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1、

(a) $f(n) = \pm n$: Not a function. (e.g. $f(1) = 1$ or -1)

(b) $f(n) = \sqrt{n^2+1}$: Yes.

(c) $f(n) = \frac{1}{(n^2-4)}$: Not a function. (e.g. $n = \pm 2$, $f(n) \to$ undefined.)

2、

(a)

assuming g isn't 1-1, meaning existed a, b s.t. $g(a) = g(b)$

$\because g(a) = g(b)$

$\therefore f(g(a)) = f(g(b)) \to f \circ g(a) = f \circ g(b)$

Since $f \circ g$ is 1-1, Therefore, by defn., g is 1-1 ※

(b)

assuming $A: \{1, 2\}$, $B: \{1\}$

$f: A \to B$, $g: B \to A$, and $\begin{cases} f(1) = 1 \\ f(2) = 1 \\ g(1) = 1 \end{cases}$

then, because $\forall b \in B$, $\exists a \in A$ s.t. $f(a) = b$, f is onto.

and because $\forall b \in B$, $\exists a \in A$ s.t. $f \circ g(a) = b$, $f \circ g$ also onto.

However, $2 \in A$ but no element s.t. $g(x) = 2$ $\forall x \in B$, g isn't onto.

3.

(a) $a_n = 6a_{n-1}$

$a_0 = 2$

$a_1 = 6 \times 2 = 12$

$a_2 = 6 \times 12 = 72$

$a_3 = 6 \times 72 = 432$

$a_4 = 6 \times 432 = 2592$

$a_5 = 6 \times 2592 = 15552$

(b) $a_n = na_{n-1} + n^2 a_{n-2}$

$a_0 = 1$

$a_1 = 1$

$a_2 = 2 \times 1 + 4 \times 1 = 6$

$a_3 = 3 \times 6 + 9 \times 1 = 27$

$a_4 = 4 \times 27 + 16 \times 6 = 204$

$a_5 = 5 \times 204 + 25 \times 27 = 1695$

(c) $a_n = a_{n-1} + a_{n-3}$

$a_0 = 1$

$a_1 = 2$

$a_2 = 0$

$a_3 = 0 + 1 = 1$

$a_4 = 1 + 2 = 3$

$a_5 = 3 + 0 = 3$

(d) $a_n = na_{n-1} + a_{n-2}^2$

$a_0 = -1$

$a_1 = 0$

$a_2 = 2 \times 0 + (-1)^2 = 1$

$a_3 = 3 \times 1 + 0^2 = 3$

$a_4 = 4 \times 3 + 1^2 = 13$

$a_5 = 5 \times 13 + 3^2 = 74$

(e) $a_n = a_{n-1} - a_{n-2} + a_{n-3}$

$a_0 = 1$

$a_1 = 1$

$a_2 = 2$

$a_3 = 2 - 1 + 1 = 2$

$a_4 = 2 - 2 + 1 = 1$

$a_5 = 1 - 2 + 2 = 1$

4.

(a) $a_n = 2a_{n-1} - 3$, $a_0 = -1$

$a_1 = 2a_0 - 3$

$a_2 = 2a_1 - 3 = 2(2a_0 - 3) - 3 = 4a_0 - 9 = 2^2 a_0 - (3 + 3\times 2)$

$a_3 = 2a_2 - 3 = 2(4a_0 - 9) - 3 = 8a_0 - 21 = 2^3 a_0 - (3 + 3\times 2 + 3\times 4)$

$\Rightarrow a_n = 2^n a_0 - \sum_{k=0}^{n-1} 3\times 2^k$

$= -2^n - 3\sum_{k=0}^{n-1} 2^k$

$= -2^n - 3\times \frac{1\times(2^n - 1)}{2-1}$

$= -2^n - 3\times(2^n - 1)$

$= -2^n - 3\times 2^n + 3$

$= -1\times 2^n - 3\times 2^n + 3$

$= -4\times 2^n + 3$

$= -2^2\times 2^n + 3$

$= -2^{n+2} + 3$ #

(b) $a_n = a_{n-1} + 2n + 3$, $a_0 = 4$

$a_1 = a_0 + 2\times 1 + 3$

$a_2 = a_1 + 2n + 3 = (a_0 + 2(n-1) + 3) + 2n + 3 = a_0 + 4n + (3\times 2 - 2)$

$a_3 = a_2 + 2n + 3 = (a_0 + 4(n-1) + (3\times 2 - 2)) + 2n + 3 = a_0 + 6n + (3\times 3 - 6)$

$\Rightarrow a_n = a_0 + 2\times \sum_{k=1}^{n} k + 3n$

$= a_0 + 2\times\left(\frac{n\times(n+1)}{2}\right) + 3n$

$= a_0 + n^2 + 4n$

$= n^2 + 4n + 4$ #

(c) $a_n = 2n\,a_{n-1}$, $a_0 = 1$

$a_1 = 2 \times n \times a_0$

$a_2 = 2 \times n \times a_1 = 2n \times (2(n-1) \times a_0) = 2^2 \times n(n-1)a_0$

$a_3 = 2 \times n \times a_2 = 2n \times (2^2(n-1)(n-2)a_0)$

$= 2^3 \times n(n-1)(n-2)a_0$

$\Rightarrow a_n = 2^n \times n! \times a_0$

$= 2^n \times n!$ ※

5,

(a) $a_n = -n+2$

<pf> $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$

$a_{n-1} = -(n-1)+2 = -n+3$

$a_{n-2} = -(n-2)+2 = -n+4$

$(-n+3)+2(-n+4)+(2n-9)$

$= -n+3\cancel{-2n}+8\cancel{+2n}-9$

$= -n+2 = a_n$ ※

(b) $a_n = 3(-1)^n + 2^n - n + 2$

<pf>: $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$

$a_{n-1} = 3(-1)^{n-1} + 2^{n-1} - (n-1) + 2$

$a_{n-2} = 3(-1)^{n-2} + 2^{n-2} - (n-2) + 2$

$3(-1)^{n-1} + 2^{n-1} - n + 3 + 2[3(-1)^{n-2} + 2^{n-2} - n + 4]$

$+2n-9$

$= 3(-1)^{n-1} + 2^{n-1} - n + 3 + 6(-1)^{n-2} + 2 \times 2^{n-2} \cancel{-2n} + 8 \cancel{+2n} - 9$

$= 3(-1)^{n-1} + 2^{n-1} + 6(-1)^{n-2} + 2^{n-1} - n + 2$

$= 3(-1)^{n-1} + 6(-1)^{n-2} + 2^n - n + 2$

$= 3(-1)^{x-2} + 2^n - n + 2$ ※

6.

(10%) (a) Show that the union of a countable number of countable sets is countable.

(Sol.) Suppose that $A_1, A_2, A_3, \ldots$, are countable sets. Because A_i is countable, we can list its elements in a sequence as $a_{i1}, a_{i2}, a_{i3}, \ldots$,. The elements of the set $\bigcup_{i=1}^{n} A_i$ can be listed by listing all terms a_{ij} with i + j = 2, then all terms a_{ij} with i + j = 3, then all terms a_{ij} with i + j = 4, and so on.

(a)

let $A_1 = \{ a_{11}, a_{12}, a_{13} \cdots \}$
$A_2 = \{ a_{21}, a_{22}, a_{23} \cdots \}$
$A_3 = \{ a_{31}, a_{32}, a_{33} \cdots \}$

Since each A_i is countable, the number of elements in each A_i is either finite or countably infinite.

let $S = \{ a_{11}, a_{21}, a_{22}, a_{31}, a_{32}, a_{33} \cdots\cdots \}$

Therefore, the elements in S is also finite or countably infinite.

Futhermore, Since S contains elements of all A_i,

S has 1-1 correspondence with the union of A_i.

According to defn., the union of A_i is also countable.

(b)

let $f: Z^+ \times Z^+ \to Z^+$, $f(m,n) = 2^m \times 3^n$

if $f(a,b) = f(m,n)$, then:

$2^a \times 3^b = 2^m \times 3^n$

$2^a = 2^m$
$3^b = 3^n$ $\Rightarrow a = m,\ b = n$

$\Rightarrow f$ is 1-1

$\because f$ is 1-1 $\therefore |Z^+ \times Z^+| \leq |Z^+|$

$\Rightarrow$ when a set is countable, iff $|S| \leq |Z^+|$

$\therefore Z^+ \times Z^+$ is countable.

(b) Show that the set $\boldsymbol{Z}^+ \times \boldsymbol{Z}^+$ is countable.

(Sol.) We can think of $\boldsymbol{Z}^+ \times \boldsymbol{Z}^+$ as the countable union of countable sets, where the i[th] set in the collection, noted as $A_i = \{(i,n) | n \in \boldsymbol{Z}^+\}$, for $i \in \boldsymbol{Z}^+$, is countable. By (a), $\boldsymbol{Z}^+ \times \boldsymbol{Z}^+ = \bigcup_{i=1}^{\infty} A_i$, the union of a countable number of countable sets is countable.

可數集定義 >ω<

definition of Countable Sets

1、) 有限的集 limited Sets

集合裡的元素是有限的。

2、) 与正整數存在一对一關係

可以找到一種方法，將集合中的每个元素与一个唯一的正整數配对，並且每个正整數也都被配对到集合中的一个元素。

6.

(a) <OFFICIAL Solution>

Suppose that $A_1, A_2, A_3 \ldots\ldots, A_n$ are countable sets. Because A_i is Countable, the list in a elements in a Sequence can be present as $a_{i1}, a_{i2}, a_{i3} \ldots\ldots, a_{in}$.

The elements of the set $\bigcup_{i=1}^{n} A_i$ can be listed by listing all terms a_{ij} with $i+j=2$, then all terms a_{ij} with $i+j=3$, then all terms a_{ij} with $i+j=4$ and so on

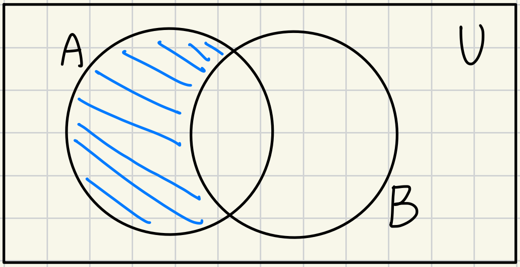
(b) <OFFICIAL Solution>

$Z^+ \times Z^+$ is countable union of countable sets where the i^{th} set in the collection, noted as $A_i = \{(i, n) \mid n \in Z^+\}$, for $i \in Z^+$, is countable. By (a). $Z^+ \times Z^+ = \bigcup_{i=1}^{\infty} A_i$, the union of countable number of countable sets is countable.

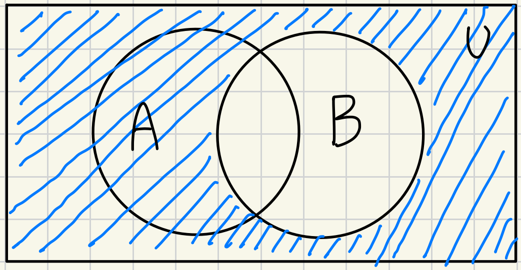
7.

(a) $A-B = A \cap \overline{B}$

$A-B$



$\overline{B}$



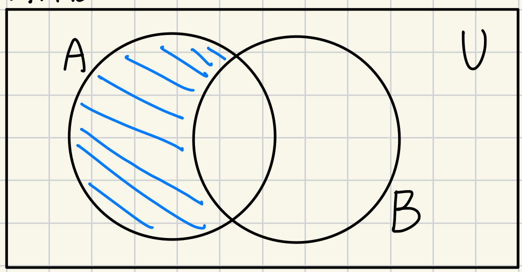
Since the Venn Diagram of $A-B$ and $A \cap \overline{B}$ are equal,

then $A-B = A \cap \overline{B}$ #

<official sol>

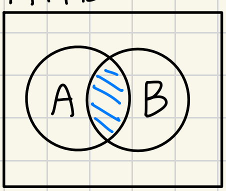
Both sides equal $\{x \mid x \in A \wedge x \notin B\}$

$A \cap \overline{B}$

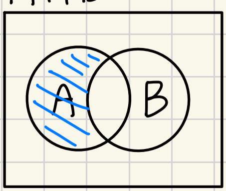


(b) $(A \cap B) \cup (A \cap \overline{B}) = A$

$A \cap B$

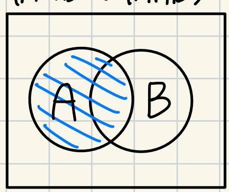


$A \cap \overline{B}$



$\longrightarrow$

$(A \cap B) \cup (A \cap \overline{B})$



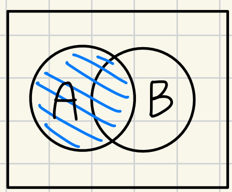
Since the Venn Diagram of $(A \cap B) \cup (A \cap \overline{B})$ and A are equal,

then $(A \cap B) \cup (A \cap \overline{B}) = A$ #

<official sol>

$A = A \cap U = A \cap (B \cup \overline{B}) = (A \cap B) \cup (A \cap \overline{B})$

A



8.

(a)

<pf> $\overline{A\cap B\cap C} = \bar{A} \cup \bar{B} \cup \bar{C}$

assuming $x \in \overline{A\cap B\cap C}$, by defn. $x \notin A\cap B\cap C$

Therefore, $x\in\bar{A}$、$x\in\bar{B}$ or $x\in\bar{C}$

∵ $x \in \bar{A}\cup\bar{B}\cup\bar{C}$

∴ $\overline{A\cap B\cap C} \subseteq \bar{A}\cup\bar{B}\cup\bar{C}$

Then, assuming $x \in \bar{A}\cup\bar{B}\cup\bar{C}$, by defn. $x\in\bar{A}$、$x\in\bar{B}$ or $x\in\bar{C}$

Therefore, x doesn't belong to $A\cap B\cap C$ at the same time.

∵ $x \in \overline{A\cap B\cap C}$

∴ $\bar{A}\cup\bar{B}\cup\bar{C} \subseteq \overline{A\cap B\cap C}$

Since we have shown both $\overline{A\cap B\cap C} \subseteq \bar{A}\cup\bar{B}\cup\bar{C}$ and $\bar{A}\cup\bar{B}\cup\bar{C} \subseteq \overline{A\cap B\cap C}$, can conclude that $\overline{A\cap B\cap C} = \bar{A}\cup\bar{B}\cup\bar{C}$ *

<official sol>

let $x \in \overline{A\cap B\cap C} \equiv x \notin A\cap B\cap C$

$\equiv x\notin A \vee x\notin B \vee x\notin C$

$\equiv x\in\bar{A} \vee x\in\bar{B} \vee x\in\bar{C}$

$\equiv x \in \bar{A}\cup\bar{B}\cup\bar{C}$

(b)

A	B	C	$\overline{A}$	$\overline{B}$	$\overline{C}$	$A \cap B \cap C$	$\overline{A \cap B \cap C}$	$\overline{A} \cup \overline{B} \cup \overline{C}$
1	1	1	0	0	0	1	0	0
1	1	0	0	0	1	0	1	1
1	0	1	0	1	0	0	1	1
1	0	0	0	1	1	0	1	1
0	1	1	1	0	0	0	1	1
0	1	0	1	0	1	0	1	1
0	0	1	1	1	0	0	1	1
0	0	0	1	1	1	0	1	1

9.

(a)

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -2 & -1 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -3 \\ 1 & 0 & 2 \\ 9 & -4 & 4 \end{bmatrix} \#$$

(b)

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

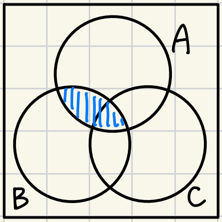
$$A \vee B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A \wedge B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

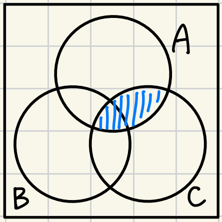
10.

(a) $(A \cap B) \cup (A \cap C)$

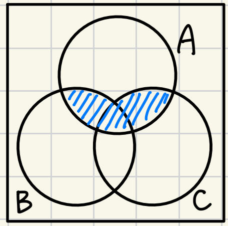
$A \cap B$



$A \cap C$

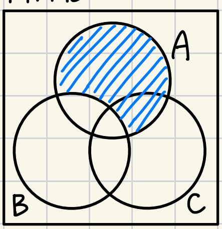


$(A \cap B) \cup (A \cap C)$

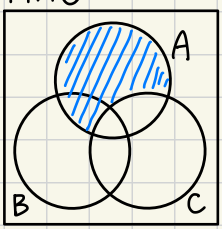


(b) $(A \cap \bar{B}) \cup (A \cap \bar{C})$

$A \cap \bar{B}$



$A \cap \bar{C}$



$(A \cap \bar{B}) \cup (A \cap \bar{C})$