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Pricing and hedging Asian basket spread options

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ABSTRACT

Asian options, basket options and spread options have been extensively studied in the literature. However, few papers deal with the problem of pricing general Asian basket spread options. This paper aims to fill this gap. In order to obtain prices and Greeks in a short computation time, we develop approximation formulae based on comonotonicity theory and moment matching methods. We compare their relative performances and explain how to choose the best approximation technique as a function of the Asian basket spread characteristics. We also give explicitly the Greeks for our proposed methods. In the last section we extend our results to options denominated in foreign currency.

1. Introduction

We consider a security market consisting of m risky assets and a risk-less asset with a constant rate of return r. We assume that under the risk neutral measure Q the price process dynamics are given by

$$dS_{jt} = rS_{jt}\,dt + \sigma_j S_{jt}\,dB_{jt}, \tag{1}$$

where $\{B_{jt} : t \geq 0\}$ is a standard Brownian motion associated with asset j and the volatility σ_j is a positive constant. Further we assume that the asset prices are correlated according to

$$\operatorname{cov}\left(B_{jt_v}, B_{it_s}\right) = \rho_{ji}\min(t_v, t_s). \tag{2}$$

Given the above dynamics, the price of the jth asset at time t_i equals

$$S_{jt_i} = S_j(0)e^{\left(r-\frac{\sigma_j^2}{2}\right)t_i+\sigma_j B_{jt_i}}. \tag{3}$$

With this in hand we can define an Asian basket spread as

$$\mathbb{S} = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}\varepsilon_j a_j S_{jt_i},$$

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where a_j is the weight given to asset j and ε_j its sign in the spread. We assume that $\varepsilon_j = 1$ for $j = 1, \dots, p$, $\varepsilon_j = -1$ for $j = p+1, \dots, m$, where p is an integer such that $1 \le p \le m-1$ and $t_0 < t_1 < t_2 < \cdots < t_n = T$. The price of an Asian basket spread with exercise price K at $t_0 = 0$ can be defined as

$$e^{-rT} E_Q(\mathbb{S} - K)_+, \tag{4}$$

with $(x)_+ = \max(x, 0)$ and where E_Q represents the expectation taken with respect to the risk neutral measure Q. In what follows we will simply write E for the expectation under the risk neutral measure.

Examples of such contracts can be found in the energy markets. The basket spread part may for example be used to cover refinement margin (crack spread) or the cost of converting fuel into energy (spark spread). While the Asian part (the temporal average) avoids the problem common to the European options, namely that speculators can increase gain from the option by manipulating the price of the assets near maturity.

Since the density function of a sum of non-independent log-normal random variables has no closed-form representation, there is no closed-form solution for the price of a security whenever $m > 1$ or $n > 1$ within the Black and Scholes framework. Therefore one has to use an approximation method when valuating such a security. It is always possible to use Monte Carlo techniques to get an approximation of the price. However such techniques are rather time-consuming. Furthermore financial institutions also need approximations of the hedge parameters in order to control the risk, which further increases the computation time. This explains why the research for a closed-form approximation has become an active area.

Some special cases of the above formula have been extensively studied. For example if we set $m = 1$ and $n > 1$ we have an Asian option. Approximation formulae for this kind of derivatives can be found in [1–8]. If $m > 1$ and $n = 1$ we have a basket option. See [9,10,8] for basket options where all the assets have a positive weight. And [11,12] for the case of basket spread options. Finally setting $m = 2$, $n = 1$ and $p = 1$ we end up with a spread option. Pretty accurate approximation formulae for spread options can be found in [13–17]. However few papers develop methods that can be used in the case of an Asian basket spread [11,12,18] are the only we are aware of.

In this paper we start by deriving approximation formulae for expression (4) using comonotonic bounds. We derive four different approximations: the upper, the improved upper, the lower and the intermediary bound. We also try to approximate the security price with the help of moment matching techniques. We improve the hybrid moment matching method of [12] and propose an extension of the method developed in [11]. We explain which method should be used depending on the basket characteristics. We also provide closed-form formulae for the Greeks of our selected approximation techniques. These methods have the advantage that they can be applied in other frameworks as well, e.g. in Lévy settings. We explain how our results can be adapted in order to deal with options written in foreign currency (compo and quanto options).

The paper is composed as follows. In Section 2, we construct a price approximation using comonotonic sums. In Section 3, we develop some moment matching methods. Section 4 studies the relative performance of the methods we developed. In Section 5 we derive the Greeks for our best performing approximation. Section 6 deals with options in foreign currency. Finally Section 7 concludes. Proofs of all propositions can be found in Appendix A. Appendix B contains numerical results for (Asian) (basket) spread options for several sets of parameters.

2. Comonotonic approximations

We start this section by recalling some results on comonotonicity from the review papers [19,20].

Definition 1. A random vector $(X_1^c, \dots, X_n^c)$ is comonotonic if each two possible outcomes $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ are ordered componentwise.

In actuarial sciences it is common to encounter sums of the form $\mathbb{S} = \sum_{i=1}^n X_i$ where the marginal distribution of each X_i is known but the dependency structure between the X_i's is unknown or too difficult to work with. In such a case comonotonicity theory allows us to find the joint distribution of the X_i's, given their marginal one, with the smaller (larger) sum in the convex order sense (we note this order as $\le_{cx}$). Put it differently, we could replace the original random vector by its comonotonic counterparts $\mathbb{S}^\ell$ and $\mathbb{S}^c$ which are such that

$$E[g(\mathbb{S}^\ell)] \le E[g(\mathbb{S})] \le E[g(\mathbb{S}^c)],$$

for any convex function $g(\cdot)$. From this it follows that

$$E(\mathbb{S}^\ell - K)_+ \le E(\mathbb{S} - K)_+ \le E(\mathbb{S}^c - K)_+,$$

for all $K \in \mathbb{R}$. Thus we see that comonotonicity allows us to find bounds for expressions like (4). Below we will see that comonotonic sums give us closed-form formulae for such approximations. Traditionally comonotonicity is used in the pricing of Asian, basket or Asian basket options (see e.g. [5,9,21]). We explicitly will discuss the different behaviour of the comonotonic bounds when dealing with positive and negative weights compared to the case where there are only positive weights. To our knowledge, this is the first time that this approach has been used in order to approximate basket spread or Asian basket spread options.

In what follows we focus on the ideas of the different approximation methods and on the numerical results, and we therefore skip the proofs here. The interested reader is referred to Appendix A.

2.1. Comonotonic upper bound

It can be shown that the convex largest sum of the components of a random vector X is given by the following comonotonic sum (see [20]):

$$\mathbb{S}^c = \sum_{i=1}^{n} F_{X_i}^{-1}(U),$$

where the distribution function of each X_i is non-decreasing and right-continuous, and $F_{X_i}^{-1}(p)$ is defined as

$$F_{X_i}^{-1}(p) = \inf\left\{x \in \mathbb{R} \mid F_{X_i}(x) \geq p\right\}, \quad p \in (0,1).$$

In [2], it is shown that the inverse distribution function of a sum of comonotonic random variables is equal to the sum of the marginal inverse distribution functions. Assuming that the marginal distributions are strictly increasing we can recover the cumulative distribution function (cdf) of the comonotonic sum using:

$$x = F_{\mathbb{S}^c}^{-1}(F_{\mathbb{S}^c}(x)) = \sum_{i=1}^{n} F_{X_i}^{-1}(F_{\mathbb{S}^c}(x)), \qquad F_{\mathbb{S}^c}^{-1}(0) < x < F_{\mathbb{S}^c}^{-1}(1).$$

The next theorem, of which the proof can be found in [20,2], will be useful in what follows.

Theorem 1. *The stop-loss premium of the comonotonic sum $\mathbb{S}^c$ of the random vector X is given by*

$$E[(\mathbb{S}^c - K)_+] = \sum_{i=1}^{n} E\left[\left(X_i - F_{X_i}^{-1}\left(F_{\mathbb{S}^c}(K)\right)\right)_+\right],$$

for $F_{\mathbb{S}^c}^{-1}(0) < K < F_{\mathbb{S}^c}^{-1}(1)$.

In what follows we will denote the cdf of a standard normal by Φ and its inverse function by Φ^{-1}.

Proposition 1. *A comonotonic upper bound to the price of a derivative of the type (4) when the underlying dynamics are described by (1) is given by*

$$e^{-rT}E(\mathbb{S}^c - K)_+ = e^{-rT}\left[\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}\varepsilon_j a_j S_j(0)e^{rt_i}\Phi\left(Y_{ji} - \Phi^{-1}\left(F_{\mathbb{S}^c}(K)\right)\right) - K\left(1 - F_{\mathbb{S}^c}(K)\right)\right],$$

where $F_{\mathbb{S}^c}(K)$ can be found by solving the following equation:

$$K = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}\varepsilon_j a_j S_j(0)e^{\left(r-\frac{\sigma_j^2}{2}\right)t_i + Y_{ji}\Phi^{-1}\left(F_{\mathbb{S}^c}(K)\right)}, \tag{5}$$

and where $Y_{ji} = \varepsilon_j\sigma_j\sqrt{t_i}$.

Remark. To compute the comonotonic upper bound we need $F_{\mathbb{S}^c}(K)$. This can be evaluated by solving the non-linear equation (5). Note that even if (5) is a non-linear equation it can easily be solved since it is monotonic in the unknown.

Remark. We can rewrite the upper bound as

$$e^{-rT}\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}\varepsilon_j a_j E\left(S_j(0)e^{\left(r-\frac{\sigma_j^2}{2}\right)t_i + \varepsilon_j\sigma_j\sqrt{t_i}\Phi^{-1}(U)} - K_{ij}\right)_{\varepsilon_j}, \tag{6}$$

with

$$K_{ij} = S_j(0)e^{\left(r-\frac{\sigma_j^2}{2}\right)t_i + \varepsilon_j\sigma_j\sqrt{t_i}\Phi^{-1}\left(F_{\mathbb{S}^c}(K)\right)}, \tag{7}$$

where $(X)_{\varepsilon_j}$ is equal to the function $\max(X,0)$ if $\varepsilon_j = 1$ and $\min(X,0)$ if $\varepsilon_j = -1$. Formulae (6)–(7) provide a natural interpretation to the comonotonic upper bound. Indeed these formulae show that we could write the upper bound as a linear combination of call and put options on the initial underlying with different maturities and strike prices given by K_{ij}. This result can be linked to the literature on static hedging, see for example [22,19,23,24].

Note that we only need to know the marginal distributions to compute the comonotonic upper bound. This means that a comonotonic upper bound could also be derived if the Brownian motion in (1) is replaced by a more general Lévy process. See [25] for a definition of comonotonic Lévy copulas and [26] for an application to Asian options.

2.2. Improved comonotonic upper bound

It is possible to sharpen the above upper bound by conditioning the distribution of the vector X on some random variable Λ. Assume that Λ is a random variable whose distribution is known, and such that the distribution of the X_i conditionally on Λ is known. If we further assume that the cumulative density functions $F_{X_i|\Lambda}$ are continuous and strictly increasing, then we have the following theorem from [20]:

Theorem 2. *Let U be a uniform $(0, 1)$ distributed random variable independent of Λ. Then we have*

$$\mathbb{S} = \sum_{i=1}^{n} X_i \leq_{cx} \mathbb{S}^{ic} = \sum_{i=1}^{n} F^{-1}_{X_i|\Lambda}(U) \leq_{cx} \mathbb{S}^{c} = \sum_{i=1}^{n} F^{-1}_{X_i}(U).$$

In what follows we will consider the conditioning variable:

$$\Lambda = \sum_{j=1}^{m} \sigma_j a_j S_j(0) B_{jT}. \tag{8}$$

Numerical results showed that in the case of positively correlated assets this conditioning variable produces the sharpest bounds. Using all this, we can derive the following proposition:

Proposition 2. *The improved comonotonic upper bound (ICUB) of the price of a derivative of the type (4) when the underlying dynamics are given by (1) is*

$$\begin{aligned}\mathrm{e}^{-rT} E\left[E(\mathbb{S}^{ic} - K)_+ |\Lambda\right] = {} & \mathrm{e}^{-rT} \int_0^1 \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m} \varepsilon_j a_j S_j(0) \mathrm{e}^{\left(r - \frac{(\gamma_{ji}\sigma_j)^2}{2}\right) t_i + A_{ji}(u)} \Phi\left(Y_{ji} - \Phi^{-1}\left(F_{\mathbb{S}^{ic}|U=u}(K)\right)\right) \mathrm{d}u \\ & - \mathrm{e}^{-rT} K(1 - F_{\mathbb{S}^{ic}}(K)),\end{aligned}$$

where we used that $\Lambda \stackrel{d}{=} E[\Lambda] + \sigma_\Lambda \Phi^{-1}(U)$ and where we can recover $F_{\mathbb{S}^{ic}|U=u}(K)$ by solving

$$K = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m} \varepsilon_j a_j S_j(0) \mathrm{e}^{\left(r - \frac{\sigma_j^2}{2}\right) t_i + A_{ji}(u) + Y_{ji} \Phi^{-1}\left(F_{\mathbb{S}^{ic}|U=u}(K)\right)},$$

with

$$F_{\mathbb{S}^{ic}}(K) = \int_0^1 F_{\mathbb{S}^{ic}|U=u}(K) \mathrm{d}u,$$

and where for γ_{ji} being the correlation between B_{jt_i} and the conditioning variable Λ,

$$A_{ji}(u) = \gamma_{ji} \sigma_j \sqrt{t_i} \Phi^{-1}(u), \qquad Y_{ji} = \varepsilon_j \sigma_j \sqrt{1 - \gamma_{ji}^2} \sqrt{t_i}.$$

2.3. Comonotonic lower bound

Assume that there exists a conditioning variable Λ such that the distribution of X_i conditionally on Λ is known for each i. Then from [2] we know that the following random variable provides a lower bound in the convex order sense:

$$\mathbb{S}^{\ell} = E\left[\mathbb{S}|\Lambda\right].$$

Furthermore assume the conditioning variable Λ is such that for all i, $E[X_i|\Lambda]$ is a non-decreasing (or non-increasing, which can be dealt with in a similar way) and a continuous function of Λ for each i. And if we further assume that the cdf of $E[X_i|\Lambda]$ is continuous and strictly increasing, then we can recover the distribution function of $\mathbb{S}^{\ell}$ from:

$$\sum_{i=1}^{n} E\left[X_i | \Lambda = F_\Lambda^{-1}(F_{\mathbb{S}^\ell}(x))\right] = x, \quad x \in \left(F^{-1}_{\mathbb{S}^\ell}(0), F^{-1}_{\mathbb{S}^\ell}(1)\right). \tag{9}$$

In such a case we can also use Theorem 1 and write the stop-loss premium as

$$E(\mathbb{S}^{\ell} - K)_+ = \sum_{i=1}^{n} E\left[(E[X_i|\Lambda] - E[X_i|\Lambda = F_\Lambda^{-1}(F_{\mathbb{S}^\ell}(K))])_+\right]$$

for all $K \in (F^{-1}_{\mathbb{S}^\ell}(0), F^{-1}_{\mathbb{S}^\ell}(1))$.

The most difficult part consists in finding a conditioning variable Λ such that all the conditional expectations are non-decreasing[2] (or non-increasing). In the case of a general Asian basket spread with positively correlated Brownian motion there is no obvious choice. Indeed finding such a conditioning variable would require a numerical optimization procedure. We would need to find the conditioning variable Λ which is a linear combination of the Brownian motion such that it maximizes the lower bound under a comonotonicity constraint. Considering the high dimensionality of the problem this would quickly become impossible. This is why we choose another approach. Instead of taking the correlation between the Brownian motions as given and start looking for a conditioning variable such that $\mathbb{S}^\ell$ is comonotone, we take a specific conditioning variable and then determine which set of correlation coefficients satisfy in order to have comonotonicity.

In what follows we will take the following conditioning variable

$$\Lambda = \sum_{j=1}^{m} \varepsilon_j B_{jT}, \tag{10}$$

which produces comonotonic conditional expectation vectors as long as the correlation coefficients (2) satisfy

$$\operatorname{sign}\left(\sum_{j=1}^{m} \varepsilon_j \rho_{jl}\right) = \varepsilon_l \quad \forall l. \tag{11}$$

One should note that such a condition always holds for simple spreads. The above discussion leads to the following proposition:

Proposition 3. *Under the assumption that the correlation coefficients satisfy (11), a comonotonic lower bound to the price of a derivative of type (4) when the underlying dynamics are described by (1) is given by:*

$$\mathrm{e}^{-rT} E(\mathbb{S}^\ell - K)_+ = \mathrm{e}^{-rT}\left[\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m} \varepsilon_j a_j S_0 \mathrm{e}^{rt_i} \Phi\left(\sigma_j\sqrt{t_i}\gamma_{ji} - \Phi^{-1}(F_{\mathbb{S}^\ell}(K))\right) - K\left(1 - F_{\mathbb{S}^\ell}(K)\right)\right]$$

where γ_{ji} is the correlation between B_{jt_i} and the conditioning variable given in (10), and $F_{\mathbb{S}^\ell}(K)$ can be found by solving (9).

For the choice in (10) we arbitrarily chose to use B_{jT}. But one could also choose any other time t_k in $[0, T]$ and then maximize the lower bound over all those t_k.

2.4. Comonotonic intermediary bound

As explained above it is indeed fairly difficult to find a conditioning variable that produces a comonotone conditional expectation vector. In such a case one can always build an approximation using the following procedure. Start by choosing a first conditioning variable,[3] denoted by Λ_1. Then construct the conditional expectation vector

$$E[X|\Lambda_1].$$

Once this is done, we choose a second conditioning variable Λ_2 and construct an ICUB cfr. Section 2.2 of this conditional expectation vector. The advantage of this procedure is that we surely end up with a comonotone vector. The drawback is that we do not know whether our approximation is an upper or a lower bound. In our computations we choose Λ_1 and Λ_2 to be

$$\Lambda_1 = \sum_{j=1}^{m} \sigma_j a_j S_0 B_{jT} \quad \text{and} \quad \Lambda_2 = \sum_{j=1}^{m} B_{jT}.$$

This choice of Λ_1 is justified by the fact that this conditioning variable seemed to provide a good first order approximation of $\mathrm{Var}(\mathbb{S}|\Lambda)$. And this choice of Λ_2 was yielding one of the best ICUB. Since it can be shown that (see [10])

$$\mathbb{S}^\ell \leq_{cx} \mathbb{S}^{\mathrm{int}} \leq_{cx} \mathbb{S}^{ic},$$

where $\mathbb{S}^\ell$ is the (non-comonotonic) lower bound based on conditioning variable Λ_1 and $\mathbb{S}^{ic}$ is the improved comonotonic upper bound based on Λ_2, we are reducing the possible range of fluctuation of our approximation.

Proposition 4. *The intermediary bound to the price of a derivative of the type (4) when the underlying dynamics are described by (1), is given by:*

[2] Note that a non-decreasing conditional expectation vector is equivalent to the requirement $\varepsilon_j = \operatorname{sign}(\gamma_{ji})$, where γ_{ji} is the correlation between the jth Brownian motion at time i and the conditioning variable.

[3] The only restriction set on this variable is that it is normally distributed and that (Λ_1, Λ_2) is bivariate normally distributed.

$$e^{-rT}\left[E\left(\mathbb{S}^{\text{int}}-K\right)_+\right] = e^{-rT}\int_0^1 \frac{1}{n}\sum_{i=1}^n\sum_{j=1}^m \varepsilon_j a_j S_j(0) e^{\left(r-\frac{(\gamma_{ji}\gamma_{\Lambda_1\Lambda_2}\sigma_j)^2}{2}\right)t_i + A_{ji}(u)}\Phi\left(Y_{ji}-\Phi^{-1}(F_{\mathbb{S}^{\text{int}}|U=u}(K))\right)\mathrm{d}u$$
$$- e^{-rT}K(1-F_{\mathbb{S}^{\text{int}}}(K)),$$

where we used that $\Lambda_2 \overset{d}{=} E[\Lambda_2]+\sigma_{\Lambda_2}\Phi^{-1}(U)$ *and where* $\gamma_{\Lambda_1\Lambda_2}$ *is the correlation between the conditioning variables* Λ_1 *and* Λ_2*, and* γ_{ji} *is the correlation between the first conditioning variable* Λ_1 *and* B_{jt_i}*.* $F_{\mathbb{S}^{\text{int}}|U=u}(K)$ *can be recovered by solving*

$$K = \frac{1}{n}\sum_{j=1}^m\sum_{i=1}^n \varepsilon_j a_j S_j(0) e^{\left(r-\frac{(\gamma_{ji}\sigma_j)^2}{2}\right)t_i + A_{ji}(u) + Y_{ji}\Phi^{-1}\left(F_{\mathbb{S}^{\text{int}}|U=u}(K)\right)},$$

where

$$A_{ji}(u) = \gamma_{\Lambda_1\Lambda_2}\gamma_{ji}\sigma_j\sqrt{t_i}\Phi^{-1}(u), \qquad Y_{ji} = \varepsilon_j\sqrt{1-\gamma^2_{\Lambda_1\Lambda_2}}|\gamma_{ji}|\sigma_j\sqrt{t_i},$$

and satisfies

$$F_{\mathbb{S}^{\text{int}}}(K) = \int_0^1 F_{\mathbb{S}^{\text{int}}|U=u}(K)\mathrm{d}u.$$

3. Moment matching methods

Another approximation technique is the so-called moment matching method. The idea is to replace the original distribution of the underlying by a law with the same p first moments as the original law (with p the number of parameters of the approximating law) and whose stop-loss premium can easily be approximated by a closed-form expression. In this section we describe two different moment matching techniques that can be used to price Asian basket spread options. These techniques could also be applied when the Brownian motion in (1) is replaced by a more general Lévy process (as long as the p first moments exist).

3.1. Hybrid moment matching method

An obvious way of attacking the problem is to use the hybrid moment matching method, see for example [12]. The idea is to reduce the Asian basket spread option pricing problem to a spread option pricing problem. To do so, we start by splitting the underlying $\mathbb{S}$ in two parts (one containing all the assets with a positive sign, denoted as $\mathbb{S}_1$, another containing those with a negative sign, denoted as $\mathbb{S}_2$) and moment match each term, separately, with a log-normal random variable. Once this is done we are left with the problem of approximating a spread option. This is a well studied problem for which pretty accurate approximations are available. In this section we improve the classical hybrid moment matching method by using new spread approximation techniques. We will see later that one of these approximations turns out to be extremely useful when we need to recover the Greeks of such an Asian basket option.

More formally, hybrid moment matching allows us to rewrite (4) as:

$$E\left(\tilde{\mathbb{S}}_1-\tilde{\mathbb{S}}_2-K\right)_+, \tag{12}$$

where $\tilde{\mathbb{S}}_j$ is a log-normal random variable with mean μ_j and variance σ_j^2 given by:

$$\mu_j = 2\ln(m_{1j}) - \frac{1}{2}\ln(m_{2j}), \qquad \sigma_j^2 = \ln(m_{2j}) - 2\ln(m_{1j}), \quad j=1,2. \tag{13}$$

Here m_{1j} and m_{2j} are the first and second moments of the sum $\mathbb{S}_j$ described above. We will also need the correlation coefficient ρ between $\ln(\tilde{\mathbb{S}}_1)$ and $\ln(\tilde{\mathbb{S}}_2)$ to compute the spread approximation. We use the following equation to recover ρ from the equality of the crossmoments (crm):

$$E[\mathbb{S}_1\mathbb{S}_2] = E[\tilde{\mathbb{S}}_1\tilde{\mathbb{S}}_2],$$

namely:

$$\text{crm} := \sum_{j=p+1}^m\sum_{l=1}^p\sum_{i,s=1}^n \frac{1}{n^2}a_j a_l E[S_{jt_i}S_{lt_s}] = e^{\mu_1+\mu_2+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_2^2+\sigma_1\sigma_2\rho}. \tag{14}$$

Finally, we have to approximate the resulting spread (12). We will use two different approximations. First, we use the method proposed in [17] (from now on called the Li et al. approximation). We choose this method since it does not require any optimization and performs remarkably well compared to other techniques like the ones of [14] or [15]. As a second approximation of the spread we choose the ICUB of expression (12). The reason for this choice will become clear in the next section where we will see that the improved comonotonic upper bound outperforms the Li et al. approximation. We derive the ICUB in the next proposition.

Proposition 5. *Consider the following stop-loss premium*

$$E(e^{\mu_1+\sigma_1 X_1} - e^{\mu_2+\sigma_2 X_2} - K)_+,$$

with X_1, X_2 two correlated standard normal random variables. The comonotonic improved upper bound of this spread is given by

$$\sum_{i=1}^{2} \varepsilon_i \int_0^1 e^{\mu_i + A_i(u) + \frac{1}{2}Y_i^2} \Phi\left(Y_i - \Phi^{-1}\left(F_{\mathbb{S}^{ic}|U=u}(K)\right)\right) du - K\left(1 - F_{\mathbb{S}^{ic}}(K)\right), \tag{15}$$

where $\varepsilon_1 = 1$ and $\varepsilon_2 = -1$, and where $F_{\mathbb{S}^{ic}|U=u}$ can be found by solving

$$\sum_{i=1}^{2} \varepsilon_i e^{\mu_i + A_i(u) + Y_i \Phi^{-1}(F_{\mathbb{S}^{ic}|U=u}(K))} = K,$$

with, for γ_i denoting the correlation between X_i and the conditioning variable $\Lambda \overset{d}{=} E[\Lambda] + \sigma_\Lambda \Phi^{-1}(U)$

$$A_i(u) = \gamma_i \sigma_i \Phi^{-1}(u), \qquad Y_i = \varepsilon_i \sigma_i \sqrt{1-\gamma_i^2}, \qquad F_{\mathbb{S}^{ic}}(K) = \int_0^1 F_{\mathbb{S}^{ic}|U=u}(K) du.$$

When performing the approximations, we used the following conditioning variable

$$\Lambda = e^{\mu_1}\sigma_1 X_1 + e^{\mu_2}\sigma_2 X_2, \tag{16}$$

where X_1, X_2 are $N(0, 1)$ with correlation coefficient ρ determined through (14) and μ_i and σ_i are given by (13).

In order to interpret the approximation error we introduce the following decomposition:

$$\begin{aligned} E(\mathbb{S} - K)_+ &= E(\tilde{\mathbb{S}}_1 - \tilde{\mathbb{S}}_2 - K)_+ + \overbrace{(E(\mathbb{S} - K)_+ - E(\tilde{\mathbb{S}}_1 - \tilde{\mathbb{S}}_2 - K)_+)}^{\Delta_1} \\ &= \tilde{\Pi} + \overbrace{E(\tilde{\mathbb{S}}_1 - \tilde{\mathbb{S}}_2 - K)_+ - \tilde{\Pi}}^{\Delta_2} + \Delta_1 \\ &= \tilde{\Pi} + \Delta_1 + \Delta_2, \end{aligned}$$

where $\tilde{\Pi}$ is the chosen spread approximation. Thus Δ_1 represents the error made by replacing our original basket with moment matched log-normal random variables, while Δ_2 is the error originating from the analytical approximation of the resulting spread. Later we will see that when approximating the price of an Asian basket spread, the first part has the highest contribution to the error.

3.2. Shifted log-extended skew normal moment matching

In this section, we develop an extension of the methodology introduced in [11,8]. We consider the possibility of using a shifted log-extended skew normal random variable instead of a shifted log-normal random variable in order to perform moment matching. In doing so, we are combining both the approach of [11] and of [8]. Compared to the traditional shifted log-normal moment matching of [11], this new method introduces two additional parameters giving us more moments to match. Compared to the [8] approach, this new method allows us to consider baskets in which some assets have a negative weight. We start this section by giving a general overview about extended skew normal random variables and introducing the shifted log-extended skew normal law, before explaining its implementation in option pricing problems.

3.2.1. Shifted log-extended skew normal law

We say that X is extended skew normally distributed with skewness parameters α and τ if it has the following probability density function:

$$\psi(x, \alpha, \tau) = \phi(x) \frac{\Phi\left(\tau\sqrt{1+\alpha^2} + \alpha x\right)}{\Phi(\tau)}, \quad \alpha, \tau \in \mathbb{R}, \tag{17}$$

where ϕ is the probability density function of a standard normal. When the parameters α and τ are both equal to zero, we recover the standard normal distribution. Whenever one of them is different from zero, we have an extended skew normal distribution. This distribution family was first introduced in [27] and studied in details in [28]. The presence of these two additional parameters allows us to model the asymmetry in the distribution.

A random variable Z is said to be shifted log-extended skew normal with location parameter μ, scale parameter σ, shift parameter η and skewness parameters α and τ if it is of the type

$$Z = \mathrm{e}^{\mu+\sigma X} + \eta,$$

where X is an extended skew normal random variable with skewness parameters α and τ. In order to allow for negative values, we thus modified the specification of [8] by adding the shift parameter η. We denote a shifted log-extended skew normal random variable by SLESN$(\mu, \sigma, \alpha, \tau, \eta)$. After some straightforward computation one can write the density of Z as:

$$\psi(z, \mu, \sigma, \alpha, \tau, \eta) = \frac{1}{(z-\eta)\sigma}\phi\left(\frac{\ln(z-\eta)-\mu}{\sigma}\right)\frac{\Phi\left(\tau\sqrt{1+\alpha^2}+\alpha\left((\ln(z-\eta)-\mu)/\sigma\right)\right)}{\Phi(\tau)}, \quad z > \eta.$$

We will need the first five moments $(\hat{m}_1, \ldots, \hat{m}_5)$ of SLESN$(\mu, \sigma, \alpha, \tau, \eta)$. After some straightforward computations we can compute and rewrite the moments as

$$\begin{aligned}
\hat{m}_1 &= M_1 + \eta \\
\hat{m}_2 &= M_2 + 2\eta\hat{m}_1 - \eta^2 \\
\hat{m}_3 &= M_3 + 3\eta\hat{m}_2 - 3\eta^2\hat{m}_1 + \eta^3 \\
\hat{m}_4 &= M_4 + 4\eta\hat{m}_3 - 6\eta^2\hat{m}_2 + 4\eta^3\hat{m}_1 - \eta^4 \\
\hat{m}_5 &= M_5 + 5\eta\hat{m}_4 - 10\eta^2\hat{m}_3 + 10\eta^3\hat{m}_2 - 5\eta^4\hat{m}_1 + \eta^5,
\end{aligned} \tag{18}$$

where M_j is the jth moment of the corresponding log-extended skew normal and is given by:

$$M_j = \mathrm{e}^{j\mu+\frac{1}{2}(j\sigma)^2}\frac{\Phi(\tau+j\delta\sigma)}{\Phi(\tau)}, \qquad \delta = \frac{\alpha}{\sqrt{1+\alpha^2}}. \tag{19}$$

Below we will need the negative SLESN$(\mu, \sigma, \alpha, \tau, \eta)$ which is defined as

$$Z = -\mathrm{e}^{\mu+\sigma X} - \eta.$$

Its density can be derived in exactly the same way as for the SLESN, and its moments can be found from those of the SLESN by replacing M_1, M_3 and M_5 by $-M_1$, $-M_3$ and $-M_5$.

3.2.2. *Pricing Asian basket spread options*

We follow the methodology introduced in [11] but instead of using a shifted log-normal law we use a shifted log-extended skew normal law as our matching distribution. Thus we proceed as follows:

1. Start by computing the first five moments of the Asian basket.
2. Compute the Asian basket skewness which is defined as

   $$E[(\mathbb{S} - E[\mathbb{S}])^3].$$

3. If the skewness is negative, moment match the Asian basket spread with a negative shifted log-extended skew normal. If the skewness is positive use a positive shifted log-extended skew normal law.
4. Adjust the shift parameter of the matching distribution if needed.
5. Compute the stop-loss premium of the matched random variable.

The following theorem gives the stop-loss premium for a SLESN random variable.

Proposition 6. *Let X be a positive* SLESN$(\mu, \sigma, \alpha, \tau, \eta)$ *then*

$$E(X-K)_+ = \mathrm{e}^{\mu+\frac{1}{2}\sigma^2}\frac{\Phi(\tau+\delta\sigma)}{\Phi(\tau)}\Psi(I_1; -\alpha; \tau+\delta\sigma) - (K-\eta)\Psi(I_2; -\alpha; \tau),$$

where for $K - \eta > 0$

$$I_1 = \frac{\mu+\sigma^2-\ln(K-\eta)}{\sigma} \qquad I_2 = I_1 - \sigma,$$

and Ψ is the cdf of an extended skew normal law with density function (17).

The following proposition states the corresponding result for a negative SLESN random variable.

Proposition 7. *Let X be a negative* SLESN$(\mu, \sigma, \alpha, \tau, \eta)$ *random variable then,*

$$E(X-K)_+ = -e^{\mu+\frac{1}{2}\sigma^2}\frac{\Phi(\tau+\delta\sigma)}{\Phi(\tau)}\Psi(I_1;\alpha;\tau+\delta\sigma) - (K+\eta)\Psi(I_2;\alpha;\tau),$$

where for $-K-\eta > 0$

$$I_1 = \frac{\ln(-K-\eta)-\mu-\sigma^2}{\sigma} \qquad I_2 = I_1+\sigma,$$

and Ψ *is the cdf of an extended skew normal law with density function (17).*

Notice that if we set $\tau = 0$ and $\alpha = 0$ we recover the formula from [11]. If we set $\eta = 0$ then we recover the formula from [8].

Remark 1. It may seem strange to look at the skewness of the Asian basket (with possible negative weights) when performing moment matching. After all, we introduced the SLESN in order to model this skewness. Thus normally all the skewness from the original distribution should be embedded in the matched parameters α and τ. Indeed the reason is more computational than theoretical. If we try to match a positive SLESN to an Asian basket whose skewness is negative our algorithm fails. The failure is due to the shift parameter which is converging to $-\infty$ in order to associate a positive density to the negative (positive) value of the axis. Taking a negative SLESN law as matching distribution resolves the problem in this case.

Remark 2. Solving the system in (18) poses some serious problems since it contains five non-linear equations. Fortunately some simple manipulations in the spirit of those introduced in [8] allow us to avoid this problem. We proceed as follows: start by taking the logarithm of the first equation in (19) for $j = 1$ and $j = 2$, which leads to

$$\mu + \frac{1}{2}\sigma^2 = -\ln\frac{\Phi(\tau+\sigma\delta)}{M_1} + \ln\Phi(\tau),$$
$$\mu + \sigma^2 = -\frac{1}{2}\ln\frac{\Phi(\tau+2\sigma\delta)}{M_2} + \frac{1}{2}\ln\Phi(\tau).$$

We can rewrite those two equations as

$$\begin{aligned}\mu &= -2\ln\frac{\Phi(\tau+\sigma\delta)}{M_1} + \frac{1}{2}\ln\frac{\Phi(\tau+2\sigma\delta)}{M_2} + \frac{3}{2}\ln\Phi(\tau),\\ \sigma^2 &= 2\ln\frac{\Phi(\tau+\sigma\delta)}{M_1} - \ln\frac{\Phi(\tau+2\sigma\delta)}{M_2} - \ln\Phi(\tau).\end{aligned} \tag{20}$$

These equations will provide us the parameters μ and σ^2. Note that the right-hand side of those equations depends on the parameters τ, the product $\sigma\delta$ and η through the moments M_j see (18). Let us denote $\theta = \sigma\delta$. Combining the remaining equations in (19) for $j = 3, 4, 5$ and using the above expression for μ and σ^2 yields

$$\begin{aligned}&\ln\frac{\Phi(\tau+3\theta)}{M_3} - 3\ln\frac{\Phi(\tau+2\theta)}{M_2} + 3\ln\frac{\Phi(\tau+\theta)}{M_1} = \ln\Phi(\tau),\\ &\ln\frac{\Phi(\tau+4\theta)}{M_4} - 6\ln\frac{\Phi(\tau+2\theta)}{M_2} + 8\ln\frac{\Phi(\tau+\theta)}{M_1} = 3\ln\Phi(\tau),\\ &\ln\frac{\Phi(\tau+5\theta)}{M_5} + 15\ln\frac{\Phi(\tau+\theta)}{M_1} - 10\ln\frac{\Phi(\tau+2\theta)}{M_2} = 6\ln\Phi(\tau).\end{aligned} \tag{21}$$

When computing the matching distribution, we start by replacing the $\hat{m}_j$ in (18) by the corresponding moments of $\mathbb{S}$. Next, we solve those Eq. (18) for M_j in terms of η and substitute in (21). We solve, simultaneously, the three non-linear equations (21) in order to find θ, η and τ. Then we insert those parameters in the expressions in (20) for μ and σ^2. Finally, we recover α using

$$\alpha = \frac{\theta}{\sqrt{\sigma^2-\theta^2}}.$$

Remark 3. If we set $\tau = 0$ then we are left with a skew normal distribution. This distribution will depend on only 4 parameters instead of 5. This means that when matching our Asian basket spread we only need to consider a system of 4 equations. These 4 equations can be resolved by solving simultaneously the first two equations in (21) in θ and η and then using the relations (20) in order to recover μ and σ. The advantage of setting $\tau = 0$ is that we need to solve only 2 non-linear equations when determining the skewness and the shift. This reduces the complexity and can have an impact on the accuracy of our numerical procedure. The drawback is that we lose a parameter, meaning that we could lose precision when performing an approximation. One always has to make a tradeoff between the sophistication of the model and the numerical complexity.

4. Numerical illustrations

In this section we give a number of numerical examples of spread, basket spread and Asian basket spread options. We study the relative performances of the approximation formulae developed above and give a general procedure for approximating spread options. When performing calculations we set the risk-less rate of return to 5%. The asset volatilities where chosen randomly between [0.1, 0.6] and their correlations between [0.1, 0.9]. The option's maturity is set equal to one year. Finally, when working with Asian options we took the average over the last 30 days, which is a common practice in energy markets. The tables containing numerical results can be found in Appendix B. Due to its high computational cost we choose not to consider the method proposed in [18] in the computations that follow.

4.1. Spread options

We start by considering spread options. We can draw the following conclusions:

- The comonotone upper bound offers a poor approximation to spread option prices. This failure is due to the combination of sign switching and a positive correlation between the assets which is generating a non-comonotonic spread.
- The lower bound yields poor approximations. Furthermore its accuracy is strongly influenced by the correlation between the assets. This failure should not come as a surprise since, as our conditioning variable choice was constrained by the comonotonicity requirement, we could not choose a conditioning variable maximizing $\mathrm{Var}(\mathbb{S}|\Lambda)$.
- The intermediary bound has some stability problems, its performance being competitive only for some parametrization of the spread.
- The ICUB performs remarkably well. Its performance is comparable with that of the Li et al. approximation. In addition, we know that the obtained approximation represents an upper bound. Thus, our preference goes to the former method when pricing spread options. Note that this justifies the introduction of ICUB in hybrid moment matching (see Section 3.1).[4]
- The shifted log-normal moment matching performs poorly compared to the ICUB and Li et al. approximation.
- The performances of SLESN moment matching varies considerably. This is due to the complexity of the system we need to solve when matching the parameters. We encountered several numerical problems when we computed the matching distribution. Fixing one of the parameters equal to zero (see Remark 3) did not solve the problem posed by optimization.

Because the problems we have pointed out concerning the comonotonic upper, lower, intermediary bounds and SLESN moment matching only worsen when working with more general Asian basket spreads, we will not mention these approximations in what follows. From now on we will only focus on four approximation techniques: the ICUB, hybrid moment matching with Li et al.'s and ICUB approximation, and shifted log-normal moment matching.

4.2. Basket spread options

A quick look at the simulation output allows us to discard two approximation techniques. First, we see that the ICUB performance is clearly declining as the number of assets increases, and since this problem only worsens when dealing with Asian options we will no longer discuss this approximation. Second, we see that the hybrid moment matching associated with the ICUB slightly outperforms hybrid moment matching associated with Li et al.'s approximation. Thus from now on when discussing, we will only consider the hybrid moment matching method associated to the ICUB.

We are left with two candidates: the shifted log-normal and hybrid moment matching associated with the ICUB. It is indeed impossible to completely exclude one of these approximation techniques. The choice of the method will depend on the underlying characteristics. We can distinguish four possible cases:

1. The underlying has a positive skewness and $\sum_{j=1}^{m} a_j \varepsilon_j S_j(0) > 0$. Then the shifted log-normal approximation tends to outperform hybrid moment matching when the correlation between the assets in the positive (negative) part is low (by low we mean below 0.8) and the volatilities are low (see Table 4).
2. The underlying has a positive skewness and $\sum_{j=1}^{m} a_j \varepsilon_j S_j(0) < 0$. In this case, numerical results tend to favor hybrid moment matching since its results clearly dominate the shifted log-normal approximation (see Table 5).
3. The underlying has a negative skewness and $\sum_{j=1}^{m} a_j \varepsilon_j S_j(0) < 0$. Then the shifted log-normal approximation tend to outperform hybrid moment matching when the correlation between the assets in the positive (negative) part is low (by low we mean below 0.8) and the volatilities are low (see Table 6).

[4] We computed the Li et al. approximation and the ICUB for several spread options. We compared their mean squared error and mean absolute deviation. Numerical results showed that the ICUB slightly outperforms the Li et al. approximation according to both criteria.

4. The underlying has a negative skewness and $\sum_{j=1}^{m} a_j\varepsilon_j S_j(0) > 0$. In this case, numerical results tend to favor hybrid moment matching, the results of which clearly dominate the shifted log-normal approximation (see Table 7).

As we see, the choice of the approximation will depend on the skewness, correlation, and initial values of the underlying. Numerical results for each of the basket configurations can be found in Appendix B.

4.3. Asian basket spread options

Finally we consider Asian basket spread options. First of all, remark that in the case of Asian spread options, hybrid moment matching associated with the ICUB works remarkably well. This may be explained by the strong correlation between the components of the positive (negative) part of such options (due to the temporal dependency). Second, when dealing with more general Asian basket spreads, hybrid moment matching seems to be slightly more efficient than shifted log-normal moment matching especially when the volatilities are low. Our preference clearly goes to hybrid moment matching when dealing with Asian basket spread options.

5. Option Greeks

In the previous section we saw that depending on the underlying characteristics, it may be preferable to use hybrid moment matching or shifted log-normal moment matching. Below we provide the Greeks for the hybrid moment matching approximation. In case of shifted log-normal moment matching, the Greeks can be computed as in [11].

5.1. Simple spread approximation

Computing the Greeks of the ICUB is a heavy task. The difficulties originate from the conditioning variable we choose. Since the conditioning variable (8) is a function of $S_j(0)$ and σ_j we need to differentiate all the covariance and variance present in the approximation when computing the Greeks. In order to simplify the computations we choose to replace this conditioning variable by

$$\Lambda = \sum_{j=1}^{2} B_{jT}. \tag{22}$$

The accuracy of the ICUB based on this conditioning variable seems to be the same as the one obtained with Li et al.'s approximation. For simple spreads we have the following proposition.

Proposition 8. *The Greeks of the ICUB (denoted by Γ) of a simple spread when the conditioning variable is (22), are given by*

$$\frac{\partial \Gamma}{\partial S_j(0)} = \int_0^1 \varepsilon_j a_j \mathrm{e}^{-\frac{A_j^2}{2}+A_j\Phi^{-1}(u)}\Phi(Y_j - F(u,K))\mathrm{d}u$$

$$\frac{\partial^2 \Gamma}{\partial S_j(0)\partial S_p(0)} = \int_0^1 \frac{\varepsilon_j\varepsilon_p a_j a_p \mathrm{e}^{\left(r-\frac{\sigma_j^2}{2}-\frac{\sigma_p^2}{2}\right)T+(A_j+A_p)\Phi^{-1}(u)+(Y_j+Y_p)F(u,K)-\frac{F(u,K)^2}{2}}}{\sqrt{2\pi}\sum_{i=1}^{2}\varepsilon_i a_i S_i(0) Y_i \mathrm{e}^{(r-\frac{\sigma_i^2}{2})T+A_i\Phi^{-1}(u)+Y_iF(u,K)}}\mathrm{d}u$$

$$\begin{aligned}\frac{\partial \Gamma}{\partial \sigma_j} &= \varepsilon_j a_j S_j(0)(-\gamma_j^2\sigma_j T + \gamma_j\sqrt{T}\Phi^{-1}(u))\int_0^1 \mathrm{e}^{-\frac{A_j^2}{2}+A_j\Phi^{-1}(u)}\Phi(Y_j - F(u,K))\mathrm{d}u \\ &\quad + a_j S_j(0)\sqrt{\frac{1-\gamma_j^2}{2\pi}}\sqrt{T}\int_0^1 \mathrm{e}^{A_j\Phi^{-1}(u)-\frac{\sigma_j^2 T}{2}-\frac{F(u,K)^2}{2}+Y_jF(u,K)}\mathrm{d}u \\ \frac{\partial \Gamma}{\partial \rho} &= \sum_{i=1}^{2}\varepsilon_i a_i S_i(0)\int_0^1 \frac{\sigma_i}{2}T\left(-\frac{\sigma_i}{2}+\frac{\Phi^{-1}(u)}{\sigma_\Lambda}\right)\mathrm{e}^{-\frac{A_i^2}{2}+A_i\Phi^{-1}(u)}\Phi(Y_i - F(u,K))\mathrm{d}u \\ &\quad - \sum_{i=1}^{2}\frac{\sigma_i\sqrt{T}}{4\sqrt{2\pi(1-\gamma_i^2)}}a_i S_i(0)\int_0^1 \mathrm{e}^{-\frac{\sigma_i^2 T}{2}+A_i\Phi^{-1}(u)-\frac{F(u,K)^2}{2}+Y_iF(u,K)}\mathrm{d}u\end{aligned}$$

where $F(u,K) = \Phi^{-1}(F_{\mathbb{S}^{ic}|U=u}(K))$, $A_i = \gamma_i\sigma_i\sqrt{T}$, $Y_i = \varepsilon_i\sigma_i\sqrt{T}\sqrt{1-\gamma_i^2}$ with $\gamma_i^2 = \frac{1+\rho}{2}$.

5.2. Asian basket spread approximation

In this case a simple trick allows us to compute the Greeks easily. Start by replacing the conditioning variable (16) by $\Lambda = X_1 + X_2$.

The parameters σ_j, μ_j and ρ are determined according to

$$\mu_j = 2\ln(m_{1j}) - \frac{1}{2}\ln(m_{2j})$$
$$\sigma_j^2 = \ln(m_{2j}) - 2\ln(m_{1j})$$
$$\rho = \frac{\ln(\mathrm{crm}) - \mu_1 - \mu_2}{\sigma_1\sigma_2} - \frac{1}{2}\left(\frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1}\right),$$

with crm given in (14). Assume we want to recover the sensitivity of the price with respect to a parameter (denote this parameter by G). First of all, remark that

$$\begin{aligned}\frac{\partial\mu_j}{\partial G} &= 2\frac{\frac{\partial m_{1j}}{\partial G}}{m_{1j}} - \frac{1}{2}\frac{\frac{\partial m_{2j}}{\partial G}}{m_{2j}}\\ \frac{\partial\sigma_j^2}{\partial G} &= \frac{\frac{\partial m_{2j}}{\partial G}}{m_{2j}} - 2\frac{\frac{\partial m_{1j}}{\partial G}}{m_{1j}}.\end{aligned} \tag{23}$$

From these computations one can easily write $\partial\rho/\partial G$ as

$$\begin{aligned}\frac{\partial\rho}{\partial G} = {} & \frac{1}{\sigma_1\sigma_2}\left(\frac{1}{\mathrm{crm}}\frac{\partial\mathrm{crm}}{\partial G} - \frac{\partial\mu_1}{\partial G} - \frac{\partial\mu_2}{\partial G}\right) - \frac{\ln(\mathrm{crm}) - \mu_1 - \mu_2}{(\sigma_1\sigma_2)^2}\left(\sigma_2\frac{\partial\,\sigma_1}{\partial G} + \sigma_1\frac{\partial\sigma_2}{\partial G}\right)\\ & - \frac{1}{2}\left(\frac{\partial\sigma_1}{\partial G}\frac{1}{\sigma_2} - \frac{\sigma_1}{\sigma_2^2}\frac{\partial\sigma_2}{\partial G} + \frac{\partial\sigma_2}{\partial G}\frac{1}{\sigma_1} - \frac{\sigma_2}{\sigma_1^2}\frac{\partial\sigma_1}{\partial G}\right).\end{aligned} \tag{24}$$

With these elements in hand we can now recover an arbitrary Greek by differentiating (15), which we denote by $\tilde{\Gamma}$, with respect to G by using the relations in (23) and (24):

$$\frac{\partial\tilde{\Gamma}}{\partial G} = \mathrm{e}^{-rT}\sum_{i=1}^{2}\int_0^1 h_i\mathrm{e}^{\mu_i + A_i\Phi^{-1}(u) - \frac{F(u,K)^2}{2} + Y_iF(u,K)}\mathrm{d}u + \mathrm{e}^{-rT}\sum_{i=1}^{2}\varepsilon_i\int_0^1 f_i(u)\mathrm{e}^{\mu_i + A_i\Phi^{-1}(u) + \frac{Y_i^2}{2}}\Phi(Y_i - F(u,K))\mathrm{d}u$$

with

$$h_i = \frac{1}{\sigma_i\sqrt{8\pi(1-\gamma_i^2)}}\left((1-\gamma_i^2)\frac{\partial\sigma_i^2}{\partial G} - \frac{\sigma_i^2}{2}\frac{\partial\rho}{\partial G}\right)$$
$$f_i(u) = \frac{\partial\mu_i}{\partial G} + \left[\frac{\sigma_i}{2\sqrt{2(1+\rho^2)}}\Phi^{-1}(u) - \frac{\sigma_i^2}{4}\right]\frac{\partial\rho}{\partial G} + \frac{1}{2}\left[1 + \gamma_i\left(\frac{\Phi^{-1}(u)}{\sigma_i} - \gamma_i\right)\right]\frac{\partial\sigma_i^2}{\partial G}$$

and $F(u,K) = \Phi^{-1}(F_{\mathbb{S}^{ic}|U=u}(K))$, $A_i = \gamma_i\sigma_i$, $Y_i = \varepsilon_i\sigma_i\sqrt{1-\gamma_i^2}$, $\gamma_i^2 = \frac{1+\rho}{2}$.
Taking G to be σ_j, ρ or $S_j(0)$ yields the Greeks.

6. Quanto and compo options

When the underlying is not denominated in the domestic currency, the option contains an additional risk, the exchange rate risk. In order to cover this risk, the buyer/seller may choose to modify the contract in order to include the exchange rate dynamics in it. Below, we consider two ways of studying these quanto and compo options. We describe how previous results can be adapted in order to price quanto and compo Asian basket spread options. To our knowledge, this is the first time that such an extension is considered.

Let r_d be the domestic risk free interest rate and r_f the foreign risk free interest rate, and assume that both are constant. The dynamics of the exchange rate and foreign assets price under the domestic risk neutral probability measure Q^d are

$$\mathrm{d}I_t = (r_d - r_f)I_t\mathrm{d}t + \sigma_eI_t\mathrm{d}B_t^e,$$

and

$$\mathrm{d}S_{jt}^f = (r_f - \hat{\rho}_j\sigma_j\sigma_e)S_{jt}^f\mathrm{d}t + \sigma_jS_{jt}^f\mathrm{d}B_{jt}$$

where $\{B_t^e : t \geq 0\}$ and $\{B_{jt} : t \geq 0\}$ are Brownian motions under the measure Q^d with correlation coefficient $\hat{\rho}_j$, and with σ_e and σ_j constant volatilities.

6.1. Quanto options

The price of a quanto option is evaluated as

$$\mathrm{e}^{-r_d T} E_{Q^d}\left(\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}\varepsilon_j a_j S^f_{jt_i} I_{t_i} - K\right)_+, \tag{25}$$

where E_{Q^d} is the expectation under the domestic risk neutral measure Q^d.

It is not hard to see that under Q^d the underlying of (25) can be written as

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}\varepsilon_j a_j S^f_j(0) I(0)\mathrm{e}^{\left(r_d-\frac{\hat{\sigma}_j^2}{2}\right)t_i+\hat{\sigma}_j\hat{B}_{jt_i}}, \tag{26}$$

where $\{\hat{B}_{jt} : t \geq 0\}$ are Q^d Brownian motions and

$$\hat{B}_{jt_i} = \frac{\sigma_j B_{jt_i} + \sigma_e B^e_{t_i}}{\hat{\sigma}_j},$$
$$\hat{\sigma}_j = \sqrt{\sigma_j^2 + \sigma_e^2 + 2\sigma_j\sigma_e\hat{\rho}_j},$$
$$\mathrm{corr}(\hat{B}_{jt_i}, \hat{B}_{lt_s}) = \frac{\sigma_j\sigma_l\rho_{jl} + \sigma_j\sigma_e\hat{\rho}_j + \sigma_l\sigma_e\hat{\rho}_l + \sigma_e^2}{\hat{\sigma}_j\hat{\sigma}_l}\,\frac{\min(t_i, t_s)}{\sqrt{t_i t_s}}.$$

We can see that in this case the underlying is still log-normally distributed. Thus we can apply the methods explained before but with the basket given in (26).

6.2. Compo options

The option price is given by

$$\mathrm{e}^{-r_e T} E_{Q^d}\left[\left(\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}\varepsilon_j a_j S^f_{jt_i} - K\right)_+ I_T\right]. \tag{27}$$

Due to the no arbitrage assumption (see [29]) (27) is equivalent to

$$\mathrm{e}^{-r_f T} I(0) E_{Q^f}\left[\left(\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}\varepsilon_j a_j S^f_j(0)\mathrm{e}^{\left(r_f-\frac{\sigma_j^2}{2}\right)t_i+\sigma_j\hat{B}_{jt_i}} - K\right)_+\right],$$

where E_{Q^f} is the foreign risk neutral measure and $\{\hat{B}_{jt} : t \geq 0\}$ are standard Brownian motions under Q^f. So, we manage to reduce the pricing of the compo option to the calculation of a stop-loss premium of a sum of log-normal random variables. Thus once more, we can use our previous methodologies in order to price this option.

7. Conclusion

In this paper, we found that the ICUB offers a good approximation of the price of spread options. We tried several approximation methods for Asian basket spread options and found that a combination of hybrid moment matching combined with the ICUB and shifted log-normal moment matching seems to work best. We developed formulae for the Greeks for the hybrid moment matching method with an ICUB approximation. We also showed how our methodology can easily be applied to the case of options in foreign currency. Extension of the above approximation techniques to Lévy market models is a topic under current investigation.

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Appendix A. Proofs

Proof of Proposition 1. The underlying is of the type

$$\mathbb{S} = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}\varepsilon_j a_j S_j(0)\mathrm{e}^{\left(r-\frac{\sigma_j^2}{2}\right)t_i+\sigma_j B_{jt_i}}.$$

Thus using Proposition 1 from [2], its comonotonic counterpart is given by

$$F_{\mathbb{S}^c}^{-1}(U) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}\varepsilon_j a_j S_j(0)\mathrm{e}^{\left(r-\frac{\sigma_j^2}{2}\right)t_i+\varepsilon_j\sigma_j\sqrt{t_i}\Phi^{-1}(U)},$$

where U is uniformly distributed over $(0, 1)$. Then by means of Theorem 1 we can rewrite the upper bound as

$$\mathrm{e}^{-rT}\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}\varepsilon_j a_j E\left(S_j(0)\mathrm{e}^{\left(r-\frac{\sigma_j^2}{2}\right)t_i+\varepsilon_j\sigma_j\sqrt{t_i}\Phi^{-1}(U)} - K_{ij}\right)_{\varepsilon_j},$$

where

$$K_{ij} = S_j(0)\mathrm{e}^{\left(r-\frac{\sigma_j^2}{2}\right)t_i+\varepsilon_j\sigma_j\sqrt{t_i}\Phi^{-1}(F_{\mathbb{S}^c}(K))},$$

where $(X)_{\varepsilon_j}$ is equal to the function $\max(X, 0)$ if $\varepsilon_j = 1$ and $\min(X, 0)$ if $\varepsilon_j = -1$. The result then follows from standard computations. □

Proof of Proposition 2. First, notice that the conditional distribution of the underlying is

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}\varepsilon_j a_j S_j(0)\mathrm{e}^{\left(r-\frac{\sigma_j^2}{2}\right)t_i+\gamma_{ji}\sigma_j\sqrt{t_i}\Phi^{-1}(U)+\varepsilon_j\sigma_j\sqrt{1-\gamma_{ji}^2}\sqrt{t_i}\Phi^{-1}(V)},$$

where the random variables $U = \Phi[(\Lambda - E[\Lambda])/\sigma_\Lambda]$ and V are independent and are uniformly distributed on $(0, 1)$. Then, using the tower property

$$E[(\mathbb{S}^{ic} - K)_+] = E[E[(\mathbb{S}^{ic} - K)_+|\Lambda]],$$

the rest follows by applying Theorem 1 and some standard computations. □

Proof of Proposition 3. Simply start by noting that

$$\mathbb{S}^\ell = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}\varepsilon_j a_j S_j(0)\mathrm{e}^{\left(r-\frac{(\gamma_{ji}\sigma_j)^2}{2}\right)t_i+\gamma_{ji}\sigma_j\sqrt{t_i}\Phi^{-1}(U)}.$$

The proof then follows from the discussion in Section 2.3, an application of Theorem 1 and some standard computations. □

Proof of Proposition 4. The proof is completely analogous to the one of Theorem 7 in [10]. □

Proof of Proposition 5. See the proof of Proposition 2. □

Proof of Proposition 6. Let, for $K - \eta > 0, A = \{x : \mathrm{e}^{\mu+\sigma x} \geq K - \eta\}$, then

$$\begin{aligned}
E(X-K)_+ &= \int_A (\mathrm{e}^{\mu+\sigma x} + \eta - K)\psi(x, \alpha, \tau)\mathrm{d}x \\
&= \mathrm{e}^{\mu+\frac{1}{2}\sigma^2}\frac{\Phi(\tau+\delta\sigma)}{\Phi(\tau)}\int_{A-\sigma}\psi(x, \alpha, \tau+\delta\sigma)\mathrm{d}x - (K-\eta)\int_A \psi(x, \alpha, \tau)\mathrm{d}x \\
&= \mathrm{e}^{\mu+\frac{1}{2}\sigma^2}\frac{\Phi(\tau+\delta\sigma)}{\Phi(\tau)}\Psi(-I_1; -\alpha; \tau+\delta\sigma) - (K-\eta)\Psi(-I_2; -\alpha; \tau)
\end{aligned}$$

where in the last step we used the property that $1 - \Psi(x, \alpha, \tau) = \Psi(-x, -\alpha, \tau)$. □

Proof of Proposition 7. Immediate from the proof of Proposition 6. □

Proof of Proposition 8. The proof follows from brute force computation. □

Appendix B. Numerical results

CUB = comonotonic upper bound
ICUB = improved comonotonic upper bound
CLB = comonotonic lower bound
CIntB = comonotonic intermediary bound
SLN = shifted log-normal approximation
SLESN = shifted log-extended skew normal approximation
Li et al. = Li et al. spread approximation
MC = Monte Carlo price
S.E. = standard error
HybMMICUB = Hybrid moment matching with improved comonotonic upper bound
HybMMLi = Hybrid moment matching with Li et al. spread approximation

B.1. Spread options

See Tables 1–3.

Table 1
$a = [1, -1]$, $S(0) = [100, 200]$, $\sigma = [0.6, 0.6]$, $\rho_{12} = 0.28$, 100 millions of paths (negative skewness).

Strike	CUB	ICUB	CLB	CIntB	SLN	SLESN	Li et al.	MC	S.E.
−70	51.4002	29.0854	26.9232	28.7313	31.0619	29.7272	29.0742	29.0846	0.0005
−80	55.8467	33.6150	30.8369	33.3442	35.8096	34.5680	33.6083	33.6142	0.0005
−90	60.5351	38.5281	35.1084	38.3007	40.8772	39.7385	38.5235	38.5273	0.0006
−100	65.4629	43.8043	39.7382	43.5861	46.2500	45.2212	43.7990	43.8034	0.0006
−110	70.6261	49.4212	44.7238	49.1844	51.9127	50.9979	49.4130	49.4203	0.0006
−120	76.0199	55.3561	50.0590	55.0786	57.8497	57.0506	55.3433	55.3552	0.0006
−130	81.6384	61.5861	55.7345	61.2516	64.0453	63.3615	61.5678	61.5851	0.0007

Table 2
$a = [1, -1]$, $S(0) = [100, 40]$, $\sigma = [0.4, 0.17]$, $\rho_{12} = 0.12$, 60 millions of paths (positive skewness).

Strike	CUB	ICUB	CLB	CIntB	SLN	SLESN	Li et al.	MC	S.E.
45	27.3131	24.5975	22.1702	24.5788	24.6096	24.6064	24.5982	24.5981	0.0006
50	24.5923	21.8240	19.0498	21.8093	21.8441	21.8328	21.8247	21.8247	0.0006
55	22.0846	19.3079	16.2331	19.2957	19.3342	19.3161	19.3086	19.3085	0.0005
60	19.7844	17.0384	13.7215	17.0275	17.0692	17.0458	17.0391	17.0391	0.0005
65	17.6840	15.0022	11.5087	14.9914	15.0355	15.0085	15.0029	15.0029	0.0005
70	15.7739	13.1835	9.5811	13.1719	13.2178	13.1887	13.1842	13.1842	0.0005
75	14.0436	11.5656	7.9202	11.5525	11.5997	11.5698	11.5663	11.5664	0.0004

Table 3
$a = [1, -1]$, $S(0) = [100, 50]$, $\sigma = [0.6, 0.2]$, $\rho_{12} = 0.89$, 60 millions of paths (positive skewness).

Strike	CUB	ICUB	CLB	CIntB	SLN	SLESN	Li et al.	MC	S.E.
35	34.8962	27.4968	17.8043	28.2332	27.5689	27.4974	27.4992	27.4964	0.0008
40	32.5542	25.1757	13.9743	25.9171	25.2255	25.1761	25.1781	25.1754	0.0008
45	30.3585	23.0587	10.6097	23.7962	23.0877	23.0591	23.0611	23.0585	0.0008
50	28.3029	21.1293	7.7776	21.8555	21.1395	21.1297	21.1317	21.1291	0.0008
55	26.3808	19.3715	5.4995	20.0808	19.3651	19.3721	19.3739	19.3715	0.0008
60	24.5854	17.7703	3.7499	18.4584	17.7496	17.7711	17.7727	17.7704	0.0007
65	22.9101	16.3117	2.4667	16.9753	16.2790	16.3128	16.3141	16.3119	0.0007

B.2. Basket spread options

See Tables 4–7.

Table 4
$a = [1, -1, -1]$, $S(0) = [100, 24; 46]$, $\sigma = [0.4, 0.22, 0.3]$, $\rho_{12} = 0.17$, $\rho_{13} = 0.91$, $\rho_{23} = 0.41$, 300 millions of paths (positive skewness).

Strike	ICUB	SLN	HybMMLi	HybMMICUB	MC	S.E.
15	19.9819	19.6925	19.5251	19.5231	19.6849	0.00009
20	17.0143	16.7345	16.5693	16.5673	16.7051	0.00008
25	14.4105	14.1460	13.9964	13.9944	14.1010	0.00008
30	12.1523	11.9059	11.7811	11.7790	11.8519	0.00007
35	10.2123	9.9851	9.8898	9.8876	9.9281	0.00007
40	8.5588	8.3506	8.2860	8.2837	8.2951	0.00006
45	7.1581	6.9683	6.9330	6.9305	6.9174	0.00006

Table 5
$a = [1, -1, -1, -1]$, $S(0) = [100, 100, 50, 70]$, $\sigma = [0.5, 0.15, 0.2, 0.17]$, $\rho_{ij} = 0.9$ for all i and j, 300 millions of paths (positive skewness).

Strike	ICUB	SLN	HybMMLi	HybMMICUB	MC	S.E.
−90	2.8088	2.4884	2.1748	2.4043	2.4089	0.00004
−100	3.8757	3.4820	3.6191	3.3098	3.3122	0.00005
−110	5.4669	4.9521	5.3296	4.6565	4.6670	0.00006
−120	7.9235	7.1616	7.3458	6.7643	6.7757	0.00006
−130	11.7246	10.5190	10.6016	10.2529	10.2660	0.00007
−140	17.2439	15.6048	16.3404	15.8233	15.8360	0.00008
−150	24.4315	22.9793	23.3843	23.4623	23.4799	0.00009

Table 6
$a = [1, -1, -1, -1]$, $S(0) = [100, 60, 40, 30]$, $\sigma = [0.16, 0.23, 0.32, 0.43]$, $\rho_{12} = 0.42$, $\rho_{13} = 0.5$, $\rho_{14} = 0.3$, $\rho_{23} = 0.24$, $\rho_{24} = 0.42$, $\rho_{34} = 0.35$, 500 millions of paths (negative skewness).

Strike	ICUB	SLN	HybMMLi	HybMMICUB	MC	S.E.
−5	5.5456	1.3847	1.5248	1.5248	1.4384	0.000009
−10	7.0286	2.2538	2.3780	2.3780	2.2796	0.00001
−20	10.7128	4.9936	5.0508	5.0508	4.9511	0.00001
−30	15.3583	9.2153	9.1940	9.1939	9.1259	0.00001
−40	20.9135	14.8764	14.8006	14.8006	14.7814	0.00003
−50	27.2823	21.7566	21.6606	21.6606	21.6868	0.00004
−60	34.3462	29.5647	29.4747	29.4747	29.5299	0.00004

Table 7
$a = [1, -1, -1]$, $S(0) = [100, 63, 12]$, $\sigma = [0.21, 0.34, 0.63]$, $\rho_{12} = 0.87$, $\rho_{13} = 0.3$, $\rho_{23} = 0.43$, 15 millions of paths (negative skewness).

Strike	ICUB	SLN	HybMMLi	HybMMICUB	MC	S.E.
2.5	24.6617	23.1681	23.5137	23.5138	23.5925	0.0009
10	18.5944	16.8591	17.1363	17.1373	17.2049	0.0008
17.5	13.0945	11.3394	11.3854	11.3873	11.4099	0.0007
25	8.4135	6.9203	6.6579	6.6584	6.6009	0.0006
32.5	4.8064	3.7629	3.3226	3.3147	3.1872	0.0004
40	2.3929	1.7925	1.3950	1.3853	1.2518	0.0003
47.5	1.0323	0.7369	0.4861	0.4913	0.4026	0.0001

B.3. Asian basket spread options (with maturity 1 year and with 30 averaging dates)

See Tables 8–11.

Table 8
$a = [1, -1]$, $S(0) = [100, 60]$, $\sigma = [0.33, 0.25]$, $\rho_{12} = 0.4$, 400 millions of paths (positive skewness).

Strike	SLN	HybMMICUB	MC	S.E.
25	20.8073	20.7637	20.7645	0.00009
30	17.7711	17.6921	17.6931	0.00009
35	15.0630	14.9580	14.9591	0.00008
40	12.6769	12.5566	12.5579	0.00008
45	10.5983	10.4734	10.4747	0.00007
50	8.8065	8.6859	8.6873	0.00007
55	7.2767	7.1671	7.1685	0.00006

Table 9
$a = [1, -1]$, $S(0) = [100, 240]$, $\sigma = [0.18, 0.35]$, $\rho_{12} = 0.9$, 400 millions of paths (negative skewness).

Strike	SLN	HybMMICUB	MC	S.E.
−200	61.7315	61.7593	61.7623	0.0002
−180	47.0642	47.0946	47.0975	0.0002
−160	33.9175	33.9449	33.9473	0.0001
−140	22.6666	22.6835	22.6853	0.0001
−120	13.6581	13.6572	13.6585	0.00008
−100	7.1111	7.0901	7.0908	0.00005
−80	2.9895	2.9558	2.9561	0.00003

Table 10
$a = [1, -1, -1]$, $S(0) = [100, 50, 25]$, $\sigma = [0.35, 0.3, 0.25]$, $\rho_{12} = 0.3$, $\rho_{13} = 0.8$, $\rho_{23} = 0.7$, 300 millions of paths (positive skewness).

Strike	SLN	HybMMICUB	MC	S.E.
10	20.7157	20.5521	20.6014	0.00008
15	17.7346	17.5209	17.5549	0.00008
20	15.0677	14.8236	14.8411	0.00007
25	12.7097	12.4559	12.4577	0.00007
30	10.6478	10.4030	10.3907	0.00007
35	8.8635	8.6425	8.6186	0.00006
40	7.3342	7.1472	7.1146	0.00006

Table 11
$a = [1, 1, -1, -1]$, $S(0) = [100, 50, 200, 20]$, $\sigma = [0.18, 0.2, 0.25, 0.3]$, $\rho_{12} = 0.6$, $\rho_{13} = 0.7$, $\rho_{14} = 0.8$, $\rho_{23} = 0.5$, $\rho_{24} = 0.75$, $\rho_{34} = 0.9$, 300 millions of paths (negative skewness).

Strike	SLN	HybMMICUB	MC	S.E.
−40	3.6613	3.6643	3.6659	0.00004
−50	6.2375	6.2174	6.2191	0.00005
−60	9.7479	9.7098	9.7103	0.00007
−70	14.2134	14.1659	14.1661	0.00008
−80	19.5929	19.5450	19.5432	0.00008
−90	25.8016	25.7600	25.7580	0.00009
−100	32.7291	32.6977	32.6946	0.0001

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