CAS 741: SRS
A numerical search for the spectrum related to travelling periodic waves

Robert White

October 4, 2018

1 Revision History

Date	Version	Notes
25-09-2018	1.0	Creation of first draft
29-09-2018	1.1	Edit of 1.1. Added a GS, responsibilites, requirenments, definitions and verification of compatibility. Updated comments .tex. My comments are in red, SS comments are in blue.

2 Reference Material

This section records information for easy reference.

2.1 Table of Units

Throughout this document SI (Système International d'Unités) is employed as the unit system. In addition to the basic units, several derived units are used as described below. For each unit, the symbol is given followed by a description of the unit and the SI name.
[Is it OK if we mark some units as unitless? My supervisor and I are more concerned about mathematical properties, not physical. —RW] [Yes, unitless symbols are common, even in specific physical problems. (For instance, strain is unitless.) I suggest that you explicitly state that you have unitless symbols. In your table of symbols, you do have concepts that are usually associated with units, like wave speed and wave amplitude. I suppose that your amplitude isn't necessarily in units of length, but your speed will be the units of amplitude divided by time. Actually, I suppose this isn't necessary either. This seems to be something you could say in words, instead of coming up with a new convention for documenting variable units. —SS]

symbol	**unit**	**SI**
m	length	metre
t	time	second

[If you aren't using units, you should change what you write in this section. You should keep the heading, but in the contents explain that your units are abstracted out. This might be the spot to mention that the units are assumed to be consistent. —SS]

2.2 Table of Symbols

The table that follows summarizes the symbols used in this document along with their units. The choice of symbols was made to be consistent with the heat transfer [is your problem in the heat transfer domain? This looks like a copy and paste error? —SS] literature and with existing documentation for solar water heating systems. The symbols are listed in alphabetical order.

symbol	**unit**	**description**
λ		spectral parameter
u		wave amplitude
ϕ		eigen function
c		wave speed
ω		angular frequency

2.3 Abbreviations and Acronyms

symbol	description
A	Assumption
$\mathbb{C}$	Complex numbers
cn	Elliptic cosine
DD	Data Definition
dn	Delta amplitude
GD	General Definition
GS	Goal Statement
i	imaginary number
IM	Instance Model
LC	Likely Change
NLS	Nonlinear Schrodinger
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
PS	Physical System Description
$\mathbb{R}$	Real number line
R	Requirement
sn	Elliptic sine
SRS	Software Requirements Specification
ProgName	SpecSearch
T	Theoretical Model

2.4 Mathematical Notation

Let u be a complex number of the form $u = m+ni$, where $m, n \in \mathbb{R}$. The complex conjugate of u is defined to be $\bar{u} = m - ni$. The modulus of u is defined to be $|u| = \sqrt{m^2 + n^2}$.

Let $u(x,t) : \mathbb{R}^2 - > \mathbb{C}$. We will let u_t denote the partial derivative of u with respect to t. Similarly, u_x is the partial derivative of u with respect to x. If u has a one independent variable than we will denote its derivative by u'.

[Why is this section before the table of contents? —RW]

[Reference material is often before the table of contents. Although other decisions could be made, the benefits of standardization outweight giving the flexibility to move this section. —SS]

Contents

3 Introduction

3.1 Purpose of Document

The purpose of this document is to describe the requirements for the spectral search program. The spectral search program will search for the location of the continuous spectrum of a particular lax equation related to the NLS equation. This abstract document will explain the mathematical models, assumptions, solution characterisitcs and goals of the software. After reading this doucment one should be able to understand the mathematical and physical context of the inputs and outputs. They should also be able to identify the constraints of the system and have an idea as to how the output is derived from the given input. The SRS is meant to be unambigious and will not provide a detailed explanation of the numerical solution or code.

The SRS details the quality attributes of the software, such as functionality, without dicussing any code, numerical algorithms or scientific computing solutions. It is a precursor to the documents required in the development of scientific software. One should read and understand this document before the VnV, design or code itself.

3.2 Scope of Requirements

The scope of SpecSearch is limited to the creation of a spectral diagram for a particular linear lax equation. The waves being investigated are solutions to the NLS equation. The spectrum will be plotted on the complex plane.

3.3 Characteristics of Intended Reader

The intended reader of this document should have taken an introductory course in partial differential equations and complex analysis. They should also have a first year undergraduate understanding of linear algebra.

3.4 Organization of Document

This document follows a template provided by Dr. Spencer Smith at McMaster University. It begins by introducing the document. This intdocution is followed by a general overview of the system and than an outline of the goals and mathematical theory. The document continues with behaviour between inputs and outputs, judging output and than foreshadows changes to the software. It concludes with useful graphs.

More details of the template can be found in Smith et al (2005) and Smith et al (2007). The latex template is available in Dr. Spencer Smith's gitlab repository CAS 741.

4 General System Description

This section identifies the interfaces between the system and its environment, describes the user characteristics and lists the system constraints.

4.1 System Context

- User Responsibilities:
  - Ensure that the input variables resemble the wave they intend to analyze.
  - Ensure that the assumptions imposed on waves in this model are reasonable for the user's reasearch
  - Understand research limitations of software model, such as strict assumptions on physical system.
- ProgName Responsibilities: [I am wondering if my responsibilities are similar to Goal Statements and functional requirenments. —RW] [This sounds correct. Your program's responsibilities provide an abstract view of the functional requirements, which are in turn related to the goal statements. —SS]
  - Detect data type mismatch.
  - Solve the lax equation associated with given wave parameters.
  - Connect the discrete spectrum in order to form a continuous spectrum.
  - Plot the continuous spectrum on the complex plane.
  - Determine whether or not the solution is stable.

4.2 User Characteristics

The end user of SpecSearch should have an introductory level understanding of complex analysis and partial differential equations. They should also be familiar with matrix alegbra and eigenvectors presented in a first year course in linear algebra.

4.3 System Constraints

The system only deals with general traveling periodic wave solutions of the NLS.

5 Specific System Description

This section first presents the problem description, which gives a high-level view of the problem to be solved. This is followed by the solution characteristics specification, which presents the assumptions, theories, definitions and finally the instance models.

5.1 Problem Description

A lax pair is a set of matrices or operators that satisfy differential equations. The NLS equation, a PDE used to model rogue waves and modulated wave packets, appears as a compatibility condition of a particular lax pair of equations. One equation is a spectral problem and the other is a time evolution problem.

SpecSearch will produce a graph of a numerical approximation of the continuous spectrum of a lax linear equation assocaited with general travelling periodic wave solutions of the NLS equation. Previous attempts have used an algebraic method to calculate the spectral parameter. These attempts have only been successful at finding a countable number of eigenvalues. This software will "connect" these points in an attempt to approximate and search for the continuous spectrum.

5.1.1 Terminology and Definitions

This subsection provides a list of terms that are used in the subsequent sections and their meaning, with the purpose of reducing ambiguity and making it easier to correctly understand the requirements:

- **Spectrum**: The set of allowable values for the spectral parameter of a matrix or operator.
- **Operator**: A mapping that transforms elements of a space int other elements of the same space.
- **Traveling Periodic Wave**: A periodic and one-dimensional wave that travels with a constant speed.
- **Compatibility Condition**: Conditions under which the lax pair of equations is guaranteed.
- **Stability**: A solution is stable if it slight perturbations lead to at most slight perturbations in the solution.
- **Initial Data**: The initial profile of a wave (or its derivative) at a fixed point in time.
- **Rogue Wave**: A wave is considered rogue if its amplitude is more than double of the average of the upper third surrounding wave amplitudes.

5.1.2 Physical System Description

The physical system of SpecSearch includes the following elements:

PS1: Unspecified body of water or laser channel with a traveling periodic wave.

5.1.3 Goal Statements

[Are these too similar to functional responsibilities and requirenments? —RW] [The answer depends on whether there is a more abstract version that would be helpful. Could you just have GS4 (why isn't it GS3?) as a goal? That is could finding the stability of the traveling wave be your goal? This would make sense if the algorithm implied by the other two goals is one of many possible algorithms. If it is a scope time decision to use this algorithm, then you can "bake" this algorithm into your goal. I would add a scope statement to that affect, if that is what you choose to do. —SS] Given the constant wave speed, c, angular frequenct, ω, and intial profile of a general traveling periodic wave, the goal statements are:

GS2: Find elements of the continuous spectrum on the complex plane for a general traveling periodic wave.

GS3: Approximate the coninuous spectrum given the points from GS1.

GS4: Determine the stability of traveling wave solutions.

5.2 Solution Characteristics Specification

The instance models that govern SpecSearch are presented in Subsection 5.2.5. The information to understand the meaning of the instance models and their derivation is also presented, so that the instance models can be verified.

5.2.1 Assumptions

This section simplifies the original problem and helps in developing the theoretical model by filling in the missing information for the physical system. The numbers given in the square brackets refer to the theoretical model [T], general definition [GD], data definition [DD], instance model [IM], or likely change [LC], in which the respective assumption is used.

A1: The wave equation is a complex hamiltonian [Hamiltonian? I've noticed you do not use capitals for the names of scientists. I believe the normal convention is to do so. —SS] evolution equation.

A2: The linear momentum densities are proportional to the integrands of the corresponding velocity functionals

A3: All waves in the model are general traveling perioidic waves with constant speed and frequency.

A4: The wave is a solution to the NLS equation.

A5: The system only deals with focusing waves.

5.2.2 Theoretical Models

This section focuses on the general equations and laws that SpecSearch is based on.

Number	T1
Label	**Nonlinear focusing shchrodinger [Schrödinger —SS] equation**
Equation	$iu_t + u_{xx} + 2\|u\|^2 u = 0$
Description	The above equation is a PDE meant to model modulated wave packets and rogue waves in physics. u is the complex envelope of the wave, t is time and x is the position in one dimensional space.
Source	`http://www.efunda.com/formulae/heat_transfer/conduction/overview_cond.cfm`
Ref. By	GD??

5.2.3 General Definitions

This section collects the laws and equations that will be used in deriving the data definitions, which in turn are used to build the instance models. [Should T2 be a data definition? —RW] [It is hard to tell how your parts should be organized until I understand how they fit together. The traceability matrix will help you. The theories use the definitions, not the other way around. My guess is that your T1 is correct. I believe what you have labelled T2 and T3 should be labelled as GD1 and GD2, respectively. —SS]

Number	T2
Label	**General travelling periodic wave**
Equation	$u(x,t) = u(x + 2ct)e^{i\omega t}$
Description	This is the standard form of a traveling periodic wave. The variables c and ω denote the wave speed and angular frequency respectively. $u(x + 2ct)$ represents an amplitude value traveling in time for a fixed x value.
Source	`http://www.efunda.com/formulae/heat_transfer/conduction/overview_cond.cfm`
Ref. By	GD??

Number	T3
Label	**General travelling periodic wave**
Equation	$u'' + 2|u|^2 u + 2icu' = \omega u$
Description	A reduction of T1 into a second order ODE. This is derived by substituting T2 into T1.
Source	http://www.efunda.com/formulae/heat_transfer/conduction/overview_cond.cfm
Ref. By	T1,T2

5.2.4 Data Definitions

This section collects and defines all the data needed to build the instance models. The dimension of each quantity is also given.

Number	DD1
Label	**Conservation Equations**
Symbols	g, c, d, ω
SI Units	$\mathrm{W\,m^{-2}}$
Equation(1)	$\bar{u}u' - u\bar{u}' + 2ic|u|^2 = 2ig$
Equation(2)	$|u'|^2 + |u|^4 + d = \omega|u|^2$
Description	Equation 1 (2) is derived by multiplying T3 by $\bar{u}$ ($\bar{u}'$) and than subtracting (adding) the conjugate equation. The resultant equations are integrated to yield 1 and 2.
Sources	Citation here
Ref. By	IM1

5.2.5 Instance Models

This section transforms the problem defined in Section 5.1 into one which is expressed in mathematical terms. It uses concrete symbols defined in Section 5.2.4 to replace the abstract symbols in the models identified in Sections 5.2.2 and 5.2.3.

The goals [reference your goals —SS] are solved by [reference your instance models —SS]. [other details, with cross-references where appropriate. —SS] [Modify the examples below for your problem, and add additional models as appropriate. —SS]

Number	IM1
Label	**Searching for the continuous spectrum**
Input	c, ω, g, d, initial wave profile
Output	λ such that: $\phi_x = U(u, \lambda)\phi$
Description	$U(u,\lambda) = \begin{pmatrix} \lambda & u \\ -\bar{u} & -\lambda \end{pmatrix}$ T3 is a compabililty condition for the lax pair comprised of the above row's ODE and $\phi_t = V(u,\lambda)\phi$ $V(u,\lambda) = \begin{pmatrix} 2\lambda^2 + \lvert u\rvert^2 & u_x + 2\lambda u \\ \bar{u}_x - 2\lambda\bar{u} & -2\lambda^2 - \lvert u\rvert^2 \end{pmatrix}$ $\phi \in \mathbb{C}$
Sources	Citation here
Ref. By	IM**??**

For a smooth C^2 function ,ϕ, we have that $\phi_{tx} = \phi_{xt}$.
From IM1 $\Rightarrow \phi_{tx} = \delta_t U\phi + UV\phi$ and $\phi_{xt} = \delta_x V\phi + VU\phi$.
Combining the above two equations :
$\Rightarrow \delta_t U\phi + UV\phi = \delta_x V\phi + VU\phi$
$\Rightarrow \delta_t U + UV = \delta_x V + VU$ (–)
Equating the two sides of (–) we arrive at 4 equalities (since each side is a 2x2 matrix) :
(1,1) : $0 + \lambda(-4\lambda^3 - 2\lambda u^2) + u(4\lambda^2 u - 2\lambda u_x + 2u^3 + u_{xx}) = -4\lambda u u_x + \lambda(-4\lambda^3 - 2\lambda u^2) - u(-4\lambda^2 u - 2\lambda u_x - 2u^3 - u_{xx})$
$\Rightarrow 0 = 0$
We have the same result for component (2,2)
(1,2) : $u_t + \lambda(-4\lambda^2 u - 2\lambda u_x - 2u^3 - u_{xx}) + u(4\lambda^3 + 2\lambda u^2) = -4\lambda^2 u_x - 2\lambda u_{xx} - 6u^2 u_x - u_{xxx} + u(-4\lambda^3 - 2\lambda u^2) - \lambda(-4\lambda^2 u - 2\lambda u_x - 2u^3 - u_{xx})$
$\Rightarrow u_t = -6u^2 u_x - u_{xxx}$
$\Rightarrow u_t + 6u^2 u_x + u_{xxx} = 0$
The above equation is satisfied when u is a solution to the NLS.
The result is the same for (2,1). Thus, the NLS is a compatability condition for IM1.

5.2.6 Data Constraints

Tables ?? and 1 show the data constraints on the input and output variables, respectively. The column for physical constraints gives the physical limitations on the range of values that can be taken by the variable. The column for software constraints restricts the range of inputs to reasonable values. The constraints are conservative, to give the user of the model the flexibility to experiment with unusual situations. The column of typical values is intended to provide a feel for a common scenario. The uncertainty column provides an estimate of the confidence with which the physical quantities can be measured. This information would be part of the input if one were performing an uncertainty quantification exercise.

The specification parameters in Table ?? are listed in Table ??.

Table 1: Output Variables

Var	Physical Constraints
λ	$\lambda \in \mathbb{C}$

5.2.7 Properties of a Correct Solution

A correct solution will be an array of numbers that appear on the continuous spectrum. The plot will connect these points in an attempt to model the entire continuous spectrum. The stability of solutions will be displayed as booleans.

6 Requirements

This section provides the functional requirements, the business tasks that the software is expected to complete, and the nonfunctional requirements, the qualities that the software is expected to exhibit.

[Are these too similar to goal statements and responsibilities? —RW] [No, I don't think so. The idea of the SRS is to document a successive refinement from abstract theories to concrete requirements. There is going to be repetition of ideas. The difference is that each time the idea is repeated, it should become less abstract (more concrete). If you simplify your goal statement, as we discussed above, you might see this refinement more clearly. —SS]

6.1 Functional Requirements

R1: SpecSearch takes in conserved quantities d, g and wave quantities c, ω as input.

R2: SpecSearch will find values of λ within a certain resolution.

R3: SpecSearch will connect the points in R2 in an attempt to approximate the continuous spectrum.

R4: SpecSearch will plot the spectrum on the complex plane.

R5: SpecSearch will determine the stability of solutions.

6.2 Nonfunctional Requirements

NFR1: The resultant spectrum is legible and previously known values from algebraic methods are highlighted.

NFR2: A legend of parameters are given and the axes are clearly labeled.

7 Likely Changes

LC1: We may be able to run the software with less input variables. In particular, we could remove g, d or c, ω.

8 Traceability Matrices and Graphs

The purpose of the traceability matrices is to provide easy references on what has to be additionally modified if a certain component is changed. Every time a component is changed, the items in the column of that component that are marked with an "X" may have to be modified as well. Table 2 shows the dependencies of theoretical models, general definitions, data definitions, and instance models with each other. Table 3 shows the dependencies of instance models, requirements, and data constraints on each other. Table 4 shows the dependencies of theoretical models, general definitions, data definitions, instance models, and likely changes on the assumptions.

[I don't know how to correctly mark row T3 and DD1 in table 2. T3 is derived by substituting T2 into T1. So T3 cannot be changed independently. I cannot just change T3. Also DD1 is a manipulation of T3. —RW]

[The fact that figuring out the traceability table is a clue that you might need to do further organization. Shouldn't you have an assumption that allows T2 to be substituted into T1. I don't think you can do this in general, but you have made an assumption that lets you make this simplification. If this assumption changes than T3 will change. —SS]

[I do not know how to mark the first row in table 3. The Instance model is essentially what the software is built around. —RW]

	T1	T2	T3	DD1	IM1
T1	X		X	X	X
T2		X	X	X	
T3			X	X	
DD1				X	
IM1					X

Table 2: Traceability Matrix Showing the Connections Between Items of Different Sections

	IM1	R1	R2	R3	R4	R5
IM1	X	X	X	X	X	
R1	X	X				
R2			X	X	X	
R3				X	X	
R4					X	
R5						X

Table 3: Traceability Matrix Showing the Connections Between Requirements and Instance Models

	A1	A2	A3	A4	A5
T1	X	X		X	
T2			X	X	
T3	X	X	X	X	
DD1	X	X	X	X	
IM1	X	X	X	X	X
LC1					

Table 4: Traceability Matrix Showing the Connections Between Assumptions and Other Items

The purpose of the traceability graphs is also to provide easy references on what has to be additionally modified if a certain component is changed. The arrows in the graphs represent dependencies. The component at the tail of an arrow is depended on by the component at the head of that arrow. Therefore, if a component is changed, the components that it points to should also be changed. Figure **??** shows the dependencies of theoretical models, general definitions, data definitions, instance models, likely changes, and assumptions on each other. Figure **??** shows the dependencies of instance models, requirements, and data constraints on each other.

9 Appendix

[Your report may require an appendix. For instance, this is a good point to show the values of the symbolic parameters introduced in the report. —SS]

9.1 Symbolic Parameters

[The definition of the requirements will likely call for SYMBOLIC_CONSTANTS. Their values are defined in this section for easy maintenance. —SS]