BIOS 755: Generalized Linear Mixed Models II

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Generalized Linear Mixed Model for an Ordinal Response

Suppose Y_{ij} is an ordinal response with K categories $(1, \ldots, K)$. A logistic mixed effects model for the *cumulative response probabilities* is given by:

1. Conditional on a vector of random effects $\boldsymbol{b}_i$, the Y_{ij} are independent and have a multinomial distribution
   - the multinomial covariance is determined by the conditional means (given below) or the conditional response probabilities.
2. The k^{th} cumulative response probability for Y_{ij} depends on fixed and random effect with

$$\alpha_{0k} + \boldsymbol{X}_{ij}\boldsymbol{\beta} + \boldsymbol{Z}_i\boldsymbol{b}_i$$

Generalized Linear Mixed Model for an Ordinal Response

2. (cont.) which is related to the conditional cumulative response probabilities with

$$\log\left\{\frac{\Pr(Y_{ij} \leq k|\boldsymbol{b}_i)}{\Pr(Y_{ij} > k|\boldsymbol{b}_i)}\right\} = \alpha_{0k} + \boldsymbol{X}'_{ij}\beta + \boldsymbol{Z}_i\boldsymbol{b}_i$$

3. The random effects have a bivariate normal distribution $\boldsymbol{b}_i \sim N(0, \boldsymbol{G})$.

This is a proportional odds mixed-effects regression model.

Generalized Linear Mixed Model for Counts

Suppose Y_{ij} is a count.

- Usually, we model counts using a Poisson distribution with a log-link.
- Conditional on the random effects $\boldsymbol{b}_i$, the Y_{ij} are independent and have a Poisson distribution with

$$Var(Y_{ij}|\boldsymbol{b}_i) = E(Y_{ij}|\boldsymbol{b}_i)$$

  or that

$$\sqrt{E(Y_{ij}|\boldsymbol{b}_i)} = StdDev(Y_{ij}|\boldsymbol{b}_i)$$

- This would mean that if the expectation is 100 the standard deviation would be 10.

The Negative Binomial Model

- The negative binomial model allows for extra variance versus what we see in the Poisson model.
- This model also uses a log link for the covariate data.
- Under the negative binomial model

$$Var(Y_{ij}|\boldsymbol{b}_i) = E(Y_{ij}|\boldsymbol{b}_i) + \theta\{E(Y_{ij}|\boldsymbol{b}_i)\}^2$$

  where $\theta \geq 0$.
- This model allows for “over-dispersion” and is often called the **over-dispersed Poisson model**.

Generalized Linear Mixed Model with over-dispersion

- We can actually add over-dispersion to any GLMM.
- Suppose that
$$g\{E(Y_{ij}|\boldsymbol{b}_i)\} = \boldsymbol{X}_i\boldsymbol{\beta} + b_{i0}$$
over-dispersion can be modeled by adding an additional random effect at the observation level.
$$g\{E(Y_{ij}|\boldsymbol{b}_i)\} = \boldsymbol{X}_i\boldsymbol{\beta} + b_{i0} + b_{ij0}$$
- This is not recommended with the Poisson, but can be used there or with the logistic, multinomial, etc.
- Over-dispersion means that the variance is larger than what we would expect.