Week 9

December 12, 2018

Linear Ordinary Differential Equation

Definition. The **general linear equation** of order n is an equation in the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_0(x)y = f(x),$$

where $a_n(x) \neq 0$ for some interval I. $a_0(x), \ldots, a_n(x)$ are **coefficient functions** of equation and $a_n(x)$ is a **leading coefficient function**. If $f(x) = 0$, the equation is called a **homogeneous equation**, otherwise it is called a **non-homogeneous equation** and $f(x)$ is the **non-homogeneous term**. We call

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_0(x)y = 0$$

the **related homogeneous equation** of the previous equation. We call that the general ODE is normal if $a_0(x), \ldots, a_n(x)$ and $f(x)$ are all continuous and $a_n(x) \neq 0$ for all $x \in I$.

Note. We can check linearity by proving that $L(cu + v) = cL(u) + L(v)$. Example: $x^2y'' + 2xy' + 4y = L(y)$.

Example

Consider

$$y'' - \frac{1}{x}y' + 2y = \sin x.$$

The equation is normal on every interval which does not contain 0.

Theorem. (Existence and Uniqueness Theorem)

Let

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_0(x)y = f(x)$$

be normal on $I, x_0 \in I$ and $y_0, \ldots, y_{n-1} \in \mathbb{R}$. Then there exists a **unique** solution $y = y(x)$ that satisfies the conditions

$$y(x_0) = y_0, y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$$

Corollary.

If $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_0(x)y = 0$ and $y(x_0) = y'(x_0) = \ldots = y^{(n-1)}(x_0) = 0$, the only solution of the equation is $y = 0$.

Example

> Show that $y = e^{2x}$ is the only solution of $y'' - 3y' + 2y = 0$, $y(0) = 1$, $y'(0) = 2$.

Substitute and use the existence and uniqueness theorem to show that the solution is the only solution.

Definition. A solution $y(x) = 0$ is called a **trivial solution**, otherwise it is called a **nontrivial solution**.

Theorem. (Superposition Principle)

Let $y_1, \ldots, y_m$ be solutions of a linear ODE

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_0(x)y = 0$$

and $c_1, \ldots, c_m$ are constants, then

$$y = c_1 y_1 + \ldots + c_m y_m$$

is a solution of the equation.

Definition. We say that $y = c_1 y_1 + \ldots + c_m y_m$ is a **linear combination** of functions $y_1, \ldots, y_m$.

Example

Let $y_1 = e^{-x}$ and $y_2 = e^{2x}$. It is easy to show that these are solutions of

$$y'' - y' - 2y' = 0$$

on $\mathbb{R}$. By the superposition principle,

$$y = c_1 e^{-x} + c_2 e^{2x}$$

are solutions of the ODE for all $c_1, c_2 \in \mathbb{R}$.

Definition. Let $f_1, \ldots, f_n$ be functions on I. We say that these functions are **linearly independent** if for every $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$c_1 f_1(x) + \ldots + c_n f_n(x) = 0$$

for all $x \in I$, then $c_1 = \ldots = c_n = 0$.

Example

> Show that $\sin x$ and $\cos x$ are linearly independent on $\mathbb{R}$.

> Show that $f_1(x) = |x|$ and $f_2(x) = x$ are linearly independent on $\mathbb{R}$.

Theorem. For every linear ODE of order n

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_0(x)y = 0$$

that is normal on an interval I, there exists solutions $y_1, \ldots, y_n$, which are linearly independent and every particular solution of the equation can be written as a linear combination of $y_1, \ldots, y_n$.

Wronskian

Definition. Let $f_1, f_2, \ldots, f_n$ be $n-1$ times differentiable functions on $\mathbb{R}$, then the determinant

$$W(x) := W(f_1, \ldots, f_n : x) := \begin{vmatrix} f_1(x) & f_2(x) & \ldots & f_n(x) \\ f_1'(x) & f_2'(x) & \ldots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \ldots & f_n^{(n-1)}(x) \end{vmatrix}$$

is called the **Wronskian** of $f_1, f_2, \ldots, f_n$.

Theorem. Let $f_1, \ldots, f_n$ be $n-1$ differentiable functions on an interval I. If $W(f_1, \ldots, f_n : x) \neq 0$ for some $x \in I$, then $f_1, \ldots, f_n$ are linearly independent.

Caution. The converse of the theorem is not true.

Corollary. If $f_1, \ldots, f_n$ are linearly dependent, then $W(f_1, \ldots, f_n : x) = 0$ for all $x \in I$.

Examples

> Show that xe^x, x^2e^x are linearly independent.

> Let f_1 and f_2 be real-valued functions on the set of real numbers by
>
> $$f_1(x) = \begin{cases} x^3 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Theorem. Let $y_1, \ldots, y_n$ be particular solutions of a linear ODE of order n

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_0(x)y = 0$$

that is normal on an interval I. Either $W(y_1, \ldots, y_n : x)$ is identically zero or $W(y_1, \ldots, y_n : x) \neq 0$ for all $x \in I$.

This theorem is equivalent to the following.

Theorem. Let $y_1, \ldots, y_n$ be particular solutions of a linear ODE of order n

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_0(x)y = 0$$

that is normal on an interval I. $y_1, \ldots, y_n$ are linearly independent if and only if $W(y_1, \ldots, y_n : x) \neq 0$ for all $x \in I$.

Complete Solution of a Homogeneous Equation

Theorem. (Review) For every linear ODE of order n

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_0(x)y = 0$$

that is normal on an interval I, there exists solutions $y_1, \ldots, y_n$, which are linearly independent and every particular solution of the equation can be written as a linear combination of $y_1, \ldots, y_n$.

Definition. Functions $y_1, \ldots, y_n$ which are linearly independent particular solutions of a linear ODE of order n on an interval I are called **fundamental solutions** or **basis** of all solutions I and a linear combination

$$y = c_1y_1 + \ldots + c_ny_n$$

where $c_1, \ldots, c_n$ are constants, is a **complete solution** or **general solution** of an equation.

Example

> Use functions x^{-2} and $x^{-2}\ln x$ to find a complete solution (or general solution) of
>
> $$x^2y'' + 5xy' + 4y = 0$$
>
> on $[0, \infty)$.

Solution Verify solution. Check Wronskian.

Theorem. Let y_1 be one of solution of

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0; a_2(x) \neq 0$$

then a complete solution is

$$y = c_1 y_1 + c_2 y_2$$

where

$$y_2 = y_1 \int y_1^{-2} e^{-\int \frac{a_1(x)}{a_2(x)} dx} dx$$

and c_1, c_2 are constants.

Example

> Find a general solution of $x^2 y'' - 3xy' + 3y = 0$, where $y = x$ is one of its fundamental solution.

> Find a general solution of $4x^2 y'' - 8xy' + 9y = 0$, where $y = x^{3/2}$ is one of its fundamental solution.

> Find a general solution of $y'' - 4xy' + 2(2x^2 - 1)$

> Find a general solution of $(2x+1)y'' - (4x+4)y' + 4y = 0$, where $y = e^{2x}$ is one of its fundamental solution.

Homogeneous Equations with Constants Coefficients

Second Order Equations

We consider the second order ODE

$$ay'' + by' + cy = 0$$

where a, b, c are constants and $a \neq 0$

Let $y = e^{rx}$, then $y' = re^{rx}$ and $y'' = r^2 e^{rx}$. Then $e^{rx}(ar^2 + br + c) = 0$. Note that $e^{rx} \neq 0$ for all $x \in \mathbb{R}$. So

$$ar^2 + br + c = 0.$$

This equation is called the **characteristic equation**.

We then have three cases

Case 1. $b^2 - 4ac > 0$.

Hence $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ are solutions. Note that $W(y1, y2 : x) \neq 0$ for all $x \in \mathbb{R}$. That is y_1 and y_2 are linearly independent, so a general solution is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

where c_1 and c_2 are constants.

Case 2. $b^2 - 4ac = 0$.

$y_1 = e^{rx}$ is a solution when $r = -\frac{b}{2a}$. We will find another solution y_2 that is linearly independent with y_1 by

$$y_2 = y_1 \int \frac{1}{y_1^2} e^{-\int \frac{b}{a} dx} dx = x e^{rx}$$

So a general solution is

$$y = c_1 e^{rx} + c_2 x e^{rx} = e^{rx}(c_1 + c_2 x)$$

where c_1 and c_2 are constants.

Case 3. $b^2 - 4ac < 0$

The equation has two complex conjugate roots, say $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$. So

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} = e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x})$$

is a general solution of the equation. Next, we will simply using Euler's identity. We have

$$y = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x$$

Let $y_1 = e^{\alpha x} \cos \beta x$ and $y_2 = e^{\alpha x} \sin \beta x$.

Note that $W(y_1, y_2 : x) \neq 0$.

So a general solution of ...(fill)

Examples

> Find a general solution of $y'' - 2y' = 0$.

From the problem, a characteristic equation is $r^2 - 2r = 0$. Thus a general solution is $y = c_1 + c_2 e^{2x}$.

> Find a general solution of $y'' - 3y' - y = 0$.

> Find a general solution of $y'' + 5y = 0$.

> Find the particular solution of $4y'' + 16y' + 17y = 0, y(0) = 1, y(\pi) = 0$