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EXERCISE 12.6 MODELLING WITH FIRST-ORDER DIFFERENTIAL EQUATIONS

1 Market research in a large city indicates that the maximum sales of a soon-to-be-released mobile device, is 10 truckloads per month (1 truckload = 10 000 devices).

Past experience with models iThingie1 through to iThingie6 indicates that the rate of growth in the truckloads of sales $\frac{ds}{dt}$, t months after the release of an iThingie, is directly proportional to the difference between the current sales and the maximum monthly sales.

(a) Find an equation for the rate of growth $\frac{ds}{dt}$ in the sales s as a function of the time t in months after the new product is first released onto the market. Express your answer in terms of the constant of proportionality r.
(b) Find the solution curve of your model. Express your answer in terms of the constant of proportionality r.
(c) If two truckloads are sold after one month, find the predicted number of truckloads per month after three months. (Express your answer correct to the nearest truckload.)

2 A simple model for the spread of a contagious illness assumes that the rate at which the illness spreads $\frac{dI}{dt}$ varies jointly with the product of the number of ill people I and the number of people still susceptible to the illness S. This means that $\frac{dI}{dt} = rIS$, $r > 0$.

Assume that one infected person is introduced into a fixed population of size P.
Then $P + 1 = I + S \therefore S = P + 1 - I$. Therefore, $\frac{dI}{dt} = rI(P+1-I)$, $I(0) = 1$ and $r > 0$.

(a) Show that $\frac{1}{I(P+1-I)} = \frac{1}{(1+P)}\left[\frac{1}{(1+P-I)} + \frac{1}{I}\right]$. **(b)** Find I as a function of time.

3 A pond initially contains 200 000 litres of unpolluted water. A stream begins to flow into the pond at rate of 10 000 litres per day. The stream is polluted with a concentration of 2 grams of pollutant per litre. The pond also has an outlet that spills 10 000 litres of well-mixed water per day.

(a) State the initial value problem that models the mass of pollutant $m(t)$ grams in the pond, t days after the polluted stream first begins to flow into the pond.
(b) Hence find a differential equation that models the concentration of pollutant $c(t) = \frac{m(t)}{200\,000}$ grams per litre in the pond, t days after the polluted stream first begins to flow into the pond.
(c) Solve the model from part **(a)**.
(d) What is the concentration of pollutant in the pond after 10 days?

4 A tank initially contains 1000 litres of salt solution of concentration 0.01 kg/L. A solution of the same salt, but concentration 0.04 kg/L, flows into the tank at a rate of 10 litres per minute. The mixture in the tank is kept uniform by stirring and the mixture flows out at a rate of 5 litres per minute.

Let Q kg be the quantity of salt in the tank after t minutes. Set up (but do not solve) the differential equation for Q in terms of t, and specify the initial conditions.

5 Carbon monoxide (chemical symbol CO) is toxic to humans. Two hours of exposure to air with a volume concentration of CO at 0.02% will cause headaches and confusion.

During World War I, some generals commanded their soldiers from inside a bombproof bunker with an internal volume of 80 m^3. Troops resting near the air intake to the bunker would often smoke cigarettes. Unfortunately, the air intake to the bunker sucked the carbon-monoxide-filled smoke from the cigarettes back into the bunker.

Assume that smoky air is sucked into the bunker at a rate of 2 m^3/min, and that 0.03% of this air (by volume) is carbon monoxide. Ventilation fans keep the air well mixed inside the bunker, and the well-mixed air is extracted from the bunker at the same rate of 2 m^3/min. It can be shown that $\frac{dv}{dt} = 0.025(0.0003 - v)$, $v(0) = 0$.

(a) Solve this differential equation to find $v(t)$.
(b) Hence find the time for the volume fraction of carbon monoxide to reach 0.02% by volume inside the bunker. Express your answer in minutes, correct to the nearest minute.

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6 A lottery winner puts \$5 000 000 in winnings into a fund that has a 5% annual rate of return, paid continuously throughout the year. Each year the winner spends \$300 000, withdrawn from the account at a continuous rate over the course of the year.

(a) Show that the differential equation to model the fund balance $x(t)$ after t years is $\frac{dx}{dt} = 0.05(x - 600000)$ and state the value of $x(0)$.

(b) Solve the differential equation in part (a).

(c) Hence determine the balance after 20 years. Express your answer correct to the nearest 5 cents.

SOLUTIONS

EXERCISE 12.6

1 (a) Maximum monthly sales = 10, Difference of current sales = $10 - s$.

$\frac{ds}{dt} = r(10 - s)$, $s(0) = 0$

(b) $s = 10 - 10e^{-rt}$

(c) $s(3) = 10 - 10 \times \left(\frac{4}{5}\right)^3 = 4.88 \approx 5$ truck loads

2 (a) $$\text{RHS} = \frac{1}{(1+P)}\left[\frac{1}{(1+P-I)} + \frac{1}{I}\right]$$

$$= \frac{1}{(1+P)}\left(\frac{I+1+P-I}{I(1+P-I)}\right)$$

$$= \frac{1}{(1+P)}\left(\frac{1+P}{I(1+P-I)}\right)$$

$$= \frac{1}{I(P+1-I)} = \text{LHS}$$

(b) $$I = \frac{1+P}{1+Pe^{-(1+P)rt}}$$

3 (a) Net rate of change of pollutant (g/day) = rate of inflow (g/day) − rate of outflow (g/day)

$$\frac{dm}{dt} = 10000 \times 2 - 10000 \times \frac{m}{200000}$$

$$= 20000 - \frac{m}{20}$$

Initially pond is unpolluted, so $m(0) = 0$

(b) $c(t) \equiv \frac{m(t)}{200000}$

$$\frac{dc(t)}{dt} = \frac{1}{200000} \times \frac{dm(t)}{dt}$$

$$\frac{dc}{dt} = \frac{1}{200000} \times \left(20000 - \frac{m}{20}\right)$$

$$= \frac{1}{10} - \frac{m}{20 \times 200000}$$

$$= \frac{1}{10} - \frac{c}{20} \text{ where } c(0) = 0$$

(c) $c = 2\left(1 - e^{-\frac{t}{20}}\right)$

(d) $c(10) = 2\left(1 - e^{-\frac{1}{2}}\right) = 0.79$ g/L

4 Net rate of change of salt (kg/min) = rate of inflow (kg/min) − rate of outflow (kg/min)

Net rate of change of volume (L/min) = rate of inflow (L/min) − rate of outflow (L/min)

$Q(0) = 1000 \times 0.01 = 10$ kg

$$\frac{dV}{dt} = \text{rate of inflow} - \text{rate of outflow}$$

$$= 10 - 5 = 5 \text{ L/min}$$

$V(0) = 1000$ L

$V(t) = 5t + 1000$

$$\frac{dQ}{dt} = \text{rate of inflow} - \text{rate of outflow}$$

$$= 0.04 \times 10 - \frac{Q}{V(t)} = 5$$

$$= 0.4 - \frac{5Q}{1000+5t}$$

$$= 0.4 - \frac{Q}{200+t}$$

$Q(0) = 10$

5 (a) $v(t) = 0.0003\left(1 - e^{-0.025t}\right)$

(b) $v(t) = 0.0002$: $t = \frac{\log_e 3}{0.025} \approx 44$ minutes

6 (a) Net rate of change of balance (\$/year) = rate of inflow (\$/year) − rate of outflow (\$/year)

$$\frac{dx}{dt} = 0.05x - 300000$$

$$= 0.05(x - 600000) \text{ where } x(0) = 5000000$$

(b) $x(t) = 1000000\left(6 - e^{0.05t}\right)$

(c) $t = 20$: $x(20) = 1000000\left(6 - e^1\right) = 3281718.17$

Balance is \$3 281 718.15