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The pricing of basket-spread options

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Since the pioneering paper of Black and Scholes was published in 1973, enormous research effort has been spent on finding a multi-asset variant of their closed-form option pricing formula. In this paper, we generalize the Kirk [*Managing Energy Price Risk*, 1995] approximate formula for pricing a two-asset spread option to the case of a multi-asset basket-spread option. All the advantageous properties of being simple, accurate and efficient are preserved. As the final formula retains the same functional form as the Black–Scholes formula, all the basket-spread option Greeks are also derived in closed form. Numerical examples demonstrate that the pricing and hedging errors are in general less than 1% relative to the benchmark results obtained by numerical integration or Monte Carlo simulation with 10 million paths. An implicit correction method is further applied to reduce the pricing errors by factors of up to 100. The correction is governed by an unknown parameter, whose optimal value is found by solving a non-linear equation. Owing to its simplicity, the computing time for simultaneous pricing and hedging of basket-spread option with 10 underlying assets or less is kept below 1 ms. When compared against the existing approximation methods, the proposed basket-spread option formula coupled with the implicit correction turns out to be one of the most robust and accurate methods.

Keywords: Multi-variate contingent; Derivatives pricing; Black-Scholes model; Portfolio hedging; Closed-form approximation; Parametric correction

1. Introduction

A basket-spread option is an option whose pay-off at maturity is given by the difference (or so-called the spread) between two baskets of aggregated underlying asset prices. For a standard European basket-spread call option, the pay-off function reads:

$$P(\mathbf{S}, T) = \left(\sum_{i=1}^{M} S_i(T) - \sum_{i=M+1}^{M+N} S_i(T) - K \right)^+ \tag{1}$$

where K is the strike price and $S_i(T)$ is the ith underlying asset price at maturity time $t = T$. This embodies a general class of options including the two-asset spread ($M = N = 1$), basket ($N = 0$) and single-asset vanilla ($M = 1$, $N = 0$) options.

A basket-spread options are prevalent in a variety of contemporary markets including the fixed-income, foreign exchange, commodity, energy and equity markets (see Carmona and Durrleman 2003b and references therein for a survey of applications of spread options). They are useful financial tools for hedging a portfolio of long and short positions in the underlying assets. A simple, accurate and efficient method to price and hedge basket-spread options is therefore inevitable. There is a growing demand for pricing a sizeable quantity of basket-spread options, each with a sizeable portfolio of underlying assets, in an almost instant manner.

Since the pioneering paper of Black and Scholes (1973) was published, enormous research effort has been spent on finding a multi-asset variant of their closed-form option pricing formula. Margrabe (1978) made the first breakthrough on pricing the exchange option, which is effectively a two-asset spread option with zero strike price. Within the Black–Scholes framework, extension beyond that is inherently difficult because the underlying asset prices are assumed to be driven by lognormal processes, but a linear combination of lognormal random variables does not follow the lognormal distribution. The lack of an exact marginal distribution for the basket-spread variable (i.e. $\sum_{i=1}^{M} S_i - \sum_{i=M+1}^{M+N} S_i$) restricts the emergence of an exact closed-form solution for the basket-spread option. Over the past decades, researchers have developed a wide variety of approximation methods to tackle the problem, which can be broadly divided into three categories: numerical, semi-analytic and analytic methods.

Numerical methods are mostly robust but inefficient. They offer the flexibility to adjust accuracy at the cost of computational time. One classical approach is the three-dimensional binomial tree method, which was first used by Boyle (1988) to price two-asset spread options. Later, Pearson (1995) simplified the spread option problem into an one-dimensional integral, which could be evaluated numerically. More recently, Dempster and Hong (2000) tackled the same problem with the fast Fourier transform. Other common numerical methods include the Monte Carlo simulation and finite-difference

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methods. Unfortunately, most numerical methods become practically infeasible when the number of underlying assets exceeds two.

For semi-analytic methods, we discuss three approaches that can be found in the recent literature. Carmona and Durrleman (2003a) derived an analytic formula to approximate the two-asset spread option price, and later (2006) they extended their analysis to the basket-spread option. But before their analytic formula can be applied, a set of coupled non-linear equations needs to be solved numerically. It was pointed out by Li *et al.* (2010) that the convergence of such a numerical solution was very sensitive to the choice of initial input values. Borovkova *et al.* (2007) proposed to approximate the basket-spread distribution with a shifted lognormal distribution by matching their first three moments. But again the matching process requires numerically solving a set of non-linear equations. Venkatramanan and Alexander (2011) transformed the spread option into two 'compound exchange options' (CEOs), which had no exact solution either. They took a lognormal approximation and derived an analytic pricing formula for the CEO. More recently, Alexander and Venkatramanan (2012) adopted a similar approach to price the basket-spread option. They transformed the multi-asset problem into a recursive sum of CEOs. Although they did not make any comparison with other existing methods, they placed their emphasis on capturing the volatility smiles and correlation frowns observed in market basket-spread option prices.

Most analytic methods rely on a well-known distribution to approximate the unknown spread, basket or basket-spread distribution, and then use the corresponding density function to construct pricing formula for the relevant multi-asset option. One class of such approximation is regarded as the Bachelier Model (see Wilcox 1990, Shimko 1994, Poitras 1998), in which the spread distribution is approximated by a normal distribution, so that the Bachelier formula can be used to price the spread option. Others proposed to approximate the basket distribution with a lognormal distribution, either through the technique of moments matching (see Turnbull and Wakeman 1991) or by replacing the arithmetic average with the corresponding geometric average (see Gentle 1993).† Milevsky and Posner (1998b) suggested approximating the basket distribution, with a reciprocal gamma distribution and they derived yet another analytic formula for pricing basket options.‡ These analytic methods, although being very simple in nature, are reported to suffer from significant pricing errors.

The most notable analytic solution appears to be given by Kirk (1995), in which an analytic formula is proposed to approximate the two-asset spread option price. It remains as one of the most popular methods amongst practitioners because it retains all the simplicity and tractability of the classical Black–Scholes formula. Another notable analytic solution was recently proposed by Li *et al.* (2008), in which they introduced the notion of 'exercise boundary' and approximated it with a quadratic function, such that the one-dimensional integral proposed by Pearson (1995) could be evaluated analytically. Later Li *et al.* (2010) extended their analysis to the pricing of multi-asset spread options. Their solution is claimed to be the most accurate existing analytic method for pricing spread options, with reported relative errors in the order of 0.1%. However, they only provided a solution for the special case of $M = 1$. For $M > 1$, they suggested to approximate the first basket variable with its geometric mean. Such crude approximation not only deteriorates the pricing accuracy, but also restricts their solution's applicability for the basket option (when $M > 1, N = 0$). Furthermore, in actual implementation, we found that their method requires a lot of linear algebraic manipulations such as solving eigensystems, matrix diagonalization and matrix decomposition.

In this paper, we generalize Kirk's approximate formula for pricing the two-asset spread option to the case of the multi-asset basket-spread option. Upon generalizing the assumption behind Kirk's approximation, the original multi-asset problem transforms into the valuation of an exchange option, which has a closed-form solution available. As the final basket-spread option formula retains the same functional form as the Black–Scholes formula, all the hedging Greeks can also be derived in closed form. In addition, we propose a simple and efficient procedure which we call the 'alpha method', to improve the overall accuracy of the approximation. It exploits the fact that the basket-spread option formula does not satisfy the pricing partial differential equation (PDE) exactly. By introducing a single correction parameter α, it enforces the approximate solution to satisfy the pricing PDE at least at $t = 0$. Numerical examples demonstrate that the basket-spread option formula coupled with the parametric correction, despite being a lot simpler, can produce more accurate and robust results than most other approximation methods.

This paper is structured as follows. Section 2 outlines the model set-up of the basket-spread option pricing problem. Section 3 proposes closed-form approximate formulae for the basket-spread option price and Greeks. Section 4 introduces an implicit correction method to improve the overall approximation. Section 5 compares the proposed approximation methods with various existing methods through a number of numerical examples. Section 6 contains the conclusion.

2. The model setup

The pricing of basket-spread option can be formulated as a multi-dimensional partial differential equation (PDE), which can be considered as a multi-asset variant of the Black–Scholes equation. Under the risk-neutral measure, we assume that the underlying asset prices are driven by the following correlated lognormal processes:

$$\frac{dS_i(t)}{S_i(t)} = (r - q_i)dt + \sigma_i dW_i(t), \qquad i = 1, 2 \ldots M + N \quad (2)$$

where r is the risk-free interest rate, q_i's are the dividend yields, σ_i's are the volatilities and W_i's are correlated Wiener processes with $\mathbb{E}[dW_i dW_j] = \rho_{ij} dt$. By Ito's lemma, the risk-neutral pricing PDE reads:

†Vorst (1992) suggested to approximate the arithmetic average of lognormal random variables with the corresponding geometric average when he was pricing Asian options. Later Gentle (1993) applied the same technique to price basket options. The pricing of Asian and basket options are often considered as parallel problems with transferable solutions between the two.

‡Milevsky and Posner (1998a), Milevsky and Posner (1998b) priced Asian and basket options separately with essentially the same technique.

$$\frac{\partial P(\mathbf{S},t)}{\partial t} + \frac{1}{2}\sum_{i=1}^{M+N}\sum_{j=1}^{M+N}\rho_{ij}\sigma_i\sigma_j S_i S_j \frac{\partial^2 P(\mathbf{S},t)}{\partial S_i \partial S_j} + \sum_{i=1}^{M+N}(r-q_i)S_i\frac{\partial P(\mathbf{S},t)}{\partial S_i} - rP(\mathbf{S},t) = 0 \quad (3)$$

In principle, if one could solve $P(\mathbf{S},t)$ from the above PDE subject to appropriate final and boundary conditions, the solution is unique and exact for the price of a basket-spread option. For a standard European basket-spread call option, the final condition is given in (1), and the boundary conditions are given by:

$$P\left(S_i, \left\{S_j = 0\,\forall_{j\neq i}\right\}, t\right) = P_{\mathrm{BS}}(S_i, t;\ r, q_i, \sigma_i, K)\,\mathbb{1}_{i\leq M}, \quad i = 1, 2 \dots M+N \quad (4)$$

where $P_{\mathrm{BS}}(\cdot)$ is the Black–Scholes formula specified in appendix 1; $\mathbb{1}_A$ is the indicator function conditional on event A.

Furthermore, basket-spread options are sometimes contingent on future prices rather than spot prices of the underlying assets. Such options are more common in the commodity and fixed-income markets. Black (1976) has shown that, for the single-asset case, the future option pricing formula is equivalent to the Black–Scholes formula if one replaces the spot underlying price with the future price and sets its drift rate equal to zero. In the language of probability theory, the future prices are martingales, with their risk-neutral dynamics given by:

$$\frac{dF_i(t)}{F_i(t)} = \sigma_i dW_i(t), \quad i = 1, 2 \dots M+N \quad (5)$$

By virtue of no-arbitrage argument, the future prices are related to the spot prices through:

$$F_i(t) = S_i(t)\,\mathrm{e}^{(r-q_i)(T-t)}, \quad i = 1, 2 \dots M+N \quad (6)$$

provided that all the underlying futures would mature at the same time as the basket-spread option at $t = T$.† By a simple change of variables using equation (6), the pricing PDE in equation (3) transforms into:

$$\frac{\partial P(\mathbf{F},t)}{\partial t} + \frac{1}{2}\sum_{i=1}^{M+N}\sum_{j=1}^{M+N}\rho_{ij}\sigma_i\sigma_j F_i F_j \frac{\partial^2 P(\mathbf{F},t)}{\partial F_i \partial F_j} - rP(\mathbf{F},t) = 0 \quad (7)$$

The final condition takes the same form as in (1) because $F_i(T) = S_i(T)$. The boundary conditions become:

$$P\left(F_i, \left\{F_j = 0\,\forall_{j\neq i}\right\}, t\right) = P_{\mathrm{BS}}(F_i, t;\ r, r, \sigma_i, K)\,\mathbb{1}_{i\leq M}, \quad i = 1, 2 \dots M+N \quad (8)$$

where the RHS is really the Black (1976) future option formula. The solutions of the two option pricing problems, one contingent on spot prices while another one contingent on future prices, are equivalent. The pricing methods to be developed in this paper can be applied to the basket-spread option of both types.

†If the underlying futures do not all mature at $t = T$, a more complex and careful implementation needs to be incorporated into the analysis, which is outside the scope of our discussion here.

3. Closed-form approximation

3.1. *Two-asset spread option*

Before we tackle the general basket-spread option, we quote the Kirk (1995) two-asset spread option approximate formula and illustrate how it is closely related to the Black–Scholes formula. Following the same notation as before, Kirk's formula is given by:

$$P_{\mathrm{Kirk}}(F_1, F_2, t) = P_{\mathrm{BS}}(F_1, t;\ r, r, \tilde{\sigma}, F_2 + K) \quad (9)$$

where $P_{BS}(\cdot)$ is the Black–Scholes formula; and

$$\tilde{\sigma} = \sqrt{\sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2\left(\frac{F_2}{F_2+K}\right) + \sigma_2^2\left(\frac{F_2}{F_2+K}\right)^2} \quad (10)$$

is the effective volatility. If the spread option is contingent on spot prices, each of the F_i's can be replaced by the spot-future relation in (6). One may recognize that equation (9) is equivalent to Margrabe's exchange option formula when K is zero. When $F_1 = 0$, its limit tends to zero; when $F_2 = 0$, it recovers the Black (1976) single-asset future option formula. Hence, it satisfies the boundary conditions in (8) exactly. It also satisfies the final condition because it approaches $(F_1 - F_2 - K)^+$ when the limit $t \to T$ is taken. If $\tilde{\sigma}$ is assumed to be a constant (which is the common assumption when practitioners calculate the spread option Greeks), it can be verified by direct substitution that equation (9) satisfies the pricing PDE in (7). However, $\tilde{\sigma}$ is actually a function dependent on F_2, and thus Kirk's formula is only an approximate solution. Appendix 2 provides a simple derivation of equation (9) and points out the key assumption behind Kirk's approximation.

3.2. *Multi-asset basket-spread option*

By inspection of equation (9), one can in fact speculate the form of a multi-asset variant. It can also be derived by generalizing the assumption behind Kirk's approximation, as done in appendix 3. To our knowledge, such simple extension cannot be found in existing literature.‡ We propose the following formula as a generalization of Kirk's formula to approximate the basket-spread option price:

$$P_{\mathrm{Mul}}(\mathbf{F}, t) = P_{\mathrm{BS}}\left(\tilde{F}, t;\ r, r, \tilde{\sigma}, \tilde{K}\right) \quad (11)$$

where

$$\tilde{F} = \sum_{i=1}^{M} F_i \quad (12a)$$

$$\tilde{K} = K + \sum_{i=M+1}^{M+N} F_i \quad (12b)$$

$$\tilde{\sigma} = \sqrt{\sum_{i=1}^{M+N}\sum_{j=1}^{M+N}\rho_{ij}\sigma_i\sigma_j m_i m_j} \quad (12c)$$

‡Li *et al.* (2010) proposed an extended Kirk's approximation, which replaced each of the basket values with its corresponding geometric average. Under such crude approximation, the pricing errors tend to be fairly large.

$$m_i = \begin{cases} F_i/\tilde{F}, & i = 1, 2...M \\ -F_i/\tilde{K}, & i = M+1, M+2...M+N \end{cases} \tag{12d}$$

Again each of the F_i's can be replaced by the spot-future relation in (6) if the option is contingent on spot prices. It can be readily verified that the above equation satisfies the boundary and final conditions. It also satisfies the pricing PDE if $\tilde{\sigma}$ is assumed to be a constant. When $M = N = 1$, it recovers Kirk's formula in equation (9).

3.3. *The Greeks*

Since the basket-spread option formula in equation (11) is in the same functional form as the Black–Scholes formula, one may express all the basket-spread option Greeks in terms of the Black–Scholes Greeks.† Here, we relax the assumption of constant $\tilde{\sigma}$ when we calculate the Greeks. This is contrary to the general perception when practitioners apply Kirk's formula to calculate the spread option Greeks (see discussion on Venkatramanan and Alexander 2011). When $\tilde{\sigma}$ is treated as a function, differentiating the basket-spread option formula would incur additional terms that involve the derivatives of $\tilde{\sigma}$. These additional terms turn out to be the primary building blocks of the implicit correction method we introduce in section 4.

By the chain rule of differentiation, it is not difficult to obtain all the Greeks in closed form. The Delta's and Gamma's are given by:

$$\begin{aligned} \Delta_i &= \frac{\partial P_{\text{Mul}}}{\partial S_i} = \mathrm{e}^{(r-q_i)(T-t)} \frac{\partial P_{\text{Mul}}}{\partial F_i} \\ &= \mathrm{e}^{(r-q_i)(T-t)} \left(\Delta_{\text{BS}} \mathbb{1}_{i \le M} + \kappa_{\text{BS}} \mathbb{1}_{i>M} + \nu_{\text{BS}} \frac{\partial \tilde{\sigma}}{\partial F_i} \right), \end{aligned} \tag{13}$$

$$\begin{aligned} \Gamma_{ij} &= \frac{\partial^2 P_{\text{Mul}}}{\partial S_i \partial S_j} = \mathrm{e}^{(2r-q_i-q_j)(T-t)} \frac{\partial^2 P_{\text{Mul}}}{\partial F_i \partial F_j} \\ &= \mathrm{e}^{(2r-q_i-q_j)(T-t)} \Bigg\{ \Gamma_{\text{BS}}^{\text{vv}} \frac{\partial \tilde{\sigma}}{\partial F_i} \frac{\partial \tilde{\sigma}}{\partial F_j} + \nu_{\text{BS}} \frac{\partial^2 \tilde{\sigma}}{\partial F_i \partial F_j} \\ &\quad + \left[\Gamma_{\text{BS}}^{\text{ss}} + \Gamma_{\text{BS}}^{\text{sv}} \left(\frac{\partial \tilde{\sigma}}{\partial F_i} + \frac{\partial \tilde{\sigma}}{\partial F_j} \right) \right] \mathbb{1}_{i \le M \cap j \le M} \\ &\quad + \left[\Gamma_{\text{BS}}^{\text{kk}} + \Gamma_{\text{BS}}^{\text{kv}} \left(\frac{\partial \tilde{\sigma}}{\partial F_i} + \frac{\partial \tilde{\sigma}}{\partial F_j} \right) \right] \mathbb{1}_{i > M \cap j > M} \\ &\quad + \left[\Gamma_{\text{BS}}^{\text{sk}} + \Gamma_{\text{BS}}^{\text{sv}} \frac{\partial \tilde{\sigma}}{\partial F_j} + \Gamma_{\text{BS}}^{\text{kv}} \frac{\partial \tilde{\sigma}}{\partial F_i} \right] \mathbb{1}_{i \le M \cap j > M} \Bigg\} \end{aligned} \tag{14}$$

for $i = 1, 2..M+N$, $j \ge i$; and $\Gamma_{ij} = \Gamma_{ji}$ otherwise. When the basket-spread option is contingent on future prices, its *Theta* is given by:

$$\Theta(\mathbf{F}, t) = \frac{\partial P_{\text{Mul}}}{\partial t} = \Theta_{\text{BS}} \tag{15}$$

When the option is contingent on spot prices, the time dependence of $\tilde{F}$, $\tilde{K}$ and $\tilde{\sigma}$ need to be incorporated as well. The *Theta* becomes:

$$\Theta(\mathbf{S}, t) = \Theta_{\text{BS}} + \nu_{\text{BS}} \frac{\partial \tilde{\sigma}}{\partial t} + \Delta_{\text{BS}} \frac{\partial \tilde{F}}{\partial t} + \kappa_{\text{BS}} \frac{\partial \tilde{K}}{\partial t} \tag{16}$$

where

$$\frac{\partial \tilde{F}}{\partial t} = -\sum_{i=1}^{M} (r - q_i) S_i \mathrm{e}^{(r-q_i)(T-t)} \tag{17a}$$

$$\frac{\partial \tilde{K}}{\partial t} = -\sum_{i=M+1}^{M+N} (r - q_i) S_i \mathrm{e}^{(r-q_i)(T-t)} \tag{17b}$$

All $\Delta_{\text{BS}}, \kappa_{\text{BS}}, \Gamma_{\text{BS}}$'s, ν_{BS} and Θ_{BS} are the Black–Scholes Greeks taking the same set of input parameters as in equation (11). To evaluate the full expressions of the Greeks, we also need the various derivatives of $\tilde{\sigma}$. First, we define three more effective volatilities:

$$\tilde{\sigma}_F^2 = \sum_{i=1}^{M} \sum_{j=1}^{M} \rho_{ij} \sigma_i \sigma_j m_i m_j \tag{18a}$$

$$\tilde{\sigma}_K^2 = \sum_{i=M+1}^{M+N} \sum_{j=M+1}^{M+N} \rho_{ij} \sigma_i \sigma_j m_i m_j \tag{18b}$$

$$\tilde{\sigma}_X^2 = \sum_{i=1}^{M} \sum_{j=M+1}^{M+N} \rho_{ij} \sigma_i \sigma_j m_i m_j \tag{18c}$$

where m_i's are defined in (12). Collectively, they are the partial sums that contribute to the double summation within $\tilde{\sigma}^2$:

$$\tilde{\sigma}^2 = \tilde{\sigma}_F^2 + 2\tilde{\sigma}_X^2 + \tilde{\sigma}_K^2 \tag{19}$$

Further introducing two sets of marginal effective volatilities:

$$\tilde{\sigma}_{F,i}^2 = \sigma_i \sum_{j=1}^{M} \rho_{ij} \sigma_j m_j \tag{20a}$$

$$\tilde{\sigma}_{K,i}^2 = \sigma_i \sum_{j=M+1}^{M+N} \rho_{ij} \sigma_j m_j \tag{20b}$$

for $i = 1, 2...M+N$, then the spatial derivatives of $\tilde{\sigma}$ are given by:

$$\begin{aligned} \frac{\partial \tilde{\sigma}}{\partial F_i} &= -\frac{\tilde{\sigma}_F^2 + \tilde{\sigma}_X^2 - \tilde{\sigma}_{F,i}^2 - \tilde{\sigma}_{K,i}^2}{\tilde{\sigma} \tilde{F}} \mathbb{1}_{i \le M} \\ &\quad - \frac{\tilde{\sigma}_K^2 + \tilde{\sigma}_X^2 + \tilde{\sigma}_{F,i}^2 + \tilde{\sigma}_{K,i}^2}{\tilde{\sigma} \tilde{K}} \mathbb{1}_{i>M} \end{aligned} \tag{21a}$$

$$\begin{aligned} \frac{\partial^2 \tilde{\sigma}}{\partial F_i \partial F_j} &= -\frac{1}{\tilde{\sigma}} \frac{\partial \tilde{\sigma}}{\partial F_i} \frac{\partial \tilde{\sigma}}{\partial F_j} - \left(\frac{\partial \tilde{\sigma}}{\partial F_i} + \frac{\partial \tilde{\sigma}}{\partial F_j} \right) \\ &\quad \times \left(\frac{\mathbb{1}_{i \le M \cap j \le M}}{\tilde{F}} + \frac{\mathbb{1}_{i > M \cap j > M}}{\tilde{K}} \right) \\ &\quad + \frac{\rho_{ij} \sigma_i \sigma_j + \tilde{\sigma}_F^2 - \tilde{\sigma}_{F,i}^2 - \tilde{\sigma}_{F,j}^2}{\tilde{\sigma} \tilde{F}^2} \mathbb{1}_{i \le M \cap j \le M} \\ &\quad + \frac{\rho_{ij} \sigma_i \sigma_j + \tilde{\sigma}_K^2 + \tilde{\sigma}_{K,i}^2 + \tilde{\sigma}_{K,j}^2}{\tilde{\sigma} \tilde{K}^2} \mathbb{1}_{i > M \cap j > M} \\ &\quad - \frac{\rho_{ij} \sigma_i \sigma_j - \tilde{\sigma}_X^2 + \tilde{\sigma}_{K,i}^2 - \tilde{\sigma}_{F,j}^2}{\tilde{\sigma} \tilde{F} \tilde{K}} \mathbb{1}_{i \le M \cap j > M} \end{aligned} \tag{21b}$$

for $i = 1, 2..M+N$, $j \ge i$; and $\frac{\partial^2 \tilde{\sigma}}{\partial F_i \partial F_j} = \frac{\partial^2 \tilde{\sigma}}{\partial F_j \partial F_i}$ otherwise. If $\tilde{\sigma}$ is expressed in terms of the spot prices, its time derivative is given by:

†Appendix 1 provides a list of the Black–Scholes Greeks in their closed form.

$$\frac{\partial \tilde{\sigma}}{\partial t} = \frac{1}{\tilde{\sigma}} \sum_{i=1}^{M+N} \sum_{j=1}^{M+N} \rho_{ij}\sigma_i\sigma_j \frac{\partial m_i}{\partial t} m_j \tag{22}$$

where

$$\frac{\partial m_i}{\partial t} = -\left[m_i \sum_{j=1}^{M} (r-q_j)(\delta_{ij} - m_j) \right] \mathbb{1}_{i\le M} - \left[m_i \sum_{j=M+1}^{M+N} (r-q_j)(\delta_{ij} + m_j) \right] \mathbb{1}_{i>M} \tag{23}$$

for $i = 1, 2...M+N$; δ_{ij} is the Kronecker delta function. If $\tilde{\sigma}$ is expressed in terms of the future prices, it is not explicitly dependent on time.

Other Greeks may also be derived by simple differentiation. Here, we present three more Greeks that are commonly used in practice. The *Kappa* is given by:

$$\kappa = \frac{\partial P_{\mathrm{Mul}}}{\partial K} = \kappa_{\mathrm{BS}} + \nu_{\mathrm{BS}} \frac{\partial \tilde{\sigma}}{\partial K} \tag{24}$$

The Vega's and Chi's are given by:

$$\nu_i = \frac{\partial P_{\mathrm{Mul}}}{\partial \sigma_i} = \nu_{\mathrm{BS}} \frac{\partial \tilde{\sigma}}{\partial \sigma_i}, \tag{25}$$

$$\chi_{ij} = \frac{\partial P_{\mathrm{Mul}}}{\partial \rho_{ij}} = \nu_{\mathrm{BS}} \frac{\partial \tilde{\sigma}}{\partial \rho_{ij}}, \tag{26}$$

for $i = 1, 2...M+N$, $j \neq i$. The relevant derivatives of $\tilde{\sigma}$ are given by:

$$\frac{\partial \tilde{\sigma}}{\partial K} = -\frac{\tilde{\sigma}_K^2 + \tilde{\sigma}_X^2}{\tilde{\sigma}\tilde{K}} \tag{27a}$$

$$\frac{\partial \tilde{\sigma}}{\partial \sigma_i} = \frac{m_i\left(\tilde{\sigma}_{F,i}^2 + \tilde{\sigma}_{K,i}^2\right)}{\tilde{\sigma}\sigma_i} \tag{27b}$$

$$\frac{\partial \tilde{\sigma}}{\partial \rho_{ij}} = \frac{\sigma_i\sigma_j m_i m_j}{\tilde{\sigma}} \tag{27c}$$

In actual implementation, one could first calculate the Black–Scholes price and Greeks with the set of input parameters specified in (12). Then, the various derivatives of $\tilde{\sigma}$ can be calculated and stored up in some arrays. Finally, the full set of basket-spread option price and Greeks can be calculated using the closed-form formulae presented above. In our calculation, the computing time for a set of 32-asset basket-spread option price and Greeks takes less than 1 ms on a PC with 2.27 GHz CPU.

3.4. Put-call parity and negative strike

Conventionally, the European put option price forms a closed-form parity with the call counterpart, as suggested by Merton (1973). Therefore, if a closed-form formula is derived for the call, a separate derivation for the put is not necessarily required. The put-call parity for single-asset vanilla option is well known and can be found on almost any textbook on financial derivatives, but the corresponding multi-asset put-call parity is not generally mentioned. Pearson (1995) suggested a two-asset put-call parity, but we did not manage to find any multi-asset put-call parity in the existing literature. Here we provide, if it is not already obvious to the readers, a put-call parity for the basket-spread option:

$$P^{\mathrm{put}}(\mathbf{S}, t) = P^{\mathrm{call}}(\mathbf{S}, t) - \left(\sum_{i=1}^{M} S_i \mathrm{e}^{-q_i(T-t)} - \sum_{i=M+1}^{M+N} S_i \mathrm{e}^{-q_i(T-t)} \right) + K\mathrm{e}^{-r(T-t)} \tag{28}$$

The above relation is exact, as can be verified by direct substitution into the pricing PDE in equation (3). Hence, the pricing errors for the basket-spread put option would be the same as the call counterpart if one uses equation (11) to approximate the value of P^{call}. The put option Greeks can be related to the call option Greeks by differentiating equation (28) directly.

Without explicitly stated, all the pricing and hedging formulae presented so far are primarily designed for basket-spread option with non-negative strike price. When $K < 0$, the effective strike price defined in (12b) may become negative, which breaches the admissible domain of the Black–Scholes formula. Alternatively, one can re-write the pay-off in the following order:

$$P(\mathbf{F}, T) = \left(\sum_{i=1}^{M} F_i(T) - K - \sum_{i=M+1}^{M+N} F_i(T) \right)^{+} \tag{29}$$

and re-define the set of effective input parameters as:

$$\tilde{F} = -K + \sum_{i=1}^{M} F_i \tag{30a}$$

$$\tilde{K} = \sum_{i=M+1}^{M+N} F_i \tag{30b}$$

$$\tilde{\sigma} = \sqrt{\sum_{i=1}^{M+N} \sum_{j=1}^{M+N} \rho_{ij}\sigma_i\sigma_j m_i m_j} \tag{30c}$$

$$m_i = \begin{cases} F_i/\tilde{F}, & i = 1, 2...M \\ -F_i/\tilde{K}, & i = M+1, M+2...M+N \end{cases} \tag{30d}$$

when $K < 0$. Then, following a similar derivation to the one in appendix 3, one obtains the same form of approximate pricing formula:

$$P_{\mathrm{Mul}}(\mathbf{F}, t) = P_{\mathrm{BS}}\left(\tilde{F}, t;\ r, r, \tilde{\sigma}, \tilde{K}\right)$$

except that the set of input parameters is given by (30) instead of (12). It is clear that the two sets of input parameters are identical when $K = 0$. Hence, equation (11) together with the two sets of input parameters in (12) and (30) provides a set of continuous approximation for the basket-spread call option with any strike price.

4. The alpha method

In this section, we propose a simple and efficient procedure which we call the 'alpha method', to improve the approximate basket-spread option price and Greeks. The method is inspired by the recognition that the pricing PDE in equation (3) can be expressed in terms of the Greeks:

$$\Theta + \frac{1}{2} \sum_{i=1}^{M+N} \sum_{j=1}^{M+N} \rho_{ij}\sigma_i\sigma_j S_i S_j \Gamma_{ij} + \sum_{i=1}^{M+N} (r-q_i) S_i \Delta_i - rP = 0 \tag{31}$$

which also represents the dynamic delta–gamma hedging relation. Under the assumption of constant $\tilde{\sigma}$, all the derivatives of $\tilde{\sigma}$ within the Greeks' formulae in (13)–(16) vanish, the relation in (31) would hold at $t = 0$.† However, when such assumption is relaxed (i.e. when $\tilde{\sigma}$ is treated as the function defined in 12c), the full expressions in (13)–(16) should be used and the relation in (31) no longer holds. To overcome such incompleteness, we propose the following anticipated form:

$$P^{\alpha}_{\text{Mul}}(\mathbf{S}, t) = P_{\text{BS}}\left(\tilde{F} + \alpha(T-t), t;\ r, r, \tilde{\sigma}, \tilde{K} + \alpha(T-t)\right) \tag{32}$$

where $\alpha(T-t)$ can be viewed as an implicit correction being imposed on the effective underlying price and effective strike price. As a leading-order approximation, α is assumed to be a constant, such that the Δ_i's and Γ_{ij}'s formulae in (13) and (14) remain valid, except that the set of input parameters is modified according to (32). The formula for Θ becomes:

$$\Theta^{\alpha}(\mathbf{S}, t) = \Theta_{\text{BS}} + \nu_{\text{BS}}\frac{\partial\tilde{\sigma}}{\partial t} + \Delta_{\text{BS}}\left(\frac{\partial\tilde{F}}{\partial t} - \alpha\right) + \kappa_{\text{BS}}\left(\frac{\partial\tilde{K}}{\partial t} - \alpha\right) \tag{33}$$

because the implicit correction is assumed to be linear in time. Hence, all the pricing and hedging solutions become dependent on α, the relation in (31) can be re-written as:

$$\begin{aligned} f(\alpha|\mathbf{S}, t) = \Theta^{\alpha} &+ \frac{1}{2}\sum_{i=1}^{M+N}\sum_{j=1}^{M+N}\rho_{ij}\sigma_i\sigma_j S_i S_j \Gamma^{\alpha}_{ij} \\ &+ \sum_{i=1}^{M+N}(r - q_i)S_i\Delta^{\alpha}_i - rP^{\alpha}_{\text{Mul}} = 0 \end{aligned} \tag{34}$$

which is a non-linear equation of α. Given a set of initial underlying asset prices, it can be solved by any feasible numerical method.‡ Once the value of α is found, the improved approximate basket-spread option price and Greeks are simultaneously available. The degree of freedom carried by the correction parameter α allows the dynamic delta–gamma hedging relation to hold at $t = 0$, even when $\tilde{\sigma}$ is treated as a function.

In principle, the optimal correction is not really constant over the entire hyperspace of $\mathbf{S}$. If α is considered as a spatial dependent function, the formulae for Δ_i's and Γ_{ij}'s would incur additional terms involving the derivatives of α. But it turns out that the optimal correction values are fairly uniform across the space, and therefore the derivatives of α are negligible compared with other terms within the Greeks' formulae. The non-linear equation in (34) can be considered as a leading-order approximation to the full pricing PDE, with terms involving the derivatives of α being neglected. The assumption of constant α simplifies the entire correction scheme and allows the improved basket-spread option price and Greeks to be computed very efficiently. Although the closed-form nature is sacrificed in the 'alpha method', the overall accuracy of the approximation can be improved substantially, as demonstrated in the next section.

†This can be verified by direct substitutions.

‡In our calculation, we use the Secant method which could find the root of α within the precision of 10^{-10} in just a few iterative steps. In principle, any root-finding method that does not require the derivative of f is feasible.

5. Numerical examples

In this section, we present a few numerical examples to illustrate the accuracy and robustness of the proposed approximation methods. The benchmark 'exact' results are obtained by numerical integration (to be denoted by NI). The multi-dimensional expectation integral of the basket-spread option price is evaluated numerically using the Gaussian quadratures. The 'exact' Greeks are evaluated similarly with the expectation integrals of their pathwise derivatives, likelihood ratios or mixtures of both estimators (see Glasserman 2003). When numerical integration is not feasible, Monte Carlo simulation (to be denoted by MC) with 10 million paths is used to evaluate the expectations.§ Having the benchmark results on hand, we compare the proposed closed-form approximation and the 'alpha method' (to be denoted by CF and Alpha respectively) with various existing approximation methods.¶

Example 1: two-asset spread option

The two-asset spread option corresponds to $M = N = 1$ in equation (1), in which case the basket-spread option formula in equation (11) is equivalent to Kirk's formula in equation (9). In this example, we compare the methods of CF and Alpha with two other existing methods, namely the semi-analytic method of Carmona and Durrleman (2006) and the second-order boundary approach of Li *et al.* (2010) (to be denoted by CD and LiSB, respectively). The latter is known as the most accurate approximation method for pricing spread option; while the former is well known for its full set of associated Greeks' formulae.‖

The numerical results presented in table 1 illustrate that the implicit correction represented by the 'alpha method' is able to reduce the pricing errors of the closed-form formula (Kirk's formula) by a significant amount. The overall accuracy of Alpha is comparable with that of LiSB, while the overall accuracy of CF is comparable with that of CD. However, the accuracies of both CF and Alpha deteriorate as the correlation becomes negative. This is the region where the assumption behind Kirk's formula does not hold so well. A negative correlation imposes downward pressure on $F_2(t)$, causing the condition of $K \ll F_2(t)$ to deteriorate from its initial state. The Greeks calculated by all four methods are very accurate except that LiSB does not support analytic solutions for some of the Greeks. Besides its excellent accuracy, the 'alpha method' is simple and efficient as it only involves solving a single non-linear equation. The computing time is almost instant for generating a full set of two-asset spread option price and Greeks on a PC with 2.27 GHz CPU.

§To achieve convergence in the order of 10^{-6}, the NI approach is found to be feasible for four or lower dimensional problems. The NI and MC methods are equivalent when estimating the same expectation under the same probability measure.

¶The Greeks of the CF method are evaluated under the assumption of constant $\tilde{\sigma}$, i.e. all derivatives of $\tilde{\sigma}$ are neglected.

‖Li *et al.* (2010) only presented analytic formulae for the Delta's and Kappa. In view of the complexity in their pricing formula, they suggested to approximate other Greeks with finite-difference estimators.

Table 1. The two-asset spread option prices and Greeks computed by the 'exact' NI method and various approximation methods. Values in parentheses are the absolute deviations from the 'exact' results. Input parameters: $T = 1$, $K = 20$, $\rho_{12} = 0.5$ (when not varied), $r = 0.05$, $\mathbf{S} = \{110, 90\}$, $\mathbf{q} = \{0.03, 0.02\}$ and $\boldsymbol{\sigma} = \{0.3, 0.2\}$ (fixed).

		NI	CF		CD		LiSB		Alpha	
T	0.5	7.8284	7.8279	(0.0005)	7.8279	(0.0005)	7.8284	(0.0000)	7.8284	(0.0000)
	1.0	10.8323	10.8310	(0.0013)	10.8309	(0.0014)	10.8323	(0.0000)	10.8323	(0.0000)
	1.5	12.9994	12.9971	(0.0023)	12.9970	(0.0024)	12.9993	(0.0001)	12.9994	(0.0000)
	2.0	14.7186	14.7152	(0.0034)	14.7150	(0.0036)	14.7185	(0.0001)	14.7186	(0.0000)
K	5	18.7282	18.7278	(0.0004)	18.7281	(0.0001)	18.7282	(0.0000)	18.7282	(0.0000)
	10	15.7457	15.7451	(0.0006)	15.7453	(0.0004)	15.7457	(0.0000)	15.7457	(0.0000)
	20	10.8323	10.8310	(0.0013)	10.8309	(0.0014)	10.8323	(0.0000)	10.8323	(0.0000)
	40	4.7033	4.7009	(0.0023)	4.7008	(0.0025)	4.7031	(0.0001)	4.7032	(0.0001)
	80	0.7336	0.7354	(0.0018)	0.7325	(0.0011)	0.7335	(0.0001)	0.7338	(0.0002)
ρ_{12}	0.7	9.0862	9.0855	(0.0008)	9.0847	(0.0015)	9.0862	(0.0000)	9.0863	(0.0001)
	0.5	10.8323	10.8310	(0.0013)	10.8309	(0.0014)	10.8323	(0.0000)	10.8323	(0.0000)
	0.3	12.3319	12.3320	(0.0001)	12.3308	(0.0011)	12.3319	(0.0000)	12.3319	(0.0000)
	0.0	14.2844	14.2884	(0.0040)	14.2838	(0.0006)	14.2844	(0.0000)	14.2839	(0.0005)
	−0.3	15.9939	16.0030	(0.0091)	15.9936	(0.0003)	15.9939	(0.0000)	15.9925	(0.0014)
	−0.5	17.0348	17.0477	(0.0129)	17.0346	(0.0002)	17.0348	(0.0000)	17.0324	(0.0024)
	−0.7	18.0126	18.0297	(0.0171)	18.0125	(0.0001)	18.0126	(0.0000)	18.0090	(0.0036)
Greeks	Δ_1	0.5288	0.5286	(0.0001)	0.5287	(0.0001)	0.5286	(0.0002)	0.5286	(0.0001)
	Δ_2	−0.4319	−0.4317	(0.0001)	−0.4318	(0.0001)	−0.4317	(0.0002)	−0.4317	(0.0001)
	κ	−0.4230	−0.4232	(0.0001)	−0.4232	(0.0001)	−0.4231	(0.0000)	−0.4232	(0.0001)
	Γ_{11}	0.0134	0.0134	(0.0000)	0.0134	(0.0000)	N/A		0.0134	(0.0000)
	Γ_{12}	−0.0135	−0.0135	(0.0000)	−0.0135	(0.0000)	N/A		−0.0135	(0.0000)
	Γ_{22}	0.0136	0.0136	(0.0000)	0.0136	(0.0000)	N/A		0.0136	(0.0000)
	Θ	−4.9633	−4.9617	(0.0016)	−4.9616	(0.0017)	N/A		−4.9636	(0.0003)
	ν_1	0.3541	0.3541	(0.0000)	0.3408	(0.0133)	N/A		0.3542	(0.0000)
	ν_2	0.0195	0.0194	(0.0001)	0.0198	(0.0003)	N/A		0.0194	(0.0001)
	χ_{12}	−8.0500	−8.0600	(0.0100)	−8.0500	(0.0000)	N/A		−8.0500	(0.0000)

All ν's are calculated with respect to 1% change in the volatility.

Example 2: three-asset basket option

The three-asset basket option corresponds to $M = 3$, $N = 0$ in equation (1). Not all existing methods are applicable for pricing basket options. Some primitive approximation methods are simply not accurate. In this example, we compare the methods of CF and Alpha with two moment-matching methods found in the literature, namely the analytic method of Milevsky and Posner (1998b) and the semi-analytic method of Borovkova *et al.* (2007) (to be denoted by RG2 and SLN3, respectively). The former matches the basket distribution with a reciprocal gamma distribution up to the second moment; the latter matches the basket distribution with a shifted lognormal distribution up to the third moment, but it requires the numerical solution of a set of non-linear equations, and thus is semi-analytic.

The numerical results presented in table 2 illustrate that two of the approximation methods, CF and SLN3, demonstrate comparable accuracies, despite that one is analytic while another one is semi-analytic. The method of RG2 appears to be the least accurate amongst the four methods, with percentage errors going up to 100% in the deeply out-of-the-money region. The Alpha method produces the most accurate prices, with errors being consistently and significantly smaller than the errors of other three methods. When compared against the next accurate method, SLN3, the errors are almost 10 times smaller in the region where SLN3 does not perform so well, i.e. high volatility and long time-to-maturity region. The Greeks are not compared in this example because no corresponding formulae exist for the two moment-matching methods.

Example 3: four-asset spread option

The four-asset spread option here corresponds to $M = 1$, $N = 3$ in equation (1). This is a special kind of multi-asset spread option commonly seen in the commodity markets. The spread is between one main asset and a basket of sub-assets. Since the method of LiSB is specifically designed for this kind of spread options, we compare the two proposed methods, CF and Alpha, with LiSB in this example.

The numerical results presented in table 3 illustrate that the methods of LiSB and Alpha exhibit comparable accuracies. While the former is more accurate when volatilities are high ($\sigma_i = 0.5$), the latter is consistently more accurate when T or K is large. Although CF is less accurate than the other two methods, it is still practically useful because it is a closed-form approximation and the errors are in general below 1%. It should also be noted that the implementation of Alpha is a lot simpler than LiSB. For the Greeks, similar to the observation made in example 1, all approximations are very accurate with absolute errors staying in the order of 10^{-4}, except that LiSB does not support analytic solutions for some of the higher order Greeks.

Example 4: one-hundred-asset basket-spread option

The 100 asset basket-spread option here refers to the case of $M = N = 50$ in equation (1). Most existing numerical and approximation methods are either inaccurate or inapplicable for this high-dimensional problem. We therefore compare CF and Alpha with only the MC simulation of 10 million paths.

Table 2. The three-asset basket option prices computed by the 'exact' NI method and various approximation methods. Input parameters: $T = 1$, $K = 150$, $\sigma_{1/2/3} = 0.3$ (when not varied), $r = 0.05$, $\mathbf{S} = \{60, 50, 40\}$, $q_{1/2/3} = 0.03$, $\rho_{12} = 0.7$, $\rho_{13} = 0.5$ and $\rho_{23} = 0.3$ (fixed).

		NI	CF		RG2		SLN3		Alpha	
T	0.5	11.0731	11.0661	(0.0070)	10.9681	(0.1050)	11.0774	(0.0043)	11.0733	(0.0002)
	1.0	15.7863	15.7670	(0.0193)	15.4986	(0.2877)	15.7982	(0.0119)	15.7869	(0.0006)
	1.5	19.3572	19.3232	(0.0340)	18.8456	(0.5116)	19.3790	(0.0218)	19.3591	(0.0019)
	2.0	22.3003	22.2490	(0.0513)	21.5370	(0.7633)	22.3328	(0.0325)	22.3031	(0.0028)
K	50	98.0054	98.0054	(0.0000)	98.0054	(0.0000)	98.0054	(0.0000)	98.0054	(0.0000)
	100	50.9624	50.9654	(0.0030)	50.7645	(0.1979)	50.9714	(0.0090)	50.9618	(0.0006)
	150	15.7863	15.7670	(0.0193)	15.4986	(0.2877)	15.7982	(0.0119)	15.7869	(0.0006)
	200	3.0026	2.9715	(0.0311)	3.2327	(0.2301)	2.9918	(0.0107)	3.0009	(0.0017)
	250	0.4357	0.4222	(0.0135)	0.6122	(0.1766)	0.4280	(0.0076)	0.4340	(0.0016)
	300	0.0563	0.0528	(0.0035)	0.1205	(0.0641)	0.0540	(0.0023)	0.0557	(0.0006)
σ_i	0.1	6.3500	6.3499	(0.0001)	6.3263	(0.0237)	6.3510	(0.0010)	6.3500	(0.0000)
	0.2	11.0488	11.0438	(0.0050)	10.9463	(0.1025)	11.0530	(0.0042)	11.0489	(0.0001)
	0.3	15.7863	15.7670	(0.0193)	15.4986	(0.2877)	15.7982	(0.0119)	15.7869	(0.0006)
	0.4	20.5250	20.4772	(0.0478)	19.8972	(0.6278)	20.5507	(0.0257)	20.5264	(0.0014)
	0.5	25.2526	25.1596	(0.0930)	24.0896	(1.1630)	25.3020	(0.0494)	25.2572	(0.0046)

Table 3. The four-asset spread option ($M = 1$, $N = 3$) prices and Greeks computed by the 'exact' NI method and various approximation methods. Input parameters: $T = 1$, $K = 40$, $\sigma_{1/2/3/4} = 0.3$ (when not varied), $r = 0.05$, $\mathbf{S} = \{160, 40, 40, 40\}$, $q_{1/2/3/4} = 0.03$, $\rho_{12/13/14} = 0.3$ and $\rho_{23/24/34} = 0.5$ (fixed).

		NI	CF		LiSB		Alpha	
T	0.5	13.0010	12.9930	(0.0080)	12.9990	(0.0020)	12.9991	(0.0019)
	1.0	18.1868	18.1682	(0.0186)	18.1846	(0.0022)	18.1851	(0.0017)
	1.5	21.9996	21.9683	(0.0313)	21.9975	(0.0021)	21.9986	(0.0010)
	2.0	25.0701	25.0248	(0.0453)	25.0683	(0.0018)	25.0705	(0.0004)
K	0	42.7728	42.7587	(0.0141)	42.7721	(0.0007)	42.7716	(0.0012)
	20	28.8179	28.7942	(0.0237)	28.8177	(0.0002)	28.8179	(0.0000)
	40	18.1868	18.1682	(0.0186)	18.1846	(0.0022)	18.1851	(0.0017)
	80	6.2063	6.1950	(0.0113)	6.2045	(0.0018)	6.2052	(0.0011)
	160	0.5306	0.5276	(0.0030)	0.5302	(0.0004)	0.5304	(0.0002)
σ_i	0.1	6.3383	6.3376	(0.0007)	6.3384	(0.0001)	6.3384	(0.0001)
	0.2	12.2704	12.2653	(0.0051)	12.2704	(0.0000)	12.2705	(0.0001)
	0.3	18.1868	18.1682	(0.0186)	18.1846	(0.0022)	18.1851	(0.0017)
	0.4	24.0694	24.0309	(0.0385)	24.0684	(0.0010)	24.0702	(0.0008)
	0.5	29.9123	29.8397	(0.0726)	29.9108	(0.0015)	29.9157	(0.0034)
Greeks	Δ_1	0.5480	0.5475	(0.0005)	0.5479	(0.0001)	0.5475	(0.0005)
	$\Delta_{2/3/4}$	−0.4306	−0.4361	(0.0055)	−0.4305	(0.0001)	−0.4301	(0.0004)
	Γ_{11}	0.008266	0.008271	(0.000005)	N/A		0.008263	(0.000003)
	$\Gamma_{12/13/14}$	−0.008288	−0.008312	(0.000024)	N/A		−0.008288	(0.000000)
	$\Gamma_{22/33/44}$	0.008574	0.008353	(0.000220)	N/A		0.008591	(0.000018)
	$\Gamma_{23/24/34}$	0.008211	0.008353	(0.000143)	N/A		0.008215	(0.000005)
	Θ	−8.6633	−8.6246	(0.0387)	N/A		−8.6634	(0.0000)

The last two columns in table 4 illustrate that the absolute differences between the two sets of approximate results and the MC results are in general smaller than the MC standard errors. This implies that both the closed-form approximation and 'alpha method' are adequate candidates to replace the brute-force MC simulation method. For the Greeks, we report the average absolute values to avoid listing all of them out. All the average deviations from the MC estimates are negligible, further verifying that the two proposed approximation methods are accurate in hedging as well as pricing. As expected, the Alpha method always generates closer results to the MC estimates than the CF method.

As mentioned earlier, there is a growing demand of pricing a sizeable quantity of basket-spread options, each with a sizeable portfolio of underlying assets. The computing time is a serious challenge to accomplish such task. In table 5, we report the total and average computing time of using the two proposed methods to generate 100 sets of basket-spread option price and Greeks on a PC with 2.27 GHz CPU.† The number of underlying assets varies from 4 to 128, with equal size in each of the sub-baskets (i.e. $M = N$). The computing time of the

†For one set of basket-spread option price and Greeks, we refer to the set of formulae listed in section 3.

Table 4. The 100-asset basket-spread option ($M = N = 50$) prices and Greeks computed by MC simulation and the two proposed approximation methods. Input parameters: $T = 1$, $K = 50$ (when not varied), $r = 0.05$, $S_{i\leq M} = 10$, $q_{i\leq M} = 0.05$, $\sigma_{i\leq M} = 0.5$, $S_{i>M} = 9$, $q_{i>M} = 0.03$, $\sigma_{i>M} = 0.3$, $\rho_{i\leq M\cap j\leq M} = 0.6$, $\rho_{i\leq M\cap j>M} = 0.4$ *and* $\rho_{i>M\cap j>M} = 0.5$ (fixed).

		MC	CF	Alpha	Deviations from MC		
					MC Std Err	CF	Alpha
T	0.25	27.0070	27.0005	27.0048	0.0328	0.0065	0.0022
	0.50	37.1500	37.1313	37.1433	0.0475	0.0187	0.0067
	0.75	44.4386	44.3902	44.4119	0.0591	0.0484	0.0267
	1.00	50.1510	50.1311	50.1641			
K	0	72.4708	72.4170	72.4545	0.0837	0.0538	0.0163
	50	50.1858	50.1311	50.1641	0.0692	0.0547	0.0217
	100	34.2689	34.2886	34.2880	0.0571	0.0197	0.0191
	150	23.3060	23.3390	23.2897	0.0472	0.0330	0.0163
Greeks - average abs values of n estimates					n		
	Δ	0.4574	0.4559	0.4571	100	0.0015	0.0003
	Γ	0.002673	0.002662	0.002666	5050	0.000011	0.000007
	Θ	20.8560	20.7732	20.7989	1	0.0828	0.0571
	κ	0.3766	0.3762	0.3762	1	0.0004	0.0004
	ν	0.0167	0.0167	0.0167	100	0.0000	0.0000

Table 5. Total and average computing time of generating 100 sets of basket-spread option price and Greeks on a PC with 2.27 GHz CPU. The number of underlying assets varies from 4 to 128; the input parameters are kept in the same structure as in table 4.

	Total time for 100 sets (sec.)		Average time per set (sec.)	
No. of assets (M+N)	CF	Alpha	CF	Alpha
128	0.81	6.43	0.0081	0.0643
64	0.20	2.11	0.0020	0.0211
32	0.05	0.73	0.0005	0.0073
16	0.00	0.24	0.0000	0.0024
8	0.00	0.06	0.0000	0.0006
4	0.00	0.00	0.0000	0.0000

closed-form approximations are almost instant for all cases. The Alpha method is semi-analytic and thus requires longer computing time. On average, the Alpha method takes less than one millisecond to compute one set of price and Greeks with 10 underlying assets or less. For the extreme case of 128 underlying assets, it takes around 0.06 seconds.

Example 5: Option-implied correlation of iTraxx indexes

In this example, we intend to illustrate a possible application of the proposed basket-spread option formula on real market data. In a previous study, Hui and the authors of this paper (see Hui *et al.* 2013) applied a similar multi-asset option pricing technique to extract option-implied correlations from the iTraxx Europe Main index. The Main index consists of two sub-indexes, namely the iTraxx Financials and Non-Financials indexes. Each of the sub-indexes consists of underlying entities' CDS spreads within the financial or non-financial sector. Therefore, option contingent on the Main index can be considered as a basket-spread option. The closed-form formula in equation (11) can be used if the underlying CDS spreads are assumed to be governed by the correlated lognormal processes in (5). This is consistent with the assumption made by Hull and White (2003), in which they modelled the corporate CDS spreads with lognormal processes.

Since the iTraxx option prices are quoted daily on the market, given enough data points, the correlations amongst the underlying CDS spreads can be implied from the Main index option prices by inverting the basket-spread option formula. Following similar procedures as in Hui *et al.* (2013)†, we obtain the option-implied correlations between the iTraxx Financials and Non-Financials indexes for the period September 2010 to January 2013. And then, we adopt the dynamic conditional correlation multi-variate GARCH model (to be denoted by *DCC GARCH*) proposed by Engle and Sheppard (2001) to obtain the corresponding realized correlations over the same period. Figure 1 shows that the two correlation time series exhibit similar levels and trends over the sample period. In fact, a more extensive analysis (as done on the paper) reveals that there exists a lead–lag relationship between the two sets of correlations. Nevertheless, our emphasis here is that the closed-form basket-spread option formula provides an accurate and convenient tool for implying correlations or even the entire covariance matrix from market basket-spread option prices.

†Quoting structure of the iTraxx market data is more complicated and needs to be dealt with carefully. Details of the procedures can be found on the paper.

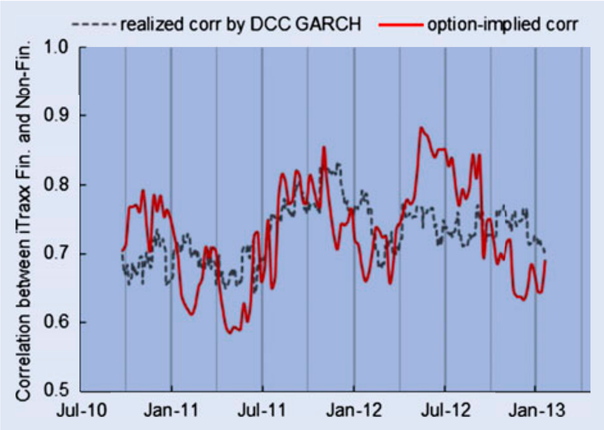

Figure 1. Correlations between the iTraxx Financials and Non-Financials indexes. The realized correlations are computed by DCC GARCH; the option-implied correlations are obtained by inverting the basket-spread option formula. Sample period: September 2010 to January 2013.

6. Conclusion

In this study, we generalize the Kirk (1995) approximate two-asset spread option formula to the case of the multi-asset basket-spread option, i.e. the spread in the option's pay-off is between two baskets of aggregated underlying asset prices. Since the final formula retains the same functional form as the Black–Scholes formula, all the basket-spread option Greeks are also derived in closed form. Numerical examples demonstrate that the pricing and hedging errors are in general less than 1% relative to the benchmark results obtained by numerical integration or Monte Carlo simulation with 10 million paths. An implicit correction method is further applied to improve the overall approximation. It involves a single unknown parameter, whose optimal value is found by solving a single non-linear equation. Owing to its simplicity, the computing time is kept below 1 ms on a PC with 2.27 GHz CPU for simultaneous pricing and hedging of basket-spread option with 10 underlying assets or less. The pricing errors are significantly reduced by factors of up to 100. When compared with the existing approximation methods, the basket-spread option formula coupled with the implicit correction turns out to be one of the most accurate and robust methods.

The approximation methods proposed in this paper can be useful for a wide scope of applications. For instance, the correction method can be used to fine-tune approximate formulae of other types of exotic options. The closed-form basket-spread option formula can be inverted to extract market implied correlations and/or volatilities. It also has the potential to be applicable for pricing interest rate derivatives under the multi-factor LIBOR market model (LMM). But before it can be applied on the LMM, time-varying parameters need to be incorporated into the pricing formula. We leave these to future research.

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Appendix 1. Black–Scholes formula and the Greeks

The Black–Scholes model assumes that the underlying asset price is driven by a lognormal process:

$$\frac{dS}{S} = (r-q)dt + \sigma dW \tag{A1}$$

where r is the risk-free interest rate, q is the dividend yield and σ is the volatility. The corresponding risk-neutral price of a vanilla call option, with strike price K and maturity time $t = T$, is given by:

$$P_{\mathrm{BS}}(S, t;\ r, q, \sigma, K) = \mathrm{e}^{-q(T-t)} S \Phi(d_1) - \mathrm{e}^{-r(T-t)} K \Phi(d_2) \tag{A2}$$

where
$$d_1 = \frac{\ln(S/K) + (r - q + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \tag{A3a}$$
$$d_2 = d_1 - \sigma\sqrt{T-t}, \tag{A3b}$$

and $\Phi(\cdot)$ is the cumulative normal distribution function. Equation (A2) is the Black–Scholes formula. It can be differentiated analytically to yield the various Black–Scholes Greeks:

$$\Delta_{\mathrm{BS}} = \frac{\partial P_{\mathrm{BS}}}{\partial S} = \mathrm{e}^{-q(T-t)}\Phi(d_1) \tag{A4}$$
$$\kappa_{\mathrm{BS}} = \frac{\partial P_{\mathrm{BS}}}{\partial K} = -\mathrm{e}^{-r(T-t)}\Phi(d_2) \tag{A5}$$
$$\nu_{\mathrm{BS}} = \frac{\partial P_{\mathrm{BS}}}{\partial \sigma} = S\mathrm{e}^{-q(T-t)}\phi(d_1)\sqrt{T-t} \tag{A6}$$
$$\Theta_{\mathrm{BS}} = \frac{\partial P_{\mathrm{BS}}}{\partial t} = S\mathrm{e}^{-q(T-t)}\left[q\Phi(d_1) - \frac{\sigma}{2\sqrt{T-t}}\phi(d_1)\right] - rK\mathrm{e}^{-r(T-t)}\Phi(d_2) \tag{A7}$$

where $\phi(\cdot)$ is the probability density function of standard normal distribution. The second derivatives are given by:

$$\Gamma^{\mathrm{ss}}_{\mathrm{BS}} = \frac{\partial^2 P_{\mathrm{BS}}}{\partial S^2} = \frac{\mathrm{e}^{-q(T-t)}\phi(d_1)}{S\sigma\sqrt{T-t}} \tag{A8}$$
$$\Gamma^{\mathrm{kk}}_{\mathrm{BS}} = \frac{\partial^2 P_{\mathrm{BS}}}{\partial K^2} = \frac{\mathrm{e}^{-r(T-t)}\phi(d_2)}{K\sigma\sqrt{T-t}} \tag{A9}$$
$$\Gamma^{\mathrm{sk}}_{\mathrm{BS}} = \frac{\partial^2 P_{\mathrm{BS}}}{\partial S\partial K} = -\frac{\mathrm{e}^{-q(T-t)}\phi(d_1)}{K\sigma\sqrt{T-t}} \tag{A10}$$
$$\Gamma^{\mathrm{sv}}_{\mathrm{BS}} = \frac{\partial^2 P_{\mathrm{BS}}}{\partial S\partial \sigma} = -\mathrm{e}^{-q(T-t)}\phi(d_1)\frac{d_2}{\sigma} \tag{A11}$$
$$\Gamma^{\mathrm{kv}}_{\mathrm{BS}} = \frac{\partial^2 P_{\mathrm{BS}}}{\partial K\partial \sigma} = \mathrm{e}^{-r(T-t)}\phi(d_2)\frac{d_1}{\sigma} \tag{A12}$$
$$\Gamma^{\mathrm{vv}}_{\mathrm{BS}} = \frac{\partial^2 P_{\mathrm{BS}}}{\partial \sigma^2} = \frac{d_1 d_2}{\sigma}\nu_{\mathrm{BS}} \tag{A13}$$

For a given set of input parameters, $d_{1/2}$, $\phi(d_{1/2})$ and $\Phi(d_{1/2})$ are often pre-computed before the option price and Greeks are calculated.

Appendix 2. Derivation of Kirk's formula

Here, we provide a simple derivation of the Kirk (1995) two-asset spread option formula and identify the condition for this approximate formula to be accurate. To begin with, we express the two-asset spread option price as a conditional expectation under the risk-neutral measure at $t = 0$:

$$P(F_1(0), F_2(0), 0) = \mathbb{E}_Q\left[\mathrm{e}^{-rT}\left(F_1(T) - F_2(T) - K\right)^+ |\mathcal{F}_0\right] \tag{B1}$$

Defining:

$$\tilde{K}(t) = F_2(t) + K \tag{B2}$$

such that, by Ito's lemma, its dynamics is given by:

$$\frac{d\tilde{K}(t)}{\tilde{K}(t)} = \sigma_2 m(t) dW_2(t) \tag{B3}$$

where

$$m(t) = \frac{F_2(t)}{F_2(t) + K} \tag{B4}$$

For $K \ll F_2(t)$, the scaling function $m(t)$ is a slowly varying function of $F_2(t)$, i.e. $\frac{F_2(t)}{m(t)}\left|\frac{\partial m(t)}{\partial F_2(t)}\right| \ll 1$, then one can approximate $m(t)$ with its initial value at $t = 0$. The assumption of $m(t)$ being time invariant is indeed the key assumption behind Kirk's approximation. It has been generally acknowledged in the literature that the accuracy of Kirk's formula tends to deteriorate as K increases. If $m(t)$ is approximated by $m(0)$, the dynamics of $\tilde{K}(t)$ becomes (approximately) a zero-drift lognormal process with volatility $\sigma_2 m(0)$:

$$\frac{d\tilde{K}(t)}{\tilde{K}(t)} \simeq \sigma_2 m(0) dW_2(t) \tag{B5}$$

The risk-neutral expectation in equation (B1) can be re-written as:

$$P(F_1(0), \tilde{K}(0), 0) = \mathbb{E}_Q\left[\mathrm{e}^{-rT}\left(F_1(T) - \tilde{K}(T)\right)^+ \middle| \mathcal{F}_0\right] \tag{B6}$$

which can be viewed as the valuation of an exchange option with $F_1(t)$ and $\tilde{K}(t)$ being the two lognormal underlying assets. Hence, upon substituting all the relevant parameters into Margrabe's exchange option formula, the approximate spread option price is given by:†

$$P(F_1(0), F_2(0), 0) \simeq \mathrm{e}^{-rT}\left[F_1(0)N(\tilde{d}_1) - (F_2(0) + K)N(\tilde{d}_2)\right] \tag{B7}$$

where

$$\tilde{d}_1 = \frac{\ln\left[F_1(0)/(F_2(0)+K)\right] + \tilde{\sigma}^2 T/2}{\tilde{\sigma}\sqrt{T}} \tag{B8a}$$
$$\tilde{d}_2 = \tilde{d}_1 - \tilde{\sigma}\sqrt{T} \tag{B8b}$$
$$\tilde{\sigma} = \sqrt{\sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2 m(0) + \sigma_2^2 m(0)^2} \tag{B8c}$$

which is exactly Kirk's formula given in equation (9). The assumption of time invariant $m(t)$ is legitimate if $K \ll F_2(t)$ holds, and thus $K \ll F_2(0)$ and relatively small σ_2 are considered as the initial requirements for Kirk's formula to be accurate. When ρ_{12} is negative, there could be a downward pressure on $F_2(t)$, causing the condition $K \ll F_2(t)$ to deteriorate over time.

Appendix 3. Derivation of the basket-spread option formula

Here, we generalize the assumption behind Kirk's approximation, and derive an approximate pricing formula for the multi-asset basket-spread option. We start by expressing the basket-spread option price as a conditional expectation under the risk-neutral measure at $t = 0$:

$$P(\mathbf{F}(0), 0) = \mathbb{E}_Q\left[\mathrm{e}^{-rT}\left(\sum_{i=1}^{M} F_i(T) - \sum_{i=M+1}^{M+N} F_i(T) - K\right)^+ \middle| \mathcal{F}_0\right] \tag{C1}$$

Defining:

$$\tilde{F}(t) = \sum_{i=1}^{M} F_i(t) \tag{C2a}$$
$$\tilde{K}(t) = K + \sum_{i=M+1}^{M+N} F_i(t) \tag{C2b}$$

such that their dynamics, by Ito's lemma and equation (5), are given by:

$$\frac{d\tilde{F}(t)}{\tilde{F}(t)} = \sum_{i=1}^{M} \sigma_i m_i(t) dW_i(t) \tag{C3a}$$
$$\frac{d\tilde{K}(t)}{\tilde{K}(t)} = \sum_{i=M+1}^{M+N} -\sigma_i m_i(t) dW_i(t) \tag{C3b}$$

†Interested readers may consult the book of Kwok (2008) section 3.4.6 for a clear and concise derivation of the exchange option formula by the change of numéraire technique.

where

$$m_i(t) = \begin{cases} F_i(t)/\tilde{F}(t), & i = 1, 2...M \\ -F_i(t)/\tilde{K}(t), & i = M+1, M+2...M+N \end{cases} \tag{C4}$$

Squaring the dynamics in (C3), we observe:

$$\left(\frac{d\tilde{F}(t)}{\tilde{F}(t)}\right)^2 = \tilde{\sigma}_F(t)^2 dt$$

$$\left(\frac{d\tilde{K}(t)}{\tilde{K}(t)}\right)^2 = \tilde{\sigma}_K(t)^2 dt$$

$$\left(\frac{d\tilde{F}(t)}{\tilde{F}(t)}\right)\left(\frac{d\tilde{K}(t)}{\tilde{K}(t)}\right) = -\tilde{\sigma}_X(t)^2 dt$$

where

$$\tilde{\sigma}_F(t)^2 = \sum_{i=1}^{M}\sum_{j=1}^{M} \rho_{ij}\sigma_i\sigma_j m_i(t) m_j(t) \tag{C5a}$$

$$\tilde{\sigma}_K(t)^2 = \sum_{i=M+1}^{M+N}\sum_{j=M+1}^{M+N} \rho_{ij}\sigma_i\sigma_j m_i(t) m_j(t) \tag{C5b}$$

$$\tilde{\sigma}_X(t)^2 = \sum_{i=1}^{M}\sum_{j=M+1}^{M+N} \rho_{ij}\sigma_i\sigma_j m_i(t) m_j(t) \tag{C5c}$$

Generalizing the assumption behind Kirk's approximation, we assume that each of the $m_i(t)$'s defined in (C4) is time invariant, so that they can be approximated by their initial values at $t = 0$. For this assumption to be reasonable, the following conditions need to hold (at least for the lifetime of the option):

$$F_i(t) \ll \sum_{i=1}^{M} F_i(t), \qquad i = 1, 2...M \tag{C6a}$$

$$F_i(t) \ll \sum_{i=M+1}^{M+N} F_i(t), \qquad i = M+1, M+2...M+N \tag{C6b}$$

$$K \ll \sum_{i=M+1}^{M+N} F_i(t) \tag{C6c}$$

so that each $m_i(t)$ is a slowly varying function in each of the $F_i(t)$'s involved. If each $m_i(t)$ is approximated by $m_i(0)$, the dynamics of $\tilde{F}(t)$ and $\tilde{K}(t)$ can be approximated by a pair of correlated lognormal processes:

$$\frac{d\tilde{F}(t)}{\tilde{F}(t)} \simeq \tilde{\sigma}_F(0) dW_F(t) \tag{C7a}$$

$$\frac{d\tilde{K}(t)}{\tilde{K}(t)} \simeq \tilde{\sigma}_K(0) dW_K(t) \tag{C7b}$$

where the two Wiener processes are correlated through:

$$\mathbb{E}[dW_F(t) dW_K(t)] = -\frac{\tilde{\sigma}_X(0)^2}{\tilde{\sigma}_F(0)\tilde{\sigma}_K(0)} dt \tag{C8}$$

The risk-neutral expectation in (C1) can be re-written as:

$$P(\tilde{F}(0), \tilde{K}(0), 0) = \mathbb{E}_Q\left[\mathrm{e}^{-rT}\left(\tilde{F}(T) - \tilde{K}(T)\right)^+ \middle| \mathcal{F}_0 \right] \tag{C9}$$

which can be viewed as the valuation of an exchange option with $\tilde{F}(t)$ and $\tilde{K}(t)$ being the two lognormal underlying assets. Hence, upon substituting all the relevant parameters into Margrabe's exchange option formula, the approximate basket-spread option price is given by:

$$P(\mathbf{F}(0), 0) \simeq \mathrm{e}^{-rT}\left[\left(\sum_{i=1}^{M} F_i(0)\right) N(\tilde{d}_1) - \left(K + \sum_{i=M+1}^{M+N} F_i(0)\right) N(\tilde{d}_2)\right] \tag{C10}$$

where

$$\tilde{d}_1 = \frac{\ln\left[\left(\sum_{i=1}^{M} F_i(0)\right) / \left(K + \sum_{i=M+1}^{M+N} F_i(0)\right)\right] + \tilde{\sigma}^2 T/2}{\tilde{\sigma}\sqrt{T}} \tag{C11a}$$

$$\tilde{d}_2 = \tilde{d}_1 - \tilde{\sigma}\sqrt{T} \tag{C11b}$$

$$\tilde{\sigma} = \sqrt{\tilde{\sigma}_F(0)^2 + 2\tilde{\sigma}_X(0)^2 + \tilde{\sigma}_K(0)^2} \tag{C11c}$$

which is exactly the basket-spread option formula given in equation (11). The above approximate formula is accurate so long as the quadratic combination of $m_i(t)$'s embedded in $\tilde{\sigma}$ is approximately time invariant. This would be the case if the conditions in (C6) hold.