The Curious Case of Ulam’s Prime Spiral

How some idle doodling at a conference inspired the March 1964 cover of Scientific American and renewed interest in quadratic number theory

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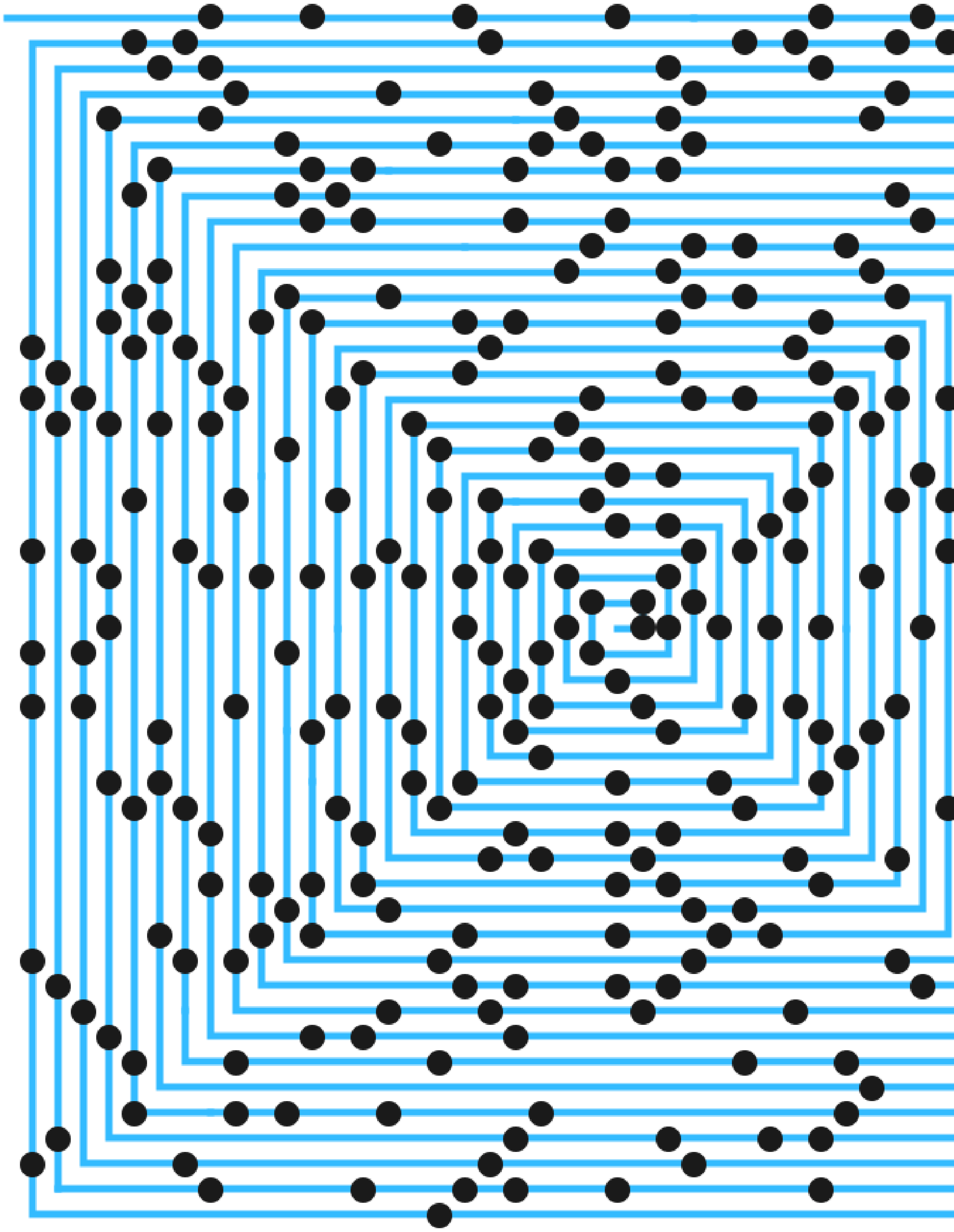

The begininning of Ulam's Spiral. (Image by Author)

"The infinite we shall do right away. The finite may take a little longer." — Stanislaw Ulam (1909–1984)

While researching Science Spectrum's newsletter, This Week in Science History, I found that April 13, 1909, is the birthdate of mathematician, computer scientist, and nuclear physicist Stanislaw Ulam. Reading about his life inspired me to revisit a common interest of ours — prime numbers. First, some background.

Ulam was one of the small set of mathematicians known for their contributions to both pure and applied mathematics. Among his many achievements, he is credited with pioneering the Monte Carlo method. This numerical technique was fundamental to the development of the hydrogen bomb —a project with which he was deeply involved. Separate from his first-class scientific accomplishments, he is most popularly known for his prime-generating spiral, "Ulam's spiral." A prime number is one that can be divided only by one and itself.

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Stanislaw Ulam circa 1945 at the Los Alamos National Laboratory (Public domain, Source)

Ulam’s Spiral

The story goes that Ulam was attending a conference in 1963 and, during a presentation of a “very long and boring paper,” he began doodling. He started arranging the counting numbers in the form of a rectangular spiral, with 1 in the center, 2 to the right, 3 above 2, and continued winding counterclockwise:

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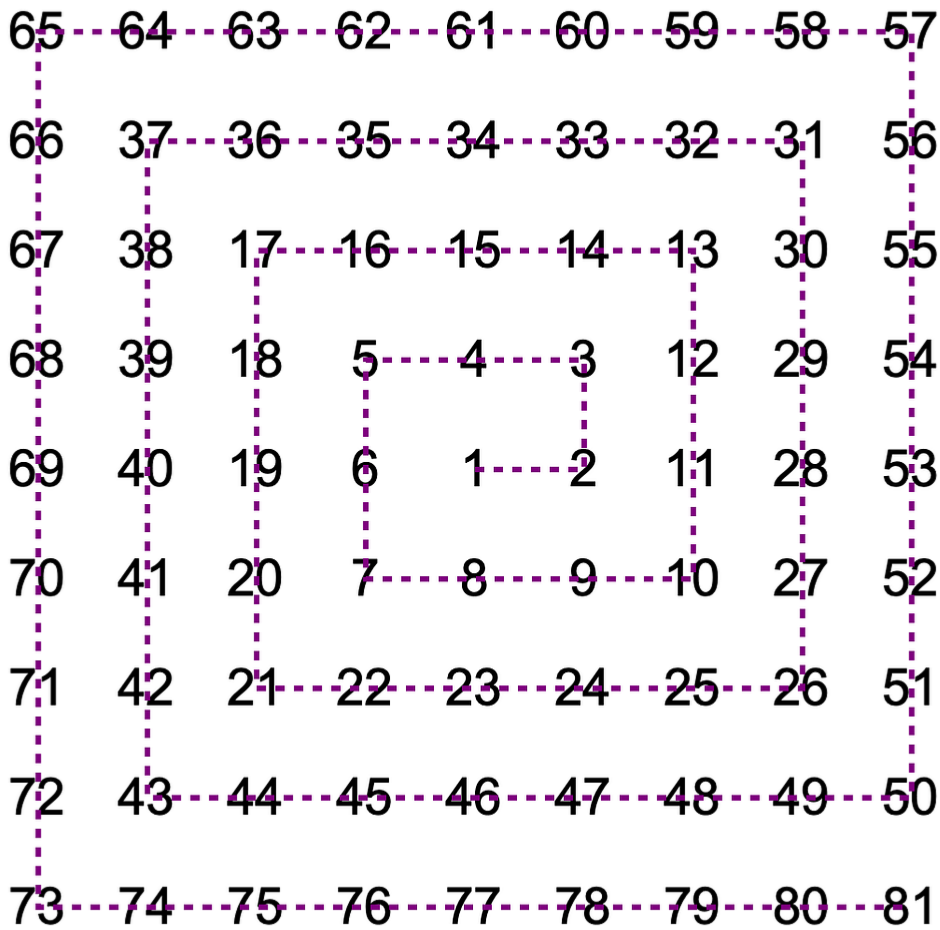

Figure 1. The structure of Ulam’s spiral. (Image by Author)

He then circled primes and noticed that they seemed to “exhibit a strongly nonrandom appearance.” Many of the diagonals contained large numbers of primes, while others contained no primes. Here, the primes appear in red, and one of the prime-rich diagonals is marked by a blue line:

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197	196	195	194	**193**	192	**191**	190	189	188	187	186	185	184	183
198	145	144	143	142	141	140	**139**	138	**137**	136	135	134	133	182
199	146	**101**	100	99	98	**97**	96	95	94	93	92	91	132	**181**
200	147	102	65	64	63	62	**61**	60	**59**	58	57	90	**131**	180
201	148	**103**	66	**37**	36	35	34	33	32	**31**	56	**89**	130	**179**
202	**149**	104	**67**	38	**17**	16	15	14	**13**	30	55	88	129	178
203	150	105	68	39	18	**5**	4	**3**	12	**29**	54	87	128	177
204	**151**	106	69	40	**19**	6	1	**2**	**11**	28	**53**	86	**127**	176
205	152	**107**	70	**41**	20	**7**	8	9	10	27	52	85	126	175
206	153	108	**71**	42	21	22	**23**	24	25	26	51	84	125	174
207	154	**109**	72	**43**	44	45	46	**47**	48	49	50	**83**	124	**173**
208	155	110	**73**	74	75	76	77	78	**79**	80	81	82	123	172
209	156	111	112	**113**	114	115	116	117	118	119	120	121	122	171
210	**157**	158	159	160	161	162	**163**	164	165	166	**167**	168	169	170
211	212	213	214	215	216	217	218	219	220	221	222	**223**	224	225

Figure 2. Ulam's spiral with primes in red. One prime-rich diagonal is marked. (Image by Author)

When he got back to the Los Alamos Laboratory, he used the lab's Maniac II "super" computer to display the primes on an oscilloscope. A photograph of the screen revealed that many lines, particularly diagonals, were more densely packed with primes. Ulam and his colleagues, Myron Stein and

Mark Wells, described their discovery in a 1964 paper entitled "A Visual Display of Some Properties of the Distribution of Primes."

For comparison, here is Ulam's spiral on the left, where the black points represent 8,769 primes, and a random distribution of 8,769 points on the right:

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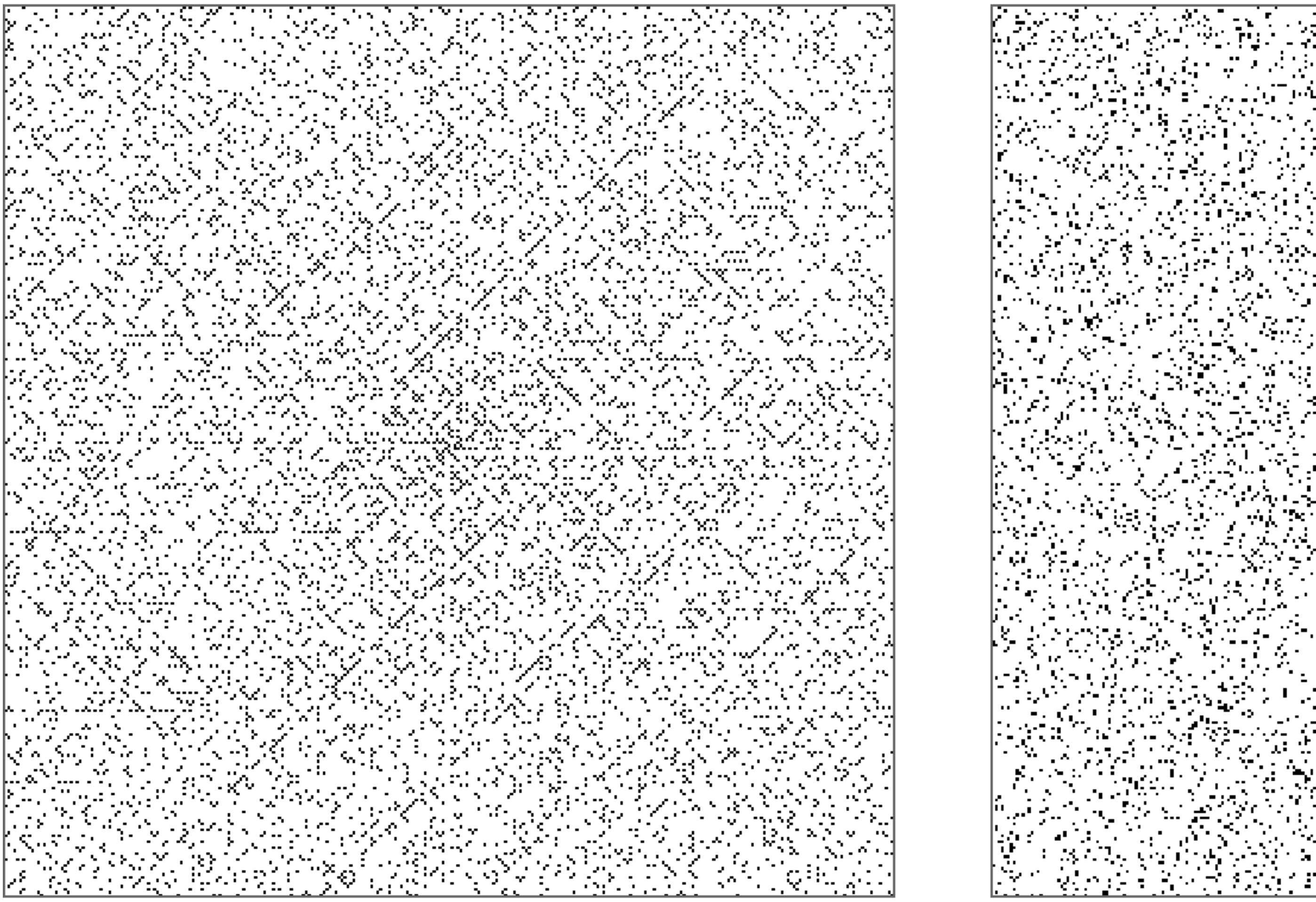

Figure 3. Ulam's spiral on the left along with a random distribution of points on the right. (Image by Author)

As we see in Figure 3, the pattern Ulam noticed was far from random. His paper and the intriguing nature of his discovery became a story in the March 1964 issue of Scientific American.

What’s Going On?

Quadratic expressions are polynomials in which the highest power of $\boldsymbol{x}$ is 2. They take the form

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$$ax^2 + bx + c$$

where $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are constants and $\boldsymbol{a} \neq 0$.

They describe the shape known as a *parabola*. For $\boldsymbol{a} > 0$, it opens upward, and when $\boldsymbol{a} < 0$, it opens downward.

So, what has this got to do with the construction of Ulam’s spiral? We can see a clue in Figure 4 below. In it, we've marked concentric squares. Following the blue diagonal line starting at 1, we see that the lower-right corners of each square correspond to a perfect square: (9, 25, 49, 81...) = (3^2, 5^2, 7^2, 9^2...).

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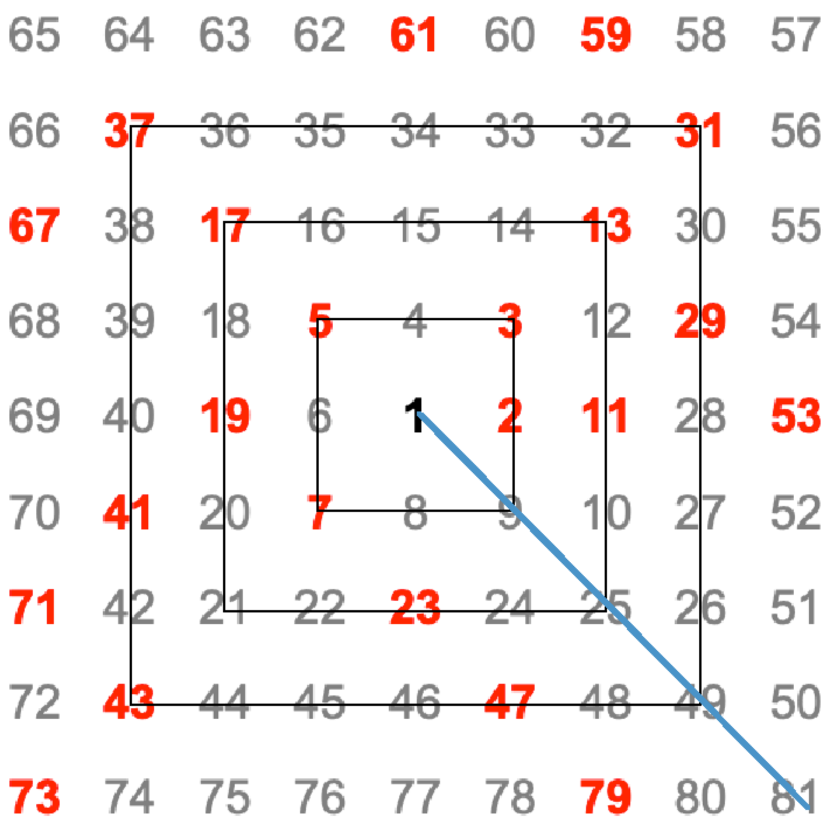

Figure 4. Illustration of the quadratic structre of Ulam's spiral. (Image by Author)

The quadratic equation corresponding to the blue line in Figure 4, and extending down to the right beginning with $\boldsymbol{x} = 0$, is $\boldsymbol{y} = 4\boldsymbol{x}^2 + 4\boldsymbol{x} + 1$.

Because it can be factored into $(2x + 1)^2$, it's easy to understand why it will never output a prime number.

This is what it looks like:
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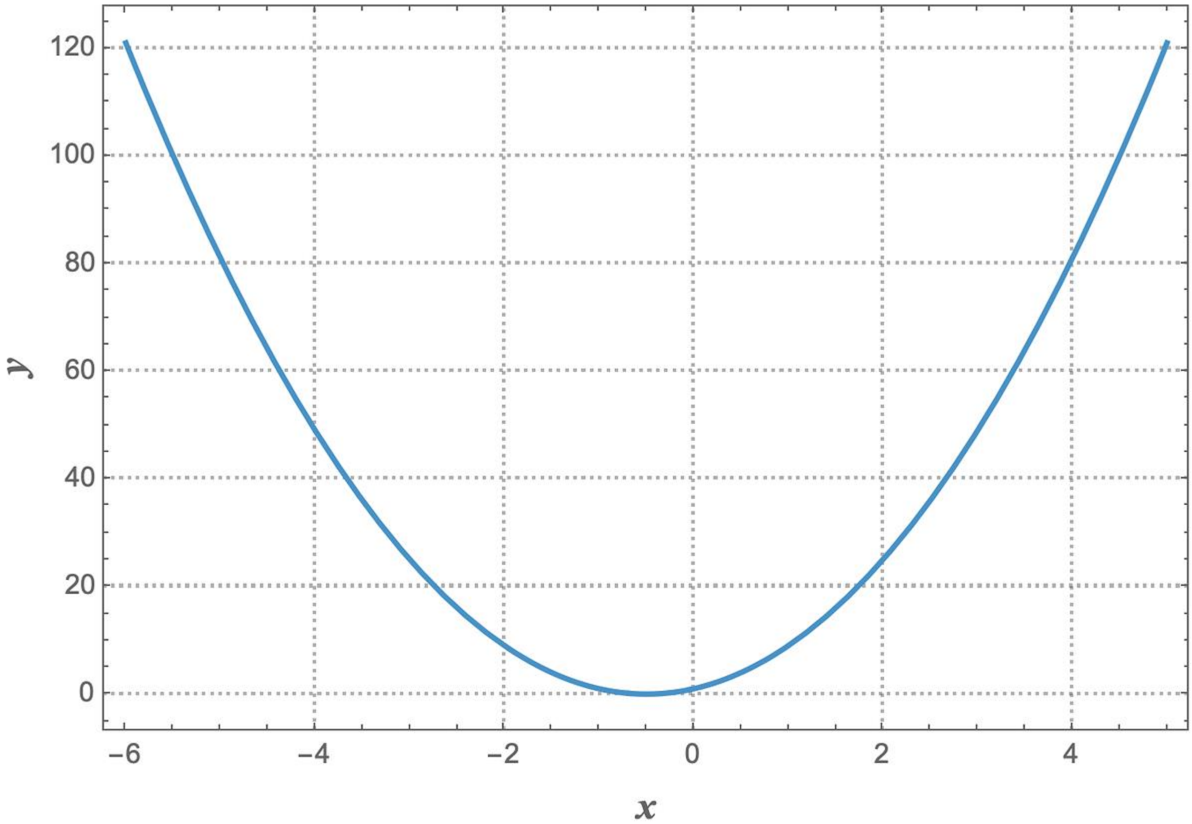

FIgure 5. Plot of the parabola $y = 4x^2 + 4x + 1$. (Image by Author)

In general, because the nested squares in Figure 4 grow in relation to the square of the side length, every diagonal or sub-diagonal can be represented by a quadratic equation.

Even though the diagonal corresponding to $y = 4x^2 + 4x + 1$ is never prime, quadratic expressions *are* known for their ability to generate a high proportion of prime numbers.

Prime Generating Quadratic Polynomials

Christian Goldbach, in 1752, showed that there are no polynomials of any sort with integer coefficients that generate only primes. However, some quadratics are good at generating many primes. The best known is due to the great Swiss mathematician Leonhard Euler. In a 1771 letter to Johann Bernoulli III, Euler introduced this formula, stating that the first 40 terms are all prime numbers:
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$$x^2 - x + 41\,.$$

Euler's polynomial.

If we try plugging in 41 to get yet another prime, it won't work because $41^2 - 41 + 41 = 41^2$, which is clearly a perfect square. Here are the primes it generates:
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{41, 43, 47, 53, 61, 71, 83, 97, 113, 131, 151, 173, 197, 223, 251, 281, 313, 347, 383, 421, 461, 503, 547, 593, 641, 691, 743, 797, 853, 911, 971, 1033, 1097, 1163, 1231, 1301, 1373, 1447, 1523, 1601}

If we offset Ulam's spiral so that it begins with 41 in the center, we can see how Euler's quadratic formula falls along the main diagonal:
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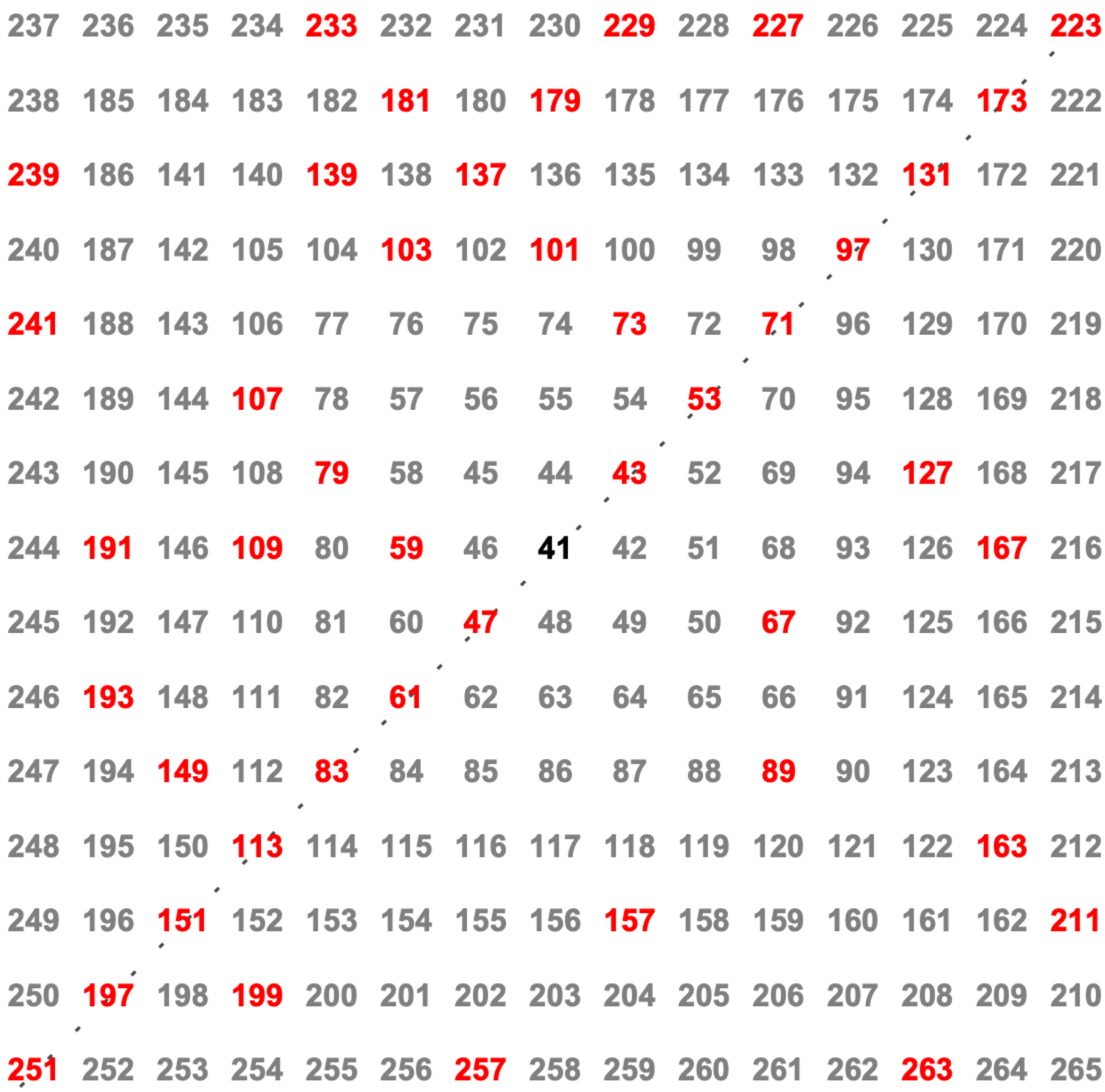

Figure 6. Ulam’s spiral offset to begin with 41 in the center. (Image by Author)

Euler’s formula can also be adjusted symmetrically to yield a total of 80 consecutive primes, but still only 40 are unique:
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$$n^2 - 79n + 1601.$$

This is what it looks like over the prime range:
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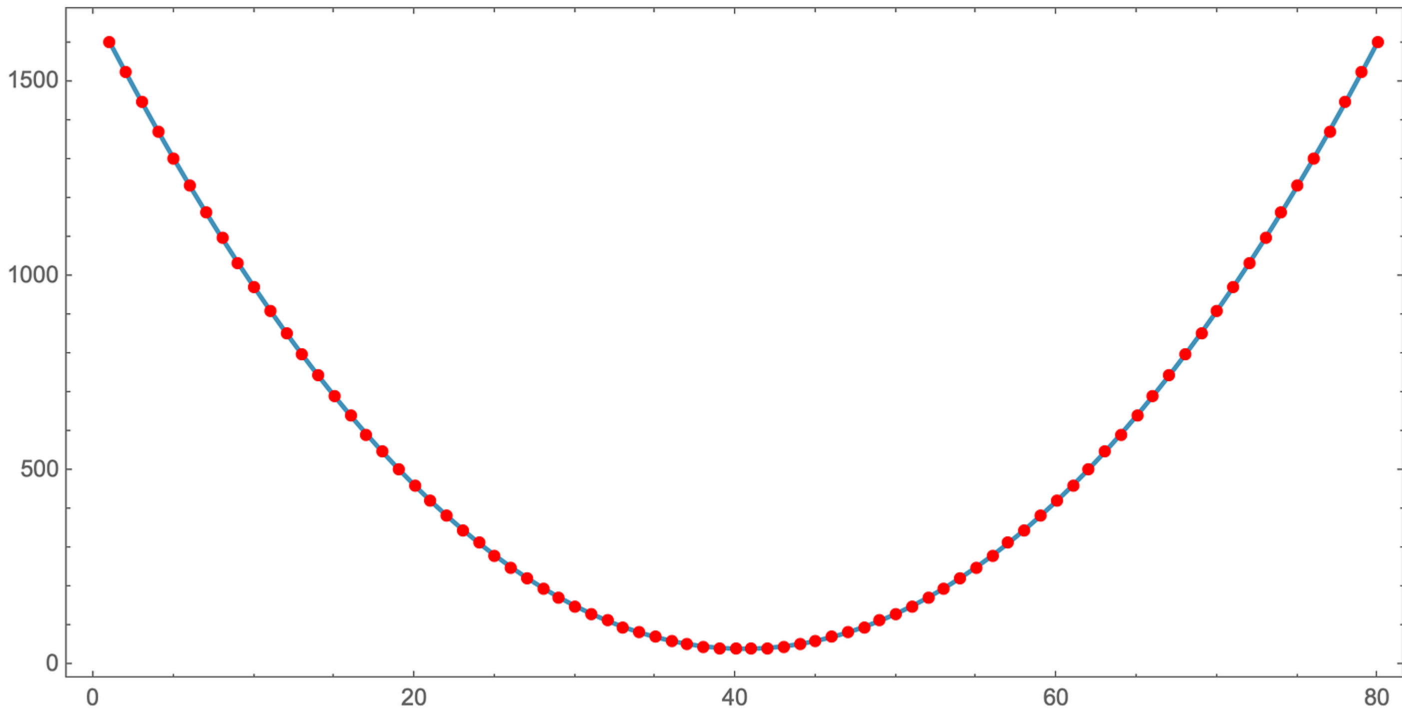

Figure 7. Euler's extended quadratic formula for primes. Red points represent primes. (Image by Author)

The record number of unique consecutive primes is held by the Ruby polynomial, devised in 1989, which is prime for x = 0, 1, 2, …, 44 — a total of 45 primes:
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$$36x^2 - 810x + 2753.$$

So, why do some quadratics work better than others? The reasons are related to the discriminant (b^2 - 4ac) and are deeply rooted in algebraic number theory. If you're well-versed in math and curious, the paper by R. A. Mollin below is a great place to start. Regardless, there are some simple observations we can make. If the coefficients all share a common factor, then the polynomial can produce, at most, 1 prime (in absolute value). If it can be factored more generally, at best it will produce mostly composite numbers. Also, if a + b is even, and c is also even, then it will generate only even numbers.

About 12 years ago, I became obsessed with finding prime-generating polynomials. I was aware of and intrigued by Ulam's spiral. I enjoyed modest success, finding several new quadratics that generate 40 consecutive unique primes. Here's one of them:
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$$137n^2 - 6643n + 77977.$$

These are the 40 primes it outputs:
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{77977, 71471, 65239, 59281, 53597, 48187, 43051, 38189, 33601, 29
14771, 11827, 9157, 6761, 4639, 2791, 1217, -83, -1109, -1861, -233
-1511, -619, 547, 1987, 3701, 5689, 7951, 10487, 13297, 16381, 1973

and this is what the prime-run section of its parabola looks like:
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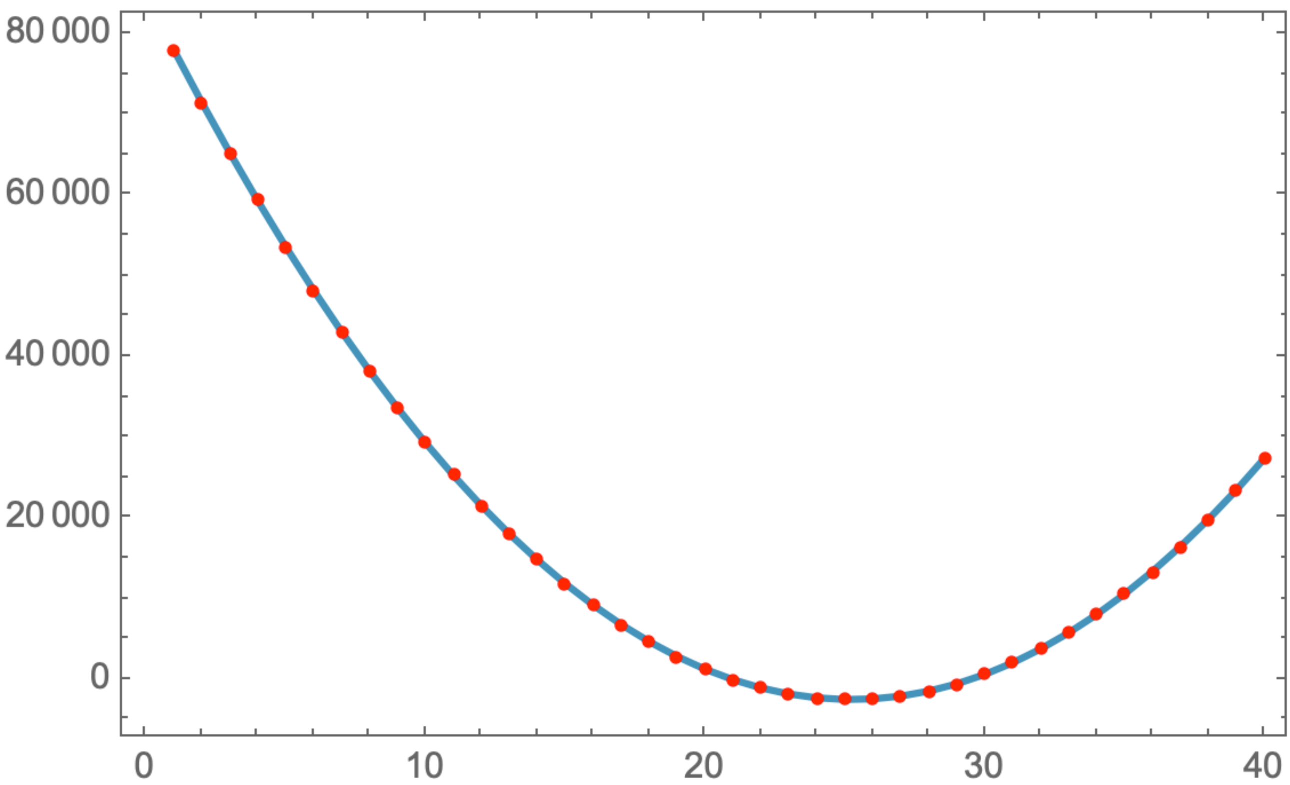

Figure 8. A segment of the quadratic 137n² — 6643n + 77977. Red points represent primes. (Image by Author)

In Figure 8, keep in mind that the x and y scales are vastly different. If they were the same, we'd be looking at a pencil-thin parabola!

Quadratics that can produce 40 consecutive primes appear to be rare. We know of only two that have exceeded that mysterious threshold: the Ruby polynomial and two others that generate 43 unique primes (one by Ruby's collaborator Gilbert Fung, the other by Speiser). To be clear, polynomials of higher degrees can produce longer runs of primes. I've included a link below to the Wolfram Research page on polynomials of this type.

While absorbed in the subject, I even went so far as to explore what three-dimensional analogs of Ulam's spiral might look like, searching for patterns related to prime numbers. Here's one of them:

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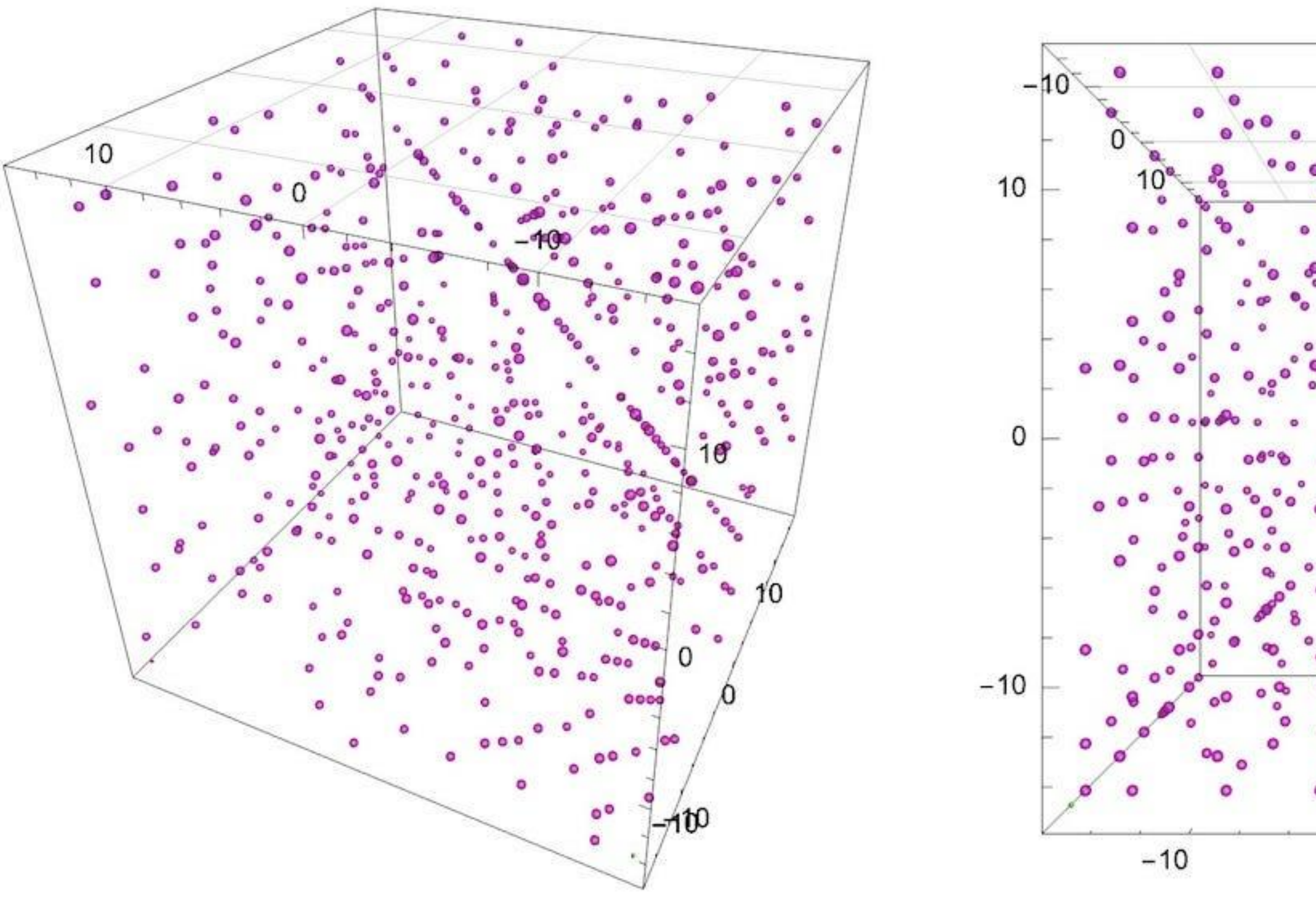

Figure 9. A 3-dimensional analog of Ulam's spiral. (Image by Aurthor)

While fascinating and fun, they'll require more time to tease out any actionable insights.

Variations

Ulam's spiral is not the only 2-dimensional approach to arranging numbers so as to reveal patterns in the primes. Ulam had a predecessor. In 1932, Laurence Klauber published a triangular arrangement of numbers where the primes tend to fall along straight lines.

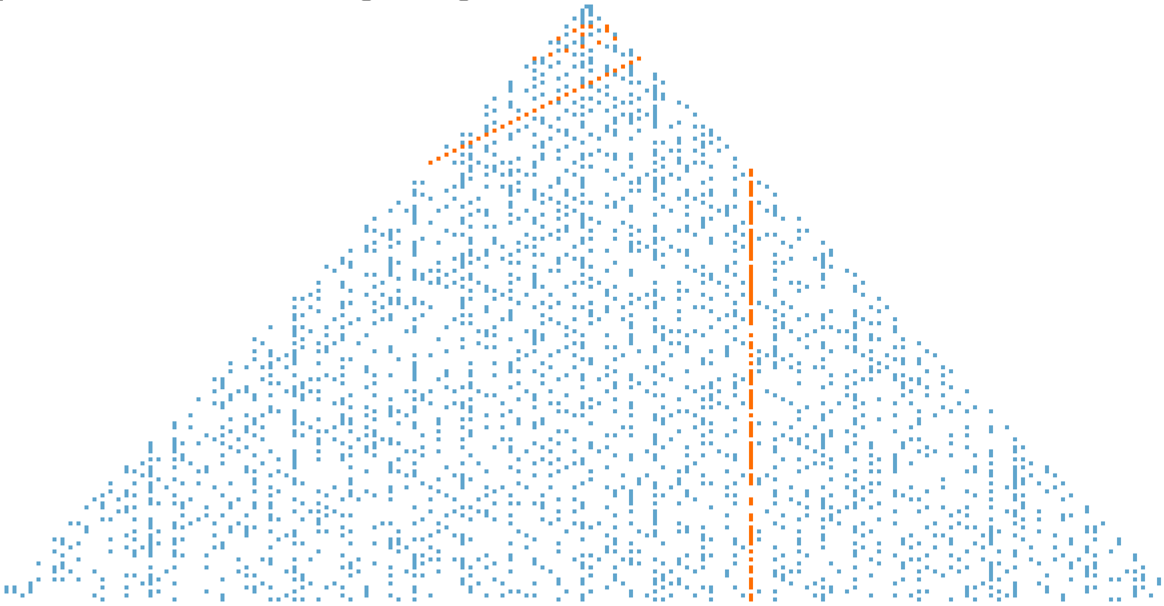

Figure 10. Klauber prime triangle. Orange points correspond to the primes generated by Euler's polynomial. (Source)

In 1994, Robert Sacks devised what is now called the "Sacks spiral." It is an Archimedean spiral wherein the numbers are spaced so that one perfect square occurs for each full rotation. It causes primes to fall along arcs.
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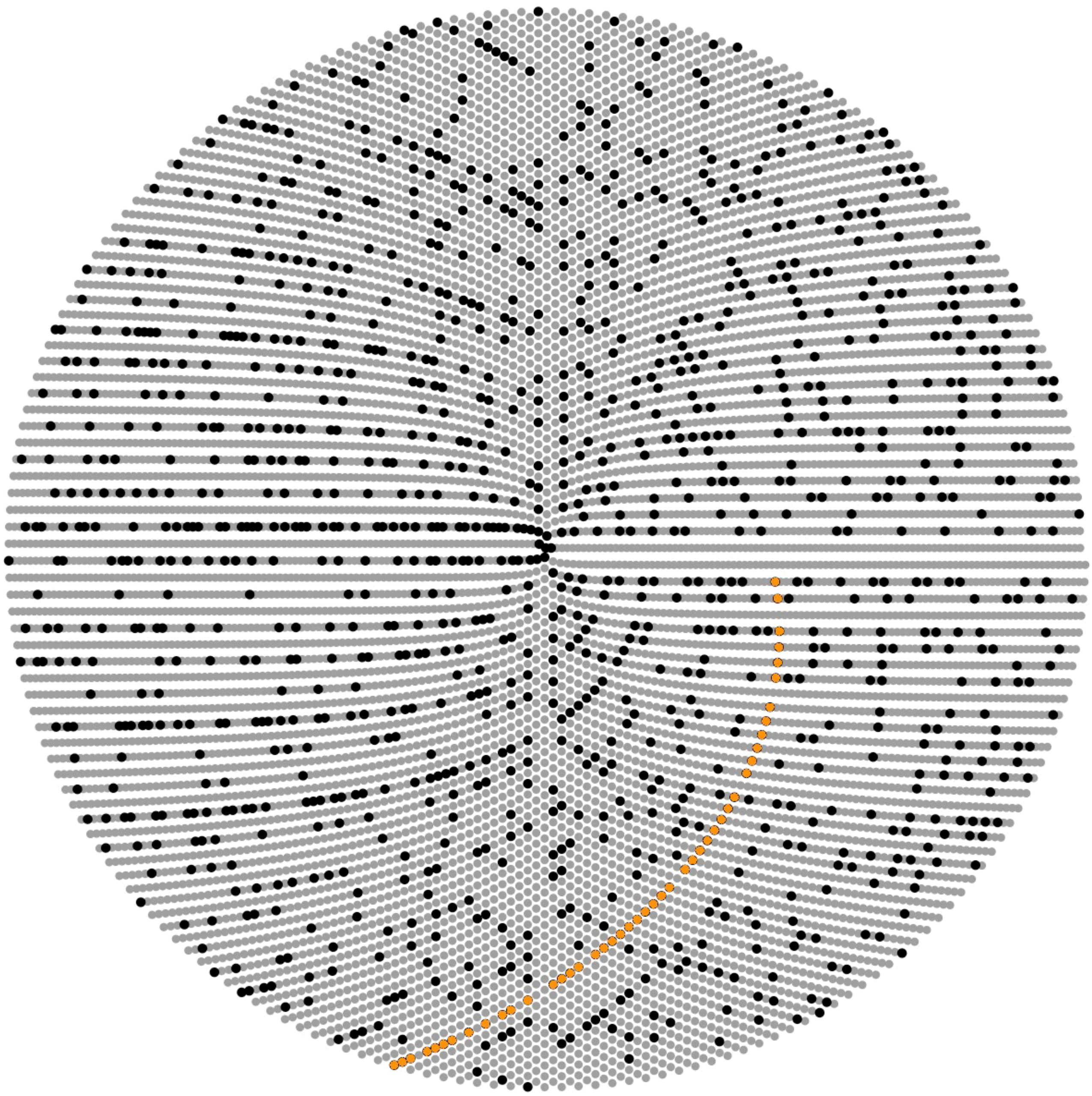

Figure 11. Sacks spiral. Orange points correspond to the primes generated by Euler’s polynomial. (Source, modified by Author)

Here, I’ve highlighted in orange the primes corresponding to Euler's polynomial.

Other number configurations are possible. This is a hexagonal number grid where prime numbers are green and darker blue indicates more factors:
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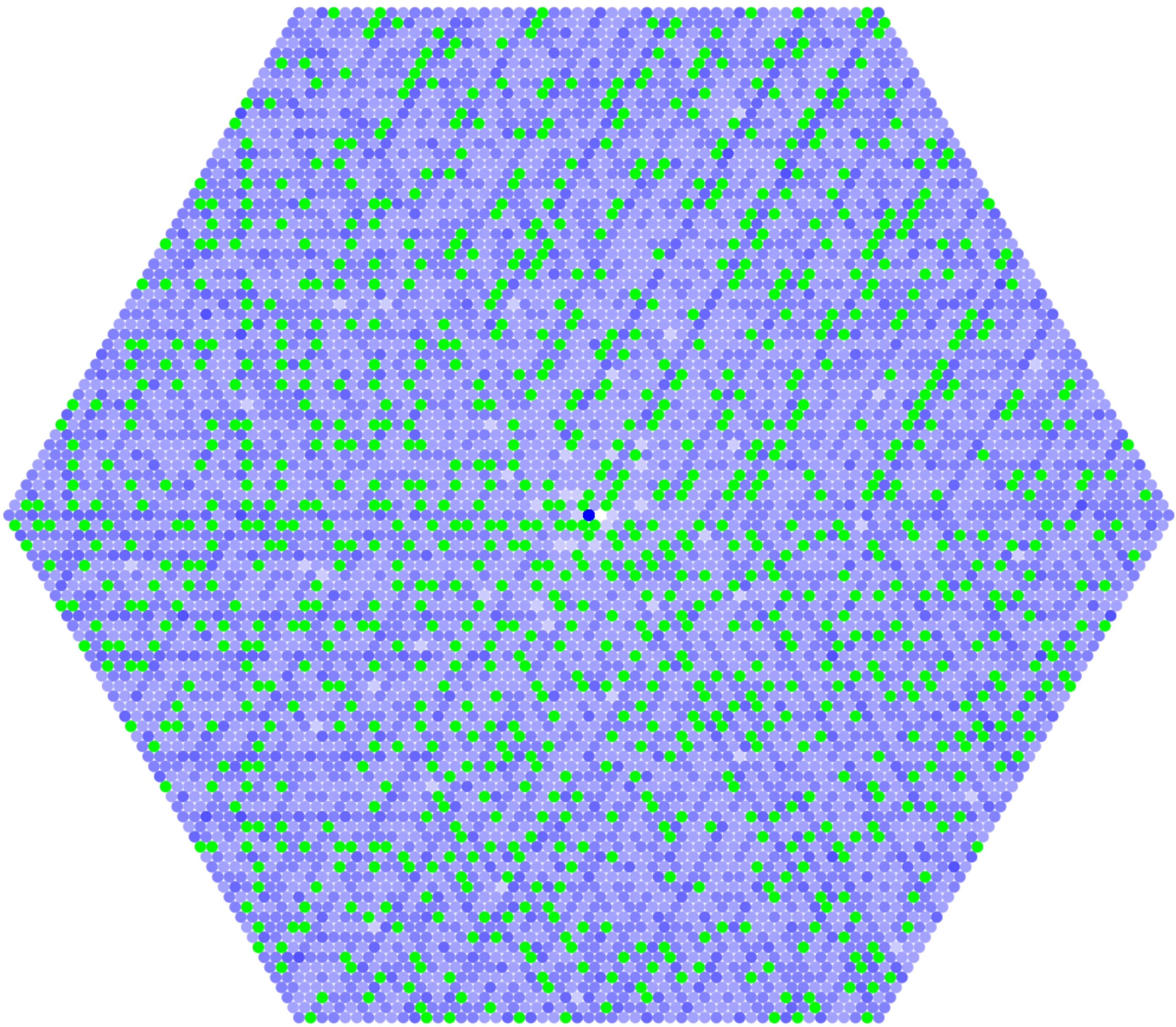

Figure 12. A hexagonal number grid where prime numbers are green and darker blue indicates more factors. (Source)

Coda

Not all progress is made through intentional thinking — history is full of unintended discoveries. Modern research in cognitive science tells us that daydreaming often has strong neural connections with creativity. Einstein is the best-known example of a scientist who gained insights from allowing his mind to wander.

What started as an intriguing observation by Ulam inspired many mathematicians to revisit quadratic number theory and investigate precisely why some quadratics yield a high density of primes. His serendipitous discovery is a wonderful reminder of the power of the idle mind.

Down The Rabbit Hole

- Linda Hall Library: Scientist of the Day — Stanislaw Ulam
- Stanislaw Ulam's research paper: "A Visual Display of Some Properties of the Distribution of Primes"
- Play with Ulam's spiral using Dario Alpern's interactive Web app: Ulam Spiral
- Wolfram Research: Prime Generating Polynomial
- Interactive spiral constructions from Complexity Explorables: Prime Time
- Research paper by R. A. Mollin: Quadratic polynomials producing consecutive, distinct primes and class groups
- Research paper: New quadratic polynomials with high densities of prime values
- Research paper: The bright side and dark side of daydreaming predict creativity together through brain functional connectivity
- The cover of the March 1964 issue of Scientific American:

Scientific American Volume 210, Issue 3
"All-Weather Aircraft Landing", "Bacterial Endotoxins", "Experimental Narcotic Addiction" and more
www.scientificamerican.com

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