# Lecture 20
## Review

*STAT 8010 Statistical Methods I*
March 31, 2020

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Inference for One Population Mean
Inference for Two Population Means
Inferences for Matched Pairs
ANOVA
Multiple Comparisons & Linear Contrasts
RCBD

# Agenda

1. Inference for One Population Mean
2. Inference for Two Population Means
3. Inferences for Matched Pairs
4. ANOVA
5. Multiple Comparisons & Linear Contrasts
6. RCBD

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# Inference for One Population Mean

# Inferences for One Population Mean $\mu$

**Goal**: To infer $\mu = \mathbb{E}(X)$ from a random sample $\{X_1, X_2, \cdots, X_n\}$

- Point estimation:

$$\bar{X}_n = \frac{\sum_{i=1}^{n} X_i}{n}$$

- Interval Estimation: $100 \times (1-\alpha)\%$ Confidence Interval (CI)
  - $\sigma = \sqrt{\text{Var}(X)}$ is known:

$$\left(\bar{X}_n - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{X}_n + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

  - $\sigma$ is unknown:

$$\left(\bar{X}_n - t_{\alpha/2, df=n-1}\frac{s}{\sqrt{n}}, \bar{X}_n + t_{\alpha/2, df=n-1}\frac{s}{\sqrt{n}}\right)$$

**Assumptions**

- Data is a random sample from the population
- i) sample size $n$ is sufficiently large (e.g., $n > 30$) or the population $X$ follows a normal distribution

# Margin of Error & Sample Size Calculation



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- Margin of error:

$$z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \qquad \text{if } \sigma \text{ known}$$

$$t_{\alpha/2,df=n-1}\frac{s}{\sqrt{n}} \qquad \text{if } \sigma \text{ unknown}$$

$\Rightarrow$ CI for $\mu = \bar{X}_n \pm$ margin of error

- Sample size determination:

$$n = \left(\frac{z_{\alpha/2} \times \sigma}{\text{margin of error}}\right)^2,$$

if $\sigma$ is given

# Hypothesis Testing for $\mu$

1. State the null and alternative hypotheses:

$$H_0 : \mu = \mu_0 \text{ vs. } H_a : \mu > \text{ or } \neq \text{ or } < \mu_0$$

2. Compute the test statistic:

$$z_{obs} = \frac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}}, \quad \sigma \text{ known; } t_{obs} = \frac{\bar{X}_n - \mu_0}{s/\sqrt{n}}, \quad \sigma \text{ unknown}$$

3. Make the decision of the test:

   Rejection Region/ P-Value Methods

4. Draw the conclusion of the test:

   We (do/do not) have enough statistical evidence to conclude that ($H_a$ in words) at $\alpha$ significant level.

# Type I, II Error & Power



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| True State | Decision | |
|---|---|---|
| | Reject $H_0$ | Fail to reject $H_0$ |
| $H_0$ is true | Type I error | Correct |
| $H_0$ is false | Correct | Type II error |

- Type I error: $\mathbb{P}(\text{Reject } H_0 | H_0 \text{ is true}) = \alpha$
- Type II error: $\mathbb{P}(\text{Fail to reject } H_0 | H_0 \text{ is false}) = \beta$
- The power (PWR): $\mathbb{P}(\text{Reject } H_0 | H_0 \text{ is false}) = 1 - \beta$.

$$\Rightarrow \text{PWR}(\mu_a) = 1 - \beta(\mu_a) = 1 - \mathbb{P}\left(z^* \le z_\alpha - \frac{|\mu_0 - \mu_a|}{\sigma/\sqrt{n}}\right)$$

# Duality of Hypothesis Test with Confidence Interval

There is an interesting relationship between CIs and hypothesis tests. If $H_0 = \mu_0$ is rejected with significance level $\alpha$ then the corresponding confidence interval does not contain $\mu_0$ with the confidence level $(1-\alpha)$, and vice versa

| Hypothesis testing at $\alpha$ level | $(1-\alpha)\times 100\%$ CI |
|---|---|
| $H_0: \mu = \mu_0$ vs. $H_a: \mu \neq \mu_0$ | $\bar{X} \pm t_{\alpha/2,n-1} s/\sqrt{n}$ |
| $H_0: \mu = \mu_0$ vs. $H_a: \mu > \mu_0$ | $(\bar{X} - t_{\alpha,n-1} s/\sqrt{n}, \infty)$ |
| $H_0: \mu = \mu_0$ vs. $H_a: \mu < \mu_0$ | $(-\infty, \bar{X} + t_{\alpha,n-1} s/\sqrt{n})$ |

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# Inferences for Two Population Means

## Statistical Inference for $\mu_1 - \mu_2$

- **Assumptions**: Populations $X_1$ and $X_2$ both follows a normal distribution (or their sample sizes are large enough); Data are random sample from their population
- Point estimation: $\bar{X}_1 - \bar{X}_2$
- Interval estimation:

$$\bar{X}_1 - \bar{X}_2 \pm \text{ margin of error},$$

where margin of error =

$$t_{\alpha/2, df^*}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}, \quad df^* = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{(s_1^2/n_1)^2}{n_1-1} + \frac{(s_2^2/n_2)^2}{n_2-1}}$$

When $s_1$ and $s_2$ "similar enough", we replace $\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$ by $s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$, where $s_p = \sqrt{\frac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2}}, df = n_1 + n_2 - 2$



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# Hypothesis Testing for $\mu_1 - \mu_2$

- State the null and alternative hypotheses:
  - Upper-tailed test: $H_0 : \mu_1 - \mu_2 = 0$ vs. $H_a : \mu_1 - \mu_2 > 0$
  - Lower-tailed test: $H_0 : \mu_1 - \mu_2 = 0$ vs. $H_a : \mu_1 - \mu_2 < 0$
  - Two-tailed test: $H_0 : \mu_1 - \mu_2 = 0$ vs. $H_a : \mu_1 - \mu_2 \neq 0$
- Compute the test statistic:

$$t_{obs} = \begin{cases} \dfrac{\bar{X}_1 - \bar{X}_2}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}, & \sigma_1 = \sigma_2 \\ \dfrac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}, & \sigma_1 \neq \sigma_2 \end{cases}$$

- Make the decision of the test:

Rejection Region/ P-Value Methods

- Draw the conclusion of the test

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# Inferences for Matched Pairs

## Paired T-Tests



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- **When to use**: before/after study, pairing subjects, study on twins, etc
- $H_0 : \mu_{diff} = 0$ vs. $H_a : \mu_{diff} > 0$ or $\mu_{diff} < 0$ or $\mu_{diff} \neq 0$, where $\mu_{diff}$ is the population mean of the paired difference
- Test statistic: $t_{obs} = \frac{\bar{X}_{diff} - 0}{\frac{s_{diff}}{\sqrt{n}}}$

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# ANOVA

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# ANOVA and Overall $F$ Test

## Overall F-Test

- $H_0 : \mu_1 = \mu_2 = \cdots = \mu_J$
  $H_a$ : At least one mean is different

- ANOVA Table:

| **Source** | df | SS | MS | F statistic |
|---|---|---|---|---|
| Treatment | $J-1$ | SSTr | MSTr = $\frac{\text{SSTr}}{J-1}$ | $F = \frac{\text{MSTr}}{\text{MSE}}$ |
| Error | $N-J$ | SSE | MSE = $\frac{\text{SSE}}{N-J}$ | |
| Total | $N-1$ | SSTo | | |

- Test Statistic: $F_{obs} = \frac{\text{MSTr}}{\text{MSE}}$. Under $H_0$, $F^* \sim F_{df_1=J-1, df_2=N-J}$

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# Multiple Comparisons & Linear Contrasts

# Family-Wise Error Rate (FWER) and Mulitple Comparisons



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- Family-Wise Error Rate (FWER) $\bar{\alpha}$: the probability of making 1 or more type I errors in a set of hypothesis tests
- Bonferroni Correction: Adjust the significant level for each of the $m$ tests to be $\frac{\alpha}{m}$ to control the **FWER**
- Fisher's LSD and Tukey's HSD

# Linear Contrasts

- **Definition**: Let $c_1, c_2, \cdots, c_J$ are constants where $\sum_{j=1}^{J} c_j = 0$, then $L = \sum_{j=1}^{J} c_j \mu_j$ is called a **linear contrast** of the population means.
- Point Estimation:

$$\hat{L} = \sum_{j=1}^{J} c_j \bar{X}_j$$

- Interval Estimation:

$$(\hat{L} - t_{\alpha/2, df=N-J} \hat{se}_{\hat{L}}, \hat{L} + t_{\alpha/2, df=N-J} \hat{se}_{\hat{L}}),$$

where $\hat{se}_{\hat{L}} = \sqrt{\text{MSE}\left(\frac{c_1^2}{n_1} + \cdots + \frac{c_J^2}{n_J}\right)}$

- Hypothesis Testing for linear contrasts

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- ANOVA table

| **Source** | df | SS | MS | F statistic |
|---|---|---|---|---|
| Treatment | $J-1$ | $SS_{trt}$ | $MS_{trt} = \frac{SS_{trt}}{J-1}$ | $F_{trt} = \frac{MS_{trt}}{MS_{err}}$ |
| Block | $B-1$ | $SS_{blk}$ | $MS_{blk} = \frac{SS_{blk}}{B-1}$ | |
| Error | $(B-1)(J-1)$ | $SS_{err}$ | $MS_{err} = \frac{SS_{err}}{(B-1)(J-1)}$ | |
| Total | $N-1$ | $SStot$ | | |

- Why we may want to do blocking (See the example in Lecture 17)?
- Use interaction plot to assess the additivity assumption (i.e., treatment effects are consistent across blocks)