Lecture 12

Hypergeometric and Poisson Random Variables

Text: Chapter 4

STAT 8010 Statistical Methods I
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Agenda

Bernoulli Trials and Binomial Random Variables

- **Bernoulli Trials**:
  - The result of each trial may be either a success or failure
  - The probability of success, p, is the same in every trial
  - The trials are **independent:** the outcome of one trial has no influence on later outcomes
- **Binomial Random Variables**
  - The number of successes in n Bernoulli trials, where the probability of success in one trial is $p \Rightarrow X \sim \text{Bin}(n,p)$
  - **Probability mass function**
    $p_X(x) = \binom{n}{x} p^x (1-p)^{n-x},\ x = 0, 1, \cdots, n$
  - **Mean:** $\mathbb{E}[X] = np$; **Variance:** $Var(X) = np(1-p)$

Example

Suppose that 95% of consumers can recognize Coke in a blind taste test. Assume consumers are independent of one another. The company randomly selects 4 consumers for a taste test. Let X be the number of consumers who recognize Coke.

1. Write out the pmf table for X
2. What is the probability that X is at least 1?
3. What is the probability that X is at most 3?

Binomial and Hypergeometric Distributions

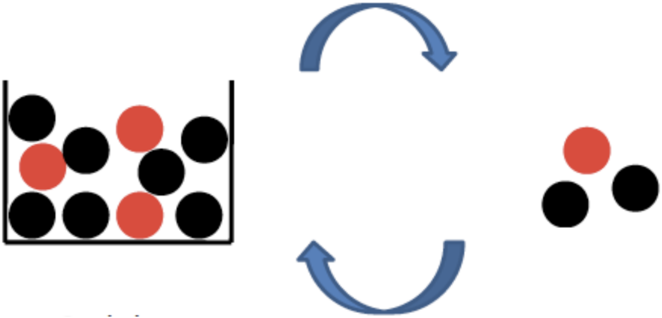
The binomial distribution describes the probability of k successes in n trials with replacement.

We want a distribution to describe the probability of k successes in n trials without replacement from a finite population of size N containing exactly K successes.

$\Rightarrow$ Hypergeometric Distribution

Important applications are **quality control** and statistical **estimation of population proportions**. The hypergeometric r.v. is the equivalent of a Binomial r.v. except that sampling is done without replacement.

Drawing balls from an urn with 2 different colors



We have an urn with 3 red (“success”) and 7 black (“failure”) balls. Suppose we select 3 balls randomly.

- With replacement: $X \sim \text{Bin}(n = 3, p = 0.3)$
- Without replacement: $X \sim \text{Hypergeo}(n = 3, N = 10, K = 3)$

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- The expected value: $\mathbb{E}[X] = n\frac{K}{N}$
- The variance: $Var(X) = n\frac{K}{N}\frac{N-K}{N}\frac{N-n}{N-1}$

Example

There are 100 identical looking 52" TVs at Best Buy in Anderson, SC. Let 10 of them be defective. Suppose we want to buy 8 of the aforementioned TVs (at random). What is the probability that we don't get any defective TVs?

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Let D be the number of defective TVs in the sample.

$D \sim \text{Hypergeo}(N = 100, n = 8, K = 10)$
$\mathbb{P}(D = 0) = \frac{\binom{10}{0}\binom{90}{8}}{\binom{100}{8}} = 0.4166$

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The Poisson distribution is a discrete probability distribution that expresses the probability of a given number of events occurring in a fixed interval of time and/or space $\Rightarrow$ **does not have a (fixed) sample size**

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- The variance: $Var(X) = \lambda$

Poisson approximation to Binomial Distribution

If $X \sim$ Bin(n, p) and $n > 100$ and $p < .01$ then we can approximate the distribution X by using $X^* \sim$ Poi$(\lambda = n \times p)$

Example

Let $X \sim$ Bin$(200, 0.005)$. Then,

$$\mathbb{E}[X] = 1; \quad \mathbb{E}[X^*] = 1$$
$$Var(X) = 0.995; \quad Var(X^*) = 1$$
$$\mathbb{P}(X = 1) = 0.3688; \quad \mathbb{P}(X^* = 1) = 0.3679$$

Example

Suppose a certain disease has a 0.14 % of occurring. Let's sample 1,000 people. Find the exact and approximate probabilities that 0 people have the disease and at most 5 people have the disease.

Solution.

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Solution.

Set-up:
Let X be the number of people have the disease in the sample ($n = 1000$).

Which distribution to use?

- The sample size is fixed $(n = 1000) \Rightarrow$ Binomial or Hypergeometric

What are the parameters? $n = 1000, p = .0014$

Example Cont'd

Any approximation?
Since $n = 1000 > 100$ and $p = .0014 < .01$. We can use Poisson X^* to approximate the Binomial distribution X

Exact probabilities
$X \sim \text{Bin}(n = 1000, p = .0014)$
$\mathbb{P}(X = 0) = \binom{1000}{0}(.0014)^0(1 - .0014)^{1000-0} = .2464$
$\mathbb{P}(X \leq 5) = \sum_{x=0}^{5} \binom{1000}{x}(.0014)^x(1 - .0014)^{1000-x} = .9986$

Approximate probabilities
$X^* \sim \text{Poi}(\lambda = n \times p = 1.4)$
$\mathbb{P}(X = 0) \approx \mathbb{P}(X^* = 0) = \frac{e^{-1.4}1.4^0}{0!} = .2466$
$\mathbb{P}(X \leq 5) \approx \mathbb{P}(X^* \leq 5) = \sum_{x=0}^{5} \frac{e^{-1.4}\lambda^x}{x!} = .9986$