CLEMSON UNIVERSITY

Lecture 10

Randomized Complete Block Designs, Factorial Designs, and Split-Plot Designs

Reading: Oehlert Chapters 8, 13, 16; Dean-Voss-Draguljić Chapters 6, 10, 19

DSA 8020 Statistical Methods II

March 8-12, 2021

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Agenda

1. Randomized Complete BlockDesigns
2. Factorial Designs
3. Split-Plot Designs

Review: Completely Randomized Design (CRD)

A CRD has

- g different treatment levels
- g known treatment level sizes $n_1, n_2, \cdots, n_g$ with $\sum_{j=1}^{g} n_j = N$ (i.e., N experimental units in total)
- Completely random assignment of treatment levels to experimental units

A key assumption of CRD is that all experimental units are (approximately) homogeneous

Review: Completely Randomized Design (CRD)

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Question: What if this assumption is violated/unreasonable?

Randomized Complete Block Design (RCBD)

- The population of experimental units is divided into a number of relatively homogeneous sub-populations (blocks), and it is assumed that all experimental units within a given block are homogeneous
- Within each block, treatments are randomly assigned to experimental units such that each treatment occurs equally often (usually once) in each block $\Rightarrow$ CRD within each block
- A key assumption in the analysis is that the effect of each level of the treatment is the same for each level of the blocking factor $\Rightarrow$ additive assumption

RCBD Notation

- g is the number of treatment levels; r is the number of blocks
- y_{ij} is the measurement on the unit in block i that received treatment j
- $N = r \times g$ is the total number of experimental units
- $\bar{y}_{.j} = \sum_{i=1}^{r} \frac{y_{ij}}{r}$ is the average of all measurements for units receiving treatment j
- $\bar{y}_{i.} = \sum_{j=1}^{g} \frac{y_{ij}}{g}$ is the average of all measurements for units in the i_{th} block
- $\bar{y}_{..} = \sum_{i=1}^{r} \sum_{j=1}^{g} \frac{y_{ij}}{N}$ is the average of all measurements

RCBD Model and Assumptions

- The model for an RCBD is:

$$Y_{ij} = \underbrace{\mu + \alpha_j}_{\mu_j} + \beta_i + \varepsilon_{ij}, \quad i = 1, \cdots, r, \quad j = 1, \cdots, g$$

  where μ is the overall mean, α_j is the effect of treatment j, β_i is the effect of block i, and $\varepsilon_{ij} \overset{iid}{\sim} \mathrm{N}(0, \sigma^2)$ are random errors

- The effect of each level of the treatment is the same across blocks $\Rightarrow$ no interaction between $\alpha's$ and $\beta's$

RCBD Sums of Squares

- **Total sum of square**:

$$SS_{tot} = \sum_{i=1}^{r}\sum_{j=1}^{g}(y_{ij} - \bar{y}_{..})^2$$

- **Treatment sum of square**:

$$SS_{trt} = \sum_{j=1}^{g} r(\bar{y}_{.j} - \bar{y}_{..})^2$$

- **Block sum of square**:

$$SS_{blk} = \sum_{i=1}^{r} g(\bar{y}_{i.} - \bar{y}_{..})^2$$

- **Error sum of square**:

$$SS_{err} = \sum_{i=1}^{r}\sum_{j=1}^{g}(y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{..})^2$$

ANOVA Table and F Test

Source	df	SS	MS	F statistic
Treatment	$g-1$	SS_{trt}	$MS_{trt} = \frac{SS_{trt}}{g-1}$	$F_{trt} = \frac{MS_{trt}}{MS_{err}}$
Block	$r-1$	SS_{blk}	$MS_{blk} = \frac{SS_{blk}}{r-1}$	$F_{blk} = \frac{MS_{blk}}{MS_{err}}$
Error	$(g-1)(r-1)$	SS_{err}	$MS_{err} = \frac{SS_{err}}{(g-1)(r-1)}$	
Total	$N-1$	SS_{tot}		

There are two hypothesis tests in an RCBD:

- $H_0 : \alpha_j = 0 \qquad j = 1, \cdots, g$
  $H_a : \alpha_j \neq 0 \qquad$ for some j
  Test Statistic: $F_{trt} = \frac{MS_{trt}}{MS_{err}}$. Under H_0,
  $F_{trt} \sim F_{df_1=g-1, df_2=(g-1)(r-1)}$

- H_0 : The means of all blocks are equal
  H_a : At least one of the blocks has a different mean
  Test Statistic: $F_{blk} = \frac{MS_{blk}}{MS_{err}}$. Under H_0,
  $F_{blk} \sim F_{df_1=r-1, df_2=(g-1)(r-1)}$

Example

Suppose you are manufacturing concrete cylinders for bridge supports. There are three ways of drying concrete (say A, B, and C), and you want to find the one that gives you the best compressive strength. The concrete is mixed in batches that are large enough to produce exactly three cylinders, and your production engineer believes that there is substantial variation in the quality of the concrete from batch to batch.

You have data from $r = 5$ batches on each of the $g = 3$ drying processes. Your measurements are the compressive strength of the cylinder in a destructive test. (So there is an economic incentive to learn as much as you can from a well-designed experiment.)

Example: Data Set

The data are:

	Batch					
Treatment	1	2	3	4	5	Trt Sum
A	52	47	44	51	42	236
B	60	55	49	52	43	259
C	56	48	45	44	38	231
Batch Mean	168	150	138	147	123	726

The primary null hypothesis is that all three drying techniques are equivalent, in terms of compressive strength.

The secondary null is that the batches are equivalent (but if they are, then we have wasted power by controlling for an effect that is small or non-existent).

Example: ANOVA Table

```
Analysis of Variance Table

Response: x
          Df Sum Sq Mean Sq F value
trt        2   89.2   44.60  7.6239
blk        4  363.6   90.90 15.5385
Residuals  8   46.8    5.85
             Pr(>F)
trt       0.0140226 *
blk       0.0007684 ***
```

Interpretation?

What If We Ignore the Block Effect?

Suppose we had not blocked for batch. Then the data would be:

Treatment		Trt Sum
A	52, 47, 44, 51, 42	236
B	60, 55, 49, 52, 43	259
C	56, 48, 45, 44, 38	231

This is the same as before except now we ignore which batch the observation came from.

ANOVA Table for CRD

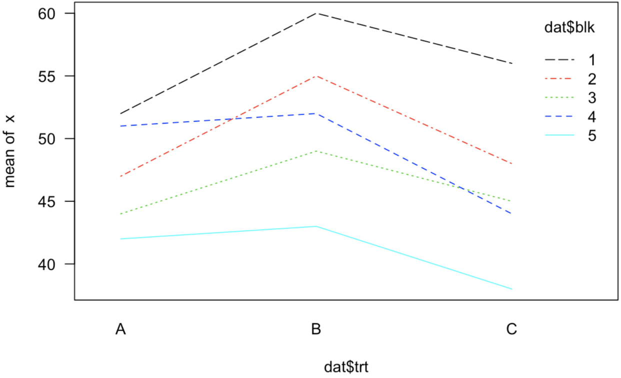

```
Analysis of Variance Table

Response: x
          Df Sum Sq Mean Sq F value Pr(>F)
trt        2   89.2    44.6  1.3041 0.3073
Residuals 12  410.4    34.2
```

We fail to reject the null $H_0 : \mu_A = \mu_B = \mu_C$ if we **ignore the block effect**

⇒ Using blocks gave us a more powerful test!

Assessing the Additivity Assumption: Interaction Plot



"Parallel lines" ⇒ No interaction occurs

The Battery Design Experiment (Example 5.1, Montgomery, 6th Ed)

An engineer would like to study what effects do material type and temperature have on the life of the battery he designed. the engineer decides to test three plate materials at three temperature levels:

Material Type	Temperature (°F) 15		70		125	
1	130	155	34	40	20	70
	74	180	80	75	82	58
2	150	188	136	122	25	70
	159	126	106	115	58	45
3	138	110	174	120	96	104
	168	160	150	139	82	60

This design is called a 3^2 factorial design

Two-Factor Factorial Design

The **effects model**:

$$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk},$$

$i = 1, \cdots, a,\ j = 1, \cdots, b,\ k = 1, \cdots, n$

- a: the number of levels in the factor A
- b: the number of levels in the factor B
- $(\alpha\beta)_{ij}$: the interaction between α_i and β_j
- $\sum_{i=1}^{a} \alpha_i = \sum_{j=1}^{b} \beta_j = \sum_{i=1}^{a} (\alpha\beta)_{ij} = \sum_{j=1}^{b} (\alpha\beta)_{ij} = 0$
- abn is the total number of the observations

ANOVA Table

Source	df	SS	MS	F
Factor A	$a-1$	SS_A	$\text{MS}_A = \frac{\text{SS}_A}{a-1}$	$F = \frac{\text{MS}_A}{\text{MS}_E}$
Factor B	$b-1$	SS_B	$\text{MS}_B = \frac{\text{SS}_B}{b-1}$	$F = \frac{\text{MS}_B}{\text{MS}_E}$
Interaction	$(a-1)(b-1)$	SS_{AB}	$\text{MS}_{AB} = \frac{\text{SS}_{AB}}{(a-1)(b-1)}$	$F = \frac{\text{MS}_{AB}}{\text{MS}_E}$
Error	$ab(n-1)$	SS_E	$\text{MS}_E = \frac{\text{SS}_E}{ab(n-1)}$	
Total	$abn-1$	SS_T		

R Output

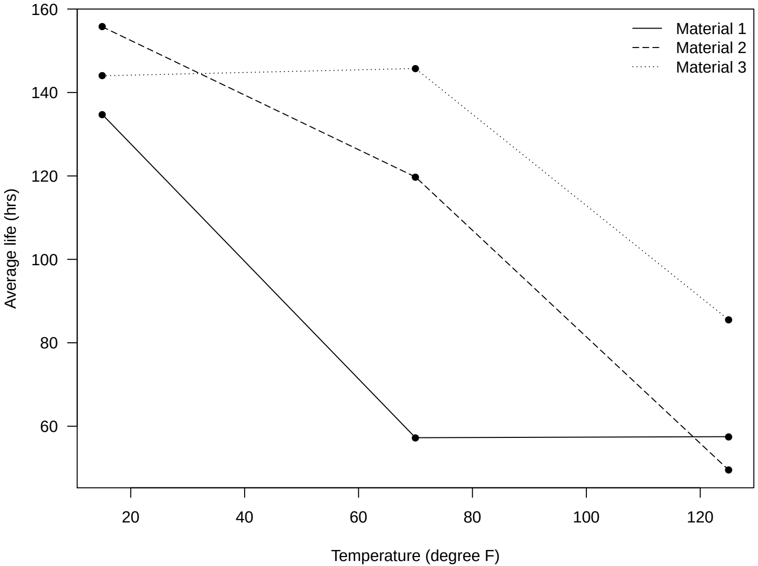

```
lm <- lm(y ~ temp * material, data = dat)
anova(lm)
```
```

```
Analysis of Variance Table

Response: y
              Df Sum Sq Mean Sq F value   Pr(>F)
temp   2 39119 19559.4 28.9677 1.909e-07 ***
material   2 10684 5341.9 7.9114 0.001976 **
temp:material 4   9614 2403.4 3.5595 0.018611 *
Residuals   27 18231   675.2

Signif. codes: 0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```
```

Interaction Plot



An Example of Split-Plot Design

Suppose we wish to determine the effects of two fertilizers (“A” and “B”) and two irrigation methods (“1” and “2”) on yield.

- **Treatments**: Fertilizer types and irrigation methods
- **Experimental Unit**: a small region

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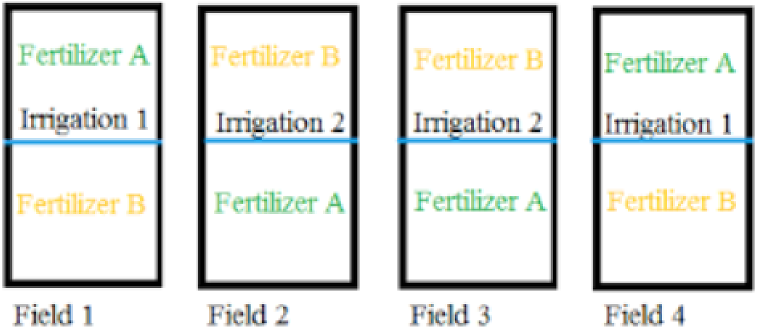

Model

$$Y_{ijk} = \mu + \alpha_i + \delta_{k(i)} + \beta_j + \alpha\beta_{ij} + \varepsilon_{k(ij)},$$

$i = 1, \cdots, a,\ j = 1, \cdots, b,\ k = 1, \cdots, n$

- μ: overall mean
- α_i: effect of i_{th} level of the whole-plot factor
- β_j: effect j_{th} split-plot factor
- $(\alpha\beta)_{ij}$: joint effect of i_{th} level of the whole-plot factor and j_{th} split-plot factor
- $\delta_{k(i)} \sim \mathrm{N}(0, \sigma_\delta^2)$ and $\varepsilon_{k(ij)} \sim \mathrm{N}(0, \sigma_\varepsilon^2)$: the whole-plot and the split-plot level random errors

ANOVA Table

Source	df	SS	MS	F
Factor A	$a-1$	SS_A	$MS_A = \frac{SS_A}{a-1}$	$F = \frac{MS_A}{MS_{E_1}}$
Error 1	$a(n-1)$	SS_{E_1}	$MS_{E_1} = \frac{SS_{E_1}}{a(n-1)}$	
Factor B	$b-1$	SS_B	$MS_B = \frac{SS_B}{b-1}$	$F = \frac{MS_B}{MS_{E_2}}$
Interaction	$(a-1)(b-1)$	SS_{AB}	$MS_{AB} = \frac{SS_{AB}}{(a-1)(b-1)}$	$F = \frac{MS_{AB}}{MS_{E_2}}$
Error 2	$a(b-1)(n-1)$	SS_{E_2}	$MS_{E_2} = \frac{SS_{E_2}}{a(b-1)(n-1)}$	
Total	$abn-1$	SS_T		