

An Analysis of Pricing Methods for Baskets Options

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Abstract:

This paper deals with the task of pricing basket options. Here, the main problem is not path-dependency but the multi-dimensionality which makes it impossible to give exact analytical representations of the option price. We review the literature and compare six different methods in a systematic way. Thereby we also look at the influence of various parameters such as strike, correlation, forwards or volatilities on the performance of the different approximations.

Keywords and Phrases:

Exotic options, basket options, numerical methods

1 Introduction

While with many exotic options it is even hard to fully understand the way their final payoff is built up, the construction of the payoff of a (European) basket option is very simple. We define the price of a basket of stocks by

$$B(T) = \sum_{i=1}^n w_i S_i(T),$$

i.e. it is the weighted average of the prices of n stocks at maturity T . Here the weights w_i are usually assumed to be positive and to sum up to 1, but also be quite arbitrary.

Our task is to determine the price of a call ($\theta = 1$) or a put ($\theta = -1$) with strike K and maturity T on the basket, i.e. to value the payoff

$$P_{\text{Basket}}(B(T), K, \theta) = [\theta(B(T) - K)]^+.$$

We price these options with the Black-Scholes Model. Note that by the form of the payoff it is not necessary to distinguish between the trading date and the valuation date to calculate the values of these options, since they are not path-dependent. Hence without loss of generality we can set $t = 0$ and denote the remaining time to maturity with T . In order to ease the calculations we use the so-called *forward notation*. The T -forward price

of stock i is given by

$$F_i^T = S_i(0) \exp \left(\int_0^T (r(s) - d_i(s)) ds \right)$$

where $r(\cdot)$ and $d_i(\cdot)$ are deterministic interest rates and dividend yields. With its help the stock prices can be represented as

$$S_i(T) = F_i^T \exp \left(- \int_0^T \frac{1}{2} \sigma_i^2 ds + \int_0^T \sigma_i dW_i(s) \right)$$

where the $W_i(\cdot)$ are correlated one-dimensional Brownian motions with correlation of ρ_{ij} . Further, we define the discount factor as

$$\text{Df}(T) = \exp \left(- \int_0^T r(s) ds \right).$$

The forward-oriented notation has two advantages: Firstly, in opposite to short rates and dividend yields, forward prices and discount factors are market-quotes. Secondly, from a computational point of view, it is less costly to work with single numbers, i.e. the forward prices and the discount factor, instead of several term-structures, namely the short rates and the dividend yields.

The problem of pricing the above basket options in the Black-Scholes Model is the following: The stock prices are modelled by geometric

Brownian motions and are therefore log-normally distributed. As the sum of log-normally distributed random variables is not log-normal, it is not possible to derive an (exact) closed-form representation of the basket call and put prices. Due to the fact that we are dealing with a multidimensional process, only Monte Carlo or Quasi-Monte Carlo (and over multi-dimensional intergration methods) are suitable numerical methods to determine the value of these options. As these methods can be very time consuming we will present alternative valuation methods which are based on analytical approximations in different senses.

2 ... and here are the candidates!

a) Beisser's conditional expectation techniques

Beisser (1999) adapts an idea of Rogers and Shi (1995) introduced for pricing Asian options. By conditioning on the random variable Z and using Jensen's inequality the price of the basket call is estimated by the weighted sum of (artificial) European call prices, more precisely

$$\begin{aligned} E([B(T) - K]^+) &= E(E([B(T) - K]^+ | Z)) \\ &\geq E(E([B(T) - K | Z]^+)) \\ &= E\left(\left[\sum_{i=1}^n w_i E[S_i(T) | Z] - K\right]^+\right) \end{aligned}$$

where

$$Z := \frac{\sigma_z}{\sqrt{T}} W(T) = \sum_{i=1}^n w_i S_i(0) \sigma_i W_i(T)$$

with σ_z appropriately chosen. Note that in contradiction to $S_i(T)$, all conditional expectations $E[S_i(T) | Z]$ are log-normally distributed with respect to one Brownian motion $W(T)$. Hence, there exists an x^* , such that

$$\sum_{i=1}^n w_i E[S_i(T) | W(T) = x^*] = K$$

By defining:

$$\tilde{K}_i := E[S_i(T) | W(T) = x^*]$$

the event $\sum_{i=1}^n w_i E[S_i(T) | Z] \geq K$ is equivalent to $E[S_i(T) | Z] \geq \tilde{K}_i$ for all $i = 1, \dots, n$.

Using this argument we conclude that

$$\begin{aligned} E\left(\left[\sum_{i=1}^n w_i E[S_i(T) | Z] - K\right]^+\right) &= \sum_{i=1}^n w_i E([E[S_i(T) | Z] - \tilde{K}_i]^+) \\ &= \sum_{i=1}^n w_i [\tilde{F}_i^T N(d_{1i}) - \tilde{K}_i N(d_{2i})] \end{aligned}$$

where \tilde{F}_i^T , \tilde{K}_i adjusted forwards and strikes and d_{1i} , d_{2i} are the usual terms with modified parameters.

b) Gentle's approximation by geometric average

Gentle (1993) approximates the arithmetic average in the basket payoff by a geometric average. The fact that a geometric average of lognormal random variables is again log-normally distributed allows for a Black-Scholes type valuation formula for pricing the approximating payoff. More precisely after rewriting the payoff of the basket option as

$$\begin{aligned} P_{Basket}(B(T), K, \theta) &= \left[\theta \left(\sum_{i=1}^n w_i S_i(T) - K \right) \right]^+ \\ &= \left[\theta \left(\left(\sum_{i=1}^n w_i F_i^T \right) \sum_{i=1}^n a_i S_i^*(T) - K \right) \right]^+, \end{aligned}$$

where

$$\begin{aligned} a_i &= \frac{w_i F_i^T}{\sum_{i=1}^n w_i F_i^T}, \\ S_i^*(T) &= \frac{S_i(T)}{F_i^T} = \exp\left(-\frac{1}{2} \int_0^T \sigma_i^2 ds + \int_0^T \sigma_i dW_i(s)\right) \end{aligned}$$

we approximate $\sum_{i=1}^n a_i S_i^*(T)$ by the geometric average, thus

$$\tilde{B}(T) = \left(\sum_{i=1}^n w_i F_i^T \right) \prod_{i=1}^n (S_i^*(T))^{a_i}.$$

To correct for the mean,

$$K^* = K - (E(B(T)) - E(\tilde{B}(T)))$$

is introduced. As approximation for $(B(T) - K)^+$, $(\tilde{B}(T) - K^*)^+$ is used, which—as $\tilde{B}(T)$ is log-normally distributed—can be valued by the Black-Scholes formula resulting in

$$V_{Basket}(T) = Df(T) \theta \left(e^{\tilde{m} + \frac{1}{2} \tilde{v}^2} N(\theta d_1) - K^* N(\theta d_2) \right), \quad (2.1)$$

where $Df(T)$ is the discount factor, $N(\cdot)$ the distribution function of a standard normal random variable and

$$d_1 = \frac{\tilde{m} - \log K^* + \tilde{v}^2}{\tilde{v}},$$

$$d_2 = d_1 - \tilde{v},$$

$$\tilde{m} = E(\log \tilde{B}(T)) = \log \left(\sum_{i=1}^n w_i F_i^T \right) - \frac{1}{2} \sum_{i=1}^n a_i \sigma_i^2 T \quad \text{and}$$

$$\tilde{v}^2 = \text{Var}(\log \tilde{B}(T)) = \sum_{i=1}^n \sum_{j=1}^n a_i a_j \sigma_i \sigma_j \rho_{ij} T.$$

c) Levy's log-normal moment matching

The basic idea of Levy (1992) is to approximate the distribution of the basket by a log-normal distribution $\exp(X)$ with mean M and variance

$V^2 - M^2$, such that the first two moments of this and of the original distribution of the weighted sum of the stock prices coincide, i.e.

$$\begin{aligned} m &= 2 \log(M) - 0.5 \log(V^2) \\ v^2 &= \log(V^2) - 2 \log(M) \quad \text{and} \\ M &\equiv E(B(T)) = \sum_{i=1}^n w_i F_i(T) \\ V^2 &\equiv E(B^2(T)) = \sum_{i=1}^n w_i w_j F_i^T F_j^T \exp(\sigma_i \sigma_j \rho_{ij} T) \end{aligned}$$

result in

$$E(B(T)) = E(e^X) = e^{m+0.5v^2} \quad \text{and} \quad E(B^2(T)) = E(e^{2X}) = e^{2m+2v^2}$$

where X is a normally distributed random variable with mean m and variance v^2 .

The basket option price is now approximated by

$$V_{Basket}(T) \approx Df(T) (MN(d_1) - KN(d_2))$$

with

$$\begin{aligned} d_1 &= \frac{m - \ln(K) + v^2}{v}, \\ d_2 &= d_1 - v. \end{aligned}$$

Note the subtle difference to Gentle's method. Here, the distribution of $B(T)$ is approximated directly by a log-normal distribution that matches the first two moments, while in Gentle's approximation only the first moment is matched.

d) Ju's Taylor expansion

Ju (2002) considers a Taylor expansion of the ratio of the characteristic function of the arithmetic average to that of the approximating lognormal random variable around zero volatility. He includes terms up to σ^6 in his closed-form solution.

Let

$$A(z) = \sum_{i=1}^n F_i^T \exp\left(-\frac{1}{2}(z\sigma_i)^2 T + z\sigma_i W_i(T)\right)$$

be the arithmetic mean were the volatilities are scaled by a parameter z . Note that for $A(1)$ we recover the original mean. Let $Y(z)$ be a normally distributed random variable with mean $m(z)$ and variance $v(z)$ such that the first two moments of $\exp(Y(z))$ match those of $A(z)$. The appropriate parameters are derived in section c), only σ_i has to be replaced by $z\sigma_i$. Let $X(z) = \log(A(z))$, then the characteristic function is given as:

$$E[e^{i\phi X(z)}] = E[e^{i\phi Y(z)}] \frac{E[e^{i\phi X(z)}]}{E[e^{i\phi Y(z)}]} = E[e^{i\phi Y(z)}] f(z),$$

where

$$\begin{aligned} E[e^{i\phi Y(z)}] &= e^{i\phi m(z) - \phi^2 v(z)/2} \\ f(z) &= E[e^{i\phi X(z)}] e^{-i\phi m(z) + \phi^2 v(z)/2} \end{aligned}$$

Ju performs a Taylor expansion of the two factors of $f(z)$ up to z^6 , leading to

$$f(z) \approx 1 - i\phi d_1(z) - \phi^2 d_2(z) + i\phi^3 d_3(z) + \phi^4 d_4(z),$$

where $d_i(z)$ are polynomials of z and terms of higher order than z^6 are ignored. Finally $E[e^{i\phi X(z)}]$ is approximated by

$$E[e^{i\phi X(z)}] \approx e^{i\phi m(z) - \phi^2 v(z)/2} (1 - i\phi d_1(z) - \phi^2 d_2(z) + i\phi^3 d_3(z) + \phi^4 d_4(z)).$$

For this approximation, an approximation of the density $h(x)$ of $X(1)$ is derived as

$$h(x) = p(x) + \left(\frac{d}{dx} d_1(1) + \frac{d^2}{dx^2} d_2(1) + \frac{d^3}{dx^3} d_3(1) + \frac{d^4}{dx^4} d_4(1) \right) p(x)$$

where $p(x)$ is the normal density with mean $m(1)$ and variance $v(1)$. The approximate price of a basket call is then given by,

$$\begin{aligned} V_{Basket}(T) &= Df(T) \left\{ \left[\left(\sum w_i F_i^T \right) N(d_1) - KN(d_2) \right] \right. \\ &\quad \left. + K \left[z_1 p(y) + z_2 \frac{dp(y)}{dy} + z_3 \frac{d^2 p(y)}{dy^2} \right] \right\}, \end{aligned}$$

where

$$y = \log(K), \quad d_1 = \frac{m(1) - y}{\sqrt{v(1)}} + \sqrt{v(1)}, \quad d_2 = d_1 - \sqrt{v(1)}$$

and $z_1 = d_2(1) - d_3(1) + d_4(1)$, $z_2 = d_3(1) - d_4(1)$, $z_3 = d_4(1)$. Note that the first summand is equal to Levy's approximation and the second summand gives the higher order corrections.

e) The reciprocal gamma approximation by Milevsky and Posner

Milevsky and Posner (1998) use the reciprocal gamma distribution as an approximation for the distribution of the basket. The motivation is the fact that the distribution of correlated log-normally distributed random variables converges to the reciprocal gamma distribution as $n \rightarrow \infty$. Consequently, the first two moments of both distributions are matched to obtain a closed-form solution. Let G_R be the reciprocal gamma distribution and G the gamma distribution with parameters α, β , then per definition:

$$G_R(y, \alpha, \beta) = 1 - G(1/y, \alpha, \beta)$$

If the random variable Y is reciprocally gamma distributed, then

$$E[Y^i] = \frac{1}{\beta^i (\alpha - 1)(\alpha - 2) \dots (\alpha - i)}$$

and with M and V^2 denoting the first two moments as defined in the previous section, we get:

$$\begin{aligned} \alpha &= \frac{2V^2 - M^2}{V^2 - M^2} \\ \beta &= \frac{V^2 - M^2}{V^2 M} \end{aligned}$$

Basic calculations yield:

$$V_{\text{Basket}}(T) \approx \text{Df}(T) (MG(1/K, \alpha - 1, \beta) - KG(1/K, \alpha, \beta))$$

Note, that we use the parametrisation of the gamma distribution found in Staunton (2002), since this produces more accurate results than that from the original paper by Milevsky and Posner (1998).

f) Milevsky and Posner's approximation via higher moments

Milevsky and Posner (1998b) use distributions from the Johnson (1994) family as state-price densities to match higher moments of distribution of the arithmetic mean. More precisely, they write the price of a call on a basket as:

$$V_{\text{Basket}}(T) = \text{Df}(T) \left[\int_0^\infty (x - K)^+ h(x) dx \right]$$

where $h(x)$ is the state price density. Note that, we would end up in Levy's approximation, if we were using the lognormal density with the first two moments matching those of the mean. Milevsky and Posner however use two members of the Johnson family, which is a collection of statistical distributions, that can be represented by a transformation of the normal distribution Z :

$$\text{Type I: } X = c + d \exp\left(\frac{Z - a}{b}\right) \quad \text{or}$$

$$\text{Type II: } X = c + d \sinh\left(\frac{Z - a}{b}\right)$$

The parameters a, b, c and d are chosen, so that the four moments of the arithmetic mean are approximated (since there are no closed-form solutions for them). If the kurtosis of the Type I is close enough to the kurtosis of the mean, they use Type I, otherwise Type II. The closed-form solution for Type I is given by:

$$V_{\text{Basket}}(T) \approx \text{Df}(T) \left[M - K + (K - c)N(Q) - d \exp\left(\frac{1 - 2ab}{2b^2}\right) N\left(Q - \frac{1}{b}\right) \right]$$

where

$$M = \sum_i^n a_i F_i^T$$

$$Q = a + b \log\left(\frac{K - c}{d}\right)$$

$$\omega = \frac{1}{2} \sqrt[3]{8 + 4\eta^2 + 4\sqrt{4\eta^2 + \eta^4}} + \frac{2}{\frac{1}{2} \sqrt[3]{8 + 4\eta^2 + 4\sqrt{4\eta^2 + \eta^4}}} - 1$$

$$a = 1/\sqrt{\log(\omega)}, \quad b = \frac{1}{2} \log(\omega(\omega - 1)/\xi^2), \quad d = \text{sign}(\eta), \quad c = dM - e^{(\frac{1}{2b} - a)/b}$$

where ξ is the variance, η the skewness and κ the kurtosis.

3 Test Results

As the advantage of analytical methods compared to Monte Carlo or numerical integration is of course speed of computations, we only have to compare the accuracy of the analytical methods presented in the foregoing section.

We will perform a systematic test by looking at the effect of varying correlations, strikes, forward and strikes and volatilities. Our standard test example is a call option on a basket with four stocks and parameters given by

$$\begin{aligned} T &= 5.0, \\ \text{Df}(T) &= 1.0, \\ \rho_{ij} &= 0.5 \quad (\text{for } i \neq j), \\ K &= 100, \\ F_i^T &= 100, \\ \sigma_i &= 40\% \quad \text{and} \\ w_i &= \frac{1}{4}. \end{aligned}$$

As reference values we compute the prices of all the options below by a Monte Carlo simulation using antithetic method and geometric mean as control variate for variance reduction. The number of simulations was always chosen large enough to keep the standard deviation below 0.05.

We did not test the method of Hyunh (1993), because it is an application of the method of Turnbull & Wakeman (1991) for Asian Options (Edgeworth expansion up to the 4th moment) and it is a well-known problem that this approximation gives really bad results for long maturities and high volatilities. See also Ju (2002), who pointed out that the Edgeworth expansion diverges if the approximating random variable is lognormal.

We also tested Curran's (1994) approximation which computes the price by conditioning on the geometric mean. But we do not show the numerical results here, because—if we transformed the forwards to one (simply by multiplying the weights with them)—the prices were *exactly* the same as those of Beisser (1999). If did not transform the forwards to one, Beisser and Curran gave different prices, but on the other hand Curran's results were mostly worse. For further reading we refer to Deelstray, Liinev, Vanmaele (2003) and Beisser (2001) who developed a general framework for the pricing of baskets and asian options via conditioning.

a) Varying the correlations

Table 1 below shows the effect of simultaneously changing all correlations from $\rho = \rho_{ij} = 0.1$ to $\rho = 0.95$. Note that, except for Milevsky and Posner's reciprocal gamma (MP-RG) and Gentle, all methods perform reasonably well. Especially for $\rho \geq 0.8$, the methods of Beisser, Ju, Levy, the four moments method of Milevsky and Posner (MP-4M) and Monte Carlo give virtually the same price.

The good performance of Beisser, Ju, Gentle and Levy for high correlations can be explained as follows: All four methods provide exactly the Black-Scholes prices for the special case that the number of stocks is one. For high correlations the distribution of the basket is approximately the sum of the *same* (for $\rho = 1$ exactly the same) log-normal distributions, which is indeed again log-normal. As Levy uses a log-normal distribution with the correct moments, it has to be a good approximation for these cases. The same argumentation applies for Gentle. If we have effectively one stock the geometric and the arithmetic average are the same. The

TABLE 1: VARYING THE CORRELATIONS SIMULTANEOUSLY

| ρ | Beisser | Gentle | Ju | Levy | MP-RG | MP-4M | Monte Carlo CV | StdDev |
|-------------------|---------|--------|-------|-------|-------|-------|----------------|----------|
| 0,10 | 20,12 | 15,36 | 21,77 | 22,06 | 20,25 | 21,36 | 21,62 | (0,0319) |
| 0,30 | 24,21 | 19,62 | 25,05 | 25,17 | 22,54 | 24,91 | 24,97 | (0,0249) |
| 0,50 | 27,63 | 23,78 | 28,01 | 28,05 | 24,50 | 27,98 | 27,97 | (0,0187) |
| 0,70 | 30,62 | 27,98 | 30,74 | 30,75 | 26,18 | 30,74 | 30,72 | (0,0123) |
| 0,80 | 31,99 | 30,13 | 32,04 | 32,04 | 26,93 | 32,04 | 32,03 | (0,0087) |
| 0,95 | 33,92 | 33,41 | 33,92 | 33,92 | 27,97 | 33,92 | 33,92 | (0,0024) |
| Dev. ¹ | 0,700 | 4,013 | 0,071 | 0,203 | 4,119 | 0,108 | | |

$$\text{Dev.} = \sqrt{\frac{1}{n} \sum_{i=1}^n (\text{Price} - \text{MC Price})^2}.$$

bad performance of MP-RG for high correlations can be explained by the fact, that with effectively one stock we are far away from “infinitely” many stocks, which is the motivation for this method. A test with fixed correlation $\rho_{12} = 0.95$ and varying the remaining correlations symmetrically shows exactly the same result.

In total the prices calculated by Ju’s approach (whose method slightly overprices) and MP-4M are overall the closest to the Monte Carlo prices. These approaches are followed by Levy’s and Beisser’s approximation (whose approach slightly underprices). The other two methods are not recommendable.

b) Varying the strikes

With all other parameters set to the default values, the strike K is varied from 50 to 150. Table 2 contains the results.

The differences between the prices calculated by Monte Carlo and the approaches of Ju and MP-4M are relatively small. The price curves of

the method of Gentle and Milevsky’s and Posner’s reciprocal gamma approach (MP-RG) run almost parallel to the Monte Carlo curve and represent an under-evaluation. The relative and absolute differences of all methods are generally increasing when K is growing, since the approximation of the real distributions in the tails is getting worse and the absolute prices are decreasing.

Again, overall Ju’s approximation and MP-4M perform best, while Ju’s slightly overprices. Levy is the third and Beisser the fourth best.

c) Varying the Forwards and Strikes

The forwards on all stocks are now set to the same value F which is varied between 50 and 150 in this set of tests. Table 3 shows that MP-4M and Ju’s method perform excellently, while the second one again typically slightly overprices. Levy and Beisser’s method also perform well and Beisser again slightly underprices. The other methods perform worse. These effects can also be seen if some forwards are fixed and the remaining ones are varied.

d) Varying the Volatilities

The next set of tests involves varying the volatilities σ_i . We start with the symmetrical situation at each step, σ_i is set to the same value σ , which is varied between 5% and 100%.

Table 4 shows the results of the test.

The prices calculated by the different methods are more or less equal for “small” values of the volatility. They start to diverge at $\sigma \approx 20\%$. The Monte Carlo, Beisser, Ju and Levy prices remain close, whereas the prices calculated by the other methods are too low.

The picture obtained so far completely changes if we have asymmetry in the volatilities, precisely if there are groups of stocks with high and with low volatilities entering the basket. This is clearly demonstrated by “Figure 1” where we fix $\sigma_1 = 5\%$ and vary the remaining volatilities symmetrically. This time the prices diverge much more. The method Levy is massively overpricing with all other methods underpricing. We note that Ju’s and Beisser’s method performs best. Particularly remarkable is the excellent performance of Ju for high volatilities. Since it is a Taylor expansion around zero volatilities, one would not expect the validity of this expansion far away from zero.

The same test but now with $\sigma_1 = 100\%$ results in Table 5 and Figure 2. Note the extremely bad performance for Levy’s method for small values of σ which it is even outperformed by Gentle’s method! Beisser the is only one who can deal with this parameters, while both Milevsky and Posner methods are also bad.

e) Implicit distributions

In addition we plot the *implicit distribution* of the particular approximations and compare them to the real ones calculated by Monte Carlo simulation. With *implicit distribution* we mean, that we derive the underlying distribution of the particular method by an appropriate portfolio of calls. Consider

TABLE 2: VARYING THE STRIKE

| K | Beisser | Gentle | Ju | Levy | MP-RG | MP-4M | Monte Carlo CV | StdDev |
|--------|---------|--------|-------|-------|-------|-------|----------------|----------|
| 50,00 | 54,16 | 51,99 | 54,31 | 54,34 | 51,93 | 54,35 | 54,28 | (0,0383) |
| 60,00 | 47,27 | 44,43 | 47,48 | 47,52 | 44,41 | 47,50 | 47,45 | (0,0375) |
| 70,00 | 41,26 | 37,93 | 41,52 | 41,57 | 38,01 | 41,53 | 41,50 | (0,0369) |
| 80,00 | 36,04 | 32,40 | 36,36 | 36,40 | 32,68 | 36,34 | 36,52 | (0,0363) |
| 90,00 | 31,53 | 27,73 | 31,88 | 31,92 | 28,22 | 31,86 | 31,85 | (0,0356) |
| 100,00 | 27,63 | 23,78 | 28,01 | 28,05 | 24,50 | 27,98 | 27,98 | (0,0350) |
| 110,00 | 24,27 | 20,46 | 24,67 | 24,70 | 21,39 | 24,63 | 24,63 | (0,0344) |
| 120,00 | 21,36 | 17,65 | 21,77 | 21,80 | 18,77 | 21,73 | 21,74 | (0,0338) |
| 130,00 | 18,84 | 15,27 | 19,26 | 19,28 | 16,57 | 19,22 | 19,22 | (0,0332) |
| 140,00 | 16,65 | 13,25 | 17,07 | 17,10 | 14,70 | 17,04 | 17,05 | (0,0326) |
| 150,00 | 14,75 | 11,53 | 15,17 | 15,19 | 13,10 | 15,14 | 15,15 | (0,0320) |
| Dev. | 0,323 | 3,746 | 0,031 | 0,065 | 3,038 | 0,030 | | |

TABLE 3: VARYING THE FORWARDS SYM.
WITH $K = 100$

| F | Beisser | Gentle | Ju | Levy | MP-RG | MP-4M | Monte Carlo CV | StdDev |
|--------|---------|--------|-------|-------|-------|-------|----------------|----------|
| 50,00 | 4,16 | 3,00 | 4,34 | 4,34 | 3,93 | 4,33 | 4,34 | (0,0141) |
| 60,00 | 7,27 | 5,53 | 7,51 | 7,52 | 6,56 | 7,50 | 7,50 | (0,0185) |
| 70,00 | 11,26 | 8,91 | 11,55 | 11,57 | 9,95 | 11,53 | 11,53 | (0,0227) |
| 80,00 | 16,04 | 13,13 | 16,37 | 16,40 | 14,10 | 16,34 | 16,35 | (0,0268) |
| 90,00 | 21,53 | 18,11 | 21,89 | 21,92 | 18,97 | 21,86 | 21,86 | (0,0309) |
| 100,00 | 27,63 | 23,78 | 28,01 | 28,05 | 24,50 | 27,98 | 27,98 | (0,0350) |
| 110,00 | 34,27 | 30,08 | 34,66 | 34,70 | 30,63 | 34,63 | 34,63 | (0,0391) |
| 120,00 | 41,36 | 36,91 | 41,75 | 41,80 | 37,32 | 41,73 | 41,71 | (0,0433) |
| 130,00 | 48,84 | 44,21 | 49,23 | 49,28 | 44,49 | 49,21 | 49,19 | (0,0474) |
| 140,00 | 56,65 | 51,92 | 57,04 | 57,10 | 52,08 | 57,03 | 57,00 | (0,0516) |
| 150,00 | 64,75 | 59,98 | 65,13 | 65,19 | 60,05 | 65,14 | 65,08 | (0,0556) |
| Dev. | 0,316 | 3,989 | 0,031 | 0,072 | 3,516 | 0,022 | | |

TABLE 4: VARYING THE VOLATILITIES SYM.
WITH $K = 100$

| σ | Beisser | Gentle | Ju | Levy | MP-RG | MP-4M | Monte Carlo CV | StdDev |
|----------|---------|--------|-------|-------|-------|-------|----------------|----------|
| 5% | 3,53 | 3,52 | 3,53 | 3,53 | 3,52 | 3,53 | 3,53 | (0,0014) |
| 10% | 7,04 | 6,98 | 7,05 | 7,05 | 6,99 | 7,05 | 7,05 | (0,0042) |
| 15% | 10,55 | 10,33 | 10,57 | 10,57 | 10,36 | 10,57 | 10,57 | (0,0073) |
| 20% | 14,03 | 13,52 | 14,08 | 14,08 | 13,59 | 14,08 | 14,08 | (0,0115) |
| 30% | 20,91 | 19,22 | 21,08 | 21,09 | 19,49 | 21,07 | 21,07 | (0,0237) |
| 40% | 27,63 | 23,78 | 28,01 | 28,05 | 24,50 | 27,98 | 27,98 | (0,0350) |
| 50% | 34,15 | 27,01 | 34,84 | 34,96 | 28,51 | 34,73 | 34,80 | (0,0448) |
| 60% | 40,41 | 28,84 | 41,52 | 41,78 | 31,56 | 41,19 | 41,44 | (0,0327) |
| 70% | 46,39 | 29,30 | 47,97 | 48,50 | 33,72 | 46,23 | 47,86 | (0,0490) |
| 80% | 52,05 | 28,57 | 54,09 | 55,05 | 35,15 | 48,39 | 54,01 | (0,0685) |
| 100% | 62,32 | 24,41 | 64,93 | 67,24 | 36,45 | 47,90 | 65,31 | (0,0996) |
| Dev. | 1,22 | 16,25 | 0,12 | 0,69 | 11,83 | 5,53 | | |

the payoff of the following portfolio consisting only of calls:

$$\begin{aligned} \Pi(B(T)) = & \alpha * \left[P_{\text{Basket}} \left(B(T), L - \frac{1}{\alpha}, 1 \right) - P_{\text{Basket}} \left(B(T), L, 1 \right) \right. \\ & \left. - \left(P_{\text{Basket}} \left(B(T), L + \Delta L, 1 \right) - P_{\text{Basket}} \left(B(T), L + \Delta L + \frac{1}{\alpha}, 1 \right) \right) \right] \end{aligned}$$

We notice that the payoff $\Pi(B(T))$ is explicitly given by

$$\Pi(B(T)) = \begin{cases} 0 & : B(T) < L - \frac{1}{\alpha} \\ \alpha [B(T) - (L - \frac{1}{\alpha})] & : L - \frac{1}{\alpha} \leq B(T) \leq L \\ 1 & : L \leq B(T) \leq L + \Delta L \\ 1 - \alpha [B(T) - (L + \Delta L)] & : L + \Delta L \leq B(T) \leq L + \Delta L + \frac{1}{\alpha} \\ 0 & : B(T) > L + \Delta L + \frac{1}{\alpha} \end{cases} \quad (3.2)$$

For $\alpha \rightarrow \infty$ it is equal to:

$$\Pi(B(T)) = \begin{cases} 0 & : B(T) < L \\ 1 & : L \leq B(T) \leq L + \Delta L \\ 0 & : B(T) > L + \Delta L \end{cases}$$

So for a sufficiently high α the *value* of our portfolio is approximately the probability that the price of the basket is at maturity in $[L, L + \Delta L]$. To calculate the whole implicit distribution, we shift the boundaries stepwise by ΔL . Instead of applying the underlying distributions, we use this procedure, because we can not directly determine the distribution for Beisser's approximation. Besides, this procedure seems to be more objective and consistent to compare the approximations.

We examined the distributions for the test cases a)-d). The results confirmed our findings from the comparison of the prices. For the cases a)-c) the implicit distributions of Ju, Levy and Beisser were consistent with Monte Carlo, and the other ones not. But only Beisser was able to deal with inhomogeneous volatilities in case d), where Levy showed massive deviations.

We plot an example with $\sigma_1 = 90\%$, $\sigma_2 = \sigma_3 = 50\%$ and $\sigma_4 = 10\%$ in Figure 3 to test if there is some "balancing" effect, i.e. observe that $(\sigma_1 + \sigma_4)/2 = \sigma_2$. We see there is one except for Levy's approach.

We did not plot the graph for the state-price density method of Milevsky and Posner, because it was running into serious problems for small K . The parameter Q is defined as $a + b \log((K - c)/d)$, hence for all $K < c$ the formula of Milevsky and Posner is not well-defined (a similar problem occurs for Type II). But for this parameter set c is around 65, so we simply couldn't calculate all necessary prices.

So which method to choose?

The tests confirm that the approximation of Ju is overall the best performing method. In addition it has the nice property, that it always overprices slightly. Ju's method shows only a little

weakness in the case of inhomogeneous volatilities, where Beisser is better. Even though it is based on a Taylor expansion around zero volatilities, it has absolutely no problems with high volatilities, which is quite contrary to both methods of Milevsky and Posner.

Beisser's approximation underprices slightly in all cases. The underpricing of Beisser's approach is not surprising since this method is essentially a lower bound on the true option price. Beisser's approach is the only method which is reliable in the case of inhomogeneous volatilities.

The performances of Milevsky and Posner's reciprocal gamma and Gentle's approach are mostly poor. A reason for the bad performance of MP-RG may be, that the sum of log-normally distributed random variables is *only* distributed like the reciprocal gamma distribution as



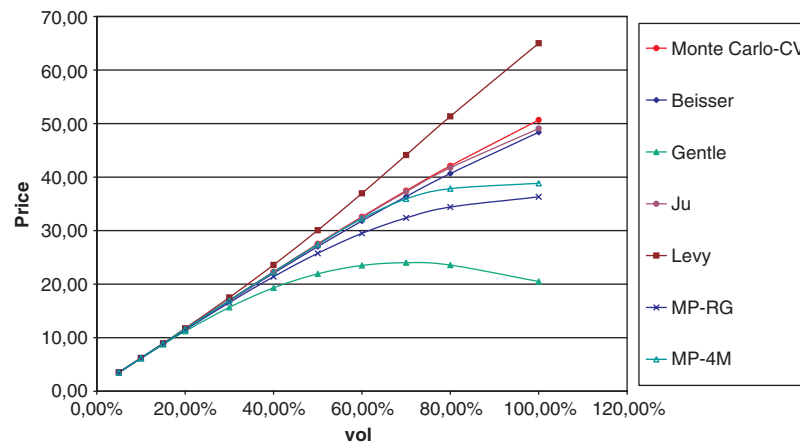


Figure 1: Varying the volatilities sym. with $\sigma_1 = 5\%$, $K = 100$

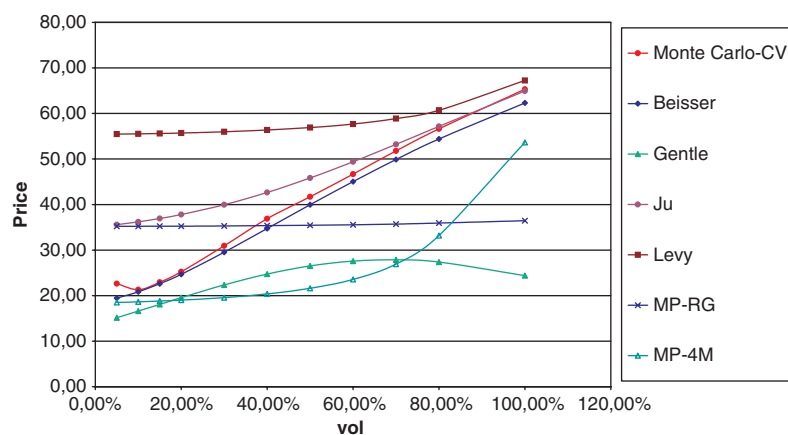


Figure 2: Varying the volatilities sym. with $\sigma_1 = 100\%$, $K = 100$ (Table 5)

TABLE 5: VARYING THE VOLATILITIES SYM. WITH $\sigma_1 = 100\%$, $K = 100$ (FIGURE 2)

| σ | Beisser | Gentle | Ju | Levy | MP-RG | MP-4M | Monte Carlo CV | Std Dev |
|----------|---------|--------|-------|-------|-------|-------|----------------|----------|
| 5,00% | 19,45 | 15,15 | 35,59 | 55,56 | 35,22 | 18,51 | 22,65 | (0,5594) |
| 10,00% | 20,84 | 16,60 | 36,19 | 55,52 | 35,23 | 18,64 | 21,30 | (0,3858) |
| 15,00% | 22,60 | 18,08 | 36,93 | 55,61 | 35,24 | 18,81 | 22,94 | (0,2660) |
| 20,00% | 24,69 | 19,56 | 37,80 | 55,71 | 35,26 | 19,01 | 25,24 | (0,2124) |
| 30,00% | 29,52 | 22,35 | 39,97 | 55,98 | 35,30 | 19,42 | 30,95 | (0,1603) |
| 40,00% | 34,72 | 24,73 | 42,66 | 56,35 | 35,36 | 20,37 | 36,89 | (0,1156) |
| 50,00% | 39,96 | 26,52 | 45,84 | 56,89 | 35,44 | 20,60 | 41,72 | (0,0894) |
| 60,00% | 45,05 | 27,59 | 49,39 | 57,68 | 35,56 | 21,72 | 46,68 | (0,0472) |
| 70,00% | 49,88 | 27,87 | 53,21 | 58,87 | 35,72 | 23,66 | 51,78 | (0,0587) |
| 80,00% | 54,39 | 27,38 | 57,17 | 60,70 | 35,93 | 27,38 | 56,61 | (0,0742) |
| 100,00% | 62,32 | 24,41 | 64,93 | 67,24 | 36,45 | 47,90 | 65,31 | (0,0996) |
| Dev. | 1,92 | 19,18 | 8,96 | 22,70 | 14,48 | 17,84 | | |

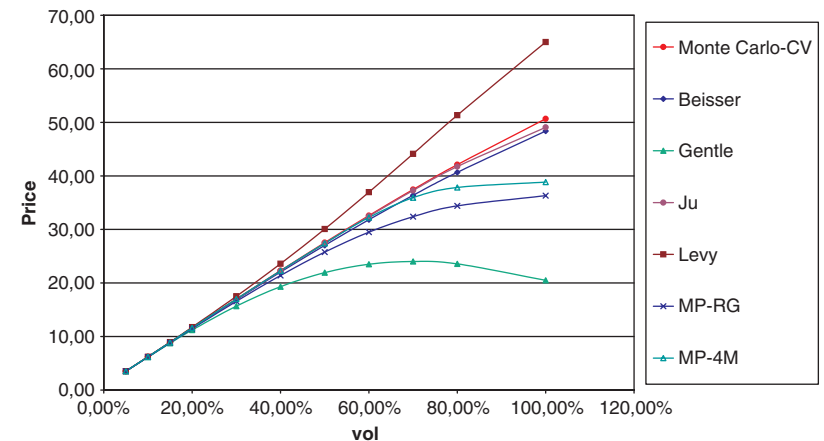


Figure 3: Densities for the standard scenario with $\sigma_1 = 90\%$, $\sigma_2 = \sigma_3 = 50\%$, $\sigma_4 = 10\%$

$n \rightarrow \infty$. But as in our case where $n = 4$ or even in practice with $n = 30$ we are far away from infinity. The geometric mean used in Gentle's approach also seems to be an inappropriate approximation for the arithmetic mean. For instance, the geometric mean of the forwards equal to 1, 2, 3 and 4 would be without mean correction 2.21 instead of 2.5. This is corrected, but the variance is still wrong. The MP-4M four moment method is recommendable only for low vols.

The Ju method is the best approximation except for the case of inhomogeneous volatilities. The reason for this drawback may be that all stocks are "thrown" together on one distribution. This is quite contrary to Beisser's approximation, where every single stock keeps a transformed log-normal distribution and the expected value of every stock is individually evaluated. This is probably the reason why this method is able to handle the case of inhomogeneous volatilities.

A rule of thumb for a practitioner would be to use Ju's method for homogeneous volatilities and Beisser's for inhomogeneous ones. But then the question occurs, how to define the switch exactly. So we suggest the following: Price the basket with Ju and Beisser: If the relative difference between the two computed values is less than 5% use Ju's price for an upper and Beisser's price for a lower bound. If it is bigger than 5% run a Monte-Carlo simulation or if this is not suitable, keep the Beisser result (keep in mind that it is only a lower bound for the prices!).

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