Homework 2.

- The file name of your homework (in PDF) should be in the format: "學號-作業編號.pdf". For example: 00957999-hw2.pdf
- Please submit your homework to Tronclass **before 23:59, October 27 (Sunday), 2024.**

(可以用 word 檔寫完後轉成 pdf 檔上傳，或是手寫後拍照後存成 pdf 檔上傳)

1. (5%) Determine whether f is a function from **Z** to **R** if

   (a) $f(n) = \pm n$

   (Sol.) No (寫出 Yes 或 No 即可). This is not a function because the rule is not well-defined. We do not know whether $f(3) = 3$ or $f(3) = -3$. For a function, it cannot be both at the same time.

   (b) $f(n) = \sqrt{n^2 + 1}$

   (Sol.) Yes (寫出 Yes 或 No 即可). This is a function. For all integers n, $\sqrt{n^2 + 1}$ is a well-defined real number.

   (c) $f(n) = 1/(n^2 - 4)$

   (Sol.) No (寫出 Yes 或 No 即可). This is not a function with domain Z, since for n = 2 (and also for n = -2) the value of $f(n)$ is not defined by the given rule. In other words, $f(2)$ and $f(-2)$ are not specified since division by 0 makes no sense.

2. (10%) (a) If f and $f \circ g$ are one-to-one, does it follow that g is one-to-one?

   (Sol.) Yes (寫出 Yes 或 No 即可). To clarify the setting, suppose that $g: A \to B$ and $f: B \to C$, so that $f \circ g: A \to C$. We will prove that if $f \circ g$ is one-to-one, then g is also one-to-one, so not only is the answer to the question "yes," but part of the hypothesis is not even needed. Suppose that g were not one-to-one. By definition this means that there are distinct elements a_1 and a_2 in A such that $g(a_1) = g(a_2)$. Then certainly $f\big(g(a_1)\big) = f(g(a_2))$, which is the same statement as $f \circ g(a_1) = f \circ g(a_2)$. By definition this means that $f \circ g$ is not one-to-one (contradiction!), and our proof is complete.

   (b) If f and $f \circ g$ are onto, does it follow that g is onto?

(Sol.) No (寫出 Yes 或 No 即可). For example, suppose that $g: \{a\} \to \{b, c\}$ and $f: \{b, c\} \to \{d\}$, then $f \circ g: \{a\} \to \{d\}$. Assume $g(a) = b, f(b) = d$, and $f(c) = d$. Then f and $f \circ g$ are onto, but g is not.

3. (10%) Find the first five terms of the sequence defined by each of these recurrence relations and initial conditions.
   (a) $a_n = 6a_{n-1}, a_0 = 2$
   (Sol.) 2, 12, 72, 432, 2592
   (b) $a_n = na_{n-1} + n^2 a_{n-2}, a_0 = 1, a_1 = 1$
   (Sol.) 1, 1, 6, 27, 204
   (c) $a_n = a_{n-1} + a_{n-3}, a_0 = 1, a_1 = 2, a_2 = 0$
   (Sol.) 1, 2, 0, 1, 3
   (d) $a_n = na_{n-1} + a_{n-2}^2, a_0 = -1, a_1 = 0$
   (Sol.) -1, 0, 1, 3, 13
   (e) $a_n = a_{n-1} - a_{n-2} + a_{n-3}, a_0 = 1, a_1 = 1, a_2 = 2$
   (Sol.) 1, 1, 2, 2, 1

4. (15%) Find the solution to each of these recurrence relations and initial condition. (請寫出計算過程)
   (a) $a_n = 2a_{n-1} - 3, a_0 = -1$
   (Sol.)

$$
\begin{aligned}
a_n &= -3 + 2a_{n-1} \\
&= -3 + 2(-3 + 2a_{n-2}) = -3 + 2(-3) + 4a_{n-2} \\
&= -3 + 2(-3) + 4(-3 + 2a_{n-3}) = -3 + 2(-3) + 4(-3) + 8a_{n-3} \\
&= -3 + 2(-3) + 4(-3) + 8(-3 + 2a_{n-4}) = -3 + 2(-3) + 4(-3) + 8(-3) + 16a_{n-4} \\
&\ \ \vdots \\
&= -3(1 + 2 + 4 + \cdots + 2^{n-1}) + 2^n a_{n-n} = -3(2^n - 1) + 2^n(-1) = -2^{n+2} + 3
\end{aligned}
$$

   (b) $a_n = a_{n-1} + 2n + 3, a_0 = 4$
   (Sol.) $a_n = n^2 + 4n + 4$
   (c) $a_n = 2na_{n-1}, a_0 = 1$
   (Sol.) $a_n = 2^n n!$

5. (10%) Show that the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$ if
   (a) $a_n = -n + 2$
   (Sol.) $a_{n-1} + 2a_{n-2} + 2n - 9 = -(n - 1) + 2 + 2[-(n - 2) + 2] +$

$2n - 9 = -n + 2 = a_n$

(b) $a_n = 3(-1)^n + 2^n - n + 2$

(Sol.) $a_{n-1} + 2a_{n-2} + 2n - 9 = 3(-1)^{n-1} + 2^{n-1} - (n-1) + 2 + 2[3(-1)^{n-2} + 2^{n-2} - (n-2) + 2] + 2n - 9 = 3(-1)^{n-2}(-1+2) + 2^{n-2}(2+2) - n + 2 = a_n$

6. (10%) (a) Show that the union of a countable number of countable sets is countable.

(Sol.) Suppose that $A_1, A_2, A_3, \ldots$, are countable sets. Because A_i is countable, we can list its elements in a sequence as $a_{i1}, a_{i2}, a_{i3}, \ldots$,. The elements of the set $\cup_{i=1}^{n} A_i$ can be listed by listing all terms a_{ij} with i + j = 2, then all terms a_{ij} with i + j = 3, then all terms a_{ij} with i + j = 4, and so on.

(b) Show that the set $\boldsymbol{Z}^+ \times \boldsymbol{Z}^+$ is countable.

(Sol.) We can think of $\boldsymbol{Z}^+ \times \boldsymbol{Z}^+$ as the countable union of countable sets, where the ith set in the collection, noted as $A_i = \{(i,n) | n \in \boldsymbol{Z}^+\}$, for $i \in \boldsymbol{Z}^+$, is countable. By (a), $\boldsymbol{Z}^+ \times \boldsymbol{Z}^+ = \cup_{i=1}^{\infty} A_i$, the union of a countable number of countable sets is countable.

7. (10%) Show that if A and B are sets, then

a) $A - B = A \cap \overline{B}$.

(Sol.) Both sides equal {x | x ∈ A ∧ x ∉ B}.

b) $(A \cap B) \cup (A \cap \overline{B}) = A$.

(Sol.) $A = A \cap U = A \cap (B \cup \overline{B}) = (A \cap B) \cup (A \cap \overline{B})$

8. (10%) Show that if A, B, and C are sets, then $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$

a) by showing each side is a subset of the other side.

(Sol.)

$x \in \overline{A \cap B \cap C} \equiv x \notin A \cap B \cap C \equiv x \notin A \vee x \notin B \vee x \notin C \equiv x \in \overline{A} \vee x \in \overline{B} \vee x \in \overline{C} \equiv x \in \overline{A} \cup \overline{B} \cup \overline{C}$

b) using a membership table.

(Sol.)

A	B	C	$A \cap B \cap C$	$\overline{A \cap B \cap C}$	$\overline{A}$	$\overline{B}$	$\overline{C}$	$\overline{A} \cup \overline{B} \cup \overline{C}$
1	1	1	1	0	0	0	0	0
1	1	0	0	1	0	0	1	1
1	0	1	0	1	0	1	0	1
1	0	0	0	1	0	1	1	1
0	1	1	0	1	1	0	0	1
0	1	0	0	1	1	0	1	1
0	0	1	0	1	1	1	0	1
0	0	0	0	1	1	1	1	1

9. (10%) (a) Find $\boldsymbol{AB}$ if $\boldsymbol{A} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 3 & -2 & -1 \\ 1 & 0 & 2 \end{bmatrix}$.

(Sol.) $\begin{bmatrix} 2 & -2 & -3 \\ 1 & 0 & 2 \\ 9 & -4 & 4 \end{bmatrix}$

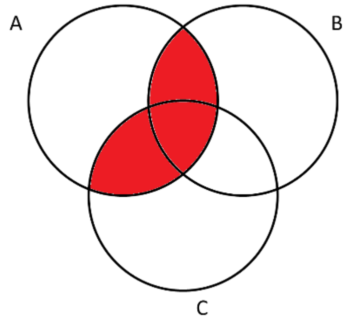
(b) Let $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. Find $\boldsymbol{A} \vee \boldsymbol{B}$ and $\boldsymbol{A} \wedge \boldsymbol{B}$.

(Sol.) $\boldsymbol{A} \vee \boldsymbol{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $\boldsymbol{A} \wedge \boldsymbol{B} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

10. (10%) Draw the Venn diagrams for each of these combinations of the sets A, B, and C.

a) $(A \cap B) \cup (A \cap C)$

(Sol.)



b) $(A \cap \overline{B}) \cup (A \cap \overline{C})$

(Sol.)