Factor Models

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Market Portfolio is unrealistic

1. According to CAPM, an investor only needs to buy market portfolio (a collection of all assets) and risk free asset. There is no need to find optimal weights for an optimal portfolio since market portfolio is the best portfolio
2. In reality, it is impossible to hold a portfolio that includes all assets. It is also impossible to have unlimited access to a risk free asset
3. Thus in practice we still want to construct an optimal portfolio with only n assets
4. When n is large, it is mathematically challenging to compute the n-by-n variance-covariance matrix, the key input for MPT
5. Factor model is a parametric approach to simplify the computation of variance-covariance matrix by imposing a structure for return

Single Factor Model (SFM)

1. A single factor model assumes the returns of all assets are related to just one common factor
$$r_J = c_J + \beta_J r_M + e_J \tag{1}$$
where the single factor r_M is the return of market portfolio according to CAPM

2. c_J is the asset-specific intercept, and β_J is the asset-specific slope. According to CAPM
$$\beta_J = \frac{\sigma_{JM}}{\sigma_M^2}$$

3. The error term e_J captures the part of r_J unexplained by the single factor

4. The SFM looks similar to a simple regression model

Assumptions of Single Factor Model

1. SFM assumes that all assets are affected by a single factor r_M that reflects effects of macro events such as changes in interest rates

2. The error e_J represents micro event that affects asset J only, such as change of management in a company

3. SFM assumes

$$E(e_J) = 0 \tag{2}$$

$$cov(r_M, e_J) = 0 \tag{3}$$

$$cov(e_J, e_K) = 0 \tag{4}$$

   The last one assumes micro events for different assets are uncorrelated. That is the key assumption to simplify math derivation

Implications

It follows that

$$\begin{aligned} \mu_J &= c_J + \beta_J \mu_M && (5) \\ \sigma_J^2 &= \beta_J^2 \sigma_M^2 + var(e_J) && (6) \\ \sigma_{JK} &= \beta_J \beta_K \sigma_M^2 && (7) \end{aligned}$$

Note that SFM is related to CAPM by letting $c_J = r_f - \beta_J r_f = (1 - \beta_J) r_f$

Easier Computation

1. One motivation for factor models is to ease computation of the variance-covariance matrix for a portfolio

2. For a portfolio with n assets, there are $(n-1)+(n-2)+...1=\frac{n(n-1)}{2}$ covariances and n variances

3. Based on SFM, we only need to compute σ_M^2, n betas, and n variances of error terms

Variance of Portfolio

1. Consider a portfolio of assets J and K. We can show its variance is

$$\begin{aligned}
\sigma_p^2 &= var(w_1 r_J + w_2 r_K) && (8)\\
&= w_1^2\sigma_J^2 + w_2^2\sigma_K^2 + 2w_1w_2\sigma_{JK} && (9)\\
&= w_1^2(\beta_J^2\sigma_M^2 + var(e_J)) + w_2^2(\beta_K^2\sigma_M^2 + var(e_K)) + 2w_1w_2\beta_J\beta_K\sigma_M^2 && (10)\\
&= (w_1^2\beta_J^2 + w_2^2\beta_K^2 + 2w_1w_2\beta_J\beta_K)\sigma_M^2 + (w_1^2 var(e_J) + w_2^2 var(e_K)) && (11)\\
&= (w_1\beta_J + w_2\beta_K)^2\sigma_M^2 + (w_1^2 var(e_J) + w_2^2 var(e_K)) && (12)
\end{aligned}$$

2. In general, the variance of a portfolio consisting of n assets is

$$\sigma_p^2 = \left(\sum_{i=1}^{n} w_i\beta_i\right)^2 \sigma_M^2 + \sum_{i=1}^{n} w_i^2 var(e_i) \qquad (13)$$

An illustrative example

Consider a special portfolio that has equal weights for homoskedastic assets—$w_i = \frac{1}{n}, var(e_i) = \sigma^2$. We can show

$$\sum_{i=1}^{n} w_i^2 var(e_i) = \sum_{i=1}^{n} n^{-2}\sigma^2 = \frac{\sigma^2}{n} \rightarrow 0 \; if \; n \rightarrow \infty$$

In short, the second term $\sum_{i=1}^{n} w_i^2 var(e_i)$ approaches zero when n rises

Diversification, Systematic and Unsystematic Risks

1. As $n \to \infty$, the second term $\sum_{i=1}^{n} w_i^2 var(e_i)$ converges to zero. Thus we call $var(e_i)$ measures the risk that can be diversified away, or it represents the unsystematic risk
2. The first term $\left(\sum_{i=1}^{n} w_i \beta_i\right)^2 \sigma_M^2$ dominates. So we call it systematic risk
3. The systematic risk is driven by β, or is driven by covariance between asset and market portfolio
4. For a portfolio with many assets, we can drop the second term and use below approximation

$$\sigma_p^2 \approx \left(\sum_{i=1}^{n} w_i \beta_i\right)^2 \sigma_M^2 \tag{14}$$

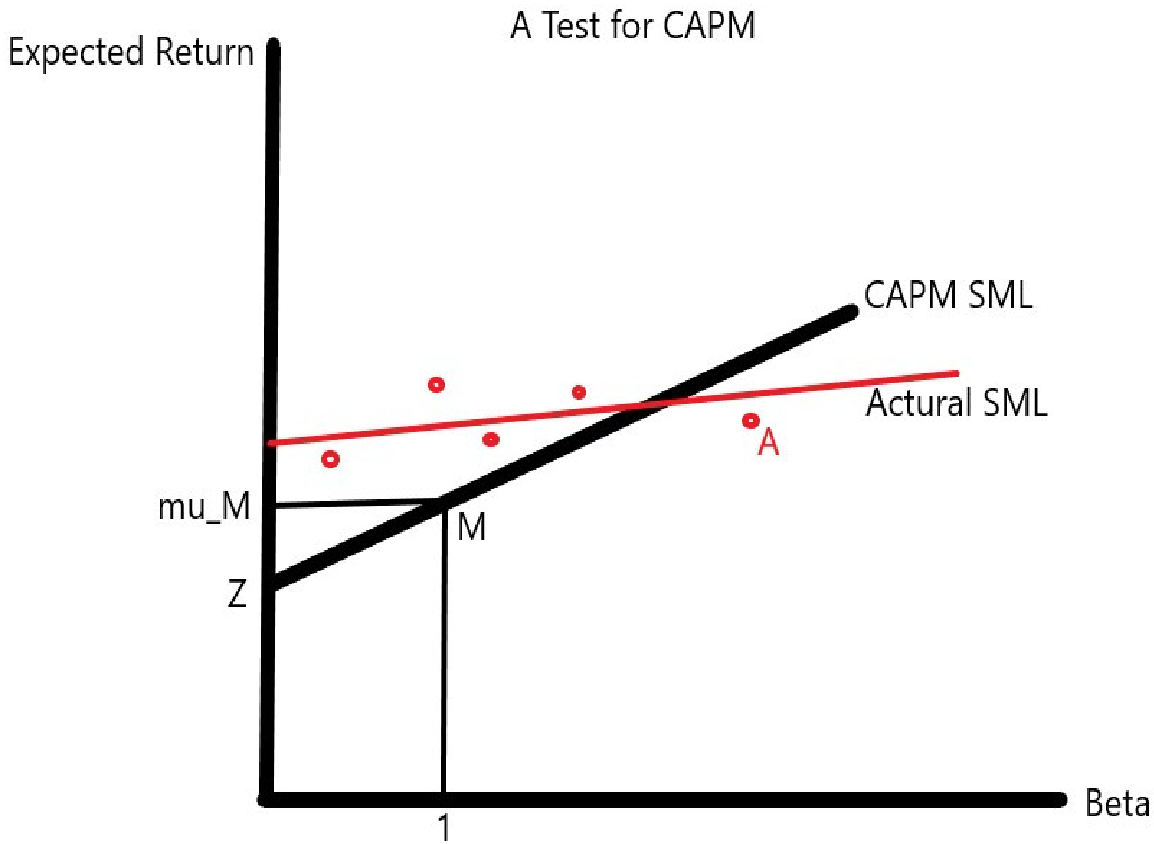
Multiple Factor Model (MFM)

1. It is easy to extend SFM to a Multiple Factor Model. For instance, consider a two factor model
$$r_J = c_J + \beta_{1J} r_M + \beta_{2J} f_2 + e_J \tag{15}$$
where the second factor f_2 may capture industry-event that affects firms in the same industry

2. Assuming $cov(r_M, f_2) = 0$, we can show the variance of portfolio is
$$\sigma_p^2 = \left(\sum_{i=1}^{n} w_i \beta_{1i}\right)^2 \sigma_M^2 + \left(\sum_{i=1}^{n} w_i \beta_{2i}\right)^2 \sigma_{f2}^2 + \sum_{i=1}^{n} w_i^2 var(e_i) \tag{16}$$

3. MFM can be seen as an extension of CAPM, which assumes the only factor is return of market portfolio

A Test of CAPM



CAPM may suffer omitted variables or factors

1. One empirical test of CAPM involves a two-step procedure
2. In step one, estimate β separately for n assets by running the time series regression

$$r_{J,t} = \beta_0 + \beta_J r_{M,t} + error, \quad t = 1, \ldots T$$

3. In the second step, use β as the sole regressor, and run the cross section regression

$$\hat{\mu}_i = c_0 + c_1 \beta_i + error, \quad i = 1, \ldots n$$

4. It is common to find that $c_1 < \mu_M - r_f$, or the estimated actual SML is flatter than the theoretical SML implied by CAPM
5. From econometrics perspective, this finding indicates that CAPM may suffer omitted factors

HW6 (See syllabus for due date)

1. (2 points) Please self-study (google, chatgpt, or youtube) and summarize the main idea of Fama-French three-factor model (FFTFM)

2. (1 point) Please describe how to use historical data to estimate the factor loadings of FFTFM

3. (2 points) Suppose the risk-free rate is 3%, the expected return of market portfolio is 9%, SMB is 2%, HML is 1%, and factor loadings of a stock are $\beta_M = 1.1, \beta_{SMB} = 0.5$, $\beta_{HML} = -0.3$. Please find the expected return on the stock according to the FFTFM.