Normal Density Curves
Standard Normal
Sums of Normal Random Variables

Lecture 14

Normal Distribiutions

Text: Chapter 4

STAT 8010 Statistical Methods I
September 20, 2019

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Clemson University

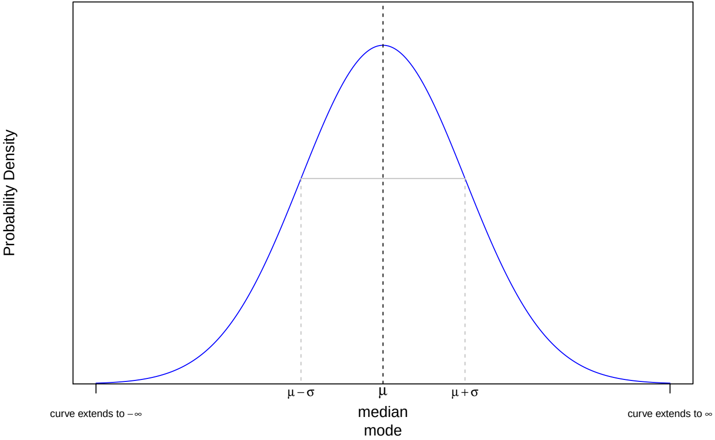
Agenda

1. **Normal Density Curves**

2. **Standard Normal**

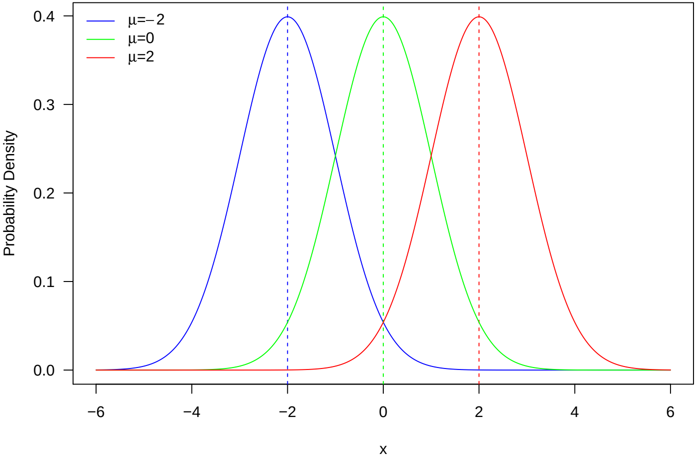
3. **Sums of Normal Random Variables**

Probability Density Curve for a Normal Random Variable



Normal Density Curves

Different μ but same σ^2



Normal Density Curves Cont'd

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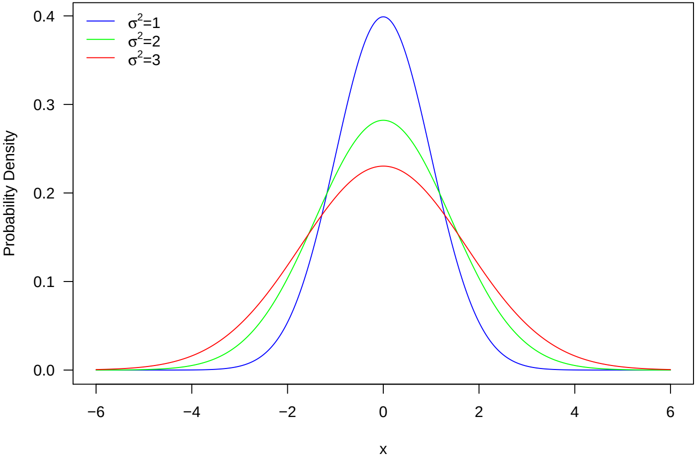

Characteristics of Normal Random Variables

Let X be a Normal r.v.

- The support for X: $(-\infty, \infty)$
- Parameters: μ : mean and σ^2 : variance
- The probability density function (pdf): $\frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$ for $-\infty < x < \infty$
- The cumulative distribution function (cdf): No explicit form, look at the value $\Phi(\frac{x-\mu}{\sigma})$ for $-\infty < x < \infty$ from standard normal table
- The expected value: $\mathbb{E}[X] = \mu$
- The variance: $Var(X) = \sigma^2$

Standard Normal $Z \sim N(\mu = 0, \sigma^2 = 1)$

- Normal random variable X with mean μ and standard deviation σ can convert to standard normal Z by the following :

$$Z = \frac{X - \mu}{\sigma}$$

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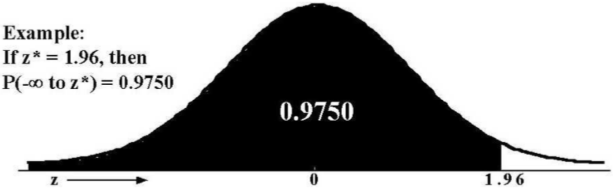
- The probability $\mathbb{P}(a \leq X \leq b)$ where $X \sim N(\mu, \sigma^2)$ can be compute

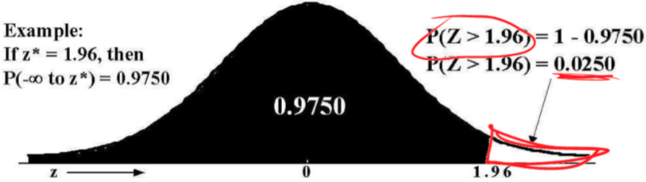
$$\mathbb{P}(a \leq X \leq b) = \mathbb{P}(\frac{a - \mu}{\sigma} \leq Z \leq \frac{b - \mu}{\sigma})$$
$$= \Phi(\frac{b - \mu}{\sigma}) - \Phi(\frac{a - \mu}{\sigma})$$

Standard Normal (Z) Table



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	**0.9750**	0.9756	0.9761	0.9767

Standard Normal (Z) Table Cont'd



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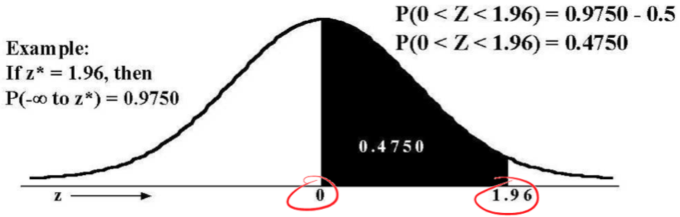

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- $\Phi(0) = .50 \Rightarrow$ Mean and Median (50_{th} percentile) for standard normal are both 0
- $\Phi(-z) = 1 - \Phi(z)$
- $\mathbb{P}(Z > z) = 1 - \Phi(z) = \Phi(-z)$

The Empirical Rules

The Empirical Rules provide a quick way to approximate certain probabilities for the Normal Distribution as the following table:

Interval	Percentage with interval
$\mu \pm \sigma$	68%
$\mu \pm 2\sigma$	95%
$\mu \pm 3\sigma$	99.7%

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Example

Let us examine Z. Find the following probabilities with respect to Z:

1. Z is at most -1.75
2. Z is between -2 and 2 inclusive
3. Z is less than .5

Example Cont'd

Solution.

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2. $\mathbb{P}(-2 \leq Z \leq 2) = \Phi(2) - \Phi(-2) = .9772 - .0228 = .9544$
3. $\mathbb{P}(Z < .5) = \Phi(.5) = .6915$

Example

Suppose a STAT-8020 exam score follows a normal distribution with mean 78 and variance 36. Let X to denote the exam score, answer the following questions:

1. What is the probability that a randomly chosen test taker got a score greater than 84?
2. Suppose the passing score for this exam is 75. What is the probability that a randomly chosen test taker got a score greater than 84 given that she/he pass the exam?
3. Using the empirical rule to find the 84_{th} percentile.

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- Let $S_n = \sum_{i=1}^n X_i$ then $S_n \sim N(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2)$
- This can be applied for any integer n

Example

Let X_1, X_2, and X_3 be mutually independent, Normal random variables. Let their means and standard deviations be $3k$ and k for $k = 1$, 2, and 3 respectively. Find the following distributions:

1. $\sum_{i=1}^{3} X_i$
2. $X_1 + 2X_2 - 3X_3$
3. $X_1 + 5X_3$

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Example Cont'd

Solution.

1. $\sum_{i=1}^{3} X_i \sim N(\mu = 3+6+9 = 15, \sigma^2 = 1^2+2^2+3^2 = 14)$
2. $X_1 + 2X_2 - 3X_3 \sim N(\mu = 3 + 12 - 27 = -12, \sigma^2 = 1^2 + 4 \times 2^2 + 9 \times 3^2 = 98)$
3. $X_1 + 5X_3 \sim N(\mu = 3 + 45 = 48, \sigma^2 = 1^2 + 25 \times 3^2 = 226)$