**Semesterthesis**

# Implementation of multi-asset spread option pricing methods

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## Abstract

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# 1 Introduction

A hybrid European basket-spread option is a financial derivative, whose maturity is given by the difference (the so called spread) between two baskets of aggregated and weighted underlying asset prices. In mathematical terms it's pay-off is given by the formula:

$$P(S,T) = (\sum_{i=0}^{M} w_i S_i(T) - \sum_{j=M+1}^{M+N} w_j S_j(T) - K)^+, \tag{1}$$

with $(x)^+ = max(x,0)$ and where $S_i$ is the ith underlying asset price, $w_i$ its weight and $K$ is the strike price. Basket-spread options play an important role in hedging a portfolio of correlated long and short positions. Especially they are very common in commodity markets, as producers are exposed to risks arising from spreads between feedstock and end products. Basket-spread options are traded over-the-counter and on exchanges. Since there is no closed-form solution available for the fair price it is inescapable to have an accurate and fast approximation method at hand. The multi-dimensionality and hence the lack of a marginal distribution for the basket-spread makes it impossible to give an exact analytical representation for the price (even not in the simplest framework where the asset prices are driven by correlated geometric Brownian motions). Numerical approaches such as Monte Carlo simulations or PDE methods become very slow and hence inpracticble as the number of underlyings increases. Therefore I closely look at two different basket-spread option pricing methods which have been introduced by S.Deng, M.Li, and J.Zhou [1] and by G.Deelstra, A.Petkovic and M.Vanmaele [2]. According to the authors both the second-order boundary approximation method [1,Chap.3,Prop.5] and the hybrid moment matching method associated to the improved comonotonic upper bound (HybMMICUB) [2,Chap.3,Prop.5] are considered to be extremely fast and accurate. Therefore I implement both methods in MATLAB and compare their numerical performances as a function of the basket spread characteristics. As a comparison benchmark I estimate the true prices with Monte Carlo simulations also implemented in MATLAB.
The paper is organized as follows.

## 2 Approximation methods

In the next subsection 2.1 I show the assumptions on which both the second-order boundary method and the HybMMICUB method are based on. Followed by subsection 2.2 and 2.3 which both formally present the two approximation methods. These subsections will closely follow the works of S.Deng, M.Li, and J.Zhou [1] and by G.Deelstra, A.Petkovic and M.Vanmaele [2].

### 2.1 Model setup

Consider M+N assets $S_1(t), S_2(t), ...,$ and $S_{M+N}(t)$ each of it following a geometric Brownian motion

$$dS_i(t) = u_i S_i(t)dt + \sigma_i S_i(t)dW_i(t).$$

The correlations of the Brownian motions $W_i(t)$ is given by the matrix $\varrho = (\rho_{i,j})$. The payoff of the hybrid European basket-spread has already been introduced in equation 1. Conditioning on the initial asset price $S_i(0), log(S_i(T))$ is jointly normally distributed with mean $\mu_i$, variance $\sigma_i^2$ and correlation matrix $\rho_{i,j}$ for i,j = 1,2,...,M+N. Further the interest rate $r$ is constant and the price of the hybrid European basket-spread option is given by

$$V = e^{-rT}\mathbb{E}^{\mathbb{Q}}[(\sum_{i=0}^{M} w_i S_i(T) - \sum_{j=M+1}^{M+N} w_j S_j(T) - K)^+], \tag{2}$$

where $\mathbb{Q}$ is the risk-neutral measure under which discounted security prices are martingales. In the GBM setting the means and variances are defined by

$$\mu_i = E^{\mathbb{Q}}[log(S_i(T))] = log(S_i(0)) + (r - \frac{1}{2}{\sigma_k}^2)T$$

and

$${\sigma_i}^2 = Var^{\mathbb{Q}}[log(S_k(T))] = \sigma_i\sqrt{T}, \quad i = 1, 2, ..., M + N.$$

Because the weights $w_i$ can be incorporated in the asset price by taking $S_i(t)' = w_i S_i(t)$ and $\mu_i = log(w_i) + \mu_i, \sigma_i' = \sigma_i$ and $\rho_{i,j}' = \rho_{i,j}$, we will assume that all weights $w_i$ equal 1 without loss of generality.

### 2.2 Second-order Boundary Method

The second-order boundary method for two-asset spread options (SOB) has been introduced in 2006 by Deng, Li and Zhou [3] and it was extended in 2007 to the multi-asset case [1]. Beside the SOB method Deng, Li and Zhou also introduced the extended Kirk approximation in [1]. Their work is a valuable contribution because numerous methods have been existing for spread options only involving two underlyings. In [1] Deng, Li and Zhou compare their two methods with a pricing method from Carmona and Durrleman (2005) [4] which also approximates the value of a multi-asset spread option. A study* of the results shows that the SOB method is the most accurate and also the fastest. The computational edge of the SOB method lies in its closed form solution

which only involves arithmetic calculations.
The SOD method approximates the price of a spread option with payoff

$$(S_0(T) - \sum_{j=1}^{N} S_j(T) - K)^+, \tag{3}$$

thus we introduce the random variables

$$H_0(t) = \sum_{i=1}^{M} S_t(t)$$

and

$$H_k(t) = S_{k+M}(t), \quad k = 1, 2, ..., N$$

which can be plugged into preceding payoff (equation 3) and replace $S_0(T)$ and $S_j(T)$. The idea is to approximate the distribution of $H_0(T)$ by the geometric averages of the corresponding $S_i$'s to extend the SOD method to price hybrid basked-spread options. Notice that $log(H_0(T))$ is not normally distributed nor are the $log(H_i)$'s jointly normally distributed. This comes from the fact that the sum of lognormal distributed random variables is no longer lognormal distributed.
The mean and the variance of the newly introduced random variable $H_0(T)$ is approximated by

$$\mu_0^H = log(\sum_{i=0}^{M} e^{\mu_i + 1/2{\sigma_i}^2}) - 1/2{\sigma_0^H}^2$$

and

$$\sigma_0^H = \frac{1}{M}\sqrt{\sum_{i=1}^{M}\sum_{j=1}^{M} \rho_{i,j}\sigma_i\sigma_j}$$

Next we define random variables $X$ and $Y_k$ by

$$X = \frac{log(H_0(T)) - \mu_0^H}{\sigma_0^H}, Y_k = \frac{log(H_k(T)) - \mu_k^H)}{\sigma_k^H}, \quad k = 1, 2, ..., N$$

whereas $\mu_k^H = \mu_k + M$ and $\sigma_k^H = \sigma_k + M$ for $k = 1, 2, ..., N$. Following that we can approximate the variables $X$ and $Y_k$ as jointly normally distributed with mean vector 0, variance vector 1 and correlation matrix $\Sigma = (q_{i,j})$ for $i, j = 0, 1, ..., N$, where

$$q_{0,0} = 1,$$

$$q_{0,k} = q_{k,0} = 1,$$

$$q_{i,j} = \rho_{M+i,M+j}.$$

Rewriting the correlation matrix $\Sigma$ as a composition of a Nx1 vector $\Sigma_{10}$ and $\Sigma_{11}$ the NxN correlation matrix of the $Y_k$'s simplifies notation for later computations:

$$\Sigma = (q_{i,j}) = \begin{pmatrix} 1 & \Sigma_{10}' \\ \Sigma_{10} & \Sigma_{11} \end{pmatrix}.$$

Before introducing the methods for valuing hybrid basket-spread option a short analysis of the exercise boundary is necessary. At time T, the hybrid basket-spread option is in-the-money if $H_0(T) - \sum_{k=1}^{N} S_k(T) - K > 0$. If $K > 0$, rewriting this inequality in terms of the random variable $X$ gives

$$X > \frac{log(\sum e^{\sigma_k Y_k - \mu_k + K} - \mu_0^H}{\sigma_0^H}$$

Conditioned on $Y_k = y_k$, the option is in-the-money if $X > x(y)$, where

$$x(y) \equiv \frac{log(\sum e^{\sigma_k y_k - \mu_k + K} - \mu_0^H)}{\sigma_0^H}.$$

Note that the exercise boundary $x(y)$ is a nonlinear function in the components of y but close to being linear around $y = 0$. The analytical computation of the expectation value given in equation 2 is shown below:

$$V = e^{-rT} \int_{\mathbb{R}^N} \int_{\mathbb{R}} \big(e^{\sigma_0^H x + \mu_0^H} - \sum_{k=1}^{N} e^{\sigma_k y_k + \mu_k} - K\big)^+ \phi(\{x, y\}; 0, \Sigma) dx dy, \qquad (4)$$

whereas $\phi(z; m; \Sigma)$ stands for the multivariate normal density function with mean vector m and covariance matrix $\Sigma$. Note that the random variables $X$ and $Y$ in equation 4 are approximately jointly normally distributed with density $\phi(x, y; 0; \Sigma)$.

Proposition 1 (Pearson (1995) [5]) allows us to reduce the (N+1)-dimensional integral from above (equation 4) to N+2 N-dimensional integrals.

**Proposition 1.** *Under the jointly-normal returns setup with $K \geq 0$ and det $\Sigma \neq 0$, the price of the spread option (equation 3) can be written as*

$$V = e^{-rT + \mu_0^H + \frac{1}{2}\sigma_0^{H2}} I_0 - \sum_{k=1}^{N} e^{-rT + \mu_k + \frac{1}{2}\sigma_k^2} I_k - K e^{-rT} I_{N+1}.$$

*The integrals $I_i$'s are given by*

$$I_0 = \int_{\mathbb{R}^N} \phi(y; 0, \Sigma_{11}) \Phi\big(A(y + \sigma_0^H \Sigma_{10}) + \sigma_0^H \sqrt{\Sigma_{x|y}}\big) dy,$$

$$I_k = \int_{\mathbb{R}^N} \phi(y; 0, \Sigma_{11}) \Phi\big(A(y + \sigma_k \Sigma_{11} e_k)\big) dy, \quad k = 1, 2, ..., N,$$

$$I_{N+1} = \int_{\mathbb{R}^N} \phi(y; 0, \Sigma_{11}) \Phi\big(A(y)\big) dy,$$

*where $e_k$ is the unit column vector (0,...,0,1,0,...,0)' with 1 at the k-th position, and*

$$A(y) = \frac{\mu_{x|y} - x(y)}{\sqrt{\Sigma_{x|y}}},$$

*with*

$$\mu_{x|y} = \Sigma_{10}' \Sigma_{11}^{-1} y, \quad \Sigma_{x|y} = 1 - \Sigma_{10}' \Sigma_{11}^{-1} \Sigma_{10}. \qquad (5)$$

In proposition 1, $\Phi(z)$ stands for the one-dimensional cumulative distribution function. Note that I already plugged in the approximated mean $\mu_0^H$ and variance $\sigma_0^H$ of the random variable $H_0(t)$ instead of the mean and variance of $S_0(t)$. From now one I'll be using the approximations $\mu_0^H$ and $\sigma_0^H$.
Recover that $\det \Sigma \neq 0$ such that $\Sigma_{x|y} \neq 0$, $\det \Sigma_{11} \neq 0$ and $A(y)$ is always well defined. In the GBM setting, the price $V$ reduces to the form

$$V = S_0(t)e^{-q_0 T} I_0 - \sum_{k=1}^{N} S_k(0)e^{-q_k T} I_k - Ke^{-rT} I_{N+1}. \tag{6}$$

The proof of Proposition 1 can be found in [1] in the appendix. The formula presented above is the starting point of our next approximation. We do a second order taylor expansion of the exercise boundary $x(y)$ and the function $A(y)$ such that the $I_k$'s can be computed in closed form. Observe that $A(y)$ is also close to being linear around $y = 0$. The second order approximation of $x(y)$ and $A(y)$ is given by Proposition 2:

**Proposition 2.** *The exercise boundary x(y) can be approximated to second order in y as*

$$x(y) \approx x(0) + \nabla x|_0' y + \frac{1}{2} y' \nabla^2 x|_0 y,$$

*where*

$$x(0) = \frac{log(R+K) - \mu_0^H}{\sigma_0^H},$$

$$(\nabla x|_0)_k = \frac{e^{\mu_k}\sigma_k}{\sigma_0^H (R+K)}, k = 1, 2, ..., N,$$

$$(\nabla^2 x|_0)_{i,j} = \frac{e^{\mu_i+\mu_j}\sigma_i\sigma_j}{\sigma_0^H (R+K)^2} + \delta_{i,j}\frac{v_j^2 e_j^{\mu}}{\sigma_0^H (R+K)}, i, j = 1, 2, ..., N$$

*with $\delta_{i,j}$ being the Kronecker delta function, and*

$$R = \sum_{k=1}^{N} e^{\mu_k}.$$

*Accordingly, the function A(y) can be approximated as*

$$A(y) = \frac{\mu_{x|y} - x(y)}{\sqrt{\Sigma_{x|y}}} \approx c + d'y + y'Ey,$$

*where*

$$c = -\frac{log(R+K) - \mu_0^H}{v_0^H \sqrt{\Sigma_{x|y}}},$$

$$d = \frac{1}{\sqrt{\Sigma_{x|y}}}(\Sigma_{11}^{-1}\Sigma_{10} - \nabla x|_0),$$

$$E = -\frac{1}{2\sqrt{\Sigma_{x|y}}}(\nabla^2 x|_0).$$

To gain a closed-form approximation of the option price (equation 6) we need to further expand $\Phi(c + d'y + y'Ey)$ into three terms to second order in $y'Ey$ around $y'Ey = \epsilon$. The approximation formula of the price of a hybrid basket-spread is presented in Proposition 3, which has been derived with the help of an identity in Li (2004) [6].

**Proposition 3.** *Let $K \geq 0$ and $det\Sigma \neq 0$. The spread option price $V$ under the general jointly-normal returns setup is given by*

$$V = e^{-rT+\mu_0^H+\frac{1}{2}\sigma_0^{H2}} I_0 - \sum_{k=1}^{N} e^{-rT+\mu_k+\frac{1}{2}\sigma_k^2} I_k - Ke^{-rT} I_{N+1}. \quad (7)$$

*The integrals $I_i$'s are approximated as*

$$I_i \approx J^0(c_i, d_i) + J^1(c_i, d_i) - \frac{1}{2} J^2(c_i, d_i), i = 0, 1, ..., N+1,$$

*where the scalar functions $J^i$ are defined as*

$$J^0(u, v) = \Phi\Big(\frac{u}{\sqrt{1+v'v}}\Big), \quad (8)$$

$$J^1(u, v) = \frac{\lambda}{\sqrt{1+v'v}} \phi\Big(\frac{u}{\sqrt{1+v'v}}\Big), \quad (9)$$

$$\begin{aligned} J^2(u, v) = & \frac{u}{(1+v'v)^{3/2}} \phi\Big(\frac{u}{\sqrt{1+v'v}}\Big) \Big\{\lambda^2 + 2tr[(PFP)^2] - \\ & 4\lambda(1+v'v)(v'P^2FP^2v) + (4u^2 - 8 - 8v'v)||PFP^2v||^2\Big\}, \end{aligned} \quad (10)$$

*with*

$$P = P(v) \equiv (I + vv')^{-1/2}, \quad (11)$$

$$\lambda = \lambda(u, v) \equiv u^2 v'P^2FP^2v + tr(PFP) - tr(F), \quad (12)$$

*where $tr$ stands for the trace operator of a matrix. The scalars $c_i$, vectors $d_i$, and matrix $F$ are given by*

$$c_0 = c + tr(F) + \sigma_0^H \sqrt{\Sigma_{x|y}} + \sigma_0^H \Sigma'_{10} d + {\sigma_0^H}^2 \Sigma'_{10} E \Sigma_{10}, \quad (13)$$

$$d_0 = \Sigma_{11}^{1/2}(d + 2\sigma_0^H E\Sigma_{10}), \quad (14)$$

$$c_k = c + tr(F) + \sigma_k e'_k \Sigma_{11} d + \sigma_k^2 e'_k \Sigma_{11} E \Sigma_{11} e_k, k = 0, 1, ..., N, \quad (15)$$

$$d_k = \Sigma_{11}^{1/2}(d + 2\sigma_k E\Sigma_{11} e_k), k = 1, 2, ..., N, \quad (16)$$

$$c_{N+1} = c + tr(F), \quad (17)$$

$$d_{N+1} = \Sigma_{11}^{1/2} d, \quad (18)$$

$$F = \Sigma_{11}^{1/2} E \Sigma_{11}^{1/2}, \quad (19)$$

*with $\Sigma_{x|y}$ given in Proposition 1.*

The computation in Proposition 3 appears to be cumbersome but most of the matrix multiplication can be bypassed which is shown in proposition 4. The proofs of propositions 3(4) and 4(5) can be found in [1] in the appendix.

**Proposition 4.** *With $P = P(v)$ as defined in equation 11, we have*

$$P = I - \theta vv', P^2 = I - \psi vv',$$

*where the scalars $\theta$ and $\psi$ are given by*

$$\theta = \theta(v) = \frac{\sqrt{1 + v'v} - 1}{v'v + \sqrt{1 + v'v}}, \psi = \psi(v) = \frac{1}{1 + v'v}.$$

*Furthermore, we have*

$$tr[(PFP)^2] = tr(F^2) - \psi(1 + \psi)v'F^2v,$$

$$v'P^2FP^2v = \psi^2 v'Fv,$$
$$||PFP^2v||^2 = \psi^2[v'F^2v - \psi(v'Fv)^2],$$
$$tr(PFP) = tr(F) - \psi v'Fv.$$

*Thus the scalar function $J^i$'s given in (8-10) can be simplified as*

$$J^0(u, v) = \Phi(u\sqrt{v}), \tag{20}$$

$$J^1(u, v) = \psi^{3/2}(\psi u^2/1)v'Fv\phi(u\sqrt{\psi}), \tag{21}$$

$$\begin{aligned} J^2(u, v) =& u\psi^{3/2}\phi(u\sqrt{v})\{2tr(F^2) - 4(1 - tr(F))\psi - \psi^2)v'Fv+ \\ & \psi^2(9 + (2 - 3u^2)\psi - u^2(4 - u^2)\psi^2)(v'Fv)^2- \\ & 2\psi(5 + (1 - 2u^2)\psi)v'F^2v\}. \end{aligned} \tag{22}$$

## 2.3 Hybrid Moment Matching associated with Improved Comontonic Upper Bound

The hybrid moment matching method associated with improved comontonic upper bound method (HybMMICUP) combines two approximation techniques. First, the underlyings are split up according their sign and aggregated in two sums $\mathbb{S}_1 = \sum_{i=1}^{M} S_i$ and $\mathbb{S}_2 = \sum_{j=M+1}^{M+N} S_j$. Then both sums are moment matched with a log-normal random variable. Second, the price of the price of the resulting spread option with payoff $(\mathbb{S}_1 - \mathbb{S}_2 - K)^+$ is approximated with the improved comontonic upper bound method. Moment matching methods, the improved comontonic upper bound and the HybMMICUP procedure have been presented by Deelstra, Petkovic and Vanmaele in 2009 [2]. The numerical results from [2] show that the HybMMICUP method performs best at approximating the price of a basket spread option. It can also price options of Asian type but this case won't be treated in my work.
More formally, the fair value of the hybrid European basket spread option will be rewritten as

$$V = e^{-rT}\mathbb{E}^{\mathbb{Q}}(\mathbb{S}_1(T) - \mathbb{S}_2(T) - K)^+, \tag{23}$$

where $\mathbb{S}_j$ is a log-normal random variable with mean $u_j = 2ln(m_{1j}) - 1/2ln(m_{2j})$ and variance $\sigma_j^2 = ln(m_{2j}) - 2ln(m_{1j})$. $S_1$ and $S_2$ represent the sums of the assets with positive and negative sign and $m_{1j}$ and $m_{2j}$ are the first and second moment of the respective sum $S_j$. There are different studies on how to approximate the moments of a sum of log-normal distributed random variables (such as [7] and [8]). The formulas I used to reproduce the results from [2] is as follows:

$$m_{11} = e^{rT}\sum_{i=1}^{M} S_i(0), \quad m_{12} = e^{rT}\sum_{i=M+1}^{M+N} S_i(0), \tag{24}$$

$$m_{2j} = e^{2rT}\sum_{i=1}^{M}\sum_{j=1}^{M+N} S_i(0)S_j(0)e^{\rho_{i,j}\sigma_i\sigma_j T} \tag{25}$$

Additionally we need the correlation coefficient $\rho$ between $ln(\tilde{\mathbb{S}}_1)$ and $ln(\tilde{\mathbb{S}}_2)$ to approximate the spread given in equation 23. Note the tildes above $\mathbb{S}$ as we are left with moment matched approximations of log-normal random variables. The coefficcent $\rho$ can be recovered from the eqiality of cross moments (crm):

$$\mathbb{E}[S_1 S_2] = \mathbb{E}[\tilde{S}_1 \tilde{S}_2],$$

namely

$$crm := \sum_{i=1}^{M}\sum_{j=1}^{M+N} \mathbb{E}[S_i(T)S_j(T)] = e^{\mu_1 + \mu_2 + 1/2\sigma_1^2 + 1/2\sigma_2^2 + \sigma_1\sigma 2\rho}, \tag{26}$$

where $\mathbb{E}[S_i(T)S_j(T)] = S_i(0)S_j(0)e^{2rT + \rho_{ij}\sigma_i\sigma_j T}$.
For an approximation of the spread (23) the improved comontonic upper bound (ICUB) is used. The ICUB is displayed in the next proposition, the proof and derivation can be found in [2]:

**Proposition 5.** *Consider the following stop-loss premium*

$$\mathbb{E}[(e^{\mu_1 + \sigma_1 X_1} - e^{\mu_2 + \sigma_2 X_2} - K)^+],$$

*with $X_1$, $X_2$ two correlated standard normal random variables. The comonotonic improved upper bound of this spread is given by*

$$V \simeq \sum_{i=1}^{2} \epsilon_i \int_0^1 e^{\mu_i + A_i(u) + 1/2Y_i^2} \Phi\big(Y_i - \Phi^{-1}\big(F_{\mathbb{S}^{ic}|U=u}(K)\big)\big) du \\ - K\big(1 - F_{\mathbb{S}^{ic}}(K)\big), \tag{27}$$

*where $\epsilon_1 = 1$ and $\epsilon_2 = -1$, and where $F_{\mathbb{S}^{ic}|U=u}$ can be found by solving*

$$\sum_{i=1}^{2} \epsilon_i e^{\mu_i + A_i(u) + Y_i \Phi^{-1}(F_{\mathbb{S}^{ic}|U=u}(K))} = K, \tag{28}$$

*with, for $\gamma_i$ denoting the correlation between $X_i$ and the conditioning variable $\Lambda \stackrel{d}{=} \mathbb{E}[\Lambda] + \sigma_\Lambda \Phi^{-1}(U)$*

$$A_i(u) = \gamma_i \sigma_i \Phi^{-1}(u), \quad Y_i = \epsilon_i \sigma_i \sqrt{1 - \gamma_i^2}, \quad F_{\mathbb{S}^{ic}}(K) = \int_0^1 F_{\mathbb{S}^{ic}|U=u}(K) du.$$

The corelation coefficient $\gamma_i$ is calculated as follows:

$$\gamma_i = \frac{e^{\mu_i}\sigma_i + e^{\mu_j}\sigma_j \rho}{\sqrt{e^{2\mu_1}\sigma_1^2 + 2e^{\mu_1+\mu_2}\sigma_1\sigma_2 + e^{2\mu_2}\sigma_2^2}}.$$

**Remark.** *In the presentation of the HybMMICUP method I also omitted the weights of the assets as they can be incorporated in the price of each asset how indicated in subsection 2.1.*

# 3 Numerical Analysis

This section enlightens the computational bottlenecks, such as complexity and instabilities, of both the SOB and the HybMMICUB method. The analysis is based on my MATLAB implementations *priceBasketSpreadOptionSOB.m* and *priceBasketSpreadOptionHybMMICUB.m*.
To experiment with my implementations I created a method *generateMarket-Params.m* to generate different sets of input parameters for my pricing methods. It allows me to create combinations of scenarios for which I can chose the number of assets $N$, the number of assets with positive sign in the spread $nP$ and two different types (*charged* and *descending*) of vectors holding the initial values of the assets:

$$S_c(0) = [10(N+1), 10, ..., 10], \quad S_{de}(0) = [10N, 10(N-1), ..., 10].$$

Also I can chose between a volatility vector with *constant* entries equal $\sigma_0$ or with *descending* entires, which means that entires are evenly spaced between $\sigma_0$ and 0.1:

$$\sigma_{const} = [\sigma_0, \sigma_0, ..., \sigma_0] \quad \sigma_{de} = [\sigma_0, \frac{(\sigma_0 - 0.1)(N-2)}{N-1}, ..., 0.1].$$

Moreover I can create three different correlation matrices, *constant* type

$$\Sigma_{const} = \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \cdots & \rho \end{pmatrix},$$

*alternating* type $\Sigma_{alt}$, where all non-diagonal elements $\Sigma_{i \neq j} = \pm\rho$ whereas the sign changes inbetween cross-diagonals. The correlation matrix of *descending* type is defined as follows:

$$\Sigma_{de} = \begin{pmatrix} 1 & \frac{1}{3} & \cdots & \frac{1}{N+1} \\ \frac{1}{3} & 1 & \cdots & \frac{1}{N+2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N+1} & \frac{1}{N+2} & \cdots & \frac{1}{2N} \end{pmatrix},$$

whereas the non-diagonal elements are set to $\Sigma_{i \neq j} = \frac{1}{i+j}$. Throughout my experiments the weights $w_i$ will be set to one, the interst rate is kept constant $r = 0.05$ and maturity $T$ is one.

## 3.1 SOB Method

There are four remarkable operations in the SOB implementaion. First, there are mainly matrix-vector operations which have a computational complexity of $\mathcal{O}(n^2)$ (ignoring scalar operations). Second, in order to compute the values $\mu_{x|y}$ and $\Sigma_{x|y}$ in equation 5 I preferably solve the linear system $\Sigma_{10} = \Sigma_{11}x$ instead of inverting the matrix $\Sigma_{11}$. In this case solving a system of linear equations has a complexity of $\mathcal{O}(n^3)$. Third, to compute $\Sigma_{11}^{1/2}$ required for all $d$'s (equation 14, 16, 18) I compute it's eigenvalues and eigenvectors for which

MATLAB choses the Cholesky decomposition having complexity $\mathcal{O}(n^3)$. Last, there are a couple of matrix-matrix multiplications in the code with complexity of $\mathcal{O}(n^3)$ also. All in all, the SOB method has a compuational complexity of $\mathcal{O}(n^3)$ up to some constant, but apart from the computational costly operations this method is very straight forward to implement and all operations can be vectorized. Runtime plots will follow up in subsection 3.3.

## 3.2 HybMMICUP

This subsection refers to my implementation *priceBasketSpreadOptionHybMMICUB.m*. At first sight, there are few computational costly operations involving matrix multiplications. But the complexity of the procedure lies in the two nested subroutines which are needed to evaluate equation 27. The eqaution includes three integrals (one of it to deternine $F_{\mathbb{S}^{ic}}(K)$) for which in each discretization step $\delta u$ one numerically needs so solve equation 28 in order to determine $F_{\mathbb{S}^{ic}|U=u}(K)$. Therefore I need a reliable integration scheme to find the improved comontonic upper bound. To find $F_{\mathbb{S}^{ic}|U=u}(K)$ I first use the zero finding subroutine *fzero* and apply it to the function

$$f(x,u) = \sum_{i=1}^{2} \epsilon_i e^{\mu_i + A_i(u) + Y_i x} - K, \quad u \in (0,1) \tag{29}$$

where $x = \Phi^{-1}(F_{\mathbb{S}^{ic}|U=u})$. In a second step I calculate $F_{\mathbb{S}^{ic}|U=u} = \Phi(x)$. Experiments have shown that splitting up the calculation provides stability while finding $F_{\mathbb{S}^{ic}|U=u}$. Figure 1 shows $f(x,u)$ (29) for different types of scenario. Indeed $f(x)$ is a monotonic function such that *fzero* is able to find a zero for

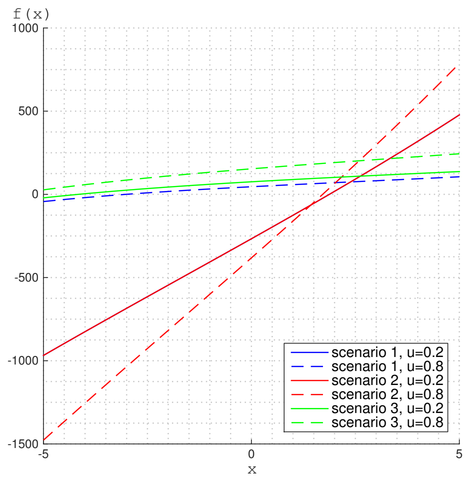


Figure 1: A picture of a gull.

given function. Denote that f(x) is not defined for $u = 1$ and for $u = 0$ there exists no zero as $f(x, 0) = -K$. This poses a problem for the integrals which are defined on $u \in [0, 1]$. To avoid this problem I limit the integration intervall to $u \in [\delta u, 1 - \delta u]$, $\delta u \to 0$.

Figure 2 shows the two integrands in equation 27. To solve the integrals I consid-

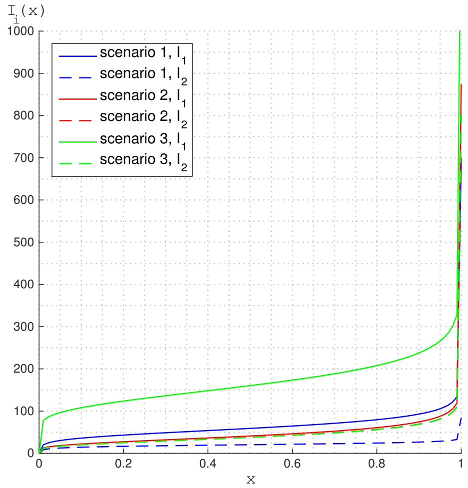


Figure 2: A picture of a gull.

ered two integration techniques, the adaptive Simpson's rule and the rectangle method. The adaptive Simpson's rule repeatedly splits an integral $I = \int_a^b g(x)dx$ in two subintervalls $I_l = \int_a^{\frac{b-a}{2}} g(x)dx$ and $I_r = \int_{\frac{b-a}{2}}^b g(x)dx$ until the criteria $|I - I_l - I_r| \leq \epsilon$ is met, whereas the integrals are numerically approximated according to the standard Simpson's rule. The rectangle method divides the interval (a,b) into $N$ equal subintervals of length $h = \frac{b-a}{N}$ and approximates the integral by adding up the areas of the $N$ rectangles under the curve. The rectangle method allows to precompute $F_{\mathbb{S}^{ic}|U=u}(K)$ for all subintervals once and reuse it in all three integrals. On the other hand the adaptive Simpson's rule might use less subintervalls in total but possibly this approach needs to recompute $F_{\mathbb{S}^{ic}|U=u}(K)$ multiple times for the same subintervall. The accuracy and runtime of the two mentioned methods strongly depend on $\epsilon$ and $N$, respectively. A performance analysis based on convergence rate versus computation time shows that the adaptive Simpson's rule is outperforming the rectangle method. Based on my analysis I set $\epsilon = 10^{-5}$.

## 3.3 Runtime

# 4 Results and Future Work

Insert Results and other text.

# References


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