Homework 4.

- The file name of your homework (in PDF) should be in the format: “學號-作業編號.pdf”. For example: 00957999-hw4.pdf
- Please submit your homework to Tronclass **before 23:59, December 22 (Sunday), 2024.**

(可以用 word 檔寫完後轉成 pdf 檔上傳，或是手寫後拍照後存成 pdf 檔上傳)

1. (3%) How many numbers must be selected from the set {1, 3, 5, 7, 9, 11, 13, 15} to guarantee that at least one pair of these numbers add up to 16 (其中有兩個數加起來大於等於 16)?

   (Sol.) 5.

   Reason: We can point out that choosing four numbers is not enough, since we could choose {1, 3, 5, 7}, and no pair of them add up to more than 12.

2. (4%) A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A man takes socks out at random in the dark.
   (a) How many socks must he take out to be sure that he has at least two socks of the same color?
   (Sol.) 3
   (b) How many socks must he take out to be sure that he has at least two black socks?
   (Sol.) 14

3. (4%) (a) How many subsets with an odd number of elements does a set with 10 elements have?

   (Sol.) 512.

   C(10, 1) + C(10, 3) + C(10, 5) + C(10, 7) + C(10, 9) = 10 + 120 + 252 + 120 + 10 = 512.

   (b) How many subsets with more than two elements does a set with 100 elements have?

   (Sol.) 2100−5051

4. (16%) How many positive integers between 100 and 999 inclusive
   (a) are divisible by 7?
   (Sol.) 128
   (b) are odd?
   (Sol.) 450
   (c) have the same three decimal digits?

(Sol.) 9

(d) are not divisible by 4?

(Sol.) 675

(e) are divisible by 3 or 4?

(Sol.) 450

(f) are not divisible by either 3 or 4?

(Sol.) 450

(g) are divisible by 3 but not by 4?

(Sol.) 225

(h) are divisible by 3 and 4?

(Sol.) 75

5. (4%) (a) What is the coefficient of x^9 in $(2 - x)^{19}$?

   (Sol.) -2^{10} C (19,9) = −94,595,072

   (b) What is the coefficient of x^8y^9 in the expansion of $(3x + 2y)^{17}$?

   (Sol.) $C(17,9)3^8 2^9$ = 24310 * 6561 * 512 = 81,662,929,920.

6. (4%) The row of Pascal's triangle containing the binomial coefficients $\binom{10}{k}$, $0 \le k \le 10$, is:

   1 10 45 120 210 252 210 120 45 10 1

   Use Pascal's identity to produce the row immediately following this row in Pascal's triangle.

   (Sol.) 1 11 55 165 330 462 462 330 165 55 11 1

7. (12%) (a) Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one stair or two stairs at a time.

   (Sol.) $a_n = a_{n-1} + a_{n-2}$ for $n \ge 2$

   (b) What are the initial conditions?

   (Sol.) $a_0 = 1,\ a_1 = 1$

   (c) In how many ways can this person climb a flight of eight stairs?

   (Sol.) 34

8. (25%) Solve these recurrence relations together with the initial conditions given.

   (a) $a_n = 2a_{n-1}$ for $n \ge 1, a_0 = 3$

   (Sol.) $a_n = 3 \cdot 2^n$

   (b) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \ge 2, a_0 = 1, a_1 = 0$

   (Sol.) $a_n = 3 \cdot 2^n - 2 \cdot 3^n$

   (c) $a_n = a_{n-2} / 4$ for $n \ge 2, a_0 = 1, a_1 = 0$

   (Sol.) $a_n = \left(\frac{1}{2}\right)^{n+1} - \left(-\frac{1}{2}\right)^{n+1}$

(d) $a_n = 6a_{n-1} - 8a_{n-2}$ for $n \geq 2, a_0 = 4, a_1 = 10$

(Sol.) $a_n = 3 \cdot 2^n + 4^n$

(e) $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with $a_0 = 5, a_1 = -9$, and $a_2 = 15$

(Sol.) $a_n = (n^2 + 3n + 5)(-1)^n$

9. (10%) (a) Find all solutions of the recurrence relation $a_n = 2a_{n-1} + 2n^2$.

(Sol.) $a_n = \alpha 2^n - 2n^2 - 8n - 12.$

b) Find the solution of the recurrence relation in part (a) with initial condition $a_1 = 4$.

(Sol.) $a_n = 13 \cdot 2^n - 2n^2 - 8n - 12$

10. (8%) How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21,$$

where x_i, $i = 1, 2, 3, 4, 5$, is a nonnegative integer such that

(a) $x_1 \geq 1$?

(Sol.) 10,626

想成 $(x_1 - 1) + x_2 + x_3 + x_4 + x_5 = 20$

(b) $x_i \geq 2$ for $i = 1, 2, 3, 4, 5$?

(Sol.) 1,365

想成 $(x_1 - 2) + (x_2 - 2) + (x_3 - 2) + (x_4 - 2) + (x_5 - 2) = 11$

(c) $0 \leq x_1 \leq 10$?

(Sol.) 11,649

想成 全部減掉x_1為 11~21 的情形

(d) $0 \leq x_1 \leq 3,\ 1 \leq x_2 < 4$, and $x_3 \geq 15$?

(Sol.) 106

先想成 $x_1 + x_2 + (x_3 - 15) + x_4 + x_5 = 6$ · 再分別處理$x_1 = 0, 1, 2, 3$配上

$x_2 = 1, 2, 3$的情形。

11. (10%) (a) Find all solutions of the recurrence relation $a_n = 2a_{n-1} + 3^n$.

(Sol.) $a_n = \alpha 2^n + 3^{n+1}$

(b) Find the solution of the recurrence relation in part (a) with initial condition $a_1 = 5$.

(Sol.) $a_n = -2 \cdot 2^n + 3^{n+1}$