Year 12 Extension 1 Maths

Worksheet. The Logistic Differential Equation

(1) In the Corona Virus pandemic of 2020, Brazil's cases followed a path which could be modelled using the logistic differential equation, that is, if N is the number of cases, and P is the total population,

$$\frac{dN}{dt} = kN(P-N)$$

(a) Show that $\dfrac{P}{N(P-N)} = \dfrac{1}{N} + \dfrac{1}{P-N}$

(b) By solving $\dfrac{dN}{dt} = kN(P-N)$, show that

$$N = \frac{P}{1+Be^{-Pkt}}$$

(c) Taking $t=0$ as June 26 when Brazil [with a population of 210 000 000] had 1 200 000 cases, find the value of B.

(d) By June 30th, Brazil had 1 390 000 cases. Find the value of k.

(e) How many cases was Brazil expected to have by July 10? [to the nearest 10 000].

(f) When was Brazil expected to reach 5 000 000 cases?

Note: (1) Corona Virus figures [current 30/6/2020] are from Worldmeter – Corona Virus. [Q.1 & 2]
(2) Population statistics [current 2020] are from Worldmeter – Population]

Y12 The Logistic Differential Equation p. 2

(2) In the same pandemic, India [population 1 400 000 000] had 470 000 cases on June 26 and 570 000 cases on June 30.

(a) Show BY SUBSTITUTION that $N = \dfrac{P}{1+Be^{-Pkt}}$ is a solution to $\dfrac{dN}{dt} = kN(P-N)$

(b) By using $t=0$ as June 26 [& $t=4$ as June 30] find the values of B and k.

(c) How many cases was India expected to have by July 10 [to the nearest 10 000].

(d) When was India expected to hit 1000 000 cases?

(3) By some estimates, Australia's maximum sustainable population [or CARRYING CAPACITY] is estimated to be 50 000 000 people.
At the start of 2000, Australia had a population of 20 million. At the start of 2020, Australia's population was 25 million.

(a) Show that $\dfrac{50\,000\,000}{N(50\,000\,000-N)} = \dfrac{1}{N} + \dfrac{1}{50\,000\,000 - N}$.

(b) By solving $\dfrac{dN}{dt} = kN(50\,000\,000 - N)$ show that

$$N = \frac{50\,000\,000}{1+Be^{-50\,000\,000kt}}$$ where B is some constant.

(c) By using $t=0$ is the year 2000, find the value of B.
i) In 2020, $t=$? Find the value of k.
What is Australia's population expected to be in 2050 by this model?

Y12 The Logistic DE p.3

Q.(4) = Q.22 Ex.12.6 New Senior Ext. 1.

22 A nature conservation group releases 21 Tasmanian devils onto a remote island off the coast of Tasmania. The group believes the island can support at most 588 Tasmanian devils. The growth rate of the Tasmanian devil population p is $\frac{dp}{dt} = rp\left(1 - \frac{p}{588}\right)$, $p(0) = 21$, with $r > 0$ and t measured in years.

(a) Show that $\frac{1}{p\left(1-\frac{p}{588}\right)} = \frac{1}{p} - \frac{1}{p-588}$.

(b) State the model for the Tasmanian devil population p in terms of r and time t.

Three years after the beginning of the breeding program, the population is 294.

(c) Find r.

(d) Use the model from part **(c)** to estimate the devil population after 6 years.

Answers / Solutions:

Q.(1) I have included SOLUTIONS to (1)(a)&(b) & 2(a) to make it clear what is required.

(1)(a) $\frac{1}{N} + \frac{1}{P-N}$

$= \frac{(P-N)+N}{N(P-N)}$

$= \frac{P}{N(P-N)}$

[Note: Ext 2 students COULD do this by partial fractions].

(b) $\frac{dN}{dt} = kN(P-N)$

$\frac{1}{N(P-N)} \cdot \frac{dN}{dt} = k$

$\frac{P}{N(P-N)} \cdot \frac{dN}{dt} = kP$ [1ST trick!]

$\int \frac{P}{N(P-N)} dN = \int kP.dt$

$\int \frac{1}{N} + \frac{1}{P-N} dN = \int kP.dt$

$\ln\left(\frac{N}{P-N}\right) = kPt + C$

$\frac{N}{P-N} = e^{kPt+C}$ [2ND trick].

$\frac{N}{P-N} = e^{C} \times e^{kPt}$

$\frac{N}{P-N} = Ae^{kPt}$ where $A = e^{C}$ [part of 2ND trick]

$N = (P-N)Ae^{kPt}$

$N = PAe^{kPt} - NAe^{kPt}$

$N(1+Ae^{kPt}) = PAe^{kPt}$

$N = \frac{PAe^{kPt}}{1+Ae^{kPt}}$ $\frac{\div Ae^{kPt}}{\div Ae^{kPt}}$

$N = \frac{P}{\left(\frac{1}{A}\right)e^{-kPt}+1}$

$N = \frac{P}{Be^{-kPt}+1}$, $B = \frac{1}{A}$ [3RD trick]

(c) $B = 174$ (d) $k = \frac{0.147892566-}{840\ 000\ 000}$

$= \frac{0.0369-}{210\ 000\ 000}$

(e) By July 10, Brazil could have 2 005 871 cases!

(f) 39.09 days i.e. by August 4TH for 5 000 000 cases.

Q.(2)(a) This time instead of deriving we demonstrate by SUBSTITUTION:

LHS:

$$N = \frac{P}{1+Be^{-Pkt}}$$

$$\frac{dN}{dt} = \frac{-P}{(1+Be^{-Pkt})^2} \times -BPke^{-Pkt}$$

$$= \frac{BP^2 e^{-Pkt}}{(1+Be^{-Pkt})^2}$$

RHS:

$$kN(P-N)$$

$$= \frac{kP}{1+Be^{-Pkt}}\left(P - \frac{P}{1+Be^{-Pkt}}\right)$$

$$= \frac{kP}{1+Be^{-Pkt}}\left[\frac{P+PBe^{-Pkt}-P}{1+Be^{-Pkt}}\right]$$

$$= \frac{kP}{(1+Be^{-Pkt})} \times \frac{PBe^{-Pkt}}{(1+Be^{-Pkt})}$$

$$= \frac{kP^2Be^{-Pkt}}{(1+Be^{-Pkt})^2}$$

= LHS QED.

(b) $B = 2977.723404$

$$k = \frac{0.04824378031}{1.4\times10^9}$$

(c) 923 177 cases (≈920 000 cases)

(d) 15.65 days so late on July 11th!

(3)(c) $B = 1.5$

(d) $k = \frac{0.02027}{50\,000\,000}$

(e) 32 376 477 or roughly 32 380 000

Note: Australia is ACTUALLY expected to have 38 million people by 2050.

22. (a) $\text{RHS} = \frac{1}{p} - \frac{1}{p-588}$

$$= \frac{p-588-p}{p(p-588)}$$

$$= \frac{-588}{p(p-588)}$$

$$= \frac{1}{p\left(1-\frac{p}{588}\right)} = \text{LHS}$$

(b) $\frac{dp}{dt} = rp\left(1-\frac{p}{588}\right)$

$$\frac{dp}{p\left(1-\frac{p}{588}\right)} = r\,dt$$

$$\int \frac{dp}{p\left(1-\frac{p}{588}\right)} = r\int dt$$

$$\int\left(\frac{1}{p} - \frac{1}{p-588}\right)dp = r\int dt$$

$$\log_e|p| - \log_e|p-588| = rt + c$$

$$\log_e\left|\frac{p}{p-588}\right| = rt + c$$

$$\left|\frac{p}{p-588}\right| = e^{rt+c}$$

$$\frac{p}{p-588} = Ae^{rt} \text{ where } A = \pm e^c$$

$$p(0) = 21: \quad \frac{21}{21-588} = A \therefore A = -\frac{1}{27}$$

$$p = \frac{-p}{27}e^{rt} + \frac{588}{27}e^{rt}$$

$$p\left(27+e^{rt}\right) = 588e^{rt}$$

$$p = \frac{588e^{rt}}{27+e^{rt}} = \frac{588}{1+27e^{-rt}}$$

(c) $p(3) = 294: \quad 294 = \frac{588}{1+27e^{-3r}}$

$$1+27e^{-3r} = 2$$

$$e^{-3r} = \frac{1}{27}$$

$$r = \tfrac{1}{3}\log_e 27 = \log_e 3$$

(d) $p(6) = \frac{588}{1+27e^{-6\log_e 3}} = \frac{588}{1+27\times\frac{1}{3^6}} = \frac{588}{1+\frac{1}{27}} = 567$