Week 4

November 7, 2018

Differential Equation

For example: $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$

Definition. An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a **differential equation**.

Definition. An **ordinary differential equation** is a differential equation containing one or more functions of *one* independent variable and its derivative.

Definition. A **partial differential equation** is a differential equation containing one or more functions of two or more independent variables and partial derivatives.

Examples of Ordinary Differential Equations

$$\frac{d^5y}{dx^5} - 2x\frac{d^4y}{dx^4} + \sin y\frac{dy}{dx} = x^2$$

$$x^3\frac{d^4y}{dx^4} - x^3\frac{dy}{dx} + x\frac{dy}{dx} = \frac{1}{1+y^2}$$

Definition. The **order** of a differential equation is the order of the highest order derivative in the equation.

Definition. The **degree** of a differential equation is the power of the *highest order derivative* in the equation.

Definition. A **linear differential equation** is a differential equation that satisfy the following equations

1. Every dependent variables and derivatives of dependent variables has the power 1.
2. No term of product of dependent variables and/or derivatives of dependent variables.
3. No term of transcendental functions (eg. $\sin, \cos, \tan$) of dependent variables or derivative of dependent variables

A differential equation is said to be a **nonlinear equation** if it is not a linear differential equation.

Examples of Linear Equations

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = x^3y + sinx$$

$$\frac{dy}{dx} = xy$$

Examples of Nonlinear Equations

$$x^2\frac{d^2y}{dx^2} + y\frac{dy}{dx} = x^3y + cosx$$

$$\sqrt{\frac{dy}{dx}} = xy$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=e^u$$

Every ODE linear equation of order n can be written in the form

$$F(x,y,y',\ldots,y^{(n)})=0$$

If the equation is linear, we can write it in the form of

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\ldots a_1(x)\frac{dy}{dx}+a_0(x)y=G(x)$$

where $a_0(x)\neq 0$.

A **solution** of a DE is any function y which satisfies the equation.

Examples

> Show that $y=ax+be^x$ is a solution of
>
> $$(1-x)y''+xy'-y=0$$

Definition. A **general solution** of a DE of order n is a solution that involves exactly n arbitrary constants. A **particular solution** of a DE of order n is a solution that has arbitrary values assigned.

Definition. An **initial-value problem** (IVP) is a DE of order n with n initial conditions at $x=x_0$:

$$y(x_0)=d_0,y'(x_0)=d1,\ldots,y^{n-1}(x_0)=d_n-1,$$

where $d_0,d_1,\ldots,d_{n-1}$ are constants and $y(x)$ is a solution of the DE when $x\geq x_0$

Example $y''+y'-x^3y=\cos x$; $y(2)=3,y'(2)=-1$

Definition. A **boundary-value problem** is a system of DE of order n with n boundary conditions specified at more than one point. A **2-point boundary-value problem** is a system of DE of order n with n boundary conditions specified at $x=a$ and $x=b$ and $y(x)$ is a solution of DE when $a\leq x\leq b$.

Example $y''+xy'-x^3y=\cos x$; $y(2)=3,y'(5)=0$

Now we will find a solution of DE of order 1 with degree 1

$$\frac{dy}{dx}=f(x,y)$$

or

$$M(x,y)dx+N(x,y)dy=0$$

Separable Equation

A DE of order 1 and degree 1

$$M(x,y)dx+N(x,y)dy=0$$

is **separable** if

$$M(x,y)=M_1(x)M_2(y), N(x,y)=N_1(x)N_2(y).$$

. To solve, we rearrange to find

$$\frac{M_1(x)}{N_1(x)}dx+\frac{N_2(y)}{M_2(y)}dy=0$$

and integrate both sides, so

$$\int\frac{M_1(x)}{N_1(x)}dx+\int\frac{N_2(y)}{M_2(y)}dy=C$$

Examples

> Find a general solution of $y'-8xy=3y$

$3x+4x^2-\ln|y|=C$

> Find a general solution of $dx+xydy=y^2dx+ydy$

$\frac{1}{2}\ln|y^2-1|=\ln|x-1|+C$

> Find a general solution of $\frac{dy}{dx}=cos^2xcos^22y$

> Find a particular solution of IVP $xdx+ye^{-x}dy=0; y(0)=1$

$y^2=-2xe^x+2e^x-1$

Definition. Let D be a domain on $\mathbb{R}^2$. A function $F(x,y)$ is a **homogeneous function** of degree n if

$$F(\lambda x,\lambda y)=\lambda^nF(x,y)$$

for all $\lambda>0$ and $(x,y)\in D$

Examples

Determine whether or not each of the following functions is homogeneous, and if so of what degree.

> $F(x,y)=\sqrt{xy}-y$

Yes and of order 1.

> $F(x,y)=\frac{y^3-xy^2}{x^3-x^2y}$

Yes and of order 0.

> $F(x,y)=x(\ln\sqrt{x^2+y^2}-\ln y)+ye^{x/y}$

Definition. A DE is a **homogeneous differential equation** if $M(x,y)$ and $N(x,y)$ are homogeneous equations of the same degree.

Strategy to solve homogeneous DE

Assume the degree is k then

$$M(x,y) = x^k M\left(1, \frac{y}{x}\right), N(x,y) = x^k N\left(1, \frac{y}{x}\right)$$

That is,

$$M(1, \frac{y}{x})dx + N(1, \frac{y}{x})dy = 0$$

Let $u = \dfrac{y}{x}$ then $dy = udx + xdu$. So

$$M(1,u)dx + N(1,u)(udx + xdu) = 0$$

and we get

$$(M(1,u) + uN(1,u))dx + xN(1,u)du = 0,$$

which is a separable equation.

Examples

> Find a general solution of $\dfrac{dy}{dx} = \dfrac{x-y}{x+y}$

> Find a general solution of $\sqrt{x^2+y^2}dx = xdy - ydx$

> Find a general solution of $(x^2y + 2xy^2 - y^3)dx - (2y^3 - xy^2 + x^3)dy = 0$

> Find a general solution of $\left(x^2 \sin\left(\dfrac{y^2}{x^2}\right) - 2y^2 \cos\left(\dfrac{y^2}{x^2}\right)\right) dx + 2xy \cos\left(\dfrac{y^2}{x^2}\right) dy = 0$