Robust Fixed Frequency Invariant Beamformer Design Subject to Norm-Bounded Errors

Yong Zhao and Wei Liu

***Abstract*—A novel robust fixed frequency invariant beamformer (FIB) against arbitrary array manifold errors is proposed, by employing the worst-case performance optimization technique with norm-bounded error matrices. A closed-form solution is provided by finding the minimum generalized eigenvector of a pair of matrices. Design examples show that our proposed design significantly outperforms previously proposed FIBs in terms of both frequency invariant property and sidelobe attenuation in the presence of array manifold errors.**

***Index Terms*—Array manifold error, frequency invariant beamformer, robust beamforming.**

I. INTRODUCTION

WIDEBAND beamforming has been studied extensively in the past and one important class of them is the frequency invariant beamformer (FIB) [1]–[8], which can form beams pointing to the signal of interest with a constant beamwidth. With the FIB technique, both adaptive wideband beamforming and wideband DOA (direction of arrival) estimation can be simplified greatly [9].

Recently, a series of least squares based FIB design methods have been proposed with the aid of a frequency invariant constraint called response variation (RV) [6]. Given exact formulation of the array manifold, such methods can achieve a satisfactory frequency invariant property and are applicable to arbitrary array geometries. However, in practice, there are always some mismatch errors in the array manifold, which can severely degrade the beamformer's performance. To tackle the problem, for general wideband beamforming, robust methods have been proposed for both data dependent (adaptive) beamforming [10]–[16], and the data independent (fixed) case [17]–[21].

For robust fixed FIB design, so far most of the methods are based on the assumption that the probability density function (PDF) of the array manifold error is known and the basic idea is to minimize the mean array pattern error with respect to a given PDF. This could be computationally very expensive since multiple integration is needed with a large number of variables. Moreover, the PDF may not be known beforehand for many applications. In this paper, we consider another case of the robust FIB design, i.e., the array manifold error is norm-bounded, which has been studied extensively in robust adaptive beamforming [14], [17]. Firstly we introduce norm-bounded error matrices into the minimization of both the RV element and the sidelobe energy, so that good response consistency and satisfactory sidelobe attenuation can be ensured. Then another norm-bounded error matrix is considered at a reference frequency point for the look direction in order to maintain the desired shape of the mainlobe. Finally a closed-form solution is obtained by finding the generalized minimum eigenvector of a pair of matrices.

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The authors are with the Communications Research Group, Department of Electronic & Electrical Engineering, University of Sheffield, Sheffield, U.K. (e-mail: yong.zhao@sheffield.ac.uk, w.liu@sheffield.ac.uk).

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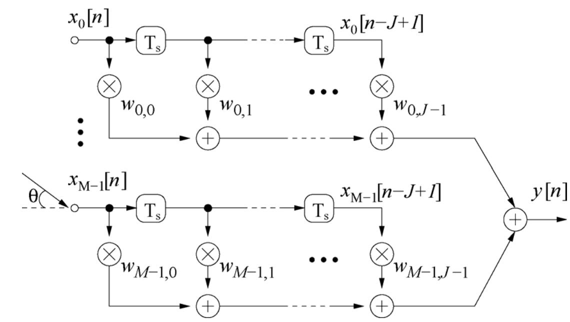

Fig. 1. A general wideband beamforming structure.

II. WIDEBAND BEAMFORMING STRUCTURE

A general structure for wideband beamforming with M sensors and a tapped delay-line (TDL) length J is shown in Fig. 1. Its response as a function of the angular frequency ω and DOA θ can be expressed as

$$\tilde{R}(\omega,\theta)=\sum_{m=0}^{M-1}\sum_{k=0}^{J-1} w_{m,k}^{*}e^{-j\omega(\tau_m+kT_s)}, \tag{1}$$

where $*$ denotes the conjugate operation, T_s is the unit delay in the attached TDLs, τ_m is the delay between the m-th sensor and the zero-phase reference point. In a vector form, we have

$$\tilde{R}(\omega,\theta)=\mathbf{w}^H\mathbf{s}(\omega,\theta), \tag{2}$$

where $\mathbf{w}$ is the coefficient vector defined as

$$\mathbf{w}=[w_{0,0},\ldots,w_{M-1,0},\ldots,w_{0,J-1},\ldots,w_{M-1,J-1}]^T, \tag{3}$$

and $\mathbf{s}(\omega,\theta)$ is the $MJ\times 1$ steering vector given by

$$\mathbf{s}(\omega,\theta)=\Big[e^{-j\omega\tau_0},\ldots,e^{-j\omega\tau_{M-1}},\ldots,e^{-j\omega(\tau_0+(J-1)T_s)},\ldots, e^{-j\omega(\tau_{M-1}+(J-1)T_s)}\Big]^T. \tag{4}$$

For a uniform linear array (ULA) with an inter-element spacing d, we have $\tau_m=\tau_0+m(d/c)\sin\theta$. With the normalized angular frequency $\Omega=\omega T_s$, we further have

$$\Delta_{m,k}=\omega(\tau_m+kT_s)=\Omega(\tau_0/T_s+m\mu\sin\theta+k) \tag{5}$$

with $\mu = d/(cT_s)$. Then we obtain the beam response as a function of Ω and θ

$$R(\Omega,\theta) = \sum_{m=0}^{M-1}\sum_{k=0}^{J-1} w_{m,k}^{*} e^{-j\frac{\Omega\tau_0}{T_s}} e^{-jm\mu\Omega\sin\theta} e^{-jk\Omega} = \mathbf{w}^H \mathbf{s}(\Omega,\theta) \tag{6}$$

with

$$\mathbf{s}(\Omega,\theta) = [e^{-j\Omega\Delta_{0,0}},\ldots,e^{-j\Omega\Delta_{M-1,0}},\ldots, e^{-j\Omega\Delta_{0,J-1}},\ldots,e^{-j\Omega\Delta_{M-1,J-1}}]^T. \tag{7}$$

III. Proposed Robust FIB

Before introducing the proposed robust FIB design, we first study the case without steering vector errors. Given perfect knowledge of the steering vector, the RV element is minimized to ensure good frequency invariant property of the designed beamformer over the frequency-angle range of interest [6]:

$$\min_{\mathbf{w}} \mathrm{RV} = \sum_{\Omega_n\in\Omega_{\mathrm{i}}}\sum_{\theta_k\in\Theta_{\mathrm{FI}}} |\mathbf{w}^H\mathbf{s}(\Omega_n,\theta_k) - \mathbf{w}^H\mathbf{s}(\Omega_{\mathrm{r}},\theta_k)|^2, \tag{8}$$

where Ω_{i} represents the operating frequency range, Θ_{FI} denotes the angle range over which the RV parameter is measured, and Ω_{r} is a reference frequency. When RV is zero, the beamformer has a consistent frequency invariant response over Ω_{i} and Θ_{FI}. Without loss of generality, we consider the case where the frequency invariant response is required over the whole angle range Θ, i.e., $\Theta_{\mathrm{FI}} = \Theta$.

The problem in (8) can be transformed to

$$\min_{\mathbf{w}} \ \mathrm{RV} = \mathbf{w}^H\mathbf{R}_{\mathrm{Q}}\mathbf{w} \tag{9}$$

with $\mathbf{R}_{\mathrm{Q}} = \mathbf{Q}^H\mathbf{Q}$, where

$$\mathbf{Q} = \begin{pmatrix} \mathbf{d}_r^H(\Omega_0,\theta_0) \\ \vdots \\ \mathbf{d}_r^H(\Omega_{N-1},\theta_{K-1}) \end{pmatrix} \tag{10}$$

with $\mathbf{d}_r(\Omega_n,\theta_k) = \mathbf{s}(\Omega_n,\theta_k) - \mathbf{s}(\Omega_{\mathrm{r}},\theta_k)$, and N and K are the number of points uniformly sampled over Ω_{i} and Θ_{FI}, respectively.

In addition to (8), the sidelobe response at the reference frequency Ω_{r} is also required, given by

$$\min_{\mathbf{w}} \sum_{\theta_k\in\Theta_s} |\mathbf{w}^H\mathbf{s}(\Omega_{\mathrm{r}},\theta_k)|^2, \tag{11}$$

where Θ_s denotes the sidelobe region. Similarly, it can be transformed to

$$\min_{\mathbf{w}} \ \mathbf{w}^H\mathbf{R}_{\mathrm{S}}\mathbf{w} \tag{12}$$

with $\mathbf{R}_{\mathrm{S}} = \mathbf{S}_r^H\mathbf{S}_r$, where

$$\mathbf{S}_r = \begin{pmatrix} \mathbf{s}^H(\Omega_{\mathrm{r}},\theta_0) \\ \vdots \\ \mathbf{s}^H(\Omega_{\mathrm{r}},\theta_{K_s-1})^H \end{pmatrix}, \tag{13}$$

and K_s are the number of points uniformly sampled over Θ_s.

Moreover, we need to maintain a unity response (or maximum response) at the desired direction θ_{d}, from which the signal of interest arrives at the array, i.e.,

$$\mathbf{w}^H\mathbf{R}_d\mathbf{w} = 1 \text{ with } \mathbf{R}_d = \mathbf{s}(\Omega_{\mathrm{r}},\theta_{\mathrm{d}})\mathbf{s}^H(\Omega_{\mathrm{r}},\theta_{\mathrm{d}}). \tag{14}$$

Combining the above two minimizations and the constraint in (14), the FIB design problem can be formulated as

$$\min_{\mathbf{w}} \mathbf{w}^H\mathbf{R}_{\mathrm{QS}}\mathbf{w} \text{ s.t. } \mathbf{w}^H\mathbf{R}_d\mathbf{w} = 1 \tag{15}$$

with $\mathbf{R}_{\mathrm{QS}} = (1-\beta)\mathbf{R}_{\mathrm{Q}} + \beta\mathbf{R}_{\mathrm{S}}$, where β is a positive trade-off parameter between the frequency invariant property and the sidelobe attenuation, and its value depends on the specific applications. A larger β will lead to a higher sidelobe attenuation level and more variation in its frequency response. This is a generalised eigenvector problem and equivalent to the ULS (unconstrained least squares) design in [6].

However, in practice, arbitrary errors in $\mathbf{s}(\Omega,\theta)$ are inevitable and the design in (15) will become less effective. Considering arbitrary array manifold errors, the assumed $\mathbf{R}_{\mathrm{QS}}$ in (15) and $\mathbf{R}_d$ in (14) change to

$$\hat{\mathbf{R}}_{\mathrm{QS}} = \mathbf{R}_{\mathrm{QS}} + \mathbf{E}_{\mathrm{QS}}, \tag{16}$$

and

$$\hat{\mathbf{R}}_d = \mathbf{R}_d + \mathbf{E}_{\mathrm{d}}, \tag{17}$$

where $\hat{\mathbf{R}}_{\mathrm{QS}}$ and $\hat{\mathbf{R}}_d$ are the real ones, and $\mathbf{E}_{\mathrm{QS}}$ and $\mathbf{E}_{\mathrm{d}}$ are the Hermitian error matrices with respect to $\mathbf{R}_{\mathrm{QS}}$ and $\mathbf{R}_d$, respectively. We assume the two error matrices are norm-bounded as

$$\|\mathbf{E}_{\mathrm{QS}}\|_F \le \gamma \text{ and } \|\mathbf{E}_{\mathrm{r}}\|_F \le \delta, \tag{18}$$

where γ and δ are both positive constants.

Thus the robust FIB design can be formulated in a similar way as in (15) using the minmax approach [22], [23]:

$$\begin{aligned} &\min_{\mathbf{w}} \max_{\mathbf{E}_{\mathrm{QS}}} \ \mathbf{w}^H[\mathbf{R}_{\mathrm{QS}} + \mathbf{E}_{\mathrm{QS}}]\mathbf{w} \\ &\text{s.t.} \ \min_{\mathbf{E}_{\mathrm{d}}} \ \mathbf{w}^H[\mathbf{R}_d + \mathbf{E}_{\mathrm{d}}]\mathbf{w} = 1 \\ &\qquad \|\mathbf{E}_{\mathrm{QS}}\|_F \le \gamma, \ \|\mathbf{E}_{\mathrm{d}}\|_F \le \delta. \end{aligned} \tag{19}$$

First we solve the following sub-problem of (19):

$$\max_{\mathbf{E}_{\mathrm{QS}}} \ \mathbf{w}^H[\mathbf{R}_{\mathrm{QS}} + \mathbf{E}_{\mathrm{QS}}]\mathbf{w} \text{ s.t. } \|\mathbf{E}_{\mathrm{QS}}\|_F \le \gamma, \tag{20}$$

Since $\mathbf{w}^H\mathbf{E}_{\mathrm{QS}}\mathbf{w}$ is a linear function of $\mathbf{E}_{\mathrm{QS}}$, the maximum of the above cost function occurs at the boundary of the inequality constraint, i.e.,

$$\|\mathbf{E}_{\mathrm{QS}}\|_F = \gamma. \tag{21}$$

Thus the problem in (20) can be solved by the standard Lagrange multipliers method. First we have

$$\frac{\partial L(\mathbf{E}_{\mathrm{QS}},\lambda)}{\partial \mathbf{E}_{\mathrm{QS}}} = 0, \quad \|\mathbf{E}_{\mathrm{QS}}\|_F^2 = \mathrm{Tr}\left(\mathbf{E}_{\mathrm{QS}}^2\right) = \gamma^2 \tag{22}$$

with $L(\mathbf{E},\lambda)$ given by

$$L(\mathbf{E}_{\mathrm{QS}},\lambda) = -\mathbf{w}^H[\mathbf{R}_{\mathrm{QS}} + \mathbf{E}_{\mathrm{QS}}]\mathbf{w} + \lambda\left[\|\mathbf{E}_{\mathrm{QS}}\|_F^2 - \gamma^2\right], \tag{23}$$

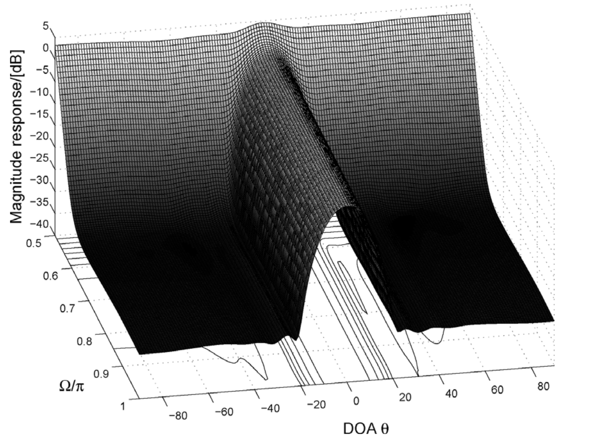

Fig. 2. Resultant average beam pattern obtained by the non-robust FIB design in (15).

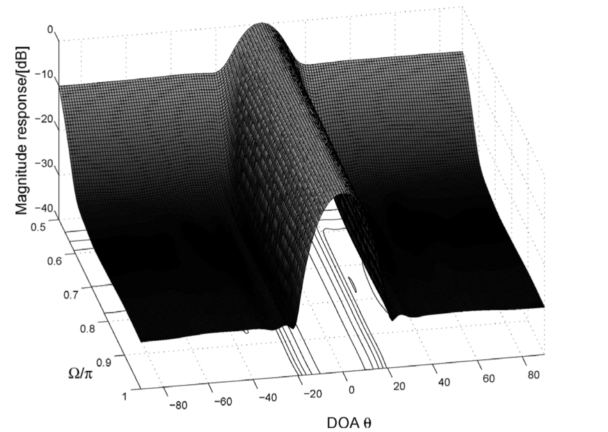

Fig. 3. Resultant average beam pattern obtained by applying the WNG constraint to the FIB design in (15).

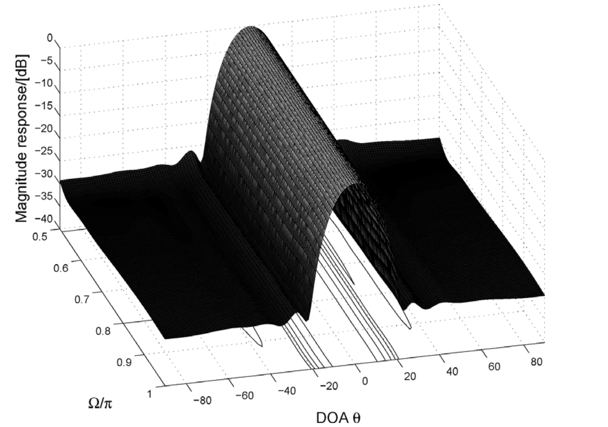

Fig. 4. Resultant average beam pattern obtained by the robust FIB design in (27).

where $\mathbf{E}_{\mathrm{QS}}$ is Hermitian as mentioned earlier and $\mathrm{Tr}\{\cdot\}$ denotes the trace of a matrix. Then we have

$$\mathbf{E}_{\mathrm{QS,opt}} = \gamma \frac{\mathbf{w}\mathbf{w}^H}{\|\mathbf{w}\|_2^2}. \tag{24}$$

Substituting (24) into (20), the objective function in (19) changes to

$$\min_{\mathbf{w}} \quad \mathbf{w}^H[\mathbf{R}_{\mathrm{QS}} + \gamma\mathbf{I}]\mathbf{w} \tag{25}$$

Now we consider the second subproblem of (19), given by

$$\min_{\mathbf{E}_{\mathrm{d}}} \ \mathbf{w}^H[\mathbf{R}_d + \mathbf{E}_{\mathrm{d}}]\mathbf{w} \ \text{s.t.} \ \|\mathbf{E}_{\mathrm{d}}\|_F \leq \delta. \tag{26}$$

With a derivation similar to solving the first subproblem in (20), we can find that when $\mathbf{E}_{\mathrm{d}} = -\delta\mathbf{I}$, $\mathbf{w}^H[\mathbf{R}_d + \mathbf{E}_{\mathrm{d}}]\mathbf{w}$ will reach its minimum, given by $\mathbf{w}^H[\mathbf{R}_d - \delta\mathbf{I}]\mathbf{w}$.

Combined with (25), the original formulation in (19) can be transformed into

$$\min_{\mathbf{w}} \quad \mathbf{w}^H[\mathbf{R}_{\mathrm{QS}} + \gamma\mathbf{I}]\mathbf{w} \ \text{s.t.} \ \mathbf{w}^H[\mathbf{R}_d - \delta\mathbf{I}]\mathbf{w} = 1 \tag{27}$$

The solution to (27) can be obtained by finding the generalized eigenvector corresponding to the smallest generalized eigenvalue of the matrices $\mathbf{R}_{\mathrm{QS}} + \gamma\mathbf{I}$ and $\mathbf{R}_d - \delta\mathbf{I}$. Note that the chosen δ should be smaller than the maximal eigenvalue of $\mathbf{R}_d$. Otherwise, $\mathbf{w}^H[\mathbf{R}_d - \delta\mathbf{I}]\mathbf{w} \leq 0$, and no solution can be found for the problem. Different from the traditional diagonal loading techniques, here the two parameters γ and δ have clear physical meanings and they are the upper bound of the norm of the two error matrices $\mathbf{E}_{\mathrm{QS}}$ and $\mathbf{E}_{\mathrm{d}}$ as given in (18). A larger value for both parameters will lead to increased robustness of the design against a wider range of errors.

Note that the proposed design can be applied to arbitrary array geometries and the TDL structure can also be replaced by the sensor delay-line (SDL) structure [24].

IV. Design Examples

In this section, we provide some design examples to show the effectiveness of the proposed design in (27). As a comparison, results based on the non-robust method in (15) are also provided. Moreover, as pointed in [25], limiting the white noise gain (WNG) of the beamformer will improve its robustness to array model errors. Following the idea, we can add the constraint $\|\mathbf{w}\| \leq \nu$ to the original FIB design in (15) and the result can be obtained easily by some optimisation toolboxes [26]. The following is a list of the design parameters:

- A ULA with $M = 14$ and $J = 20$; $\Omega_{\mathrm{i}} = [0.5\pi, \pi]$, $\Omega_{\mathrm{r}} = 0.8\pi$ and $\theta_{\mathrm{d}} = 0^\circ$.
- d is half the wavelength corresponding to the maximum normalized signal frequency π so that $\mu = 1$.
- $\beta = 0.5$, $\gamma = 1$, $\delta = 0.001$ and $\nu = 2$.
- Regarding the array manifold errors, we consider a scenario with arbitrary phase errors.

With the phase error, the beam response changes to

$$R(\Omega, \theta) = \sum_{m=0}^{M-1}\sum_{k=0}^{J-1} w_{m,k}^{*} e^{-j\frac{\Omega\tau_0}{T_s}} e^{-j(m\Omega\sin\theta + \phi_m)} e^{-jk\Omega} \tag{28}$$

where ϕ_m, $m = 0, \ldots, M-1$ are the phase errors with an independent identical uniform distribution within the range $[-10^\circ, 10^\circ]$.

Figs. 2–4 show the resultant beam patterns for the non-robust FIB design in (15), the FIB design with WNG constraint and the proposed robust design in (27), respectively, obtained by averaging 200 simulation results with respect to the change of phase error. We can see clearly that the robust design exhibits a much better frequency invariant property and sidelobe

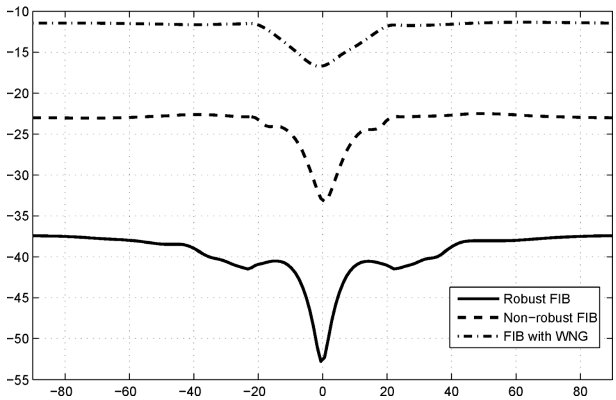

Fig. 5. Variance of magnitude response versus DOA range.

attenuation compared with the other two. In order to measure the frequency invariant property in a quantitative way, we also illustrate the variance of the magnitude response of the resultant beam patterns for the three methods at each sampled DOA angle in Fig. 5, where the proposed method has a very low variance with respect to the change of steering vector in the form of phase errors and therefore gives a very robust design result.

V. Conclusion

A robust FIB against arbitrary array manifold errors has been derived with a closed-form solution provided. Different from the existing robust design where the PDF of the steering vector error is required, in our method we have considered the case with norm-bounded errors. Compared with a previously proposed FIB design method for the ideal case and a design with a simple WNG constraint, significant improvement in both frequency invariant property and sidelobe attenuation in the resultant beamformer has been achieved in the presence of array manifold errors.

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