Lecture 8

Seasonal Time Series Models

Readings: Cryer & Chan Ch 10; Brockwell & Davis Ch 6.5; Shumway & Stoffer Ch 3.9

MATH 8090 Time Series Analysis
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Modeling Trend, Seasonality, and Noise

Recall the trend, seasonality, noise decomposition mentioned in Week 2:

$$Y_t = \mu_t + s_t + \eta_t,$$

where

- μ_t: (deterministic) trend component;
- s_t: (deterministic) seasonal component with mean 0;
- η_t: random noise with $\mathbb{E}(\eta_t) = 0$

We have already described ways to estimate each component both separately and jointly (via likelihood-based method). But what about if $\{s_t\}$ is a “random” function of t?

⇒ The seasonal ARIMA model allows us to model the case when s_t itself varies randomly from one cycle to the next

The Seasonal ARIMA (SARIMA) Model

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Let d and D be non-negative integers. Then $\{X_t\}$ is a seasonal ARIMA$(p, d, q) \times (P, D, Q)$ process with period s if

$$Y_t = \nabla^d \nabla_s^D X_t = (1 - B)^d (1 - B^s)^D X_t,$$

is a casual ARMA process define by

$$\phi(B)\Phi(B^s)Y_t = \theta(B)\Theta(B^s)Z_t,$$

where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$.

$\{Y_t\}$ is causal if $\phi(z) \neq 0$ and $\Phi(z) \neq 0$, for $|z| \leq 1$, where

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p;$$
$$\Phi(z) = 1 - \Phi_1 z - \cdots - \Phi_P z^P.$$

An Illustration of Seasonal Model

Consider a monthly time series $\{X_t\}$ with both a trend, and a seasonal component of period $s = 12$.

- Suppose we know the values of d and D such that $Y_t = (1 - B)^d(1 - B^{12})^D X_t$ is stationary
- We can arrange the data this way:

	Month 1	Month 2	$\cdots$	Month 12
Year 1	Y_1	Y_2	$\cdots$	Y_{12}
Year 2	Y_{13}	Y_{14}	$\cdots$	Y_{24}
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
Year r	$Y_{1+12(r-1)}$	$Y_{2+12(r-1)}$	$\cdots$	$Y_{12+12(r-1)}$

The Inter-annual Model

Here we view each column (month) of the data table from the previous slide as a separate time series

- For each month m, we assume the same ARMA(P, Q) model. We have

$$Y_{m+12s} - \sum_{i=1}^{P} \Phi_i Y_{m+12(s-i)} = U_{m+12s} + \sum_{j=1}^{Q} \Phi_j U_{m+12(s-j)},$$

  for each $s = 0, \cdots, r-1$, where $\{U_{m+12s:s=0,\cdots,r-1}\} \sim \mathrm{WN}(0, \sigma_U^2)$ for each m

- We can write this as

$$\Phi(B^{12}) Y_t = \Theta(B^{12}) U_t,$$

  and this defines the inter-annual model

The Intra-Annual Model

We induce correlation between the months by letting the process $\{U_t\}$ follow an ARMA(p, q) model,

$$\phi(B)U_t = \theta(B)Z_t,$$

where $Z_t \sim \mathrm{WN}(0, \sigma^2)$

- This is the intra-annual model
- The combination of the inter-annual and intra-annual models for the differenced stationary series,

  $$Y_t = (1 - B)^d(1 - B^{12})^D X_t,$$

  yields a SARIMA model for $\{X_t\}$

Steps for Modeling SARIMA Processes

1. Transform data is necessary

2. Find d and D so that

$$Y_t = (1 - B)^d (1 - B^s)^D X_t$$

is stationary

3. Examine the sample ACF/PACF of $\{Y_t\}$ at lags that are multiples of s for plausible values for P and Q

4. Examine the sample ACF/PACF at lags $\{1, 2, \cdots, s - 1\}$, to identify possible values for p and q

Modeling SARIMA Processes (Cont'd)

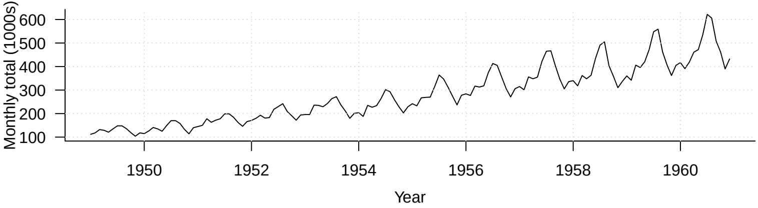
5. Use maximum likelihood method to fit the models

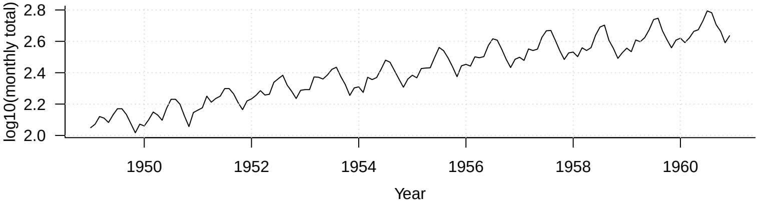
6. Use model summaries, diagnostics, AIC (AICC) to determine the best SARIMA model

7. Conduct forecast

Airline Passengers Example

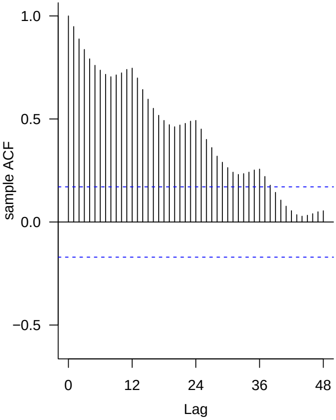
We consider the data set `airpassengers`, which are the monthly totals of international airline passengers from 1949 to 1960, taken from Box and Jenkins [1970]

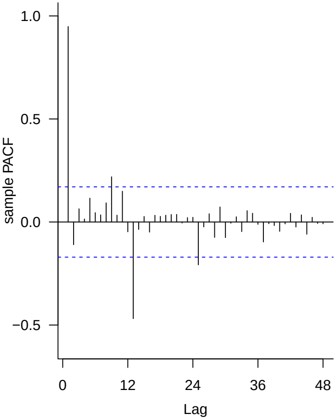




Here we stabilize the variance with a $\log_{10}$ transformation

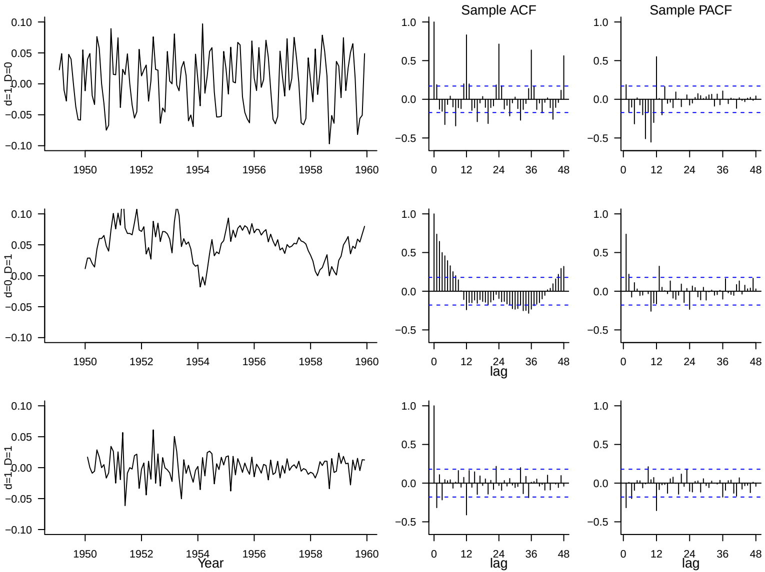
Sample ACF/PACF Plots





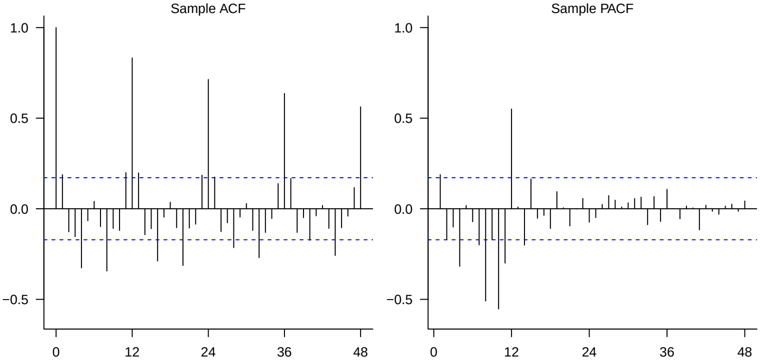
- The sample ACF decays slowly with a wave structure ⇒ seasonality
- The lag one PACF is close to one, indicating that differencing the data would be reasonable

Trying Different Orders of Differencing



Choosing Candidate SARIMA Models

We choose a SARIMA$(p, 1, q) \times (P, 0, Q)$ model. Next we examine the sample ACF/PACF of the process $Y_t = (1 - B)X_t$

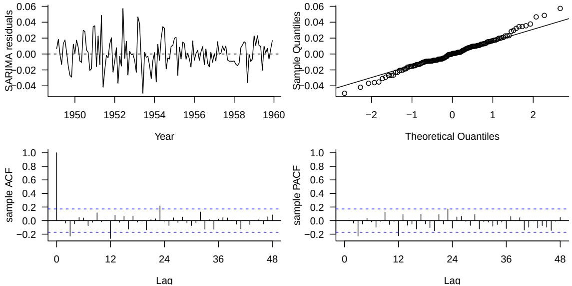


Now we need to choose P , Q, p, and q

Fitting a SARIMA$(1,1,0) \times (1,0,0)$ model

```
> fit1 <- arima(diff.1.0, order = c(1, 0, 0), seasonal = list(order = c(1, 0, 0), period = 12))
> fit1

Call:
arima(x = diff.1.0, order = c(1, 0, 0), seasonal = list(order = c(1, 0, 0),
    period = 12))

Coefficients:
          ar1    sar1  intercept
      -0.2667  0.9291     0.0039
s.e.   0.0865  0.0235     0.0096

sigma^2 estimated as 0.0003298:  log likelihood = 327.27,  aic = -646.54
```

```
> Box.test(fit1$residuals, lag = 48, type = "Ljung-Box")

        Box-Ljung test

data:  fit1$residuals
X-squared = 55.372, df = 48, p-value = 0.2164
```



A Discussion of the Model Fit

- The spread of the residuals is larger in 1949-1955 compared to the later years and the residual distribution has heavy tails
- The Ljung-Box test result indicates the fitted SARIMA $(1, 1, 0) \times (1, 0, 0)$ has sufficiently account for the temporal dependence
- 95% CI for ϕ_1 and Φ_1 do not contain zero $\Rightarrow$ no need to go with simpler model

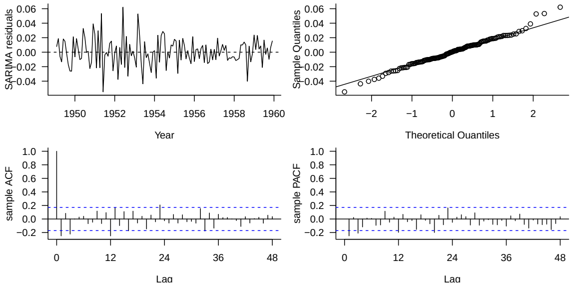
Our estimated model is

$$(1 + 0.2667B)(1 - 0.9291B^{12})(X_t - 0.0039) = Z_t,$$

where $\{Z_t\} \overset{i.i.d.}{\sim} \mathrm{N}(0, \sigma^2)$ with $\hat{\sigma}^2 = 0.00033$

Comparing with a SARIMA$(0,1,0)\times(1,0,0)$ Model


```
> (fit2 <- arima(diff.1.0, seasonal = list(order = c(1, 0, 0), period = 12)))

Call:
arima(x = diff.1.0, seasonal = list(order = c(1, 0, 0), period = 12))

Coefficients:
        sar1  intercept
      0.9081     0.0040
s.e.  0.0278     0.0108

sigma^2 estimated as 0.0003616:  log likelihood = 322.75,  aic = -639.51
> Box.test(fit2$residuals, lag = 48, type = "Ljung-Box")

        Box-Ljung test

data:  fit2$residuals
X-squared = 80.641, df = 48, p-value = 0.002209
```



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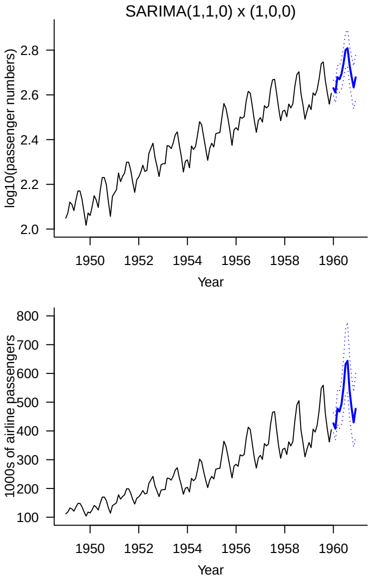
A Discussion of Model Fit2

Here we drop the AR(1) term

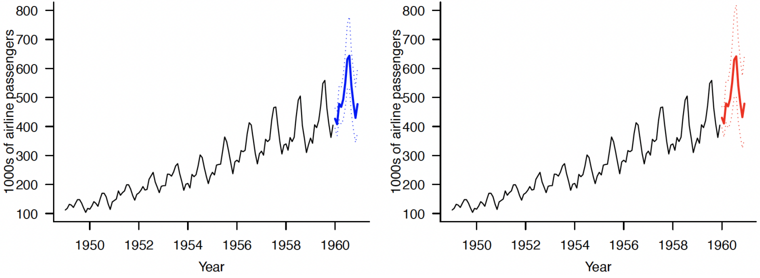
- The residual plots looks quite similar to before: The spread of the residuals is larger in 1949-1955 compared to the later years and the residual distribution has heavy tails
- Both $\hat{\sigma}^2$ and AIC increase (compared with model fit1)
- The lag 1 of ACF and PACF now lies outside the IID noise bounds. The Ljung-Box P-value of 0.0022, leads us to reject the IID residual assumption

In conclusion, the SARIMA$(1,1,0) \times (1,0,0)$ model fits better than SARIMA$(0,1,0) \times (1,0,0)$

Forecasting the 1960 Data



Evaluating Forecast Performance



Metrics	Model Fit1	Model Fit2
Root Mean Square Error	30.36	31.32
Mean Relative Error	0.057	0.060
Empirical Coverage	0.917	1.000

The SARIMA$(1,1,0) \times (1,0,0)$ Model is Equivalent To?

Out model for the log passenger series $\{X_t\}$ is

$$\phi(B)\Phi(B^{12})(1-B)X_t = Z_t,$$

where $\phi(B) = 1 - \phi_1 B$ and $\Phi(B) = 1 - \Phi_1(B)$

Note that

$$\begin{aligned}\phi(B)\Phi(B^{12}) &= (1-\phi B)(1-\Phi_1 B^{12}) \\ &= 1 - \phi_1 B - \Phi_1 B^{12} + \phi_1\Phi_1 B^{13}\end{aligned}$$

Question: What is this SARIMA model equivalent to?

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Unit Root Tests

Suppose we have $X_1, \cdots, X_n$ that follow the model

$$(1 - \phi B)(X_t - \mu) = (X_t - \mu) - \phi(X_{t-1} - \mu) = Z_t,$$

where $\{Z_t\}$ is a $\mathrm{WN}(0, \sigma^2)$ process

- A unit root test considers the following hypotheses:

$$H_0 : \phi = 1 \text{ versus } H_a : |\phi| < 1$$

- Note that where $|\phi| < 1$ the process is stationary (and causal) while $\phi = 1$ leads to a nonstationary process

- Exercise: Letting $Y_t = \nabla X_t$, show that

$$\begin{aligned} Y_t &= (1 - \phi)\mu + (\phi - 1)X_{t-1} + Z_t \\ &= \phi_0^* + \phi_1^* X_{t-1} + Z_t, \end{aligned}$$

where $\phi_0^* = (1 - \phi)\mu$ and $\phi_1^* = (\phi - 1)$

Unit Root Tests (Cont'd)

- We can estimate ϕ_0^* and ϕ_1^* using ordinary least squares

- Using the estimate of ϕ_1^*, $\hat{\phi}_1^*$, and its standard error, $\hat{\mathrm{SE}}(\hat{\phi}_1^*)$, the Dickey-Fuller statistics is

$$T = \frac{\hat{\phi}_1^*}{\hat{\mathrm{SE}}(\hat{\phi}_1^*)}$$

- Under H_0 this statistic follows a Dickey-Fuller distribution. For a level α test we reject if the observed test statistic is smaller than a critical value C_α

α	0.01	0.05	0.10
C_α	-3.43	-2.86	-2.57

- We can extend to other processes (AR(p), ARMA(p, q), and MA(q))–see Brockwell and Davis [2002, Section 6.3] for further details

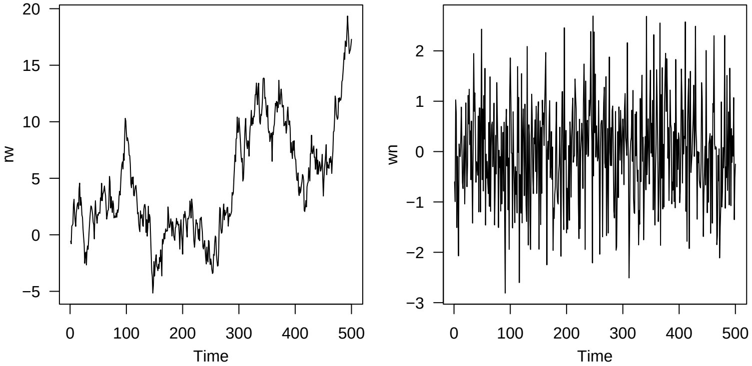
Unit Root Test: Simulated Examples

Recall

$$\nabla = \phi_0^* + \phi_1^* X_{t-1} + Z_t,$$

where $\phi_0^* = (1 - \phi)\mu$ and $\phi_1^* = (\phi - 1)$

Let's demonstrate the test with a simulated random walk (rw, $\phi = 1$) and a simulated white noise (wn, $\phi = 0$)



Unit Root Test: Simulated Examples Cont'd

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```
> diff.rw <- diff(rw); n <- length(rw)
> ys <- diff.rw; xs <- rw[1:(n-1)]
> ols.rw <- lm(ys ~ xs); summary(ols.rw)
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.10125    0.05973   1.695   0.0906 .
xs          -0.01438    0.00899  -1.600   0.1102
> diff.wn <- diff(wn)
> ys <- diff.wn; xs <- wn[1:(n-1)]
> ols.wn <- lm(ys ~ xs); summary(ols.wn)
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.001138   0.045329  -0.025     0.98
xs          -1.002420   0.044843 -22.354   <2e-16
```