Lecture 12

Hypergeometric and Poisson Random Variables

Text: Chapter 4

STAT 8010 Statistical Methods I
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Notes

Agenda

1. Bernoulli Trials and Binomial Random Variables
2. Hypergeometric Random Variables
3. Poisson Random Variables

Notes

Bernoulli Trials and Binomial Random Variables

- **Bernoulli Trials**:
  - The result of each trial may be either a success or failure
  - The probability of success, p, is the same in every trial
  - The trials are **independent:** the outcome of one trial has no influence on later outcomes
- **Binomial Random Variables**
  - The number of successes in n Bernoulli trials, where the probability of success in one trial is $p \Rightarrow X \sim \text{Bin}(n, p)$
  - **Probability mass function** $p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}, \; x = 0, 1, \cdots, n$
  - **Mean:** $\mathbb{E}[X] = np$; **Variance:** $Var(X) = np(1-p)$

Notes

Example

Suppose that 95% of consumers can recognize Coke in a blind taste test. Assume consumers are independent of one another. The company randomly selects 4 consumers for a taste test. Let X be the number of consumers who recognize Coke.

1. Write out the pmf table for X
2. What is the probability that X is at least 1?
3. What is the probability that X is at most 3?

Notes

Binomial and Hypergeometric Distributions

The binomial distribution describes the probability of k successes in n trials with replacement.

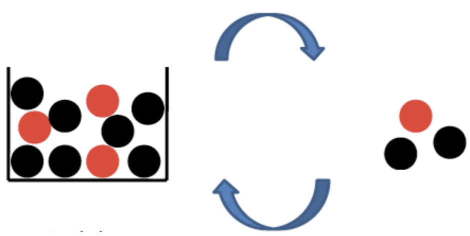
We want a distribution to describe the probability of k successes in n trials without replacement from a finite population of size N containing exactly K successes.

$\Rightarrow$ Hypergeometric Distribution

> Important applications are **quality control** and statistical **estimation of population proportions**. The hypergeometric r.v. is the equivalent of a Binomial r.v. except that sampling is done without replacement.

Notes

Drawing balls from an urn with 2 different colors



We have an urn with 3 red ("success") and 7 black ("failure") balls. Suppose we select 3 balls randomly.

- With replacement: $X \sim \text{Bin}(n = 3, p = 0.3)$
- Without replacement: $X \sim \text{Hypergeo}(n = 3, N = 10, K = 3)$

Notes

Hypergeometric Distributions

Let X be a hypergeometric r.v.

- The definition of X: The number of successes in n trials of a random experiment, where sampling is done without replacement
- The support: $x \in \{\max(0, n+K-N), \cdots, \min(n, K)\}$
- Its parameter(s) and definition(s): N: the population size, n: the sample size, and K: number of success in the population
- The probability mass function (pmf): $p_X(x) = \frac{\binom{K}{k}\times\binom{N-K}{n-k}}{\binom{N}{n}}$
- The expected value: $\mathbb{E}[X] = n\frac{K}{N}$
- The variance: $Var(X) = n\frac{K}{N}\frac{N-K}{N}\frac{N-n}{N-1}$

Notes

Example

There are 100 identical looking 52" TVs at Best Buy in Anderson, SC. Let 10 of them be defective. Suppose we want to buy 8 of the aforementioned TVs (at random). What is the probability that we don't get any defective TVs?

Solution.

Let D be the number of defective TVs in the sample.

$D \sim \text{Hypergeo}(N = 100, n = 8, K = 10)$
$\mathbb{P}(D = 0) = \frac{\binom{10}{0}\binom{90}{8}}{\binom{100}{8}} = 0.4166$

Notes

Poisson Random Variables

So far, we have seen discrete probability distributions of the number of successes in a sequence of random experiments with specified sample size.

- Bernoulli distribution: independent trial (sampling with replacement), sample size $= 1$
- Binomial distribution: independent trials (sampling with replacement), sample size $= n$
- Hypergeometric distribution: dependent trials (sampling without replacement), sample size $= n$

The Poisson distribution is a discrete probability distribution that expresses the probability of a given number of events occurring in a fixed interval of time and/or space $\Rightarrow$ **does not have a (fixed) sample size**

Notes

Poisson Distributions

Let X be a Poisson r.v.

- The definition of X: The number of successes
- The support: $x = 0, 1, 2, \cdots$
- Its parameter(s) and definition(s): λ: the average number of successes
- The probability mass function (pmf): $p_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}$
- The expected value: $\mathbb{E}[X] = \lambda$
- The variance: $Var(X) = \lambda$

Notes

Poisson approximation to Binomial Distribution

If $X \sim \text{Bin}(n, p)$ and $n > 100$ and $p < .01$ then we can approximate the distribution X by using $X^* \sim \text{Poi}(\lambda = n \times p)$

Example

Let $X \sim \text{Bin}(200, 0.005)$. Then,

$$\begin{aligned} \mathbb{E}[X] = 1; &\quad \mathbb{E}[X^*] = 1 \\ Var(X) = 0.995; &\quad Var(X^*) = 1 \\ \mathbb{P}(X = 1) = 0.3688; &\quad \mathbb{P}(X^* = 1) = 0.3679 \end{aligned}$$

Notes

Example

Suppose a certain disease has a 0.14 % of occurring. Let's sample 1,000 people. Find the exact and approximate probabilities that 0 people have the disease and at most 5 people have the disease.

Solution.

Set-up:
Let X be the number of people have the disease in the sample ($n = 1000$).
Which distribution to use?

- The sample size is fixed ($n = 1000$) $\Rightarrow$ Binomial or Hypergeometric

What are the parameters? $n = 1000, p = .0014$

Notes

Example Cont'd

Any approximation?

Since $n = 1000 > 100$ and $p = .0014 < .01$. We can use Poisson X^* to approximate the Binomial distribution X

Exact probabilities

$X \sim \text{Bin}(n = 1000, p = .0014)$

$\mathbb{P}(X = 0) = \binom{1000}{0}(.0014)^0(1 - .0014)^{1000-0} = .2464$

$\mathbb{P}(X \leq 5) = \sum_{x=0}^{5} \binom{1000}{x}(.0014)^x(1 - .0014)^{1000-x} = .9986$

Approximate probabilities

$X^* \sim \text{Poi}(\lambda = n \times p = 1.4)$

$\mathbb{P}(X = 0) \approx \mathbb{P}(X^* = 0) = \frac{e^{-1.4}1.4^0}{0!} = .2466$

$\mathbb{P}(X \leq 5) \approx \mathbb{P}(X^* \leq 5) = \sum_{x=0}^{5} \frac{e^{-1.4}\lambda^x}{x!} = .9986$

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