

# Lecture 5

## Law of Total Probability, Bayes' Rule, and Random Variables

Text: Chapter 4

STAT 8010 Statistical Methods I  
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### Agenda

- 1 Law of Total Probability
- 2 Bayes' Rule
- 3 Random Variables



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### Law of Partitions & Multiplication Rule

#### Law of partitions

Let  $A_1, A_2, \dots, A_k$  form a partition of  $\Omega$ . Then, for all events  $B$ ,

$$\mathbb{P}(B) = \sum_{i=1}^k \mathbb{P}(A_i \cap B)$$

#### Multiplication rule

- 2 events:

$$\mathbb{P}(B \cap A) = \mathbb{P}(A) \times \mathbb{P}(B|A) = \mathbb{P}(B) \times \mathbb{P}(A|B)$$

- More than 2 events:

$$\mathbb{P}(\cap_{i=1}^n A_i) = \mathbb{P}(A_1) \times \mathbb{P}(A_2|A_1) \times \mathbb{P}(A_3|A_1 \cap A_2) \\ \times \dots \times \mathbb{P}(A_n|A_{n-1} \cap \dots \cap A_1)$$



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Law of Total Probability

Let  $A_1, A_2, \dots, A_k$  form a partition of  $\Omega$ . Then, for all events  $B$ ,

$$\begin{aligned} \mathbb{P}(B) &= \sum_{i=1}^k \underbrace{\mathbb{P}(A_i \cap B)}_{\text{Law of partitions}} \\ &= \sum_{i=1}^k \underbrace{\mathbb{P}(B|A_i) \times \mathbb{P}(A_i)}_{\text{Multiplication rule}} \end{aligned}$$

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Example

Suppose that two factories supply light bulbs to the market. Factory X's bulbs work for over 5000 hours in 99% of cases, whereas factory Y's bulbs work for over 5000 hours in 95% of cases. It is known that factory X supplies 60% of the total bulbs available and Y supplies 40% of the total bulbs available. What is the chance that a purchased bulb will work for longer than 5000 hours?

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Bayes' Rule: Motivating example

The Monty Hall Problem

There was an old television show called Let's Make a Deal, whose original host was named Monty Hall. The set-up is as follows. You are on a game show and you are given the choice of three doors. Behind one door is a car, behind the others are goats. You pick a door, and the host, who knows what is behind the doors, opens another door (not your pick) which has a goat behind it. Then he asks you if you want to change your original pick. The question we ask you is, "Is it to your advantage to switch your choice?"

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The Monty Hall Problem



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The Monty Hall Problem Solution

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Bayes' Rule

General form

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}$$

Let  $A_1, A_2, \dots, A_k$  form a partition of the sample space.  
Then for every event  $B$  in the sample space,

$$\mathbb{P}(A_j|B) = \frac{\mathbb{P}(B|A_j) \times \mathbb{P}(A_j)}{\sum_{i=1}^k \mathbb{P}(B|A_i) \times \mathbb{P}(A_i)}, \quad j = 1, 2, \dots, k$$

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Example

Let us assume that a specific disease is only present in 5 out of every 1,000 people. Suppose that the test for the disease is accurate 99% of the time a person has the disease and 95% of the time that a person lacks the disease. What is the probability that the person has the disease given that they tested positive?

Solution.

$$\mathbb{P}(D|+) = \frac{\mathbb{P}(D \cap +)}{\mathbb{P}(+)} = \frac{.005 \times .99}{.005 \times .99 + .995 \times .05} = \frac{.00495}{.0547} = .0905$$

The reason we get such a surprising result is because the disease is so rare that the number of false positives greatly outnumbers the people who truly have the disease.

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Review of Probability (we learned so far)

Basic Concepts:

- Random Experiment, Sample Space, Outcome, Event
- Frequentist Interpretation of Probability and Equally Likely Framework
- Union and Intersection, Mutually Exclusive, Exhaustive, Partition
- Venn Diagram
- Independence and Conditional Probability

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Review: Probability Rules

- $0 \leq \mathbb{P}(A) \leq 1$  for any event  $A$ ,  $\mathbb{P}(\emptyset) = 0$ ,  $\mathbb{P}(\Omega) = 1$
- Complement rule:  $\mathbb{P}(A) = 1 - \mathbb{P}(A^c)$
- General addition rule:  
 $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$
- Multiplication rule:  
 $\mathbb{P}(A \cap B) = \mathbb{P}(A|B) \times \mathbb{P}(B) = \mathbb{P}(B|A) \times \mathbb{P}(A)$
- Conditional probability:  $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$
- Law of total probability:  
 $\mathbb{P}(B) = \sum_{i=1}^k \mathbb{P}(B \cap A_i) = \sum_{i=1}^k \mathbb{P}(B|A_i) \times \mathbb{P}(A_i)$
- Independence: if  $A$  and  $B$  are independent, then  
 $\mathbb{P}(A|B) = \mathbb{P}(A)$ ,  $\mathbb{P}(B|A) = \mathbb{P}(B)$ , and  
 $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$

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# Random Variables

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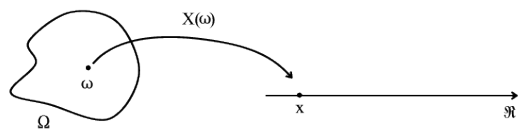
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Random Variables

A **random variable** is a real-valued function whose domain is the sample space of a random experiment. In other words, a random variable is a function

$$X : \Omega \mapsto \mathbb{R}$$

where  $\Omega$  is the sample space of the random experiment under consideration and  $\mathbb{R}$  represents the set of all real numbers.



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Example

The following is a chart describing the number of siblings each student in a particular class has.

Siblings ( $X$ )	Frequency	Relative Frequency
0	8	.200
1	17	.425
2	11	.275
3	3	.075
4	1	.025
Total	40	1

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Example cont'd

Let's define the event  $A$  as the event that a randomly chosen student has 2 or more siblings. What is  $\mathbb{P}(X \in A)$ ?  
**Solution.**

$$\begin{aligned}\mathbb{P}(X \in A) &= \mathbb{P}(X \geq 2) = \mathbb{P}(X = 2) + \mathbb{P}(X = 3) + \mathbb{P}(X = 4) \\ &= .275 + 0.75 + 0.25 = .375\end{aligned}$$

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Types of Random Variables

There are two main types of quantitative random variables: **discrete** and **continuous**. A discrete random variable often involves a count of something. Examples may include number of cars per household, etc.

**Discrete random variable**  
A random variable  $X$  is called a discrete random variable if the outcome of the random variable is limited to a countable set of real numbers (usually integers).

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Probability Mass Function

Let  $X$  be a discrete random variable. Then the probability mass function (pmf) of  $X$  is the real-valued function defined on  $\mathbb{R}$  by

$$p_X(x) = \mathbb{P}(X = x)$$

The capital letter,  $X$ , is used to denote random variable. Lowercase letter,  $x$ , is used to denote possible values of the random variable.

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Example

Flip a fair coin 3 times. Let  $X$  denote the number of heads tossed in the 3 flips. Create a pmf for  $X$

Solution.

The random variable  $X$  maps any outcome to an integer (e.g.  $X((T, T, T)) = 0$ )

$X$	0	1	2	3
$p_X(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

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Properties of a PMF

- $0 \leq p_X(x) \leq 1, \forall x \in \mathcal{R}$
- $\{x \in \mathcal{R} : p_X(x) \neq 0\}$  is countable
- $\sum_x p_X(x) = 1$

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Example

Let  $X$  be a random variable with pmf defined as follows:

$$p_X(x) = \begin{cases} k(5-x) & \text{if } x = 0, 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}$$

- 1 Find the value of  $k$  that makes  $p_X(x)$  a legitimate pmf.
- 2 What is the probability that  $X$  is between 1 and 3 inclusive?
- 3 If  $X$  is not 0, what is the probability that  $X$  is less than 3?

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Mean of Discrete Random Variables

The mean of a discrete r.v.  $X$ , denoted by  $\mathbb{E}[X]$ , is defined by

$$\mathbb{E}[X] = \sum_x x \times p_X(x)$$

Remark:

The mean of a discrete r.v. is a weighted average of its possible values, and the weight used is its probability. Sometimes we refer to the expected value as the **expectation (expected value)**, or the **first moment**.

For any function, say  $g(X)$ , we can also find an expectation of that function. It is

$$\mathbb{E}[g(X)] = \sum_x g(x) \times p_X(x)$$

Example

$$\mathbb{E}[X^2] = \sum_x x^2 \times p_X(x)$$

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Properties of Mean

Let  $X$  and  $Y$  be discrete r.v.s defined on the same sample space and having finite expectation (i.e.  $\mathbb{E}[X], \mathbb{E}[Y] < \infty$ ). Let  $a$  and  $b$  be constants. Then the following hold:

- $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$
- $\mathbb{E}[aX + b] = a \times \mathbb{E}[X] + b$

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Number of Siblings Example Revisited

Siblings ( $X$ )	Frequency	Relative Frequency
0	8	.200
1	17	.425
2	11	.275
3	3	.075
4	1	.025
Total	40	1

Find the expected value of the number of siblings

Solution.

$$\mathbb{E}[X] = \sum_x x p_X(x) = 0 \times .200 + 1 \times .425 + 2 \times .275 + 3 \times .075 + 4 \times .025 = 1.3$$

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Variance/Standard Deviation of Discrete r.v.'s

The **variance** of a (discrete) r.v., denoted by  $Var(X)$ , is a measure of the spread, or variability, in the r.v.  $Var(X)$  is defined by

$$Var(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

or

$$Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

The **standard deviation**, denoted by  $Sd(X)$ , is the square root of its variance

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Properties of Variance

Let  $c$  be a constant. Then the following hold:

- $Var(cX) = c^2 \times Var(X)$
- $Var(X + c) = Var(X)$

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Example

Suppose  $X$  and  $Y$  are random variables with  $\mathbb{E}[X] = 3$ ,  $\mathbb{E}[Y] = 4$  and  $Var(X) = 4$ . Find:

- 1  $\mathbb{E}[2X + 1]$
- 2  $\mathbb{E}[X - Y]$
- 3  $\mathbb{E}[X^2]$
- 4  $\mathbb{E}[X^2 - 4]$
- 5  $\mathbb{E}[(X - 4)^2]$
- 6  $Var(2X - 4)$

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