Beamforming using compressive sensing

Geoffrey F. Edelmann[a] and Charles F. Gaumond
Naval Research Laboratory, 4555 Overlook Avenue West, Code 7140, Washington, DC 20375
geoffrey.edelmann@nrl.navy.mil, charles.gaumond@nrl.navy.mil

Abstract: Compressive sensing (CS) is compared with conventional beamforming using horizontal beamforming of at-sea, towed-array data. They are compared qualitatively using bearing time records and quantitatively using signal-to-interference ratio. Qualitatively, CS exhibits lower levels of background interference than conventional beamforming. Furthermore, bearing time records show increasing, but tolerable, levels of background interference when the number of elements is decreased. For the full array, CS generates signal-to-interference ratio of 12 dB, but conventional beamforming only 8 dB. The superiority of CS over conventional beamforming is much more pronounced with undersampling.

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1. Introduction

Beamforming has achieved wide success in passive sonar applications such as the detection and localization of sound sources.[1,2] Plane wave beamforming is commonly applied to ship-towed horizontal arrays, where it is used to detect targets, determine target bearing, and enhance signal-to-noise ratio. Specifically, it is a technique for the estimation of the spatial Fourier wavenumber spectrum from measurements of a spatially varying acoustic field. Discrete peaks in the spectrum are associated with sources of sound localized in space.[3] However, beamforming is a conventional l_2 technique and requires that measurements be properly sampled; i.e., the array must have at least twice as many samples as the highest wavenumber in order to prevent aliasing. In this paper, the compressive sensing technique is applied to bearing estimation using a towed array. Results demonstrate that compressive sensing provides better estimates of target wavenumbers (bearing) while only requiring approximately one eighth of the number of measured samples as conventional techniques.

In compressive sensing theory, l_1-minimization has proven to be an efficient method for exactly reconstructing the sparsest solution to underdetermined problems of linear equations.[4] Let there be an unknown signal of interest, $x_o \in \mathbb{R}^N$. The signal is sparse: it has only a few values that are significantly larger than the rest. It is possible to generate a vector of measured data, $y \in \mathbb{R}^M$, via the linear projection $\mathbf{A}x = y$, where the sensing matrix $\mathbf{A}$ is chosen such that it has rank M and dimension $M < N$. Unlike conventional l_2 techniques, this problem can be solved even though the measurements in y are fewer than the dimension of x when x_o is sufficiently sparse—namely that the number of nonzero entries of x is K, and $K \ll M$. The sparsity-sensing properties of this technique have made it ideal for such applications as bandwidth compression, image recovery, and signal recovery.[5,6] In this paper an alternating direction algorithm is used to solve this convex optimization problem.[7] Specifically, the YALL1 algorithm was selected after comparison with other currently available algorithms, and found to produce repeatable inversion results for beamforming using compressive sensing.[8] The interface of YALL1 enabled simpler implementation with existing code.

[a]Author to whom correspondence should be addressed.

2. Beamforming using compressive sensing

An algorithm is proposed for passive bearing estimation where the number of sources and their source signals are considered unknown in the presence of noise. An unknown but limited number of targets is assumed to be distributed in bearing (look angle) with unknown bearings and magnitudes. The receiver consists of a linear array (either horizontal or vertical) that has arbitrary inter-element spacing. Let r be the vector of receiver positions. The classical discretized equation for a beamformer is

$$A_{nm} = w_m \exp[ik \sin(\phi_n) r_m] \tag{1}$$

where $\mathbf{A}$ is the phase and amplitude weights calculated for any frequency $f = kc/2\pi$, and c is the soundspeed of the medium. The look directions, ϕ_n, range from 0 to π radians with $\pi/2$ being defined as broadside to the array. Although not necessary, the array may be shaded by w (e.g., hanning, hamming, or box car). For an M element array, N look directions are assumed in the beam domain. Here the seemingly underdetermined problem where $N > M$ is considered. The sparsity solution x comprises the angle and amplitude information of the $K < M$ targets present and can be estimated by the following l_1 minimization problem:

$$\hat{x} = \min \| x \|_{1,} \quad \text{subject to } \mathbf{A}x = y, \tag{2}$$

where y is a vector of measurements $y = [y_1, y_2, \ldots, y_m]^T$ and $\hat{x}$ is the solution of the unknown x.

2.1 Broadband estimation of an unknown source signal

In passive acoustic problems, the source function of the target of interest is often considered unknown. Therefore, a coherent broadband $\mathbf{A}$ matrix cannot be constructed to search for the sparse look direction vector $\hat{x}$. The approached used here is to solve each frequency independently, and to average incoherently the solutions producing a final beamformer output of $\hat{x} = \sum_{j=1}^{J} \left| \hat{x}(f_j) \right|$ for all J frequencies.

2.2 Stability of A

The restricted isometry property (RIP) relates to the orthonormality of the columns of the $\mathbf{A}$ matrix for sparse recovery.[9] The RIP condition is a necessary, though not sufficient, condition for signal recovery. The k-restricted isometry constant is defined as the smallest δ_k value that satisfies

$$(1 - \delta_k) \| b \|_2^2 \leq \| Ab \|_2^2 \leq (1 + \delta_k) \| b \|_2^2 \tag{3}$$

for every possible k-sparse vector b. Additionally, the k, k'-restricted orthogonality constant $\Theta_{k,k'}$ is defined as the smallest value that satisfies

$$|\langle Ab, Ab' \rangle| \leq \Theta_{\kappa\kappa'} \| b \|_2 \| b' \|_2 \tag{4}$$

for all b and b' such that they are k-sparse and k'-sparse, respectively, and have disjoint supports. For the data analysis presented in this paper, the RIP condition always satisfied the inversion-without-noise criteria, namely, $\delta_{1.75k} < \sqrt{2} - 1$, where $k = 4$ sparse targets were present.[10] The recovery of sparse signals in the presence of bounded error is possible if $\delta_{1.25k} + \Theta_{k,1.25k} < 1$. This condition was almost met; a value of 1.05 is obtained for the fully populated array case. However, numerical inversions remained stable to noise without use of the l_2 constraint (i.e., noisy compressive sensing[11]).

Most linear arrays have inter-element spacing designed for a specific frequency. When such an array is used at a lower-than-design frequency, $\mathbf{A}$ becomes ill-conditioned. Compressive sensing inversions are seeded with an l_2 estimate, therefore even in the case of an arbitrarily spaced array, the rank of $\mathbf{A}$ may be insufficient, leading to l_2 inversion instabilities and a poorly defined starting point. In order to improve the rank of **A**, while retaining potentially valuable data in y, the use of a singular value decomposition (SVD) to intelligently precondition the basis set and data is used here. The SVD of **A** (where ^{H} indicates the conjugate transpose) produces:

$$\mathbf{U}\mathbf{S}\mathbf{V}^H x = y. \tag{5}$$

The diagonal of **S** contains the singular values of **A**. For the data analysis presented in this paper, all singular values < 1/100 of the maximum value are set to zero. Thus, the size of the **A** matrix will be reduced from $N \times M$ to $N \times P$, where P is the number of retained singular values of **S**. After reducing the **U** and **S** matrices to their non-zero columns and rows (denoted by $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{S}}$, respectively):

$$\tilde{\mathbf{U}}^H \tilde{\mathbf{U}} \tilde{\mathbf{S}} V^H x = \tilde{\mathbf{U}}^H y, \quad \text{or equivalently } \tilde{\mathbf{A}} x = \tilde{y}, \tag{6}$$

$$x = \arg \|x\|_1, \quad \text{subject to } \tilde{\mathbf{A}} x = \tilde{y}. \tag{7}$$

Thus, the $\tilde{\mathbf{A}}$ matrix is ensured to have a stable condition number for inversion and all of the (redundant) data are maximally utilized. Additionally, reducing the size of **A** and y also has the effect of decreasing the computational time required for inversion results. The SVD can also generate orthonormality, and thereby improve the RIP condition, though, in this case, the beam matrix **A** is normal as defined.

3. Underwater acoustic results from the BASE07 experiment

Beamforming using compressive sensing is demonstrated on at-sea data measured during the BASE07 experiment.[12,13] During this experiment, the Five Octave Research Array (FORA)[14] was towed behind a ship near the Malta Plateau. The FORA consists of several nested arrays of which, for the purpose of this paper, the mid frequency element spacing array is considered (design frequency 1000 Hz, 0.75 m spacing, $M = 64$, $f_s = 8000$ Hz). During Julian Day 135 at approximately 13:15 UTC, the towed array received a loud, broadband FM broadcast by another ship. A 60 s spectrogram of the first element of the FORA array is shown in the upper left panel of Fig. 1, which depicts the FM arrival at approximately 14 s. Though not apparent in the background noise of the spectrogram, at least four other ships may be present.

3.1 Conventional beamforming results

Conventional beamforming results are shown in the upper right panel of Fig. 1. Beamformer output is shown as a function of time and look angle. The 60 s time series is divided into time steps of 2048 points (0.256 s) with no overlap. At each time step, $N = M = 64$ number of beams were calculated and applied to the measured data. The final beamformer output is the incoherent average from 800–1000 Hz ($df = 3.9$ Hz). The loud FM sweep can be seen at approximately 14.15 s and at -31°. Additionally, there appears to be at least 4 sources of opportunity made at bearing angles -61°, -44°, 7°, and 32°. One of the three sources (7°) clearly changes bearing by the end of the minute. The signal strength of the FM sweep is approximately 30 dB stronger than the sources, thus the dynamic range of the plot is set to 60 dB.

3.2 Compressive sensing beamforming results

Beamforming results using compressive sensing are shown in the lower right panel of Fig. 1. The same time steps are used; however, at each time step, $N = 128 > M$ number of beams were calculated and applied to the measured data. Conventionally, this is a highly underdetermined problem; however the compressive sensing assumption of sparseness and use of l_1 minimization requires significantly less information to achieve a unique and meaningful solution ($2N$ vs $K \log N$). Compressive sensing can yield greater angular resolution than is conventionally possible for a given array of M

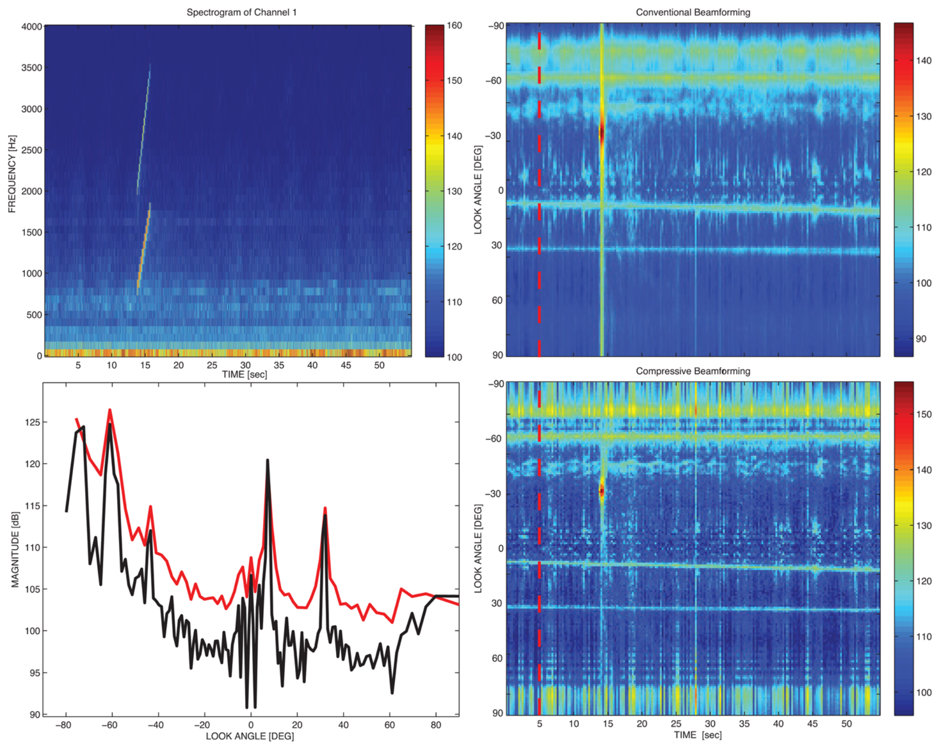

Fig. 1. (Color online) BASE07 processed results. The spectrogram of the first element of the towed FORA array shows a loud FM arrival at approximately 15 s. The right most panels show beamformer outputs incoherently averaged from 800 to 1000 Hz. The upper panel depicts conventional beamforming results and the bottom shows compressive sensing results both over 60 dB dynamic range. Compressive sensing yields smaller angular resolution, reduced angular spillage of the strong FM sweep, and tighter endfire energy. A time slice of the beamformer outputs at 5 s is depicted in the bottom left panel. Comparing the peaks and valleys of the compressive sensing (lower) and conventional (upper) results show that compressive sensing has superior interference and noise suppression (note conventional results raised by +9 dB to align peak).

elements. This is seen in the beam-former outputs in the right two panels in Fig. 1: the ship lines at about 7$^\mathrm{o}$ and 32$^\mathrm{o}$ are more narrow and focused in the lower right hand panel than the upper right hand panel. Compared with the conventional beamformer, the loud FM is more tightly focused in angle with less leakage into other beams. Additionally, the tow ship noise measured at endfire appears more tightly contained at higher angles (the endfire lobe being effectively smaller). Note that the compressive sensing beamforming results yield an accurate total receive level, and it is possible to "listen" to a bearing angle by taking an inverse Fourier transform across frequencies.

Noise (distributed in angle) and interference are significantly suppressed with compressive sensing beamforming. The bottom left hand panel of Fig. 1 depicts the beamformer output at 5 s (dashed lines on the right panels). The conventional beamforming results were offset by 9 dB to align the peak at 7.3$^\mathrm{o}$. Comparing peaks to valleys, compressive sensing provides a greater main to interference (and noise) ratio. Conceptually, this is possible because the total energy of the input signal is not required to be preserved with l_1 beamforming. The improvement is quantified for the target at bearing 32$^\mathrm{o}$ using the definition of signal-to-interference ratio (SIR),

$$\mathrm{SIR} = 20\log_{10}\left(\frac{\max[|x(30 \le \phi \le 35)|]}{\mathrm{std}[|x(25 \le \phi \le 30, 35 \le \phi \le 40)|]}\right). \tag{8}$$

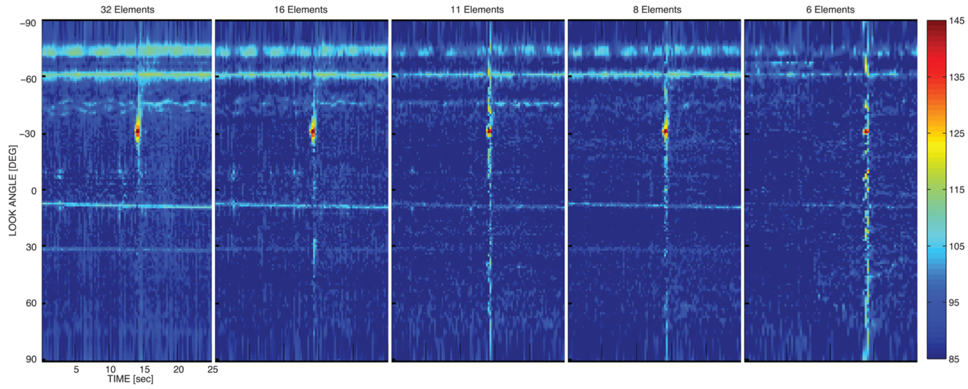

Fig. 2. (Color online) Bearing versus time results for the first 25 s of data. The total number of array elements (not aperture) is reduced toward the right. Despite the array being reduced from the original 64 elements (half wavelength spacing) to as little as 6 randomly selected elements, compressive beamforming is able to detect targets of interest.

The SIR produced with compressive sensing is approximately 12 dB, while the SIR with conventional processing is approximately 8 dB. Thus, the gain generated through compressive sensing is approximately 4 dB. Note also that at the peak (FM sweep), compressive sensing produced a 9 dB gain. Thus, with sufficient signal-to-noise ratio, compressive sensing can strongly outperform conventional processing results.

Compressive sensing is also able to reconstruct a sparse signal with fewer measurements than would be expected from traditional sampling theory.[6] Figure 2 shows the performance degradation due to reduced numbers of towed array hydrophones. Each panel depicts the first 25 s of broadband compressive beamforming with different numbers of randomly chosen hydrophones. Note that the first and last hydrophones are always included to maintain the array aperture. The first panel shows results using 32 out of 64 possible hydrophones, with the subsequent panels depicting 16, 11, 8, and 6 hydrophones, respectively. The results show time-bearing estimates without aliasing even when the array is reduced to 8 hydrophones. For the cases shown in Fig. 1, the values of SIR are 12, 9, 10, 11, and 12 dB respectively. Note that the color scale ranges from 85 dB to 145 dB and shows the relative amplitudes of the various components. This large dynamic range does not clearly show SIR with fewer hydrophones. Not shown are the results from conventional beamforming: This method predictably fails to isolate the line at 32^{o} with SIR values of 1, 1, 1, 0, and 0 dB, respectively. Note that if the elements of the array were to be equally spaced, then the spacing would have equivalently produced an array of approximately λ, 2λ, 3λ, 4λ, and 5λ spacing, respectively. Compressive sensing inversion became unstable when using the 6 element array, therefore reducing the array further became to intractable. Also of interest, the 32 element results are slightly more superior than the 64 element results. It is hypothesized that this is due to the equal spacing of the full 64 hydrophone array; compressive sensing works best with an array that has been sampled uniformly at random so that a robust uncertainty principle prevails.[5]

4. Conclusion

Compressive sensing is a novel approach useful to exactly reconstruct a sparse signal with fewer measurements than defined by traditional sampling theory. This paper presents the application of compressive sensing to beamforming of measured underwater acoustic data that is sufficiently sparse in bearing angle. The implemented algorithms have the ability to improve visibility of lines in time-bearing displays. Compared with conventional beamforming, compressive sensing shows finer angular resolution and greater interference suppression when applied to measured at-sea data.

Furthermore, the technique is demonstrated to work on a sub-sampled array with no loss in SIR performance, even with a number of elements that would equivalently produce a three-wavelength spaced array (where conventional methods require half wavelength). Therefore, the method allows the use of arrays with dead hydrophones, or allows the possibility of acoustic arrays to be manufactured with fewer elements.

Acknowledgments

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