Lecture 8

Introduction to Probability II

Text: Chapter 4

STAT 8010 Statistical Methods I
September 6, 2019

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Agenda

1. **Union and Intersection**
2. **Logical Relationships among Events**
3. **Complement Rule and General Addition Rule**
4. **Venn Diagram**

Intersection and Union

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Suppose we flipped 3 fair coins. Let A be the event of **exactly 2 tails**. Let B be the event that the **first 2 tosses are tails**. Let C be the event that **all 3 tosses are tails**. What are $A \cap B$, $A \cup C$, and $(A \cap B) \cup C$?

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Solution.

$$A = \{(T,T,H),(T,H,T),(H,T,T)\}$$
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3. $(A \cap B) \cup C = \{T,T,H\} \cup \{T,T,T\} = \{(T,T,H),(T,T,T)\}$

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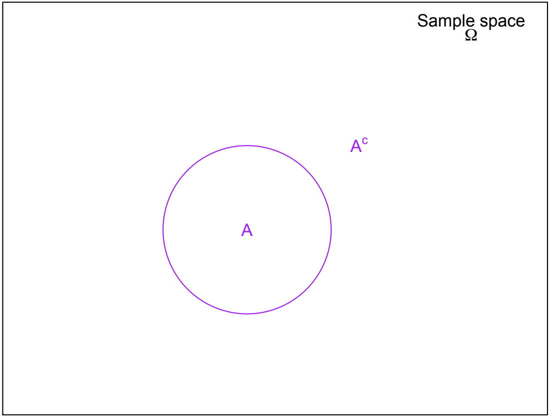
$$A \cap B = \emptyset \quad \text{and} \quad A \cup B = \Omega$$

Complement



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4. Hence we get $\boxed{\mathbb{P}(A) = 1 - \mathbb{P}(A^c)}$

Example

Suppose we rolled a fair, six–sided die 10 times. Let T be the event that we roll at least 1 three. If one were to calculate T you would need to find the probability of 1 three, 2 threes, $\cdots$, and 10 threes and add them all up. However, you can use the complement rule to calculate $\mathbb{P}(T)$

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Let X be the times that we rolled a 3, then

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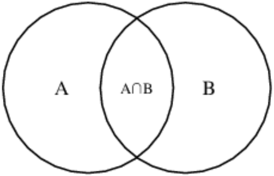
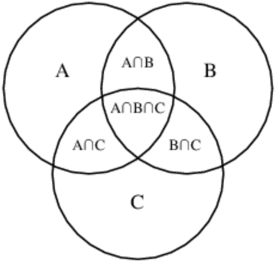
If we apply the complement rule

$$\mathbb{P}(T) = 1 - \mathbb{P}(T^c) = 1 - \mathbb{P}(X = 0)$$

Venn Diagram

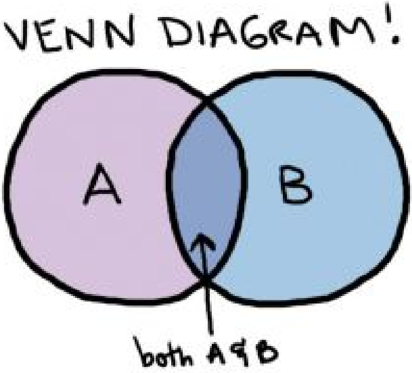
A Venn diagram is a diagram that shows all possible logical relations between a finite collection of events.





General Addition Rule

The general addition rule is a way of finding the probability of a union of 2 events. It is $\boxed{\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)}$

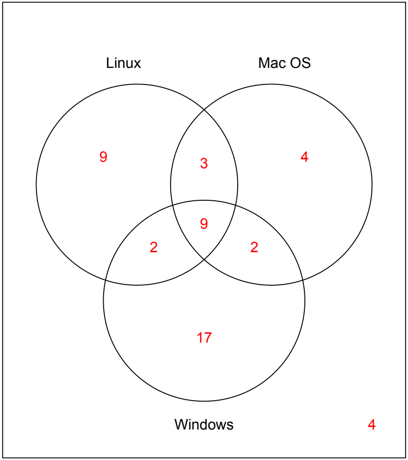


Example

Three of the major commercial computer operating systems are Windows, Mac OS, and Red Hat Linux Enterprise. A Computer Science professor selects 50 of her students and asks which of these three operating systems they use. The results for the 50 students are summarized below.

- 30 students use Windows
- 16 students use at least two of the operating systems
- 9 students use all three operating systems
- 18 students use Mac OS
- 46 students use at least one of the operating systems
- 11 students use both Windows and Linux

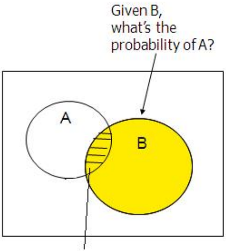
Example cont'd



Conditional Probability

Let A and B be events. The probability that event A occurs given (knowing) that event B occurs is called a conditional probability and is denoted by $P(A|B)$. The formula of conditional probability is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$



In a conditional probability problem, the sample space is "reduced" to the "space" of the given outcome (e.g. if given B, we now just care about the probability of A occurring "inside" of B)

Summary

In this lecture, we learned

- Union , Intersection
- **Logical Relationships among Events** Mutually Exclusive, Exhaustive, Partition
- Complement Rule and General Addition Rule
- Venn Diagram