y^2+1)^2+2xy^{-4}(2y^2+1)^2+g(y) \end{aligned}$$

for some function g of y.

So,

$$-6xy^{-7}(2y^2+1)^2+g'(y)=\frac{\partial f}{\partial y}=-y^{-4}(2y^2+1)^2-6xy^{-7}(2y^2+1)^2.$$

Hence,

$$g'(y)=-y^{-4}(2y^2+1)^2 \quad \rightarrow \quad g(y)=-4y+4y^{-1}+\frac{y^{-3}}{3}+C.$$

Thus, a general solution of this DE is

$$xy^{-6}(2y^2+1)^2+2xy^{-4}(2y^2+1)^2-4y+4y^{-1}+\frac{y^{-3}}{3}=C,$$

where C is a constant. □

Example

Find a general solution of

$$(3xy + y^2)dx + (x^2 + xy)dy = 0.$$

Solution Note that

$$M(x,y) = 3xy + y^2 \quad \text{and} \quad N(x,y) = x^2 + xy$$

and

$$\frac{\partial M}{\partial y} = 3x + 2y, \quad \frac{\partial N}{\partial x} = 2x + y.$$

So, this equation is nonexact.

Since,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{x}$$

is a function of x, the integration factor is

$$\mu(x)=e^{\int \frac{1}{x}dx}=e^{\ln|x|}=x.$$

Next, we multiply with x on both sides of DE, we have

$$(3x^2y+xy^2)dx+(x^3+x^2y)dy=0.$$

Then

$$\begin{aligned}
(3x^2y + xy^2)dx + (x^3 + x^2y)dy &= 0 \\
yd(x^3) + \frac{1}{2}y^2d(x^2) + x^3dy + \frac{1}{2}x^2d(y^2) &= 0 \\
d(x^3y) + d\left(\frac{1}{2}x^2y^2\right) &= 0 \\
d\left(x^3y + \frac{1}{2}x^2y^2\right) &= 0.
\end{aligned}$$

So, a general solution of this DE is

$$x^3y + \frac{1}{2}x^2y^2 = C,$$

where C is a constant.

Example

Find a general solution of

$$x^2y' + 4xy = e^x.$$

Example

Find a general solution of

$$x^2y' + 4xy = e^x.$$

Since $y' = \frac{\mathrm{d}y}{\mathrm{d}x}$, so

$$(e^x - 4xy)dx - x^2dy = 0.$$

Note that

$$M(x,y) = e^x - 4xy, \quad N(x,y) = -x^2$$

and

$$\frac{\partial M}{\partial y} = -4x, \quad \frac{\partial N}{\partial x} = -2x.$$

So, this equation is nonexact.

Since

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{2}{x}$$

is a function of x, the integration factor is

$$\mu(x)=e^{\int \frac{2}{x}dx}=e^{2\ln|x|}=x^2.$$

Next, we multiply x^2 on both sides to get

$$(x^2e^x-4x^3y)dx-x^4dy=0.$$

Hence, there exists a function f such that

$$\frac{\partial f}{\partial x} = x^2e^x - 4x^3y \quad \text{and} \quad \frac{\partial f}{\partial y} = -x^4.$$

Then

$$\begin{aligned} f(x,y) &= \int (-x^4)dy + h(x) \\ &= -x^4y + h(x) \end{aligned}$$

for some function h of x. So,

$$-4x^3y + h'(x) = \frac{\partial f}{\partial y} = x^2e^x - 4x^3y$$

That is

$$h'(x) = x^2e^x \quad \rightarrow \quad h(x) = x^2e^x - 2xe^x + 2e^x + C$$

for some constant C.

Thus, a general solution of this DE is

$$-x^4y + x^2e^x - 2xe^x + 2e^x = C,$$

where C is a constant. $\square$

Extra : Tokyo University (1991)

Let $f(x)$ be a continuous function defined for $x > 0$ such that $f(x_1) > f(x_2) > 0$ whenever $0 < x_1 < x_2$. Let

$$S(x) = \int_x^{2x} f(t)dt$$

and $S(1) = 1$. For any $\alpha > 0$, the area bounded by the following is $3S(a)$.

1. the line joining the origin and the point $(a, f(a))$,
2. the line joining the origin and the point $(2a, f(2a))$,
3. the curve $y = f(x)$.

Extra : Tokyo University (1991)

1. Express $S(x), f(x) - 2f(2x)$ as a function of x.
2. For $x > 0$, let
$$a(x) = \lim_{n\to\infty} 2^n f(2^n x).$$
Find the value of the integral
$$\int_x^{2x} a(t)dt.$$
3. Determine the function $f(x)$.

Definition

The **first-order linear equation** is a differential equation of a form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x).$$

Strategy to solve first-order linear equation

First, we can change the equation in the form

$$M(x,y)dx + N(x,y)dy = 0,$$

where $M(x,y) = P(x)y - Q(x)$ and $N(x,y) = 1$. Then

$$\frac{\partial M}{\partial y} = P(x) \quad \text{and} \quad \frac{\partial N}{\partial x} = 0.$$

So, this equation is non exact. Note that

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = P(x).$$

So, the integration factor of the DE is $\mu(x) = e^{\int P(x)dx}$.

Strategy to solve first-order linear equation (cont.)

So, we multiply with $\mu(x)$ on both sides,

$$(P(x)y - Q(x))e^{\int P(x)}dx + e^{\int P(x)dx}dy = 0.$$

$$yP(x)e^{\int P(x)dx}dx + e^{\int P(x)dx}dy = Q(x)e^{\int P(x)dx}dx.$$

Note that $d\mu(x) = e^{\int P(x)dx}P(x)dx$. So

$$\begin{aligned} yd\mu(x) + \mu(x)dy &= Q(x)\mu(x)dx \\ d(y\mu(x)) &= Q(x)\mu(x)dx. \end{aligned}$$

That is $y\mu(x) = \displaystyle\int Q(x)\mu(x)dx + C.$

Strategy to solve first-order linear equation (cont.)

So, the solution of the first-order linear equation is

$$y = \frac{1}{\mu(x)}\left(\int Q(x)\mu(x)dx + C\right),$$

where $\mu(x) = e^{\int P(x)dx}$.

Example

Find a general solution of

$$x^2y' + 4xy = e^x.$$

Example

Find a general solution of

$$x^2y' + 4xy = e^x.$$

Solution First, we divided by x^2 on both sides,

$$y' + 4x^{-1}y = x^{-2}e^x.$$

So, this DE is a linear equation with $P(x) = 4x^{-1}$ and $Q(x) = x^{-2}e^x$. Hence

$$\mu(x) = e^{\int \frac{4}{x}dx} = e^{4\ln|x|} = x^4$$

and a general solution is

$$y = \frac{1}{x^4}\left(\int x^2e^x dx + C\right) = \frac{1}{x^4}\left(x^2e^x - 2xe^x + 2e^x + C\right). \quad \square$$

Example

Find the particular solution of

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^2 + 1, \quad y(1) = 1.$$

Example

Find the particular solution of

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^2 + 1, \quad y(1) = 1.$$

Solution We divided by x on both sides,

$$y' + x^{-1}y = x + x^{-1}.$$

So, this DE is a linear equation with $P(x) = x^{-1}$ and $Q(x) = x + x^{-1}$. Hence

$$\mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = x$$

and a general solution is

$$y = \frac{1}{x}\left(\int (x^2+1)dx + C\right) = \frac{1}{x}\left(\frac{x^3}{3} + x + C\right).$$

From a constraint $y(1) = 1$, we have

$$1 = \frac{1}{3} + 1 + C \quad \rightarrow \quad C = -\frac{1}{3}.$$

Thus, the particular solution of this DE is

$$y = \frac{1}{x}\left(\frac{x^3}{3} + x - \frac{1}{3}\right). \quad \square$$

Example

Find a general solution of

$$y' + 3x^2y = 6x^2.$$

Example

Find a general solution of

$$y' + 3x^2y = 6x^2.$$

Solution This is a linear equation with $P(x) = 3x^2$ and $Q(x) = 6x^2$. So

$$\mu(x) = e^{\int 3x^2dx} = e^{x^3}$$

and a general solution is

$$y = \frac{1}{e^{x^3}}\left(6\int x^2e^{x^3}dx + C\right) = \frac{1}{e^{x^3}}\left(2e^{x^3} + C\right). \quad \square$$

Example

Find the particular solution of

$$(1+x^2)(dy-dx)=2xydx, \quad y(0)=1.$$

Example

Find the particular solution of

$$(1+x^2)(dy-dx)=2xydx, \quad y(0)=1.$$

Solution First, we simplify this equation to get

$$y'-\frac{2x}{1+x^2}y=1.$$

So, this is a linear equation with $P(x)=-\frac{2x}{1+x^2}$ and $Q(x)=\frac{1}{1+x^2}$.
Hence

$$\mu(x)=e^{-\int\frac{2x}{1+x^2}dx}=e^{-\ln\left(1+x^2\right)}=\frac{1}{1+x^2}.$$

That is a general solution is

$$y = (1+x^2)\left(\int \frac{1}{1+x^2}dx + C\right) = (1+x^2)(\arctan x + C).$$

From $y(0) = 1$, we obtain

$$C = 1.$$

Thus, the particular solution of this DE is

$$y = (1+x^2)\Big(1 + \arctan x\Big). \quad \square$$

Example

Find a general solution of

$$4y' + 12y = 80 \sin 11x.$$

Example

Find a general solution of

$$4y' + 12y = 80\sin 11x.$$

<u>Solution</u> First, we divided by 4 on both sides,

$$y' + 3y = 20\sin 11x.$$

So, this is a linear equation with $P(x) = 3$ and $Q(x) = 20\sin 11x$. Then

$$\mu(x) = e^{\int 3dx} = e^{3x}$$

and a general solution is

$$y = e^{-3x}\left(20\int e^{3x}\sin 11x dx + C\right).$$

It is easy to find that

$$\int e^{3x} \sin 11x dx = \frac{e^{3x}}{130}(3 \sin 11x - 11 \cos 11x) + C.$$

Hence, a general solution of this DE is

$$y = \frac{20}{130}(3 \sin 11x - 11 \cos 11x) + Ce^{-3x},$$

where C is a constant. $\square$

Definition

Bernoulli differential equation is an equation of a form

$$y' + P(x)y = Q(x)y^n,$$

where $n \in \mathbb{R}$.

Definition

Bernoulli differential equation is an equation of a form

$$y' + P(x)y = Q(x)y^n,$$

where $n \in \mathbb{R}$.

Remark

Bernoulli DE is a linear equation if $n = 0$ or $n = 1$.

Strategy to solve Bernoulli DE

First, we divided with y^n on both sides,

$$y^{-n}y' + P(x)y^{1-n} = Q(x).$$

Next, we let $z = y^{1-n}$, then $z' = (1-n)y^{-n}y'$, so

$$\begin{aligned} \frac{1}{1-n}z' + P(x)z &= Q(x) \\ z' + (1-n)P(x)z &= (1-n)Q(x). \end{aligned}$$

So, this equation becomes to a linear equation with

$$\tilde{P}(x) = (1-n)P(x) \quad \text{and} \quad \tilde{Q}(x) = (1-n)Q(x).$$

Strategy to solve Bernoulli DE (Cont.)

Thus, a general solution of DE is

$$y^{1-n} = z = \frac{1}{\mu(x)}\left(\int (1-n)Q(x)\mu(x)dx + C\right),$$

where

$$\mu(x) = e^{\int (1-n)P(x)dx}$$

and C is a constant.

Example

Find a general solution of

$$3xy' + y + x^2y^4 = 0.$$

Example

Find a general solution of

$$3xy' + y + x^2y^4 = 0.$$

Solution From problem, we have

$$y^{-4}y' + \frac{1}{3}x^{-1}y^{-3} = -\frac{x}{3}.$$

Let $z = y^{1-4} = y^{-3}$, then $z' = -3y^{-4}y'$. Then

$$-\frac{z'}{3} + \frac{1}{3}x^{-1}z = -\frac{x}{3}$$
$$z' - x^{-1}z = x.$$

Hence $\mu(x) = e^{-\int \frac{1}{x} dx} = e^{-\ln|x|} = \dfrac{1}{x}$ and a general solution is

$$y^{-3} = z = x\left(\int dx + C\right) = x\left(x + C\right), \quad \square$$

where C is a constant. $\square$

Example

Find a general solution of

$$3xy^2y' = 3x^4 + y^3.$$

Example

Find a general solution of

$$3xy^2y' = 3x^4 + y^3.$$

Solution From problem, we get

$$y^2y' - \frac{1}{3}x^{-1}y^3 = x^3.$$

Let $z = y^3$, then $z' = 3y^2y'$, so

$$\begin{aligned} \frac{z'}{3} - \frac{1}{3}x^{-1}z &= x^3 \\ z' - x^{-1}z &= 3x^3. \end{aligned}$$

Hence $\mu(x) = e^{-\int \frac{1}{x}dx} = e^{-\ln|x|} = \frac{1}{x}$ and a general solution is

$$y^3 = z = x\left(3\int x^2 dx + C\right) = x(x^3 + C),$$

where C is a constant. $\square$

Example

Find the particular solution of

$$xy' = y + 2x^{\frac{1}{2}}y^{\frac{1}{2}}, \quad y(1) = 16.$$

Example

Find the particular solution of

$$xy' = y + 2x^{\frac{1}{2}}y^{\frac{1}{2}}, \quad y(1) = 16.$$

<u>Solution</u> From problem, we obtain

$$y^{-\frac{1}{2}}y' - x^{-1}y^{\frac{1}{2}} = 2x^{-\frac{1}{2}}.$$

Let $z = y^{\frac{1}{2}}$, then $z' = \frac{1}{2}y^{-\frac{1}{2}}y'$, so

$$\begin{aligned} 2z' - x^{-1}z &= 2x^{-\frac{1}{2}} \\ z' - \frac{1}{2}x^{-1}z &= x^{-\frac{1}{2}}. \end{aligned}$$

Hence, $\mu(x) = e^{-\frac{1}{2}\int \frac{1}{x}dx} = e^{-\frac{1}{2}\ln|x|} = x^{-\frac{1}{2}}$ and a general solution is

$$y^{\frac{1}{2}} = z = x^{\frac{1}{2}}\left(\int \frac{1}{x}dx + C\right) = x^{\frac{1}{2}}(\ln|x| + C),$$

where C is a constant.
From $y(1) = 16$, we get $C = 4$. Thus, the particular solution of this DE is

$$y^{\frac{1}{2}} = x^{\frac{1}{2}}(\ln|x| + 4). \quad \square$$

A general equation for second-order ODE is

$$F\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) = 0.$$

For reducible second-order equation. We can separate into 2 cases

1. y disappear in an equation. That is $F\left(x, \frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) = 0.$
2. x disappear in an equation. That is $F\left(y, \frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) = 0.$

Strategy to solve $F\left(x, \frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) = 0$

Let $v = \frac{\mathrm{d}y}{\mathrm{d}x}$, then $\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. So

$$F\left(x, \frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) = 0 \quad \Rightarrow \quad F\left(x, v, \frac{\mathrm{d}v}{\mathrm{d}x}\right) = 0.$$

Hence, the second-order ODE becomes to first-order ODE. After we get a solution, we substitute $v = \frac{\mathrm{d}v}{\mathrm{d}x}$ to solve y again.

Strategy to solve $F\left(y, \dfrac{\mathrm{d}y}{\mathrm{d}x}, \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) = 0$

Let $v = \dfrac{\mathrm{d}y}{\mathrm{d}x}$. By using chain rule, we have

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = v\frac{\mathrm{d}v}{\mathrm{d}y}.$$

So

$$F\left(y, \frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) = 0 \quad \Rightarrow \quad F\left(y, v, v\frac{\mathrm{d}v}{\mathrm{d}y}\right) = 0.$$

Hence, the second-order ODE becomes to first-order ODE. After we get a solution, we substitute $v = \dfrac{\mathrm{d}y}{\mathrm{d}x}$ to solve y again.

Example

Find a general solution of $y'' + (y')^2 = y'$.

Example

Find a general solution of $y'' + (y')^2 = y'$.

Solution Let $v = \frac{\mathrm{d}y}{\mathrm{d}x}$, then $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = v\frac{\mathrm{d}v}{\mathrm{d}y}$. So

$$v\frac{\mathrm{d}v}{\mathrm{d}y} + v^2 = v.$$

Hence

$$\frac{\mathrm{d}v}{\mathrm{d}y} = 1 - v$$

$$\frac{dv}{1-v} = dy.$$

We integrate on both side to obtain

$$-\ln|1-v| = y + C \quad \rightarrow \quad v = 1 - C_1e^{-y}.$$

Since $v = \dfrac{\mathrm{d}y}{\mathrm{d}x}$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - C_1 e^{-y}.$$

So

$$\begin{aligned} \frac{dy}{1 - C_1 e^{-y}} &= dx \\ \frac{e^y}{e^y - C_1} dy &= dx. \end{aligned}$$

Hence, a general solution is

$$\ln|e^y - C_1| = x + C_2,$$

where C_1 and C_2 are constants. $\square$

Example

Find a general solution of $xy'' + 2y' = \frac{1}{x}$.

Example

Find a general solution of $xy'' + 2y' = \dfrac{1}{x}$.

<u>Solution</u> Let $v = y'$, then $v' = y'$. So

$$xv' + 2v = \frac{1}{x} \quad \rightarrow \quad v' + \frac{2}{x}v = \frac{1}{x^2}.$$

This is a linear equation with $P(x) = 2x^{-1}$ and $Q(x) = x^{-2}$. Then

$$\mu(x) = e^{2\int \frac{1}{x}dx} = e^{2\ln|x|} = x^2.$$

Hence, a general solution of v is

$$y' = v = \frac{1}{x^2}\left(\int dx + C_1\right) = \frac{1}{x} + \frac{C_1}{x^2}.$$

Thus, a general solution is

$$y = \int \left(\frac{1}{x} + \frac{C_1}{x^2}\right) dx + C_2 = \ln|x| - \frac{C_1}{x} + C_2,$$

where C_1 and C_2 are constant. $\square$

Example

Find a general solution of $xy'' + 2y' = 6x$.

Example

Find a general solution of $xy'' + 2y' = 6x$.

Solution Let $v = y'$, then $v' = y''$. So

$$xv' + 2v = 6x \quad \rightarrow \quad v' + 2x^{-1}v = 6.$$

This is a linear equation with $P(x) = 2x^{-2}$ and $Q(x) = 6$. Then

$$\mu(x) = e^{2\int \frac{1}{x}dx} = e^{2\ln|x|} = x^2.$$

Hence, a general solution of v is

$$y' = v = \frac{1}{x^2}\left(6\int x^2 dx + C_1\right) = 2x + \frac{C_1}{x^2}.$$

Thus, a general solution is

$$y = \int \left(2x + \frac{C_1}{x^2}\right) + C_2 = x^2 - \frac{C_1}{x} + C_2,$$

where C_1 and C_2 are constants. $\square$

Example

Find a general solution of $y'' = 2y(y')^3$.

Example

Find a general solution of $y'' = 2y(y')^3$.

Solution Let $v = y'$, then $y'' = v\frac{\mathrm{d}v}{\mathrm{d}y}$. So

$$\begin{aligned} v\frac{\mathrm{d}v}{\mathrm{d}y} &= 2yv^3 \\ \frac{dv}{v^2} &= 2ydy \\ -\frac{1}{v} &= y^2 + C_1 \\ v &= -\frac{1}{y^2 + C_1}. \end{aligned}$$

Since $v = y'$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{y^2 + C_1}$$
$$(y^2 + C_1)dy = -dx.$$

Hence, a general solution is

$$\frac{y^3}{3} + C_1 x + C_2 = -x,$$

where C_1 and C_2 are constants. □

Example

Find the particular solution of IVP

$$y'' + 2y = 2y^3, \quad y(0) = 0, y'(0) = 1.$$

Example

Find the particular solution of IVP

$$y'' + 2y = 2y^3, \quad y(0) = 0, y'(0) = 1.$$

<u>Solution</u> Let $v = y'$, then $y'' = v\dfrac{\mathrm{d}v}{\mathrm{d}y}$. So

$$v\frac{\mathrm{d}v}{\mathrm{d}y} + 2y = 2y^3$$

$$vdv = (2y^3 - 2y)dy.$$

We integrate on both sides to obtain

$$\frac{(y')^2}{2} = \frac{v^2}{2} = \frac{y^4}{2} - y^2 + C_1.$$

From $y(0) = 0$ and $y'(0) = 1$, we have

$$\frac{1}{2} = 0 - 0 + C_1 \quad \rightarrow \quad C_1 = \frac{1}{2}.$$

Hence

$$\frac{(y')^2}{2} = \frac{y^4}{2} - y^2 + \frac{1}{2} \quad \rightarrow \quad y' = \sqrt{(y^2 - 1)^2} = |y^2 - 1|.$$

Since $y'(0) = 1$ and $y(0) = 0$, $y' = 1 - y^2$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - y^2$$
$$\frac{dy}{1 - y^2} = dx.$$

Hence, a general solution is

$$-\frac{1}{2}\ln|1-y| + \frac{1}{2}\ln|1+y| = x + C_2,$$

where C_2 is a constant.
From $y(0) = 0$, we obtain $C_2 = 0$. Thus, the particular solution is

$$-\frac{1}{2}\ln|1-y| + \frac{1}{2}\ln|1+y| = x. \quad \square$$