Week 6

November 21, 2018

Exact Equations

If

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x},$$

we will build a function $\mu = \mu(x, y)$ that, when multiplied to both sides of the equation, changes the first order DE to an exact equation. This is called the **integration factor.**

To find μ,

we multiply both sides by μ. Thus

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

and this implies

$$\mu \frac{\partial M}{\partial y} + M \frac{\partial \mu}{\partial y} = \mu \frac{\partial N}{\partial x} + N \frac{\partial \mu}{\partial x}.$$

Case I $\mu = \mu(x)$, so $\frac{\partial \mu}{\partial y} = 0$ and

$$\mu \frac{\partial M}{\partial y} = \mu \frac{\partial N}{\partial x} + N \frac{\partial \mu}{\partial x}.$$

Hence,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{\mu} \frac{d\mu}{dx}$$

and we integrate with respect to x on both sides, so

$$\int \left[\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\right] dx = \int \frac{1}{\mu} \frac{d\mu}{dx} dx$$

$$\ln|\mu| = \int \left[\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\right] dx$$

and solve for μ.

Case II $\mu = \mu(y)$, so $\frac{\partial \mu}{\partial x} = 0$.

Similar to case 1.

Examples

> Find a general solution of $y^2 \cos x dx + (4 + 5y \sin x)dy = 0.$

> Find a general solution of $y + 2y^3 = (y^3 + 6x)y'$.

$$xy^{-6}(2y^2 + 1)^2 \ldots$$

> Find a general solution of $(3xy + y^2)dx + (x^2 + xy)dy = 0$

$$x^3 y + \frac{1}{2}x^2 y^2 = C$$

> Find a general solution of $x^2 y' + 4xy = e^x$

Extra: Tokyo University Entrance Exam (1991)

> Let $f(x)$ be a continuous function defined for $x > 0$ such that $f(x_1) > f(x_2) > 0$ whenever $0 < x_1 < x_2$. Let
>
> $$S(x) = \int_x^{2x} f(t)dt$$
>
> and $S(1) = 1$. For any $a > 0$, the area bounded by the following is $3S(a)$.
>
> - the line joining the origin and the point $(a, f(a))$,
> - the line joining the origin and the point $(2a, f(2a))$,
> - the curve $y = f(x)$.
>
> 1. Express $S(x)$ and $f(x) - 2f(2x)$ as a function of x
> 2. For $x > 0$, let
>
> $$a(x) = \lim_{n \to \infty} 2^n f(2^n x)$$
>
> Find the value of the integral
>
> $$\int_x^{2x} a(t)dt.$$
>
> 3. Determine the function $f(x)$.