Lecture 12

Hypothesis Testing & Inference on Two Population Means

Text: Chapters 5, 6

STAT 8010 Statistical Methods I
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Agenda

1. Hypothesis Testing
2. Type I & Type II Errors
3. Duality of Hypothesis Test with Confidence Interval
4. Inference on Two Population Means

Example (taken from The Cartoon Guide To Statistics)

New Age Granola Inc claims that average weight of its cereal boxes is 16 oz. The Genuine Grocery Corporation will send back a shipment if the average weight is any less.

Suppose Genuine Grocery Corporation takes a random sample of 49 boxes, weight each one, and compute the sample mean $\bar{X} = 15.90$ oz and sample standard deviation $s = 0.35$ oz.

Perform a hypothesis test at 0.05 significant level to determine if they would reject H_0, and therefore, this shipment

Cereal Weight Example Cont'd

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2. Test Statistic: $t_{obs} = \frac{15.9-16}{0.35/\sqrt{49}} = -2$

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3. Rejection Region Method: $-t_{0.05,48} = -1.68 \Rightarrow$ Rejection Region is $(-\infty, -1.68]$. Since t_{obs} is in rejection region, we reject H_0

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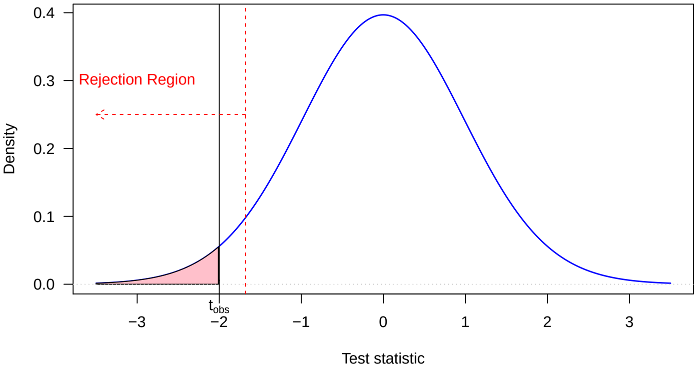
4. P-Value Method: $\mathbb{P}(t^* \leq -2) = 0.0256 < \alpha = 0.05 \Rightarrow$ reject H_0
5. Draw a Conclusion: We do have enough statistical evidence to conclude that the average weight is less than 16 oz at 0.05% significant level

Cereal Weight Example Cont'd

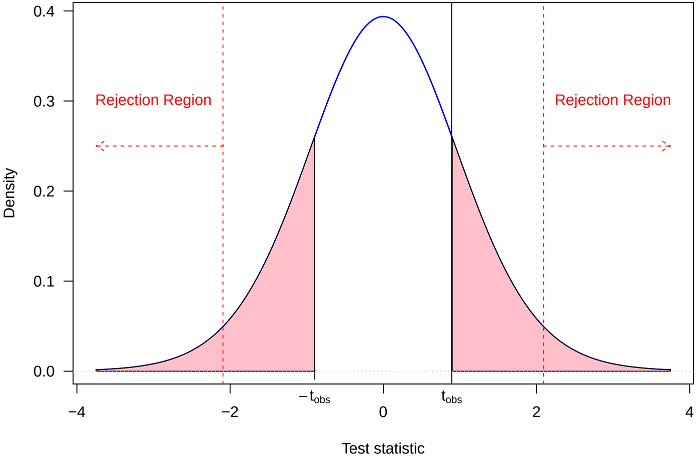


Example

A series of blood tests were run on a particular patient over five days. It is of interest to determine if the mean blood protein for this patient differs from 7.25, the value for healthy adults. Suppose the sample mean (n=20) is 7.35 and sample standard deviation is 0.5. Perform a hypothesis test using significance level of 0.05

1. $H_0 : \mu = 7.25$ vs. $H_a : \mu \neq 7.25$
2. $t_{obs} = \frac{7.35-7.25}{0.5/\sqrt{20}} = 0.8944$
3. P-value: $2 \times \mathbb{P}(t^* \geq 0.8944) = 0.3823 > 0.05$
4. We do not have enough statistical evidence to conclude that the mean blood protein is different from 7.25 at 5% significant level

Example Cont'd



Recap: Hypothesis Testing

1. State the null H_0 and the alternative H_a hypotheses

   - $H_0 : \mu = \mu_0$ vs $H_a : \mu > \mu_0 \Rightarrow$ Upper-tailed
   - $H_0 : \mu = \mu_0$ vs $H_a : \mu < \mu_0 \Rightarrow$ Lower-tailed
   - $H_0 : \mu = \mu_0$ vs $H_a : \mu \neq \mu_0 \Rightarrow$ Two-tailed

2. Compute the test statistic

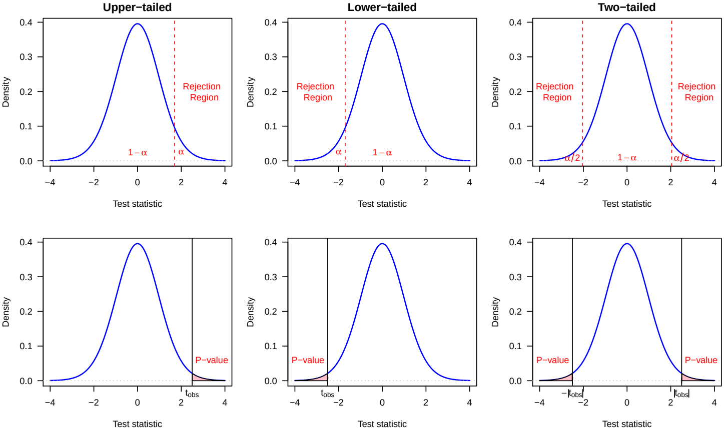
   $t^* = \frac{\bar{X}_n - \mu_0}{s/\sqrt{n}}$ (σ unknown); $z^* = \frac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}}$ (σ known)

3. Identify the rejection region(s) (or compute the P-value)

4. Draw a conclusion

   We do/do not have enough statistical evidence to conclude H_a at α significant level

Region Region and P-Value Methods



The 2×2 Decision Paradigm for Hypothesis Testing

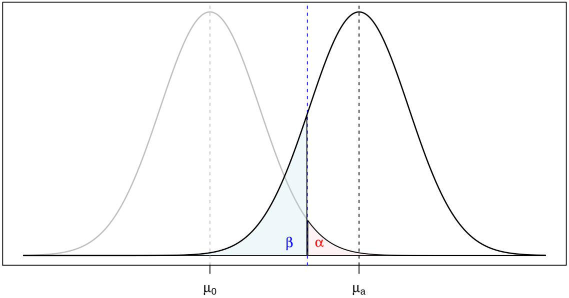
True State	Decision	
	Reject H_0	Fail to reject H_0
H_0 is true	Type I error	Correct
H_0 is false	Correct	Type II error

Errors in Hypothesis Testing

- The probability of a type I error is denoted by α
- The probability of a type II error is denoted by β

Type I & Type II Errors

- Type I error: $\mathbb{P}(\text{Reject } H_0 | H_0 \text{ is true}) = \alpha$
- Type II error: $\mathbb{P}(\text{Fail to reject } H_0 | H_0 \text{ is false}) = \beta$



$\alpha \downarrow \beta \uparrow$ and vice versa

Type II Error and Power

- The type II error, β, depends upon the true value of μ (let's call it μ_a)
- We use the formula below to compute β

$$\beta(\mu_a) = \mathbb{P}(z^* \leq z_\alpha - \frac{|\mu_0 - \mu_a|}{\sigma/\sqrt{n}})$$

- The power (PWR): $\mathbb{P}(\text{Reject } H_0 | H_0 \text{ is false}) = 1 - \beta$. Therefore PWR$(\mu_a) = 1 - \beta(\mu_a)$

Question: What increases Power?

Sample Size Determination

Suppose that we wish to determine what sample size is required to detect the difference between a hypothesized mean and true mean $\mu_0 - \mu_a$, denoted by Δ, with a given power $1 - \beta$ and specified significance level α and known standard deviation σ. We can use the following formulas

$$n = \sigma^2 \frac{(z_\alpha + z_\beta)^2}{\Delta^2} \text{ for a one-tailed test}$$

$$n \approx \sigma^2 \frac{(z_{\alpha/2} + z_\beta)^2}{\Delta^2} \text{ for a two-tailed test}$$

Example

An existing manufacturing process produces, on average, 100 units of output per day. A pilot plant is used to evaluate a possible process change. Suppose the Company CEO wants to know if yield is increased. The CEO uses $\alpha = 0.05$ and the sample mean $(n = 25)$ is 103. Do we have sufficient evidence to conclude that the mean yield exceeds 100 if $\sigma = 10$?

1. $H_0 : \mu = 100$ vs. $H_a : \mu > 100$
2. $z_{obs} = \frac{103-100}{10/\sqrt{25}} = 1.5$
3. The cutoff value of the rejection region is $z_{0.05} = 1.645$. Therefore we do not have enough evidence to conclude that the new process mean yield exceeds 100

Example Cont'd

Suppose the true true mean yield is 104.

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$$\begin{aligned}\beta(\mu = 104) &= \mathbb{P}\left(Z \le z_{0.05} - \frac{|100 - 104|}{10/\sqrt{25}}\right) \\ &= \mathbb{P}(Z \le 1.645 - 4/2) = \mathbb{P}(Z \le -0.355) \\ &= \Phi(-0.355) = 0.3613\end{aligned}$$

Therefore, the power is $1 - 0.3613 = 0.6387$

- What sample size is required to yield a power of 0.8 with a significance level of 0.05?

Example Cont'd

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$$= \Phi(-0.355) = 0.3613$$

Therefore, the power is $1 - 0.3613 = 0.6387$

- What sample size is required to yield a power of 0.8 with a significance level of 0.05?

$$n = \sigma^2 \frac{(z_{0.05} + z_{0.2})^2}{\Delta^2} = 10^2 \frac{(1.645 + 0.8416)^2}{4^2} = 38.6324$$

Therefore, the required sample size is 39

Duality of Hypothesis Test with Confidence Interval

There is an interesting relationship between CIs and hypothesis tests. If H_0 is rejected with significance level α then the corresponding confidence interval does not contain the value μ_0 targeted in the hypotheses with the confidence level $(1-\alpha)$, and vice versa

Hypothesis test at α level	$(1-\alpha)$ level CI
$H_0 : \mu = \mu_0$ vs. $H_a : \mu \neq \mu_0$	$\bar{X} \pm t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}$
$H_0 : \mu = \mu_0$ vs. $H_a : \mu > \mu_0$	$\left(\bar{X} - t_{\alpha/2,n-1}s/\sqrt{n}, \infty\right)$
$H_0 : \mu = \mu_0$ vs. $H_a : \mu < \mu_0$	$\left(-\infty, \bar{X} + t_{\alpha/2,n-1)s/\sqrt{n}}\right)$

Comparing Two Population Means

- We often interested in comparing two groups (e.g.)
  - Does a particular pesticide increase the yield of corn per acre?
  - Do men and women in the same occupation have different salaries?
- The common ingredient in these questions: They can be answered by conducting statistical inferences of two populations using two (independent) samples, one from each of two populations

Notation

- Parameters:
  - Population means: μ_1, μ_2
  - Population standard deviations: σ_1, σ_2
- Statistics:
  - Sample means: $\bar{X}_1, \bar{X}_2$
  - Sample standard deviations: s_1, s_2
  - Sample sizes: n_1, n_2

Statistical Inference for $\mu_1 - \mu_2$

- Point estimate: $\bar{X}_1 - \bar{X}_2$
- Interval estimate: Need to figure out $\sigma_{\bar{X}_1 - \bar{X}_2}$
- Hypothesis Testing:
  - Upper-tailed test: $H_0 : \mu_1 - \mu_2 = 0$ vs. $H_a : \mu_1 - \mu_2 > 0$
  - Lower-tailed test: $H_0 : \mu_1 - \mu_2 = 0$ vs. $H_a : \mu_1 - \mu_2 < 0$
  - Two-tailed test: $H_0 : \mu_1 - \mu_2 = 0$ vs. $H_a : \mu_1 - \mu_2 \neq 0$

Confidence Intervals for $\mu_1 - \mu_2$

If we are willing to **assume** $\sigma_1 = \sigma_2$, then we can "pool" these two (independent) samples together to estimate the common σ using s_p:

$$s_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$

We can then derive the (estimated) standard error of $\bar{X}_1 - \bar{X}_2$, which takes the following form

$$\sqrt{\frac{s_p^2}{n_1} + \frac{s_p^2}{n_2}} = s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

With CLT (assuming sample sizes are sufficiently large), we obtain the $(1 - \alpha) \times 100\%$ CI for $\mu_1 - \mu_2$:

$$\underbrace{\bar{X}_1 - \bar{X}_2}_{\text{point estimate}} \pm \underbrace{t_{\alpha/2, n_1+n_2-1} s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}_{\text{margin of error}}$$

Confidence Intervals for $\mu_1 - \mu_2$: What if $\sigma_1 \neq \sigma_2$?

- We will use s_1^2, s_2^2 as the estimates for σ_1^2 and σ_2^2 to obtain the standard error:

$$\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

- The formula for the degrees of freedom is somewhat complicated:

$$\frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{(s_1^2/n_1)^2}{n_1 - 1} + \frac{(s_2^2/n_2)^2}{n_2 - 1}}$$

- We can then construct the $(1-\alpha)\times 100\%$ CI for $\mu_1 - \mu_2$:

$$\underbrace{\bar{X}_1 - \bar{X}_2}_{\text{point estimate}} \pm \underbrace{t(\alpha/2, \text{ df calculated from above }) \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}_{\text{margin of error}}$$