Week 3

October 31, 2018

Reduction Formula

Find a reduction formula for the following integrals:

> $\int (lnx)^n \ dx$

Let $u = (lnx)^n$ and $dv = dx$

Thus $du = \frac{(lnx)^{n-1}}{x}$ and $v = x$

Thus $\int (lnx)^n dx = (lnx)^n x - \int (lnx)^{n-1} dx$

> $\int \sin^n x \ dx$

Let $u = \sin^{n-1} x$ and $dv = \sin x dx$. Reduction formula reduces the power by 2.

> $\int x^n e^{ax} \ dx$

Let $u = x^n$ and $dv = e^{ax} dx$

> $\int \cos^n xdx$

> $\int e^{ax} \sin^n bxdx$ (Difficult)

Trigonometric Integrals (Continued)

> Find $\int cot^2 x csc^2 x dx$

Method 1 Substitute $u = cotx$

Method 2 Substitute $u = \frac{\pi}{2} - x$

Method 3 Let $u = \cot x$ and $du = -\csc^2 xdx$

Strategies for evaluating $\int \sin mx \cos nx dx$, $\int \sin mx \sin nx dx$, or $\int \cos mx \cos nx dx$

Use the following identities

- $2 \sin A \cos B = \sin(A - B) + sin(A + B)$
- $2 \cos A \sin B = \cos(A - B) + \cos(A + B)$

- $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$

Examples

> Evaluate $\int \sin 4x \cos 5x dx$

The integral becomes $\frac{1}{2} \int \sin(-x) + \sin(9x) dx$

> Evaluate $\int \sin x \cos 2x \sin 3x dx$

The integral becomes $\frac{1}{2} \int (\sin(-x) + \sin(3x)) \sin(3x) dx$

Use the half angle formula $sin^2 x = \frac{1-\cos(2x)}{2}$ with the latter term.

> Evaluate $\int e^x \cos(3e^x) \cos(4e^x) dx$

Substitute $u = e^x$

Trigonometric Substitution

> Find $\int \sqrt{x^2 + 16}\, dx$

> Find $\int \sqrt{r^2 - x^2}\, dx$

There are three kinds of trigonometric substitution. We can use them with

$$\sqrt{a^2 - x^2}, \sqrt{a^2 + x^2}, \sqrt{x^2 - a^2}$$

where $a > 0$.

Expression $\sqrt{a^2 - x^2}$

Let $x = a \sin \theta$, where $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. $dx = a \cos \theta d\theta$.

$\sqrt{a^2 - x^2} = a|\cos \theta| = a \cos \theta$

Expression $\sqrt{x^2 + a^2}$

Let $x = atan\theta$, where $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

$\sqrt{x^2 + a^2} = a|\tan \theta| = a \tan \theta$

Expression $\sqrt{x^2 - a^2}$

Let $x = a \sec \theta$, where $\theta \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$

$\sqrt{x^2-a^2} = \ldots$

Examples

> Find $\int \frac{\sqrt{100-x^2}}{x^2} dx$

Let $x = 10\sin\theta$

$\int \cot^2\theta d\theta = \int (\csc^2\theta - 1) d\theta$

> Find $\int e^x \sqrt{25e^{2x} - 16} dx$

> Find the area enclosed by the ellipse
>
> $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

> Find $\int x^2 \sqrt{x^2+16} dx$

> Find $\int \frac{dx}{x^5\sqrt{9x^2-1}}$

Integration of Rational Functions by Partial Fractions

> Find $\int \frac{dx}{(x^2+1)(x-1)^2}$

How to find

$$\int \frac{P(x)}{Q(x)} dx$$

, where $P(x)$ and $Q(x)$ are polynomials?

Definition. The **partial fraction decomposition** or **partial fraction expression** of rational function is the operation that consists in expressing the fraction as the sum of a polynomial and one or several fractions with a simple denominator.

Definition. The rational function

$$\frac{P(x)}{Q(x)}$$

is called **proper** if $\deg P < \deg Q$. A rational function which is not proper is **improper**.

Theorem: Division Algorithm of Polynomial There exists a unique $S(x)$ and $R(x)$ such that

$$P(x) = Q(x)S(x) + R(x)$$

where $R \equiv 0$ or $\deg R < degP$

So, if $\frac{P(x)}{Q(x)}$ is improper, there exists $S(x)$ and $R(x)$ such that

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where $\deg R < \deg Q$

Partial Fraction for Proper Rational Function

1. $Q(x)$ is a product of distinct linear factors. That is

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\dots(a_kx + b_k)$$

   where no factors are repeated. There exists constants $A_1, A_2, \dots, A_k$ such that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \dots + \frac{A_k}{a_kx + b_k}$$

2. $Q(x)$ is a product of distinct linear factors, some of which are repeated.

   If $(a_1x + b_1)^r$ occurs in the factorization of $Q(x)$, then instead of the single term $\frac{A_1}{a_1x+b_1}$, we would use

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \dots + \frac{A_r}{(a_1x + b_1)^r}$$

3. $Q(x)$ contains an irreducible quadratic factor, none of which are repeated.

   The expression will have a term of the form

$$\frac{Ax + B}{ax^2 + bx + c}$$

4. $Q(x)$ contains a repeated an irreducible quadratic factor.

   Use

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \dots + \frac{A_r + B_r}{(ax^2 + bx + c)^r}$$

Recall

1. $\int \frac{dx}{1+x^2} = \arctan x + C$
2. $\int \frac{dx}{x} = \ln x + C$

We have two ways to find the coefficients of partial fractions.

Plug in the x-values

Rewrite the equation and plug in a suitable x-value.

For example,

$$3x = A(x + 2)(x - 1) + B(x + 2)(x + 3) + C(x - 1)(x + 3)$$

. To find A, substitute $x = 3$.

Compare the Coefficients

Expand.

Examples

> Find $\int \frac{dx}{x^2-\pi^2}$

The expression becomes $\frac{1}{2\pi(x-\pi)} - \frac{1}{2\pi(x+\pi)}$. Substitute and solve.

> Find $\int \frac{3x^2-8x+13}{x^3+x^2-5x+3}dx$