Report on Curvature estimation and Visualization

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Abstract

This document summarized some concepts about the discrete differential geometry, and visualized the curvature of the given mesh.

1 Local Averaging Region

Barycentric Cell
using the triangle barycenters edge midpoints to find the cell
Voronoi Cell
using the triangle circumcenter to find the cell
Mixed Voronoi Cell
using edge midpoints to replace circumcenter for obtuse triangles to find the cell

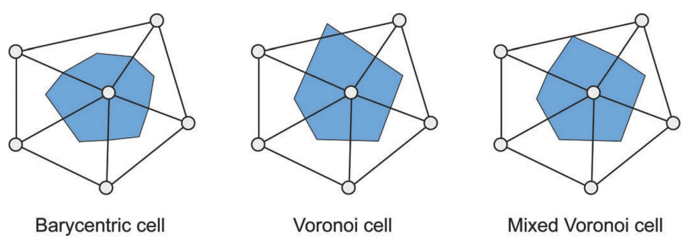

Figure 1: Local Averaging Region

2 Barycentric Coordinate

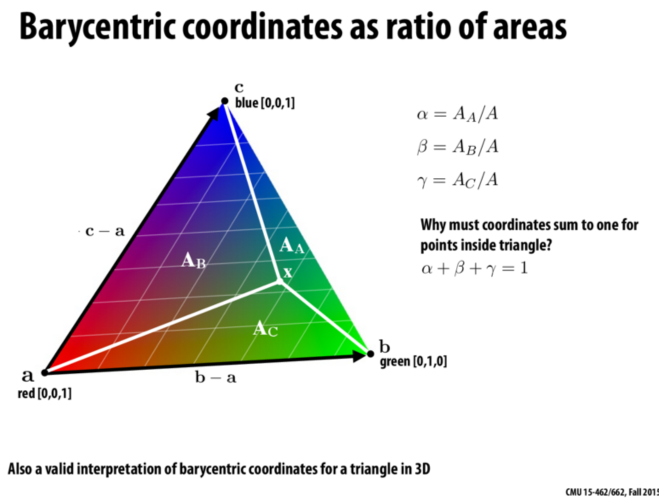

Figure 2: Diagram of Barycentric Coordinates from CMU

3 Gradients

Given a piecewise liner function

$$f(x) = \alpha f_i + \beta f_j + \gamma f_k$$

Gradient:

$$\nabla_x f(x) = f_i \nabla_x \alpha + f_j \nabla_x \beta + f_k \nabla_x \gamma$$

By Caculating the Barycentric Coordinate, we have

$$\nabla_x f(x) = f_i \frac{(x_k - x_j)^\perp}{2A_T} + f_j \frac{(x_i - x_k)^\perp}{2A_T} + f_k \frac{(x_j - x_i)^\perp}{2A_T}$$

Because

$$(x_k - x_j)^\perp + (x_i - x_k)^\perp + (x_j - x_i)^\perp = 0$$

Then, we have

$$\nabla_x f(x) = (f_j - f_i) \frac{(x_i - x_k)^\perp}{2A_T} + (f_k - f_i) \frac{(x_j - x_i)^\perp}{2A_T}$$

4 Laplace Beltrami Operator

Uniform Laplacian assume $\Omega(i)$ represent the neighbour of i

$$\Delta f_i = \sum_{j \in \Omega(i)} (f_j - f_i)$$

or,

$$\Delta f_i = \frac{1}{N_i} \sum_{j \in \Omega(i)} (f_j - f_i)$$

Cotangent Formula let A_i be the area of the local averaging region of i, using the mixed voronoi cell, we have

$$\int_{A_i} \Delta f dA = \frac{1}{2} \sum_{j \in \Omega(i)} (cot\alpha_{ij} + cot\beta_{ij})(f_j - f_i)$$

discrete average of the Laplace Beltrami operator of a function f at vertex v_i is given as:

$$\Delta f(v_i) = \frac{1}{2A_i} \sum_{j \in \Omega(i)} (cot\alpha_{ij} + cot\beta_{ij})(f_j - f_i)$$

5 Discrete Curvature

When applied to the coordinate function x, the Laplace Beltrami operator provides a discrete approximation of the mean curvature normal:

$$\Delta \boldsymbol{x} = -2H\boldsymbol{n}$$

So absolute discrete mean curvature at vertex i:

$$H_i = \frac{1}{2} \|\Delta \boldsymbol{x}\|$$

A discrete operator for Gaussian curvature:

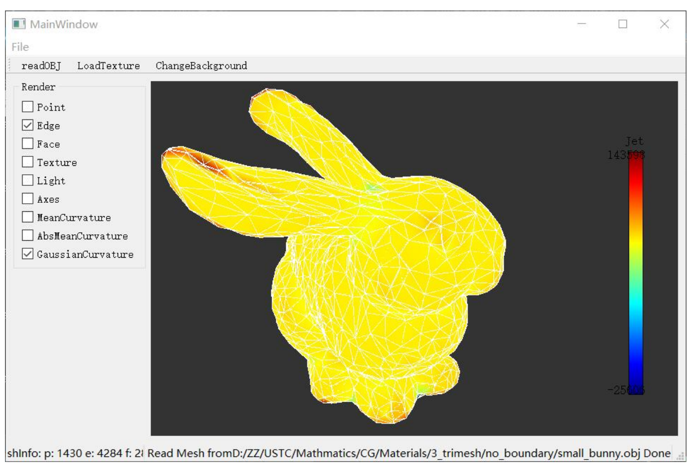

Figure 3: discrete gaussian curvature of bunny

$$K_i = \frac{1}{A_i}(2\pi - \sum_{j \in \Omega(i)} \theta_j)$$

6 Implement

Next, I will show some visualization of the discrete curvature:

No boundary & One boundary:

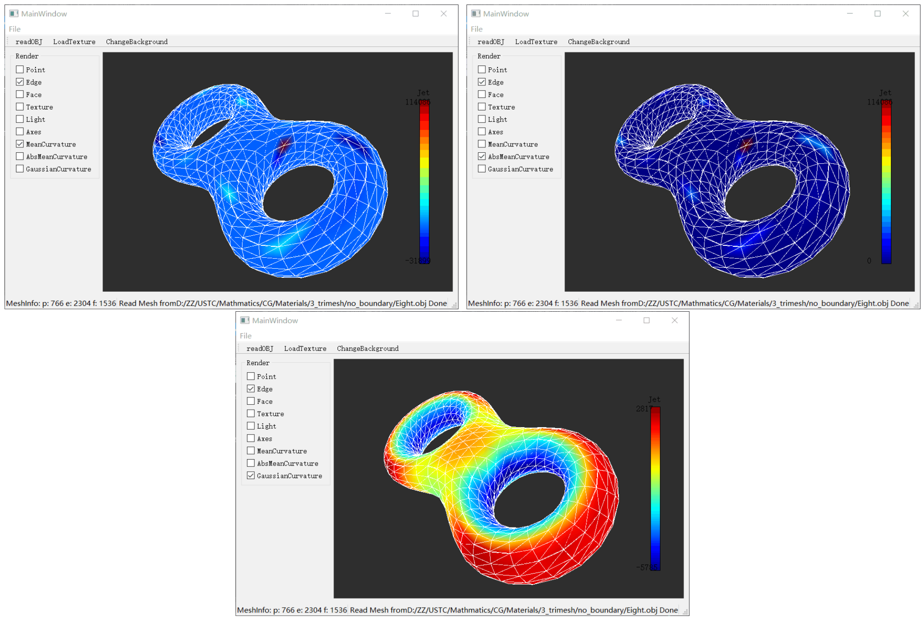

Figure 4: eight

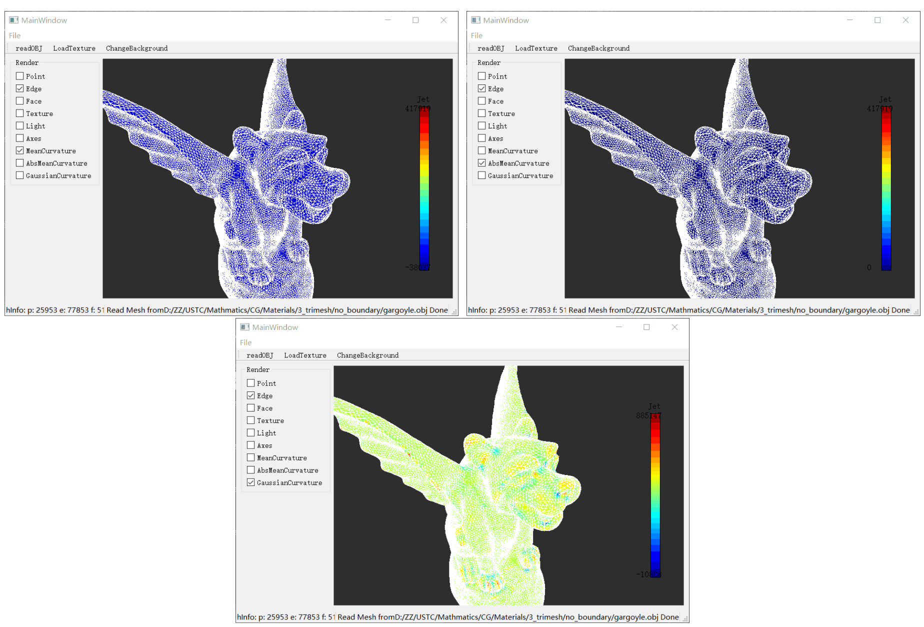

Figure 5: gargoyle

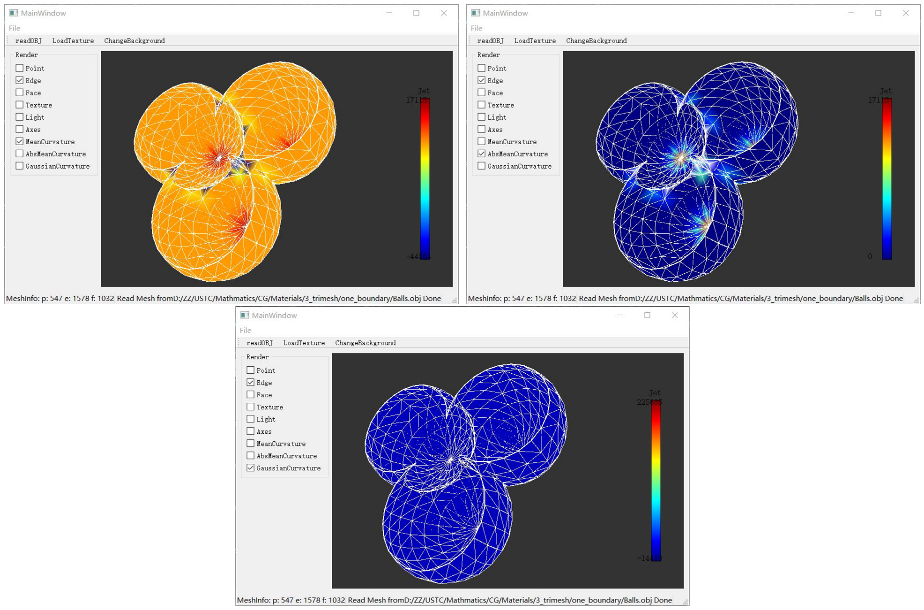

Figure 6: balls

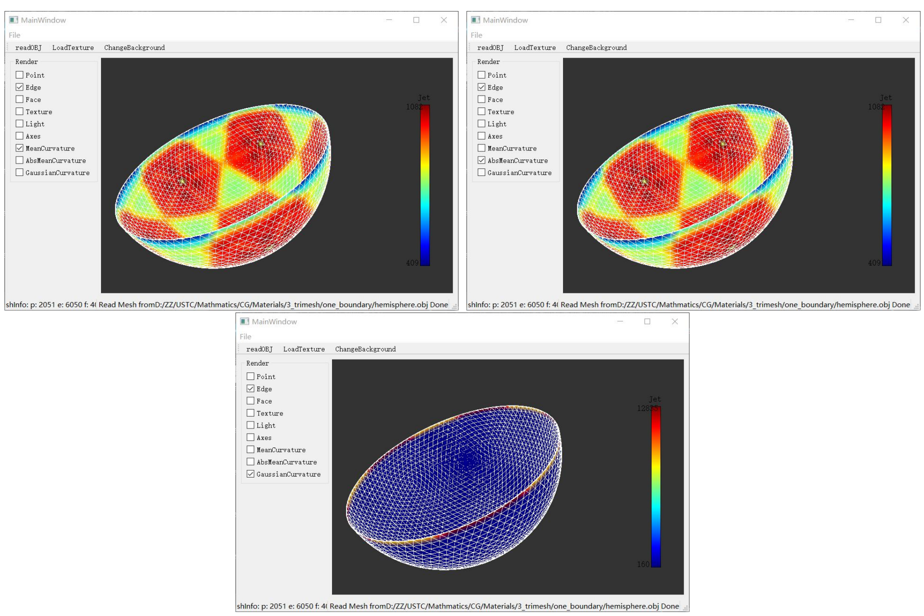

Figure 7: hemisphere