Lecture 7

Logistic Regression and Poisson Regression

Reading: Faraway 2016 Chapters 2.1-2.5; 5.1; 8.1; ISLR 2021 Chapter 4.2; 4.3.1-4.3.4; 4.6

DSA 8020 Statistical Methods II

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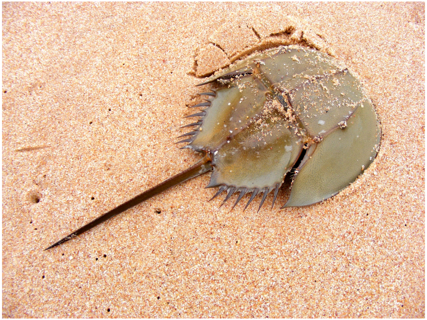
Agenda

A Motivating Example: Horseshoe Crab Mating [Brockmann, 1996; Agresti, 2013]



sat	y	weight	width
8	1	3.05	28.3
0	0	1.55	22.5
9	1	2.30	26.0
0	0	2.10	24.8
4	1	2.60	26.0
0	0	2.10	23.8
0	0	2.35	26.5
0	0	1.90	24.7
0	0	1.95	23.7
0	0	2.15	25.6

Source: `https://www.britannica.com/story/horseshoe-crab-a-key-player-in-ecology-medicine-and-more`

We are going to use this dataset to illustrate logistic regression. The response variable is $y \in \{0, 1\}$, indicates whether males cluster around the female

Logistic Regression

Let $\mathrm{P}(y = 1) = \pi \in [0, 1]$, and x be the predictor (e.g., `weight` in the previous example). In SLR we have

$$\pi(x) = \beta_0 + \beta_1 x,$$

which will lead to invalid estimate of π (i.e., > 1 or < 0).

Logistic Regression

$$\log\left(\frac{\pi(x)}{1 - \pi(x)}\right) = \beta_0 + \beta_1 x.$$

- $\log(\frac{\pi}{1-\pi})$: the log-odds or the logit
- $\pi(x) = \frac{e^{\beta_0 + \beta_1 x}}{1 + e^{\beta_0 + \beta_1 x}} \in (0, 1)$

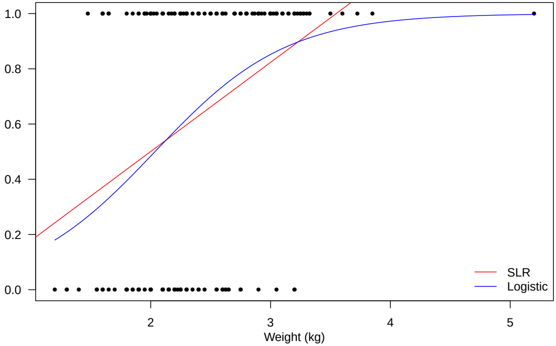
Linear and Logistic Regression Fits of Horseshoe Crab Mating Data

Linear regression:

$$\hat{y}(x) = \hat{\beta}_0 + \hat{\beta}_1 x, \hat{\beta}_0 = -0.1449(0.1472), \hat{\beta}_1 = 0.3227(0.0588)$$

Logistic regression:

$$\hat{\pi}(x) = \frac{e^{\hat{\beta}_0 + \hat{\beta}_1 x}}{1+e^{\hat{\beta}_0 + \hat{\beta}_1 x}}, \hat{\beta}_0 = -3.6947(0.8802), \hat{\beta}_1 = 1.8151(0.3767)$$



Properties of Logistic Regression

- Similar to sinple linear regression, the sign of β_1 indicates whether $\pi(x) \uparrow$ or $\downarrow$ as $x \uparrow$
- If $\beta_1 = 0$, then $\pi(x) = e^{\beta_0}/(1 + e^{\beta_0})$ is a constant w.r.t x (i.e., $\pi = \mathrm{P}(y = 1)$ does not depend on x)
- Logistic curve can be approximated at fixed x by straight line to describe rate of change: $\frac{d\pi(x)}{dx} = \beta_1 \pi(x)(1 - \pi(x))$
- $\pi(-\beta_0/\beta_1) = 0.5$
- $1/\beta_1$ is approximately equal to the distance between the x values where $\pi(x) = 0.5$ and $\pi(x) = 0.75$ (or $\pi(x) = 0.25$)

Odds Ratio Interpretation

Recall $\log\left(\frac{\pi(x)}{1-\pi(x)}\right) = \beta_0 + \beta_1 x$, we have the odds

$$\frac{\pi(x)}{1-\pi(x)} = \exp(\beta_0 + \beta_1 x).$$

If we increase x by 1 unit, the the odds becomes

$$\exp(\beta_0 + \beta_1(x+1)) = \exp(\beta_1) \times \exp(\beta_0 + \beta_1 x).$$

$$\Rightarrow \frac{\text{Odds at } x+1}{\text{Odds at } x} = \exp(\beta_1),\ \forall x$$

In the horseshoe crab example, we have

$$\hat{\beta}_1 = 1.8151 \Rightarrow e^{1.8151} = 6.14$$

$\Rightarrow$ Estimated odds of satellite multiply by 6.1 for 1 kg increase in weight.

Parameter Estimation

In logistic regression we use the method of maximum likelihood to estimate the parameters:

- **Statistical model**: $y_i \sim$ Bernoulli$(\pi(x_i))$ where $\pi(x_i) = \frac{\exp(\beta_0+\beta_1 x_i)}{1+\exp(\beta_0+\beta_1 x_i)}$.

- **Likelihood function**: We can write the joint probability density of the data $\{x_i, y_i\}_{i=1}^n$ as

$$\prod_{i=1}^{n} \left[\frac{\exp(\beta_0 + \beta_1 x_i)}{1 + \exp(\beta_0 + \beta_1 x_i)}\right]^{y_i} \left[\frac{1}{1 + \exp(\beta_0 + \beta_1 x_i)}\right]^{(1-y_i)}.$$

  We treat this as a function of parameters (β_0, β_1) given data.

- **Maximum likelihood estimate**: The maximizer $\hat{\beta}_0, \hat{\beta}_1$ is the maximum likelihood estimate. This maximization (for logistic regression) can only be solved numerically.

Horseshoe Crab Logistic Regression Fit

```
> logitFit <- glm(y ~ weight, data = crab, family = "binomial")
> summary(logitFit)

Call:
glm(formula = y ~ weight, family = "binomial", data = crab)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-2.1108  -1.0749   0.5426   0.9122   1.6285

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept)  -3.6947     0.8802  -4.198 2.70e-05 ***
weight        1.8151     0.3767   4.819 1.45e-06 ***
---
Signif. codes:
0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 225.76  on 172  degrees of freedom
Residual deviance: 195.74  on 171  degrees of freedom
AIC: 199.74

Number of Fisher Scoring iterations: 4
```

Inference: Confidence Interval

A 95% confidence interval of the parameter β_i is

$$\hat{\beta}_i \pm z_{0.025} \times \mathrm{SE}(\hat{\beta}_i), \quad i = 0, 1$$

Horseshoe Crab Example
A 95% (Wald) confidence interval of β_1 is

$$1.8151 \pm 1.96 \times 0.3767 = [1.077, 2.553]$$

Therefore, a 95% CI of e^{β_1}, the multiplicative effect on odds of 1-unit increase in x, is

$$[e^{1.077}, e^{2.553}] = [2.94, 12.85]$$

Inference: Hypothesis Test

Null and Alternative Hypotheses:

$H_0 : \beta_1 = 0 \Rightarrow y$ is independent of $x \Rightarrow \pi(x)$ is a constant
$H_a : \beta_1 \neq 0$

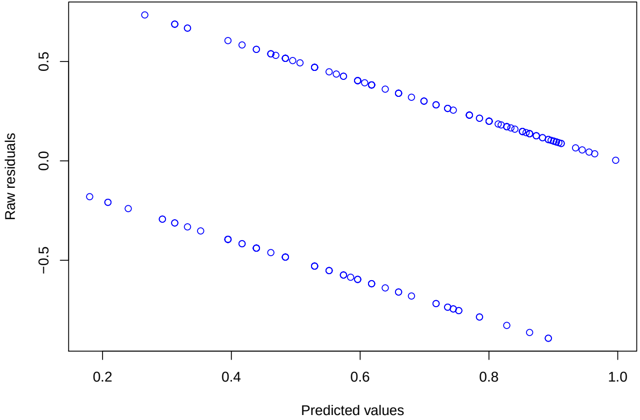
Test Statistics:

$$z_{obs} = \frac{\hat{\beta}_1}{\text{SE}(\hat{\beta}_1)} = \frac{1.8151}{0.3767} = 4.819.$$

$\Rightarrow p$-value $= 1.45 \times 10^{-6}$

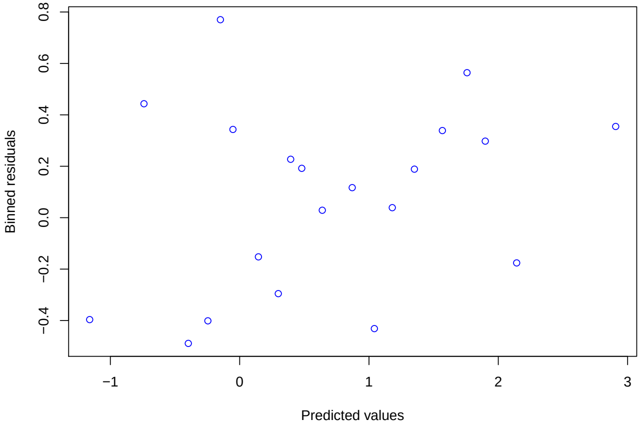
> We have sufficient evidence that `weight` has positive effect on π, the probability of having satellite male horse-shoe crabs

Diagnostic: Raw Residual Plot



The raw residual plot is not very informative because the response variable, y, only takes two possible values

Diagnostic: Binned Residual Plot



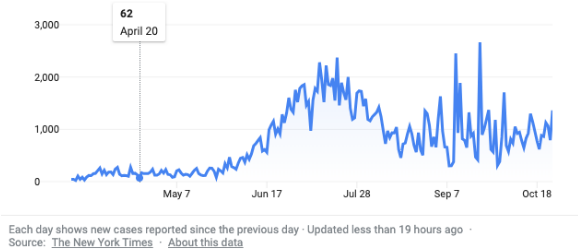
- Grouping the residuals into bins and calculating the average for each bin
- $\log\left(\frac{\hat{\pi}(x)}{1-\hat{\pi}(x)}\right)$ is plotted on the horizontal axis (rather than the $\hat{\pi}(x)$) to provide better spacing

Model Selection

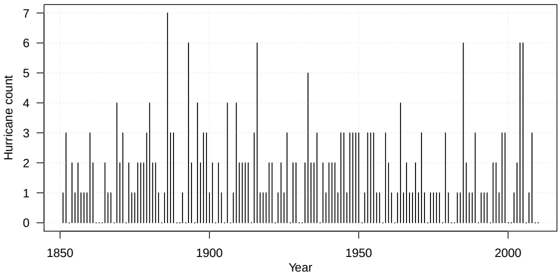
```
> logitFit2 <- glm(y ~ weight + width, data = crab, family = "binomial")
> step(logitFit2)
Start:  AIC=198.89
y ~ weight + width

         Df Deviance    AIC
- weight  1   194.45 198.45
<none>        192.89 198.89
- width   1   195.74 199.74

Step:  AIC=198.45
y ~ width

        Df Deviance    AIC
<none>       194.45 198.45
- width  1   225.76 227.76

Call:  glm(formula = y ~ width, family = "binomial", data = crab)

Coefficients:
(Intercept)        width
   -12.3508       0.4972

Degrees of Freedom: 172 Total (i.e. Null);  171 Residual
Null Deviance:      225.8
Residual Deviance: 194.5        AIC: 198.5
```

Count Data

- Daily COVID-19 Cases in South Carolina



- Number of landfalling hurricanes per hurricane season



Modeling Count Data

So far we have talked about:

- Linear regression: $y = \beta_0 + \beta_1 x + \varepsilon$, $\varepsilon \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0, \sigma^2)$
- Logistic Regression: $\log(\frac{\pi}{1-\pi}) = \beta_0 + \beta_1 x, \quad \pi = \mathrm{P}(Y = 1)$

Count data

- Counts typically have a right skewed distribution
- Counts are not necessarily binary

We can use Poisson Regression to model count data

Poisson Distribution

- If Y follow a Poisson distribution, then we have

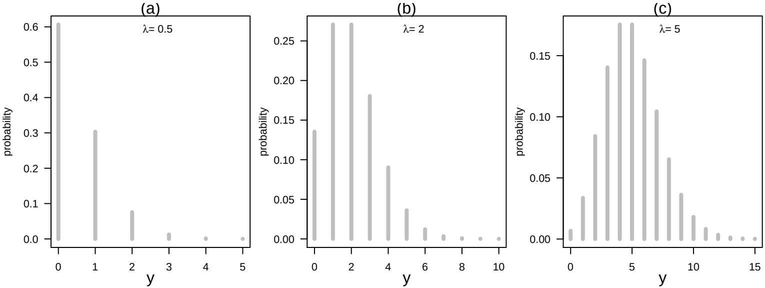
$$\mathrm{P}(Y = y) = \frac{e^{-\lambda}\lambda^{y}}{y!}, \quad y = 0, 1, 2, \cdots,$$

  where λ is the rate parameter that represents the event occurrence frequency

- $\mathrm{E}(Y) = \mathrm{Var}(Y) = \lambda$ if $Y \sim \mathrm{Pois}(\lambda), \quad \lambda > 0$

- A useful model to describe the probability of a given number of events occurring in a fixed interval of time or space

Poisson Probability Mass Function



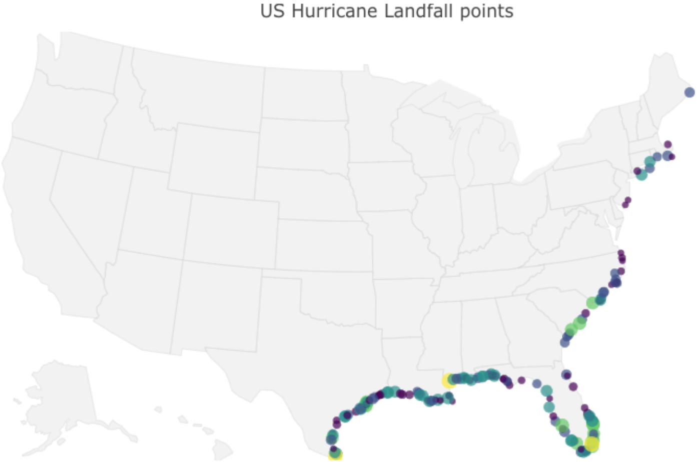
- (a): $\lambda = 0.5$: distribution gives highest probability to $y = 0$ and falls rapidly as y ↑
- (b): $\lambda = 2$: a skew distribution with longer tail on the right
- (c): $\lambda = 5$: distribution become more normally shaped

Flying-Bomb Hits on London During World War II [Clarke, 1946; Feller, 1950]

The City of London was divided into 576 small areas of one-quarter square kilometers each, and the number of areas hit exactly k times was counted. There were a total of 537 hits, so the average number of hits per area was $\frac{537}{576} = 0.9323$. The observed frequencies in the table below are remarkably close to a Poisson distribution with rate $\lambda = 0.9323$

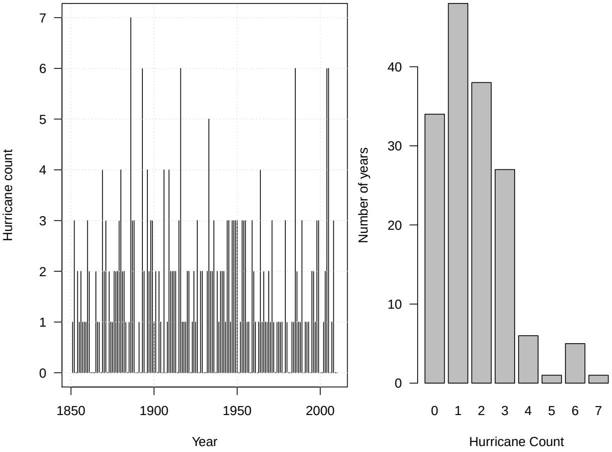
Hits	0	1	2	3	4	5+
Observed	229	211	93	35	7	1
Expected	226.7	211.4	98.5	30.6	7.1	1.6

US Landfalling Hurricanes



Source: `https://www.kaggle.com/gi0vanni/analysis-on-us-hurricane-landfalls`

Number of US Landfalling Hurricanes Per Hurricane Season

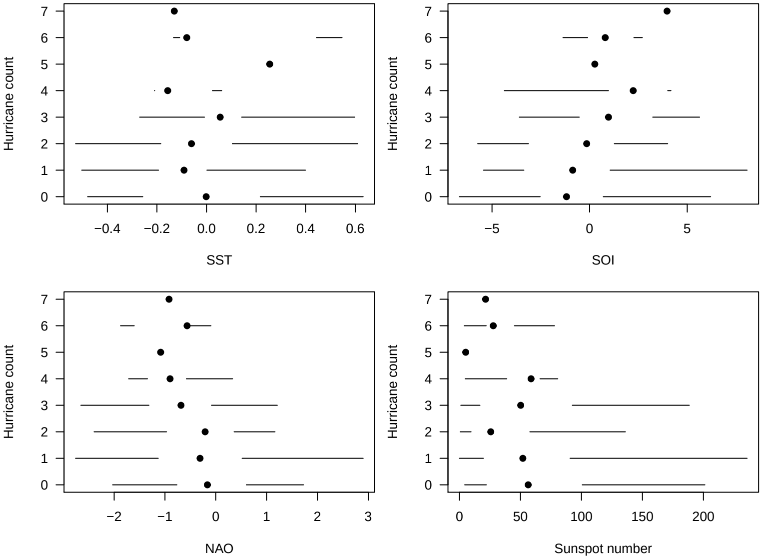


Research question: Can the variation of the annual counts be explained by some environmental variable, e.g., Southern Oscillation Index (SOI)?

Some Potentially Relevant Predictors

- Southern Oscillation Index (SOI): an indicator of wind shear
- Sea Surface Temperature (SST): an indicator of oceanic heat content
- North Atlantic Oscillation (NAO): an indicator of steering flow
- Sunspot Number (SSN): an indicator of upper air temperature

Hurricane Count vs. Environmental Variables



Poisson Regression

$$\log(\lambda) = \beta_0 + \beta_1 x_1 + \cdots + \beta_{p-1} x_{p-1}$$

$$\Rightarrow y \sim \text{Pois}(\lambda = \exp(\beta_0 + \beta_1 x_1 + \cdots + \beta_{p-1} x_{p-1}))$$

- Model the logarithm of the mean response as a linear combination of the predictors
- Parameter estimation is carry out using the maximum likelihood method
- Interpretation of $\beta's$: every one unit increase in x_j, given that the other predictors are held constant, the λ increases by a factor of $\exp(\beta_j)$

US Hurricane Count: Poisson Regression Fit

Poisson Regression Model:

$$\log(\lambda_{\texttt{Count}}) \sim \texttt{SOI} + \texttt{NAO} + \texttt{SST} + \texttt{SSN}$$

Table: Coefficients of the Poisson regression model.

	Estimate	Std. Error	z value	Pr(>|z|)
(Intercept)	0.5953	0.1033	5.76	0.0000
SOI	0.0619	0.0213	2.90	0.0037
NAO	−0.1666	0.0644	−2.59	0.0097
SST	0.2290	0.2553	0.90	0.3698
SSN	−0.0023	0.0014	−1.68	0.0928

$\Rightarrow$ every one unit increase in $\texttt{SOI}$, the hurricane rate increases by a factor of $\exp(0.0619) = 1.0639$ or 6.39%.

Issue with Linear Regression Fit

Linear Regression Model:

$$\mathrm{E}(\texttt{Count}) \sim \texttt{SOI} + \texttt{NAO} + \texttt{SST} + \texttt{SSN}$$

Table: Coefficients of the linear regression model.

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	1.8869	0.1876	10.06	0.0000
SOI	0.1139	0.0402	2.83	0.0053
NAO	−0.2929	0.1173	−2.50	0.0137
SST	0.4314	0.4930	0.88	0.3830
SSN	−0.0039	0.0024	−1.66	0.1000

If we use this fitted model to predict the mean hurricane count, say `SOI` = -3, `NAO`=3, `SST` = 0, `SSN`=250

```
> predict(lmFull, newdata = data.frame(SOI = -3, NAO = 3, SST = 0, SSN = 250))
        1
-0.318065
```

This negative number does not make sense

Model Selection

```
> step(PoiFull)
Start:  AIC=479.64
All ~ SOI + NAO + SST + SSN

       Df Deviance    AIC
- SST   1   175.61 478.44
<none>      174.81 479.64
- SSN   1   177.75 480.59
- NAO   1   181.58 484.41
- SOI   1   183.19 486.02

Step:  AIC=478.44
All ~ SOI + NAO + SSN

       Df Deviance    AIC
<none>      175.61 478.44
- SSN   1   178.29 479.12
- NAO   1   183.57 484.41
- SOI   1   183.91 484.74

Call:  glm(formula = All ~ SOI + NAO + SSN, family = "poisson", data = df)

Coefficients:
(Intercept)          SOI          NAO          SSN
   0.584957     0.061533    -0.177439    -0.002201

Degrees of Freedom: 144 Total (i.e. Null);  141 Residual
Null Deviance:      197.9
Residual Deviance: 175.6        AIC: 478.4
```

Generalized Linear Model

- **Gaussian Linear Model:**

$$y \sim \mathrm{N}(\mu, \sigma^2), \quad \mu = \boldsymbol{X}^T \boldsymbol{\beta}$$

- **Bernoulli Linear Model:**

$$y \sim \mathrm{Bernoulli}(\pi), \quad \log\left(\frac{\pi}{1-\pi}\right) = \boldsymbol{X}^T \boldsymbol{\beta}$$

- **Poisson Linear Regression:**

$$y \sim \mathrm{Poisson}(\lambda), \quad \log \lambda = \boldsymbol{X}^T \boldsymbol{\beta}$$

These models fall into the family of generalized linear models [Nelder and Wedderburn (1972); McCullagh and Nelder (1989)] with the **distributional assumptions** (normal, Bernoulli, Poisson) and the **link functions** (identity, logit, and log)

Summary

These slides cover:

- Logistic Regression
- Poisson Regression

Both of which, as well as the linear regression models covered in the past 6 weeks, can be unified into a single framework of Generalized Linear Model

`R` functions to know:

- **Logistic and Poisson Regressions**: `glm` with `family` being `"binomial"` and `"poisson"`, respectively
- Many `lm` utility functions can still be used; for example, `predict` can still be used for prediction, and `step` can still be used for model selection