Lecture 22

Simple Linear Regression I

Readings: IntroStat Chapter 11; OpenIntro Chapter 8

STAT 8010 Statistical Methods I
June 16, 2023

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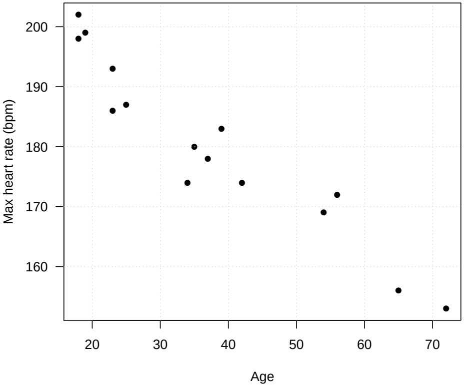
Agenda

1. **Simple Linear Regression (SLR)**
2. **Parameter Estimation in SLR**
3. **Residual Analysis**

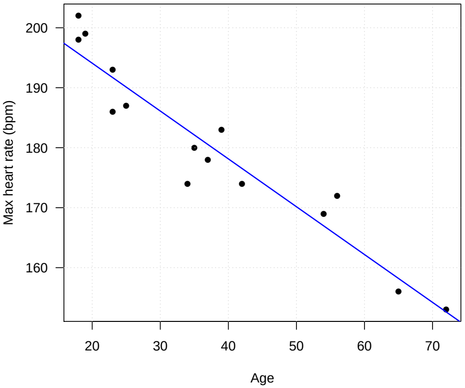
What is Regression Analysis?

Regression analysis: A set of statistical procedures for estimating the relationship between response variable and predictor variable(s)



We will focus on simple linear regression in this class

Scatterplot: Is Linear Trend Reasonable?



The relationship appears to be linear. What about the **direction** and **strength** of this linear relationship?

```
> cov(age, maxHeartRate)
[1] -243.9524
```

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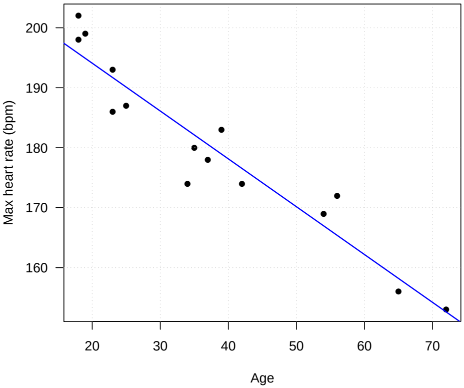

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> cov(age, maxHeartRate)
[1] -243.9524
```

```
> cor(age, maxHeartRate)
[1] -0.9534656
```

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Simple Linear Regression (SLR)

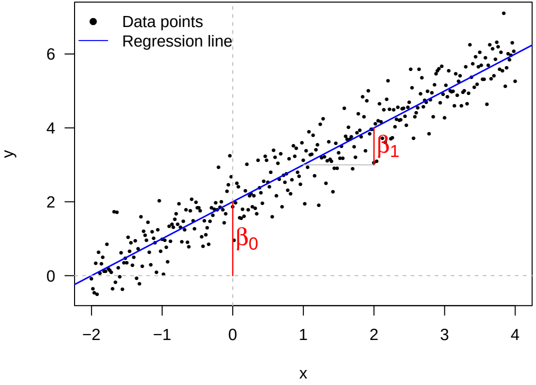
Y: dependent (response) variable; X: independent (predictor) variable

- In SLR we **assume** there is a **linear relationship** between X and Y:

$$Y = \beta_0 + \beta_1 X + \varepsilon$$

- We need to estimate β_0 (intercept) and β_1 (slope)
- We can use the estimated regression equation to
  - make predictions
  - study the relationship between response and predictor
  - control the response
- Yet we need to quantify our estimation uncertainty regarding the linear relationship (will talk about this next time)

Regression equation: $Y = \beta_0 + \beta_1 X$



- β_0: $\mathrm{E}[Y]$ when $X = 0$
- β_1: $\mathrm{E}[\Delta Y]$ when X increases by 1

Assumptions about the Random Error ε

In order to estimate β_0 and β_1, we make the following assumptions about ε

- $\mathrm{E}[\varepsilon_i] = 0$
- $\mathrm{Var}[\varepsilon_i] = \sigma^2$
- $\mathrm{Cov}[\varepsilon_i, \varepsilon_j] = 0, \quad i \neq j$

Therefore, we have

$$\mathrm{E}[Y_i] = \beta_0 + \beta_1 X_i, \text{ and}$$
$$\mathrm{Var}[Y_i] = \sigma^2$$

The regression line $\beta_0 + \beta_1 X$ represents the **conditional expectation curve** whereas σ^2 measures the magnitude of the **variation** around the regression curve

Estimation: Method of Least Square

For the given observations $(x_i, y_i)_{i=1}^n$, choose β_0 and β_1 to minimize the *sum of squared errors*:

$$\mathrm{L}(\beta_0, \beta_1) = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2$$

Solving the above minimization problem requires some knowledge from Calculus....

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We also need to **estimate** σ^2

$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2}{n-2}$, where $\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i$

Properties of Least Squares Estimates

- **Gauss-Markov** theorem states that in a linear regression these least squares estimators

1. **Are unbiased**, i.e.,
   - $\mathrm{E}[\hat{\beta}_1] = \beta_1; \mathrm{E}[\hat{\beta}_0] = \beta_0$
   - $\mathrm{E}[\hat{\sigma}^2] = \sigma^2$
2. Have **minimum variance** among all unbiased linear estimators

Note that we do not make any distributional assumption on ε_i

Example: Maximum Heart Rate vs. Age

The maximum heart rate `MaxHeartRate` of a person is often said to be related to age `Age` by the equation:

$$\text{MaxHeartRate} = 220 - \text{Age}.$$

Suppose we have 15 people of varying ages are tested for their maximum heart rate (bpm) (link to the “dataset": http://whitneyhuang83.github.io/maxHeartRate.csv)

1. Compute the estimates for the regression coefficients
2. Compute the fitted values
3. Compute the estimate for σ

Estimate the Parameters β_1, β_0, and σ^2

Y_i and X_i are the Maximum Heart Rate and Age of the i^{th} individual

- To obtain $\hat{\beta}_1$
  1. Compute $\bar{Y} = \frac{\sum_{i=1}^{n} Y_i}{n}$, $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$
  2. Compute $Y_i - \bar{Y}$, $X_i - \bar{X}$, and $(X_i - \bar{X})^2$ for each observation
  3. Compute $\sum_i^n (X_i - \bar{X})(Y_i - \bar{Y})$ divived by $\sum_i^n (X_i - \bar{X})^2$
- $\hat{\beta}_0$: Compute $\bar{Y} - \hat{\beta}_1 \bar{X}$
- $\hat{\sigma}^2$
  1. Compute the fitted values: $\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i, \quad i = 1, \cdots, n$
  2. Compute the **residuals** $e_i = Y_i - \hat{Y}_i, \quad i = 1, \cdots, n$
  3. Compute the **residual sum of squares (RSS)** $= \sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2$ and divided by $n - 2$ (why?)

Let's Do the Calculations

$$\bar{X} = \sum_{i=1}^{15} \frac{18 + 23 + \cdots + 39 + 37}{15} = 37.33$$

$$\bar{Y} = \sum_{i=1}^{15} \frac{202 + 186 + \cdots + 183 + 178}{15} = 180.27$$

X	18	23	25	35	65	54	34	56	72	19	23	42	18	39	37
Y	202	186	187	180	156	169	174	172	153	199	193	174	198	183	178
	-19.33	-14.33	-12.33	-2.33	27.67	16.67	-3.33	18.67	34.67	-18.33	-14.33	4.67	-19.33	1.67	-0.33
	21.73	5.73	6.73	-0.27	-24.27	-11.27	-6.27	-8.27	-27.27	18.73	12.73	-6.27	17.73	2.73	-2.27
	-420.18	-82.18	-83.04	0.62	-671.38	-187.78	20.89	-154.31	-945.24	-343.44	-182.51	-29.24	-342.84	4.56	0.76
	373.78	205.44	152.11	5.44	765.44	277.78	11.11	348.44	1201.78	336.11	205.44	21.78	373.78	2.78	0.11
	195.69	191.70	190.11	182.13	158.20	166.97	182.93	165.38	152.61	194.89	191.70	176.54	195.69	178.94	180.53

- $\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{n}(X_i - \bar{X})^2} = -0.7977$
- $\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X} = 210.0485$
- $\hat{\sigma}^2 = \frac{\sum_{i=1}^{15}(Y_i - \hat{Y}_i)^2}{13} = 20.9563 \Rightarrow \hat{\sigma} = 4.5778$

Let's Double Check

Output from R (RStudio)

```
> fit <- lm(MaxHeartRate ~ Age)
> summary(fit)

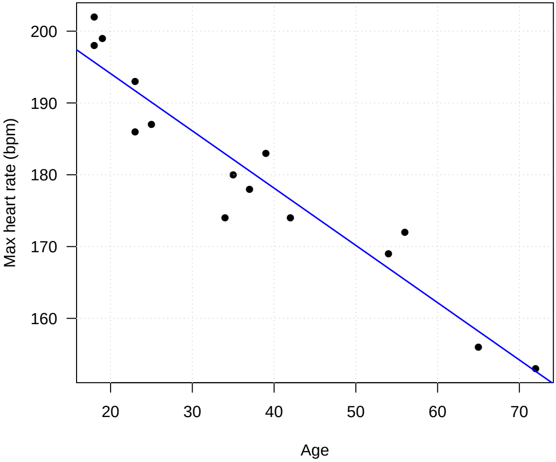
Call:
lm(formula = MaxHeartRate ~ Age)

Residuals:
    Min      1Q  Median      3Q     Max
-8.9258 -2.5383  0.3879  3.1867  6.6242

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 210.04846    2.86694   73.27  < 2e-16 ***
Age          -0.79773    0.06996  -11.40 3.85e-08 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 4.578 on 13 degrees of freedom
Multiple R-squared:  0.9091,	Adjusted R-squared:  0.9021
F-statistic:   130 on 1 and 13 DF,  p-value: 3.848e-08
```

Linear Regression Fit



Question: Is linear relationship between max heart rate and age reasonable? ⇒ Residual Analysis

Residuals

- The residuals are the differences between the observed and fitted values:

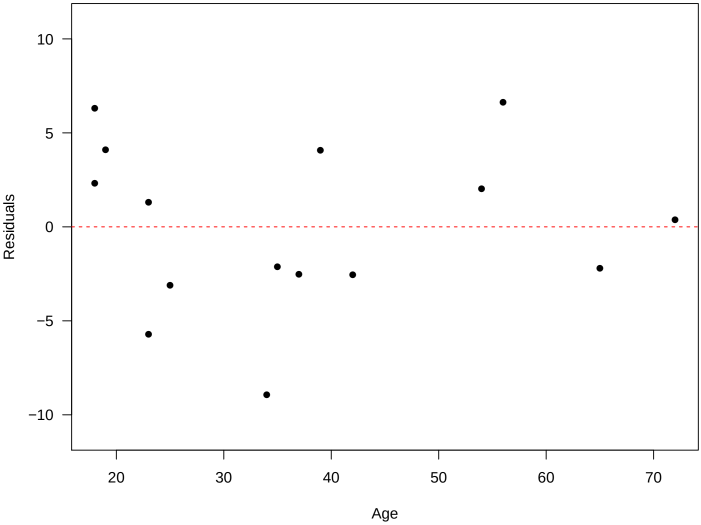
$$e_i = Y_i - \hat{Y}_i,$$

where $\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i$

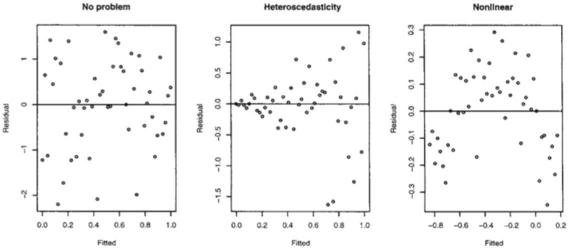
- e_i is NOT the error term $\varepsilon_i = Y_i - \mathrm{E}[Y_i]$

- Residuals are very useful in assessing the appropriateness of the assumptions on ε_i. Recall
  - $\mathrm{E}[\varepsilon_i] = 0$
  - $\mathrm{Var}[\varepsilon_i] = \sigma^2$
  - $\mathrm{Cov}[\varepsilon_i, \varepsilon_j] = 0, \quad i \neq j$

Maximum Heart Rate vs. Age Residual Plot: ε vs. X



Interpreting Residual Plots



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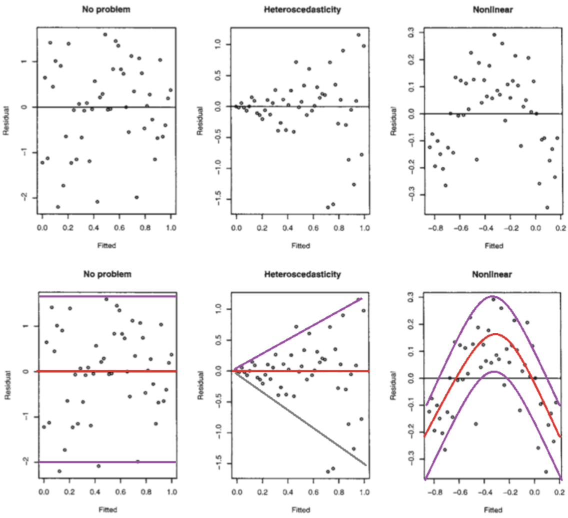

Figure: Figure courtesy of Faraway's Linear Models with R (2005, p. 59).

Summary

In this lecture, we learned

- Simple Linear Regression
- Least Squares Estimation
- Residual Analysis