Moments of the Sum of Correlated Log-Normal Random Variables

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Abstract: **This paper presents a comparison of Fenton's and Schwartz and Yeh's methods concerning their capability of predicting the mean and the variance of the sum of a finite number of correlated log-normal random variables. The predictions by the two methods differ more with decreased correlation between the log-normal components, with increasing number of components and increasing component variance. Schwartz and Yeh's method seems to provide more realistic predictions for the mean and the standard deviation of the power sum by showing a closer agreement with the results of an exact formulation for the outage probability in mobile radio systems .**

I. INTRODUCTION

Fading with log-normal statistics is important in many areas of radar and communication systems, where the sum of multiple log-normal random signals may either act as interference to the system, as in cellular telephony, compete with each other to capture a common receiver, as in the case of mobile radio data networks using random access protocols or add up to form the ultimate received signal, as in multi-hop troposcatter systems. In some of these areas of interest, as in cellular mobile radio systems, the individual log-normal signal components may be correlated, when they are shadowed by the same obstacles[1].

The analysis of such systems requires the statistics of the sum of the powers of these signals. There are basically two analytical methods available for determining the mean and the variance of the sum of a number of log-normal random signals: Fenton's method [2] and Schwartz and Yeh's method [3]. These methods are both based on the assumption that the power sum of a number of log-normal signals is also described by a log-normal probability distribution function (pdf). In Fenton's method, the mean and the variance of the power sum are matched with the sum of, respectively, the means and the variances of the individual components. Schwartz and Yeh's method employs a recursive procedure to obtain approximations for the mean and variance of the power sum of a finite number of uncorrelated log-normal variables, based on the exact expressions for the mean and variance of the power sum of two uncorrelated log-normal random variables [3]. This method has been recently extended to the correlated case by Şafak [4]. This paper presents a comparison of these two methods for predicting the mean and the variance of the power sum of a number of correlated log-normal random variables.

II. FENTON'S METHOD

Assume that we want to determine the mean and the variance of the power sum, $\exp(s_k)$, of k log-normal random variables of the form $\exp(y_i)$, $i=1..k$ where y_i is normally distributed with mean m_{yi} and variance σ^2_{yi} :

$$e^{s_k} = \sum_{i=1}^{k} e^{y_i} \qquad (1)$$

where it is implicitly assumed that the sum is also log-normally distributed. Using

$$E[e^{v y_i}] = e^{v m_{y_i} + \frac{1}{2} v^2 \sigma^2_{y_i}} \qquad (2)$$

the first two moments of $\exp(s_k)$ in (1) may be written as

$$e^{m_{s_k} + \frac{1}{2}\sigma_{s_k}^2} = \sum_{i=1}^{k} e^{m_{y_i} + \frac{1}{2}\sigma^2_{y_i}} \qquad (3)$$

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$$e^{2m_{s_k}+2\sigma^2_{s_k}}=E^2[e^{s_k}]+\sum_{i=1}^{k}(e^{\sigma^2_{y_i}}-1)\,e^{2m_{y_i}+\sigma^2_{y_i}}+2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}E[e^{y_i}]\,E[e^{y_j}]\,(e^{\rho_{y_i,y_j}\sigma_{y_i}\sigma_{y_j}}-1) \quad (4)$$

Here, the first term on the right-hand side is equal to the square of the mean of $\exp(s_k)$, the second term denotes the sum of the variances of individual log-normal components and the last term takes into account the correlation between them. For the uncorrelated case, where the last term on the right-hand side disappears, the variance of $\exp(s_k)$ becomes equal to the sum of the variances of the individual log-normal random variables. It is thus evident that either matching the first two moments of $\exp(s_k)$ with those of the individual log-normal variables or writing the mean and the variance of $\exp(s_k)$, respectively, as the sum of the means and variances of the individual log-normal random components give identical results.

Solving the last two equations for m_{sk} and σ_{sk} leads to the following expression for the variance of the power sum of k log-normal random variables :

$$\sigma^2_{s_k}=\ln\left(1+\frac{\sum_{i=1}^{k}\left[e^{\sigma^2_{y_i}}-1\right]e^{2m_{y_i}+\sigma^2_{y_i}}+2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}e^{m_{y_i}+\frac{1}{2}\sigma_{y_i}{}^2}e^{m_{y_j}+\frac{1}{2}\sigma_{y_j}{}^2}\left[e^{\rho_{y_i,y_j}\sigma_{y_i}\sigma_{y_j}}-1\right]}{\left[\sum_{i=1}^{k}e^{m_{y_i}+\frac{1}{2}\sigma^2_{y_i}}\right]^2}\right) \quad (5)$$

which reduces to Fenton's formula [5] for the uncorrelated case, $\rho_{yi,yj}=0$ for $i,j=1\ldots k$ with $i\neq j$, which denotes the correlation coefficient between y_i and y_j.

The mean of the sum of k log-normal random variables is obtained from (3) and (5):

$$m_{s_k}=-\frac{1}{2}\sigma^2_{s_k}+\ln\left(\sum_{i=1}^{k}e^{m_{y_i}+\frac{1}{2}\sigma^2_{y_i}}\right) \quad (6)$$

Note from (5) that the correlation between the components increases the variance of the power sum because of the double summation term, which is positive and accounts for the effect of correlation between the components. Therefore, the variance of the sum, given by (5), will get higher as the components become more correlated with each other. Conversely, the mean of the sum, given by (6), will be lower with increasing correlation between the components, because of the negative first term.

III. SCHWARTZ AND YEH'S METHOD

Using (1), the logarithm of the power sum of k log-normally distributed variables of the form $\exp(y_i)$ may also be written as:

$$s_k=\ln(e^{s_{k-1}}+e^{y_k})=s_{k-1}+\ln(1+e^{w_k}) \quad (7)$$

where $w_k=y_k-s_{k-1}$.

Obviously, (7) lends itself to an iterative procedure by allowing one to find the moments of the sum of k log-normal components in terms of those of the previous k-1 components and of the k'th component. For this purpose, using the last term in (7), the mean value of s_k, $E(s_k)$, may be written as :

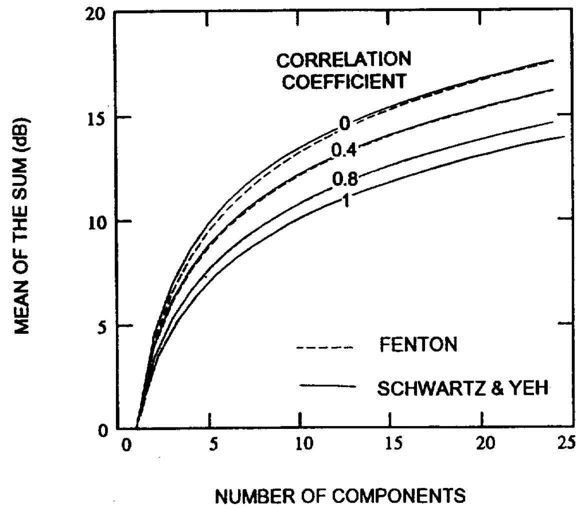
$$m_{s_k}=E(s_k)=m_{s_{k-1}}+G_1(\sigma_{w_k},m_{w_k}) \quad (8)$$

where m_{wk} and σ_{wk} denote, respectively, the mean value and the variance of w_k. Similarly, the variance of s_k is found as follows [4]:

$$\sigma^2{}_{s_k}=\sigma^2_{s_{k-1}}-G_1^2(\sigma_{w_k},m_{w_k})+G_2(\sigma_{w_k},m_{w_k})+2\frac{\sigma_{s_{k-1}}}{\sigma^2_{w_k}}(\rho_{s_{k-1},y_k}\sigma_{y_k}-\sigma_{s_{k-1}})\,G_3(\sigma_{w_k},m_{w_k}) \quad (9)$$

where the functions G_1 ,G_2 , G_3 and the correlation coefficient between s_{k-1} and y_k may be found in [4].

In Schwartz and Yeh's method, the mean and the variance of s_k are obtained directly, instead of those of $\exp(s_k)$ as in Fenton's method. Therefore, Schwartz and Yeh's method is exact for calculating the mean and the variance of two log-normal components, since there is no assumption as to the pdf of their power sum. However, for determining the mean and the variance of, for example, three components, the power sum of the first two components is assumed to be log-normally distributed. Note, however, that Schwartz and Yeh's method is more susceptible to numerical errors since it involves lengthy mathematical computations.



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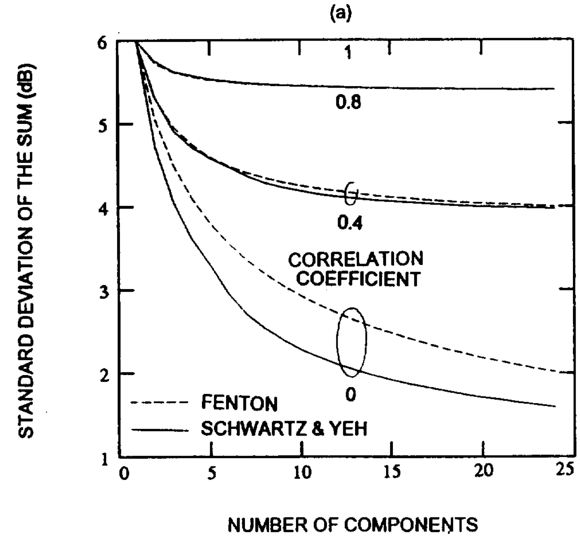
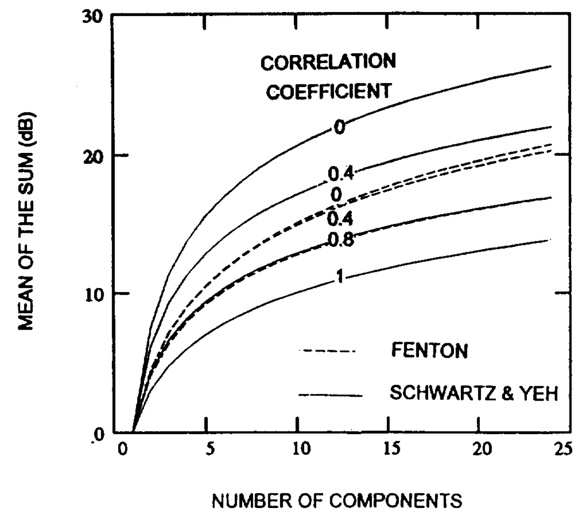


(b)

Figure 1 Mean and standard deviation of up to 24 correlated log-normal random variables each with 0 dB mean and 6 dB standard deviation.

The mean, m_{sk}, and the standard deviation, σ_{sk}, of the sum of k log-normal random variables are shown in Fig.1 as a function of the number of components, each with 6 dB standard deviation and zero dB mean. Fig.2 shows the corresponding results for 12 dB component standard deviation and zero dB mean. Fenton's and Schwartz and Yeh's results were observed to agree better with each other for lower component standard deviation, for smaller number of components, and for higher values of the correlation coefficient between the components. The



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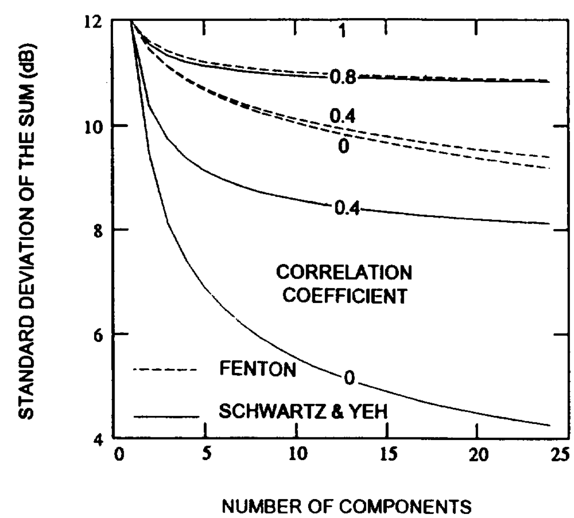

(b)

Figure 2 Mean and standard deviation of up to 24 correlated lognormal random variables each with 0 dB mean and 12 dB standard deviation.

disagreement is maximum when the components are uncorrelated and have large standard deviation. The two methods yield identical results when they are perfectly correlated. In this particular case, the sum signal has a standard deviation equal to the average value of the component standard deviations and a mean equal to the sum of the component means. For identical signal components, the standard deviation and the mean of the sum becomes equal to, respectively, that of a single component and k times the mean of a single component.

V. OUTAGE PROBABILITY

The two methods are compared when they are used in calculating the probability of outage in cellular radio systems. Outage probability is used as a measure of quality of the received signals in these systems. The results are compared with the exact results [6] for the probability of outage due to n uncorrelated co-channel interferers undergoing log-normal shadowing and Rayleigh fading in mobile radio systems (see (11) in [5]):

$$P_n = 1-\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2}dx \prod_{i=1}^{n} \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-y^2}\,dy}{1+\dfrac{\alpha\xi_i}{\xi_d}e^{\sqrt{2}(y\sigma_i - x\sigma_d)}} \quad (10)$$

where α denotes the co-channel protection ratio, and ξ_d and σ_d denote , respectively, the area mean power and the logarithmic standard deviation of the shadowing of the desired signal. Similarly, ξ_i and σ_i denote the corresponding parameters for the i'th interfering signal. In case of a single interferer (n=1), (10) reduces to

$$P_1=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-x^2}\,dx}{1+\dfrac{\xi_d}{\alpha\xi_1}e^{-\sqrt{2\sigma_d^2+2\sigma_1^2}x}} \quad (11)$$

The area mean power has a r^{β} type dependence on the distance between the transmitter and a receiver where β is taken equal to 4. Therefore, by defining a normalized frequency reuse distance, R_u, as the ratio of the distance between the co-channel interferer and the base station to the distance between the user and the base station, the ratio of area mean powers of the desired and interfering signals can simply be expressed as

$$\frac{\xi_d}{\xi_1} = R_u^{\beta}\, e^{m_d - m_1} \quad (12)$$

where m_d and m_1, respectively, denote logarithmic mean powers of the desired and the interfering signals.

Note that the outage probability due to n co-channel interferers is usually calculated by using (11) with ξ_1 and σ_1 being replaced by the area mean power and the standard deviation of the sum of n co-channel interferers (see (22) in [7]). In this case, one can use Fenton's or Schwartz and Yeh's methods to determine these parameters; the logarithmic mean power of the cumulative interference due to n co-channel interferers, m_1, is determined by (6) or (8) and, similarly, the standard deviation due to n co-channel interferers is calculated by using (5) or (9).

The outage probability is plotted as a function of the normalized frequency reuse distance in Fig. 3 for a protection ratio of 10 dB and a standard deviation of 6 dB and 12 dB due to shadowing. Since the exact outage probability calculated by (10) and the approximate outage probability from (11) should be identical in case of a single co-channel

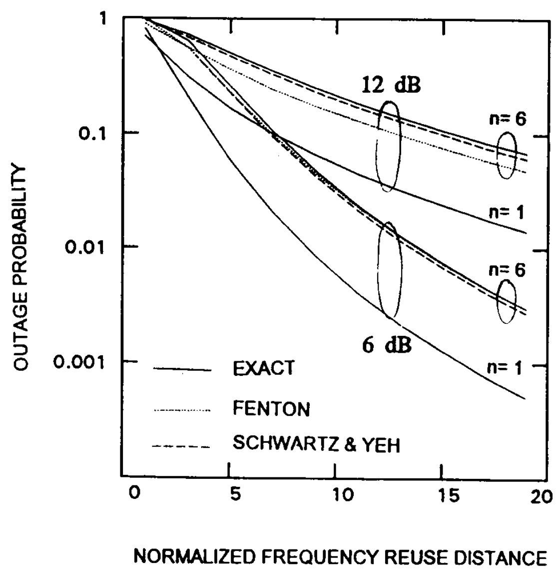

Figure 3 Outage probability versus normalized frequency reuse distance for one and six uncorrelated co-channel interferers.

interferer, the results obtained by Fenton's and Schwartz and Yeh's methods are first observed to be identical with the exact result for one interferer. Therefore, Fig. 2 of [6], showing that exact and approximate outage probabilities are different in case of a single co-channel interferer, is erroneous. In case of six interferers, the predictions by Schwartz and Yeh's method are much closer to the exact outage probability than those by Fenton's method for higher component standard deviations. Approximate results by using Schwartz and Yeh's and Fenton's methods become poorer when the component standard deviation due to shadowing is increased from 6 dB to 12 dB (Figs.1 and 2). In this respect as well, Fig 2 of [5] is erroneous since it shows closer agreement between exact and approximate methods with increased shadowing.

Table I provides a summary of the comparison of Fenton's and Schwartz and Yeh's results with the exact outage probability due to six uncorrelated co-channel interferers for 6 dB and 12 dB component standard deviation. The outage probability found by using Fenton's method is closer to the exact outage probability for lower component standard deviations. However, Schwartz and Yeh's method shows closer agreement with exact results for higher component standard deviations.

TABLE I

EXACT AND APPROXIMATE OUTAGE PROBABILITY

Component standard deviation (dB)	6	12
Exact (10)	$3.36\ 10^{-3}$	$2.16\ 10^{-1}$
Schwartz &Yeh (11)	$3.03\ 10^{-2}$	$1.99\ 10^{-1}$
Fenton (11)	$3.21\ 10^{-2}$	$1.52\ 10^{-1}$

One may thus conclude that, even under worst-case conditions, where the components are uncorrelated and have high standard deviation, (11) yields very accurate results and shows a very close agreement with the exact outage probability, given by (10). In view of Figs.1 and 2, the predictions by (11) will be closer to the exact outage probability, when the signals are correlated. Since the exact solution is suitable for calculating the outage probability for the uncorrelated case only, one may use (11) successfully for determining the outage probability and some other performance parameters like bit error rate, throughput and outage probability in systems such as indoor communications, cellular radio etc. where the signals are likely to be correlated.

V. CONCLUSIONS

This paper presents a comparison of Fenton's and Schwartz and Yeh's methods, concerning their capability of predicting the mean and the variance of the sum of a finite number of log-normally distributed random variables. Schwartz and Yeh's method provides better predictions for the mean and the variance of the power sum for higher component variances.. The difference between the two methods becomes less with increased correlation between the log-normal components, with decreasing number of components, and with decreasing component variance. For unequal component variances, the predictions by the two methods were observed to differ significantly.

VI. REFERENCES

[1] W.C.Y. Lee, Mobile communications design fundamentals, Howard W. Sams Co.: Indianapolis, 1986.

[2] L.F. Fenton, "The sum of log-normal probability distributions in scatter transmission systems", *IRE Trans.*, Vol.CS-8, March 1960, pp.57-67.

[3] S.C. Schwartz and Y.S. Yeh, "On the distribution function and moments of power sums with log-normal components", *Bell System Tech. Journal*, Vol.61, No.7, September 1982, pp.1441-1462.

[4] A.Şafak, "Statistical analysis of the power sum of multiple correlated log-normal components", *IEEE Trans. Veh. Technol.*, Vol.VT-42, No.1, pp.58-61, Feb. 1993.

[5] R. Prasad and J. C. Arnbak, "Enhanced throughput in packet radio channels with shadowing", *Electron.Lett.*, Vol.24, No.16, pp.986-987, 4 August 1988.

[6] J.P.M.G.Linnartz, "Exact analysis of the outage probability in multiple-user mobile radio", *IEEE Trans. Commun.*, Vol.40, No. 1, pp.20-23, January 1992.

[7] R. Prasad and A. Kegel, "Improved assessment of interference limits in cellular radio performance", *IEEE Trans. Veh. Technol.*, Vol. VT-40, No. 2, May 1991, pp.412-419. (see also R.Prasad and A.Kegel, " Correction to 'Improved assessment of interference limits in cellular radio performance' ", IEEE Trans. Vehicular Technology, vol.VT-41, no.4, p.551, November 1992).