Week 10

December 19, 2018

Chapter 3: Laplace Transformation

Laplace Transform

Definition. Let $f : [0, \inf) \to \mathbb{R}$. We define the function $F : S \to R$ by

$$F(s) = \int_0^\infty e^{-st} f(t)dt, s \in S$$

such that F converges for all $s \in S$, when S is a subset of $\mathbb{R}$. We say that $F(s)$ is the **Laplace Transform** of $f(t)$. We denote it by $\mathcal{L}\{f(t)\}(s)$ or $\mathcal{L}\{f(t)\}$.

Examples

> Find $\mathcal{L}\{1\}$.

$\frac{1}{s}$. (We can use this directly without having to prove.)

> Find $\mathcal{L}\{e^{at}\}$.

$\frac{1}{s-a}$. (Substitute $u = -(s-a)t$.)

> Find $\mathcal{L}\{\cos bt\}$

$\frac{s}{s^2+b^2}$

> Find $\mathcal{L}\{\sin bt\}$

$\frac{b}{s^2+b^2}$

> Find $\mathcal{L}\{t^n\}$, where n is a positive integer.

$\frac{n!}{s^{n+1}}$

If n is not a positive integer, the answer is $\frac{\Gamma(r+1)}{s^{r+1}}$, where $\Gamma(x)$ is the Gamma function, defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$$

> Find $\mathcal{L}\{f(t)\}$, where $f(t) = \begin{cases} 1 & \text{if } 0 < t < 4 \\ t & \text{otherwise} \end{cases}$

$\frac{1}{s} + \frac{(3s+1)e^{-4s}}{s^2}$

Existence of Laplace Transform and Inverse Laplace Transform

Definition. Let $f : [a, b] \to \mathbb{R}$ be a function. We say that f is **piecewise continuous** on $[a, b]$ if there exists $t_1, \ldots, t_i \in (a, b)$, where $t_1 < \ldots < t_i$ that satisfy the following conditions

1. f is continuous on (t_{k-1}, t_k) for $k = 1, \ldots, i + 1$. (We let $t_0 = a$ and $t_{i+1} = b$).
2. For each subinterval $[t_{k-1}, t_k]$, f is discontinuous at the endpoint of subinterval t_{k-1} and t_k. Moreover the limits $\lim_{t \to t_{k-1}^+} f(t)$ and $\lim_{t \to t_k^-} f(t)$ exist.

Definition. We say that $f(t)$ is of **exponential order** if there exist $\alpha, t_0 \in \mathbb{R}$ and $M > 0$ such that

$$|f(t)| \leq Me^{\alpha t},$$

for $t \geq t_0$, denoted by $f(t) = O(e^{\alpha t})$

Theorem. (Existence of Laplace Transform). Let $f(t)$ be a function such that

$$\int_0^{t_0} e^{-st} f(t) dt$$

exists for all $t_0 > 0$ and there exists $\alpha > 0$ such that $f(t) = O(e^{\alpha t})$. Then $\mathcal{L}\{f(t)\}$ exists for $s > \alpha$.

Definition. (Function of Class A)

1. f is a continuous function on $[0, T]$ for all $T > 0$
2. $f(t) = O(e^{\alpha t})$ for some $\alpha \in \mathbb{R}$

Theorem. Let f be a function of class A, then $F(s) = \mathcal{L}\{f(t)\}$ exists and $\lim_{s \to \infty} F(s) = 0$

Examples

> Let $f(t) = t^{-1/2}$. Show that $\mathcal{L}\{f(t)\}$ exists and find its Laplace transform.

$\frac{\sqrt{\pi}}{2s^{1/2}}$

Question. Is it true that $f_1(t) = f_2(t)$ if $\mathcal{L}\{f_1(t)\} = \mathcal{L}\{f_2(t)\}$

Theorem. Let $f, g : [0, T] \to \mathbb{R}$ be a continuous function for every $T > 0$ and there exists $\alpha \in \mathbb{R}$ such that $\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\}, s > \alpha$, then $f(t) = g(t)$ for every $t \in [0, \infty)$ such that f and g are continuous at t. Moreover, if f and g are continuous on $[0, \infty)$, then $f = g$.

Theorem. (Inverse Laplace Transform) Let $F : (s_0, \infty) \to \mathbb{R}$ for some $s_0 \in \mathbb{R}$ and $lim_{s \to \infty} F(s) = 0$, then there exists a unique continuous function f on $[0, \infty)$ and f is of exponential order such that $\mathcal{L}\{f(t)\} = F(s)$. We say that f is the **inverse Laplace Transform** of $F(s)$, denoted by $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

Theorem. (Linearity Property). Let $f(t)$ and $g(t)$ be functions such that $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{g(t)\}$ exist for $s > \alpha$ and $s > \beta$ respectively, and a and b are constants, then

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\},$$

where $s > \max\{\alpha, \beta\}$.

Theorem. (First Shifting Theorem) Let $f(t)$ be a function that the Laplace transform $F(s)$ exists for $s > \alpha$ and a is a constant, then

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a).$$

We use the symbol for this operation by $\mathcal{L}\{f(t)\}_{s\to s-a}$.

Examples

> Find $\mathcal{L}\{te^{2t}\}$.

= $\mathcal{L}\{t\}_{s\to s-2}$

> Find $\mathcal{L}\{e^{\pi t}\cos t\}$

Theorem. (Multiplication by t^n Property)

Let $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s),$$

where $n \in \mathbb{N}$.

Examples

> Find $\mathcal{L}\{t\sin bt\}$

$\dfrac{2bs}{(s^2+b^2)^2}$

> Find $\mathcal{L}\{t^4 e^{2t}\}$

$\dfrac{4!}{(s-2)^5}$

Theorem. $\mathcal{L}\{P(t)f(t)\} = P(-D)F(s)$, where $D = \dfrac{d}{ds}$ and $P(t) = a_n t^n + \ldots + a_1 t + a_0$.

Examples

> Find $\mathcal{L}\{(t^2+2t+1)\cos t\}$

$= ((-D)^2 + 2(-D) + I)(\mathcal{L}\{\cos t\})$

> Find $\mathcal{L}^{-1}\{(4+5t-t^2)\cos t\}$

$\dfrac{4s}{s^2+1} - \dfrac{5-5s^2}{(s^2+1)^2} - \dfrac{2s^3-6s}{(s^2+1)^3}$

> Find $\mathcal{L}^{-1}\{\ln(1+\dfrac{c^2}{s^2})\}$

$\dfrac{2\cos ct - 1}{t}$

> Find $\mathcal{L}^{-1}\{\dfrac{s}{(s^2+1)^2}\}$

$\dfrac{1}{2}t\sin t$

Theorem. (Laplace Transform of the nth derivative) Let f be n times continuous differentiable function on $[0,\infty)$, then

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots sf^{(n-2)}(0) - f^{(n-1)}(0).$$

Example

> Solve $y'' + 2y' + y = \sin x$