MATH 8090: Stationary processes

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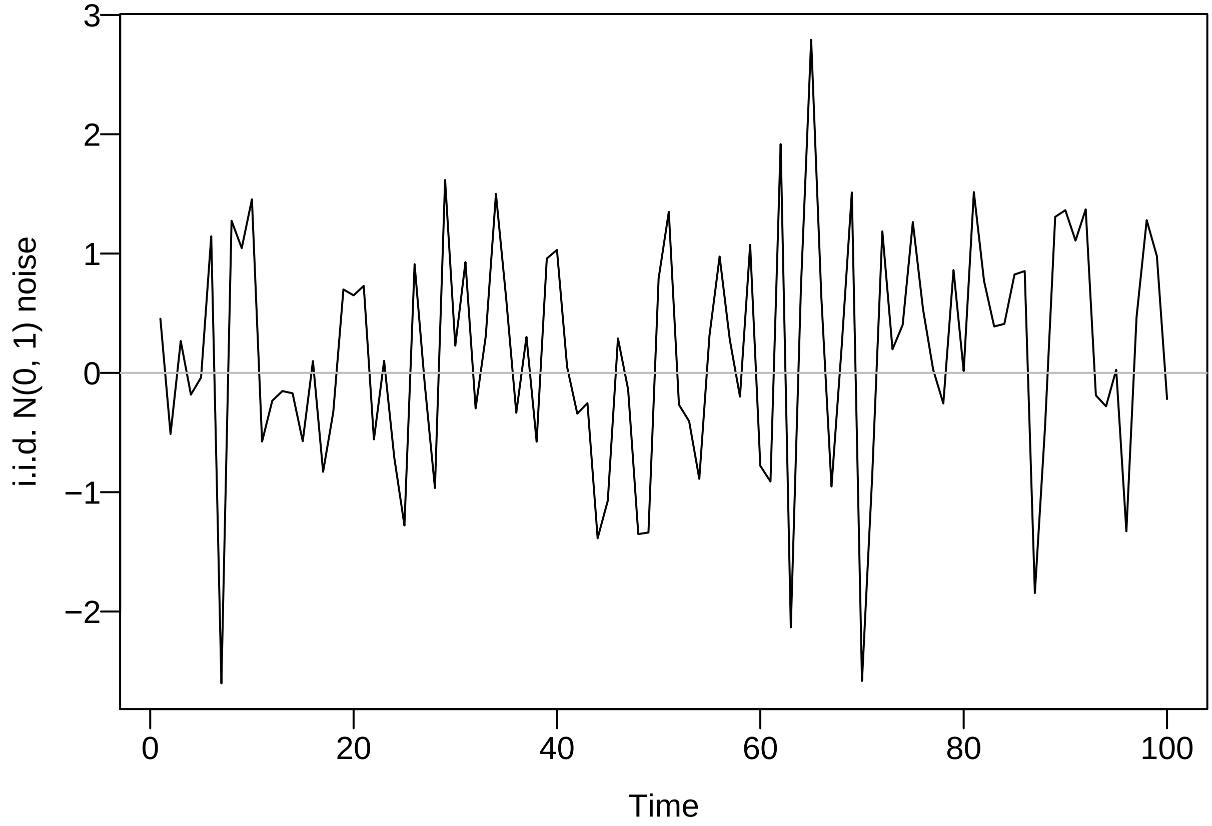
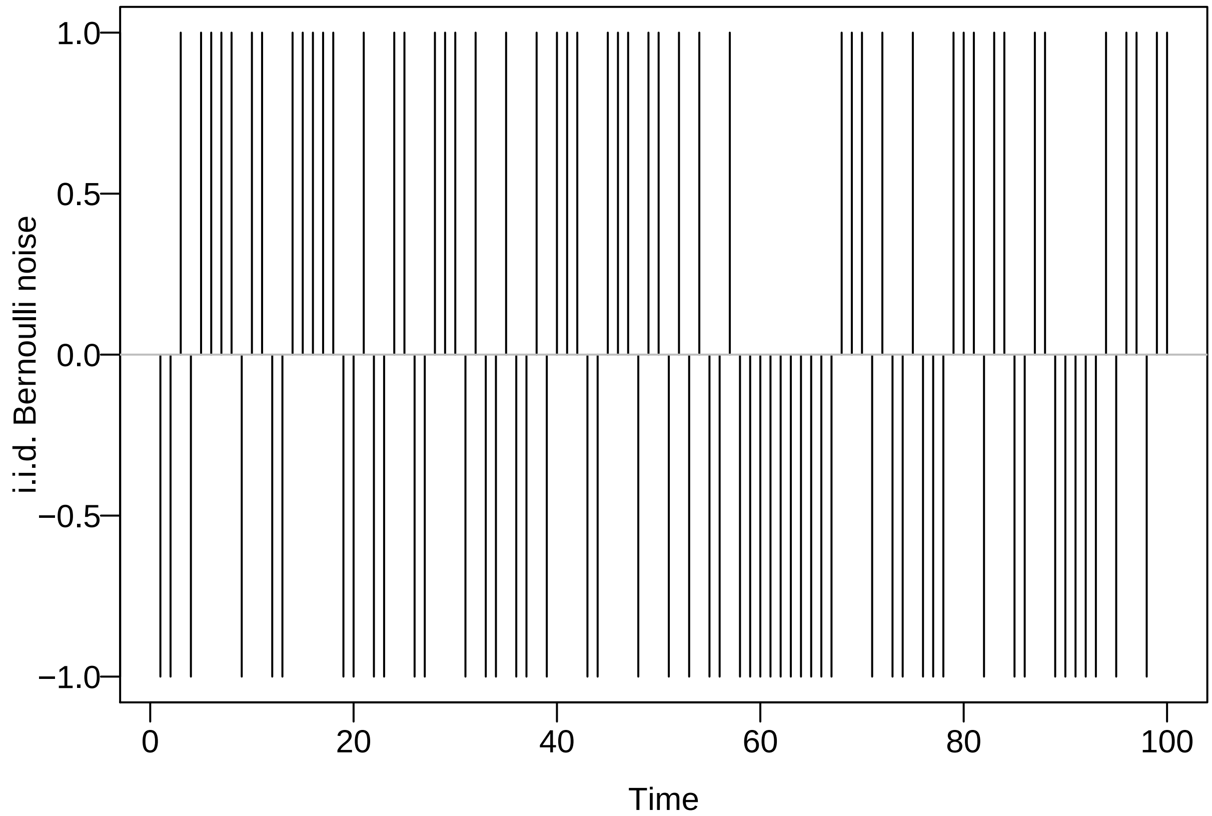
Contents

Examples of i.i.d. Noise

```
T = 100
t <- 1:T

## i.i.d. normal
normal_iid <- rnorm(T)
par(las = 1, mgp = c(2, 0.5, 0), mar = c(3.6, 3.6, 0.8, 0.6))
plot(t, normal_iid, type = "l", las = 1,
     xlab = "Time", ylab = "i.i.d. N(0, 1) noise")
abline(h = 0, col = "gray")
```



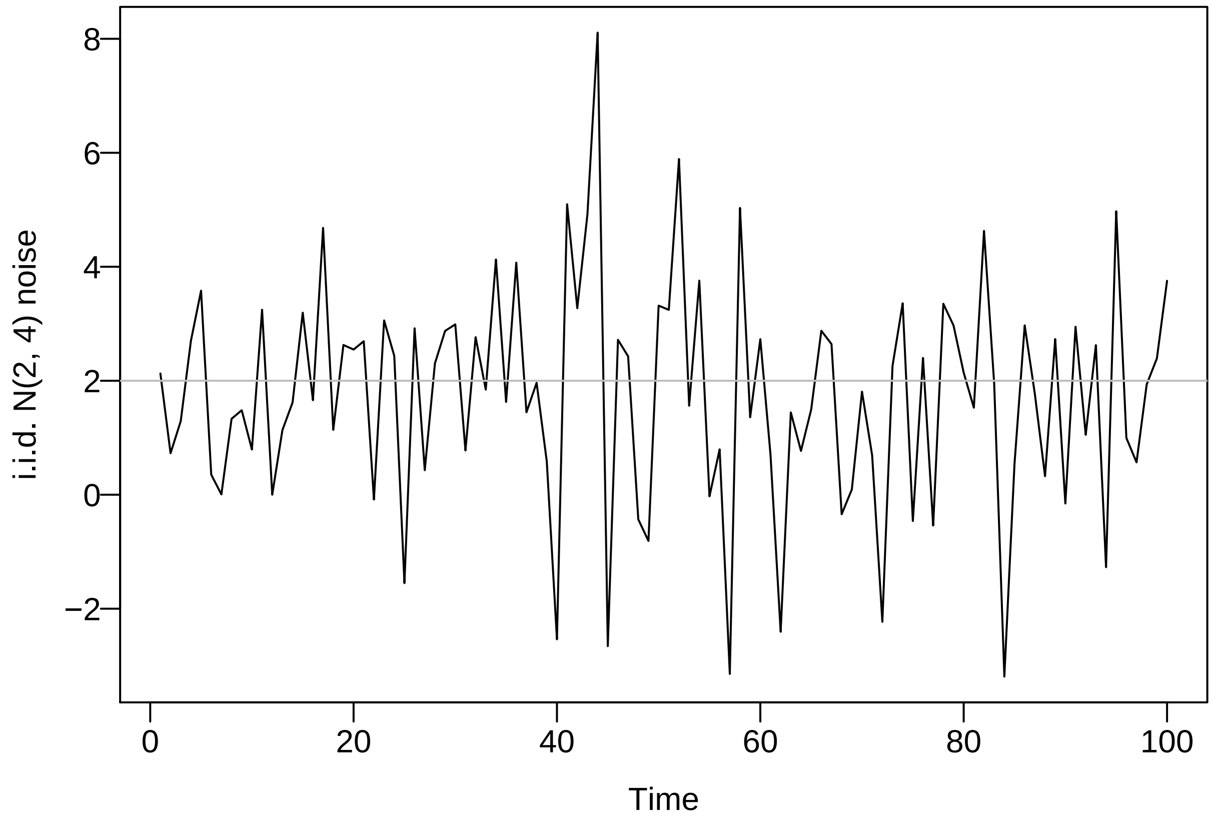

```r
## i.i.d. Binary
ber_iid <- replicate(T, rbinom(1, 1, 0.5))
ber_iid <- ifelse(ber_iid == 0, -1, 1)
plot(t, ber_iid, type = "h", las = 1,
     xlab = "Time", ylab = "i.i.d. Bernoulli noise")
abline(h = 0, col = "gray")
```



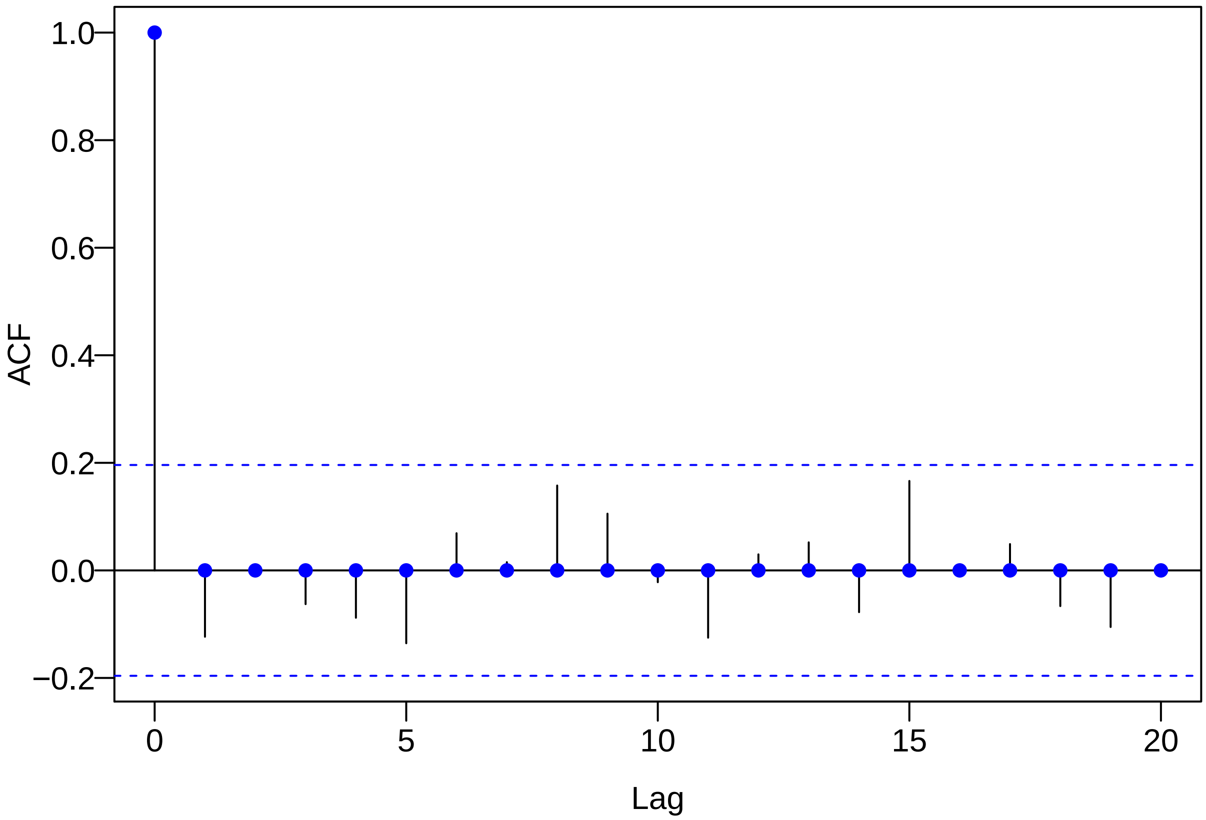
Examples realizations of white noise processes

Note: here we do not require the sequence follow the same distribution.

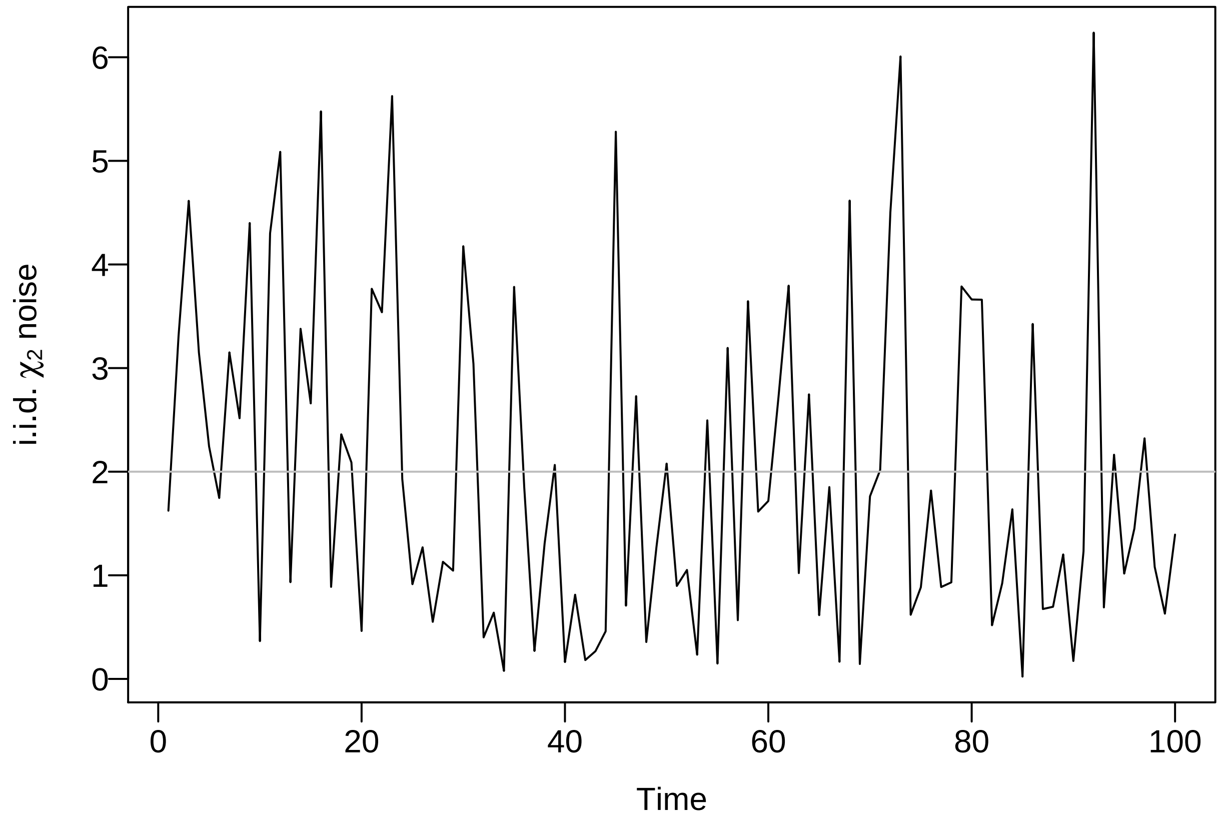
```
T = 100
t <- 1:T
WN1 <- rnorm(n = T, mean = 2, sd = 2)

par(las = 1, mgp = c(2, 0.5, 0), mar = c(3.6, 3.6, 0.8, 0.6))
plot(t, WN1, type = "l", xlab = "Time", ylab = "i.i.d. N(2, 4) noise")
abline(h = 2, col = "gray")
```



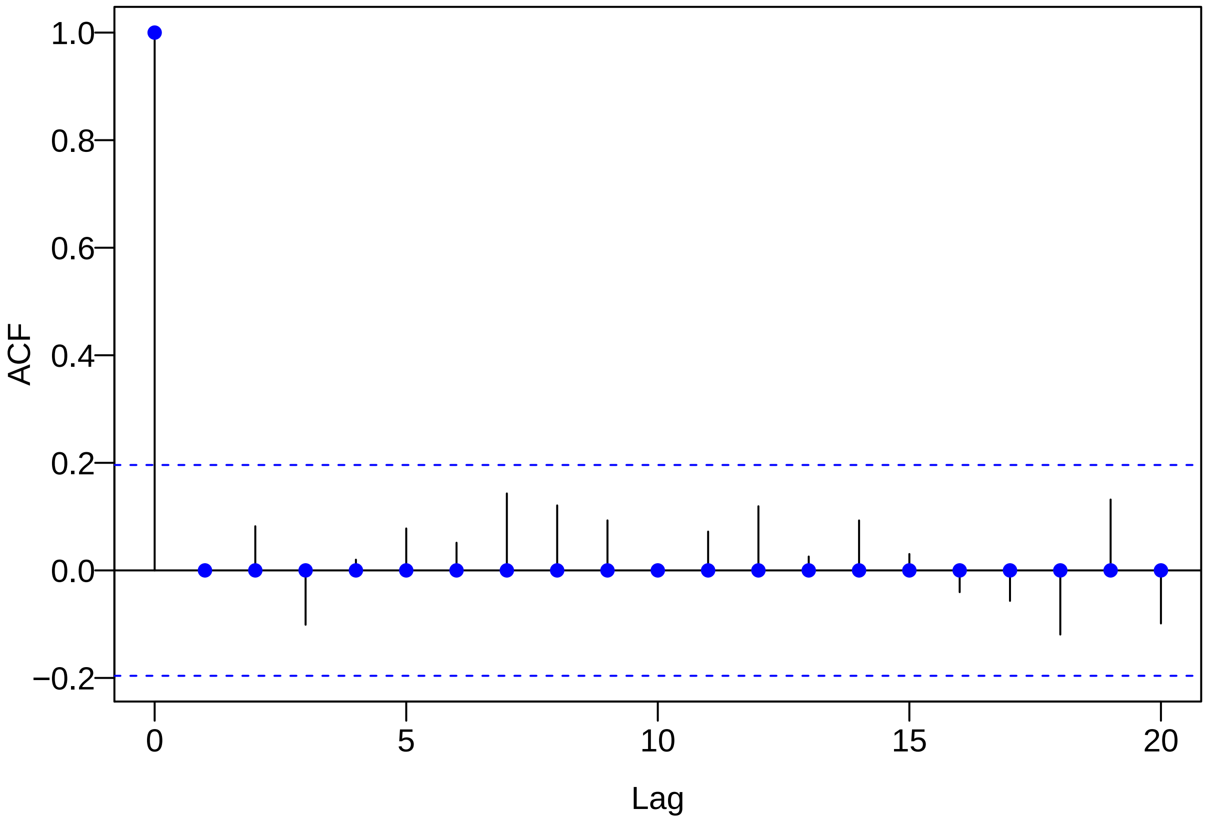

```
acf(WN1)
points(0:20, c(1, rep(0, 20)), pch = 16, col = "blue")
```



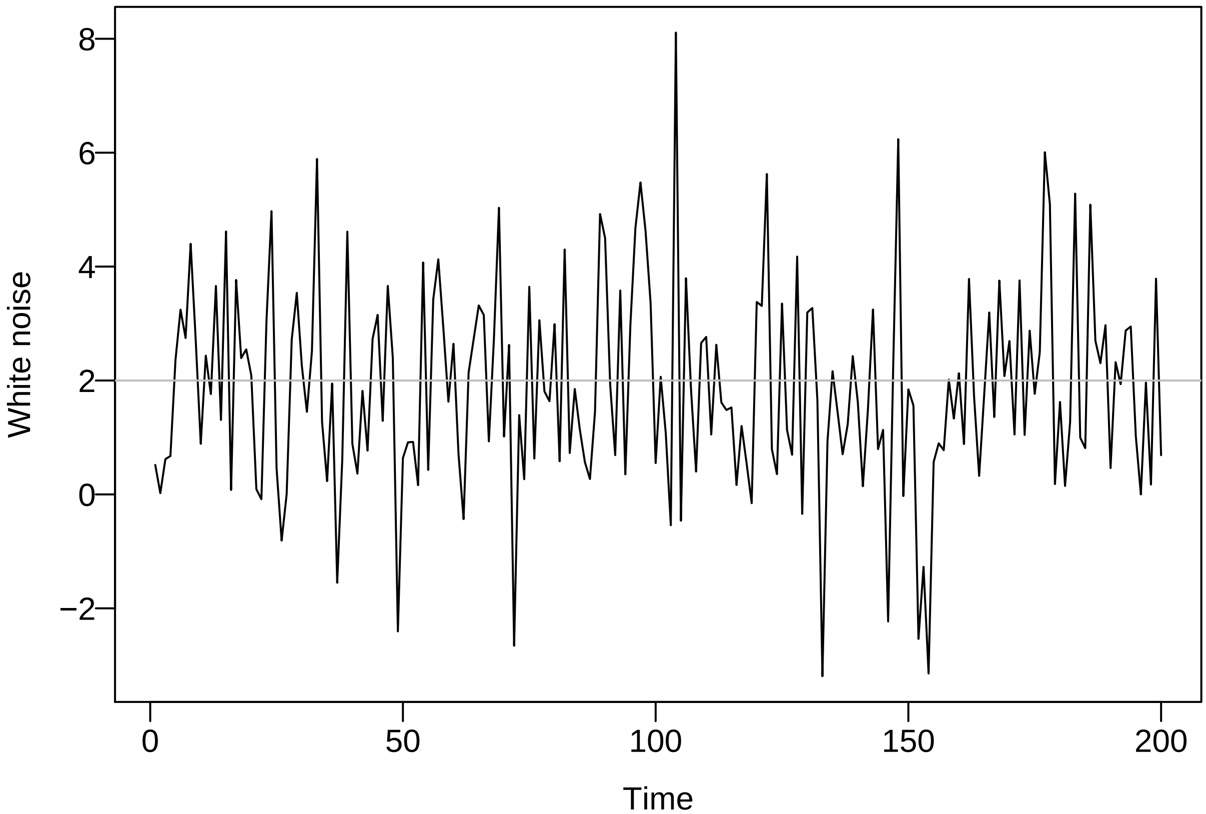

```
WN2 <- rchisq(n = T, df = 2)

plot(t, WN2, type = "l", xlab = "Time", ylab = expression(paste("i.i.d. ", chi[2], " noise")))
abline(h = 2, col = "gray")
```



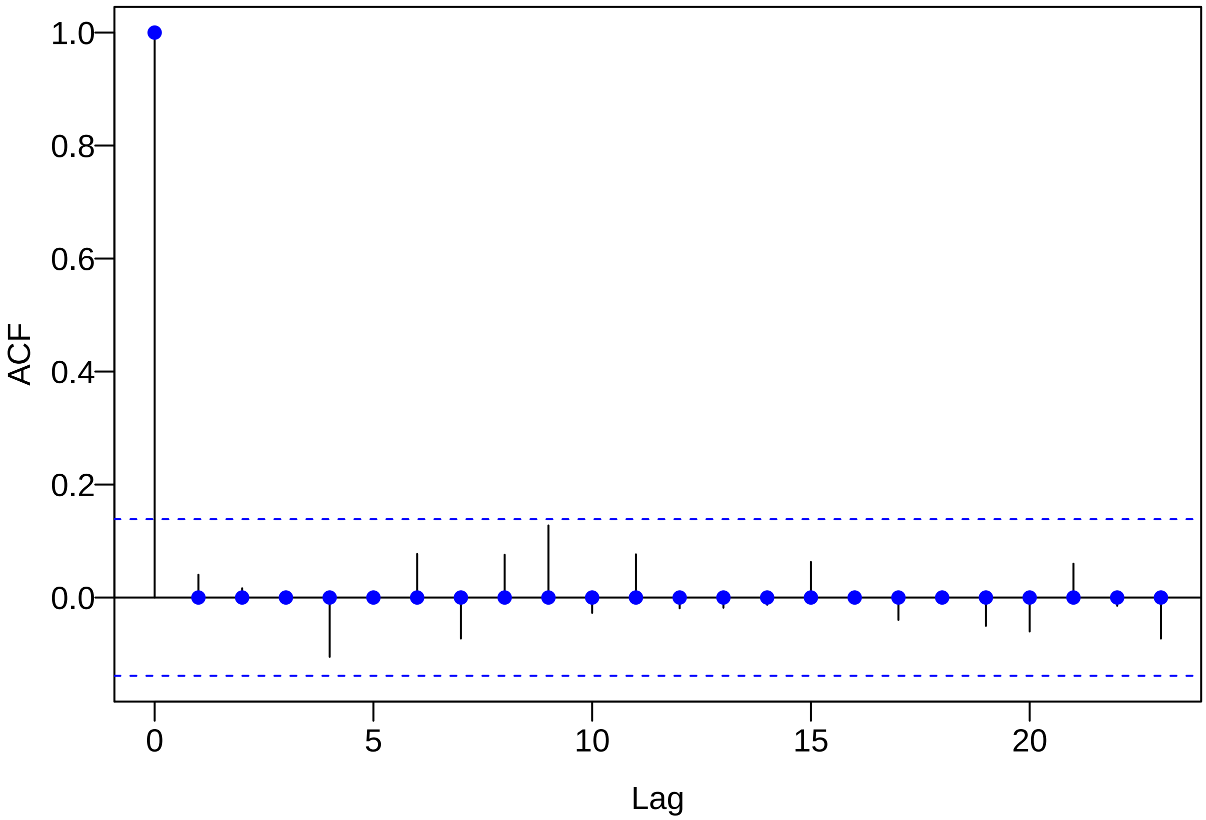

```
acf(WN2)
points(0:20, c(1, rep(0, 20)), pch = 16, col = "blue")
```




```
WN3 <- c(WN1, WN2)[sample(1:200)]
plot(1:200, WN3, type = "l", xlab = "Time", ylab = expression(paste("White noise")))
abline(h = 2, col = "gray")
```




```
acf(WN3)
points(0:23, c(1, rep(0, 23)), pch = 16, col = "blue")
```



MA(1) processes

$$\eta_t = Z_t + \theta Z_{t-1},$$

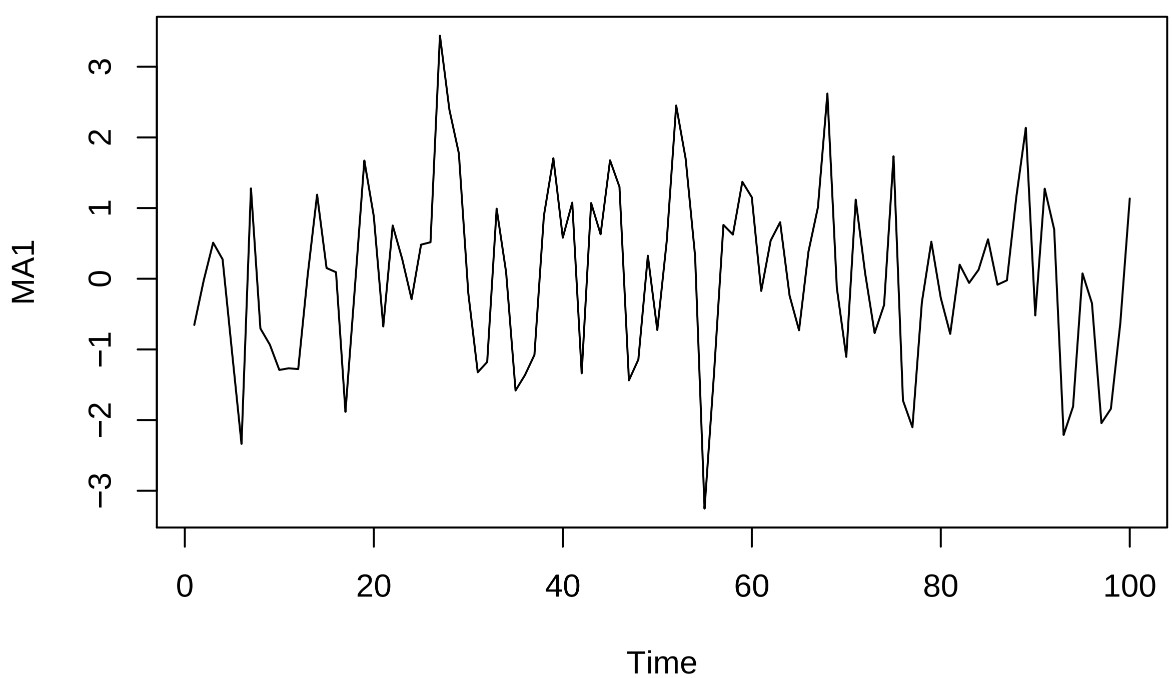
where $Z \sim \text{WN}(0, \sigma^2)$.

```
library(animation)
T = 100
t <- 1:T
z <- rnorm(110)
theta <- c(0.25, 1, -1, -0.25)
saveLatex({
  par(las = 1, mgp = c(2, 0.5, 0), mar = c(3.6, 3.6, 0.8, 0.6),
      mfrow = c(2, 1))
  for (i in 1:4){
    MA1 <- filter(z, sides = 1, c(1, theta[i]))[-(1:10)]
    plot(t, MA1, type = "l", xlab = "Time", ylab = "MA(1)")
    abline(h = 0, col = "gray")
    legend("topleft", legend = theta[i], title = expression(theta),
           bty = "n")
    acf(MA1)
    points(0:20, c(1, theta[i] / (1 + theta[i]^2), rep(0, 19)),
           pch = 16, col = "blue")
  }
}, img.name = "MA1", ani.opts = "controls,width=0.95\\textwidth",
latex.filename = ifelse(interactive(), "MA1_realizations.tex", ""),
nmax = 4, ani.dev = "pdf", ani.type = "pdf", ani.width = 8,
```

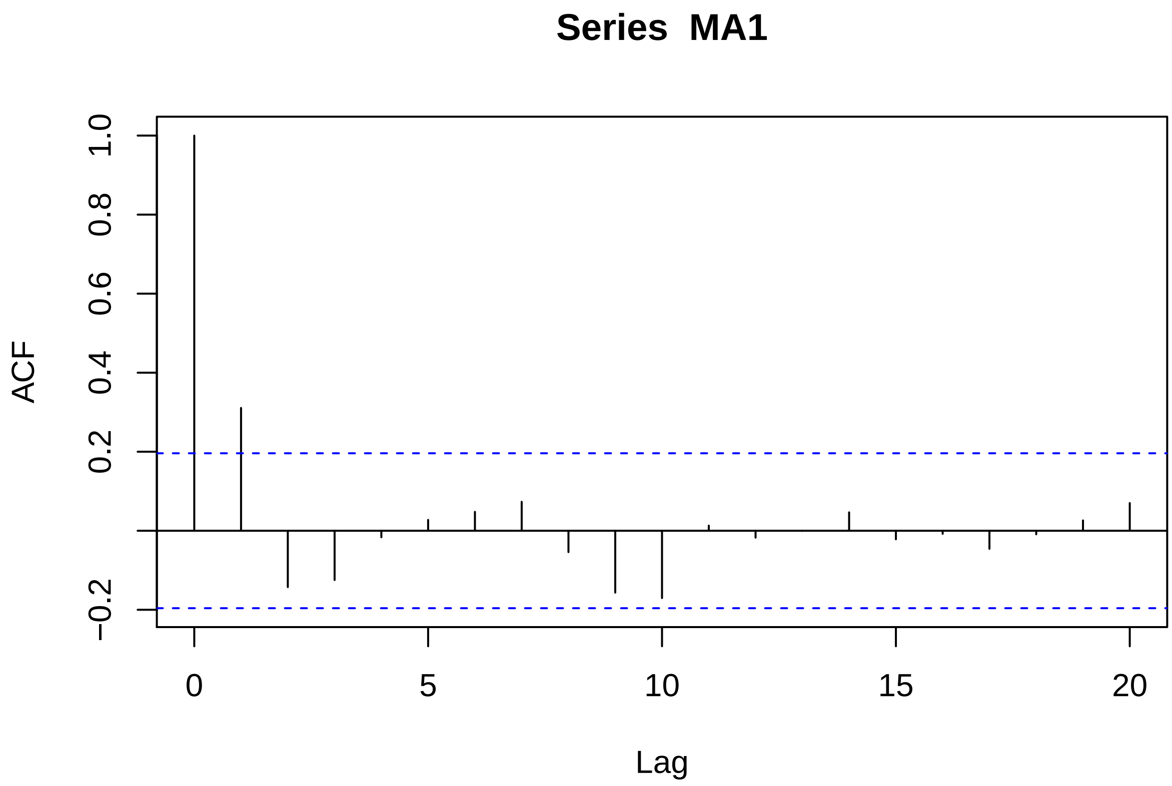
```
ani.height = 6,documentclass = paste("\\documentclass{article}",
                                      "\\usepackage[papersize={8in,6in},margin=0.1in]{geometry}",
                                      sep = "\n"))
```

```
## \documentclass{article}
## \usepackage[papersize={8in,6in},margin=0.1in]{geometry}
##                 \usepackage{animate}
##                 \begin{document}
##                 \begin{figure}
##                 \begin{center}
##                 \animategraphics[controls,width=0.95\textwidth]{1}{MA1}{0}{3}
##                 \end{center}
##                 \end{figure}
##                 \end{document}
##
```

```
##another way to simulate MA(1)
MA1 <- arima.sim(n = 100, list(ma = c(0.5)))
plot(MA1)
```




```
acf(MA1)
```



AR(1) processes

$$\eta_t = \phi\eta_{t-1} + Z_t,$$

where $|\rho| < 1$ is a constant and η_s and Z_t are uncorrelated for all $s < t \Rightarrow$ future noise is uncorrelated with the current value.

```
rho <- c(0.25, 0.9, -0.5)
saveLatex({
  par(las = 1, mgp = c(2, 0.5, 0), mar = c(3.6, 3.6, 0.8, 0.6),
      mfrow = c(2, 1))
  for (i in 1:3){
    AR1 <- arima.sim(n = 100, list(ar = c(rho[i])))
    plot(t, AR1, type = "l", xlab = "Time",
         ylab = paste("AR(1), rho = ", rho[i]))
    abline(h = 0, col = "gray")
    acf(AR1)
    points(0:20, c(1, rho[i]^(1:20)), pch = 16, col = "blue")
  }
}, img.name = "AR1", ani.opts = "controls,width=0.95\\textwidth",
latex.filename = ifelse(interactive(), "AR1_realizations.tex", ""),
nmax = 3, ani.dev = "pdf", ani.type = "pdf", ani.width = 8,
ani.height = 6,documentclass = paste("\\documentclass{article}",
                                     "\\usepackage[papersize={8in,6in},margin=0.1in]{geometry}",
                                     sep = "\n"))
```

```
## \documentclass{article}
## \usepackage[papersize={8in,6in},margin=0.1in]{geometry}
```

```
##                 \usepackage{animate}
##                 \begin{document}
##                 \begin{figure}
##                 \begin{center}
##                 \animategraphics[controls,width=0.95\textwidth]{1}{AR1}{0}{2}
##                 \end{center}
##                 \end{figure}
##                 \end{document}
##
```

Random walk

$$\eta_t = \sum_{s=1}^{t} Z_s.$$

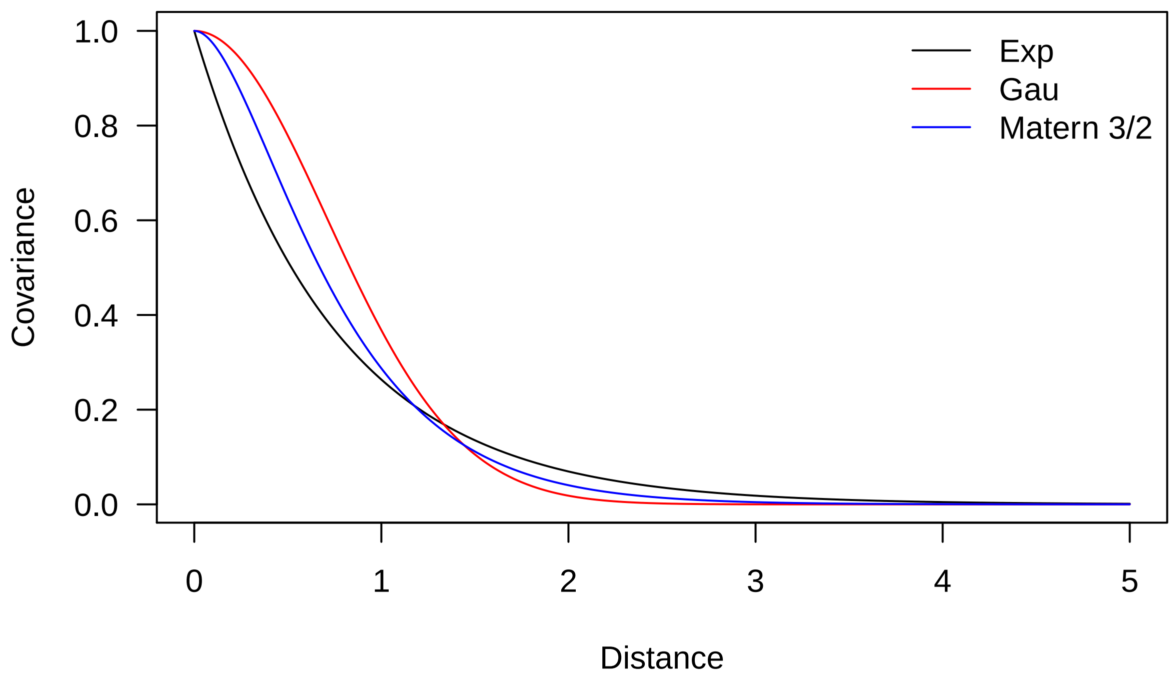
```
saveLatex({
  par(las = 1, mgp = c(2, 0.5, 0), mar = c(3.6, 3.6, 0.8, 0.6))
  for (i in 1:5){
    z <- rnorm(500)
    plot(1:500, cumsum(z), type = "l", xlab = "Time",
         ylab = "Random Walk")
    abline(h = 0, col = "gray")
  }
}, img.name = "RW", ani.opts = "controls,width=0.95\\textwidth",
latex.filename = ifelse(interactive(), "RW_realizations.tex", ""),
nmax = 5, ani.dev = "pdf", ani.type = "pdf", ani.width = 8,
ani.height = 4,documentclass = paste("\\documentclass{article}",
                                     "\\usepackage[papersize={8in,4in},margin=0.1in]{geometry}",
                                     sep = "\n"))
```

```
## \documentclass{article}
## \usepackage[papersize={8in,4in},margin=0.1in]{geometry}
##                 \usepackage{animate}
##                 \begin{document}
##                 \begin{figure}
##                 \begin{center}
##                 \animategraphics[controls,width=0.95\textwidth]{1}{RW}{0}{4}
##                 \end{center}
##                 \end{figure}
##                 \end{document}
##
```

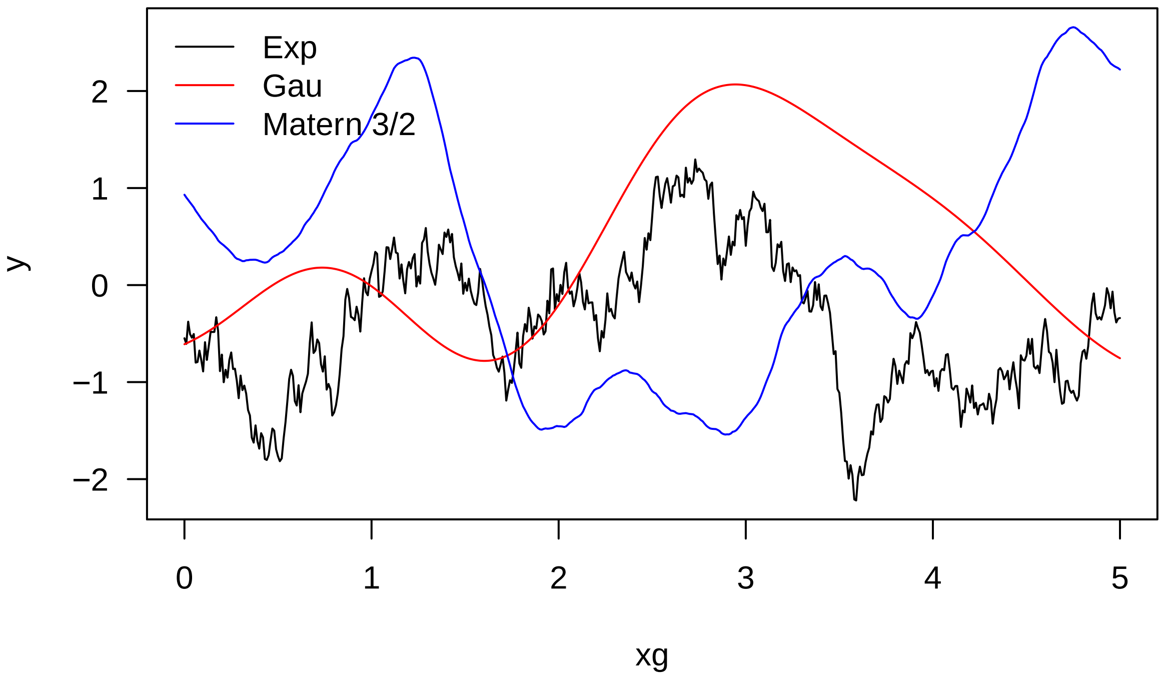
Examples realizations of Gaussian process

```
library(fields)
# Commonly used covariance functions
cov.exp <- function(h, pars) pars[1] * exp(-h / pars[2])
cov.doubleExp <- function(h, pars) pars[1] * exp(-(h / pars[2])^2)
cov.Matern <- function(h, pars) Matern(h, phi = pars[1], range = pars[2], smoothness = pars[3])
```

```r
xg <- seq(0, 5, 0.01)
c_exp <- cov.exp(xg, c(1, 0.75))
c_doubleExp <- cov.doubleExp(xg, c(1, 1))
c_Matern <- cov.Matern(xg, c(1, 0.4, 1.5))

plot(xg, c_exp, type = "l", ylab = "Covariance", xlab = "Distance", las = 1)
lines(xg, c_doubleExp, col = "red")
lines(xg, c_Matern, col = "blue")
legend("topright", legend = c("Exp", "Gau", "Matern 3/2"),
       col = c("black", "red", "blue"), lty = 1, bty = "n")
```




```r
Sigma_exp <- cov.exp(rdist(xg), c(1, 0.75))
Sigma_doubleExp <- cov.doubleExp(rdist(xg), c(1, 1))
Sigma_Matern <- cov.Matern(rdist(xg), c(1, 0.4, 1.5))
library(MASS)
set.seed(123)
sim_exp_1d <- mvrnorm(n = 1, rep(0, 501), Sigma_exp)
sim_doubleExp_1d <- mvrnorm(n = 1, rep(0, 501), Sigma_doubleExp)
sim_Matern_1d <- mvrnorm(n = 1, rep(0, 501), Sigma_Matern)

plot(xg, sim_exp_1d, type = "l", ylim = range(sim_exp_1d, sim_doubleExp_1d,
                                              sim_Matern_1d), ylab = "y", las = 1)
lines(xg, sim_doubleExp_1d, col = "red")
lines(xg, sim_Matern_1d, col = "blue")
legend("topleft", legend = c("Exp", "Gau", "Matern 3/2"),
       col = c("black", "red", "blue"), lty = 1, bty = "n")
```



ACF

Population ACF

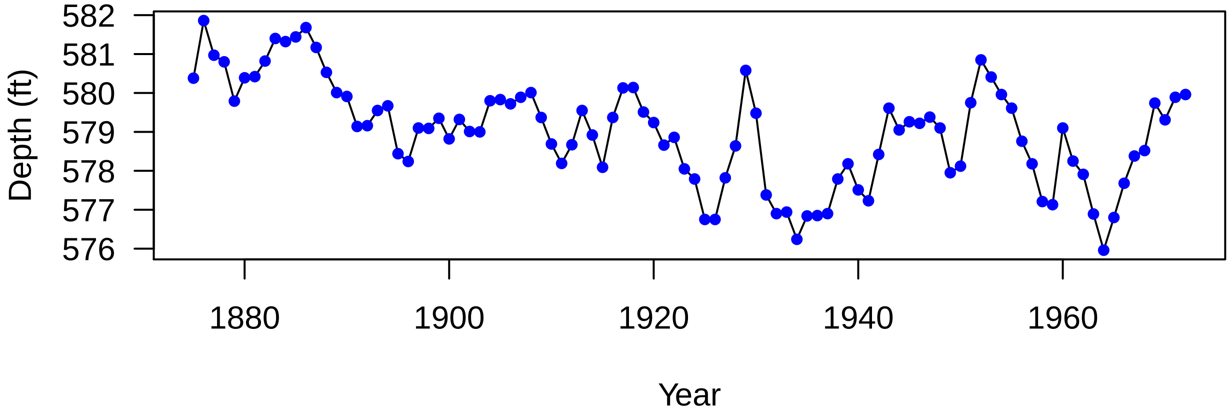
$$\rho(h) = \mathrm{Cor}(\eta_t, \eta_{t+h}) = \frac{\mathrm{E}\left[(\eta_t - \mu)(\eta_{t+h} - \mu)\right]}{\sqrt{\mathrm{Var}(\eta_t)\mathrm{Var}(\eta_{t+h})}}$$

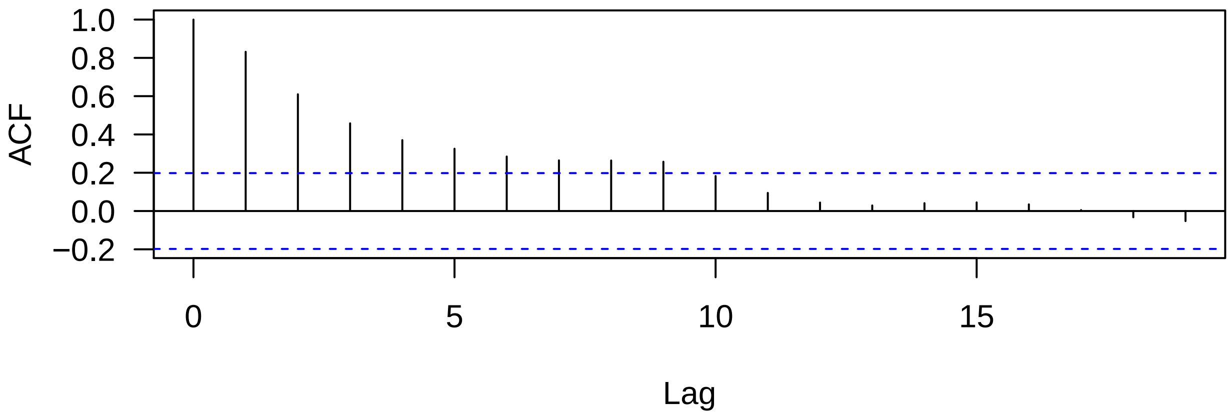
Sample ACF

$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\gamma(\hat{0})},$$

where $\hat{\gamma}(h) = \frac{1}{T}\sum_{t=1}^{T-|h|}(\eta_t - \bar{\eta})(\eta_{t+h} - \bar{\eta})$.

```
data(LakeHuron)
par(las = 1, mfrow = c(2, 1), mar = c(4, 4, 0.8, 0.6))
plot(LakeHuron, ylab = "Depth (ft)", xlab = "Year")
points(LakeHuron, cex = 0.8, col = "blue", pch = 16)
acf(LakeHuron)
```





Box test for temporal independence

Box and Pierce test Box and Pierce (1970)

We wish to test:

$$H_0 : \{\eta_1, \eta_2, \cdots, \eta_T\} \text{ is an i.i.d. noise sequence}$$
$$H_1 : H_0 \text{ is false}$$

1. Under H_0,

$$\hat{\rho}(h) \dot{\sim} \mathrm{N}(0, \frac{1}{T}) \stackrel{d}{=} \frac{1}{\sqrt{n}}\mathrm{N}(0,1)$$

2. Hence

$$Q = T\sum_{i=1}^{k} \hat{\rho}^2(h) \dot{\sim} \chi^2_{df=k}$$

3. We reject H_0 if $Q > \chi^2_k(1-\alpha)$, the $1-\alpha$ quatile of the chi-squared distribution with k degrees of freedom

Ljung-Box Test

$$Q_{LB} = T(T-2)\sum_{h=1}^{k} \frac{\hat{\rho^2}(h)}{n-h} \dot{\sim} \chi^2_k.$$

The Ljung-Box test Ljung and Box (1978) can be more powerful than the Box and Pierce test

```r
Box.test(rnorm(100), 20)
```

```
##
##  Box-Pierce test
##
## data:  rnorm(100)
## X-squared = 12.104, df = 20, p-value = 0.9125
```

```r
Box.test(LakeHuron, 20)
```

```
##
##  Box-Pierce test
##
## data:  LakeHuron
## X-squared = 182.43, df = 20, p-value < 2.2e-16
```

```r
Box.test(LakeHuron, 20, type = "Ljung")
```

```
##
##  Box-Ljung test
##
## data:  LakeHuron
## X-squared = 192.6, df = 20, p-value < 2.2e-16
```

References

Box, George EP, and David A Pierce. 1970. "Distribution of Residual Autocorrelations in Autoregressive-Integrated Moving Average Time Series Models." *Journal of the American Statistical Association* 65 (332): 1509–26.

Ljung, Greta M, and George EP Box. 1978. "On a Measure of Lack of Fit in Time Series Models." *Biometrika* 65 (2): 297–303.