Lecture 9

Multiple Linear Regression III

Reading: Chapter 12

STAT 8020 Statistical Methods II

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Agenda

1. **Review: General Linear Test**

2. **Review: Multicollinearity**

Review: General Linear Test

- Comparison of a "full model" and "reduced model" that involves a subset of full model predictors
- Consider a full model with k predictors and reduced model with ℓ predictors ($\ell < k$)
- Test statistic: $F^* = \frac{\text{SSE(R)} - \text{SSE}(F)/(k-\ell)}{\text{SSE}(F)/(n-k-1)}$ $\Rightarrow$ Testing H_0 that the regression coefficients for the extra variables are all zero
  - Example 1: $X_1, X_2, \cdots, X_{p-1}$ vs. intercept only $\Rightarrow$ Overall F test
  - Example 2: $X_j, 1 \leq j \leq p-1$ vs. intercept only $\Rightarrow$ t test for β_j
  - Example 3: X_1, X_2, X_3, X_4 vs. $X_1, X_3 \Rightarrow H_0 : \beta_2 = \beta_4 = 0$

Species Diversity on the Galapagos Islands Revisited: Full Model

```
> full <- lm(Species ~ Area + Elevation + Nearest + Scruz + Adjacent,
 data = gala)
> anova(full)
Analysis of Variance Table

Response: Species
          Df Sum Sq Mean Sq F value    Pr(>F)    
Area       1 145470  145470 39.1262 1.826e-06 ***
Elevation  1  65664   65664 17.6613 0.0003155 ***
Nearest    1     29      29  0.0079 0.9300674    
Scruz      1  14280   14280  3.8408 0.0617324 .  
Adjacent   1  66406   66406 17.8609 0.0002971 ***
Residuals 24  89231    3718                      
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

Species Diversity on the Galapagos Islands Revisited: Reduced Model

```
> reduced <- lm(Species ~ Elevation + Adjacent)
> anova(reduced)
Analysis of Variance Table

Response: Species
          Df Sum Sq Mean Sq F value    Pr(>F)
Elevation  1 207828  207828  56.112 4.662e-08 ***
Adjacent   1  73251   73251  19.777 0.0001344 ***
Residuals 27 100003    3704
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Perform a General Linear Test

- $H_0 : \beta_{\texttt{Area}} = \beta_{\texttt{Nearest}} = \beta_{\texttt{Scruz}}$ vs.
  $H_a :$ at least one of the three coefficients $\neq 0$
- $F^* = \frac{(100003-89231)/(5-2)}{89231/(30-5-1)} = 0.9657$
- P-value: $\mathrm{P}[F > 0.9657] = 0.425$, where $F \sim \mathrm{F}(3, 24)$

```
> anova(reduced, full)
Analysis of Variance Table

Model 1: Species ~ Elevation + Adjacent
Model 2: Species ~ Area + Elevation + Nearest + Scruz + Adjacent
  Res.Df    RSS Df Sum of Sq      F Pr(>F)
1     27 100003
2     24  89231  3     10772 0.9657  0.425
```

Multicollinearity

Multicollinearity is a phenomenon of high inter-correlations among the predictor variables

- Numerical issue $\Rightarrow$ the matrix $X^T X$ is nearly singular
- Statistical issue
  - β's are not well estimated
  - Spurious regression coefficient estimates
  - R^2 and predicted values are usually OK

Example

- Consider a two predictor model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \varepsilon$$

- We can show

$$\hat{\beta}_{1|2} = \frac{\hat{\beta}_1 - \sqrt{\frac{\hat{\sigma}_Y^2}{\hat{\sigma}_{X_1}^2}} r_{X_1,X_2} r_{Y,X_2}}{1 - r_{X_1,X_2}^2},$$

where $\hat{\beta}_{1|2}$ is the estimated partial regression coefficient for X_1 and $\hat{\beta}_1$ is the estimate for β_1 when fitting a simple linear regression model $Y \sim X_1$

An Simulated Example

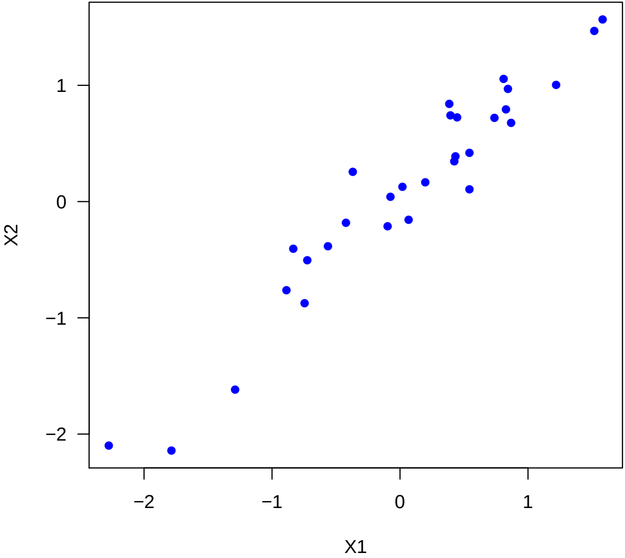
Suppose the true relationship between response Y and predictors (X_1, X_2) is

$$Y = 4 + 0.8X_1 + 0.6X_2 + \varepsilon,$$

where $\varepsilon \sim \mathrm{N}(0, 1)$ and X_1 and X_2 are positively correlated with $\rho = 0.95$. Let's fit the following models:

- Model 1: $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \varepsilon$
- Model 2: $Y = \beta_0 + \beta_1 X_1 + \varepsilon^1$
- Model 3: $Y = \beta_0 + \beta_2 X_2 + \varepsilon^2$

Scatter Plot: X_1 vs. X_2



Model 1 Fit


```
Call:
lm(formula = Y ~ X1 + X2)

Residuals:
     Min       1Q   Median       3Q      Max 
-1.91369 -0.73658  0.05475  0.87080  1.55150 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)   4.0710     0.1778  22.898  < 2e-16 ***
X1            2.2429     0.7187   3.121  0.00426 ** 
X2           -0.8339     0.7093  -1.176  0.24997    
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.9569 on 27 degrees of freedom
Multiple R-squared:  0.673,	Adjusted R-squared:  0.6488 
F-statistic: 27.78 on 2 and 27 DF,  p-value: 2.798e-07
```

Model 2 Fit


```
Call:
lm(formula = Y ~ X1)

Residuals:
     Min       1Q   Median       3Q      Max
-2.09663 -0.67031 -0.07229  0.87881  1.49739

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   4.0347     0.1763  22.888  < 2e-16 ***
X1            1.4293     0.1955   7.311 5.84e-08 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.9634 on 28 degrees of freedom
Multiple R-squared:  0.6562,    Adjusted R-squared:  0.644
F-statistic: 53.45 on 1 and 28 DF,  p-value: 5.839e-08
```

Model 3 Fit


```
Call:
lm(formula = Y ~ X2)

Residuals:
    Min      1Q  Median      3Q     Max 
-2.2584 -0.7398 -0.3568  0.8795  2.0826 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)   3.9882     0.2014   19.80  < 2e-16 ***
X2            1.2973     0.2195    5.91 2.33e-06 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.096 on 28 degrees of freedom
Multiple R-squared:  0.555,     Adjusted R-squared:  0.5391 
F-statistic: 34.92 on 1 and 28 DF,  p-value: 2.335e-06
```