MATHEMATICS EXTENSION 2
NSBHS-2019

Inequalities

1 Introduction

Definition 1.1 *Every real number x has one and only one of the following properties:*

1. $x = 0$,
2. $x > 0$,
3. $-x > 0$ *or* $x < 0$

Definition 1.2 *We write $a > b$, which read "a is greater than b", if and only if $a - b$ is a positive number; and $a < b$, read "a is less than b", if and only if $a - b$ is a negative number.*

Corollary 1.1 *Given two numbers a and b, exactly one of the following assertions is satisfied, $a = b$, $a > b$ or $a < b$.*

Corollary 1.2 *$a > b$ if and only if $b < a$.*

Theorem 1.1 *$a > 0$ if and only if a is a positive number, and $a < 0$ if and only if a is a negative number.*

Proof: $a > 0$ if and only if $a - 0$ is a positive number (1.2). But $a - 0 = a$. Therefore $a > 0$ if and only if a is a positive number. The proof of the second part is analogous.

Theorem 1.2 *If $x > 0$ and $y > 0$ then $x + y > 0$.*

Theorem 1.3 *If $x > 0$ and $y > 0$ then $xy > 0$.*

Theorem 1.4 *[Transitive Law of Inequality] If $a > b$ and $b > c$ then $a > c$.*

Proof: Since $a > b$ and $b > c$, $a - b$ and $b - c$ are both positive numbers (1.2). Thus $(a - b) + (b - c) > 0$ which implies that $a - c > 0$. Therefore $a > c$. Hence the result.

Theorem 1.5 *Let a, b and c be any real numbers. Then $a > b$ if and only if $a + c > b + c$.*

Proof: $a > b$ if and only if $a - b > 0$ (by 1.2 and 1.1). But $a - b = (a + c) - (b + c)$. Therefore $a > b$ if and only if $(a + c) - (b + c) > 0$, that is, if and only if $a + c > b + c$.

Theorem 1.6 *Let $c > 0$. Then $a > b$ if and only if $ac > bc$.*

Proof: Left as an exercise.

Theorem 1.7 *Let $c < 0$. Then $a > b$ if and only if $ac < bc$.*

Proof: Left as an exercise.

Theorem 1.8 *If $a > b$ and $c > d$, then $a + c > b + d$.*

Proof: $a - b > 0$ and $c - d > 0$ (1.2). Thus $a - b$ and $c - d$ are positive (1.1). Therefore $(a - b) + (c - d) > 0$. But this may be written $(a + c) - (b + d) > 0$, which implies that $a + c > b + d$.

The proofs of the following four theorems are easy and are left as exercises.

Theorem 1.9 *If $a \neq 0$, then $a^2 > 0$.*

Theorem 1.10 *If $a \neq 0$, $\dfrac{1}{a}$ has the same sign as a.*

Theorem 1.11 *If a and b have the same sign, and if $a < b$, then $\dfrac{1}{a} > \dfrac{1}{b}$.*

Theorem 1.12 *Let $a < b$. Then if a and b are both positive, $a^2 < b^2$; and if a and b are both negative, $a^2 > b^2$.*

1.1 Useful Identities

1. $a^2 - b^2 = (a-b)(a+b)$
2. $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
3. $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
4. $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$
5. $abc = (a+b+c)(ab+bc+ca) - (a+b)(b+c)(a+c)$

1.2 Exercise

Prove the following assertions:

1. $a < 0, \quad b < 0 \implies ab > 0.$
2. $a < 0, \quad b > 0 \implies ab < 0.$
3. $a < b, \quad b < c \implies a < c.$
4. $a < b, \quad c > d \implies a + c < b + d.$
5. $a < b \implies -b < -a.$
6. $a > 0 \implies \dfrac{1}{a} > 0.$
7. $a < 0 \implies \dfrac{1}{a} < 0.$
8. $a > 0, \quad b > 0 \implies \dfrac{a}{b} > 0.$
9. $0 < a < b, \quad 0 < c < d \implies ac < bd.$
10. $a > 1 \implies a^2 > a.$
11. $0 < a < 1 \implies a^2 < a.$

2 AM-GM inequality of two numbers

Consider $a > 0$ and $b > 0$,

$$(a-b)^2 > 0 \quad \text{for} \quad \text{all} \quad a \quad \text{and} \quad b. \tag{1}$$

Expanding (1), we obtain:

$$a^2 - 2ab + b^2 > 0. \tag{2}$$

Hence,

$$a^2 + b^2 > 2ab. \tag{3}$$

Similarly,

$$(\sqrt{a} - \sqrt{b})^2 > 0. \tag{4}$$

leads to

$$a + b > 2\sqrt{ab}. \tag{5}$$

Inequation (5) is called the AM-GM inequality, note that the arithmetic mean of a and b is $\dfrac{a+b}{2}$ and their geometric mean is $\sqrt{ab}$.

2.1 Application of the AM-GM inequality with two numbers

1. Let $a > 0$ and $b > 0$, prove that
$$(a+b)(1+ab) \geq 4ab$$

2. If $x > 0$, $y > 0$, $a > 0$ and $b > 0$, show that
$$(ab+xy)(ax+by) \geq 4abxy$$

3. Let $a > 0$ and $b > 0$, prove that
$$a + b + 1 \geq 2\sqrt{a+b}$$

4. Let $a > 0$ and $b > 0$, prove that
$$a^2 + b^2 + 1 \geq ab + a + b$$

5. Let a, b and c be real numbers. Prove that
$$a^2 + b^2 + c^2 \geq ab + ac + bc$$
Deduce that if $a + b + c = 1$, then $ab + bc + ca \leq \dfrac{1}{3}$.

6. Let $a > 0$ and $b > 0$, prove that
$$\frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$$

7. Show that if a, b, c and d are positive, then
$$a^4 + b^4 + c^4 + d^4 \geq 4abcd$$

8. if $a > 0$, $b > 0$ and $a + b = t$, show that
   (a)
   $$\frac{1}{a} + \frac{1}{b} \geq \frac{4}{t}$$
   (b)
   $$\frac{1}{a^2} + \frac{1}{b^2} \geq \frac{8}{t^2}$$

9. Let $x > 0$, $y > 0$ and $z > 0$, prove that
$$(x+y)(y+z)(z+x) \geq 8xyz$$

10. Let $x > 0$, $y > 0$ and $z > 0$ and given that $x + y + z = 1$, prove that
$$(1-x)(1-y)(1-z) \geq 8xyz$$

11. Three positive numbers a,b,c satisfy the conditions that $a \geq b \geq c$ and that $a + b + c \leq 1$. By considering $(a+b+c)^2$ otherwise, prove that
$$a^2 + 3b^2 + 5c^2 \leq 1$$

12. For $x > 0$, $y > 0$ and $z > 0$, show that
$$x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 6$$

13. For $a > 0$, $b > 0$, $c > 0$ and $d > 0$ and given that $\dfrac{a+b}{2} \geq \sqrt{ab}$, show that
$$\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$$

14. For $a > 0$, $b > 0$, $c > 0$ and $d > 0$ and given that $\dfrac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$, show that
$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geq 4$$

15. If a, b and c are positive, show that
    (a) $a + \dfrac{1}{a} \geq 2$
    (b) $(a+b)(\dfrac{1}{a} + \dfrac{1}{b}) \geq 4$ and $(a+b+c)(\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}) \geq 9$.
    (c) Hence show that
$$\frac{9}{a+b+c} \leq \frac{2}{b+c} + \frac{2}{c+a} + \frac{2}{a+b} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

16. Let a_1, a_2, $\ldots$, a_n be positive real numbers such that $a_1a_2\ldots a_n = 1$. Prove that
$$(1+a_1)(1+a_2)\ldots(1+a_n) \geq 2^n.$$

17. Prove that $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, where a, b and c are real.

18. Prove that $x^2 + x + 1 > 0$ for all real x. Hence, or otherwise prove that:
    (a) $a^2 + ab + b^2 > 0$.
    (b) $a^4 + b^4 \geq a^3b + ab^3$.

19. Given that $x^2 + y^2 + z^2 \geq xy + yz + zx$ for $x > 0$, $y > 0$ and $z > 0$. Prove that
$$(x+y+z)^2 \geq 3(xy+yz+zx).$$

20. Given a, b and c are positive real numbers. Prove that
$$a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a+b+c).$$

21. Let $x_0 > x_1 > x_2 > \cdots > x_n$ be real numbers. Prove that
$$x_0 + \frac{1}{x_0 - x_1} + \frac{1}{x_1 - x_2} + \cdots + \frac{1}{x_{n-1} - x_n} \geq x_n + 2n.$$

22. Prove that if $a + b = 1$, then $a^3 + b^3 \geq \dfrac{1}{4}$.

23. If a and b are real numbers, show that $a^4 + b^4 \geq ab(a^2+b^2)$.

3 AM-GM inequality of three numbers

Consider $a > 0$, $b > 0$ and $c > 0$, expand the following product:

$$(a+b+c)(a^2+b^2+c^2-ab-bc-ac)$$

, we obtain

$$a^3+b^3+c^3-3abc.$$

Hence

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ac)$$

Using (5), we obtain: $a^2+b^2 \geq 2ab$, $a^2+c^2 \geq 2ac$ and $b^2+c^2 \geq 2bc$. Hence

$$a^2+b^2+c^2 \geq ab+bc+ca.$$

Since $a+b+c \geq 0$ and $a^2+b^2+c^2-ab-bc-ca \geq 0$, we obtain that

$$a^3+b^3+c^3-3abc \geq 0.$$

Hence

$$a^3+b^3+c^3 \geq 3abc.$$

By a suitable substitution, we obtain

$$a+b+c \geq 3\sqrt[3]{abc}.$$

3.1 Application of the AM-GM inequality with three numbers

1. Given a, b and c are positive real numbers. Prove that $\frac{1}{3}(a^3-b^3)(a-b) \geq ab(a-b)^2$.

2. Given a, b and c are positive real numbers such that $abc=1$. Prove that $a^2+b^2+c^2 \geq a+b+c$.

3. Given a, b and c are positive real numbers. Prove that

$$\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a} \geq ab+bc+ca.$$

4. Given a, b and c are positive real numbers. Prove that

$$(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2) \geq 9a^2b^2c^2.$$

5. (Nesbitt's Inequality) Given a, b and c are positive real numbers. Prove that

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b} \geq \frac{3}{2}.$$

6. Given a, b and c are positive real numbers such that $abc=1$. Prove that

$$\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+ac}{1+c} \geq 3.$$

7. (Ireland 1998) Given a, b and c are positive real numbers. Prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 2\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right) \geq \frac{9}{a+b+c}.$$

8. Given a, b and c are positive real numbers. Prove that

$$\frac{1}{b(a+b)}+\frac{1}{c(b+c)}+\frac{1}{a(c+a)} \geq \frac{27}{2(a+b+c)^2}.$$

9. Given a, b and c are positive real numbers such that $abc = 1$. Prove that

$$\frac{a}{(a+1)(b+1)} + \frac{b}{(b+1)(c+1)} + \frac{c}{(c+1)(a+1)} \geq \frac{3}{4}.$$

10. Given a, b and c are positive real numbers such that $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 1$. Prove that

$$abc \geq 8.$$

11. ΔABC has side length a, b and c. If $a^2 + b^2 + c^2 = ab + bc + ca$, show that ΔABC is an equilateral triangle.

4 Mixed Problems

1. If m, n, p and q are positive numbers. Show that:
   (a) $m + n \geq 2\sqrt{mn}$
   (b) $(m+n)(n+p)(m+p) \geq 8mnp$
   (c) $\frac{m}{n} + \frac{n}{p} + \frac{p}{q} + \frac{q}{m} \geq 4.$
2. Let $a > 0$ and $b > 0$. Show that $a^3 + b^3 \geq a^2b + ab^2$.
3. Given that a, b and c are real positive numbers.
   (a) Prove that $a^2 + b^2 + c^2 \geq ab + bc + ca$.
   (b) Show that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{9}{a+b+c}$.
   (c) Given that $a^2 + b^2 + c^2 = 9$, prove that

$$\frac{1}{1+ab} + \frac{1}{1+bc} + \frac{1}{1+ac} \geq \frac{3}{4}.$$

4. Given that a, b and c are real numbers.
   (a) Show that $a^2 + 9b^2 \geq 6ab$.
   (b) Hence or otherwise, show that $a^2 + 5b^2 + 9c^2 \geq 3(ab + bc + ca)$
   (c) Hence if $a > b > c > 0$, show that $a^2 + 5b^2 + 9c^2 > 9bc$.
5. (NSBHS 2019, Task 2)
   Given that a, b and c are real positive numbers such that $a + b + c = 1$ and $a + b + c \geq 3\sqrt[3]{abc}$
   (a) If x, y and z are all positives. Show that $\frac{1}{xy} + x + y \geq 3$.
   (b) Hence or otherwise, prove that

$$\frac{1}{a(a+1)} + \frac{1}{b(b+1)} + \frac{1}{c(c+1)} \geq 4.$$

6. Given that a and b are real numbers.
   (a) Prove that $a^2 + b^2 \geq 2ab$.
   (b) If $x > 0$ and $y > 0$, prove that $\dfrac{x}{y} + \dfrac{y}{x} \geq 2$.
   (c) Prove by induction, or otherwise that

$$(x_1 + \cdots + x_n)(\frac{1}{x_1} + \cdots + \frac{1}{x_n}) \geq n^2,$$

   where x_i are all real and positives.

7. Given a, b and c are positive real numbers. Prove that

$$(a^3 + 2)(b^3 + 2)(c^3 + 2) \geq (a + b + c)^3.$$

$$\frac{a}{(a+1)(b+1)} + \frac{b}{(b+1)(c+1)} + \frac{c}{(c+1)(a+1)} \geq \frac{3}{4}.$$

8. (Jomaa 2019) Given that a, b and c are positive real numbers such that $a + b + c = 1$, prove that

$$\frac{2}{(1+a)^2} + \frac{2}{(1+b)^2} + \frac{2}{(1+c)^2} \leq \frac{1}{(a+b)(a+c)(b+c)}.$$

9. Given that a_1, b_1, a_2, and b_2 are real numbers.
   (a) Prove that $(a_1^2 + a_2^2)(b_1^2 + b_2^2) \geq (a_1 b_1 + a_2 b_2)^2$
   (b) Hence prove by induction that

$$(a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2) \geq (a_1 b_1 + \cdots + a_n b_n)^2,$$

   where $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ are real numbers.
   (c) Hence or otherwise, prove that

$$\frac{a^2}{y+z} + \frac{b^2}{x+z} + \frac{c^2}{x+y} \geq \frac{a+b+c}{2},$$

   where $a, b, c, x, y$and z are positive real numbers and $a + b + c = x + y + z$.

10. (NSBHS, Mar 2019) Given a, b and c are positive real numbers.
   (a) Prove that $a^2 + b^2 \geq 2ab$.
   (b) Given that

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca),$$

   prove that $\dfrac{a^3 + b^3 + c^3}{3} \geq abc$.
   (c) Explain why $\dfrac{a^3 + b^3 + 1}{3} \geq ab$.
   (d) Hence or otherwise, prove that

$$a^3 + b^3 + c^3 + 1 \geq ab + bc + ca + abc.$$

11. Given a, b and c are positive real numbers.

   (a) Prove that $a^2 + b^2 \geq 2ab$.

   (b) Show that
   $$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca).$$

   (c) Show that $a^2 + b^2 + c^2 \geq ab + bc + ca$.

   (d) Deduce that $a^3 + b^3 + c^3 \geq 3abc$.

   (e) Show that
   $$2(a^3 + b^3 + c^3) + \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \geq 3abc + 6.$$

   (f) Show that
   $$a + b + c + a^2 + b^2 + c^2 + a^3 + b^3 + c^3 \geq 3(ab + bc + ca).$$

12. Given a, b and c are positive real numbers such that $abc = 1$. Prove that
   $$a^3 + b^3 + c^3 + (ab)^3 + (bc)^3 + (ca)^3 \geq 2(a^2b + b^2c + c^2a).$$

13. Given a, b and c are positive real numbers.Prove that
   $$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{b}{a} + \frac{c}{b} + \frac{a}{c}.$$

14. Given a, b and c are positive real numbers. Prove that
   $$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{a + b + c}{abc}.$$

15. Given that a, b, c, x, y and z are real numbers.

   (a) Prove that $(a^2 + 1)(b^2 + 1) \geq (a + b)^2$.

   (b) Prove that $(a^2 + 1)(b^2 + 1) \geq (ab + 1)^2$.

   (c) Prove that $(a^2 + b^2)(c^2 + d^2) \geq (ac + bd)^2$.

   (d) Prove that $(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) \geq (ax + by + cz)^2$.

5 HSC inequalities questions

1. (2015 HSC, 15c)
   For positive real numbers x and y, $\sqrt{xy} \leq \dfrac{x + y}{2}$.

   (a) Prove $\sqrt{xy} \leq \sqrt{\dfrac{x^2 + y^2}{2}}$, for positive real numbers x and y.

   (b) Prove $\sqrt[4]{abcd} \leq \sqrt{\dfrac{a^2 + b^2 + c^2 + d^2}{4}}$, for positive real numbers a, b, c and d.

2. (2014 HSC, 15a)
   Three positive real numbers a, b and c are such that $a + b + c = 1$ and $a \leq b \leq c$.
   By Considering the expansion of $(a + b + c)^2$, or otherwise, show that
   $$5a^2 + 3b^2 + c^2 \leq 1.$$

3. (2012 HSC, 15a)

   (a) Prove that $\sqrt{ab} \leq \dfrac{a+b}{2}$, where $a \geq 0$ and $b \geq 0$.

   (b) If $1 \leq x \leq y$, show that $x(y-x+1) \geq y$.

   (c) Let n and j be positive integers with $1 \leq j \leq n$. Prove that

$$\sqrt{n} \leq \sqrt{j(n-j+1)} \leq \frac{n+1}{2}.$$

   (d) For integers $n \geq 1$, prove that

$$\left(\sqrt{n}\right)^n \leq n! \leq \left(\frac{n+1}{2}\right)^n.$$

4. (2011 HSC, 5b)
   If p, q and r are positive real numbers and $p+q \geq r$, prove that

$$\frac{p}{1+p} + \frac{q}{1+q} - \frac{r}{1+r} \geq 0.$$

5. (2004 HSC, 7a)

   (a) Let a be a positive real number. SHow that $a + \dfrac{1}{a} \geq 2$.

   (b) Let n be a positive integer and $a_1, a_2, \ldots, a_n$ be n positive real numbers.
   Prove by induction that

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq n^2.$$

   (c) Hence show that $\operatorname{cosec}^2\theta + \sec^2\theta + \cot^2\theta \geq 9\cos^2\theta$.

6. (2003 HSC, 6c)

   (a) Le x and y be real numbers such that $x \geq 0$ and $y \geq 0$. Prove that $\dfrac{x+y}{2} \geq \sqrt{xy}$.

   (b) Suppose that a, b, c are real numbers. Prove that $a^4 + b^4 + c^4 \geq a^2b^2 + a^2c^2 + b^2c^2$.

   (c) Show that $a^2b^2 + a^2c^2 + b^2c^2 \geq a^2bc + b^2ac + c^2ab$.

   (d) Deduce that if $a+b+c = d$, then $a^4 + b^4 + c^4 \geq abcd$.

7. (2001 HSC, 8a)

   (a) Show that $2ab \leq a^2 + b^2$ for all real numbers a and b.
   Hence deduce that $3(ab+bc+ca) \leq (a+b+c)^2$ for all real numbers a, b and c.

   (b) Suppose that a, b and c are the sides of a triangle. Explain why $(b-c)^2 \leq a^2$.
   Deduce that $(a+b+c)^2 \leq 4(ab+bc+ca)$.

6 Trial exams inequality problems

1. (NSBHS 2019)
   Given that a, b and c are positive real numbers.
   (a) Prove that $a^2 + (bc)^2 \geq 2abc$.
   (b) Prove that $a^2 + b^2 + c^2 \geq ab + bc + ca$.
   (c) Prove that $a^2(1+b^2) + b^2(1+c^2) + c^2(1+a^2) \geq 6abc$.
   (d) Hence, or otherwise prove that $\mathrm{a}^2(1+a^2) + b^2(1+b^2) + c^2(1+c^2) \geq 6abc$.

2. (NSGHS 2018, 15a)
   (a) Prove that, if $c \geq a$ and $d \geq b$ then $ab + cd \geq bc + ad$.
   (b) Use part (2a) to show that if $x \geq y$, then $x^2 + y^2 \geq 2xy$.
   (c) Without the use of further algebra, explain why the result in (2b) is in fact true for all the values of x and y.

3. (CSSA 2018, 13d)
   (a) If a, b and c are positive real numbers, show that
   $$a^2 \geq (a+b-c)(a+c-b).$$
   (b) Hence show that
   $$abc \geq (a+b-c)(b+c-a)(a+c-b).$$

4. (BBHS 2018, 16b)
   If $m > 0, n > 0, p > 0, q > 0$, show that
   (a) m+n$\geq 2\sqrt{mn}$.
   (b) (m+n)(n+p)(p+m)$\geq 8mnp$.
   (c) $\dfrac{m}{n} + \dfrac{n}{p} + \dfrac{p}{q} + \dfrac{q}{m} \geq 4$.

5. (Caringbah HS 2018,15c)
   If $a > 0$ and $b > 0$, show that $a^3 + b^3 \geq a^2 b + ab^2$.

6. (Hornsby Girls 2018,16b)
   It is known that n is a positive integer, where $n \geq 2$ and that a and b are real and positive.
   (a) Prove that $\dfrac{a+b}{2} \geq \sqrt{ab}$.
   It is know that for positive real numbers $a_1, a_2, \ldots, a_n$ that $\sqrt[n]{a_1 a_2 a_3 \ldots a_n} \leq \dfrac{a_1 + a_2 + \cdots + a_n}{n}$.
   (b) Prove that $n! \leq \left(\dfrac{n+1}{2}\right)^n$.

7. (JRAHS 2018, 12c)

   (a) Prove that for all real a and b,
   $$a^2 + b^2 \geq 2ab.$$

   (b) Prove that, for any positive, real x and y,
   $$\frac{x}{y} + \frac{y}{x} \geq 2.$$

   (c) Prove by induction, or otherwise, that
   $$(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right) \geq n^2,$$
   where the x_i are all real and positive.

8. (NSBHS 2018, 16b)
   Given that a, b and c are real positive numbers such that
   $$a + b + c = 1 \quad \text{and} \quad a + b + c \geq 3\sqrt[3]{abc}$$

   (a) If x, y and z are all positives. Show that $\dfrac{1}{xy} + x + y \geq 3$.

   (b) Hence or otherwise, prove that
   $$\frac{1}{a(a+1)} + \frac{1}{b(b+1)} + \frac{1}{c(c+1)} \geq 4.$$

9. (Ind 2017,12c)
   If $a > 0$, $b > 0$, $c > 0$ and $d > 0$ are real numbers, show that

   (a) $a^2 + b^2 + c^2 + d^2 \geq \dfrac{2}{3}(ab + ac + ad + bc + bd + cd)$

   (b) $\left(\dfrac{a+b+c+d}{4}\right)^2 \geq \dfrac{1}{6}(ab + ac + ad + bc + bd + cd)$

10. (JRAHS 2017, 15c)
    Given that
    $$\frac{1}{3}(x + y + z) \geq (xyz)^{\frac{1}{3}},$$
    for all x, y, z non-negative real numbers.

    (a) Show that
    $$xy + yz + zx \geq 3(xyz)^{\frac{2}{3}}.$$

    (b) Hence show that
    $$xyz \leq (a-1)^3.$$
    If $1 + (x + y + z) + (xy + yz + zx) + xyz = a^3$, where $a \geq 2$ is a real number.

11. (Hornsby GHS 2017, 16b)

    (a) Prove that if a and b are any two positive real numbers, then

    $$ab \leq \left(\frac{a+b}{2}\right)^2 \leq \frac{a^2+b^2}{2}.$$

    (b) Given also that $a+b=1$, prove that

    $$\left(a+\frac{1}{a}\right)^2 + \left(b+\frac{1}{b}\right)^2 \geq \frac{25}{2}.$$

12. (Ascham 2016, 12b)
    Use the fact that $x^2+y^2 \geq 2xy$, or otherwise, to prove that

    (a) $\frac{a}{b}+\frac{b}{a} \geq 2$ [You may assume $a, b \geq 0$.]

    (b) $p^2+q^2+r^2 \geq pq+qr+rp$

    (c) $p^3+q^3 \geq pq(p+q)$ (assume that $p, q \geq 0$)
    and hence

    (d) $2(p^3+q^3+r^3) \geq pq(p+q)+qr(q+r)+rp(r+p)$.

13. (Cringbah 2016, 16c)
    If a and b are positive numbers, prove that $a+b \geq 2\sqrt{ab}$.
    Hence, or otherwise, prove that when a, b and c are all positive numbers and $a+b+c=1$, then

    $$\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right) \geq 8.$$

14. (Hunters Hill 2016, 14b)

    (a) Prove that $a^2+b^2 \geq 2ab$.

    (b) Hence, or otherwise, Prove that

    $$(p+2)(q+2)(p+q) \geq 16pq.$$

    where p and q are positive real numbers.

15. (Ind 2016, 13b)
    If $a>0$, $b>0$ and $c>0$ are real numbers, show that

    (a)
    $$(a+b+c)^2 \geq 3(ab+bc+ca).$$

    (b)
    $$(ab+bc+ca)^2 \geq 3abc(a+b+c).$$

    (c)
    $$(a+b+c)^3 \geq 27abc.$$

16. (Knox Grammar 2016, 13d)

   (a) using the identity $(p+q)^2 = (p-q)^2 + 4pq$, show that for $p, q > 0$
   $$\frac{p+q}{2} \geq \sqrt{pq}.$$

   (b) Hence show that if p, q, r and s are greater than zero then
   $$\frac{p+q+r+s}{4} \geq \sqrt[4]{pqrs}.$$

17. (NSBHS 2016, 16a)

   (a) Given that $\dfrac{a+b}{2} \geq \sqrt{ab}$.
   Prove that $\dfrac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$.

   (b) Using part (17a) and the fact that $\dfrac{a+b+c}{3} = \dfrac{1}{4}\left(a+b+c+\dfrac{a+b+c}{3}\right)$,
   Prove that $\dfrac{a+b+c}{3} \geq^3 \sqrt{abc}$

18. (NSGHS 2016, 14c)

   (a) If a and b are real numbers, prove that $a^2 + b^2 \geq 2ab$.

   (b) Show that $a^2 + b^2 + c^2 \geq ab + bc + ca$, where a, b and c are real.

   (c) Hence or otherwise show that the equation $x^3 - 3x^2 + 4x + k = 0$ can not have three real roots for any real constant k.

19. (Penrith HS 2016, 14d)
   If $0 < x < y < \dfrac{1}{2}$. Prove that
   $$\sqrt{xy} < x + y < \sqrt{x+y}.$$

20. (Sam el Hosri 2016, 15a)

   (a) Given that a, b and c are three positive integers, show that
   $$a^2 + b^2 + c^2 \geq ab + bc + ca.$$

   (b) Given that
   $$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ac).$$
   Show that
   $$\frac{p^2}{q^2} + \frac{q^2}{r^2} + \frac{r^2}{p^2} \geq \frac{p}{r} + \frac{q}{p} + \frac{r}{q} \geq 3,$$
   where p, q and r are positive integers.

21. (Western Region 2016, 14d)

   (a) If x, y, z are real and unequal, show that $x^2 + y^2 \geq 2xy$
   and hence deduce that
   $x^2 + y^2 + z^2 \geq xy + yz + zx$.

   (b) if $x + y + z = 3$, show that $xy + yz + zx < 3$.

22. (ACE 2015, 16b)
   Show that
   $$(a+b+c)^2 \leq 3(a^2 + b^2 + c^2).$$

23. (Ind 2015, 16c)
a, b and c are real numbers such that $a > b > c > 1$.

   (a) Show that $a^a b^b c^c > a^b b^c c^a$.

   (b) Hence show that $a \ln a + b \ln b + c \ln c > b \ln a + c \ln b + a \ln c$.

24. (Knox Grammar 2015, 15c)

   (a) It can be shown that for positive real numbers a and b that

   $$a^2 + b^2 \geq 2ab \quad \text{(DO NOT PROVE THIS).}$$

   Hence show for positive real numbers a, b, c and d that

   $$3(a^2 + b^2 + c^2 + d^2) \geq 2(ab + ac + ad + bc + bd + cd).$$

   (b) Hence show for positive real numbers a, b, c and d that if $a + b + c + d = 1$ then

   $$ab + ac + ad + bc + bd + cd \leq \frac{3}{8}.$$

25. (NSGHS 2015, 16b)

   (a) Show that if a and b are real, then $a^2 + b^2 \geq 2ab$.

   (b) Hence show that if a, b, c and d are real, then $a^4 + b^4 + c^4 + d^4 \geq 4abcd$.

26. (Sam el Hosri 2015, 16c)
The three positive real numbers p, q, and r are such that

$$(p + 2)(q + 2)(r + 2) = 64.$$

Given that $a + b + c \geq 3\sqrt[3]{abc}$, for any three real positive numbers a, b and c, show that

$$pqr \leq 8.$$

27. (Ind 2014, 15b)

   (a) If $a > 0$ is a real number, show that $a + \dfrac{1}{a} \geq 2$.

   (b) Hence show that if $a > 0, b > 0, c > 0$ are real numbers, then

      i. $$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 6.$$

      ii. $$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

28. (SBHS 2014, 14c)
In each of the following parts,$x, y, z, w, a, b, c, d > 0$:

   (a) Show that $(x + y)^2 \geq 4xy$.

   (b) Show that $[(x + y)(z + w)]^2 \geq 16xyzw$.

   (c) Deduce that $\dfrac{x + y + z + w}{4} \geq \sqrt[4]{xyzw}$.

   (d) Hence show that (using (28c)):

   $$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geq 4.$$

29. (ACE 2013, 16c)
Show that $\dfrac{a^2+b^2}{a-b} \geq 2\sqrt{2}$ for $a > b$ and $ab > 1$.

30. (Ind 2013, 16b)

   (a) If a and b are positive real numbers, show that $a^2 - ab + b^2 \geq \left(\dfrac{a+b}{2}\right)^2$.

   (b) In ΔABC, if $\angle BCA \geq 60^{\circ}$ show that $c^2 \geq a^2 - ab + b^2$ and hence deduce that $2c^3 \geq a^3 + b^3$ with equality if and only if ΔABC is equilateral.

31. (NHBHS 2013, 16b)

   (a) By squaring, or otherwise, show that for $k \geq 0$,
$$2k+3 \geq 2\sqrt{k+2}\sqrt{k+1}.$$

   (b) By decomposing $2k+3$ and factorising $2\sqrt{k+2}\sqrt{k+1} - 2(k+1)$ show that for $k \geq 1$
$$\frac{1}{\sqrt{k+1}} > 2\left(\sqrt{k+2} - \sqrt{k+1}\right).$$

   (c) Hence, or otherwise, show for $n \geq 1$,
$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > 2\left(\sqrt{n+1} - 1\right).$$

32. (NSGHS 2013, 13a)

   (a) Show that $x^2 + y^2 \geq xy$, where x and y are real numbers.

   (b) If $x + y = 3z$, show that $x^2 + y^2 \geq 3z^2$.

33. (Penrith 2013, 15d)

   (a) Prove that $x^2 + x + 1 \geq 0$ for all real x.

   (b) Hence or otherwise, prove that $a^2 + ab + b^2 \geq 0$.

34. (Sam el Hosri 2013, 15a)

   (a) Show that $a + \dfrac{1}{a} \geq 2$ for $a > 0$.

   (b) Given a, b, c are three positive real numbers such that $a + b + c = 1$. Show that
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 5.$$

35. (NSGHS 2012, 15c)

   (a) Show that $a^2 + b^2 \geq 2ab$ for any values of a and b.

   (b) Hence show that $\tan^2\theta + \cos^2\theta + \operatorname{cosec}^2\theta \geq \sin\theta + \sec\theta + \cot\theta$ for all values of θ.

36. (SBHS 2012, 15b)
For positive real numbers x and y

(a) Prove that $\dfrac{x+y}{2} \geq \sqrt{xy}$.
When is there equality?

(b) Hence by considering $\dfrac{1}{a} + \dfrac{1}{b}$, or otherwise, prove that $\dfrac{2ab}{a+b} \leq \sqrt{ab}$ for positive real numbers a, b.

(c) Hence, or otherwise prove that $\dfrac{1}{x-1} + \dfrac{1}{x} + \dfrac{1}{x+1} \geq \dfrac{3}{x}$ for any $x > 1$.

(d) If $H = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \cdots + \dfrac{1}{n}$, where n is an integer $n > 1$, use (36c) to show that H has no limit as $n \to \infty$.

37. (Manly SC 2011, 6b)

(a) Show that $a^2 + b^2 \geq 2ab$ where a and b are distinct positive real numbers.

(b) Hence show that $a^2 + b^2 + c^2 \geq ab + ac + bc$.

(c) Hence show that $\sin^2 \alpha + \cos^2 \alpha \geq \sin 2\alpha$.

(d) Hence show that $\sin^2 \alpha + \cos^2 \alpha + \tan^2 \alpha \geq \sin \alpha - \cos \alpha + \sec \alpha + \dfrac{1}{2} \sin 2\alpha$.

38. (NSGHS 2011, 8b)

(a) If $a > 0, b > 0$ and $c > 0$, show that $a^2 + b^2 \geq 2ab$ and hence deduce that $a^2 + b^2 + c^2 \geq ab + be + ca$.

(b) If $a + b + c = 9$, show that $ab + bc + ca \leq 27$ and $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} \leq \dfrac{27}{abc}$.

39. (CSSA 2009, 6b)

(a) Given that $x > 0$ and $y > 0$ show that

$$x + y \geq 2\sqrt{xy}.$$

(b) Hence show that for $x > 0$, $y > 0$, $z > 0$ and $w > 0$

$$x + y + z + w \geq 4^4\sqrt{xyzw}.$$

(c) Consider x, y, z and $w = \dfrac{x+y+z}{3}$. Apply the result in (39b) to show that

$$\frac{x+y+z}{3} \geq^3 \sqrt{xyz}.$$

Application of the AM-GM inequality with two numbers.

1. $(a+b)(1+ab) \geq 4ab$

$(\sqrt{a}-\sqrt{b})^2 \geq 0 \quad \therefore \quad a+b \geq 2\sqrt{ab} \quad ①$

$(\sqrt{ab}-1)^2 \geq 0 \quad \therefore \quad 1+ab \geq 2\sqrt{ab} \quad ②$

$① \times ② \quad \therefore \quad (a+b)(1+ab) \geq 4ab.$

2. $(ab+xy)(ax+by) \geq 4abxy$

$(\sqrt{ab}-\sqrt{xy})^2 \geq 0 \quad \therefore \quad ab+xy \geq 2\sqrt{abxy} \quad ①$

$(\sqrt{ax}-\sqrt{by})^2 \geq 0 \quad \therefore \quad ax+by \geq 2\sqrt{abxy} \quad ②$

$① \times ② \quad \therefore \quad (ab+xy)(ax+by) \geq 4abxy$

3. $a+b+1 \geq 2\sqrt{a+b}$

$(\sqrt{a+b}-1)^2 \geq 0 \quad \therefore \quad a+b+1 \geq 2\sqrt{a+b}.$

4. $a^2+b^2+1 \geq ab+a+b$

$(a-b)^2+(a-1)^2+(b-1)^2 \geq 0$

$a^2+b^2-2ab+a^2-2a+1+b^2-2b+1 \geq 0$

$2(a^2+b^2+1) \geq 2(ab+a+b)$

$\therefore \quad a^2+b^2+1 \geq ab+a+b.$

5. $a^2+b^2+c^2 \geq ab+ac+bc$

$(a-b)^2+(b-c)^2+(c-a)^2 \geq 0$

$a^2-2ab+b^2+b^2-2bc+c^2+c^2-2ac+a^2 \geq 0$

$2(a^2+b^2+c^2) \geq 2(ab+bc+ca)$

$\therefore \quad a^2+b^2+c^2 \geq ab+bc+ca.$

$a+b+c=1 \quad \therefore \quad (a+b+c)^2=1 \quad \therefore$

$a^2+b^2+c^2+2(ab+bc+ca)=1 \geq ab+bc+ca+2(ab+bc+ca)$

$\therefore \quad 3(ab+bc+ca) \leq 1 \quad \therefore \quad ab+bc+ca \leq \frac{1}{3}.$

6. $(a-b)^2 \geq 0$

$$(a-b)^2 + (a+b)^2 \geq (a+b)^2$$

$$2(a^2+b^2) \geq (a+b)^2$$

$$\frac{a^2+b^2}{2} \geq \left(\frac{a+b}{2}\right)^2$$

$$\sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2}.$$

7. $(a^2-b^2)^2 \geq 0 \quad \therefore \quad a^4+b^4 \geq 2a^2b^2 \quad (1)$

Similarly $c^4+d^4 \geq 2c^2d^2 \quad (2)$

$(1) + (2) \quad \therefore$

$$a^4+b^4+c^4+d^4 \geq 2(a^2b^2+c^2d^2)$$

But $(ab-cd)^2 \geq 0 \therefore a^2b^2+c^2d^2 \geq 2abcd$

Now $a^4+b^4+c^4+d^4 \geq 2(2abcd) = 4abcd.$

8. (a) $(a-b)^2 \geq 0 \quad \therefore \quad (a+b)^2 - 4ab \geq 0$

$$\frac{(a+b)^2}{ab(a+b)} - \frac{4ab}{ab(a+b)} \geq 0$$

$$\frac{a+b}{ab} - \frac{4}{t} \geq 0$$

$$\frac{1}{a} + \frac{1}{b} \geq \frac{4}{t}.$$

Alternatively

$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4$$

$$\therefore \frac{1}{a}+\frac{1}{b} \geq \frac{4}{a+b}.$$

(b)

$$(a+b)(a+b)\left(\frac{1}{a^2}+\frac{1}{b^2}\right) \geq 4ab \times \frac{2}{ab} = 8$$

$$\frac{1}{a^2}+\frac{1}{b^2} \geq \frac{8}{t^2}.$$

9.

$$\left.\begin{aligned} x+y &\geq 2\sqrt{xy} \\ y+z &\geq 2\sqrt{yz} \\ z+x &\geq 2\sqrt{zx} \end{aligned}\right\} (x+y)(y+z)(z+x) \geq 8xyz.$$

10. $x+y+z=1$

$1-x=y+z$

$1-y=x+z$

$1-z=x+y$

$1-x=y+z\geqslant 2\sqrt{yz}$

$1-y=x+z\geqslant 2\sqrt{xz}$

$1-z=x+y\geqslant 2\sqrt{xy}$

$(1-x)(1-y)(1-y)\geqslant 8xyz$

11. $a\geqslant b\geqslant c$

$a\geqslant b \therefore ab\geqslant b^2$

$a\geqslant c \therefore ac\geqslant c^2$

$b\geqslant c \therefore bc\geqslant c^2$

$a+b+c\leq 1 \quad \therefore \quad (a+b+c)^2\leq 1$

$\therefore \quad a^2+b^2+c^2+2(ab+bc+ca)\leq 1 \quad ①$

$2(ab+bc+ca)\geqslant 2b^2+2c^2+2c^2 \quad ②$

Combining ① + ② we obtain:

$\therefore a^2+3b^2+5c^2\leq 1$

12. $x+y+z+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geqslant 6$

$x+\frac{1}{x}\geqslant 2$

$y+\frac{1}{y}\geqslant 2$

$z+\frac{1}{z}\geqslant 2$

$x+y+z+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geqslant 6.$

13. $\frac{a+b}{2}\geqslant\sqrt{ab}$, similarly $\frac{c+d}{2}\geqslant\sqrt{cd}$

$$\frac{a+b+c+d}{4}=\frac{1}{2}\left(\frac{a+b}{2}+\frac{c+d}{2}\right)\geqslant\frac{1}{2}\left(\sqrt{ab}+\sqrt{cd}\right)\geqslant\sqrt{\sqrt{ab}\sqrt{cd}}$$

$$=\sqrt[4]{abcd}$$

14. Given that $\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$

$$\frac{\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}}{4} \geqslant \sqrt[4]{\frac{a}{b}\frac{b}{c}\frac{c}{d}\frac{d}{a}} = 1$$

$$\therefore \quad \frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a} \geqslant 4 .$$

15.

(a) $\left(\sqrt{a}-\frac{1}{\sqrt{a}}\right)^2 \geqslant 0 \quad \therefore \quad a-2+\frac{1}{a} \geqslant 0 \quad \therefore \quad a+\frac{1}{a} \geqslant 2$

(b) $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4$?

$$a+b \geqslant 2\sqrt{ab}$$

$$\frac{1}{a}+\frac{1}{b} \geqslant 2\sqrt{\frac{1}{a}\frac{1}{b}} = \frac{2}{\sqrt{ab}}$$

$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geqslant 2\sqrt{ab} \times \frac{2}{\sqrt{ab}} = 4 .$$

$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$?

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$= (a+b)\left(\frac{1}{a}+\frac{1}{b}\right) + \frac{a}{c}+\frac{b}{c}+\frac{c}{a}+\frac{c}{b}+\frac{c}{c}$$

$$= (a+b)\left(\frac{1}{a}+\frac{1}{b}\right) + \frac{a}{c}+\frac{c}{a}+\frac{b}{c}+\frac{c}{b}+1$$

$$\geqslant \quad 4+2+2+1 = 9 .$$

(c) $2(a+b+c)\left[\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right]$

$$= (b+c+c+a+a+b)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right) \geq 9$$

By the previous exercise. (see 15).

$$\therefore \quad \frac{2}{b+c}+\frac{2}{c+a}+\frac{2}{a+b} \geqslant \frac{9}{a+b+c} .$$

$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4$, $(a+c)\left(\frac{1}{a}+\frac{1}{c}\right) \geq 4$, $(b+c)\left(\frac{1}{b}+\frac{1}{c}\right) \geq 4$

$\therefore \quad \frac{1}{a}+\frac{1}{b} \geqslant \frac{4}{a+b}$, $\quad \frac{1}{a}+\frac{1}{c} \geq \frac{4}{a+c}$, $\quad \frac{1}{b}+\frac{1}{c} \geqslant \frac{4}{b+c}$

$2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geqslant \frac{4}{a+b}+\frac{4}{a+c}+\frac{4}{b+c} \quad \therefore \quad \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geqslant \frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a} .$

16. $1 + a_i \geq 2\sqrt{a_i}$, $i = 1, \ldots, n$

$(1-\sqrt{a_i})^2 \geq 0 \therefore 1 - 2\sqrt{a_i} + a_i \geq 0$

$\therefore 1 + a_i \geq 2\sqrt{a_i}$

$$\prod_{i=1}^{n} (1+a_i) \geq \prod_{i=1}^{n} 2\sqrt{a_i} = 2^n \sqrt{a_1 a_2 \cdots a_n}$$

But $a_1 a_2 \ldots a_n = 1$ (given)

$\therefore (1+a_1)(1+a_2) \ldots (1+a_n) \geq 2^n$.

17. $3(a^2+b^2+c^2)$

$= a^2+b^2+c^2 + (a^2+b^2) + (b^2+c^2) + (c^2+a^2)$

$\geq a^2+b^2+c^2 + 2ab + 2bc + 2ca$

$= (a+b+c)^2$.

18. $x^2+x+1 = (x+\frac{1}{2})^2 + \frac{3}{4} > 0$

(a) $a^2+ab+b^2 = (a+\frac{b}{2})^2 + \frac{3}{4}b^2 > 0$

(b) $a^4+b^4-a^3b-ab^3$

$= a^3(a-b) + b^3(b-a)$

$= (a-b)(a^3-b^3)$

$= (a-b)(a-b)(a^2+ab+b^2)$

$= (a-b)^2(a^2+ab+b^2) > 0$.

19. Given $x^2+y^2+z^2 \geq xy+yz+zx$

$(x+y+z)^2 = x^2+y^2+z^2+2xy+2yz+2zx$

$\geq xy+yz+zx+2xy+2yz+2zx$

$= 3(xy+yz+zx)$

20 over the page

20. $x_0 > x_1 > x_2 > \ldots > x_n$

$x_0 - x_n = x_0 - x_1 + x_1 - x_2 + x_2 - x_3 + \ldots + x_{n-1} - x_n$

$x_0 - x_1 + \frac{1}{x_0-x_1} \geq 2$ and $x_i - x_{i+1} + \frac{1}{x_i - x_{i+1}} \geq 2$, $i = 0, \ldots, n-1$

$$x_0 - x_n + \frac{1}{x_0-x_1} + \frac{1}{x_1-x_2} + \ldots + \frac{1}{x_{n-1}-x_n} \geq \underbrace{2 + \ldots + 2}_{n \text{ times}} = 2n$$

Hence the result.

20. $a^2b^2+b^2c^2+c^2a^2 \geqslant abc(a+b+c)$

$$a^2b^2+b^2c^2 \geqslant 2ab^2c$$
$$a^2b^2+c^2a^2 \geqslant 2a^2bc$$
$$b^2c^2+c^2a^2 \geqslant 2abc^2$$
$$2(a^2b^2+b^2c^2+c^2a^2) \geqslant 2(ab^2c+a^2bc+abc^2)$$
$$a^2b^2+b^2c^2+c^2a^2 \geqslant abc(a+b+c).$$

22. $a^2+b^2 \geq 2ab$

$a^2+b^2 = (a+b)^2 - 2ab \geq 2ab$

$(a+b)^2 \geq 4ab$

$ab \leq \frac{(a+b)^2}{4} = \frac{1}{4}$

$(a+b)^2 = a^2+b^2+2ab$

$1 \leq a^2+b^2+\frac{1}{2}$

$\therefore \quad a^2+b^2 \geq \frac{1}{2}$

$a^3+b^3 = (a+b)(a^2-ab+b^2)$

$= a^2-ab+b^2$

$= a^2+b^2-ab$

$\geq \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

23. $a^4+b^4 = (a^2+b^2)(a^2+b^2) - 2a^2b^2$

$\geq 2ab(a^2+b^2) - 2a^2b^2$

$= ab(2a^2+2b^2-2ab)$

$= ab(a^2+b^2+a^2+b^2-2ab)$

$= ab(a^2+b^2+(a-b)^2)$

$\geq ab(a^2+b^2)$

since $a^2+b^2+(a-b)^2 \geq a^2+b^2$.

5. Application of the AM-GM inequality with three numbers

1. $\frac{1}{3}(a^3-b^3)(a-b) \geq ab(a-b)^2$

$$\frac{1}{3}(a^3-b^3)(a-b) = \frac{1}{3}(a-b)^2(a^2+ab+b^2)$$
$$= (a-b)^2\left(\frac{a^2+ab+b^2}{3}\right)$$
$$\geq (a-b)^2\sqrt[3]{a^2abb^2}$$
$$= (a-b)^2 ab.$$

2. Given $abc=1$ $\quad a^2+b^2+c^2 \geq a+b+c$

Method 2 on the back page

First prove that $3(a^2+b^2+c^2) \geq (a+b+c)^2$ (see question 17/section 3)

$$3(a^2+b^2+c^2) \geq (a+b+c)^2$$
$$= (a+b+c)\times(a+b+c)$$
$$\geq (a+b+c)\times 3\sqrt[3]{abc} = 3(a+b+c)$$
$$\therefore \quad a^2+b^2+c^2 \geq a+b+c.$$

3.
$$\frac{a^3}{b}+\frac{b^3}{c}+bc \geq 3ab$$
$$\frac{a^3}{b}+\frac{c^3}{a}+ab \geq 3ac$$
$$\frac{b^3}{c}+\frac{c^3}{a}+ac \geq 3bc$$
$$2\left(\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}\right) \geq 2(ab+bc+ca)$$
$$\therefore \quad \frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a} \geq ab+bc+ca.$$

4.
$$a^2b+b^2c+c^2a \geq 3\sqrt[3]{a^2bb^2cc^2a} = 3abc$$
$$ab^2+bc^2+ca^2 \geq 3\sqrt[3]{ab^2bc^2ca^2} = 3abc$$

$$(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2) \geq (3abc)(3abc)$$
$$= 9a^2b^2c^2$$

5. $$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\geqslant\frac{3}{2}$$

$$2\left(\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\right)=2\left(\frac{a+b+c}{b+c}+\frac{a+b+c}{a+c}+\frac{a+b+c}{a+b}\right)-6$$

$$=2(a+b+c)\left(\frac{1}{b+c}+\frac{1}{a+c}+\frac{1}{a+b}\right)-6$$

$$=(a+b+b+c+c+a)\left(\frac{1}{b+c}+\frac{1}{a+c}+\frac{1}{a+b}\right)-6$$

$$\geqslant 3\sqrt[3]{(a+b)(b+c)(c+a)}\times 3\sqrt[3]{\frac{1}{b+c}\times\frac{1}{a+c}\times\frac{1}{a+b}}-6$$

$$\geqslant 9-6=3$$

$$\therefore\quad\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\geqslant\frac{3}{2}.$$

6. $abc=1$

$$\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+ac}{1+c}$$

$$=\frac{abc+ab}{1+a}+\frac{abc+bc}{1+b}+\frac{abc+ac}{1+c}$$

$$=\frac{ab(1+c)}{1+a}+\frac{bc(1+a)}{1+b}+\frac{ac(1+b)}{1+c}$$

$$\geqslant 3\sqrt[3]{\frac{ab(1+c)}{1+a}\times\frac{bc(1+a)}{1+b}\times\frac{ac(1+b)}{1+c}}$$

$$=3\sqrt[3]{a^2b^2c^2}=3.$$

7. see section 3 question 15.

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geqslant 3\sqrt[3]{abc}\times 3\sqrt[3]{\frac{1}{a}\frac{1}{b}\frac{1}{c}}=9$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geqslant\frac{9}{a+b+c}$$

$$2(a+b+c)\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)=(a+b+b+c+c+a)\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)\geqslant 9$$

(see previous).

8. $\dfrac{1}{b(a+b)}+\dfrac{1}{c(b+c)}+\dfrac{1}{a(c+a)} \geq \dfrac{27}{2(a+b+c)^2}$

$2(a+b+c)^2\left(\dfrac{1}{b(a+b)}+\dfrac{1}{c(b+c)}+\dfrac{1}{a(c+a)}\right)$

$=(a+b+c)(a+b+b+c+c+a)\left(\dfrac{1}{b(a+b)}+\dfrac{1}{c(b+c)}+\dfrac{1}{a(c+a)}\right)$

$\geq 3\sqrt[3]{abc} \times 3\sqrt[3]{(a+b)(b+c)(c+a)} \times 3\sqrt[3]{\dfrac{1}{b(a+b)}\times\dfrac{1}{c(b+c)}\times\dfrac{1}{a(c+a)}}$

$=27$

$\therefore \dfrac{1}{b(a+b)}+\dfrac{1}{c(b+c)}+\dfrac{1}{a(c+a)} \geq \dfrac{27}{2(a+b+c)^2}$.

9. $abc=1$

$\dfrac{a}{(a+1)(b+1)}+\dfrac{b}{(b+1)(c+1)}+\dfrac{c}{(c+1)(a+1)} \geq \dfrac{3}{4}$

Multiply both sides by $4(a+1)(b+1)(c+1)$

$4\left(a(c+1)+b(a+1)+c(b+1)\right) \geq 3(a+1)(b+1)(c+1)$

which is equivalent to

$4(ab+bc+ac+a+b+c) \geq 3(abc+ab+bc+ca+a+b+c+1)$

or

$ab+bc+ac+a+b+c \geq 3(abc+1)=3(1+1)=6$

<u>Alternatively</u>: $\dfrac{a}{(a+1)(b+1)}+\dfrac{b}{(b+1)(c+1)}+\dfrac{c}{(c+1)(a+1)}$

$=1-\dfrac{2}{(a+1)(b+1)(c+1)}$

But $(a+1)(b+1)(c+1) \geq 8 \quad \therefore \dfrac{-2}{(a+1)(b+1)(c+1)} \geq -\dfrac{1}{4}$

$\therefore \dfrac{a}{(a+1)(b+1)}+\dfrac{b}{(b+1)(c+1)}+\dfrac{c}{(c+1)(a+1)} \geq 1-\dfrac{1}{4}=\dfrac{3}{4}$.

10. Given that $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 1$. $abc \geqslant 8$?

$$(a+1+b+1+c+1)\left(\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}\right) \geqslant 9$$

$a+b+c+3 \geqslant 9$ $\therefore$ $a+b+c \geqslant 6$

$\boxed{a+b+c+2 \geqslant 8}$ (1)

But $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 1$

$\therefore$ $(b+1)(c+1) + (a+1)(c+1) + (a+1)(b+1) = (a+1)(b+1)(c+1)$

$\therefore$ $bc+b+c+1+ac+a+c+1+ab+a+b+1 = abc+ab+bc+ca+a+b+c+1$

$\boxed{a+b+c+2 = abc}$ (2)

Combining (1) and (2), we obtain $abc \geqslant 8$.

11. $a^2+b^2+c^2 = ab+bc+ca$

$\therefore$ $2(a^2+b^2+c^2) = 2(ab+bc+ca)$

$\therefore$ $(a^2+b^2-2ab) + (a^2+c^2-2ac) + (b^2+c^2-2bc) = 0$

$\therefore$ $(a-b)^2 + (a-c)^2 + (b-c)^2 = 0$

$\therefore$ $a-b=0$, $a-c=0$ and $b-c=0$

$\therefore$ $a=b$, $a=c$ and $b=c$

$\therefore$ $a=b=c$

and the triangle ΔABC is an equilateral triangle.

6. Mixed problems:

1. $m, n, p, q > 0$

(a) $(\sqrt{m} - \sqrt{n})^2 \geq 0$

$\therefore \quad m - 2\sqrt{mn} + n \geq 0$

$\therefore \quad m + n \geq 2\sqrt{mn}$

(b) $m + n \geq 2\sqrt{mn}$

$n + p \geq 2\sqrt{np}$

$m + p \geq 2\sqrt{mp}$

$(m+n)(n+p)(p+m) \geq 8\sqrt{mnnpmp} = 8mnp$

(c) $\frac{m}{n} + \frac{n}{p} + \frac{p}{q} + \frac{q}{m} \geq 4$

$\frac{m}{n} + \frac{n}{p} \geq 2\sqrt{\frac{m}{n}\frac{n}{p}} = 2\sqrt{m/p}$

$\frac{p}{q} + \frac{q}{m} \geq 2\sqrt{p/q \times q/m} = 2\sqrt{p/m}$

$$\frac{m}{n} + \frac{n}{p} + \frac{p}{q} + \frac{q}{m} \geq 2\left(\sqrt{\frac{m}{p}} + \sqrt{p/m}\right)$$
$$\geq 2\left(2\sqrt[4]{\frac{m}{p} \times \frac{p}{m}}\right)$$
$$= 4.$$

2. $a > 0, b > 0, \quad a^3 + b^3 \geq a^2b + ab^2$

$(a-b)^2 \geq 0 \quad \therefore \quad a^2 - 2ab + b^2 \geq 0$

$a^2 - ab + b^2 \geq ab$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$
$$\geq (a+b)ab$$
$$= a^2b + ab^2.$$

3. $a, b, c > 0$

(a) $a^2 + b^2 + b^2 + c^2 + c^2 + a^2 \geq 2ab + 2bc + 2ac$

$\therefore \quad a^2 + b^2 + c^2 \geq ab + bc + ac$

(b) $(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 3\sqrt[3]{abc} \times 3\sqrt[3]{\frac{1}{a}\frac{1}{b}\frac{1}{c}}$

$= 9$

$\therefore \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{9}{a+b+c}$

(c) $a^2+b^2+c^2=9$

$$\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ac} \geqslant \frac{3}{4}.$$

$$(1+ab+1+bc+1+ac)\left(\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ac}\right)$$

$$\geqslant 3\sqrt[3]{(1+ab)(1+bc)(1+ac)} \times 3\sqrt[3]{\frac{1}{1+ab}\times\frac{1}{1+bc}\times\frac{1}{1+ac}}$$

$$= 9$$

$$\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ac} \geqslant \frac{9}{3+ab+bc+ca}$$

from (a) $a^2+b^2+c^2 \geqslant ab+bc+ca$

$9 \geqslant ab+bc+ca$

$12 \geqslant 3+ab+bc+ca$

$\therefore \dfrac{1}{3+ab+bc+ca} \geqslant \dfrac{1}{12}$

$$\therefore \frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ac} \geqslant 9\times\frac{1}{12}=\frac{3}{4}.$$

4. (a) $(a-3b)^2 \geqslant 0 \quad \therefore a^2-6ab+9b^2 \geqslant 0$

$\therefore a^2+9b^2 \geqslant 6ab.$

(b) $a^2+9b^2 \geqslant 6ab$

$a^2+9c^2 \geqslant 6ac$

$b^2+9c^2 \geqslant 6bc$

$2a^2+10b^2+18c^2 \geqslant 6(ab+bc+ac)$

$a^2+5b^2+9c^2 \geqslant 3(ab+bc+ca).$

(c) $a>b>c \quad \therefore ab>ac>bc$

$\therefore a^2+5b^2+9c^2 \geqslant 3(bc+bc+bc)$

$= 9bc.$

5. $a+b+c=1$ and $a+b+c \geq 3\sqrt[3]{abc}$

(a) let $a=\frac{1}{xy}$, $b=x$ and $c=y$

$$\frac{1}{xy}+x+y \geq 3\sqrt[3]{\frac{1}{xy}xy} = 3$$

(b)
$$\frac{1}{a(a+1)} + a + a+1 \geq 3$$

$$\frac{1}{b(b+1)} + b + b+1 \geq 3$$

$$\frac{1}{c(c+1)} + c + c+1 \geq 3$$

$$\frac{1}{a(a+1)}+\frac{1}{b(b+1)}+\frac{1}{c(c+1)}+2(a+b+c)+3 \geq 9$$

$$\frac{1}{a(a+1)}+\frac{1}{b(b+1)}+\frac{1}{c(c+1)} \geq 9-5 = 4.$$

6. (a) $(a-b)^2 \geq 0 \quad \therefore \quad a^2-2ab+b^2 \geq 0$

$$\therefore \quad a^2+b^2 \geq 2ab$$

(b) $\left(\sqrt{\frac{x}{y}}-\sqrt{\frac{y}{x}}\right)^2 \geq 0 \quad \therefore \quad \frac{x}{y}-2\sqrt{\frac{x}{y}\frac{y}{x}}+\frac{y}{x} \geq 0$

$$\therefore \quad \frac{x}{y}+\frac{y}{x} \geq 2$$

(c) prove by induction that

$$(x_1+\cdots+x_n)\left(\frac{1}{x_1}+\cdots+\frac{1}{x_n}\right) \geq n^2$$

For $n=1$, $x_1 \times \frac{1}{x_1} = 1 \geq 1^2$ true for $n=1$

Assume it is true for $n=k$

$$(x_1+\cdots+x_k)\left(\frac{1}{x_1}+\cdots+\frac{1}{x_k}\right) \geq k^2 \qquad (*)$$

Show it is true for $n=k+1$.

$$(x_1+\cdots+x_k+x_{k+1})\left(\frac{1}{x_1}+\cdots+\frac{1}{x_k}+\frac{1}{x_{k+1}}\right)$$

$$= (x_1+\cdots+x_k)\left(\frac{1}{x_1}+\cdots+\frac{1}{x_k}\right) + (x_1+\cdots+x_k+x_{k+1})\times\frac{1}{x_{k+1}} + x_{k+1}\times\left(\frac{1}{x_1}+\cdots+\frac{1}{x_k}\right)$$

$$\geq k^2 + \left(\frac{x_1}{x_{k+1}}+\frac{x_{k+1}}{x_1}\right)+\left(\frac{x_2}{x_{k+1}}+\frac{x_{k+1}}{x_2}\right)+\cdots+\left(\frac{x_k}{x_{k+1}}+\frac{x_{k+1}}{x_k}\right)+\frac{x_{k+1}}{x_{k+1}}$$

$$\geq k^2 + \underbrace{2+\cdots+2}_{k \text{ times}} + 1 = k^2+2k+1 = (k+1)^2$$

Hence By Mathematical induction it is true for all $n \geq 1$.

7. $(a^3+2)(b^3+2)(c^3+2) \geq (a+b+c)^3$

$$\frac{a^3}{a^3+2} + \frac{1}{b^3+2} + \frac{1}{c^3+2} \geq \frac{3a}{\sqrt[3]{(a^3+2)(b^3+2)(c^3+2)}}$$

$$\frac{1}{a^3+2} + \frac{b^3}{b^3+2} + \frac{1}{c^3+2} \geq \frac{3b}{\sqrt[3]{(a^3+2)(b^3+2)(c^3+2)}}$$

$$\frac{1}{a^3+2} + \frac{1}{b^3+2} + \frac{c^3}{c^3+2} \geq \frac{3c}{\sqrt[3]{(a^3+2)(b^3+2)(c^3+2)}}$$

$$\frac{a^3+2}{a^3+2} + \frac{b^3+2}{b^3+2} + \frac{c^3+2}{c^3+2} \geq \frac{3(a+b+c)}{\sqrt[3]{(a^3+2)(b^3+2)(c^3+2)}}$$

$$1 \geq \frac{a+b+c}{\sqrt[3]{(a^3+2)(b^3+2)(c^3+2)}}$$

$$\therefore \quad (a^3+2)(b^3+2)(c^3+2) \geq (a+b+c)^3.$$

8. (Jomaa 2019)

$$\begin{aligned} 1+a &= a+a+b+c \\ &= a+b+a+c \geq 2\sqrt{(a+b)(a+c)} \end{aligned}$$

$$1+b = a+b+b+c \geq 2\sqrt{(a+b)(b+c)}$$

$$1+c = a+c+b+c \geq 2\sqrt{(b+c)(a+c)}$$

$$\frac{2}{1+a} \leq \frac{1}{\sqrt{(a+b)(a+c)}} \quad \therefore \quad \frac{4}{(1+a)^2} \leq \frac{1}{(a+b)(a+c)}$$

Similarly $\dfrac{4}{(1+b)^2} \leq \dfrac{1}{(a+b)(b+c)}$

and $\dfrac{4}{(1+c)^2} \leq \dfrac{1}{(a+c)(b+c)}$

$$\frac{4}{(a+1)^2}+\frac{4}{(b+1)^2}+\frac{4}{(c+1)^2}\le\frac{1}{(a+b)(a+c)}+\frac{1}{(a+b)(b+c)}+\frac{1}{(a+c)(b+c)}$$

$$=\frac{b+c+a+c+a+b}{(a+b)(b+c)(c+a)}$$

$$=\frac{2}{(a+b)(b+c)(c+a)}$$

$$\therefore\ \frac{2}{(1+a)^2}+\frac{2}{(1+b)^2}+\frac{2}{(1+c)^2}\le\frac{1}{(a+b)(b+c)(c+a)}.$$

4. (a) $(a_1b_2-a_2b_1)^2\geq 0$

$$a_1^2b_2^2-2a_1b_2a_2b_1+a_2^2b_1^2\geq 0$$

$$a_1^2b_2^2+a_2^2b_1^2\geq 2a_1a_2b_1b_2$$

$$a_1^2b_2^2+a_2^2b_1^2+a_1^2b_1^2+a_2^2b_2^2\geq a_1^2b_1^2+2a_1a_2b_1b_2+a_2^2b_2^2$$

$$a_1^2(b_1^2+b_2^2)+a_2^2(b_1^2+b_2^2)\geq(a_1b_1+a_2b_2)^2$$

$$(a_1^2+a_2^2)(b_1^2+b_2^2)\geq(a_1b_1+a_2b_2)^2.$$

(b) prove by induction that

$$(a_1^2+\cdots+a_n^2)(b_1^2+\cdots+b_n^2)\geq(a_1b_1+\cdots+a_nb_n)^2$$

for $n=1$, $a_1^2b_1^2\geq(a_1b_1)^2=a_1^2b_1^2$

Assume it is true for $n=k$

$$(a_1^2+\cdots+a_k^2)(b_1^2+\cdots+b_k^2)\geq(a_1b_1+\cdots+a_kb_k)^2 \quad (*)$$

prove it true for $n=k+1$

i.e $(a_1^2+\cdots+a_k^2+a_{k+1}^2)(b_1^2+\cdots+b_k^2+b_{k+1}^2)\geq(a_1b_1+\cdots+a_{k+1}b_{k+1})^2$

$$(a_1^2+\dots+a_k^2+a_{k+1}^2)(b_1^2+\dots+b_k^2+b_{k+1}^2)$$

$$= (a_1^2+\dots+a_k^2)(b_1^2+\dots+b_k^2) + (a_1^2+\dots+a_k^2)b_{k+1}^2 + a_{k+1}^2(b_1^2+\dots+b_k^2)+a_{k+1}^2b_{k+1}^2$$

$$\geq (a_1b_1+\dots+a_kb_k)^2 + a_1^2b_{k+1}^2+a_{k+1}^2b_1^2+\dots+a_k^2b_{k+1}^2+a_{k+1}^2b_k^2+a_{k+1}^2b_{k+1}^2 \quad \text{(using *)}$$

$$\geq (a_1b_1+\dots+a_kb_k)^2 + 2a_1b_1a_{k+1}b_{k+1}+\dots+2a_kb_ka_{k+1}b_{k+1}+a_{k+1}^2b_{k+1}^2$$

$$= (a_1b_1+\dots+a_kb_k)^2+2a_{k+1}b_{k+1}(a_1b_1+\dots+a_kb_k)+a_{k+1}^2b_{k+1}^2$$

$$= (a_1b_1+\dots+a_kb_k+a_{k+1}b_{k+1})^2$$

$\therefore$ Hence by mathematical induction, it is true for all $n \geq 1$.

(c) $$\frac{a^2}{y+z}+\frac{b^2}{x+z}+\frac{c^2}{x+\cancel{z}\,y} \geq \frac{a+b+c}{2}.$$

$$2(x+y+z)\left[\frac{a^2}{y+z}+\frac{b^2}{x+z}+\frac{c^2}{x+\cancel{z}\,y}\right]$$

$$= (y+z+x+z+x+y)\left(\frac{a^2}{y+z}+\frac{b^2}{x+z}+\frac{c^2}{x+y}\right)$$

$$\geq (a+b+c)^2$$

But $x+y+z = a+b+c$

$$\therefore \frac{a^2}{y+z}+\frac{b^2}{x+z}+\frac{c^2}{x+y} \geq \frac{a+b+c}{2}.$$

10 (NSBHS, Mar 19):

(a) $(a-b)^2 \geq 0 \quad \therefore \quad a^2 - 2ab + b^2 \geq 0 \quad \therefore \quad a^2 + b^2 \geq 2ab$

(b) $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

from (a) $\quad a^2+b^2 \geq 2ab$

Similarly $\quad b^2+c^2 \geq 2bc$

$c^2+a^2 \geq 2ac$

$$a^2+b^2+b^2+c^2+c^2+a^2 \geq 2ab+2bc+2ac$$

$$\therefore \quad 2(a^2+b^2+c^2) \geq 2(ab+bc+ac)$$

$$\therefore \quad a^2+b^2+c^2 \geq ab+bc+ac$$

$$\therefore \quad a^2+b^2+c^2-ab-bc-ac \geq 0$$

but $\quad a+b+c \geq 0$

$$\therefore \quad (a+b+c)(a^2+b^2+c^2-ab-bc-ac) \geq 0$$

$$\therefore \quad a^3+b^3+c^3-3abc \geq 0$$

$$\therefore \quad \frac{a^3+b^3+c^3}{3} \geq abc.$$

(c) From part (b), let $c=1$

$$\frac{a^3+b^3+1^3}{3} \geq ab(1)$$

$$\therefore \quad \frac{a^3+b^3+1}{3} \geq ab$$

(d) From (c) $\quad \frac{a^3+b^3+1}{3} \geq ab$

Similarly $\quad \frac{a^3+c^3+1}{3} \geq ac$

and $\quad \frac{b^3+c^3+1}{3} \geq bc$

From (b) $\quad \frac{a^3+b^3+c^3}{3} \geq abc$

$$\frac{a^3+b^3+c^3}{3} + \frac{a^3+b^3+1}{3} + \frac{a^3+c^3+1}{3} + \frac{b^3+c^3+1}{3} \geq abc+ab+ac+bc$$

$$\frac{3(a^3+b^3+c^3+1)}{3} \geq abc+ab+ac+bc \quad \therefore \quad a^3+b^3+c^3+1 \geq abc+ab+bc+ca$$

11. (a) $(a-b)^2 \geq 0 \therefore a^2-2ab+b^2 \geq 0 \therefore a^2+b^2 \geq 2ab$

(b) $(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

$= a^3+ab^2+ac^2-a^2b-abc-ca^2+$
$ba^2+b^3+bc^2-ab^2-b^2c-abc+$
$ca^2+cb^2+c^3-abc-bc^2-c^2a$

$= a^3+b^3+c^3-3abc$

(c) Use (a) $\therefore a^2+b^2 \geq 2ab$

Similarly $a^2+c^2 \geq 2ac$

$b^2+c^2 \geq 2bc$

$a^2+b^2+b^2+c^2+c^2+a^2 \geq 2ab+2bc+2ac$

$\therefore 2(a^2+b^2+c^2) \geq 2(ab+bc+ca)$

$\therefore a^2+b^2+c^2 \geq ab+bc+ca$

(d) From (c) $a^2+b^2+c^2 \geq ab+bc+ca$

$\therefore a^2+b^2+c^2-ab-bc-ca \geq 0$

and $a+b+c \geq 0$

Use (b) $\therefore a^3+b^3+c^3-3abc \geq 0$

$\therefore a^3+b^3+c^3 \geq 3abc$

(e) $2(a^3+b^3+c^3)+\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}$

$= a^3+b^3+c^3+(a^3+b^3+c^3)+\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}$

$= a^3+b^3+c^3+\left(a^3+\frac{1}{a^3}\right)+\left(b^3+\frac{1}{b^3}\right)+\left(c^3+\frac{1}{c^3}\right)$

Use (a) for a^3 and $\frac{1}{a^3}$ $\therefore a^3+\frac{1}{a^3} \geq 2\sqrt{a^3 \times \frac{1}{a^3}} = 2$

Similarly $b^3+\frac{1}{b^3} \geq 2$ and $c^3+\frac{1}{c^3} \geq 2$

Use (d) $a^3+b^3+c^3 \geq 3abc$

Now

$2(a^3+b^3+c^3)+\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3} = a^3+b^3+c^3+\left(a^3+\frac{1}{a^3}\right)+\left(b^3+\frac{1}{b^3}\right)+\left(c^3+\frac{1}{c^3}\right)$

$\geq 3abc+2+2+2$

$= 3abc+6.$

(f) $a+b+c+a^2+b^2+c^2+a^3+b^3+c^3 \geq 3(ab+bc+ca)$

$a+b+c+a^3+b^3+c^3 = a+a^3+b+b^3+c+c^3$

$a+a^3 \geq 2\sqrt{aa^3} = 2a^2$

$b+b^3 \geq 2b^2$

$c+c^3 \geq 2c^2$

$\therefore a+b+c+a^3+b^3+c^3 \geq 2(a^2+b^2+c^2)$

$\therefore a+b+c+a^2+b^2+c^2+a^3+b^3+c^3 \geq 2(a^2+b^2+c^2)+a^2+b^2+c^2$

$= 3(a^2+b^2+c^2)$

$\geq 3(ab+bc+ca)$.

---

12. $a^3+b^3+a^3 \geq 3a^2b$

$b^3+c^3+b^3 \geq 3b^2c$

$a^3+c^3+c^3 \geq 3ac^2$

$3a^3+3b^3+3c^3 \geq 3(a^2b+b^2c+c^2a)$

$\therefore a^3+b^3+c^3 \geq a^2b+b^2c+c^2a$

$(ab)^3+(ac)^3+(ab)^3 \geq 3a^3b^2c = 3a^2b$ (since $abc=1$)

Similarly $(bc)^3+(ac)^3+(ac)^3 \geq 3a^2bc^3 = 3ac^2$ (since $abc=1$)

and $(bc)^3+(bc)^3+(ab)^3 \geq 3b^3c^2a = 3b^2c$ (since $abc=1$)

$\therefore 3(ab)^3+3(bc)^3+3(ca)^3 \geq 3(a^2b+b^2c+ac^2)$

$\therefore (ab)^3+(bc)^3+(ca)^3 \geq a^2b+b^2c+ac^2$

$\therefore a^3+b^3+c^3+(ab)^3+(bc)^3+(ca)^3 \geq a^2b+b^2c+c^2a+a^2b+b^2c+ac^2$

$= 2(a^2b+b^2c+c^2a)$.

---

13. $$\frac{a^2}{b^2}+\frac{b^2}{c^2} \geqslant 2\sqrt{\frac{a^2}{b^2}\times\frac{b^2}{c^2}} = \frac{2a}{c}$$

$$\frac{a^2}{b^2}+\frac{c^2}{a^2} \geqslant 2\frac{c}{b}$$

$$\frac{b^2}{c^2}+\frac{c^2}{a^2} \geqslant 2\frac{b}{a}$$

$$2\frac{a^2}{b^2}+2\frac{b^2}{c^2}+2\frac{c^2}{a^2} \geqslant 2\frac{b}{a}+2\frac{a}{c}+2\frac{c}{b}$$

$$\therefore \frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2} \geqslant \frac{b}{a}+\frac{c}{b}+\frac{a}{c}.$$

---

14. $$\frac{1}{a^2}+\frac{1}{b^2} \geqslant 2\sqrt{\frac{1}{a^2}\frac{1}{b^2}} = \frac{2}{ab} = \frac{2c}{abc}$$

$$\frac{1}{a^2}+\frac{1}{c^2} \geqslant \frac{2}{ac} = \frac{2b}{abc}$$

$$\frac{1}{b^2}+\frac{1}{c^2} \geqslant \frac{2}{bc} = \frac{2a}{abc}$$

$$\frac{2}{a^2}+\frac{2}{b^2}+\frac{2}{c^2} \geqslant \frac{2a}{abc}+\frac{2b}{abc}+\frac{2c}{abc}$$

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \geqslant \frac{a+b+c}{abc}.$$

---

15. (a) $(a^2+1)(b^2+1) = a^2b^2 + a^2 + b^2 + 1$

$a^2b^2+1 \geq 2\sqrt{a^2b^2} = 2ab$

$$\begin{aligned}(a^2+1)(b^2+1) &= a^2b^2 + a^2 + b^2 + 1 \\ &\geq a^2 + b^2 + 2ab \\ &= (a+b)^2.\end{aligned}$$

(b) $(a^2+1)(b^2+1) = a^2b^2 + a^2 + b^2 + 1$

$a^2 + b^2 \geq 2ab$

$$\begin{aligned}(a^2+1)(b^2+1) &= a^2b^2 + a^2 + b^2 + 1 \\ &\geq a^2b^2 + 2ab + 1 \\ &= (ab+1)^2.\end{aligned}$$

(c) $(a^2+b^2)(c^2+d^2) = a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2$

$a^2d^2 + b^2c^2 \geq 2\sqrt{a^2d^2b^2c^2} = 2abcd$

$$\begin{aligned}(a^2+b^2)(c^2+d^2) &= a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2 \\ &\geq a^2c^2 + b^2d^2 + 2abcd \\ &= (ac+bd)^2.\end{aligned}$$

(d)

$$\begin{aligned}&(a^2+b^2+c^2)(x^2+y^2+z^2) \\ &= (a^2+b^2)(x^2+y^2) + z^2(a^2+b^2) + c^2(x^2+y^2) + c^2z^2 \\ &= (a^2+b^2)(x^2+y^2) + (a^2z^2+c^2x^2) + (b^2z^2+c^2y^2) + c^2z^2 \\ &\geq (ax+by)^2 + 2aczx + 2bczy + c^2z^2 \\ &= (ax+by)^2 + 2cz(ax+by) + c^2z^2 \\ &= (ax+by+cz)^2.\end{aligned}$$

HSC inequalities questions

1. (2015, 15c)

(a) Given that $\frac{x+y}{2} \geq \sqrt{xy}$ (1)

Sub x^2 for x and y^2 for y in (1) we obtain

$$\frac{x^2+y^2}{2} \geq \sqrt{x^2y^2} = xy$$

Take the square root of both sides, we obtain

$$\sqrt{\frac{x^2+y^2}{2}} \geq \sqrt{xy}$$

(b) Use (a) for a and b, we obtain

$$\sqrt{\frac{a^2+b^2}{2}} \geq \sqrt{ab} \quad \therefore \quad \frac{a^2+b^2}{2} \geq ab$$

Similarly $\sqrt{\frac{c^2+d^2}{2}} \geq \sqrt{cd} \quad \therefore \quad \frac{c^2+d^2}{2} \geq cd$

$$\frac{a^2+b^2+c^2+d^2}{4} = \frac{1}{2}\left(\left(\frac{a^2+b^2}{2}\right) + \left(\frac{c^2+d^2}{2}\right)\right)$$

$$\geq \frac{1}{2}(ab+cd)$$

$$\geq \sqrt{abcd} \quad \text{using (part a)}$$

$$\therefore \sqrt{\frac{a^2+b^2+c^2+d^2}{4}} \geq \sqrt{\sqrt{abcd}} = \sqrt[4]{abcd}.$$

2. (2014, 15a)

$$a+b+c=1$$

$$\therefore \quad (a+b+c)^2=1$$

$$\therefore \quad 1=a^2+b^2+c^2+2ab+2ac+2bc$$

$$a\le b \quad \therefore \quad a^2\le ab$$

$$a\le c \quad \therefore \quad a^2\le ac$$

$$b\le c \quad \therefore \quad b^2\le bc$$

$$1=a^2+b^2+c^2+2ab+2ac+2bc$$

$$\geq a^2+b^2+c^2+2a^2+2a^2+2b^2$$

$$=5a^2+3b^2+c^2.$$

---

3. (2012, 15a)

(a) $(\sqrt{a}-\sqrt{b})^2\geq 0 \quad \therefore \quad a-2\sqrt{ab}+b\geq 0$

$$\therefore \quad a+b\geq 2\sqrt{ab}$$

$$\therefore \quad \frac{a+b}{2}\geq \sqrt{ab}.$$

(b)

$$1\le x\le y \quad \therefore \quad x-1\geq 0$$

and $y-x\geq 0$

$$\therefore \quad (x-1)(y-x)\geq 0$$

$$xy-x^2-y+x\geq 0$$

$$xy-x^2+x\geq y$$

$$x(y-x+1)\geq y.$$

(c) $1 \le j \le n$

from (b) $\therefore$ $j(n-j+1) \geq n$

$\therefore \sqrt{j(n-j+1)} \geq \sqrt{n}$

$\frac{n+1}{2} \geq \sqrt{n \times 1} = \sqrt{n}$ from (a)

$$\frac{n+1}{2} = \frac{(n-j+1)+j}{2} \geq \sqrt{j(n-j+1)}.$$

(d) $\frac{n+1}{2} \geq \sqrt{n}$ $\therefore$ $\left(\frac{n+1}{2}\right)^n \geq (\sqrt{n})^n$

using $\frac{n+1}{2} \geq \sqrt{j(n-j+1)}$ for $j=1 \cdots, n$, we obtain

$\frac{n+1}{2} \geq \sqrt{1(n-1+1)} = \sqrt{n}$

$\frac{n+1}{2} \geq \sqrt{2(n-2+1)} = \sqrt{2(n-1)}$

$\vdots$

$\frac{n+1}{2} \geq \sqrt{n(n-n+1)} = \sqrt{n}$

$$\left(\frac{n+1}{2}\right)^n \geq \sqrt{1 \cdot n \times 2 \cdot (n-1) \times 3(n-2) \times \cdots \times n \cdot 1} = \sqrt{(n!)^2} = n!$$

use (c) $\sqrt{j(n-j+1)} \geq \sqrt{n}$ for $j = 1, \cdots, n$

$$\prod_{j=1}^{n} \sqrt{j(n-j+1)} \geq \prod_{j=1}^{n} \sqrt{n}$$

$$n! \geq (\sqrt{n})^n.$$

4. (2011, 5b) $p+q \geq r$

$$\frac{p}{1+p} + \frac{q}{1+q} - \frac{r}{1+r}$$

$$= \frac{p(1+q)(1+r) + q(1+p)(1+r) - r(1+p)(1+q)}{(1+p)(1+q)(1+r)}$$

$$= \frac{p(1+q+r+qr) + q(1+p+r+pr) - r(1+p+q+pq)}{(1+p)(1+q)(1+r)}$$

$$= \frac{p+pq+pr+pqr+q+qp+qr+pqr-r-rp-rq-rpq}{(1+p)(1+q)(1+r)}$$

$$= \frac{p+q-r+2pq+pqr}{(1+p)(1+q)(1+r)}$$

Give p, q, r all positive and $p+q-r \geq 0$

$$\therefore \frac{p+q-r+2pq+pqr}{(1+p)(1+q)(1+r)} \geq 0$$

$$\therefore \frac{p}{1+p} + \frac{q}{1+q} - \frac{r}{1+r} \geq 0.$$

---

5. (2004, 7a)

(a) $\left(\sqrt{a} - \frac{1}{\sqrt{a}}\right)^2 \geq 0 \therefore a - 2 \times \sqrt{a} \times \frac{1}{\sqrt{a}} + \frac{1}{a} \geq 0$

$\therefore a + \frac{1}{a} \geq 2.$

(b) $(a_1 + \cdots + a_n)\left(\frac{1}{a_1} + \cdots + \frac{1}{a_n}\right) \geq n^2$

prove by induction.

for $n=1$, $a_1 \times \frac{1}{a_1} \geq 1$

Assume it is true for $n=k$

$$(a_1+a_2+\dots+a_k)\left(\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_k}\right) \geq k^2 \qquad (*)$$

prove it true for $n=k+1$

ie: $$(a_1+\dots+a_k+a_{k+1})\left(\frac{1}{a_1}+\dots+\frac{1}{a_k}+\frac{1}{a_{k+1}}\right) \geq (k+1)^2$$

$$(a_1+\dots+a_k+a_{k+1})\left(\frac{1}{a_1}+\dots+\frac{1}{a_k}+\frac{1}{a_{k+1}}\right)$$

$$= (a_1+\dots+a_k)\left(\frac{1}{a_1}+\dots+\frac{1}{a_k}\right)+$$
$$(a_1+\dots+a_k)\frac{1}{a_{k+1}} + a_{k+1}\left(\frac{1}{a_1}+\dots+\frac{1}{a_k}\right)+a_{k+1}\times\frac{1}{a_{k+1}}$$

$$\geq k^2+\left(\frac{a_1}{a_{k+1}}+\frac{a_{k+1}}{a_1}\right)+\left(\frac{a_2}{a_{k+1}}+\frac{a_{k+1}}{a_2}\right)+\dots+\left(\frac{a_k}{a_{k+1}}+\frac{a_{k+1}}{a_k}\right)+1$$

$$\geq k^2+\underbrace{2+2+\dots+2}_{k \text{ times}}+1 = k^2+2k+1 = (k+1)^2$$

Hence by mathematical induction, it is true for all $n \geq 1$.

(c) use $(a_1+a_2+a_3)\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\right) \geq 3^2$

$$(\operatorname{cosec}^2\theta+\sec^2\theta+\cot^2\theta)(\sin^2\theta+\cos^2\theta+\tan^2\theta) \geq 9$$

$$(\operatorname{cosec}^2\theta+\sec^2\theta+\cot^2\theta)(1+\tan^2\theta) \geq 9$$

$$(\operatorname{cosec}^2\theta+\sec^2\theta+\cot^2\theta)\sec^2\theta \geq 9$$

$$\operatorname{cosec}^2\theta+\sec^2\theta+\cot^2\theta \geq \frac{9}{\sec^2\theta} = 9\cos^2\theta.$$

6. (2003, 6c)

(a) $(\sqrt{x}-\sqrt{y})^2 \geq 0 \therefore x - 2\sqrt{xy} + y \geq 0$

$$\frac{x+y}{2} \geq \sqrt{xy}.$$

(b) $a^4 + b^4 \geq 2\sqrt{a^4b^4} = 2a^2b^2$ (using (a))

similarly $a^4+c^4 \geq 2a^2c^2$

$b^4+c^4 \geq 2b^2c^2$

$$a^4+b^4+a^4+c^4+b^4+c^4 \geq 2a^2b^2+2a^2c^2+2b^2c^2$$

$$2(a^4+b^4+c^4) \geq 2(a^2b^2+a^2c^2+b^2c^2)$$

$$\therefore \quad a^4+b^4+c^4 \geq a^2b^2+a^2c^2+b^2c^2$$

(c) $a^2b^2 + a^2c^2 \geq 2\sqrt{a^2b^2a^2c^2} = 2a^2bc$

similarly $a^2b^2+b^2c^2 \geq 2b^2ac$

$a^2c^2+b^2c^2 \geq 2c^2ab$

$$a^2b^2+a^2c^2+a^2b^2+b^2c^2+a^2c^2+b^2c^2 \geq 2a^2bc+2b^2ac+2c^2ab$$

$$2(a^2b^2+b^2c^2+a^2c^2) \geq 2(a^2bc+b^2ac+c^2ab)$$

$$a^2b^2+b^2c^2+a^2c^2 \geq a^2bc+b^2ac+c^2ab$$

(d) from (b) and (c) we obtain

$$a^4+b^4+c^4 \geq a^2b^2+a^2c^2+b^2c^2 \geq a^2bc+b^2ac+c^2ab$$

$$= abc(a+b+c)$$

$$= abcd$$

since $a+b+c=d$ (given).

7. (2001, 8a)

(a) $(a-b)^2 \geqslant 0 \therefore a^2 - 2ab + b^2 \geqslant 0$

$\therefore a^2 + b^2 \geqslant 2ab$

$$a^2 + b^2 \geqslant 2ab$$

Similarly $a^2 + c^2 \geqslant 2ac$

$$b^2 + c^2 \geqslant 2bc$$

$\therefore a^2 + b^2 + a^2 + c^2 + b^2 + c^2 \geqslant 2ab + 2ac + 2bc$

$$2(a^2 + b^2 + c^2) \geqslant 2(ab + ac + bc)$$

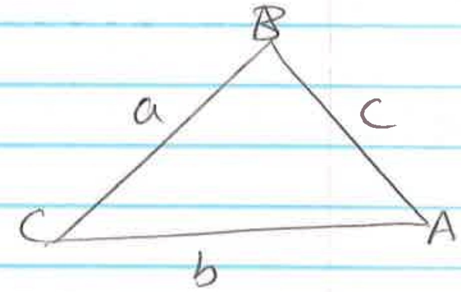
$\therefore a^2 + b^2 + c^2 \geqslant ab + ac + bc$

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$$

$$\geqslant ab + ac + bc + 2ab + 2ac + 2bc$$

$$= 3(ab + bc + ca).$$

(b) In ΔABC



$$b \leq a + c$$

$\therefore b - c \leq a$

$\therefore (b-c)^2 \leq a^2$

$\therefore (b-c)^2 + 4bc \leq a^2 + 4bc$

$\therefore (b+c)^2 \leq a^2 + 4bc$

Similarly $(a+c)^2 \leq b^2 + 4ac$

and $(a+b)^2 \leq c^2 + 4ab$

$$(a+b)^2 + (b+c)^2 + (c+a)^2 \leq c^2 + 4ab + a^2 + 4bc + b^2 + 4ac$$

$$a^2 + 2ab + b^2 + b^2 + 2bc + c^2 + c^2 + 2ac + a^2 \leq a^2 + b^2 + c^2 + 4(ab + bc + ca)$$

$$a^2 + b^2 + c^2 + 2ab + 2bc + 2ac + a^2 + b^2 + c^2 \leq a^2 + b^2 + c^2 + 4(ab + bc + ca)$$

$$a^2 + b^2 + c^2 + 2ab + 2bc + 2ac \leq 4(ab + bc + ca)$$

6. Trial exams inequality problems

1. NSBHS 2019

(a) $(a-bc)^2 \geq 0 \quad \therefore \quad a^2 - 2abc + b^2c^2 \geq 0.$

$\therefore \quad a^2 + (bc)^2 \geq 2abc$

(b) $(a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0$

$\therefore \quad a^2 - 2ab + b^2 + b^2 - 2bc + c^2 + c^2 - 2ac + a^2 \geq 0$

$\therefore \quad a^2 + b^2 + c^2 \geq ab + bc + ca$

(c) $a^2(1+b^2) + b^2(1+c^2) + c^2(1+a^2)$

$= a^2 + a^2b^2 + b^2 + b^2c^2 + c^2 + a^2c^2$

$= (a^2 + b^2c^2) + (b^2 + a^2c^2) + (c^2 + a^2b^2)$

$\geq 2abc + 2bac + 2cab$ (using (a))

$= 6abc$

(d) $a^2(1+a^2) + b^2(1+b^2) + c^2(1+c^2)$

$= a^2 + a^4 + b^2 + b^4 + c^2 + c^4$

$= a^2 + b^2 + c^2 + a^4 + b^4 + c^4$

$\geq a^2 + b^2 + c^2 + a^2b^2 + b^2c^2 + c^2a^2$ (using (b))

$\geq 6abc$ (using (c)).

2. NSGHS 2018, 15a

(a) $\left.\begin{array}{l} c \geq a \;\therefore\; (c-a) \geq 0 \\ d \geq b \;\therefore\; (d-b) \geq 0 \end{array}\right\} (c-a)(d-b) \geq 0$

$\therefore$

$cd - cb - ad + ab \geq 0$

$ab + cd \geq bc + ad$

(b) $x \geq y, \; x \geq y$ use (a) $\therefore$

$x^2 + y^2 \geq xy + xy = 2xy.$

(c) The result is symmetric in x and y, so it does not matter which is larger.

3. (CSSA 2018, 13d)

(a) $(a+b-c)(a+c-b)$

$= (a+b-c)(a-(b-c))$

$= (a^2-(b-c)^2)$

$\leq a^2$.

(b) $a^2 \geq (a+b-c)(a+c-b)$

$b^2 \geq (b+a-c)(b+c-a)$

$c^2 \geq (c+a-b)(c+b-a)$

$a^2b^2c^2 \geq (a+b-c)(a+c-b)(b+a-c)(b+c-a)(c+a-b)(c+b-a)$

$= (a+b-c)^2(b+c-a)^2(a+c-b)^2$

$(abc)^2 \geq [(a+b-c)(b+c-a)(a+c-b)]^2$

$\therefore \; abc \geq (a+b-c)(b+c-a)(a+c-b)$.

4. BBHS 2018, 16b

(a) $(\sqrt{m}-\sqrt{n})^2 \geq 0 \quad \therefore \; m - 2\sqrt{m}\sqrt{n} + n \geq 0$

$\therefore \; m+n \geq 2\sqrt{mn}$.

(b) $\left.\begin{array}{l} m+n \geq 2\sqrt{mn} \\ n+p \geq 2\sqrt{np} \\ p+m \geq 2\sqrt{pm} \end{array}\right\} (m+n)(n+p)(p+m) \geq 2\sqrt{mn}\,2\sqrt{np}\,2\sqrt{pm}$

$= 8mnp$.

(c) $\frac{m}{n} + \frac{n}{p} \geq 2\sqrt{\frac{m}{n}}\sqrt{\frac{n}{p}} = 2\sqrt{\frac{m}{p}}$

$\frac{p}{q} + \frac{q}{m} \geq 2\sqrt{\frac{p}{q}}\sqrt{\frac{q}{m}} = 2\sqrt{\frac{p}{m}}$

$\frac{m}{n} + \frac{n}{p} + \frac{p}{q} + \frac{q}{m} \geq 2\sqrt{\frac{m}{p}} + 2\sqrt{\frac{p}{m}}$

$= 2\left(\sqrt{\frac{m}{p}} + \sqrt{\frac{p}{m}}\right)$

$\geq 2 \times 2\sqrt{\sqrt{\frac{p}{m}} \times \sqrt{\frac{m}{p}}} = 4\sqrt{\sqrt{\frac{mp}{mp}}} = 4$.

5. (Caringbah HS 2018, 15c)

$$(a+b)(a-b)^2 \geq 0$$
$$(a+b)(a^2-2ab+b^2) \geq 0$$
$$a^3-2a^2b+ab^2+ba^2-2ab^2+b^3 \geq 0$$
$$a^3+b^3-a^2b-ab^2 \geq 0$$
$$a^3+b^3 \geq a^2b+ab^2.$$

6. (Hornsby Girls 2018, 16b)

(a) $(\sqrt{a}-\sqrt{b})^2 \geq 0$

$a-2\sqrt{ab}+b \geq 0 \quad \therefore \quad a+b \geq 2\sqrt{ab}$

$\therefore \quad \frac{a+b}{2} \geq \sqrt{ab}.$

(b) $n! = 1 \times 2 \times 3 \times \cdots \times n$

$$(n!)^{1/n} = (1\times 2\times 3\times \cdots \times n)^{1/n} \leq \frac{1+2+3+\cdots+n}{n}$$
$$= \frac{n(n+1)}{2n} = \frac{n+1}{2}$$

$$\therefore \quad n! \leq \left(\frac{n+1}{2}\right)^n.$$

7. (JRAHS 2018, 12c)

(a) $(a-b)^2 \geq 0 \quad \therefore \quad a^2-2ab+b^2 \geq 0 \quad \therefore \quad a^2+b^2 \geq 2ab$

(b) $\left(\sqrt{\frac{x}{y}}-\sqrt{\frac{y}{x}}\right)^2 \geq 0 \quad \therefore \quad \frac{x}{y}-2\sqrt{\frac{xy}{yx}}+\frac{y}{x} \geq 0$

$\therefore \quad \frac{x}{y}+\frac{y}{x} \geq 2$

(c) $x_1 \times \frac{1}{x_1} \geq 1 = 1^2$ true for $n=1$

Assume it is true for $n=k$

$$(x_1+x_2+\cdots+x_k)\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_k}\right) \geq k^2 \quad (*)$$

prove it for $n=k+1$

$$(x_1+x_2+\cdots+x_k+x_{k+1})\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_k}+\frac{1}{x_{k+1}}\right) \geq (k+1)^2 ?$$

$$(x_1+x_2+\cdots+x_k+x_{k+1})\left(\frac{1}{x_1}+\cdots+\frac{1}{x_k}+\frac{1}{x_{k+1}}\right)$$
$$= (x_1+\cdots+x_k)\left(\frac{1}{x_1}+\cdots+\frac{1}{x_k}\right)+x_{k+1}\left(\frac{1}{x_1}+\cdots+\frac{1}{x_k}+\frac{1}{x_{k+1}}\right)+(x_1+\cdots+x_k)\frac{1}{x_{k+1}}$$
$$= (x_1+\cdots+x_k)\left(\frac{1}{x_1}+\cdots+\frac{1}{x_k}\right)+x_{k+1}\left(\frac{1}{x_1}+\cdots+\frac{1}{x_k}\right)+(x_1+\cdots+x_k)\frac{1}{x_{k+1}}+1$$

→ Continue (c)

$$= (x_1 + \cdots + x_k)\left(\frac{1}{x_1} + \cdots + \frac{1}{x_k}\right) + \left(\frac{x_1}{x_{k+1}} + \frac{x_{k+1}}{x_1}\right) + \left(\frac{x_2}{x_{k+1}} + \frac{x_{k+1}}{x_2}\right) + \cdots + \left(\frac{x_k}{x_{k+1}} + \frac{x_{k+1}}{x_k}\right) + 1$$

$$\geq K^2 + \underbrace{2 + 2 + \cdots + 2}_{K \text{ times.}} + 1 \qquad \text{using } (*) \text{ and using (b)}$$

$$= K^2 + 2K + 1 = (K+1)^2$$

Hence By mathematical induction it is true for all $n \geq 1$.

8. NSBHS 2018, 16b

$a+b+c=1$ and $a+b+c \geq 3\sqrt[3]{abc}$

(a) $\frac{1}{xy} + x + y \geq 3\sqrt[3]{\frac{1}{xy}xy} = 3$

(b) $\frac{1}{a(a+1)} + a + a+1 \geq 3$ using (a)

similarly

$\frac{1}{b(b+1)} + b + b+1 \geq 3$

$\frac{1}{c(c+1)} + c + c+1 \geq 3$

$\frac{1}{a(a+1)} + \frac{1}{b(b+1)} + \frac{1}{c(c+1)} + 2(a+b+c) + 3 \geq 3+3+3$

$\frac{1}{a(a+1)} + \frac{1}{b(b+1)} + \frac{1}{c(c+1)} \geq 9-5 = 4$ since $(a+b+c=1)$.

9. (Ind 2017, 12c)

(a) $a^2+b^2+c^2 \geq ab+bc+ca$

$a^2+b^2+d^2 \geq ab+ad+bd$

$a^2+c^2+d^2 \geq ac+ad+cd$

$b^2+c^2+d^2 \geq bc+bd+cd$

$3(a^2+b^2+c^2+d^2) \geq 2(ab+bc+ca+ad+bd+cd)$

$a^2+b^2+c^2+d^2 \geq \frac{2}{3}(ab+ac+ad+bc+bd+cd)$

(b) $$\left(\frac{a+b+c+d}{4}\right)^2 = \frac{(a+b+c+d)^2}{16}$$

$$= \frac{a^2+b^2+c^2+d^2+2(ab+ac+ad+bc+bd+cd)}{16}$$

$$\geq \frac{1}{16}\times\frac{2}{3}(ab+ac+ad+bc+bd+cd)+\frac{2}{16}(ab+ac+ad+bc+bd+cd)$$

$$= \frac{1}{6}(ab+ac+ad+bc+bd+cd).$$

10. (JRAHS 2017, 15c).

$$\frac{1}{3}(x+y+z) \geq (xyz)^{1/3}$$

(a) $$\frac{xy+yz+zx}{3} \geq (xyyzzx)^{1/3} = (xyz)^{2/3}$$

$$\therefore \quad xy+yz+zx \geq 3(xyz)^{2/3}$$

(b) $$a^3 = 1+(x+y+z)+(xy+yz+zx)+xyz$$

$$\geq 1+3(xyz)^{1/3}+3(xyz)^{2/3}+xyz$$

$$= \left(1+(xyz)^{1/3}\right)^3$$

$$a \geq 1+(xyz)^{1/3} \quad \therefore (xyz)^{1/3} \leq a-1$$

$$\therefore \quad xyz \leq (a-1)^3.$$

11. Hornsby GHS 2017, 16b

(a) $$\left(\frac{a}{2}-\frac{b}{2}\right)^2 \geq 0 \quad \therefore \quad \frac{a^2}{4}-\frac{2ab}{4}+\frac{b^2}{4} \geq 0$$

$$\frac{a^2}{4}+\frac{b^2}{4} \geq \frac{2ab}{4} \qquad (1)$$

$$\frac{a^2}{4}+\frac{b^2}{4} \geq \frac{a^2}{4}+\frac{b^2}{4} \qquad (2)$$

$$(1)+(2) \quad \therefore \quad \frac{a^2}{2}+\frac{b^2}{2} \geq \left(\frac{a+b}{2}\right)^2$$

$$a+b \geq 2\sqrt{ab} \quad \therefore \quad \frac{a+b}{2} \geq \sqrt{ab} \quad \therefore \quad \left(\frac{a+b}{2}\right)^2 \geq ab.$$

$$\therefore \quad ab \leq \left(\frac{a+b}{2}\right)^2 \leq \frac{a^2+b^2}{2}.$$

(b) $a+b=1$

$$ab \le \left(\frac{1}{2}\right)^2 = \frac{1}{4} \quad \therefore \quad \frac{1}{ab} \geq 4$$

$$\left(a+\frac{1}{a}\right)^2 + \left(b+\frac{1}{b}\right)^2 \geq 2\left(\frac{a+\frac{1}{a}+b+\frac{1}{b}}{2}\right)^2$$

$$= 2\left(\frac{a+b+\frac{a+b}{ab}}{2}\right)^2$$

$$= 2\left(\frac{1+4}{2}\right)^2 = \frac{25}{2}.$$

12. Ascham 2016, 12b

(a) $\frac{a}{b}+\frac{b}{a} \geq 2\sqrt{\frac{a}{b}\frac{b}{a}} = 2.$

(b)
$$\left.\begin{aligned} p^2+q^2 &\geq 2pq \\ p^2+r^2 &\geq 2pr \\ q^2+r^2 &\geq 2qr \end{aligned}\right\} \quad p^2+q^2+r^2 \geq pq+qr+rp.$$

(c)
$$\begin{aligned} p^3+q^3 &= (p+q)(p^2+q^2-pq) \\ &\geq (p+q)(2pq-pq) \\ &= pq(p+q) \end{aligned}$$

(d)
$$\begin{aligned} p^3+q^3 &\geq pq(p+q) \\ p^3+r^3 &\geq pr(p+r) \\ q^3+r^3 &\geq qr(q+r) \end{aligned}$$

$$2(p^3+q^3+r^3) \geq pq(p+q)+pr(p+r)+qr(q+r)$$

13. (Gringbah 2016, 16c) $\quad a+b \geq 2\sqrt{ab}$

$a+b+c=1$

$$\frac{1}{a}-1 = \frac{1-a}{a} = \frac{b+c}{a}, \quad \frac{1}{b}-1 = \frac{a+c}{b}, \quad \frac{1}{c}-1 = \frac{a+b}{c}$$

$$\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right) = \frac{b+c}{a}\times\frac{a+c}{b}\times\frac{a+b}{c} \geq \frac{2\sqrt{bc}}{a}\times\frac{2\sqrt{ac}}{b}\times\frac{2\sqrt{ab}}{c}$$

$$= 8.$$

14. Hunters Hill 2016, 14b

(a) $(a-b)^2 \geq 0 \quad \therefore \quad a^2 - 2ab + b^2 \geq 0 \quad \therefore \quad a^2 + b^2 \geq 2ab$

(b)
$$p+2 \geq 2\sqrt{2p}$$
$$q+2 \geq 2\sqrt{2q}$$
$$p+q \geq 2\sqrt{pq}$$
$$(p+2)(q+2)(p+q) \geq 8\sqrt{2p\,2q\,pq} = 16pq.$$

15. Ind 2016, 13b

(a) $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$

$$\left.\begin{array}{l} a^2+b^2 \geq 2ab \\ a^2+c^2 \geq 2ac \\ b^2+c^2 \geq 2bc \end{array}\right\} \quad a^2+b^2+c^2 \geq ab+bc+ca$$

$$\therefore \quad (a+b+c)^2 \geq 3(ab+bc+ca).$$

(b) use (a)

$$(ab+bc+ca)^2 \geq 3(ab^2c + bc^2a + a^2bc)$$
$$= 3abc(a+b+c)$$

(c) $(a+b+c)^3 = (a+b+c)(a+b+c)^2$
$$\geq 3(a+b+c)(ab+bc+ca)$$

(a) & (b) $\therefore \left[(a+b+c)(ab+bc+ca)\right]^2 \geq 9\,abc\,(a+b+c)(ab+bc+ca)$

$$(a+b+c)(ab+bc+ca) \geq 9\,abc$$

$$(a+b+c)^3 \geq 3 \times 9abc = 27abc.$$

16. (Knox Grammar 2016, 13d)

(a) $(p+q)^2 = (p-q)^2 + 4pq \geq 4pq$

$p+q \geq 2\sqrt{pq}$

$\frac{p+q}{2} \geq \sqrt{pq}$

(b) $\left.\begin{array}{l} \frac{p+q}{2} \geq \sqrt{pq} \\ \frac{r+s}{2} \geq \sqrt{rs} \end{array}\right\}$ $\frac{\frac{p+q}{2} + \frac{r+s}{2}}{2} \geq \sqrt{\frac{p+q}{2} \times \frac{r+s}{2}}$

$\geq \sqrt{\sqrt{pq} \times \sqrt{rs}}$

$= \sqrt[4]{pqrs}$

$\therefore \quad \frac{p+q+r+s}{4} \geq \sqrt[4]{pqrs}$

17. (NSBHS 2016, 16a)

(a) $\frac{a+b}{2} \geq \sqrt{ab}$

$\frac{c+d}{2} \geq \sqrt{cd}$

$\frac{\frac{a+b}{2} + \frac{c+d}{2}}{2} \geq \frac{\sqrt{ab} + \sqrt{cd}}{2} \geq \sqrt{\sqrt{ab}\sqrt{cd}} = \sqrt[4]{abcd}$

(b) $\frac{a+b+c}{3} = \frac{1}{4}\left(a+b+c + \frac{a+b+c}{3}\right) \geq \sqrt[4]{abc\left(\frac{a+b+c}{3}\right)}$

$\therefore \left(\frac{a+b+c}{3}\right)^4 \geq abc\left(\frac{a+b+c}{3}\right)$

$\therefore \left(\frac{a+b+c}{3}\right)^3 \geq abc$

$\therefore \frac{a+b+c}{3} \geq \sqrt[3]{abc}$

18. (NSGHS 2016, 14C)

(a) $(a-b)^2 \geq 0 \therefore a^2-2ab+b^2 \geq 0 \therefore a^2+b^2 \geq 2ab$

(b) $a^2+b^2 \geq 2ab$

$a^2+c^2 \geq 2ac$

$b^2+c^2 \geq 2bc$

$\therefore a^2+b^2+a^2+c^2+b^2+c^2 \geq 2ab+2ac+2bc$

$\therefore 2(a^2+b^2+c^2) \geq 2(ab+ac+bc)$

$\therefore a^2+b^2+c^2 \geq ab+bc+ca$

(c) $x^3-3x^2+4x+k=0$

$\alpha+\beta+\gamma=3$

$\alpha\beta+\alpha\gamma+\beta\gamma=4$

$\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$

$= 3^2-2(4)=1$

But $\alpha^2+\beta^2+\gamma^2 \geq \alpha\beta+\beta\gamma+\gamma\alpha$ if α, β and γ are real (see part b)

$\therefore 1 \geq 4$ Contradiction

$\therefore$ α, β and γ Can not be all real.

19. (Penrith HS 2016, 14d)

If $0<x<y<\frac{1}{2}$

$(\sqrt{x}-\sqrt{y})^2 \geq 0 \therefore x+y-2\sqrt{xy} \geq 0 \therefore \frac{x+y}{2} \geq \sqrt{xy}$

$\therefore x+y > \frac{x+y}{2} \geq \sqrt{xy}$

$x \leq \frac{1}{2}$ and $y \leq \frac{1}{2} \therefore x+y \leq 1 \therefore \sqrt{x+y} \geq x+y$

as the square root of a number between 0 and 1 is greater than the number.

$\therefore \sqrt{xy} < x+y < \sqrt{x+y}$.

20. (Sam al Hosn 2016, 15a)

(a) $(a-b)^2+(a-c)^2+(b-c)^2 \geq 0$

$a^2-2ab+b^2+a^2-2ac+c^2+b^2-2bc+c^2 \geq 0$

$2(a^2+b^2+c^2)-2(ab+bc+ca) \geq 0$

$\therefore \quad a^2+b^2+c^2 \geq ab+bc+ca$

(b) $\frac{p^2}{q^2}+\frac{q^2}{r^2}+\frac{r^2}{p^2} \geq \frac{p}{q}\frac{q}{r}+\frac{p}{q}\frac{r}{p}+\frac{q}{r}\frac{r}{p}$

$= \frac{p}{r}+\frac{r}{q}+\frac{q}{p}$

$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ac)$

a, b, c are positive $\therefore a+b+c \geq 0$

and from (a) $a^2+b^2+c^2-ab-bc-ac \geq 0$.

$\therefore \quad a^3+b^3+c^3-3abc \geq 0$.

Let $a=\sqrt[3]{\frac{p}{r}}, \; b=\sqrt[3]{\frac{q}{p}}, \; c=\sqrt[3]{\frac{r}{q}}$

$\therefore \quad \frac{p}{r}+\frac{q}{p}+\frac{r}{q}-3\sqrt[3]{\frac{p}{r}\frac{q}{p}\frac{r}{q}} \geq 0$

$\therefore \quad \frac{p}{r}+\frac{q}{p}+\frac{r}{q} \geq 3$.

21. (western region 2016, 14d)

(a) $(x-y)^2 \geq 0 \therefore x^2-2xy+y^2 \geq 0 \therefore \quad x^2+y^2 \geq 2xy$

$$\left.\begin{array}{l} x^2+y^2 \geq 2xy \\ y^2+z^2 \geq 2yz \\ x^2+z^2 \geq 2xz \end{array}\right\} \quad \begin{array}{l} x^2+y^2+y^2+z^2+z^2+x^2 \geq 2xy+2xy+2yz \\ \therefore \; x^2+y^2+z^2 \geq xy+yz+zx. \end{array}$$

(b) $x+y+z=3$

$(x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx) \geq 3(xy+yz+zx)$

$9 \geq 3(xy+yz+zx) \therefore \quad xy+yz+zx \leq 3$.

22. (ACE 2015, 16b)

$$(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$$

but $ab+bc+ca \le a^2+b^2+c^2$

$$\therefore (a+b+c)^2 \le 3(a^2+b^2+c^2).$$

23. (Ind 2015, 16c)

(a) $a>b>c>1$

$$a^{a-b}b^{b-c} > c^{a-b}c^{b-c} = c^{a-c} \quad \text{since } a>b>c>1$$

$$(a^b b^c c^c)\, a^{a-b} b^{b-c} > a^b b^c c^c\, c^{a-c}$$

$$a^a b^b c^c > a^b b^c c^a$$

(b) since $f(x) = \ln(x)$ is monotonic increasing function,

$$\ln(a^a b^b c^c) > \ln(a^b b^c c^a)$$

$$\ln a^a + \ln b^b + \ln c^c > \ln a^b + \ln b^c + \ln c^a$$

$$\therefore a\ln a + b\ln b + c\ln c > b\ln a + c\ln b + a\ln c.$$

24. (Knox Grammar 2015, 15c)

(a) $a^2+b^2 \ge 2ab$, $a^2+c^2 \ge 2ac$, $a^2+d^2 \ge 2ad$

$b^2+c^2 \ge 2bc$, $b^2+d^2 \ge 2bd$ and $c^2+d^2 \ge 2cd$

$$3(a^2+b^2+c^2+d^2) \ge 2(ab+ac+bc+ad+bd+cd)$$

(b) $(a+b+c+d)^2 = a^2+b^2+c^2+d^2+2(ab+ac+bc+ad+bd+cd)$

$$\begin{aligned} 3(a+b+c+d)^2 &= 3(a^2+b^2+c^2+d^2)+6(ab+ac+bc+ad+bd+cd) \\ &\ge 2(ab+ac+bc+ad+bd+cd)+6(ab+ac+bc+ad+bd+cd) \\ &= 8(ab+ac+bc+ad+bd+cd) \end{aligned}$$

$$\therefore ab+ac+ad+bc+bd+cd \le \frac{3}{8}(a+b+c+d)^2 = \frac{3}{8}$$

since $a+b+c+d=1$.

25. (NSGHS 2015, 16b)

(a) $(a-b)^2 \geq 0 \quad \therefore \quad a^2 - 2ab + b^2 \geq 0 \quad \therefore \quad a^2 + b^2 \geq 2ab$

(b) $a^4 + b^4 \geq 2a^2b^2$

$c^4 + d^4 \geq 2c^2d^2$

$a^4 + b^4 + c^4 + d^4 \geq 2(a^2b^2 + c^2d^2)$

$\geq 2 \times 2\sqrt{a^2b^2c^2d^2} = 4abcd$

26. (Sam El Hosri 2015, 16c)

$a+b+c \geq 3\sqrt[3]{abc}$

$(p+2)(q+2)(r+2) = (p+2)(qr+2q+2r+4)$

$= pqr + 2pq + 2pr + 2qr + 4(p+q+r) + 8$

$64 = pqr + 2(pq+pr+qr) + 4(p+q+r) + 8$

$\geq pqr + 6\sqrt[3]{p^2q^2r^2} + 12\sqrt[3]{pqr} + 8$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Let $a = \sqrt[3]{pqr}$ and $b = 2$

$\left(\sqrt[3]{pqr} + 2\right)^3 = pqr + 6\sqrt[3]{p^2q^2r^2} + 12\sqrt[3]{pqr} + 8$

$\therefore \quad 64 \geq \left(\sqrt[3]{pqr} + 2\right)^3$

$\therefore \quad 4 \geq \sqrt[3]{pqr} + 2$

$\therefore \quad 2 \geq \sqrt[3]{pqr}$

$\therefore \quad pqr \leq 8.$

27. (Ind 2014, 15b)

(a) $\left(\sqrt{a} - \frac{1}{\sqrt{a}}\right)^2 \geq 0 \quad \therefore \quad a - 2\sqrt{a} \times \frac{1}{\sqrt{a}} + \frac{1}{a} \geq 0$

$\therefore \quad a + \frac{1}{a} \geq 2$

(b)(i) $\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c}$

$= \frac{b}{a} + \frac{c}{a} + \frac{c}{b} + \frac{a}{b} + \frac{a}{c} + \frac{b}{c}$

$= \left(\frac{b}{a} + \frac{a}{b}\right) + \left(\frac{c}{a} + \frac{a}{c}\right) + \left(\frac{c}{b} + \frac{b}{c}\right)$

$\geq 2 + 2 + 2$ (using part a)

$= 6.$

(ii) use part (i)

Replacing $a \to b+c$

$b \to c+a$

and $c \to a+b$

$$\frac{(c+a)+(a+b)}{b+c} + \frac{(a+b)+(b+c)}{c+a} + \frac{(b+c)+(c+a)}{a+b} \geq 6$$

$$1 + \frac{2a}{b+c} + 1 + \frac{2b}{c+a} + 1 + \frac{2c}{a+b} \geq 6$$

$$2\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right) \geq 3$$

$$\therefore \quad \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

28. (SBHS 2014, 14c)

(a) $(x+y)^2 = x^2+y^2+2xy \geq 2xy + 2xy = 4xy$

since $(x-y)^2 \geq 0 \therefore x^2+y^2 \geq 2xy$.

(b) $[(x+y)(z+w)]^2 = (x+y)^2(z+w)^2$

$\geq 4xy \times 4zw = 16\,xyzw$

(c) $$\frac{x+y+z+w}{4} = \frac{\frac{x+y}{2} + \frac{z+w}{2}}{2} \geq \frac{\sqrt{xy} + \sqrt{zw}}{2}$$ using (a)

$$\geq \sqrt{\sqrt{xy}\,\sqrt{zw}}$$

$$= \sqrt[4]{xyzw}$$

(d) $\frac{1}{4}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}\right) \geq \sqrt[4]{\frac{a}{b}\frac{b}{c}\frac{c}{d}\frac{d}{a}} = 1$ using (c)

$$\therefore \quad \frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geq 4.$$

29. (Ace 2013, 16c)

$a^2+b^2 = (a-b)^2 + 2ab \geq (a-b)^2 + 2$ since $ab>1$

$$\frac{a^2+b^2}{a-b} \geq a-b + \frac{2}{a-b}$$

$$\geq 2\sqrt{(a-b)\frac{2}{a-b}} = 2\sqrt{2}.$$

30. (a) $a^2-ab+b^2 = \frac{1}{4}(a+b)^2 + \frac{3}{4}(a-b)^2$ where

$\frac{3}{4}(a-b)^2 \geq 0$ since a and b are real.

$a^2-ab+b^2 \geq \left(\frac{a+b}{2}\right)^2$ with equality iff $a=b$.

(b) using Cosine rule

$c^2 = a^2+b^2-2ab\cos\angle BCA$

If $\angle BCA \geq 60$ $\therefore$ $\cos\angle BCA \leq \frac{1}{2}$

$\therefore$ $c^2 \geq a^2+b^2-2ab\times\frac{1}{2}$

$c^2 \geq a^2+b^2-ab$

with equality iff $\angle BCA = 60°$.

Using part (a)

$c \geq \sqrt{a^2-ab+b^2} \geq \frac{a+b}{2}$ with equality if

$\angle BCA = 60°$ and $a=b$ (equilateral triangle).

$\therefore$ $c^3 \geq \left(\frac{a+b}{2}\right)\left(a^2+b^2-ab\right) = \frac{a^3+b^3}{2}$

$\therefore$ $2c^3 \geq a^3+b^3$ with equality iff ΔABC is an equilateral triangle.

(31) (NHBHS 2013, 16b)

(a) $\left(\sqrt{k+2} - \sqrt{k+1}\right)^2 \geqslant 0$

$k+2+k+1-2\sqrt{(k+2)(k+1)} \geqslant 0$

$2k+3 \geqslant 2\sqrt{k+2}\,\sqrt{k+1}$.

(b) $1 \geq 2\sqrt{k+2}\,\sqrt{k+1} - 2(k+1)$ (use (a)).

$\frac{1}{\sqrt{k+1}} \geqslant 2\sqrt{k+2} - 2\sqrt{k+1}$

$\frac{1}{\sqrt{k+1}} \geqslant 2\left(\sqrt{k+2} - \sqrt{k+1}\right)$.

(c) $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} = \sum_{k=0}^{n-1} \frac{1}{\sqrt{k+1}}$

$\geqslant 2\sum_{k=0}^{n-1}\left(\sqrt{k+2} - \sqrt{k+1}\right)$

$= 2\left(\sqrt{2} - \sqrt{1} + \sqrt{3} - \sqrt{2} + \sqrt{4} - \sqrt{3} + \cdots + \sqrt{n+1} - \sqrt{n}\right)$

$= 2\left(\sqrt{n+1} - 1\right)$.

(32). (NSGHS 2013, 13a).

(a) $x^2+y^2 \geqslant 2xy > xy$

(b) $x+y = 3z$.

$(x+y)^2 = x^2+y^2+2xy \leq x^2+y^2+2(x^2+y^2)$

$= 3(x^2+y^2)$

$(3z)^2 \leq 3(x^2+y^2)$

$3z^2 \leq x^2+y^2$.

(33) (Penrith 2013, 15d)

(a) $x^2+x+1 \geq 0$

$\Delta = 1-4=-3<0$. $a>0$. concave up.

~~$x^2+x+1>0$ for all real x.~~

(b) Let $x = \frac{a}{b}$.

$\left(\frac{a}{b}\right)^2 + \frac{a}{b} + 1 \geq 0$

$a^2 + ab + b^2 \geq 0$.

(34) (Sam el Hosni 2013, 15a).

(a) $\left(\sqrt{a} - \frac{1}{\sqrt{a}}\right)^2 \geq 0 \Rightarrow a + \frac{1}{a} \geq 2$.

(b) $a+b+c=1$.

$$\left.\begin{aligned} a+\frac{1}{a} &\geq 2 \\ b+\frac{1}{b} &\geq 2 \\ c+\frac{1}{c} &\geq 2 \end{aligned}\right\} \quad \begin{aligned} &a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 6 \\ &\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 5 \text{ since } a+b+c=1. \end{aligned}$$

(35) (NSGHS 2012, 15c)

(a) $(a-b)^2 \geq 0 \therefore a^2+b^2 \geq 2ab$.

(b) $\tan^2\theta + \cos^2\theta + \operatorname{cosec}^2\theta$

$\tan^2\theta + \cos^2\theta \geq 2\tan\theta\cos\theta = 2\sin\theta$

$\tan^2\theta + \operatorname{cosec}^2\theta \geq 2\tan\theta\operatorname{cosec}\theta = 2\sec\theta$

$\cos^2\theta + \operatorname{cosec}^2\theta \geq 2\cos\theta\operatorname{cosec}\theta = 2\cot\theta$

$\therefore \tan^2\theta + \cos^2\theta + \operatorname{cosec}^2\theta \geq \sin\theta + \sec\theta + \cot\theta$.

36. (SBHS 2012, 15b)

(a) $(\sqrt{x}-\sqrt{y})^2 \geq 0 \quad \therefore \quad x - 2\sqrt{xy} + y \geq 0 \quad \therefore \quad \frac{x+y}{2} \geq \sqrt{xy}$

(b) let $x = \frac{1}{a}$ and $y = \frac{1}{b}$

$$\frac{\frac{1}{a}+\frac{1}{b}}{2} \geq \sqrt{\frac{1}{ab}} \quad \therefore \quad \frac{a+b}{2ab} \geq \frac{1}{\sqrt{ab}}$$

$$\therefore \quad \frac{2ab}{a+b} \leq \sqrt{ab}$$

(c) $\frac{1}{x-1} + \frac{1}{x} + \frac{1}{x+1} \geq \frac{1}{x} + \frac{2}{\sqrt{x^2-1}}$

using (b) $\frac{2(x^2-1)}{2x} \leq \sqrt{x^2-1} \quad \therefore \quad \frac{2\sqrt{x^2-1}}{2x} \leq 1 \quad \therefore \quad \frac{1}{\sqrt{x^2-1}} \geq \frac{1}{x}$

$$\therefore \quad \frac{1}{x-1} + \frac{1}{x} + \frac{1}{x+1} \geq \frac{1}{x} + \frac{2}{x} = \frac{3}{x}.$$

(d) $H = 1 + \left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right) + \left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right) + \cdots + \left(\frac{1}{n-2}+\frac{1}{n-1}+\frac{1}{n}\right)$

$\geq 1 + \frac{3}{3} + \frac{3}{6} + \frac{3}{9} + \cdots + \frac{3}{n-1}$ using (c)

$= 1 + 1 + \left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right) + \cdots + \frac{1}{k}$ (letting $3k = n-1$)

$\geq 1 + 1 + 1 + \cdots + \frac{3}{k-1}$ (let $k-1 = 3m$)

$\geq 1 + 1 + 1 + \cdots + \frac{1}{m}$

$H \geq 1 + 1 + 1 + \cdots + \frac{1}{m}$

As $n \to \infty$, $H \geq 1 + 1 + 1 + \cdots$

$\therefore$ H has no limit.

37 (Manly SC 2011, 6b)

(a) $a^2+b^2 \geq 2ab$

(b) $a^2+b^2+c^2 \geq ab+bc+ca$

(c) $\sin^2\alpha+\cos^2\alpha \geq 2\sin\alpha\cos\alpha = \sin 2\alpha$

(d) $\sin^2\alpha+\cos^2\alpha+\tan^2\alpha$

$\sin^2\alpha+\cos^2\alpha \geq \sin 2\alpha$

$\cos^2\alpha+\tan^2\alpha \geq 2\sin\alpha$

$\sin^2\alpha+\tan^2\alpha \geq 2\sin^2\alpha\sec\alpha = 2(1-\cos^2\alpha)\sec\alpha = 2\sec\alpha-2\cos\alpha$

$\sin^2\alpha+\cos^2\alpha+\cos^2\alpha+\tan^2\alpha+\tan^2\alpha+\sin^2\alpha \geq \sin 2\alpha+2\sin\alpha+2\sec\alpha-2\cos\alpha$

$\sin^2\alpha+\cos^2\alpha+\tan^2\alpha \geq \frac{1}{2}(\sin 2\alpha+2\sin\alpha+2\sec\alpha-2\cos\alpha)$

$= \sin\alpha-\cos\alpha+\sec\alpha+\frac{1}{2}\sin 2\alpha.$

38 (NSGHS 2011, 8b)

(a) $a^2+b^2 \geq 2ab$, $a^2+b^2+c^2 \geq ab+bc+ca$

(b) $a+b+c=9$

$ab+bc+ca \leq 27$

$(abc)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = bc+ac+ab \leq 27$

$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \leq \frac{27}{abc}$

39. (CSSA 2009, 6b)

(a) $x+y \geq 2\sqrt{xy}$

(b) $x+y+z+w \geq 4\sqrt[4]{xyzw}$

(c) $x+y+z+\frac{x+y+z}{3} \geq 4\sqrt[3]{xyz\left(\frac{x+y+z}{3}\right)}$

$\therefore \left(\frac{x+y+z}{3}\right)^4 \geq xyz\left(\frac{x+y+z}{3}\right)$

$\therefore \left(\frac{x+y+z}{3}\right)^3 \geq xyz$

$\therefore \frac{x+y+z}{3} \geq \sqrt[3]{xyz}$