Week 7

November 28, 2018

First-Order Linear Equation

Definition. The first-order linear equation is a differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Strategy to Solve First-Order Linear Equation

First, we change the equation into the form

$$[P(x)y - Q(x)]dx + dy = 0$$

Note that $\frac{\partial M}{\partial y} = P(x)$ and $\frac{\partial N}{\partial x} = 0$ and that $\mu = e^{\int P(x)dx}$

Multiply with $\mu(x)$ on both sides.

$$yP(x)e^{\int P(x)dx}dx + e^{\int P(x)dx}dy = Q(x)e^{\int P(x)dx}dx$$

Note that $d\mu(x) = e^{\int P(x)dx}P(x)dx$. So

$$\begin{gathered} yd\mu(x) + \mu(x)dy = Q(x)\mu(x)dx \\ d(y\mu(x)) = Q(x)\mu(x)dx \end{gathered}$$

That is $y\mu(x) = \int Q(x)\mu(x)dx + C$

So the solution of the first-order linear equation is

$$y = \frac{1}{\mu(x)}\left(\int Q(x)\mu(x)dx + C\right)$$

where $\mu(x) = e^{\int P(x)dx}$. Beware of the position of C.

Examples

> Find a general solution of $x^2y' + 4xy = e^x$

> Find a general solution of $y' + 3x^2y = 6x^2$

$y = 2 + \frac{C}{e^{x^3}}$

> Find a particular solution of $(1+x^2)(dy - dx) = 2xydx$, $y(0) = 1$.

> Find a general solution of $4y' + 12y = 80\sin 11x$

$y = \frac{20}{130}(3\sin 11x - 11\cos 11x) + Ce^{-3x}$

Bernoulli Differential Equation

Definition. The Bernoulli differential equation is an equation of the form

$$y' + P(x)y = Q(x)y^n$$

where $n \in \mathbb{R}$. Note that if $n = 0$ or $n = 1$, the DE is a linear equation.

Strategy to solve Bernoulli DE

Divide by y^n on both sides.

$$y^{-n}y' + P(x)y^{1-n} = Q(x)$$

Let $z = y^{1-n}$, then $z' = (1-n)y^{-n}y'$ (Note that $z' = \dfrac{dz}{dx}$)

$$\begin{gathered}\frac{1}{1-n}z' + P(x)z = Q(x) \\ z' + (1-n)P(x)z = (1-n)Q(x)\end{gathered}$$

So, this equation becomes a linear equation.

The general solution of this DE is

$$y^{1-n} = z = \frac{1}{\mu(x)}\left(\int (1-n)Q(x)\mu(x)dx + C\right),$$

where

$$\mu(x) = e^{\int (1-n)P(x)dx}$$

Examples

> Find a general solution of $3xy' + y + x^2y^4 = 0$

$y^{-3} = z = x(x + C)$

> Find a general solution of $3xy^2y' = 3x^4 + y^3$

Divide by $3x$ on both sides. Let $z = y^3$. The answer is $y^3 = x(x^3 + C)$.

> Find a particular solution of $xy' = y + 2x^{1/2}y^{1/2}$, $y(1) = 16$

$y^{1/2} = x^{1/2}(\ln|x| + 4)$

Reducible Second-Order Equation

A general equation for second-order ODE is

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$$

We can separate these into 2 cases

1. y disappears in the equation. $F\left(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right)$ = 0

   Let $v = \frac{dy}{dx}$, then $\frac{dv}{dx} = \frac{d^2y}{dx^2}$.

   The equation becomes $F\left(x, v, \frac{dv}{dx}\right) = 0$

   Hence, the second-order ODE becomes first-order ODE. After we obtain a solution, we substitute $v = \frac{dy}{dx}$ to solve for y again.

2. x disappears in the equation. $F\left(y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$

   Let $v = \frac{dy}{dx}$. By the chain rule, we have

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy} \cdot \frac{dy}{dx} = v\frac{dv}{dy}.$$

   So the equation becomes $F\left(y, v, v\frac{dv}{dy}\right) = 0$

   After we obtain a solution, we substitute $v = \frac{dy}{dx}$ to solve for y again.

Examples

> Find a general solution of $y'' + (y')^2 = y'$

$y' = 1 + \frac{C_1}{e^y}$. $\ln|e^y + C_1| = x + C_2$

> Find a general solution of $xy'' + 2y' = \frac{1}{x}$.

> Find a general solution of $xy'' + 2y' = 6x$.

> Find a general solution of $y'' = 2y(y')^3$.

> Find a particular solution of the IVP
>
> $$y'' + 2y = 2y^3, y(0) = 0, y'(0) = 1$$