Parameters Estimation using a Random Linear Array and Compressed Sensing

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Abstract—**Existing sensor array signal processing techniques always use linear arrays sampled at Nyquist rate which demands the inter-element spacing is less than or equal to half of the system wavelength. Only in this way can algorithms such as MUSIC, Capon's beamformer estimate the direction-of-arrival (DOA) of the sources unambiguously. However, in some practical use, it seems difficult to satisfy the spatial sampling theorem.**

Compressed Sensing (CS) is a novel theory which enables perfect recovery of signals and data from what appear highly sub-Nyquist-rate samples on the condition that the signals or data are sparse or compressible in some domain. This implies that for spatially sparse signals, we can design a linear array with sparse property that will enable us to implement DOA estimates or even reconstruct the original signals accurately.

In this paper, we proposed a parameters estimation method based on arrays with sub-Nyquist spatio-temporal sampling. We first sample the signals randomly in spatial domain, which means extracting a finite number of elements from a conventional uniform linear array. Then the signal in each channel is sampled by a random demodulator. By performing CS reconstruction algorithms, not only the DOAs can be estimated, but the original sources' waveforms will be recovered accurately.

Keywords-array signal processing; random array; compressed sensing; DOA estimation

I. INTRODUCTION

Sensor array signal processing problems [1] have long been of great research interest given their importance in a great variety of application such as radar, sonar, mobile communication and .etc. The primary focus of sensor array signal processing lies in the estimation of parameters by fusing temporal and spatial information, captured via sampling a wave field with a set of judiciously placed antenna sensors. The wave field is assumed to be generated by a finite number of emitters, and contains information about signal parameters characterizing the emitters.

A number of parameter estimation techniques in sensor array signal processing use spectral-based approaches by which one forms some spectrum-like function of parameters of interest, e.g., the DOA. The locations of the highest peaks of the function in question are recorded as DOA estimates. Such techniques include beamforming methods like capon's beamformer [2] and subspace-based methods like MUSIC [3] .

In practice, to avoid ambiguity, a uniform linear array with inter-element spacing less than or equal to half of the system wavelength is often used due to the spatial Nyquist sampling theorem. However, for a positioning system with linear arrays whose working frequency ranges from several hundred MHz to several GHz or even to tens of GHz, to adapt to the high frequency means to shorten the distance of two adjacent elements to a very small value, the consequences of which may bring about severe coupling in-between antenna elements. Moreover, the difficulty imposed by installing large-aperture element will also lead to a failure of Nyquist sampling.

Compressed Sensing (CS) theory [4], [5] is a novel data collection and coding theory under the condition that signal is sparse or compressible. It has been shown that when the measurement matrix satisfies certain random properties, the original signal can be reconstructed even when the number of observation is far less than the number of Nyquist rate samples.

In this paper, we proposed a new estimation method using a random linear array and CS technology. [6] shows that under some circumstances, a random sensor array provides sufficient random properties of the spatial sensing matrix that CS demands. Also, we assume that the data model enjoys a high spatial sparsity which means that at one unique time instant, only a finite number of sources impinge on the array from distinct directions. Furthermore, certain emitting signal in itself is sparse, for example, a linear frequency modulation (LFM) signal commonly used in radar application is sparse in a Gabor basis. By making full use of all the information mentioned above, not only can the correct DOAs be estimated, but the waveform corresponds to each azimuth angle will be reconstructed by CS. Compared to other estimation methods, the new method will lead to a reduction of burden in design of the array as well as less strain on the A/D converter used to obtain each snapshot.

II. RELEVANT BACKGROUND

A. Compressed Sensing Theory

Consider the case when a signal $f \in \mathbb{R}^N$ is represented by the columns of $\mathbf{\Psi} \in \mathbb{R}^{N\times N}$, i.e.

$$f = \mathbf{\Psi} x \tag{1}$$

Formally , consider x_s obtained by keeping only the terms corresponding to the S largest components of x in the expansion (1), then for $S \ll N$, x is compressible in the sense that $\| x-x_S \|_2 / \| x \|_2$ is negligibly small; in the case that $x = x_s$ which means only a small set of its components are nonzero, we say that x is S -sparse. In CS, rather than measuring f directly, we acquire it via linear measurements

$$y = \mathbf{\Phi} f \tag{2}$$

where $\mathbf{\Phi}$ is an $M \times N$ matrix representing the sampling system with $M < N$ and $y \in \mathbb{R}^M$ is the vector of measurements. We therefore have

$$y = \mathbf{\Phi\Psi} x_s + \mathbf{\Phi\Psi}(x - x_s) = \mathbf{A} x_s + z \tag{3}$$

An somewhat surprising result of [4] is that there exist explicit designs for $\mathbf{\Phi}$ and hence $\mathbf{A}$ such that x_S can be exactly recovered using a simple and computationally tractable algorithm (l_1 minimization).

To examine the properties of $\mathbf{A}$ that guarantee reliable recovery of x_S , in [7], Candes and Tao introduced the restricted isometry property (RIP) of a matrix. The RIP ensures that a variety of practical algorithms can successfully recover any compressible signal from noisy measurements which is further manifested in ([8], theorem 1.2).

CS theory has an enormous impact in areas where conventional hardware design has significant limitations. For example, wideband analog signals put severe strain on contemporary ADC systems and approach their performance limits. Devices based on CS help to provide alternative solutions. An architecture named random demodulator described in [9] is applicable to a wide variety of sparsity domains, most notably those signals having a sparse signature in time-frequency plane. The idea of a random demodulator is to multiply a signal by a pseudorandom sequence of +-1s, integrate the product over time windows, and digitize the integral at the end of each time interval. An equivalent mathematical description of the system is a sensing matrix $\mathbf{\Phi}$ satisfying RIP which has been proved incoherent with any fixed time-frequency dictionary such as the Gabor dictionary.

The method we proposed utilizes the priori that the signal distribution model enjoys a unique sparsity pattern which makes CS an operational tool for analysis and processing.

B. Signal Modal and Formulation

Consider an antenna array composed of M independent elements labeled 1 to M . Let element 1 be the reference point receiver and the locations of these elements relevant to the reference point be represented as $\vec{r}_i \quad (i = 1, ..., M; r_1 = 0)$. Assuming that all elements have the same directivity and the signal measured at reference element due to a source at azimuthal DOA θ is given by $s(t)e^{j2\pi f_c t}$ .Then the receiving signal at element i takes the form:

$$x_i(t) = s(t - \vec{r}_i^{\,T} \vec{\alpha} / c) e^{j(2\pi f_c t - \vec{r}_i^{\,T} \vec{k})} \tag{4}$$

where the constant c is the speed of propagation and λ is the wave length; $\vec{k}$ is the wave-vector with its magnitude $|\vec{k}| = 2\pi f / c = 2\pi / \lambda$ being the wave number; the unit vector $\vec{\alpha} = \vec{k} / |\vec{k}|$ points in the direction of propagation θ, so the wave-vector can be written as $\vec{k} = |\vec{k}| [\cos\theta, \sin\theta]$; $\vec{r}_i^{\,T} \vec{\alpha} / c$ represents the time delay of the baseband signal measured at the i th element with respect to reference point and $\vec{r}^{\,T} \vec{k}$ is the phase lag. Generally speaking, $s(t)$ is a narrow-banded signal which is slowly time-varying compared to the carrier $e^{j2\pi f_c t}$, for $\vec{r}^{\,T} \vec{\alpha} / c \ll 1 / B$, we can write $s(t - \vec{r}^{\,T} \vec{\alpha} / c) \simeq s(t)$.

Dropping the carrier term $e^{j2\pi f_c t}$ for convenience (in practice, the signal is usually down-converted to baseband before sampling), the output can be modeled as a vector:

$$\begin{aligned} x(t) &\triangleq [x_1(t), x_2(t), ..., x_M(t)]^T \\ &= s(t)[e^{-j\vec{r}_1^{\,T}\vec{k}}, e^{-j\vec{r}_2^{\,T}\vec{k}}, ..., e^{j\vec{r}_M^{\,T}\vec{k}}]^T = a(\theta)s(t) \end{aligned} \tag{5}$$

where $a(\theta) = [e^{-j\vec{r}_1^{\,T}\vec{k}}, e^{-j\vec{r}_2^{\,T}\vec{k}}, ..., e^{j\vec{r}_M^{\,T}\vec{k}}]^T$ is called steering vector. Given fixed array geometry and wave length, the steering vector is only dependent on θ. If N signals impinge on the M -dimensional array from distinct DOAs $\theta_i (i = 1, ..., N)$, the output vector takes the form

$$x(t) = \sum_{n=1}^{N} a(\theta_n) s_n(t) \tag{6}$$

where $s_n(t)$ with $i = 1, ..., N$ denote the baseband signal waveforms. This output equation can be put in a more compact form by defining a steering matrix and a vector of signal waveforms as

$$\begin{aligned} &\mathbf{A}(\theta) = [a(\theta_1), ..., a(\theta_N)] \; (M \times N) \\ &\mathbf{s}(t) = [s_1(t), ..., s_M(t)]^T \end{aligned} \tag{7}$$

Take an additive noise $n(t)$ into consideration, the model becomes

$$x(t) = \mathbf{A}(\theta) s(t) + n(t) \tag{8}$$

III. ESTIMATION METHOD IN THIS PAPER

Figure 1 shows the block diagram of the signal receiving and processing model which describes the method we proposed. We focuse on a class of signals having sparse spatial distributions and the waveform of each signal having an sparse representation in terms of some basis, e.g. Gabor basis. This method uses arrays with sub-Nyquist spatio-temporal sampling, with the steps stated below:

1) Design a random linear array composed of M elements with the inter-spacing larger than or equal to $\lambda / 2$.
2) Apply random demodulator to each element channel.

3) Record the output from all random demodulators and use CS reconstruction algorithms to recover the waveforms corresponding to each source.

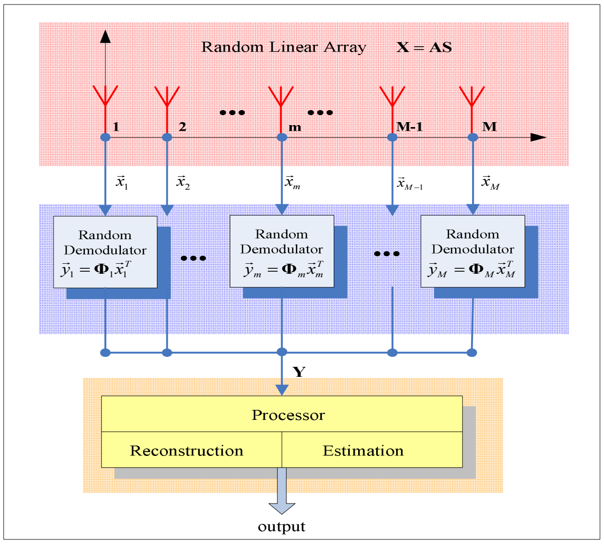

Figure 1. Signal receiving and processing model of the method in this paper.

A detailed description and formulation about this is represented in the following paragraphs.

If the Nyquist rate samples number of the signal received by the m th element is L, that is for $t = 1,2,...L$, we have L snapshots from each element. Let a row vector $\vec{s}_m$ denote $s_m(t)$ in (5) where $m = 1,2,...M$. Then $\vec{s}_m$ can be represented in term of its sparse basis $\mathbf{\Psi}$ as:

$$\vec{s}_m^T = [\mathbf{\Psi}]_{L\times L}\vec{\alpha}_m^T \tag{9}$$

where $\vec{\alpha}_m^T$ is the corresponding column coefficient vector. Then the sources waveforms can be represented as an $N\times L$ matrix $\mathbf{S}$:

$$[\mathbf{S}]_{N\times L} = \begin{bmatrix} \vec{s}_1 \\ \vec{s}_2 \\ \vdots \\ \vec{s}_N \end{bmatrix} = \begin{bmatrix} \vec{\alpha}_1\mathbf{\Psi}^T \\ \vec{\alpha}_2\mathbf{\Psi}^T \\ \vdots \\ \vec{\alpha}_N\mathbf{\Psi}^T \end{bmatrix} = \begin{bmatrix} \vec{\alpha}_1 \\ \vec{\alpha}_2 \\ \vdots \\ \vec{\alpha}_N \end{bmatrix}\mathbf{\Psi}^T = \boldsymbol{\alpha}\mathbf{\Psi}^T \tag{10}$$

where $\boldsymbol{\alpha}$ is a matrix containing the sparse coefficients. The received signals by all M sensor elements can be modeled as

$$[\mathbf{X}]_{M\times L} = \begin{bmatrix} \vec{x}_1^T \\ \vec{x}_2^T \\ \vdots \\ \vec{x}_M^T \end{bmatrix} = [\mathbf{A}]_{M\times N}\bullet[\mathbf{S}]_{N\times L} = \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vdots \\ \vec{b}_M \end{bmatrix}\bullet\mathbf{S} \tag{11}$$

where $\vec{x}_m = \vec{b}_m\bullet\mathbf{S}$ is a $1\times L$ vector received by the mth element. The subsequent step of our method is to simultaneously apply random demodulator to each receiving channel. As has been mentioned in section Ⅱ (A), the random demodulator can be viewed as a linear system acting on the discrete form $\vec{x}_m$ of the continuous-time signal $x_m(t)$. Let a $K\times L$ matrix $\mathbf{\Phi}_m$ ($K < L$) denote the equivalent sensing matrix for each channel, then the measurement vector $\vec{y}_m$ ($K\times 1$ column vector) corresponding to each channel is:

$$\vec{y}_m = \mathbf{\Phi}_m\vec{x}_m^T = \mathbf{\Phi}_m(\vec{b}_m\bullet\mathbf{S})^T = \mathbf{\Phi}_m\mathbf{S}^T\vec{b}_m^T \tag{12}$$

Recalling (10), we can rewrite (12) as

$$\vec{y}_m = \mathbf{\Phi}_m\mathbf{\Psi}\boldsymbol{\alpha}^T\vec{b}_m^T = (\mathbf{\Phi}_m\mathbf{\Psi})\bullet\boldsymbol{\alpha}^T\bullet\vec{b}_m^T \tag{13}$$

We now refer to a theorem in the theory of matrices [10]: Given $\mathbf{A}_{m\times p}$, $\mathbf{B}_{p\times q}$, $\mathbf{C}_{q\times n}$,then

$$vec(\mathbf{ABC}) = (\mathbf{C}^T\otimes\mathbf{A})vec(\mathbf{B}) \tag{14}$$

where vec{.} is a column vector whose elements are taken column-wise from the matrix in the brace; $\otimes$ is the Kronecker product operator. Applying this theorem to (13) ,we get

$$vec(\vec{y}_m) = \left[\vec{b}_m\otimes(\mathbf{\Phi}_m\mathbf{\Psi})\right]\bullet vec(\boldsymbol{\alpha}^T) \tag{15}$$

Combing (15) and $vec(\vec{y}_m) = \vec{y}_m$ (because $\vec{y}_m$ is a column vector), and considering an additive noise term $\vec{n}_m \sim N(0,\sigma^2 I_K)$ for each channel. By stacking all the M measurement vectors, we have

$$[\mathbf{Y}]_{MK\times 1} = \begin{bmatrix} \vec{y}_1 \\ \vec{y}_2 \\ \vdots \\ \vec{y}_M \end{bmatrix} + \begin{bmatrix} \vec{n}_1 \\ \vec{n}_2 \\ \vdots \\ \vec{n}_M \end{bmatrix} = \begin{bmatrix} \vec{b}_1\otimes(\mathbf{\Phi}_1\mathbf{\Psi}) \\ \vec{b}_2\otimes(\mathbf{\Phi}_2\mathbf{\Psi}) \\ \vdots \\ \vec{b}_M\otimes(\mathbf{\Phi}_M\mathbf{\Psi}) \end{bmatrix}_{MK\times LN} [vec(\boldsymbol{\alpha}^T)]_{LN\times 1} + \mathbf{N} \tag{16}$$

$$= \mathbf{H}\bullet[vec(\boldsymbol{\alpha}^T)]_{LN\times 1} + \mathbf{N}$$

As stated in [5], when a measurement matrix $\mathbf{\Phi}\in R^{M\times N}$ in CS is formed by extracting M rows uniformly random from an orthogonal matrix $\mathbf{U}\in R^{N\times N}$, the original signal will be reconstructed with overwhelming probability if the samples number M, the sparsity S and signal length N satisfies

$$M \geq C\mu^2(\mathbf{U},\mathbf{\Psi})S\log(N) \tag{17}$$

for some positive constant C, where μ is the mutual coherence which measures the largest correlation between any row of $\mathbf{U}$ and column of the representation basis $\mathbf{\Psi}$. However, in the case of (16), $\mathbf{H}$ may not be orthogonal any longer, so the condition of (17) does not apply. Equation (16) is generally considered as the sparsest representation in terms of the dictionary $\mathbf{\Xi} = \mathbf{H}$. For dictionary $\mathbf{\Xi}$, similar to the conclusion above, [11], [12] presents the relations of samples number M, the sparsity S and signal length N based on a generalized dictionary representation:

Assume $\mu^2 S \leq [8\ln(N/\varepsilon)]^{-1}, \varepsilon\in(0,1)$, and that for $p \geq 1$,

$3\left(p\ln S/\left(2\ln\frac{N}{\varepsilon}\right)\right)^{1/2}+\frac{S}{N}\|\Xi\|_2^2\le\frac{1}{4e^{1/4}}$, then by using Basis Pursuit [13], the problem under the constraints such as (16) will lead to exact recovery of $\boldsymbol{\alpha}$ with probability $1-2\varepsilon-S^{-p}$. Hence the source waveforms $\mathbf{S}$ will simply be obtained by(10).

IV. SIMULATION RESULT

To validate the method in this paper, we use simulated data with the following parameters setting: The working frequency is set to 1 GHz; Assuming that there are 41 locations uniformly distributed on a line with an interval of half-wavelength. We first form a uniform linear array by putting elements at all the 41 locations. And then we select 30 locations uniformly random whereby to give rise to a random linear array. With the uniform linear array, conventional methods MUSIC and Capon's beamformer were used whereas the method in this paper was applied to the random linear array.

A. *Resolving performance*

We first consider three sources with the same time duration T_r = 2 μs: the first source (labeled source 1) is an LFM signal with bandwidth $B_{r1}=200$ MHz and positive FM rate K_1 (an up-chirp); the second one (source 2) which is very closed to source 1 (only 4 degrees) has a bandwidth $B_{r2}=100$ MHz and a negative FM rate K_2 (a down-chirp); the third one (source 3) has a sinusoidal waveform with frequency $f_c=50$ MHz. The DOAs of these three sources are 17, 21, and 49 degree respectively. Noise corresponding to each channel is assumed to be Gaussian white noise, here we set SNR to 5 dB. Figure 2 shows the spatial spectrum calculated by Capon's beamformer and MUSIC as well as estimation result given by the method in this paper. For 4-degree separation, source 1 and 2 cannot be resolved by Capon's beamformer. However, MUSIC shows its significant performance improvement by giving rise to two peaks corresponding to the correct DOAs. For method in this paper, we use Gabor basis as the sparse basis in (10), and apply TwIST [14] reconstruction algorithm. Then the energy of signal was plotted versus azimuth angle. We also see successfully resolved peaks as MUSIC did.

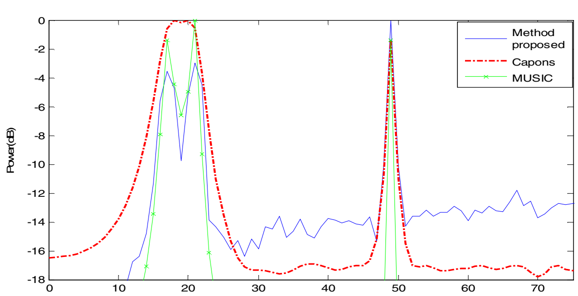

Figure 2. DOA estimation of three sources at 17, 21, 49 degree by conventional approaches with a uniform linear array and the method in this paper with a random linear array where SNR equals 5 dB. The two closely spaced sources are resolved for all algorithms except Capon's beamformer.

Figure 3 shows the case when SNR is decreased to -1.5 dB. Apparently, the two peaks of MUSIC spectrum appeared in Figure 2 has fused into one, however, the method in this paper is not subject to this low SNR limitation and still gives the correct DOA estimation values.

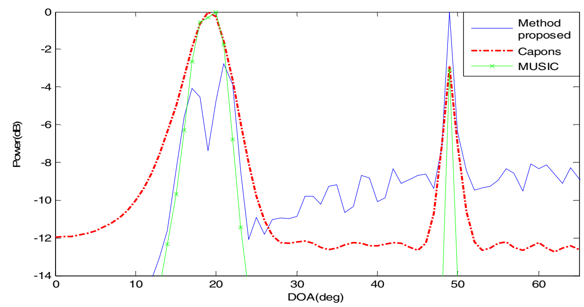

Figure 3. DOA estimation of three sources at 17, 21, 49 degree when SNR equals -1.5 dB. The two closely spaced sources can only be resolved by method proposed in this paper.

To see another unique advantage of our method over MUSIC, we consider the case of closely spaced coherent sources which is not uncommon as either a natural result of a multipath propagation, or intentional unfriendly jamming. Now we place two sources (source 3 and source 4) at the location 17 and 27 degree. and set their waveforms the same as source 1. Again we set SNR to 5 dB. Figure 4 gives the results in this situation. In Figure 4, we find that the MUSIC spectrum fails to produce peaks at the DOA locations, however, for method we proposed, the energy versus angle plot exhibits exquisite peaks at the locations of true DOAs.

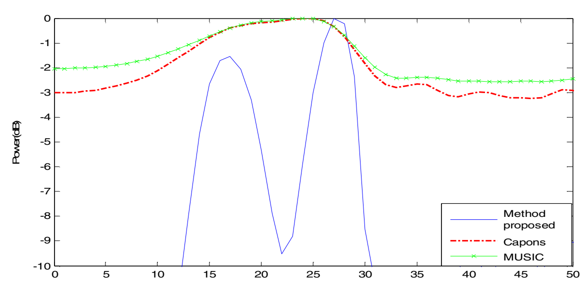

Figure 4. Resolving two coherent sources at 17, 27 degree with Capon's beamformer, MUSIC and method in this paper when SNR equals 5 dB. The method using CS reconstruction has a much weeker sensitivity to coherent sources compared to MUSIC.

B. *Estimation of signal characteristics*

Remember that the output of the method in this paper is not limited to mere DOA estimates. A useful byproduct is the reconstructed signal which contains characteristics that we may want to obtain. Figure 5 shows the reconstructed version of signals corresponding to source 1~3. The waveforms of source 1 and source 2 exhibit some FM rate characteristics though in the presence of noise introduced by receiving channels and reconstruction process, as shown in Figure 5 (a) and (b). Figure 5(c) is the reconstructed waveform of source 3 which accords with a sinusoidal signal.

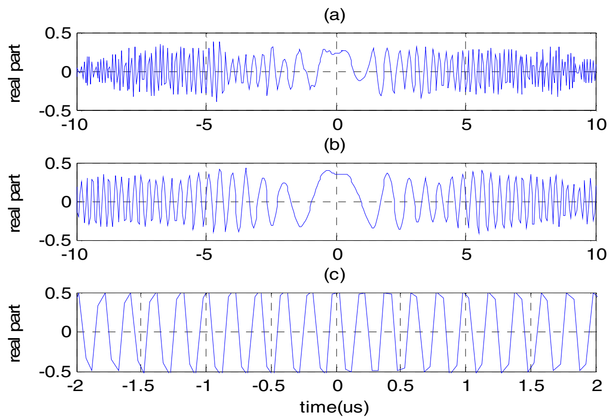

Figure 5. Reconstructed waveforms at different DOA locations. (a) source 1 at DOA 17 degree. (b) source 2 at DOA 21 degree. (c) source 3 at DOA 49 degree.

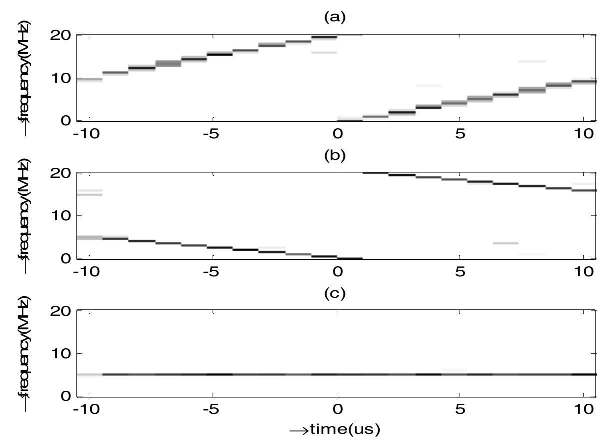

Figure 6. Time-frequency representation of three signals in Figure 4 by Gabor transform. (a) source 1 at DOA 17 degree. (b) source 2 at DOA 21 degree. (c) source 3 at DOA 49 degree.

A somewhat precise extraction of characteristics of interest can be done with existing analytic tool like time-frequency analysis. Figure 5 illustrate the case when Gabor transform is applied to the three reconstructed signals. In Figure 6(a), a linear frequency versus time characteristics can clearly be seen with the slope signifying the FM rate. This slope can further be estimated with the assistant of straight line extraction techniques, such as the Hough or Radon transforms. Thus the bandwidth of the signal is simply the product of the FM rate and the time duration. Here, it is worth noting that the frequencies with respect to negative time instants of source 1 are actually negative whereas they are aliased in our observing frequency region as shown in the left half part of Figure 6(a). Similar conclusions can be drawn from Figure 6(b). As in the case of Figure 6(c), a constant frequency excursion clearly indicates the essence of a single tone used by source 3 and hence the frequency of this monochromatic signal can be estimated.

V. CONCLUSIONS AND FUTURE WORK

In this paper, we apply the CS scheme to the sensor array signal processing problem and proposed an estimation method with a random linear array. Simulation result demonstrated the effectiveness of this method. Compared to conventional estimation algorithms, this method will not merely estimate DOAs but also reconstruct the original signals. By utilizing proper approaches, the information embedded in the reconstructed signals will be further dug out.

By using a random linear array, the limitation of a uniform array has disappeared. Combined with CS, the new method will not only lead to a reduction of burden in design of the array but also result in less strain on the A/D converter used to obtain each snapshot.

A random linear array provides sufficient sampling of spatial signals for latter CS reconstruction. The linkage allows the use of existing CS theory to quantify the performance of random arrays and thus give explicit and quantitative guide lines to design random arrays. A further discussion of this will also be involved in future work.

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