Lecture 35
Correlation & Regression

STAT 8010 Statistical Methods I
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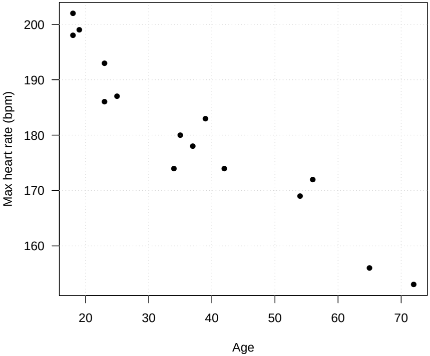
Motivated Example: Maximum Heart Rate vs. Age

Suppose we have 15 people of varying ages are tested for their maximum heart rate (bpm):

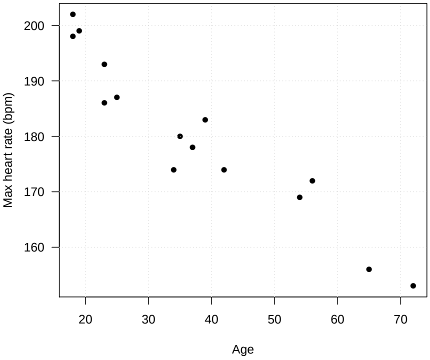
Age	18	23	25	35	65	54	34	56	72	19	23	42	18	39	37
MaxHeartRate	202	186	187	180	156	169	174	172	153	199	193	174	198	183	178



Question: How to describe the relationship between maximum heart rate and age?

Scatterplot

A scatterplot is a useful tool to graphically display the relationship between two numerical variables. Each dot on the scatterplot represents one observation from the data



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In the next few slides we will learn how to quantify the strength and direction of the linear relationship between two variables

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- Covariance is a measure of how much two quantitative random variables change together
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- The normalized version of the covariance, the correlation shows both the the direction and the strength of the linear relation

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  - If $\rho = 0$: we say the two variables have no linear relationship

Scatterplot & Pearson Correlation Coefficient

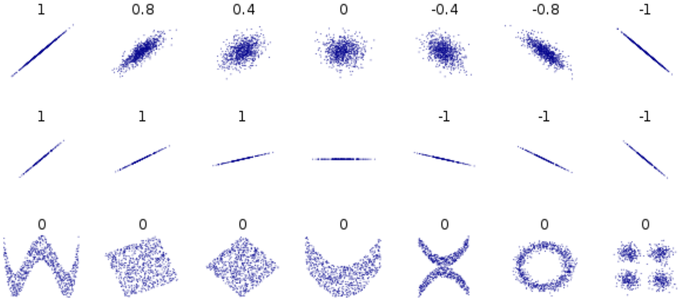

Figure: Image courtesy of Wikipedia at https://en.wikipedia.org/wiki/Correlation_and_dependence

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A Toy Example

You wonder how sleep affects productivity. You take a sample of 4 of your friends and measure last night's sleep and today's productivity in hours. Here are the results:

Sleep (X)	Productivity (Y)
2	4
4	12
6	14
10	10

Calculate the means, variances, and standard deviations of each variable and the correlation coefficient of these two variables

Toy Example Cont'd

Solution.

Let X denote last night's sleep in hours and Y denote today's productivity in hours

- $\bar{X} = \frac{2+4+6+10}{4} = 5.5, \qquad \bar{Y} = \frac{4+12+14+10}{4} = 10$

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  $s_{X,Y} = \frac{(2-5.5)(4-10)+(4-5.5)(12-10)+(6-5.5)(14-10)+(10-5.5)(10-10)}{3}$
  $= \frac{20}{3} \Rightarrow r_{X,Y} = \frac{\frac{20}{3}}{\sqrt{\frac{35}{3}}\sqrt{\frac{56}{3}}} = \frac{20}{\sqrt{35\times 56}} = 0.4518$

Inference/Hypothesis Test on ρ

1. $H_0 : \rho = 0$ vs. $H_a : \rho \neq 0$
2. Test statistic: $t^* = r\sqrt{\frac{n-2}{1-r^2}}$
3. Under H_0: $t^* \sim t_{df=n-2}$

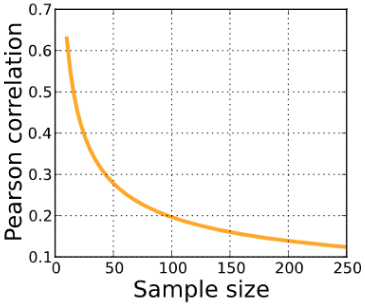
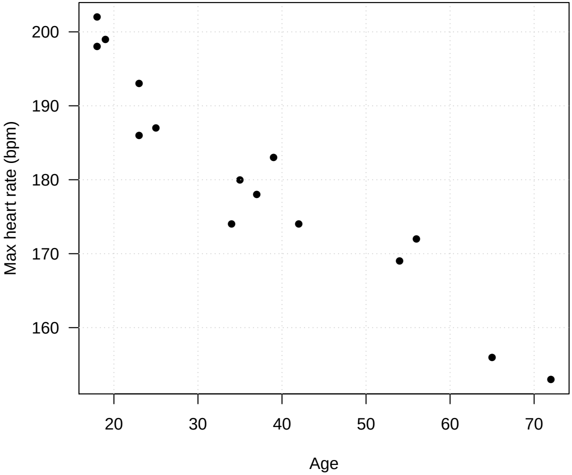


Figure: Image courtesy of Wikipedia

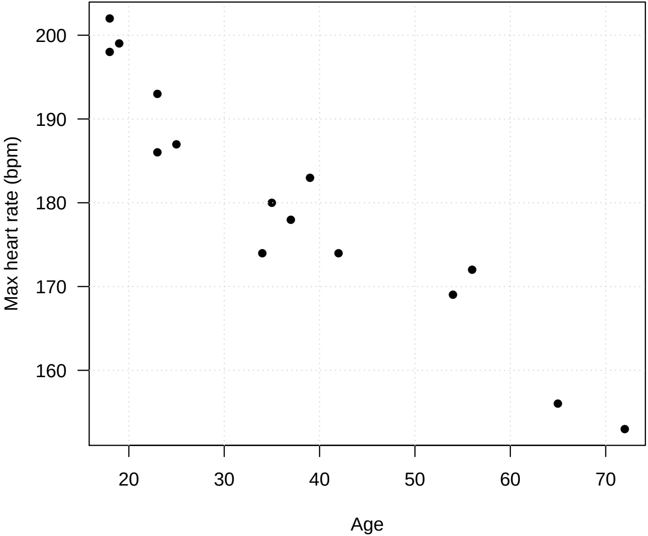
Maximum Heart Rate Example Revisited



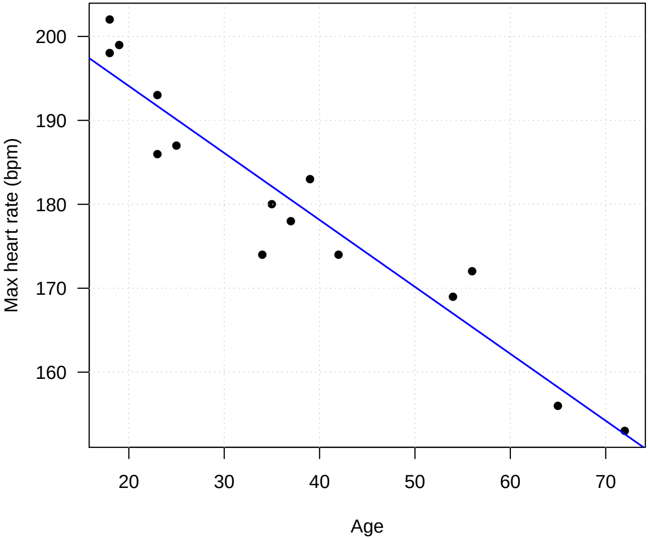
We may want to predict maximum heart rate for an individual based on his/her age ⇒ Regression Analysis

What is Regression Analysis?

Regression analysis: A set of statistical procedures for estimating the relationship between response variable and predictor variable(s)



Scatterplot: Is Linear Trend Reasonable?



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Simple Linear Regression (SLR)

Y: dependent (response) variable; X: independent (predictor) variable

- In SLR we **assume** there is a **linear relationship** between X and Y:

$$Y = \beta_0 + \beta_1 X + \varepsilon$$

- We will need to estimate β_0 (intercept) and β_1 (slope)

- Then we can use the estimated regression equation to
  - make predictions
  - study the relationship between response and predictor
  - control the response

Next lecture we will learn how to estimate the regression parameters β_0, β_1 and how to quantify the estimation uncertainty