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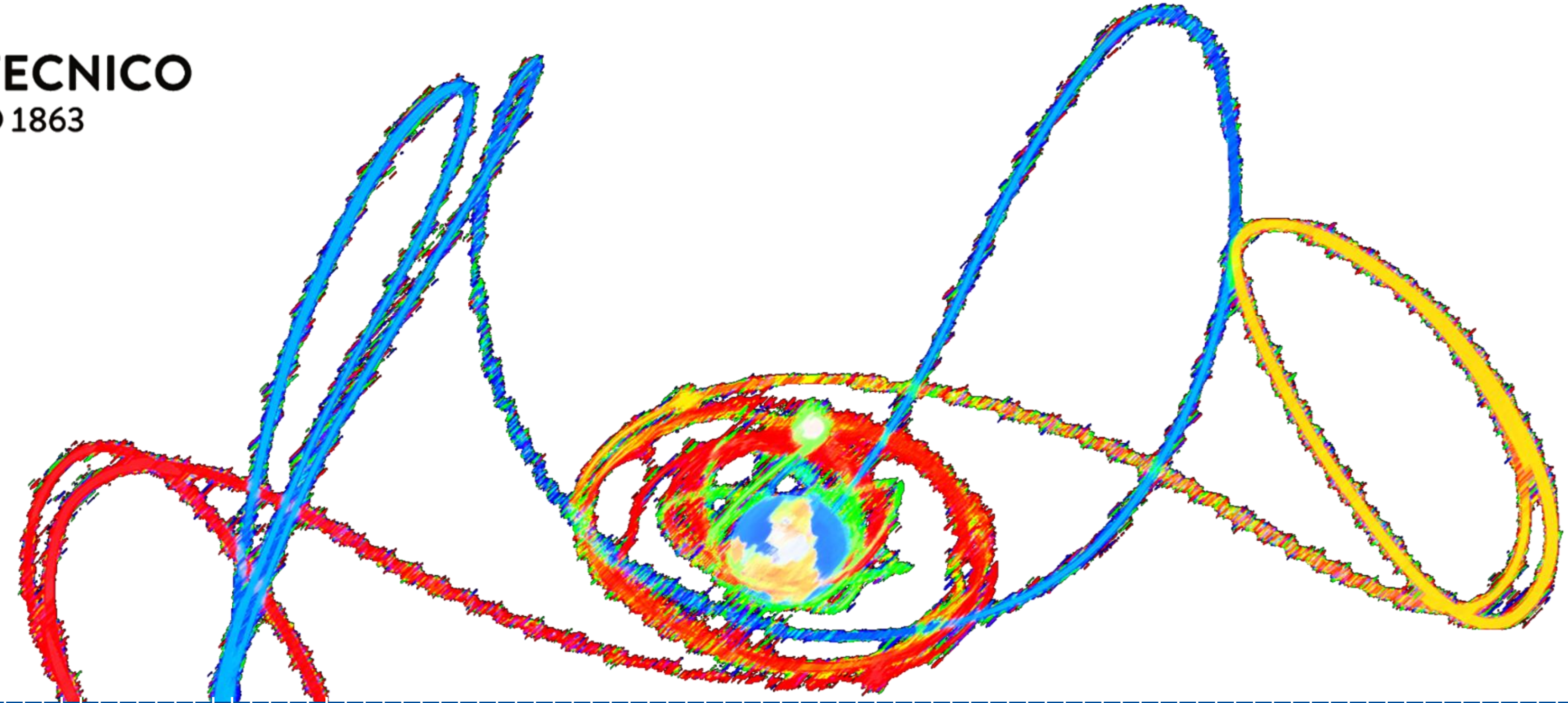

Orbital Mechanics
Module 5: Orbit perturbations

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Orbit perturbations

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References

1. Curtis, H. D.. *Orbital mechanics for engineering students,* Butterworth-Heinemann , 2014. Chapter 12
2. Vallado, D.A. *Fundamental of Astrodynamics and Applications*, 4th Ed, Microcosm Press, 2013. Chapters 8 and 9
3. Battin, R., *An Introduction to the Mathematics and Methods of Astrodynamics*, AIAA Education Series, 1999. Chapter 10
4. Colombo, C., lectures notes and slides

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PROPAGATION OF PERTURBED ORBITS

Orbital perturbations

Orbital perturbations: Any effect that causes an orbit to deviate from a Keplerian orbit

- In previous labs, we have worked with the second zonal harmonic of Earth's gravitational potential, J_2
- Other perturbations (you will study them in the lectures) [1,2,4]
  - Gravity anomalies
  - Solar radiation pressure
  - Atmospheric drag
  - Third body (e.g., Sun, Moon)

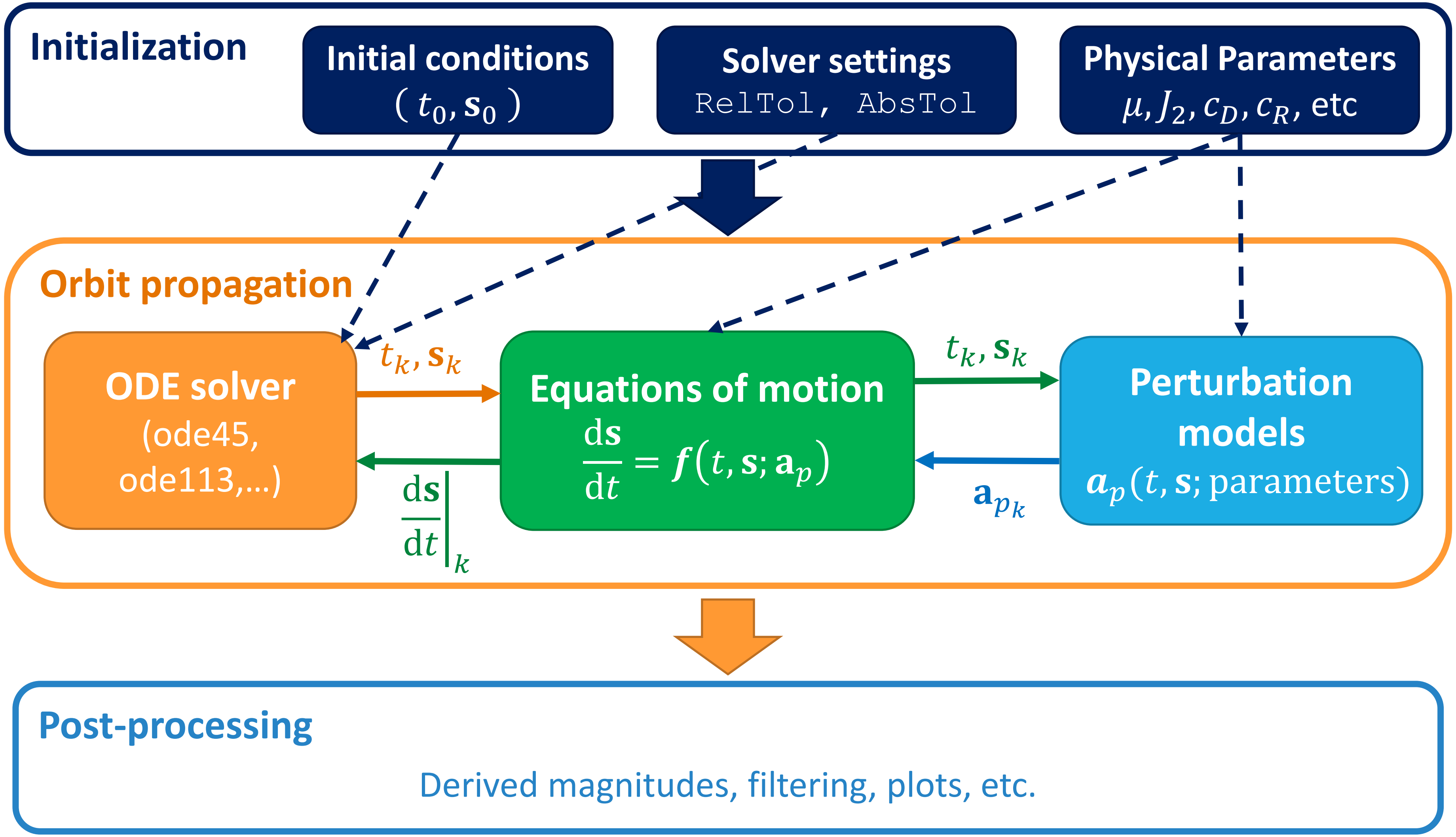
We can **propagate a perturbed orbit** by numerically integrating the **equations of motion**, together with models for the **perturbations**

- We have done this in previous labs using the equations of motion in Cartesian coordinates

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = -\frac{\mu}{r^3}\mathbf{r} + \sum \mathbf{a}_p$$

Numerical Orbit Propagation

Code structure



Try to make each block as generic and reusable as possible (e.g., the same equations of motion can be used with different ODE solvers, the same perturbation models can be used for different formulations of the equations of motion)

Gauss planetary equations

Equations of motion for the Keplerian elements

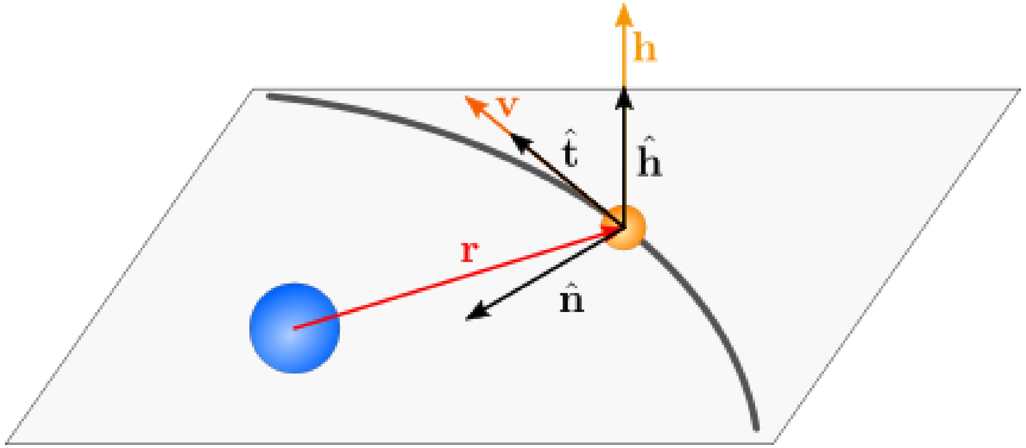
Gauss planetary equations describe the motion in terms of the variations of the Keplerian elements

- Different formulations depending on the elements considered:
  - Some formulations use h instead of a
  - For the anomaly, it is possible to use f or M (or even E for eccentric orbits)

- They also depend on the reference frame used for the perturbing acceleration

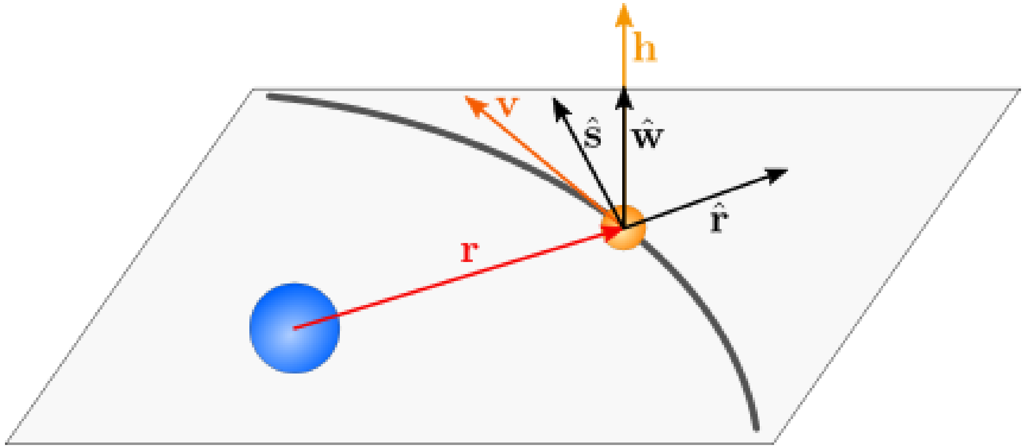
TNH (tangential – normal – out-of-plane) [3]

$$\hat{\mathbf{t}} = \frac{\mathbf{v}}{v}\ ,\ \hat{\mathbf{h}} = \frac{\mathbf{r}\times\mathbf{v}}{\|\mathbf{r}\times\mathbf{v}\|}\ ,\ \hat{\mathbf{n}} = \hat{\mathbf{h}} \times \hat{\mathbf{t}}$$



RSW (radial – transversal – out-of-plane) [1,2]

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r}\ ,\ \hat{\mathbf{w}} = \frac{\mathbf{r}\times\mathbf{v}}{\|\mathbf{r}\times\mathbf{v}\|}\ ,\ \hat{\mathbf{s}} = \hat{\mathbf{w}} \times \hat{\mathbf{r}}$$



Gauss planetary equations

Formulation for perturbing accelerations in TNH frame

From [3], with perturbing acceleration $\mathbf{a} = a_t\hat{\mathbf{t}} + a_n\hat{\mathbf{n}} + a_h\hat{\mathbf{h}}$:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2a^2 v}{\mu} a_t$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{1}{v}\left(2(e+\cos f)a_t - \frac{r}{a}\sin f\, a_n\right)$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{r\cos(f+\omega)}{h} a_h$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{r\sin(f+\omega)}{h\sin i} a_h$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{1}{ev}\left(2\sin f\, a_t + \left(2e + \frac{r}{a}\cos f\right) a_n\right) - \frac{r\sin(f+\omega)\cos i}{h\sin i} a_h$$

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{h}{r^2} - \frac{1}{ev}\left(2\sin f\, a_t + \left(2e + \frac{r}{a}\cos f\right) a_n\right)$$

or

$$\frac{\mathrm{d}M}{\mathrm{d}t} = n - \frac{b}{eav}\left(2\left(1 + \frac{e^2 r}{p}\right)\sin f\, a_t + \frac{r}{a}\cos f\, a_n\right)$$

Some useful relations:

$$v = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}}$$

$$b = a\sqrt{1-e^2}$$

$$p = \frac{b^2}{a} = \frac{h^2}{\mu} = a(1-e^2)$$

$$n = \sqrt{\frac{\mu}{a^3}}$$

$$h = \sqrt{p\mu} = nab$$

$$r = \frac{p}{1+e\cos f}$$

Gauss planetary equations

Formulation for perturbing accelerations in RSW frame

From [1,2,3], with perturbing acceleration $\mathbf{a} = a_r \hat{\mathbf{r}} + a_s \hat{\mathbf{s}} + a_w \hat{\mathbf{w}}$:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2a^2}{h}\left(e \sin f \, a_r + \frac{p}{r} a_s\right) \quad \text{or}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = r \, a_s$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{1}{h}\left(p \sin f \, a_r + \left((p+r)\cos f + re\right)a_s\right)$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{r\cos(f+\omega)}{h} a_w$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{r\sin(f+\omega)}{h \sin i} a_w$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{1}{he}\left(-p\cos f \, a_r + (p+r)\sin f \, a_s\right) - \frac{r\sin(f+\omega)\cos i}{h \sin i} a_w$$

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{h}{r^2} + \frac{1}{eh}\left(p\cos f \, a_r - (p+r)\sin f \, a_s\right) \quad \text{or}$$

$$\frac{\mathrm{d}M}{\mathrm{d}t} = n + \frac{b}{ahe}\left((p\cos f - 2re)a_r - (p+r)\sin f \, a_s\right)$$

Some useful relations:

$$v = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}}$$

$$b = a\sqrt{1-e^2}$$

$$p = \frac{b^2}{a} = \frac{h^2}{\mu} = a(1-e^2)$$

$$n = \sqrt{\frac{\mu}{a^3}}$$

$$h = \sqrt{p\mu} = nab$$

$$r = \frac{p}{1+e\cos f}$$

J_2 acceleration in different frames

- As function of Cartesian position [1,2]:

$$\mathbf{a}_{J_2}^{xyz} = \frac{3}{2}\frac{J_2\mu R_e^2}{r^4}\left[\frac{x}{r}\left(5\frac{z^2}{r^2}-1\right)\hat{\mathbf{i}} + \frac{y}{r}\left(5\frac{z^2}{r^2}-1\right)\hat{\mathbf{j}} + \frac{z}{r}\left(5\frac{z^2}{r^2}-3\right)\hat{\mathbf{k}}\right]$$

- As function of Keplerian elements, in RSW frame ([1], Ex. 12.5, Eq. 12.88):

$$\mathbf{a}_{J2}^{\mathrm{rsw}} = -\frac{3}{2}\frac{J_2\mu R^2}{r^4}\begin{bmatrix} 1-3\sin^2 i\sin^2(f+\omega) \\ \sin^2 i\sin 2(f+\omega) \\ \sin 2i\sin(f+\omega) \end{bmatrix}$$

- Rotation from TNH to RSW frame, as function of Keplerian elements ([3], Problem 10-7):

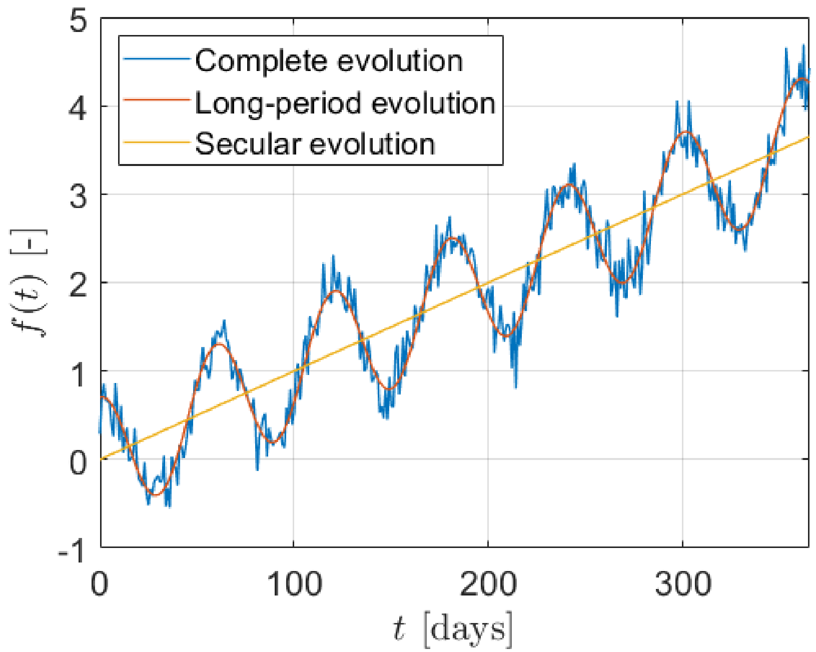
$$\begin{bmatrix} a_r \\ a_s \end{bmatrix} = \frac{h}{pv}\begin{bmatrix} e\sin f & -(1+e\cos f) \\ 1+e\cos f & e\sin f \end{bmatrix}\begin{bmatrix} a_t \\ a_n \end{bmatrix}$$

FILTERING

Timescales in orbital elements evolution

Secular, long-periodic, and short-periodic

- Different **orbital perturbations** affect the orbital elements in different timescales
  - Each perturbation can have several characteristic periods (e.g., 1 orbit, 1 year, and 1 solar cycle for SRP)
  - Different perturbations may have some common and some different frequencies (e.g., most of them include a 1-orbit frequency)
  - Linked to physical properties



- We can analyse these frequencies/periods filtering the results from the **numerical propagation**
  - Filtering the results from a full numerical propagation is not a semi-analytical method.
    In semi-analytical methods, fast frequencies are removed from the **equations of motion** **before** **numerical integration** (details on the lectures)

Filtering

Challenges and best practices

- Low-pass filters remove frequencies (periods) higher (lower) that a given threshold, the **cut-off frequency (period)**
  - We can remove the short-periodic and leave long-periodic + secular, remove short- and long-periodic and keep only secular, etc.
- **But filtering is no trivial endeavour**
  - No filter is perfect. Part of the signal above the cut-off frequency will remain, and the signal below the cut-off frequency will be affected (e.g., attenuation)
  - Important to choose properly the cut-off frequency/period
    - Consider physical aspects (i.e., your perturbations) and your numerical results
  - **Gibbs phenomenon**: Filters produce mathematical artifacts around **discontinuities**. Gibbs phenomenon depends on filter type and cut-off frequency, but is always present.
    - Some discontinuities cannot be avoided, like at the beginning and end of the data set, or physical discontinuities (e.g. impulsive manoeuvre).
    - Others are mathematical and must be avoided, like jumps in Ω, ω, M, f when we express them in $[0,360]$ deg. `unwrap` the data before filtering.

Filtering

Moving mean as a low-pass filter

- Given a set of data points, a **moving mean** computes the mean at each point **considering only the neighbouring values**
  - It can be seen as a 'sliding window' that moves along the data set computing the mean value (hence the name)
  - The window does not need to be centred at the point
  - It acts as a **low-pass filter** (*the time width of the window must correspond to the desired cut-off period*)

- Matlab includes a moving mean implementation: `movmean`

```
m = movmean( data_vector, npoints_window )
m = movmean( data_vector, [npoints_before npoints_after] )
```

  - The values of `data_vector` must be uniformly spaced in time
  - Check the documentation for additional functionalities (e.g., optional input `'Endpoints'` to decide what to do at the boundaries of the dataset)

EXERCISE: ORBIT PROPAGATION WITH GAUSS EQUATIONS

Exercise: Orbit propagation with Gauss Equations

You can reuse this for Assignment 2!

Exercise: Propagate an Earth orbit perturbed by J_2 using Gauss planetary equations and study the results.

1. Implement the code for orbit propagation with Gauss planetary equations
   a. Implement a function with the equations of motion
      - Inputs: time, vector of Keplerian elements, μ of the primary, and a generic function `a_per(t,kep)` that returns the vector of perturbing accelerations
      - You can use the formulation of Gauss equations that you prefer, just be consistent with the set of Keplerian elements and the reference frame for the perturbations
   b. Implement a function for the perturbing acceleration due to J_2
      - Inputs: time, state (Cartesian or Keplerian), and required parameters
   c. Implement a function for orbit propagation using the previous 2 functions
      - Inputs: Initial conditions, time span, physical parameters
      - This function may need to contain additional functions or anonymous functions to adapt the interfaces of the different functions involved

Exercise: Orbit propagation with Gauss Equations

You can reuse this for Assignment 2!

Exercise: Propagate an Earth orbit perturbed by J_2 using Gauss planetary equations and study the results.

2. Validate your new code by comparing it with the propagator in Cartesian coordinates you have from previous labs
   a. Propagate the given orbit using Gauss planetary equations
   b. Propagate the given orbit in Cartesian coordinates, and convert the results to Keplerian elements
   c. For each element:
      - Make a plot showing both solutions together
      - Make a plot showing the error between both solutions (absolute or relative error). For computing the error, you must propagate both orbits at the same time steps.
      - Remember to use appropriate units, ranges, labels, etc. for the plots

Exercise: Orbit propagation with Gauss Equations

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You can reuse this for Assignment 2!

Exercise: Propagate an Earth orbit perturbed by J_2 using Gauss planetary equations and study the results.

3. Filter your results to isolate the secular (linear) evolution
   a. Choose an appropriate cut-off period to remove oscillations
   b. Filter all elements. Remember that, by default, `movmean` uses datasets uniformly spaced in time.
   c. Plot together the filtered and unfiltered results for each element.
   d. Compare the slopes of the filtered Ω and ω with the analytical J_2 approximations seen in Module 2.
      - The goal is not to get an accurate numerical value of the slope, but to verify that the results from the numerical propagation are consistent with the approximate analytic models.

Data:

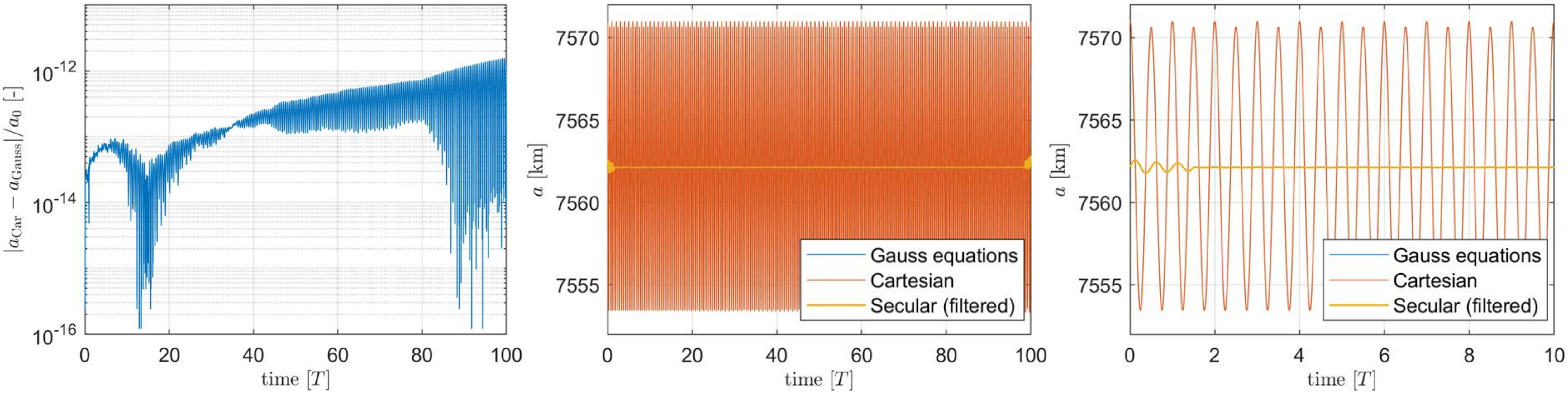
$\mu_\oplus$, $R_\oplus$, and J_2 from `astroConstants` (identifiers `13`, `23`, and `9`, respectively)

$$\mathbf{kep}_0 = [a, e, i, \Omega, \omega, f] = [\,7571\ \mathrm{km}, 0.01, 87.9\ \mathrm{deg}, 180\ \mathrm{deg}, 180\ \mathrm{deg}, 0\ \mathrm{deg}\,]$$

Propagation time: up to 100 periods

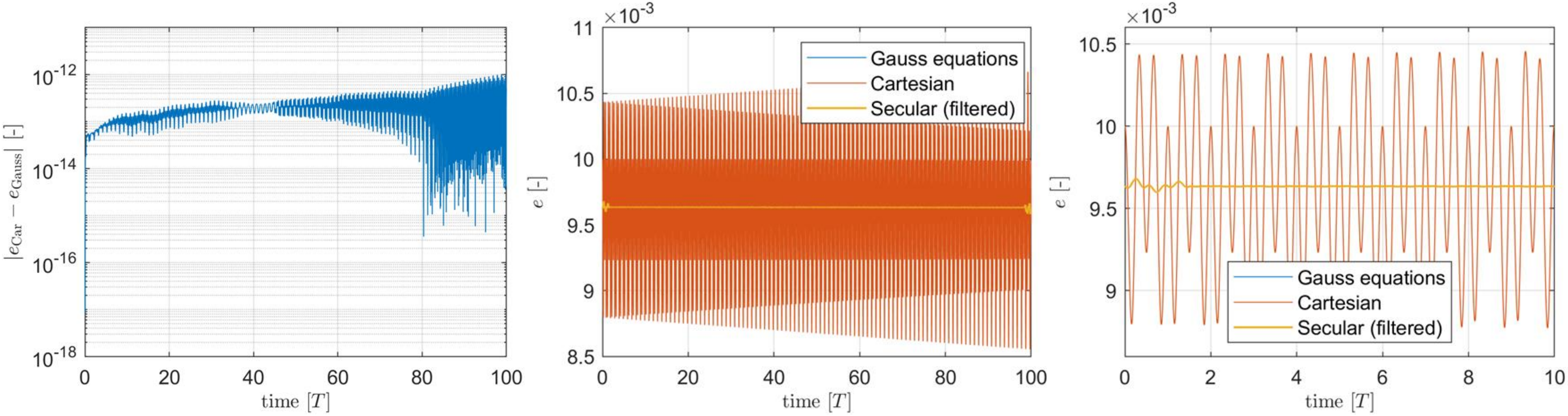
Exercise: Orbit propagation with Gauss Equations

Sample results – a



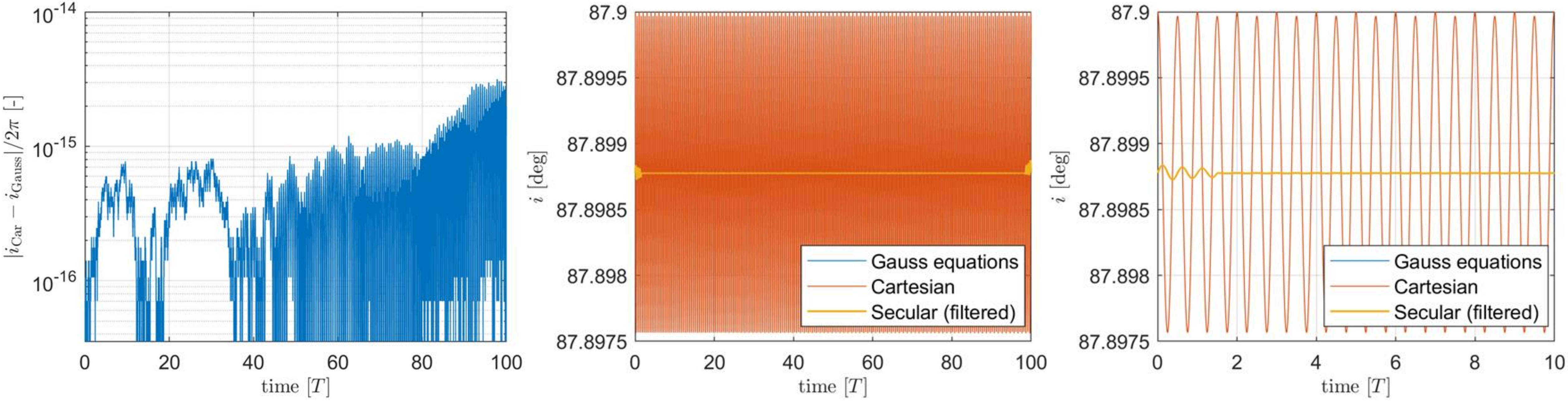
Exercise: Orbit propagation with Gauss Equations

Sample results – e



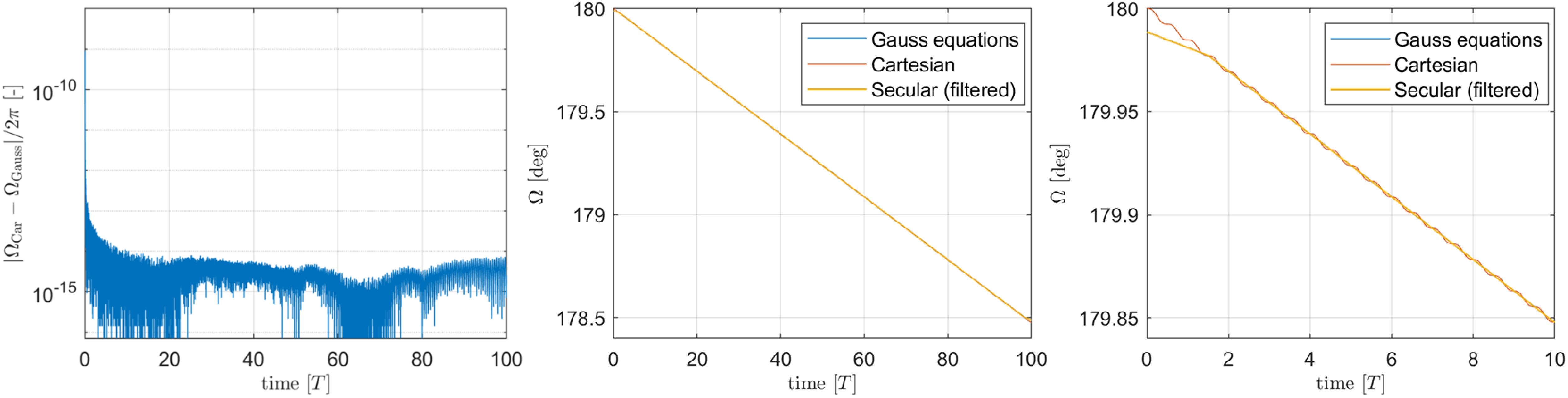
Exercise: Orbit propagation with Gauss Equations

Sample results – i



Exercise: Orbit propagation with Gauss Equations

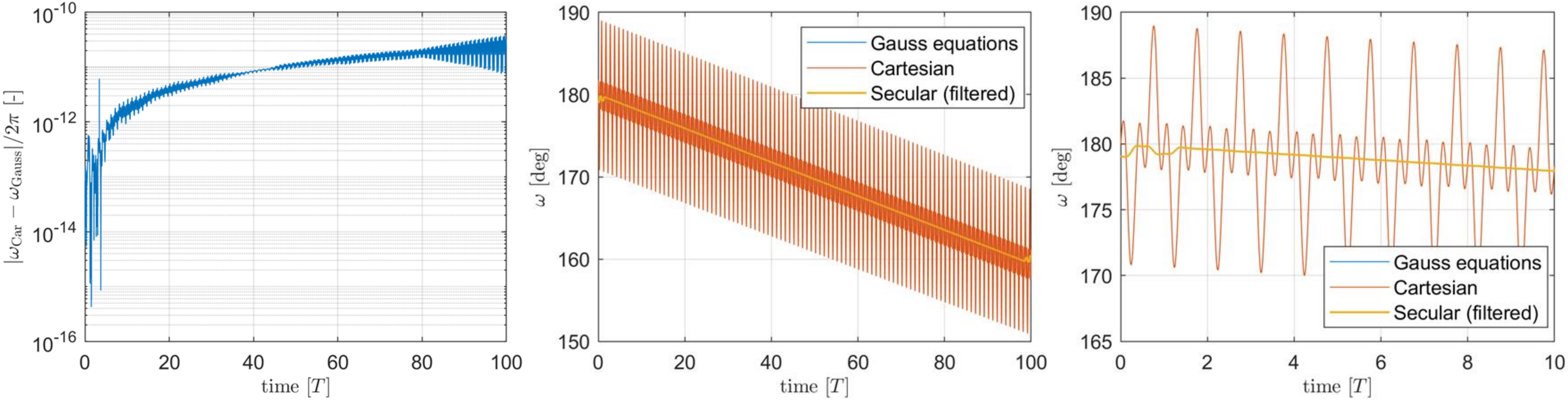
Sample results – Ω



$$\dot{\Omega}_{\text{analytic}} = -\left[\frac{3}{2}\frac{\sqrt{\mu_\oplus} J_2 R_\oplus^2}{(1-e^2)^2 a^{\frac{7}{2}}}\right]\cos i = -4.0393\ 10^{-08}\ \frac{\text{rad}}{\text{s}} = -1.5173\ 10^{-2}\ \frac{\text{deg}}{\text{orbit}}$$

Exercise: Orbit propagation with Gauss Equations

Sample results – ω



$$\dot{\omega}_{\text{analytic}} = -\left[\frac{3}{2}\frac{\sqrt{\mu_{\oplus}}J_2 R_{\oplus}^2}{(1-e^2)^2 a^{\frac{7}{2}}}\right]\left(\frac{5}{2}\sin^2 i - 2\right) = -5.4746\ 10^{-07}\ \frac{\text{rad}}{\text{s}} = -2.0564\ 10^{-1}\ \frac{\text{deg}}{\text{orbit}}$$

Exercise: Orbit propagation with Gauss Equations

Sample results – f