

Chapter 9.

Relations

關 聯

9.1 Relations and Their Properties

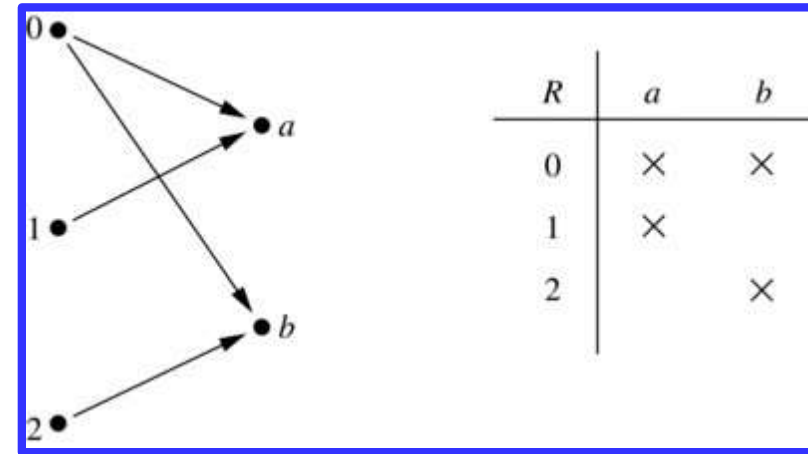
9.3 Representing Relations

9.5 Equivalence Relations

9.6 Partial Orderings

Binary Relations

Definition: A **binary relation** R from a set A to a set B is a subset $R \subseteq A \times B$.



Example:

- Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$
- $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B .
- We can represent relations from a set A to a set B graphically or using a table:

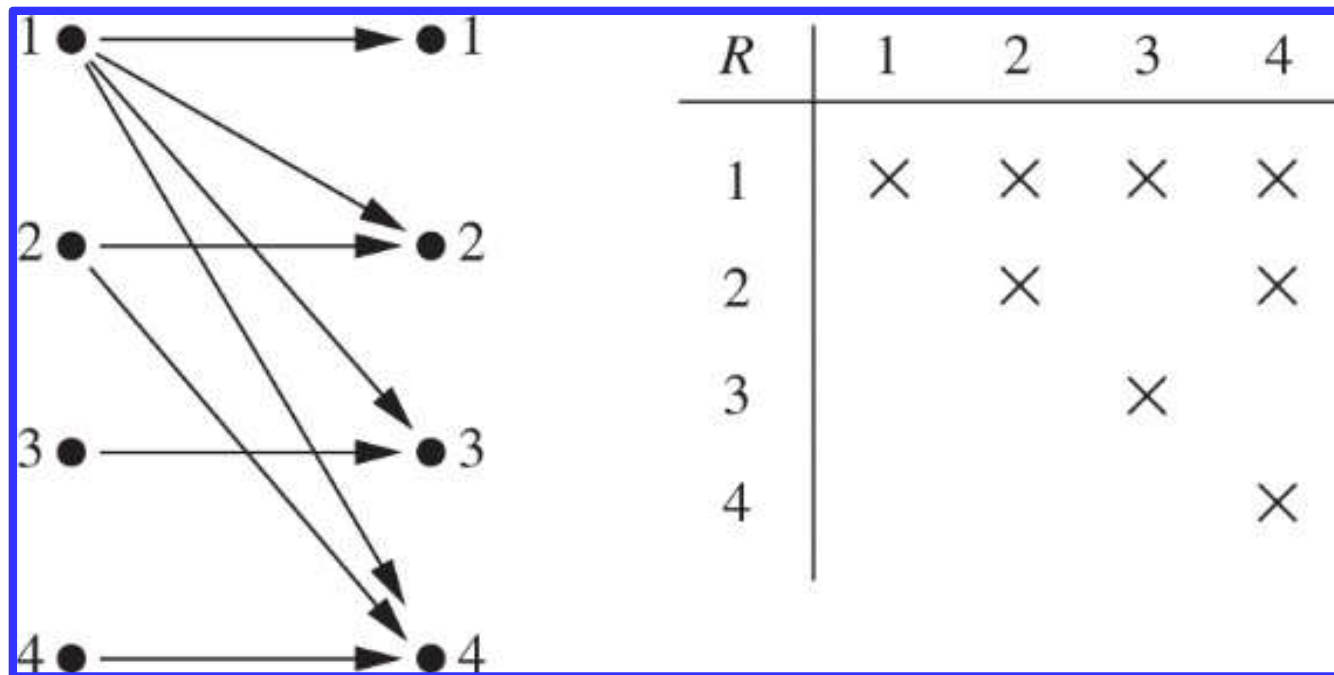
We use the notation aRb to denote that $(a, b) \in R$ and $a \not R b$ to denote that $(a, b) \notin R$. Moreover, when $(a, b) \in R$, a is said to be related to b by R .

Relations are more general than functions. A **function** is a relation where **exactly one element of B is related to each element of A .**

Binary Relations on a Set₁

Definition: A **binary relation** R on a set A is a subset of $A \times A$ or a relation from A to A .

Example:



Binary Relations on a Set₂

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Example:

- Suppose that $A = \{a, b, c\}$. Then $R = \{(a, a), (a, b), (a, c)\}$ is a relation on A .
- Let $A = \{1, 2, 3, 4\}$. The ordered pairs in the relation $R = \{(a, b) \mid a \text{ divides } b\}$ are
 $(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3),$ and $(4, 4)$.

Binary Relations on a Set₃

Question: How many relations are there on a set A with n elements?

Solution: Because a relation on A is the same thing as a subset of $A \times A$, we count the subsets of $A \times A$.

Since $A \times A$ has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are 2^{n^2} subsets of $A \times A$. Therefore, **there are 2^{n^2} relations on a set A .**

For example, there are $2^{3^2} = 2^9 = 512$ relations on the set $\{a, b, c\}$

Binary Relations on a Set₄

Example: Consider these relations on the set of integers:

$$R_1 = \{(a, b) \mid a \leq b\},$$

$$R_4 = \{(a, b) \mid a = b\},$$

$$R_2 = \{(a, b) \mid a > b\},$$

$$R_5 = \{(a, b) \mid a = b + 1\},$$

$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\},$$

$$R_6 = \{(a, b) \mid a + b \leq 3\}.$$

Note that these relations are on an infinite set and each of these relations is an infinite set.

Which of these relations contain each of the pairs

(1,1), (1, 2), (2, 1), (1, -1), and (2, 2)?

Solution: Checking the conditions that define each relation, we see that the pair (1,1) is in R_1 , R_3 , R_4 , and R_6 ; (1,2) is in R_1 and R_6 ; (2,1) is in R_2 , R_5 , and R_6 ; (1, -1) is in R_2 , R_3 , and R_6 ; (2,2) is in R_1 , R_3 , and R_4 .

Reflexive Relations

Definition: R is **reflexive** (自反關係) iff $(a,a) \in R$ for every element $a \in A$. Written symbolically, R is reflexive if and only if

$$\forall x [x \in U \rightarrow (x, x) \in R]$$

Example: The following relations on the integers are reflexive:

$$R_1 = \{(a, b) \mid a \leq b\},$$

$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\},$$

$$R_4 = \{(a, b) \mid a = b\}.$$

If $A = \emptyset$ then the empty relation is reflexive vacuously. That is, the empty relation on an empty set is reflexive!

The following relations are not reflexive:

$$R_2 = \{(a, b) \mid a > b\} \quad (\text{note that } 3 \not> 3),$$

$$R_5 = \{(a, b) \mid a = b + 1\} \quad (\text{note that } 3 \neq 3 + 1),$$

$$R_6 = \{(a, b) \mid a + b \leq 3\} \quad (\text{note that } 4 + 4 \not\leq 3).$$

Symmetric Relations

Definition: R is **symmetric** (對稱關係) iff $(b,a) \in R$ whenever $(a,b) \in R$ for all $a,b \in A$. Written symbolically, R is symmetric if and only if

$$\forall x \forall y [(x,y) \in R \rightarrow (y,x) \in R]$$

Example: The following relations on the integers are symmetric:

$$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\},$$

$$R_4 = \{(a,b) \mid a = b\},$$

$$R_6 = \{(a,b) \mid a + b \leq 3\}.$$

The following are not symmetric:

$$R_1 = \{(a,b) \mid a \leq b\} \quad (\text{note that } 3 \leq 4, \text{ but } 4 \not\leq 3),$$

$$R_2 = \{(a,b) \mid a > b\} \quad (\text{note that } 4 > 3, \text{ but } 3 \not> 4),$$

$$R_5 = \{(a,b) \mid a = b + 1\} \quad (\text{note that } 4 = 3 + 1, \text{ but } 3 \neq 4 + 1).$$

Antisymmetric Relations

Definition: A relation R on a set A such that for all $a, b \in A$ if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called **antisymmetric** (反對稱關係).
Written symbolically, R is antisymmetric if and only if

$$\forall x \forall y [(x, y) \in R \wedge (y, x) \in R \rightarrow x = y]$$

Example: The following relations on the integers are antisymmetric:

$$R_1 = \{(a, b) \mid a \leq b\},$$

$$R_2 = \{(a, b) \mid a > b\},$$

$$R_4 = \{(a, b) \mid a = b\},$$

$$R_5 = \{(a, b) \mid a = b + 1\}.$$

← For any integer, if $a \leq b$ and $a \leq b$, then $a = b$.

The following relations are not antisymmetric

$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$$

(note that both $(1, -1)$ and $(-1, 1)$ belongs to R_3),

$$R_6 = \{(a, b) \mid a + b \leq 3\} \text{ (note that both } (1, 2) \text{ and } (2, 1) \text{ belongs to } R_6).$$

Transitive Relations

Definition: A relation R on a set A is called **transitive** (遞移性) if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for all $a,b,c \in A$. Written symbolically, R is transitive if and only if

$$\forall x \forall y \forall z [(x, y) \in R \wedge (y, z) \in R \rightarrow (x, z) \in R]$$

Example: The following relations on the integers are transitive:

$$R_1 = \{(a, b) | a \leq b\},$$

$$R_2 = \{(a, b) | a > b\},$$

$$R_3 = \{(a, b) | a = b \text{ or } a = -b\},$$

$$R_4 = \{(a, b) | a = b\}.$$

For every integer, $a \leq b$
and $b \leq c$, then $b \leq c$.

The following are not transitive:

$$R_5 = \{(a, b) | a = b + 1\} \text{ (note that both } (4, 3) \text{ and } (3, 2) \text{ belongs to } R_5, \text{ but not } (4, 2)),$$

$$R_6 = \{(a, b) | a + b \leq 3\} \text{ (note that both } (2, 1) \text{ and } (1, 2) \text{ belongs to } R_6, \text{ but not } (2, 2)).$$

Combining Relations

Given two relations R_1 and R_2 , **we can combine them** using basic set operations to form new relations **such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, and $R_2 - R_1$.**

Example: Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The relations $R_1 = \{(1,1),(2,2),(3,3)\}$ and $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$ can be combined using basic set operations to form new relations:

$$R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}$$

$$R_1 \cap R_2 = \{(1,1)\} \qquad R_1 - R_2 = \{(2,2),(3,3)\}$$

$$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$$

Composition

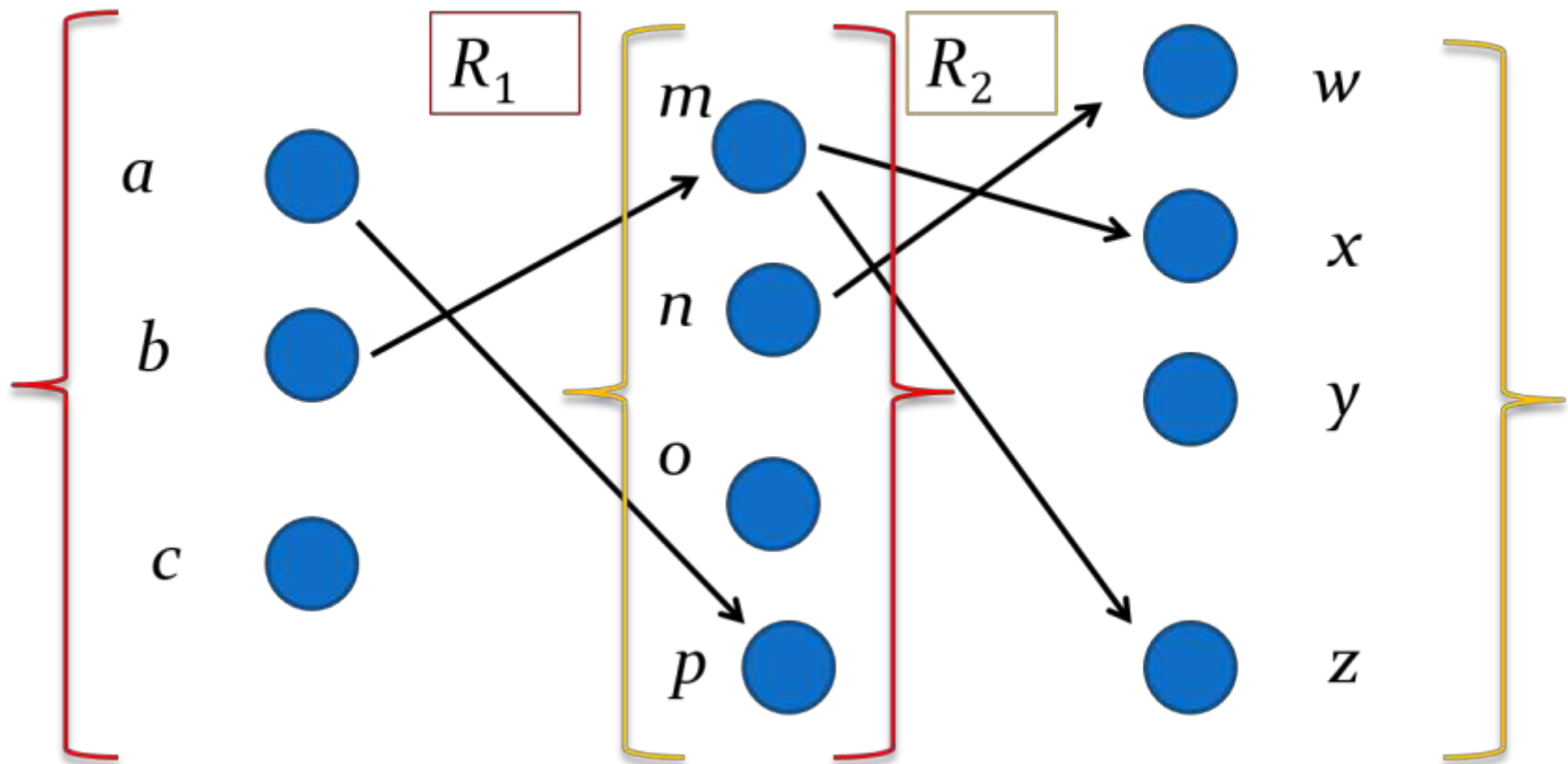
Definition: Suppose

- R_1 is a relation from a set A to a set B .
- R_2 is a relation from B to a set C .

Then the **composition (or composite) (合成)** of R_2 with R_1 , is a relation from A to C where

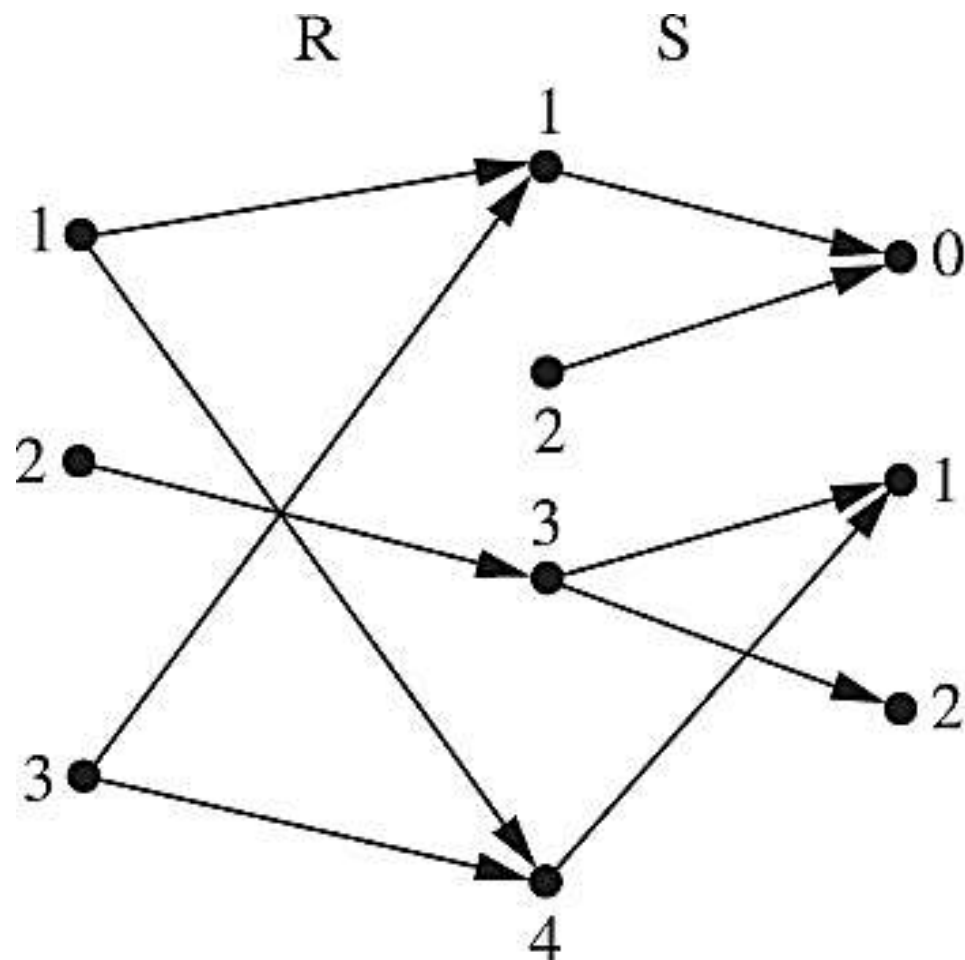
- if (x,y) is a member of R_1 and (y,z) is a member of R_2 , then (x,z) is a member of $R_2 \circ R_1$.

Representing the Composition of Relations₁



$$R_2 \circ R_1 = \{(b, x), (b, z)\}$$

Representing the Composition of Relations₂



$S \circ R$

1 → 1 → 0 (1, 0)

1 → 4 → 1 (1, 1)

2 → 3 → 1 (2, 1)

2 → 3 → 2 (2, 2)

3 → 1 → 0 (3, 0)

3 → 4 → 1 (3, 1)

Powers of a Relation

Definition: Let R be a binary relation on A . Then **the powers R^n** of the relation R can be defined inductively by:

- **Basis Step:** $R^1 = R$
- **Inductive Step:** $R^{n+1} = R^n \circ R$

Example.

Let $R = \{(1,1), (2,1), (3,2), (4,3)\}$. Find the power $R^n, n = 2, 3, 4, \dots$

$$R^2 = R \circ R = \{(1,1), (2,1), (3,1), (4,2)\}$$

$$R^3 = R^2 \circ R = \{(1,1), (2,1), (3,1), (4,1)\}$$

$$R^4 = R^3 \circ R = \{(1,1), (2,1), (3,1), (4,1)\}$$

It also follows that $R^n = R^3$ for $n = 5, 6, 7, \dots$

Powers of a Relation

Definition: Let R be a binary relation on A . Then **the powers R^n** of the relation R can be defined inductively by:

- **Basis Step:** $R^1 = R$
- **Inductive Step:** $R^{n+1} = R^n \circ R$

The powers of a transitive relation are subsets of the relation. This is established by the following theorem:

Theorem 1: The relation R on a set A is **transitive** iff $R^n \subseteq R$ for $n = 1, 2, 3, \dots$

(It can be proved via mathematical induction)

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9.6 Partial Orderings

Representing Relations Using Matrices

A relation between finite sets can be represented using a zero-one matrix.

Suppose R is a relation from $A = \{a_1, a_2, \dots, a_m\}$ to $B = \{b_1, b_2, \dots, b_n\}$.

- The elements of the two sets can be listed in any particular arbitrary order. When $A = B$, we use the same ordering.

The relation R is represented by the matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R, \\ 0 & \text{if } (a_i, b_j) \notin R. \end{cases}$$

The matrix representing R has a 1 as its (i,j) entry when a_i is related to b_j and a 0 if a_i is not related to b_j .

Examples of Representing Relations Using Matrices₁

Example 1: Suppose that $A = \{1,2,3\}$ and $B = \{1,2\}$. Let R be the relation from A to B containing (a,b) if $a \in A$, $b \in B$, and $a > b$. What is the **matrix representing** R (assuming the ordering of elements is the same as the increasing numerical order)?

Solution: Because $R = \{(2,1), (3,1), (3,2)\}$, the matrix is

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Examples of Representing Relations

Using Matrices₂

Example 2: Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

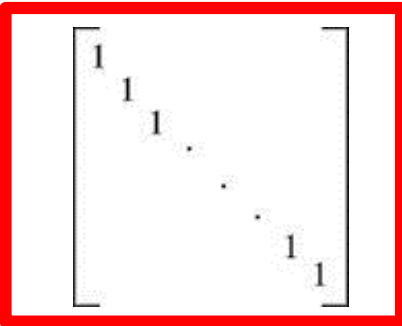
$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} ?$$

Solution: Because R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$, it follows that:

$$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}.$$

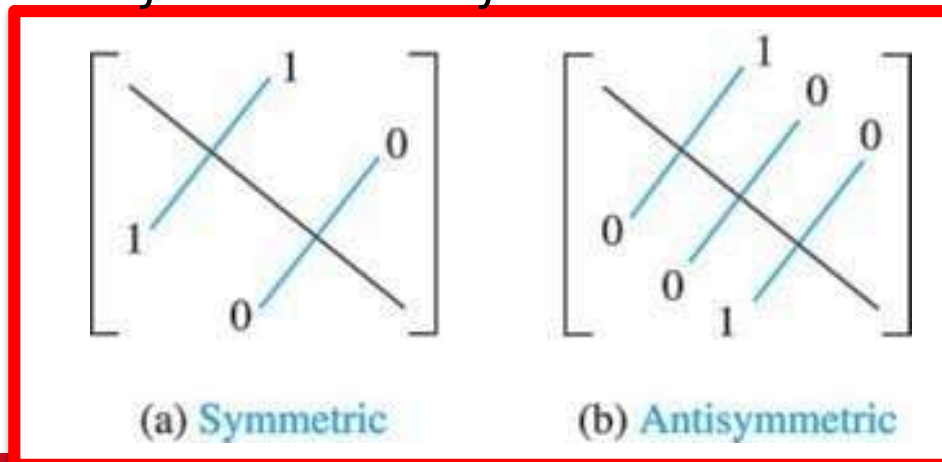
Matrices of Relations on Sets

If R is a **reflexive relation**, all the elements on the main diagonal of M_R are equal to 1.



$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & \ddots & \\ & & & & & 1 & 1 \end{bmatrix}$$

R is a **symmetric relation**, if and only if $m_{ij} = 1$ whenever $m_{ji} = 1$. R is an antisymmetric relation, if and only if $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$.



Example of a Relation on a Set

Example 3: Suppose that the relation R on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Is R reflexive, symmetric, and/or antisymmetric?

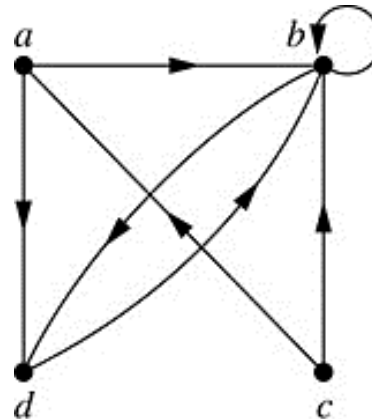
Solution: Because all the diagonal elements are equal to 1, **R is reflexive**. Because M_R is symmetric, **R is symmetric** and **not antisymmetric** because both $m_{1,2}$ and $m_{2,1}$ are 1.

Representing Relations Using Digraphs

Definition: A **directed graph** (有向圖), or **digraph**, consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex a is called the **initial vertex** of the edge (a,b) , and the vertex b is called the **terminal vertex** of this edge.

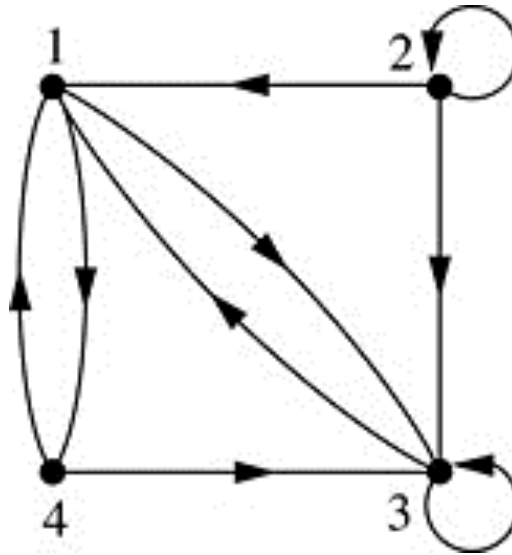
- An edge of the form (a,a) is called a **loop**.

Example 7: A drawing of the directed graph with vertices a , b , c , and d , and edges (a, b) , (a, d) , (b, b) , (b, d) , (c, a) , (c, b) , and (d, b) is shown here.



Examples of Digraphs Representing Relations

Example 8: What are the ordered pairs in the relation represented by this directed graph?



Solution: The ordered pairs in the relation are $(1, 3)$, $(1, 4)$, $(2, 1)$, $(2, 2)$, $(2, 3)$, $(3, 1)$, $(3, 3)$, $(4, 1)$, and $(4, 3)$

Determining which Properties a Relation has from its Digraph

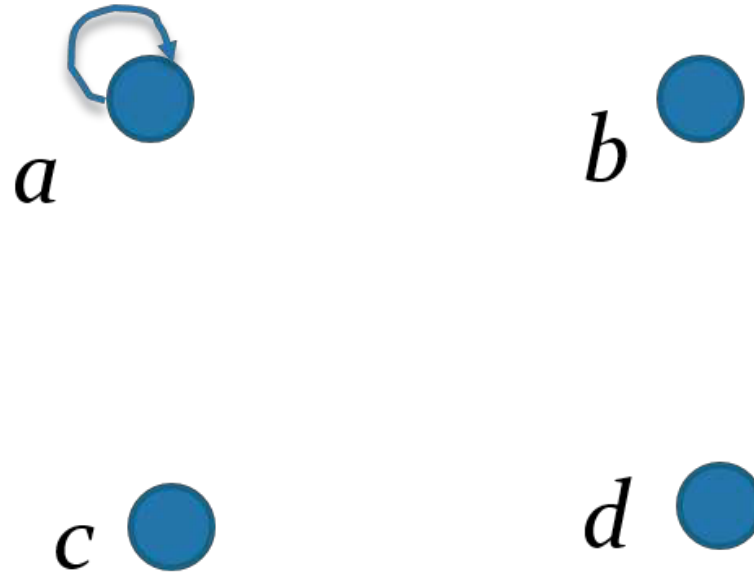
Reflexivity: A **loop** must be present at all vertices in the graph.

Symmetry: If (x,y) is an edge, then **so is (y,x)** .

Antisymmetry: If (x,y) with $x \neq y$ is an edge, then **(y,x) is not an edge**.

Transitivity: If (x,y) and (y,z) are edges, then **so is (x,z)** .

26 Determining which Properties a Relation has from its Digraph – Example 1



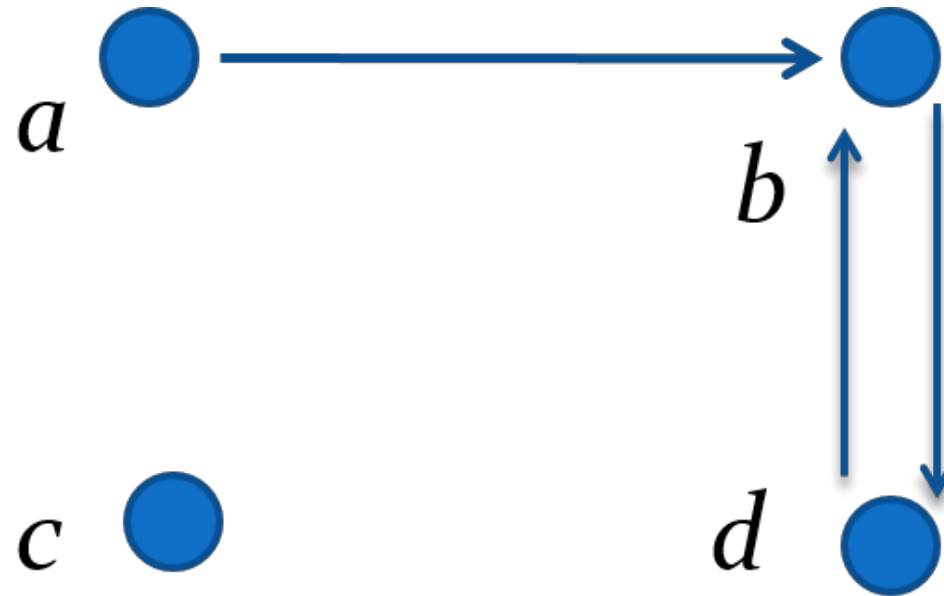
Reflexive? No, not every vertex has a loop

Symmetric? **Yes** (trivially), there is no edge from one vertex to another

Antisymmetric? **Yes** (trivially), there is no edge from one vertex to another

Transitive? **Yes**, (trivially) since there is no edge from one vertex to another

27 Determining which Properties a Relation has from its Digraph – Example 2



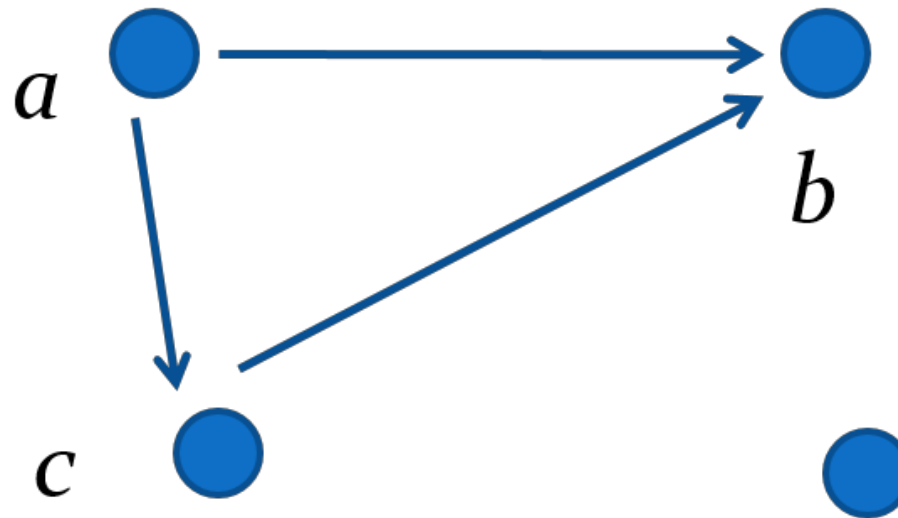
Reflexive? No, there are no loops

Symmetric? No, there is an edge from a to b , but not from b to a

Antisymmetric? No, there is an edge from d to b and b to d

Transitive? No, there are edges from a to b and from b to d , but there is no edge from a to d

28 Determining which Properties a Relation has from its Digraph – Example 3



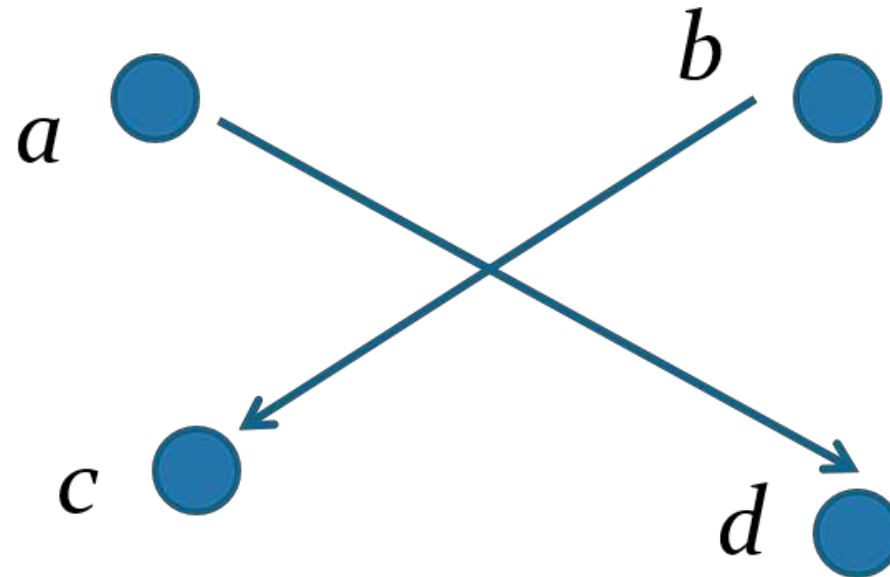
Reflexive? No, there are no loops

Symmetric? No, for example, there is no edge from c to a

Antisymmetric? **Yes**, whenever there is an edge from one vertex to another, there is not one going back

Transitive? **Yes**.

29 Determining which Properties a Relation has from its Digraph – Example 4



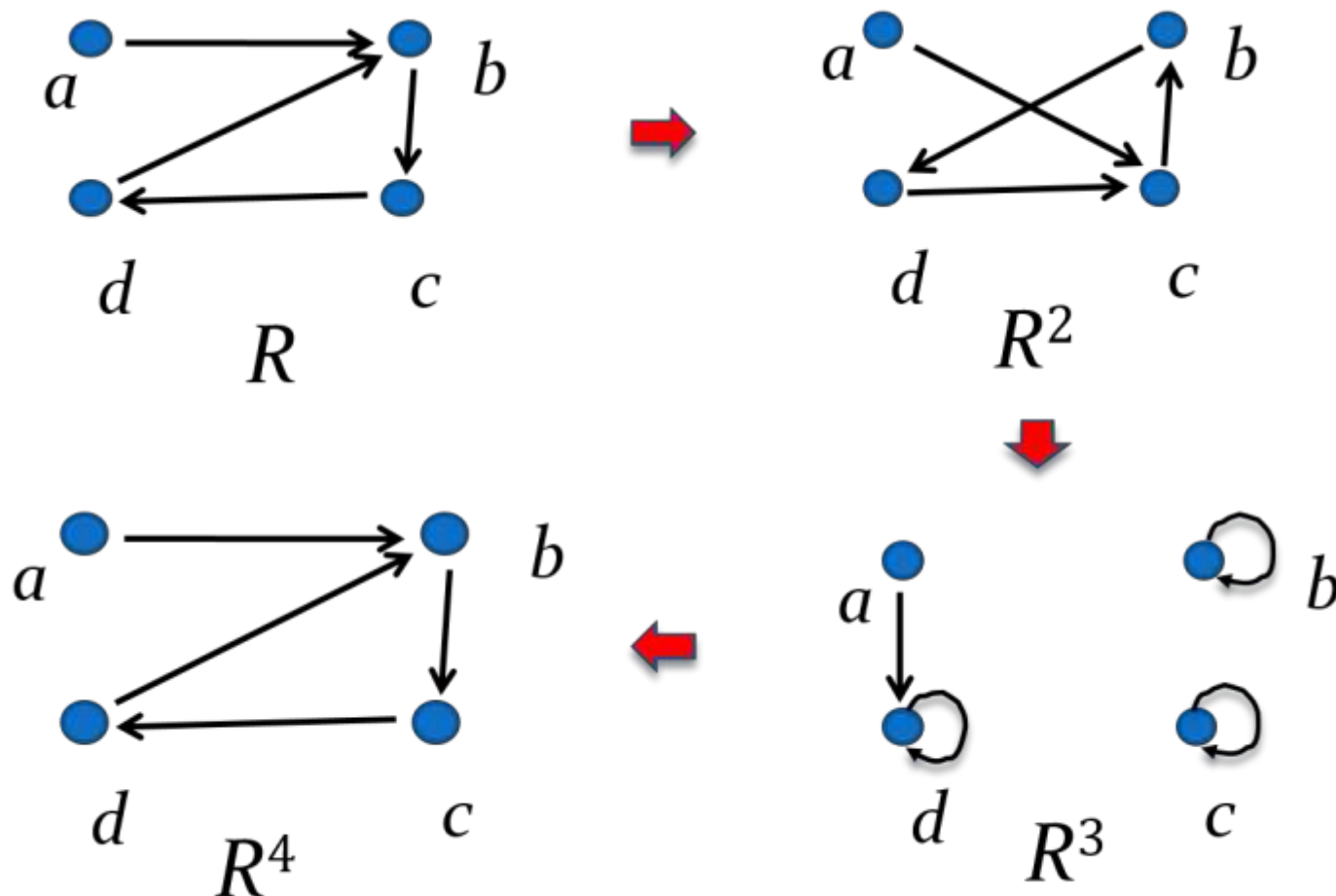
Reflexive? No, there are no loops

Symmetric? No, for example, there is no edge from d to a

Antisymmetric? **Yes**, whenever there is an edge from one vertex to another, there is not one going back

Transitive? **Yes** (trivially), there are no two edges where the first edge ends at the vertex where the second edge begins

Example of the Powers of a Relation



The pair (x, y) is in R^n if there is a path of length n from x to y in R (following the direction of the arrows).

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Equivalence Relations

Definition 1: A relation on a set A is called an *equivalence relation* (等價關係) if it is reflexive, symmetric, and transitive.

Definition 2: Two elements a , and b that are related by an equivalence relation are called *equivalent*. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Strings

Example: Suppose that R is the relation on the set of strings of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x . Is R an equivalence relation?

Solution: Show that all of the properties of an equivalence relation hold.

- **Reflexivity:** Because $l(a) = l(a)$, it follows that aRa for all strings a .
- **Symmetry:** Suppose that aRb . Since $l(a) = l(b)$, $l(b) = l(a)$ also holds and bRa .
- **Transitivity:** Suppose that aRb and bRc . Since $l(a) = l(b)$, and $l(b) = l(c)$, $l(a) = l(c)$ also holds and aRc .

Congruence Modulo m

Example: Let m be an integer with $m > 1$. Show that the relation

$$R = \{(a,b) \mid a \equiv b \pmod{m}\}$$

is an equivalence relation on the set of integers.

Solution: Recall that $a \equiv b \pmod{m}$ if and only if m divides $a - b$.

- **Reflexivity:** $a \equiv a \pmod{m}$ since $a - a = 0$ is divisible by m since $0 = 0 \cdot m$.
- **Symmetry:** Suppose that $a \equiv b \pmod{m}$. Then $a - b$ is divisible by m , and so $a - b = km$, where k is an integer. It follows that $b - a = (-k)m$, so $b \equiv a \pmod{m}$.
- **Transitivity:** Suppose that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$. Then m divides both $a - b$ and $b - c$. Hence, there are integers k and l with $a - b = km$ and $b - c = lm$. We obtain by adding the equations: $a - c = (a - b) + (b - c) = km + lm = (k + l)m$.

$$a - c = (a - b) + (b - c) = km + lm = (k + l)m.$$

Therefore, $a \equiv c \pmod{m}$.

Divides

Example: Show that the “divides” relation on the set of positive integers **is not an equivalence relation**.

Solution: The properties of reflexivity, and transitivity do hold, but there relation is not transitive. Hence, “divides” is not an equivalence relation.

- **Reflexivity:** $a \mid a$ for all a .
- **Not Symmetric:** For example, $2 \mid 4$, but $4 \nmid 2$. Hence, the relation is not symmetric.
- **Transitivity:** Suppose that a divides b and b divides c . Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c . Therefore, the relation is transitive.

Equivalence Classes

Definition 3: Let R be an equivalence relation on a set A . The set of all elements that are related to an element a of A is called the **equivalence class** of a . The equivalence class of a with respect to R is denoted by $[a]_R$.

When only one relation is under consideration, we can write $[a]$, without the subscript R , for this equivalence class.

Note that $[a]_R = \{s \mid (a, s) \in R\}$.

If $b \in [a]_R$, then b is called a **representative of this equivalence class**. Any element of a class can be used as a representative of the class.

The equivalence classes of the relation congruence modulo m are called the **congruence classes modulo m** . The congruence class of an integer a modulo m is denoted by $[a]_m$, so $[a]_m = \{\dots, a - 2m, a - m, a + m, a + 2m, \dots\}$.

For example,

$$\begin{aligned} [0]_4 &= \{\dots, -8, -4, 0, 4, 8, \dots\} & [1]_4 &= \{\dots, -7, -3, 1, 5, 9, \dots\} \\ [2]_4 &= \{\dots, -6, -2, 2, 6, 10, \dots\} & [3]_4 &= \{\dots, -5, -1, 3, 7, 11, \dots\} \end{aligned}$$

Equivalence Classes and Partitions

Theorem 1: let R be an equivalence relation on a set A . These statements for elements a and b of A are equivalent:

$$(i) \quad aRb \longrightarrow (a, b) \in R$$

$$(ii) \quad [a] = [b] \longrightarrow [a]_R = \{s \mid (a, s) \in R\}$$

$$(iii) \quad [a] \cap [b] \neq \emptyset$$

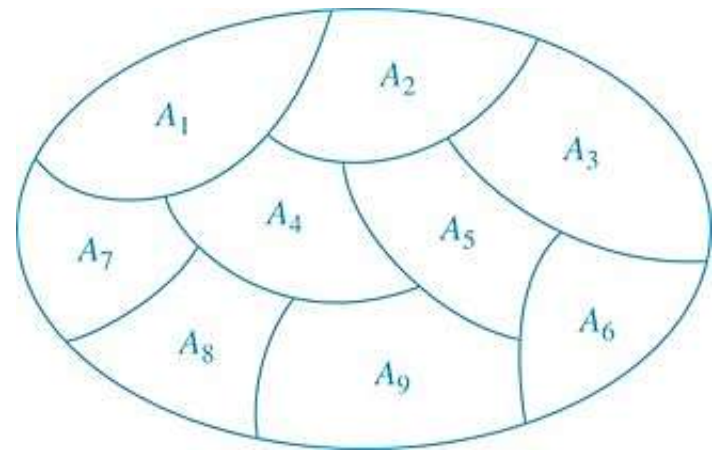
Proof: We show that **(i) implies (ii)**. Assume that aRb . Now **suppose that $c \in [a]$** . Then aRc . Because aRb and R is symmetric, bRa . Because R is transitive and bRa and aRc , **it follows that bRc** . Hence, **$c \in [b]$** . Therefore, $[a] \subseteq [b]$. A similar argument (omitted here) shows that $[b] \subseteq [a]$. Since $[a] \subseteq [b]$ and $[b] \subseteq [a]$, we have shown that $[a] = [b]$.

*(see text for proof that **(ii) implies (iii)** and **(iii) implies (i)**)*

Partition of a Set

Definition: A **partition** (分割) of a set S is a collection of **disjoint nonempty** subsets of S that have **S as their union**. In other words, the collection of subsets A_i , where $i \in I$ (where I is an index set), forms a partition of S if and only if

- $A_i \neq \emptyset$ for $i \in I$,
- $A_i \cap A_j = \emptyset$ when $i \neq j$,
- and $\bigcup_{i \in I} A_i = S$.



A Partition of a Set

An Equivalence Relation Partitions a Set₁

Let R be an **equivalence relation** on a set A . **The union of all the equivalence classes of R is all of A** , since an element a of A is in its own equivalence class $[a]_R$. In other words,

$$\bigcup_{a \in A} [a]_R = A.$$

From Theorem 1, it follows that these equivalence classes are either equal or disjoint, so $[a]_R \cap [b]_R = \emptyset$ when $[a]_R \neq [b]_R$.

Therefore, the equivalence classes form a partition of A , because they split A into disjoint subsets.

An Equivalence Relation Partitions a Set₂

Theorem 2: Let R be an **equivalence relation** on a set S . Then the equivalence classes of R form a partition of S . Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S , **there is an equivalence relation R that has the sets A_i , $i \in I$, as its equivalence classes.**

Proof: We have already shown the first part of the theorem.

For the second part, assume that $\{A_i \mid i \in I\}$ is a partition of S . Let R be the relation on S consisting of the pairs (x, y) where x and y belong to the same subset A_i in the partition. We must show that R satisfies the properties of an equivalence relation.

- *Reflexivity:* **For every $a \in S$, $(a, a) \in R$** , because a is in the same subset as itself.
- *Symmetry:* **If $(a, b) \in R$** , then b and a are in the same subset of the partition, so **$(b, a) \in R$** .
- *Transitivity:* **If $(a, b) \in R$ and $(b, c) \in R$** , then a and b are in the same subset of the partition, as are b and c . Since the subsets are disjoint and b belongs to both, the two subsets of the partition must be identical. Therefore, **$(a, c) \in R$** since a and c belong to the same subset of the partition.

An Equivalence Relation Partitions a Set₂

Theorem 2: Let R be an **equivalence relation** on a set S . Then the equivalence classes of R form a partition of S . Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S , **there is an equivalence relation R that has the sets A_i , $i \in I$, as its equivalence classes.**

Example.

List the ordered pairs in the equivalence relation R produced by the partition $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$, $A_3 = \{6\}$ of $S = \{1, 2, 3, 4, 5, 6\}$.

Solution.

The pairs $(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)$ belongs to R because $A_1 = \{1, 2, 3\}$ is an equivalence class.

The pairs $(4, 4), (4, 5), (5, 4), (5, 5)$ belongs to R because $A_2 = \{4, 5\}$ is an equivalence class.

The pair $(6, 6)$ belongs to R because $A_3 = \{6\}$ is an equivalence class.

Chapter 9.

Relations

9.1 Relations and Their Properties

9.3 Representing Relations

9.5 Equivalence Relations

9.6 Partial Orderings

Partial Orderings₁

Definition 1: A relation R on a set S is called a **partial ordering (偏序)**, or **partial order**, if it is reflexive, antisymmetric, and transitive. A set together with a partial ordering R is called a **partially ordered set**, or **poset**, and is denoted by **(S, R)** . Members of S are called **elements** of the poset.

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Antisymmetric Relations

Definition: A relation R on a set A such that for all $a, b \in A$ if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called **antisymmetric (反對稱關係)**. Written symbolically, R is antisymmetric if and only if

$$\forall x \forall y [(x, y) \in R \wedge (y, x) \in R \rightarrow x = y]$$

Example: The following relations on the integers are **antisymmetric**:

- $R_1 = \{(a, b) \mid a \leq b\}$
- $R_2 = \{(a, b) \mid a > b\}$
- $R_3 = \{(a, b) \mid a = b\}$
- $R_4 = \{(a, b) \mid a = b + 1\}$

The following relations are **not antisymmetric**:

- $R_5 = \{(a, b) \mid a = b \text{ or } a = -b\}$
(note that both $(1, -1)$ and $(-1, 1)$ belongs to R_5).
- $R_6 = \{(a, b) \mid a + b \leq 3\}$ (note that both $(1, 2)$ and $(2, 1)$ belongs to R_6).

For any integer, if $a \leq b$ and $a \leq b$, then $a = b$.

Partial Orderings₂

Example 1: Show that the “greater than or equal” relation (\geq) is a **partial ordering** on the set of integers.

- **Reflexivity:** $a \geq a$ for every integer a .
- **Antisymmetry:** If $a \geq b$ and $b \geq a$, then $a = b$.
- **Transitivity:** If $a \geq b$ and $b \geq c$, then $a \geq c$.

Partial Orderings₃

Example 2: Show that the divisibility relation ($|$) is a **partial ordering** on the set of integers.

- **Reflexivity:** $a | a$ for all integers a . (see Example 9 in Section 9.1)
- **Antisymmetry:** If a and b are positive integers with $a | b$ and $b | a$, then $a = b$. (see Example 12 in Section 9.1)
- **Transitivity:** Suppose that a divides b and b divides c . Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c . Therefore, the relation is transitive.

$(\mathbb{Z}^+, |)$ is a poset.

Partial Orderings₄

Example 3: Show that the inclusion relation (\subseteq) is a **partial ordering** on the power set of a set S .

- **Reflexivity:** $A \subseteq A$ whenever A is a subset of S .
- **Antisymmetry:** If A and B are positive integers with $A \subseteq B$ and $B \subseteq A$, then $A = B$.
- **Transitivity:** If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Comparability (可比性)

Definition 2: The elements a and b of a poset (S, \preceq) are **comparable** if **either $a \preceq b$ or $b \preceq a$** . When a and b are elements of S so that **neither $a \preceq b$ nor $b \preceq a$** , then a and b are called **incomparable**.

The symbol \preceq is used to denote the relation in any poset.

Definition 3: If (S, \preceq) is a poset and **every two elements of S are comparable**, S is called a **totally ordered (全序) or linearly ordered set**, and \preceq is called a **total order or a linear order**. A totally ordered set is also called a **chain**.

Definition 4: (S, \preceq) is a **well-ordered set (良序)** if it is a poset such that \preceq is a **total ordering** and **every nonempty subset of S has a least element (最小元素)**.

(\mathbb{N}, \leq) is a well-ordered set

(\mathbb{Z}, \leq) is a not well-ordered set

Lexicographic Order

Definition: Given two posets (A_1, \leq_1) and (A_2, \leq_2) , the **lexicographic ordering** (字典序) on $A_1 \times A_2$ is defined by specifying that (a_1, a_2) is less than (b_1, b_2) , that is,

$$(a_1, a_2) < (b_1, b_2),$$

either if $a_1 <_1 b_1$ or if $a_1 = b_1$ and $a_2 <_2 b_2$.

This definition can be easily extended to a lexicographic ordering on strings.

Example: Consider strings of lowercase English letters. A lexicographic ordering can be defined using the ordering of the letters in the alphabet. This is the same ordering as that used in dictionaries.

- **discreet** < **discrete**, because these strings differ in the seventh position and $e < t$.
- **discreet** < **discreetness**, because the first eight letters agree, but the second string is longer.

Hasse Diagrams

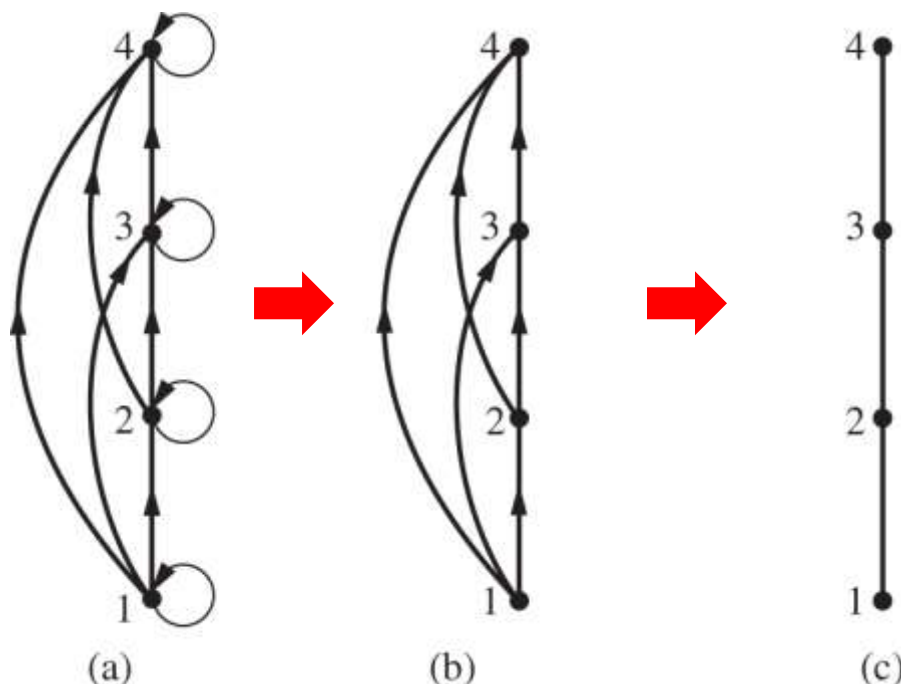
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Partial Orderings

Definition 1: A relation R on a set S is called a **partial ordering** (偏序), or **partial order**, if it is **reflexive, antisymmetric, and transitive**. A set

Definition: A **Hasse diagram** is a visual representation of a **partial ordering** that leaves out edges that must be present because of the reflexive and transitive properties.

The partial ordering $\{(a, b) \mid a \leq b\}$ on the set $\{1, 2, 3, 4\}$



A partial ordering is shown in (a) of the figure above. **The loops due to the reflexive property are deleted** in (b).

The edges that must be present due to **the transitive property are deleted** in (c).

The Hasse diagram for the partial ordering (a), is depicted in (c).

Hasse diagram

Hasse Diagrams

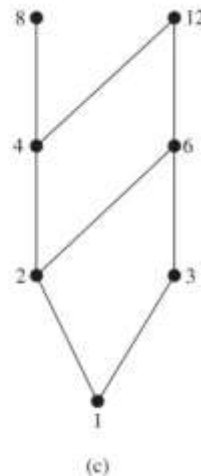
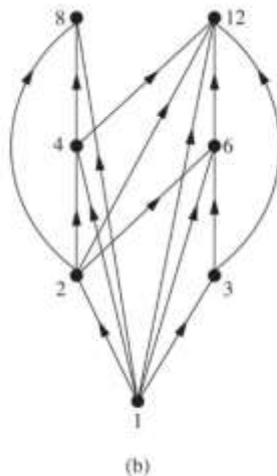
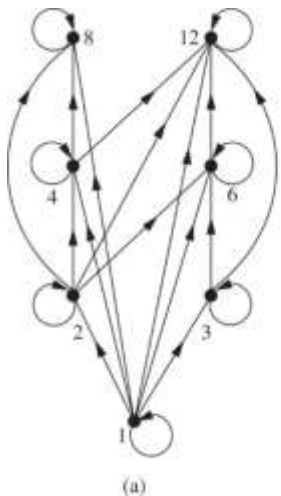
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Partial Orderings

Definition 1: A relation R on a set S is called a **partial ordering** (偏序), or **partial order**, if it is **reflexive, antisymmetric, and transitive**. A set

Definition: A **Hasse diagram** is a visual representation of a **partial ordering** that leaves out edges that must be present because of the reflexive and transitive properties.

The partial ordering $\{(a, b) \mid a \text{ divides } b\}$ on the set $\{1, 2, 3, 4, 6, 8, 12\}$



Hasse diagram

A partial ordering is shown in (a) of the figure above. **The loops due to the reflexive property are deleted** in (b).

The edges that must be present due to **the transitive property are deleted** in (c).

The Hasse diagram for the partial ordering (a), is depicted in (c).

Procedure for Constructing a Hasse Diagram

To represent a **finite poset** (S, \leq) using a Hasse diagram, start with the directed graph of the relation:

- **Remove the loops (a, a)** present at every vertex due to the reflexive property.
- **Remove all edges (x, y)** for which there is an element $z \in S$ such that $x < z$ and $z < y$. These are the edges that must be present **due to the transitive property**.
- Arrange each edge so that its initial vertex is below the terminal vertex. **Remove all the arrows**, because **all edges point upwards toward their terminal vertex**.