Lecture 13
Model Diagnostics

STAT 8020 Statistical Methods II
September 18, 2019

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Agenda

1. Leverage Values
2. Studentized & Studentized Deleted Residuals
3. DFFITS
4. Variance Inflation Factor
5. Non-Constant Variance & Transformation

Leverage

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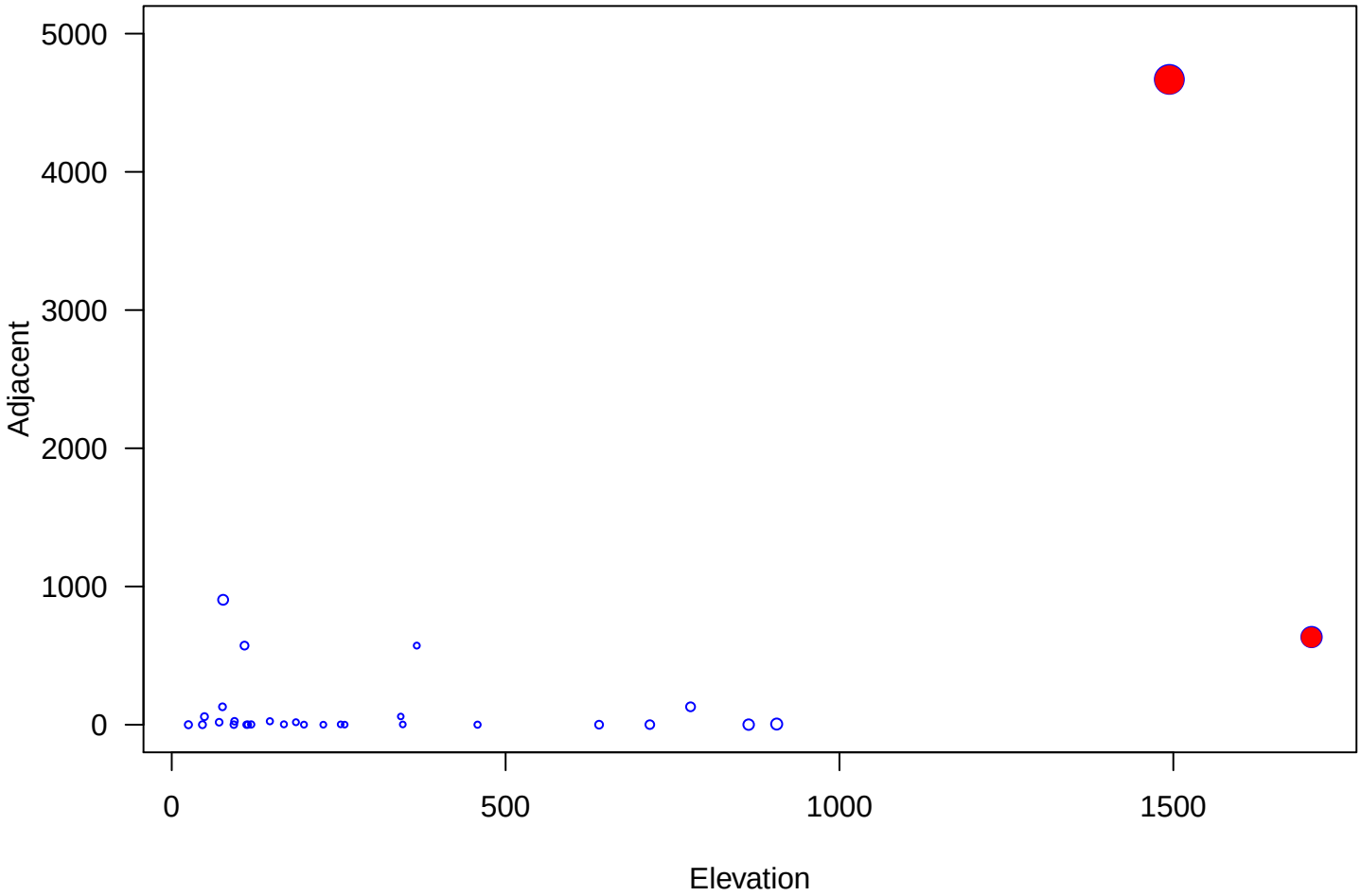
- $\frac{1}{n} \leq h_i \leq 1, \quad 1 \leq i \leq n$ and $\bar{h} = \sum_{i=1}^{n} \frac{h_i}{n} = \frac{p}{n} \Rightarrow$ a “rule of thumb" is that leverages of more than $\frac{2p}{n}$ should be looked at more closely

Leverage Values of `Species` $\sim$ `Elev` $+$ `Adj`



Studentized Residuals

As we have seen $\text{Var}(e_i) = \sigma^2(1 - h_i)$, this suggests the use of $r_i = \frac{e_i}{\hat{\sigma}\sqrt{(1-h_i)}}$

- r_i's are called **studentized residuals**. r_i's are sometimes preferred in residual plots as they have been standardized to have equal variance.

- If the model assumptions are correct then $\text{Var}(r_i) = 1$ and $\text{Corr}(e_i, e_j)$ tends to be small

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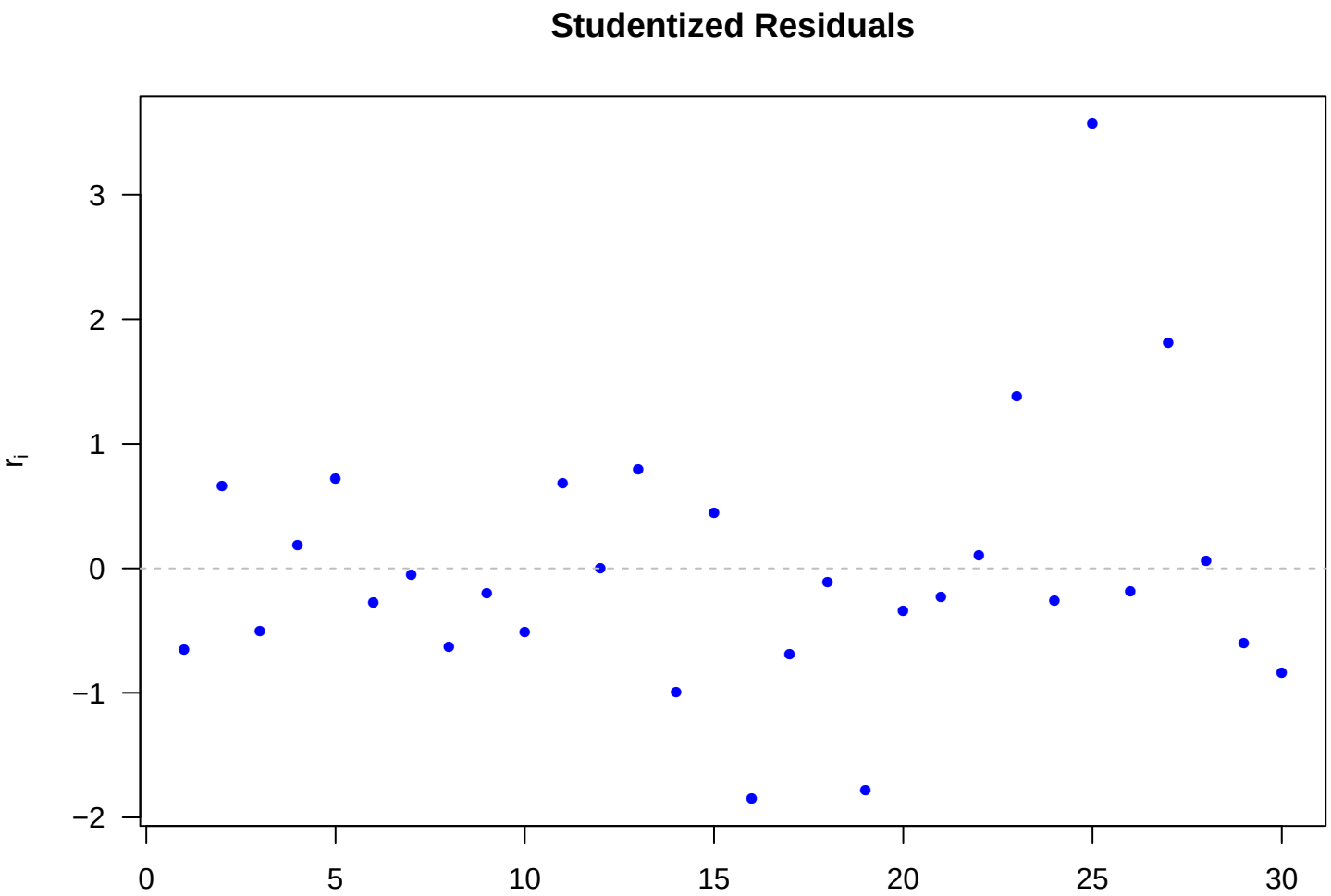

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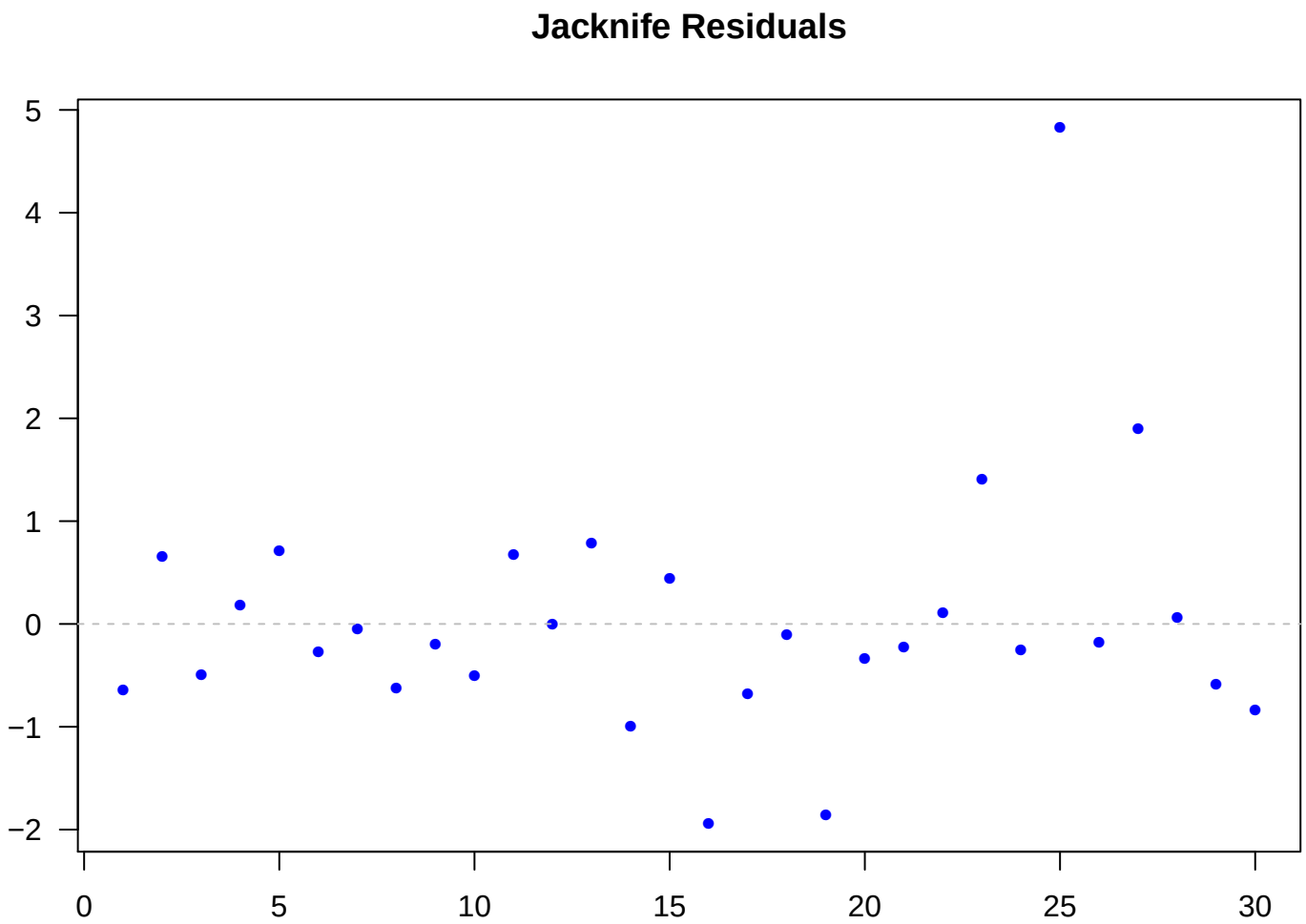
- Define the **Studentized Deleted Residuals** as

$$t_i = \frac{\hat{Y}_{i(i)} - Y_i}{\hat{\sigma}^2_{(i)}/1 - h_i} = \frac{\hat{Y}_{i(i)} - Y_i}{\text{MSE}_{(i)}(1-h_i)^{-1}}$$

  which are distributed as a t_{n-p-1} if the model is correct and $\varepsilon \sim \text{N}(\mathbf{0}, \sigma^2 \boldsymbol{I})$

Jackknife Residuals of `Species` $\sim$ `Elev` $+$ `Adj`



Influential Observations

DFFITS

- Difference between the fitted values $\hat{Y}_i$ and the predicted values $\hat{Y}_{i(i)}$
- $\text{DFFITS}_i = \frac{\hat{Y}_i - \hat{Y}_{i(i)}}{\sqrt{\text{MSE}_{(i)} h_i}}$
- Concern if absolute value greater than 1 for small data sets, or greater than $2\sqrt{p/n}$ for large data sets

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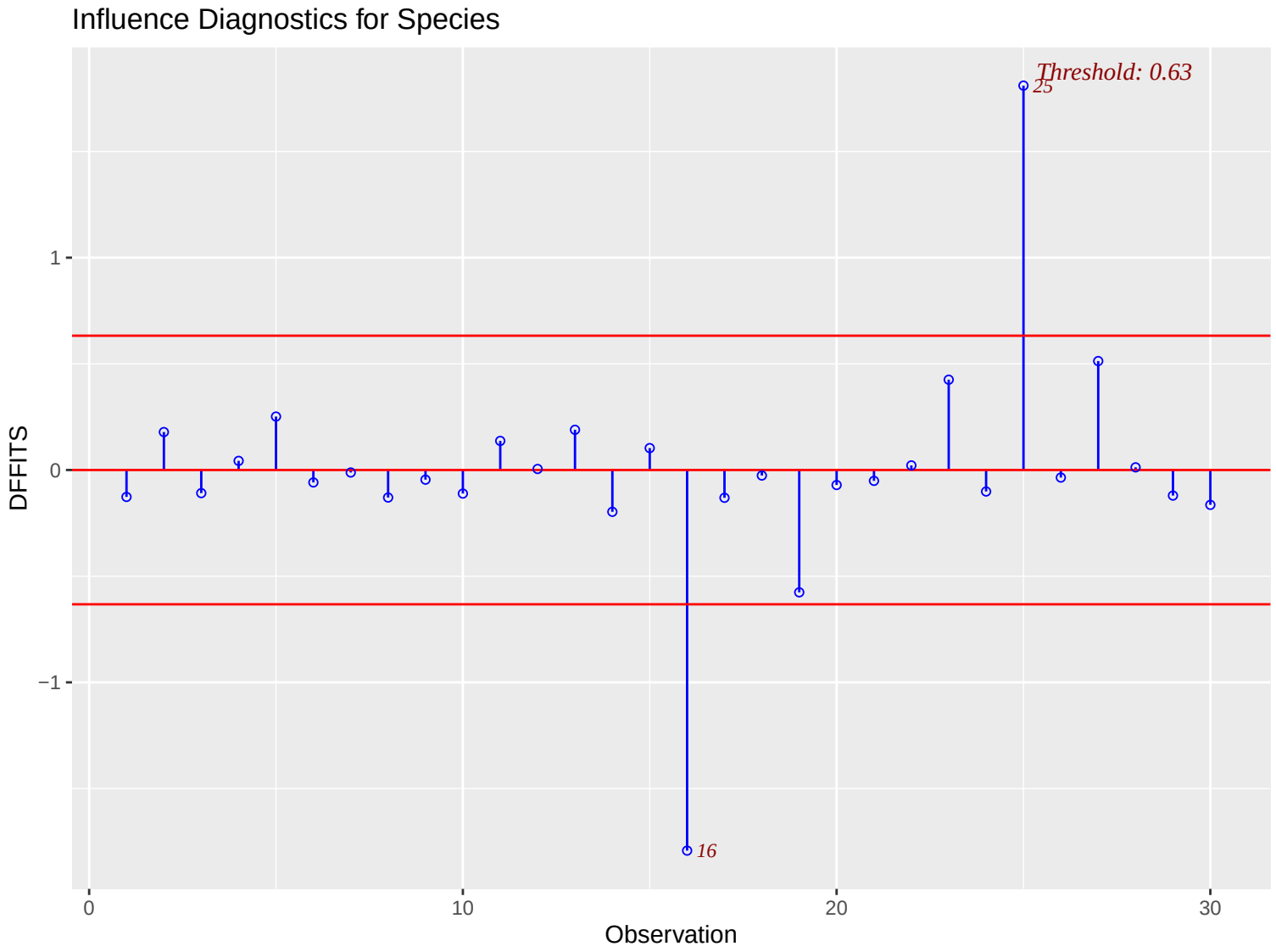


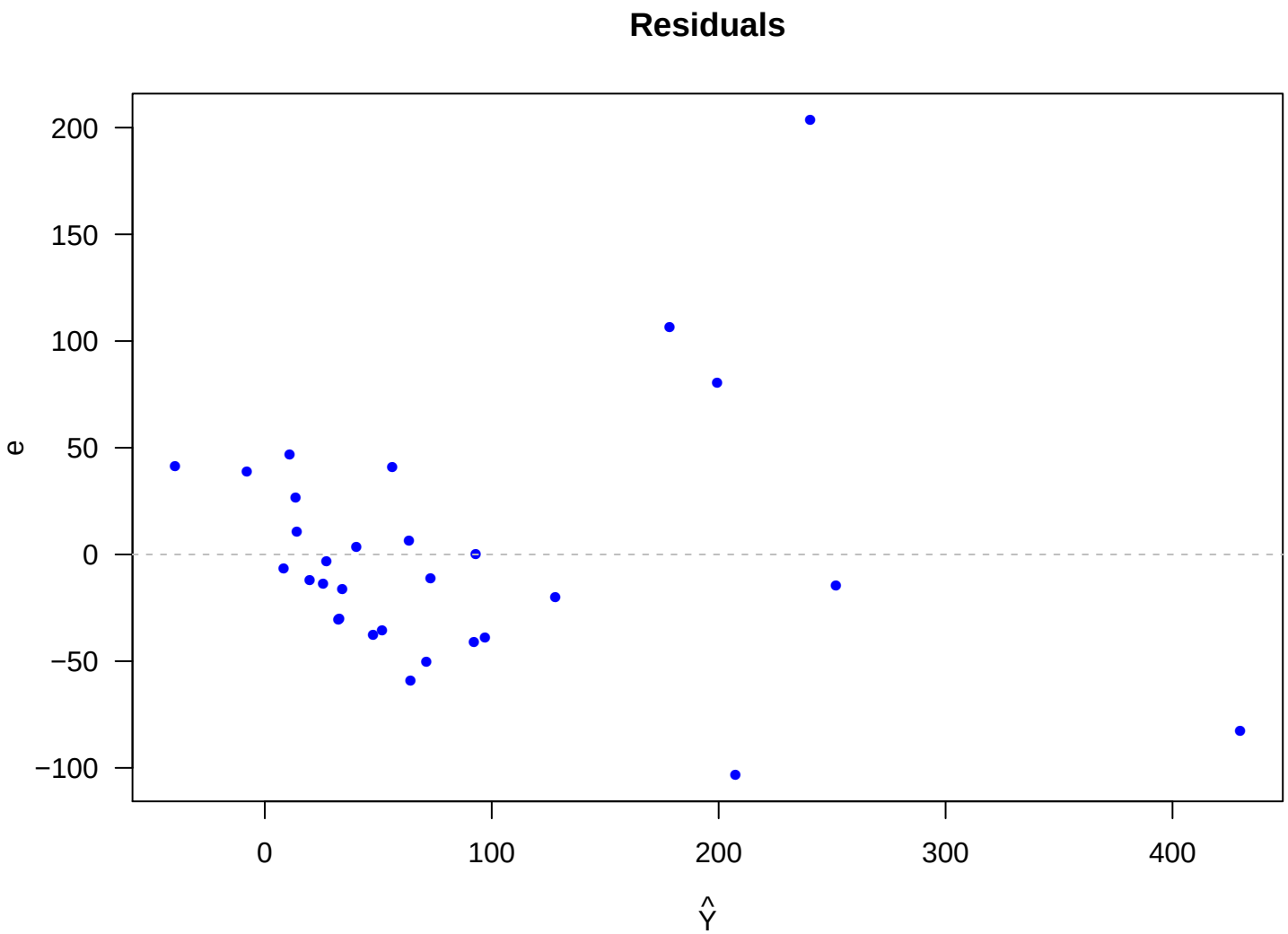
Variance Inflation Factor (VIF)

$$\text{VIF}_k = \frac{1}{1 - R_k^2},$$

where R_k^2 is the coefficient of determination when X_k is regressed on the remaining $p - 2$ other predictors.

```
> vif(step_gala)
Elevation  Adjacent
 1.404074  1.404074
> vif(full)
     Area Elevation   Nearest
 2.928145  3.992545  1.766099
    Scruz  Adjacent
 1.675031  1.826403
```

Residual Plot of `Species` $\sim$ `Elev` $+$ `Adj`



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Residual Plot After Square Root Transformation