Week 5

November 14, 2018

Exact Equation

Definition Let f be a function of $x_1, x_2, \ldots, x_n$. The **total differential** of f is defined by

$$df(x_1, \ldots, x_n) = \frac{\partial f}{\partial x_1} dx_1 + \ldots + \frac{\partial f}{\partial x_1} dx_1$$

where $\dfrac{\partial f}{\partial x}$ is the partial differentiation with respect to x.

Example $f(x, y) = x \sin y$. $\dfrac{\partial f}{\partial x} = \sin y$ and $\dfrac{\partial f}{\partial y} = x \cos y$

Definition Let R be a rectangle region. An equation

$$M(x, y)dx + N(x, y)dy = 0$$

is **exact** if and only if there exits a function $f(x, y)$ such that $M(x, y) = \dfrac{\partial f}{\partial x}$ and $N(x, y) = \dfrac{\partial f}{\partial y}$ for all (x, y) in the region R.

Theorem Let R be rectangle region. If $\dfrac{\partial M}{\partial y}$ and $\dfrac{\partial N}{\partial x}$ are continuous on R, then

$$M(x, y)dx + N(x, y)dy = 0$$

is exact if and only if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ on R.

Strategy to solve Exact equations

By definition, there exists f such that

$$\frac{\partial f}{\partial x} = M(x, y), \frac{\partial f}{\partial y} = N(x, y)$$

Since $\dfrac{\partial f}{\partial x} = M(x, y)$, we integrate with respect to x on both sides. Thus

$$f(x, y) = \int M(x, y)dx + g(y)$$

for some function g of y.

Next, we differentiate with respect to y on both sides, so

$$N(x, y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \int M(x, y)dx + g'(y)$$

Rearranging gives

$$g(y)=\int\left[N(x,y)-\frac{\partial}{\partial y}\int M(x,y)dx\right]dy$$

Thus

$$f(x,y)=\int M(x,y)dx+\int\left[N(x,y)-\frac{\partial}{\partial y}\int M(x,y)dx\right]dy$$

By the definition of an exact equation, the total differential of f is 0. Thus $f=C$ for some constant C. Thus the solution to this DE is

$$\int M(x,y)dx+\int\left[N(x,y)-\frac{\partial}{\partial y}\int M(x,y)dx\right]dy=C$$

Example

> Find a general solution of $(y^3-2x)dx+(3xy^2-1)dy=0$

Note that $M(x,y)=y^3-2x$ and $N(x,y)=3xy^2-1$. Since $\dfrac{\partial M}{\partial y}=3y^2=\dfrac{\partial N}{\partial x}$, this DE is exact. Thus there exists a function f such that $\dfrac{\partial f}{\partial x}=M$ and $\dfrac{\partial f}{\partial y}=N$. Thus

$$f(x,y)=\int M(x,y)dx+g(y)=\int(y^3-2x)dx+g(y)=xy^3-x^2+g(y)$$

for some function g of y. We differentiate with respect to y and obtain

$$3xy^2-1=\frac{\partial f}{\partial y}=3xy^2+g'(y)$$

Thus $g'(y)=-1$, so $g(y)=-y+C$. Thus $f(x,y)=xy^3-x^2-y+C$.

A general solution is $xy^3-x^2-y=C$. $\square$

An alternative solution is

$$\begin{gathered}y^3dx-2xdx+3xy^2dy-dy=0\\ y^3dx+xd(y^3)-d(x^2)-dy=0\\ d(xy^3)-d(x^2)-dy=0\\ d(xy^3-x^2-y)=0\end{gathered}$$

So, a general solution of this DE is

$$xy^3-x^2-y=C$$

> Find a general solution of $(2x+\cosh xy)dx+\left(\dfrac{xy\cosh xy-\sinh xy}{y^2}\right)dy=0$
>
> where $\cosh x=\dfrac{e^x+e^{-x}}{2}$ and $\sinh x=\dfrac{e^x-e^{-x}}{2}$

It is easy to show that $\dfrac{d\cosh x}{dx}=\sinh x$, $\dfrac{d\sinh x}{dx}=\cosh x$

We first show that this equation is exact. Thus there exists an f.

$$f(x,y)=\int M(x,y)dx+g(y)=x^2+\frac{\sinh xy}{y}+g(y)$$

Hence,

$$\frac{xy\cosh xy-\sinh xy}{y^2}=\frac{xy\cosh xy-\sinh xy}{y^2}+g'(y)$$

This implies $g'(y)=0$ and $g(y)=C$

Thus a general solution of this DE is

$$x^2+\frac{\sinh xy}{y}=C.\ \square$$

> Find the particular solution of $(ye^{xy}\cos 2x-2e^{xy}\sin 2x+2x)dx+(xe^{xy}\cos 2x-3)dy=0$, $y(3)=7$.

$$e^{xy}\cos 2x-3y+x^2=C$$

(Substitute to find C using the condition.)