A Network Formation Game

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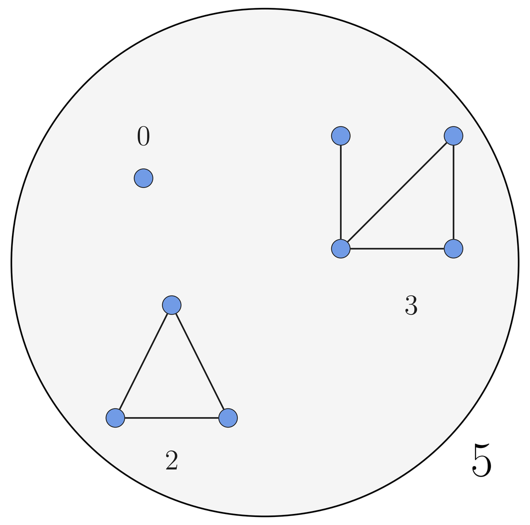

Figure 1: A graph with labelled coalition values. The graph has value 5.

In cooperative games, players separate into groups called coalitions, which are assigned value. We study a cooperative game in which players are vertices of a graph and coalitions are connected components of the graph. A coalition has value equal to its size minus one, rewarding players for connecting large communities. A graph has value equal the sum of its components' values, which is the number of players minus the number of components.

A player can change their coalition by altering their connections, however, cooperative games do not sufficiently describe such individual behaviour. To allow for unilateral decision making, we define an *allocation function*, which allocates a coalition's utility to its members. Thus, players can decide to join the coalition that provides them with the most value.

A Competitive Game

To understand our allocation function, consider the players in a given graph showing up in a random order. Given an ordering, a player's marginal value is the amount they affect the value of the pre-existing graph, upon their arrival. As our allocation function, we define a player's value as the average of their marginal value over all possible orderings. This is known as the Shapley-Myerson Value.

The figure below shows Shapley-Myerson values for a graph on four vertices, and an example calculation for the permutation (4,1,2,3). Player 4 arrives first and connects no components, and thus has marginal value 0. Player 1 arrives second, but doesn't connect any components either, and also has marginal value 0. Player 2 arrives third, and connects up 2 components, given him marginal value 2. Player 3 arrives last and connects to 1 component, and has marginal value 1. Making similar marginal value calculations for the other 23 permutations and averaging these marginal values yields the values shown in the leftmost graph below.

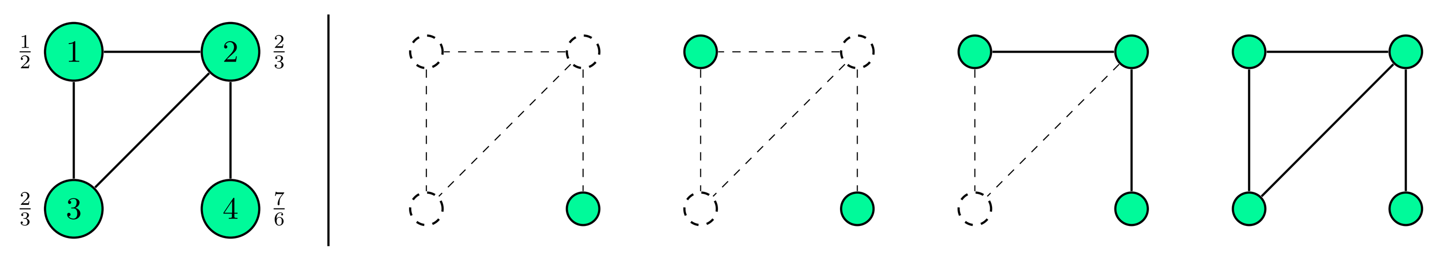

Figure 2: A Small Example

For a given player i, this process is equivalent to calculation of the following formula;

$$\varphi_i(G) = \sum_{S\subset N\setminus\{i\}} \left(\frac{|S|!(|N|-|S|-1)!}{|N|!}\cdot\left[v(G|_{S\cup\{i\}}) - v(G|_S)\right]\right),$$

where N is the set of all players, and $v(G_S)$ is the value of the graph G where only the players in S show up.

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A Network Formation Game

Realistically, players want to form and break connections to other players to improve their value. When players do this, a network emerges. In general, games where players selfishly and myopically form connections are known as *network formation games*. The sequence of improving moves made by players is known as *best response dynamics*. When (and if) best response dynamics terminates, we are at a graph in which every player is satisfied with their position. These graphs are known as *equilibria*, and studying their properties is a major goal in the study of any new game.

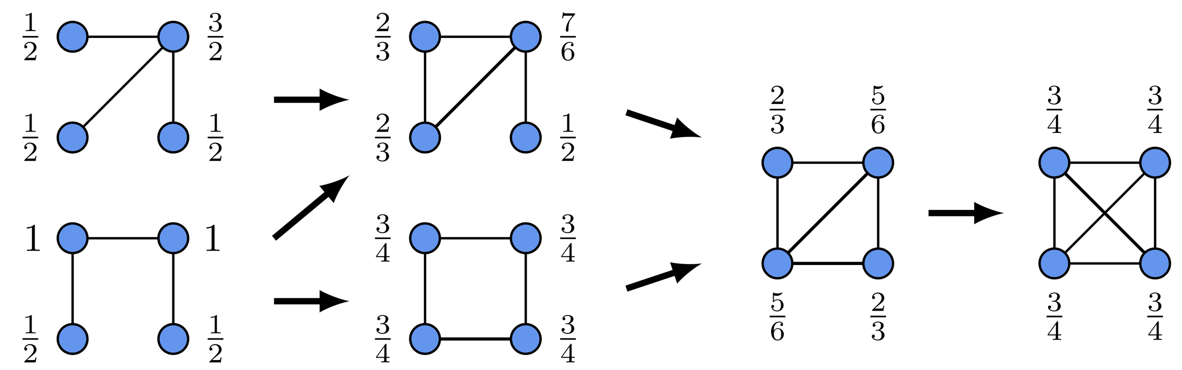

Figure 3: An example of best response dynamics, with Shapley-Myerson values shown. Note that the order in which players move may affect best response dynamics.

Characterizing Equilibria

In the game above, it is not hard to show that each player wants to make and keep connections to every other player, so best response dynamics terminates in the complete graph. However, in real world networks, making connections is seldom free. If players incur a cost per edge, describing equilibria becomes much more difficult. A player's value is now their Shapley-Myerson Value minus $e\cdot\alpha$, where the cost per edge is α and e is the number of edges incident to that player. Different α produce different equilibria; a lower cost encourages players to build more edges.

Using the Shapley-Myerson Value as our allocation function guarantees the existence of equilibria for any fixed cost α. However, there are several different ways to run best response dynamics, and different methods produce different equilibria. We describe two such models.

A Round Robin Model

In this model, players take turns moving. On a player's turn, they consider adding or removing edges to all other players. Within a given turn later moves may affect the value of earlier moves, so a player's turn ends only when satisfied with all of their edges.

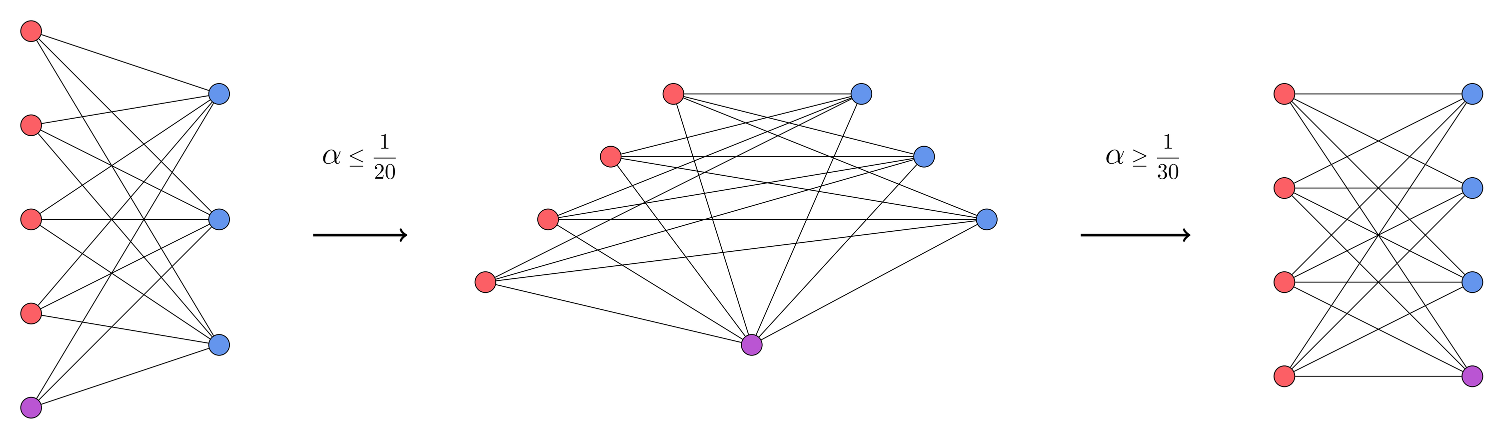

Figure 4: A representative best response move. If $\alpha \leq 1/20$, the purple player adds the edges to the red players, if, furthermore, $\alpha \geq 1/30$, then the purple player will drop the edges to the blue players.

Starting from the empty graph, this model of best response dynamics produces an easily described class of graphs, though it is a non-trivial fact that these graphs are in fact the equilibria that occur. We define these graphs in the next section.

A Class of Equilibria

The equilibria produced by our round robin model form a subclass of *complete multi-partite graphs*. In general, a multi-partite graph has some number of independent sets; sets of vertices within which no edges exist. Complete multi-partite graphs contain *all* edges *between* independent sets. A complete multi-partite graph with independent sets of size $v_1, v_2, \ldots, v_r$ is denoted $K_{v_1,v_2,\ldots,v_r}$. For example the graph on the right would be $K_{1,3,3,3}$.

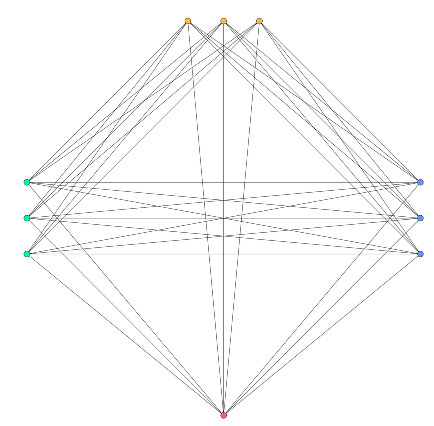

Figure 5: The graph produced by round robin best response dynamics on 10 vertices at $\frac{1}{72} < \alpha < \frac{1}{56}$.

All equilibria produced by our model follow the form $K_{n \pmod j),j,j,\ldots,j}$ where j depends on α in the following way:

$$\frac{1}{(n-j+2)(n-j+1)} < \alpha < \frac{1}{(n-j+1)(n-j)}$$

Intuitively, as independent sets get larger, they get "more unstable" so players only form independent sets up to a certain size.

A Slight Modification: Restricted Round Robin

In this model, players still take turns, but on a turn, a player may choose to drop any number of edges, and then may add only his or her most valuable edge, if one exists.

While the class of graphs produced by this model of best response dynamics is harder to classify, we have seen empirically that this model leads to equilibria with fewer edges than those previously described, as demonstrated below.

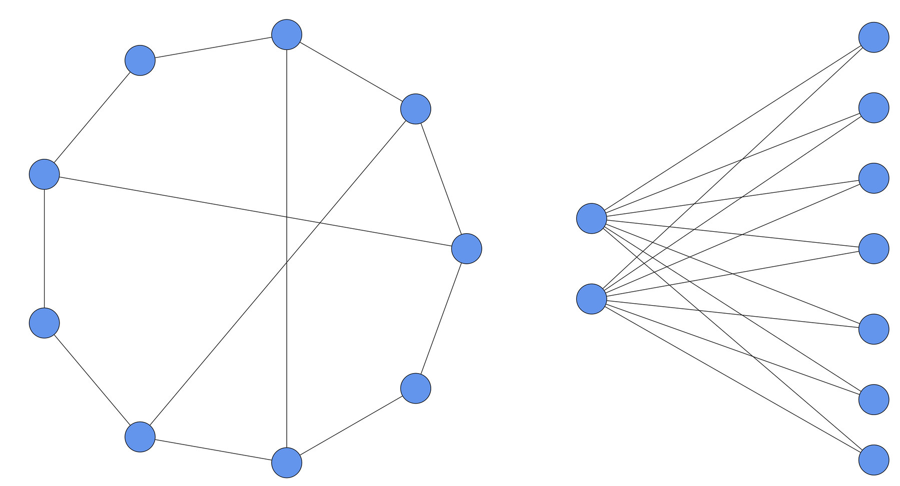

Figure 6: The graph on the left is the result of restricted round robin and the graph on the right is the result of regular round robin. The former has fewer edges. Here, α is just below 1/6.

Quality of Equilibria

Often selfish behaviour results in sub-optimal networks. However, such selfish behaviour exists in real-world networks. Game theorists are interested in the social value of equilibria created by this selfish behaviour, usually defined as the sum of player utilities. We can measure the quality of an equilibrium through the ratio of the social value of the equilibrium to that of an optimal graph.

In our game, we quickly see that if it's worthwhile to build any edges ($\alpha < 1/2$), then any tree provides a maximum social value. As such, at a fixed α, the ratio of any given equilibrium is affected solely by the number of edges. As we have seen, equilibria have different numbers of edges at the same edge cost and are thus of different quality.

Equilibria produced by our original round robin model are in fact the worst equilibria we have found, with the value of the optimal graph being at most twice the value of these graphs. We conjecture that these are, in fact, the worst quality equilibria, although it is possible that worse equilibria exist.