# Compressed Sensing Recovery Algorithms

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**There are many compressed sensing (CS) recovery algorithms that have been proposed. Here I list some of them, and provide the corresponding experimental results. *Essentially, the recovery algorithms are similar to the sparse coding based on the over-complete dictionary.* The corresponding Matlab Codes of the following algorithms can be downloaded from my homepage, http://home.ustc.edu.cn/~roy.**

## 1. Orthogonal Matching Pursuit (OMP) [1]

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**Algorithm 1: CS recovery using OMP**

**Input:** The CS observation $\boldsymbol{y}$, and a measurement matrix $\boldsymbol{\Omega} = \boldsymbol{\Phi}\boldsymbol{\Psi} = \{\boldsymbol{\omega}_i, i = 1, 2, \cdots, m\}$ where $\boldsymbol{\Phi} \in R^{n \times m}$ and $\boldsymbol{\Psi} \in R^{m \times m}$.

**Initialization:** Index $\boldsymbol{I} = \varnothing$, residual $\boldsymbol{r} = \boldsymbol{y}$, sparse representation $\boldsymbol{\theta} = \boldsymbol{0} \in R^m$.

**Iteration:**

**while** (stopping criterion false)

$i = \arg\max_j |< \boldsymbol{r}, \boldsymbol{\omega}_j >|$;

$\boldsymbol{I} = \boldsymbol{I} \cup \{i\}$;

$\boldsymbol{r} = \boldsymbol{y} - \boldsymbol{\Omega}(:, \boldsymbol{I})[\boldsymbol{\Omega}(:, \boldsymbol{I})]^{\dagger} \boldsymbol{y}$;

**end while**

$\boldsymbol{\theta}(\boldsymbol{I}) = [\boldsymbol{\Omega}(:, \boldsymbol{I})]^{\dagger} \boldsymbol{y}$;

**Output:** Sparse representation $\boldsymbol{\theta}$, and the original signal $\boldsymbol{x} = \boldsymbol{\Psi}\boldsymbol{\theta}$.

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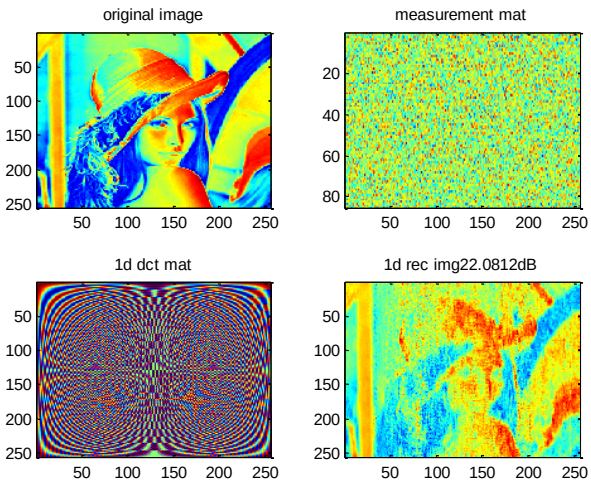


Fig. 1. Recovery result by using OMP

I employ the OMP to recovery an image in the column-by-column fashion, i.e. process each column separately. The recovery result is shown in Fig. 1. The advantages of OMP are easy implementation and fast speed.

## 2. Compressive Sampling Matching Pursuit (CoSaMP) [2]

The foundation of this algorithm is that "*For an s-sparse signal* $\boldsymbol{x}$*, the vector* $\boldsymbol{y} = \boldsymbol{\Phi}^T\boldsymbol{\Phi}\boldsymbol{x}$ *can serve as a proxy for the signal because the energy in each set of s components of* $\boldsymbol{y}$ *approximates the energy in the corresponding s components of* $\boldsymbol{x}$*. In particular, the largest s entries of the proxy* $\boldsymbol{y}$ *point toward the largest s entries of the signal* $\boldsymbol{x}$."

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**Algorithm 2: CS recovery using CoSaMP**

**Input:** The CS observation $\boldsymbol{y}$, a measurement matrix $\boldsymbol{\Omega} = \boldsymbol{\Phi}\boldsymbol{\Psi} = \{\boldsymbol{\omega}_i, i = 1,2,\cdots,m\}$ where $\boldsymbol{\Phi} \in R^{n\times m}$ and $\boldsymbol{\Psi} \in R^{m\times m}$, and the sparsity of signal $\boldsymbol{x}$ based on $\boldsymbol{\Psi}$ is $s$.

**Initialization:** Residual $\boldsymbol{r} = \boldsymbol{y}$, sparse representation $\boldsymbol{\theta} = \boldsymbol{0} \in R^m$.

**Iteration:**

**while** (stopping criterion false)

$\hat{\boldsymbol{\theta}} = \boldsymbol{\Omega}^T\boldsymbol{r}$;

$\boldsymbol{p} = \sup(\hat{\boldsymbol{\theta}}|_{2s})$;

$\bar{\boldsymbol{p}} = \boldsymbol{p} \cup \sup(\boldsymbol{\theta})$;

$\bar{\boldsymbol{\theta}}(\bar{\boldsymbol{p}}) = \boldsymbol{\Omega}^{\dagger}(:, \bar{\boldsymbol{p}})\boldsymbol{y}$;

$\boldsymbol{\theta} = \bar{\boldsymbol{\theta}}|_s$;

$\boldsymbol{r} = \boldsymbol{y} - \boldsymbol{\Omega}\boldsymbol{\theta}$;

**end while**

**Output:** Sparse representation $\boldsymbol{\theta}$, and the original signal $\boldsymbol{x} = \boldsymbol{\Psi}\boldsymbol{\theta}$.

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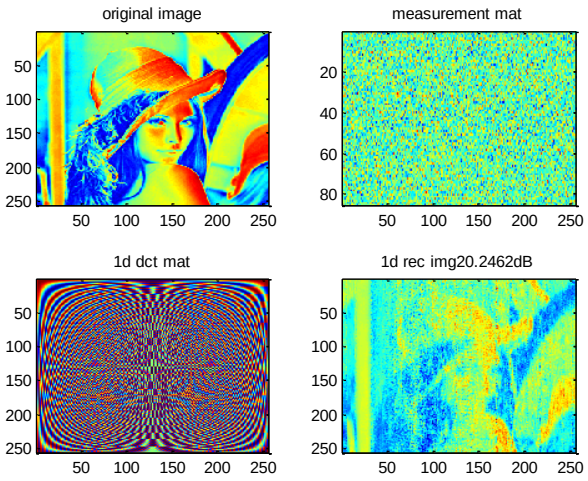

Fig. 2. Recovery result by using CoSaMP

Fig. 2 illustrates the recovered result of CoSaMP. Specifically, the image is recovered in column-by-column fashion, so as to reduce the complexity. My experimental results depict that the OMP has a better performance than the CoSaMP, while the CoSaMP is much more complex than the OMP in the sense of computation.

## 3. Greedy Basis Pursuit (GBP) [3]

The authors declared that the GBP could discard some inappropriate atoms from the selected set of atoms. Note that, this property is not reflected in the following description.

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**Algorithm 3: CS recovery using GBP**

**Input:** The CS observation $\boldsymbol{y}$, and a measurement matrix $\boldsymbol{\Omega} = \boldsymbol{\Phi}\boldsymbol{\Psi} = \{\boldsymbol{\omega}_i, i = 1, 2, \cdots, m\}$ where $\boldsymbol{\Phi} \in R^{n \times m}$ and $\boldsymbol{\Psi} \in R^{m \times m}$.

**Initialization:** Index $\boldsymbol{I} = \varnothing$, sparse representation $\boldsymbol{\theta} = \boldsymbol{0} \in R^m$.

**Procedure:**

$\boldsymbol{n} = \boldsymbol{y} / \| \boldsymbol{y}_2 \|$;

$i = \arg\max_j |< \boldsymbol{\omega}_j, \boldsymbol{n} >|$;

$\boldsymbol{I} = \{i\}$;

$\boldsymbol{\theta}(i) = < \boldsymbol{\omega}_i, \boldsymbol{y} >$;

$\tilde{\boldsymbol{y}} = \boldsymbol{\theta}(i)\boldsymbol{\omega}_i$;

**while** (stopping criterion false)

$d_H = < \boldsymbol{\omega}_{\boldsymbol{I}(1)}, \boldsymbol{n} >$;

$\tilde{\boldsymbol{y}}_H = (d_H / < \tilde{\boldsymbol{y}}, \boldsymbol{n} >)\tilde{\boldsymbol{y}}$;

$\boldsymbol{r} = \boldsymbol{y} - \tilde{\boldsymbol{y}}$;

$\boldsymbol{v} = \dfrac{\boldsymbol{r} - < \boldsymbol{r}, \boldsymbol{n} > \boldsymbol{n}}{\| \boldsymbol{r} - < \boldsymbol{r}, \boldsymbol{n} > \boldsymbol{n} \|}$;

$k = \arg\min_j (\arctan(< \boldsymbol{\omega}_j - \tilde{\boldsymbol{y}}_H, \boldsymbol{n} > / < \boldsymbol{\omega}_j - \tilde{\boldsymbol{y}}_H, \boldsymbol{v} >))$;

$\boldsymbol{I} = \boldsymbol{I} \cup \{k\}$;

$\boldsymbol{n} = - < \boldsymbol{\omega}_k - \boldsymbol{y}_H, \boldsymbol{n} > \boldsymbol{v} + < \boldsymbol{\omega}_k - \boldsymbol{y}_H, \boldsymbol{v} > \boldsymbol{n}$;

$\boldsymbol{\theta}(i) = < \boldsymbol{y}, \tilde{\boldsymbol{\omega}}_i^{\perp} >$, where $i \in I$; // refer the reference [3] for details

$\tilde{\boldsymbol{y}} = \boldsymbol{\Omega}\boldsymbol{\theta}$;

**end while**

**Output:** Sparse representation $\boldsymbol{\theta}$, and the original signal $\boldsymbol{x} = \boldsymbol{\Psi}\boldsymbol{\theta}$.

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According to my experimental results, the performance of GBP, shown in Fig. 3, is better than that of OMP and CoSaMP. However, the GBP is much more complex than OMP and CoSaMP as well.

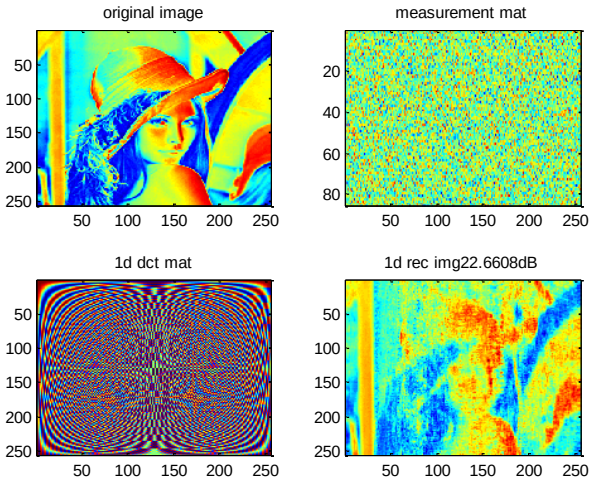


Fig. 3. Recovery result by using GBP

## 4. Iteratively Reweighted Least Square (IRLS) [4]

This algorithm is easy to understand and implement, with the objective function defined as,

$$\min_{\boldsymbol{\theta}} \sum \boldsymbol{w}_i \boldsymbol{\theta}_i^2, \quad s.t. \ \boldsymbol{y} = \boldsymbol{\Omega}\boldsymbol{\theta}.$$

Specifically, the iterative formula of calculating the sparse representation coefficients $\boldsymbol{\theta}$ is,

$$\boldsymbol{\theta}^{(t)} = \mathbf{Q}^{(t)}\boldsymbol{\Omega}^T inv(\boldsymbol{\Omega}\mathbf{Q}^{(t)}\boldsymbol{\Omega}^T)\boldsymbol{y},$$

where $\mathbf{Q}^{(i)}$ is the diagonal matrix with entries $1/\boldsymbol{w}_i$. Here the weight $\boldsymbol{w}_i$ is calculated by

$$\boldsymbol{w}_i = ((\boldsymbol{\theta}_i^{t-1})^2 + \varepsilon)^{p/2-1}.$$

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**Algorithm 4: CS recovery using IRLS**

**Input:** The CS observation $\boldsymbol{y}$, and a measurement matrix $\boldsymbol{\Omega} = \boldsymbol{\Phi}\boldsymbol{\Psi} = \{\boldsymbol{\omega}_i, i = 1, 2, \cdots, m\}$ where $\boldsymbol{\Phi} \in R^{n \times m}$ and $\boldsymbol{\Psi} \in R^{m \times m}$.

**Initialization:** sparse representation $\boldsymbol{\theta} = \boldsymbol{\Omega}^T \boldsymbol{y}$, $p = 1$, $\varepsilon = 1$.

**Iteration:**

**while** (stopping criterion false)

**for** *i* from 1 to end

$\boldsymbol{w}_i = (\boldsymbol{\theta}_i^2 + \varepsilon)^{p/2-1}$;

**end**

$\mathbf{Q} = diag(1./\boldsymbol{w})$;

$\boldsymbol{\theta} = \mathbf{Q}\boldsymbol{\Omega}^T inv(\boldsymbol{\Omega}\mathbf{Q}\boldsymbol{\Omega}^T)\boldsymbol{y}$;

**end while**

**Output:** Sparse representation $\boldsymbol{\theta}$, and the original signal $\boldsymbol{x} = \boldsymbol{\Psi}\boldsymbol{\theta}$.

Fig. 4 shows the recovery result based on this algorithm, as can be seen, there are some improvements comparing with the previous algorithms, in the sense of PSNR.

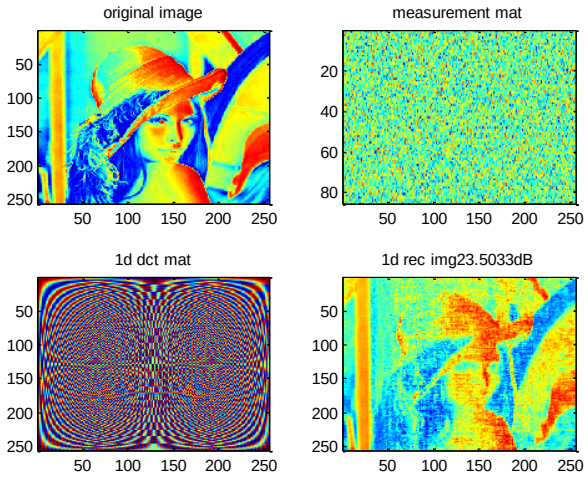


Fig. 4. Recovery result by using IRLS

## 5. Suspace Pursuit (SP) [5]

As declared by the authors, the main difference between SP and CoSaMP is the manner of adding new candidates. More precisely, SP only adds *K* new candidates in each iteration, while CoSaMP adds 2*K*, which makes the SP computationally more efficient, but the underlying analysis more complex.

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**Algorithm 5: CS recovery using SP**

**Input:** The CS observation $\boldsymbol{y}$, a measurement matrix $\boldsymbol{\Omega} = \boldsymbol{\Phi}\boldsymbol{\Psi} = \{\boldsymbol{\omega}_i, i = 1, 2, \cdots, m\}$ where $\boldsymbol{\Phi} \in R^{n \times m}$ and $\boldsymbol{\Psi} \in R^{m \times m}$, and the sparsity of signal $\boldsymbol{x}$ based on $\boldsymbol{\Psi}$ is *s*.

**Initialization:** Residual $\boldsymbol{r} = \boldsymbol{y}$, sparse representation $\boldsymbol{\theta} = \boldsymbol{0} \in R^m$.

**Iteration:**

**while** (stopping criterion false)

$\hat{\boldsymbol{\theta}} = \boldsymbol{\Omega}^T \boldsymbol{r}$;

$\boldsymbol{p} = \sup(\hat{\boldsymbol{\theta}}|_s)$;

$\bar{\boldsymbol{p}} = \boldsymbol{p} \cup \sup(\boldsymbol{\theta})$;

$\bar{\boldsymbol{\theta}}(\bar{\boldsymbol{p}}) = \boldsymbol{\Omega}^{\dagger}(:, \bar{\boldsymbol{p}})\boldsymbol{y}$;

$\boldsymbol{\theta} = \bar{\boldsymbol{\theta}}|_s$;

$\boldsymbol{r} = \boldsymbol{y} - \boldsymbol{\Omega}\boldsymbol{\theta}$;

**end while**

**Output:** Sparse representation $\boldsymbol{\theta}$, and the original signal $\boldsymbol{x} = \boldsymbol{\Psi}\boldsymbol{\theta}$.

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Fig. 5 illustrates the recovery result by using SP, as can be seen, the performance is better than that of CoSaMP in the sense of PSNR.

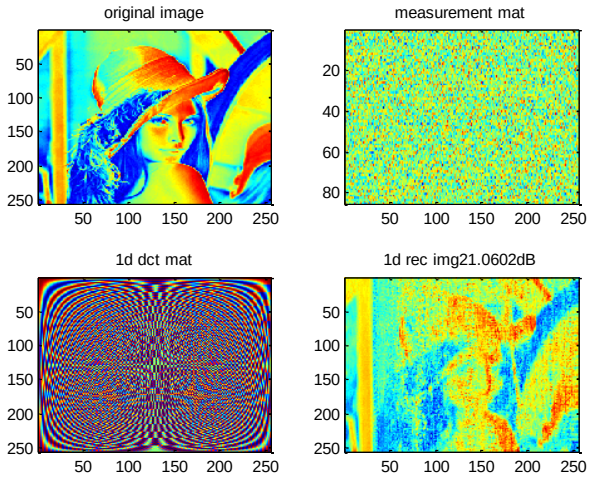


Fig. 5. Recovery result by using SP

## 6. Regularized Orthogonal Matching Pursuit (ROMP) [6]

This algorithm is similar to OMP and SP, however, the main difference is that ROMP has a regularization stage. This stage is described by the authors as "Among all subsets $J_0 \subset J$ with comparable coordinates: $|u_i| \leq 2|u_j|$ for all $i, j \in J_0$, choose $J_0$ with the maximal energy $\|u|_{J_0}\|_2$.". Furthermore, the authors explain the regularization more specifically as "The regularization step of ROMP, i.e. selecting $J_0 \subset J$, can be done fast by observing that $J_0$ is an interval in the decreasing rearrangement of coefficients. Moreover, the analysis of the algorithm shows that instead of searching over all intervals $J_0$, it suffices to look for $J_0$ among $O(\log n)$ consecutive intervals with endpoints where the magnitude of coefficients decreases by a factor of 2".

Here I can't understand the above description well, and can't implement properly.

## 7. Iterative Hard Thresholding (IHT) [7]

With the initialization $\boldsymbol{x}^{(0)} = \boldsymbol{0}$, the iterative procedure of this scheme can be described as,

$$\boldsymbol{x}^{(t+1)} = H_s(\boldsymbol{x}^{(t)} + u\Omega^T(y - \Omega\boldsymbol{x}^{(t)})),$$

where $H_s()$ is the non-linear operator that sets all but the largest s elements to zero.

I implement this scheme according to the above formula. However, the recovery result is terrible, which is depicted in Fig. 6. The reason may be that the operator norm $\|\Omega\|_2$ is not smaller than one, i.e. it doesn't meet the requirement of IHT theoretically.

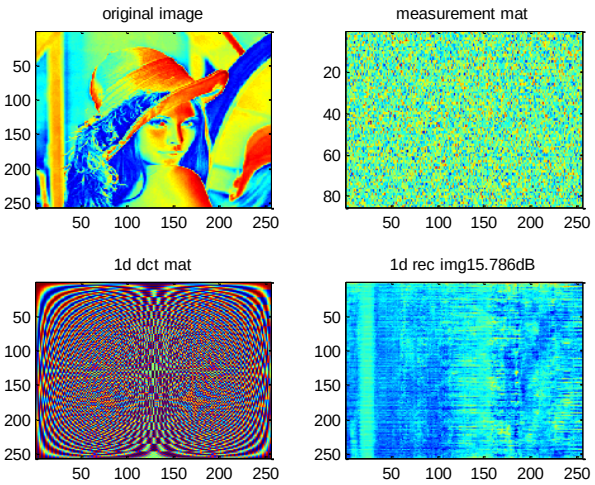


Fig. 6. Recovery result by using IHT

**All the above descriptions and experiments are based on my perception, if you find somewhere inappropriate, please contact with me <roy@mail.ustc.edu.cn>.**

**Thanks.**

## Reference

[1] J. Tropp and A. Gilbert, "Signal Recovery from Random Measurements via Orthogonal Matching Pursuit," 2007.

[2] D. Deedell andJ. Tropp, "COSAMP: Iterative Signal Recovery from Incomplete and Inaccurate Samples," 2008.

[3] P. Huggins and S. Zucker, "Greedy Basis Pursuit," 2006.

[4] R. Chartrand and W. Yin, "Iteratively Reweighted Algorithms for Compressed Sensing," 2008.

[5] W. Dai and O. Milenkovic, "Subspace Pursuit for Compressive Sensing Signal Reconstruction," 2009.

[6] D. Needell and R. Vershynin, "Uniform Uncertainty Principle and Signal Recovery via Regularized Orthogonal Matching Pursuit," 2007.

[7] T. Blumensath and M. Davies, "Iterative Hard Thresholding for Compressed Sensing," 2008.