

Lecture 23

Simple Linear Regression: Estimation and Model Assumptions

Text: Chapter 11

STAT 8010 Statistical Methods I

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Agenda

Simple Linear
Regression:
Estimation and
Model Assumptions



Simple Linear
Regression (SLR)

Parameter Estimation
in SLR

Residual Analysis

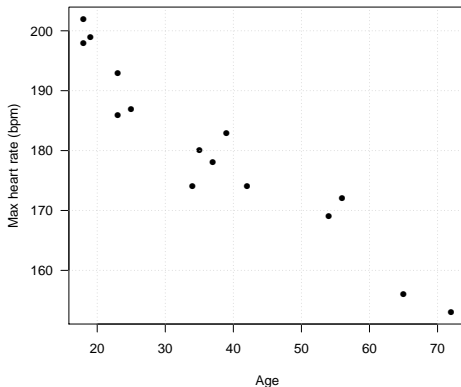
1 Simple Linear Regression (SLR)

2 Parameter Estimation in SLR

3 Residual Analysis

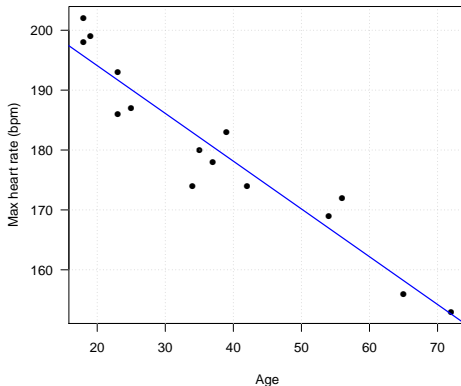
What is Regression Analysis?

Regression analysis: A set of statistical procedures for estimating the relationship between **response variable** and **predictor variable(s)**



We will focus on **simple linear regression** in the next few lectures

Scatterplot: Is Linear Trend Reasonable?

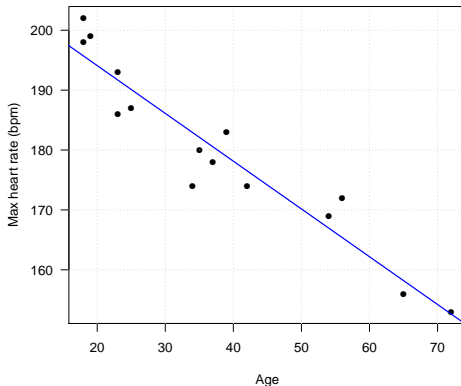


The relationship appears to be linear. What about the **direction** and **strength** of this linear relationship?

```
> cov(age, maxHeartRate)
```

```
[1] -243.9524
```

Scatterplot: Is Linear Trend Reasonable?



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> cov(age, maxHeartRate)
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```
> cor(age, maxHeartRate)
```

```
[1] -0.9534656
```

Simple Linear Regression (SLR)

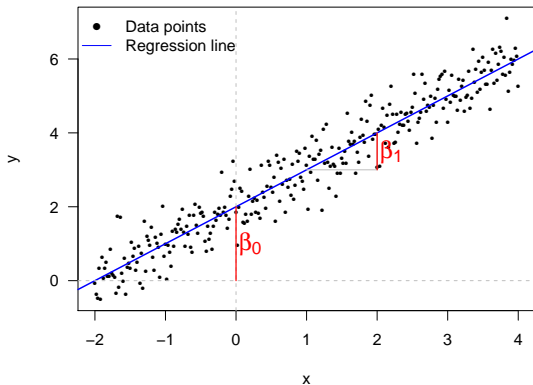
Y : dependent (response) variable; X : independent (predictor) variable

- In SLR we **assume** there is a **linear relationship** between X and Y :

$$Y = \beta_0 + \beta_1 X + \varepsilon$$

- We need to estimate β_0 (intercept) and β_1 (slope)
- We can use the estimated regression equation to
 - make predictions
 - study the relationship between response and predictor
 - control the response
- Yet we need to quantify our **estimation uncertainty** regarding the linear relationship (will talk about this next time)

Regression equation: $Y = \beta_0 + \beta_1 X$



● β_0 : $E[Y]$ when $X = 0$

● β_1 : $E[\Delta Y]$ when X increases by 1

Assumptions about the Random Error ε

In order to estimate β_0 and β_1 , we make the following assumptions about ε

- $E[\varepsilon_i] = 0$
- $\text{Var}[\varepsilon_i] = \sigma^2$
- $\text{Cov}[\varepsilon_i, \varepsilon_j] = 0, \quad i \neq j$

Therefore, we have

$$E[Y_i] = \beta_0 + \beta_1 X_i, \text{ and}$$

$$\text{Var}[Y_i] = \sigma^2$$

The regression line $\beta_0 + \beta_1 X$ represents the **conditional expectation curve** whereas σ^2 measures the magnitude of the **variation** around the regression curve

Estimation: Method of Least Square

For the given observations $(x_i, y_i)_{i=1}^n$, choose β_0 and β_1 to minimize the *sum of squared errors*:

$$L(\beta_0, \beta_1) = \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2$$

Solving the above minimization problem requires some knowledge from Calculus....

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We also need to **estimate** σ^2

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^n (Y_i - \hat{Y}_i)^2}{n-2}, \text{ where } \hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i$$

- **Gauss-Markov** theorem states that in a linear regression these least squares estimators

1 ● **Are unbiased**, i.e.,

- $E[\hat{\beta}_1] = \beta_1; E[\hat{\beta}_0] = \beta_0$

- $E[\hat{\sigma}^2] = \sigma^2$

2 ● Have **minimum variance** among all unbiased linear estimators

Note that we do not make any distributional assumption on ε_i

Example: Maximum Heart Rate vs. Age

The maximum heart rate MaxHeartRate of a person is often said to be related to age Age by the equation:

$$\text{MaxHeartRate} = 220 - \text{Age}.$$

Suppose we have 15 people of varying ages are tested for their maximum heart rate (bpm) (link to the “dataset”: <http://whitneyhuang83.github.io/maxHeartRate.csv>)

- 1 Compute the estimates for the regression coefficients
- 2 Compute the fitted values
- 3 Compute the estimate for σ

Estimate the Parameters β_1 , β_0 , and σ^2

Y_i and X_i are the Maximum Heart Rate and Age of the i^{th} individual

- To obtain $\hat{\beta}_1$

- ① Compute $\bar{Y} = \frac{\sum_{i=1}^n Y_i}{n}$, $\bar{X} = \frac{\sum_{i=1}^n X_i}{n}$

- ② Compute $Y_i - \bar{Y}$, $X_i - \bar{X}$, and $(X_i - \bar{X})^2$ for each observation

- ③ Compute $\sum_i^n (X_i - \bar{X})(Y_i - \bar{Y})$ divided by $\sum_i^n (X_i - \bar{X})^2$

- $\hat{\beta}_0$: Compute $\bar{Y} - \hat{\beta}_1 \bar{X}$

- $\hat{\sigma}^2$

- ① Compute the fitted values: $\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i$, $i = 1, \dots, n$

- ② Compute the **residuals** $e_i = Y_i - \hat{Y}_i$, $i = 1, \dots, n$

- ③ Compute the **residual sum of squares (RSS)**
 $= \sum_{i=1}^n (Y_i - \hat{Y}_i)^2$ and divided by $n - 2$ (why?)

Let's Do the Calculations

$$\bar{X} = \sum_{i=1}^{15} \frac{18 + 23 + \dots + 39 + 37}{15} = 37.33$$

$$\bar{Y} = \sum_{i=1}^{15} \frac{202 + 186 + \dots + 183 + 178}{15} = 180.27$$

X	18	23	25	35	65	54	34	56	72	19	23	42	18	39	37
Y	202	186	187	180	156	169	174	172	153	199	193	174	198	183	178
	-19.33	-14.33	-12.33	-2.33	27.67	16.67	-3.33	18.67	34.67	-18.33	-14.33	4.67	-19.33	1.67	-0.33
	21.73	5.73	6.73	-0.27	-24.27	-11.27	-6.27	-8.27	-27.27	18.73	12.73	-6.27	17.73	2.73	-2.27
	-420.18	-82.18	-83.04	0.62	-671.38	-187.78	20.89	-154.31	-945.24	-343.44	-182.51	-29.24	-342.84	4.56	0.76
	373.78	205.44	152.11	5.44	765.44	277.78	11.11	348.44	1201.78	336.11	205.44	21.78	373.78	2.78	0.11
	195.69	191.70	190.11	182.13	158.20	166.97	182.93	165.38	152.61	194.89	191.70	176.54	195.69	178.94	180.53

- $\hat{\beta}_1 = \frac{\sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^n (X_i - \bar{X})^2} = -0.7977$
- $\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X} = 210.0485$
- $\hat{\sigma}^2 = \frac{\sum_{i=1}^{15} (Y_i - \hat{Y}_i)^2}{13} = 20.9563 \Rightarrow \hat{\sigma} = 4.5778$

Let's Double Check

Output from  ( Studio)

```
> fit <- lm(MaxHeartRate ~ Age)
> summary(fit)
```

```
Call:
lm(formula = MaxHeartRate ~ Age)
```

```
Residuals:
```

Min	1Q	Median	3Q	Max
-8.9258	-2.5383	0.3879	3.1867	6.6242

```
Coefficients:
```

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	210.04846	2.86694	73.27	< 2e-16 ***
Age	-0.79773	0.06996	-11.40	3.85e-08 ***

```
---
```

```
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
Residual standard error: 4.578 on 13 degrees of freedom
```

```
Multiple R-squared:  0.9091,    Adjusted R-squared:  0.9021
```

```
F-statistic: 130 on 1 and 13 DF,  p-value: 3.848e-08
```

Linear Regression Fit

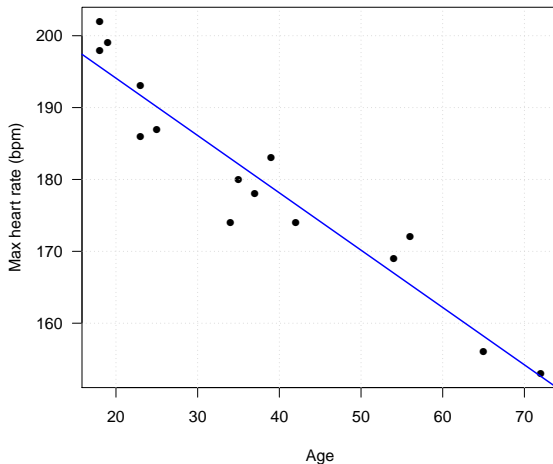
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Residual Analysis



Question: Is linear relationship between max heart rate and age reasonable? \Rightarrow [Residual Analysis](#)

- The **residuals** are the differences between the observed and fitted values:

$$e_i = Y_i - \hat{Y}_i,$$

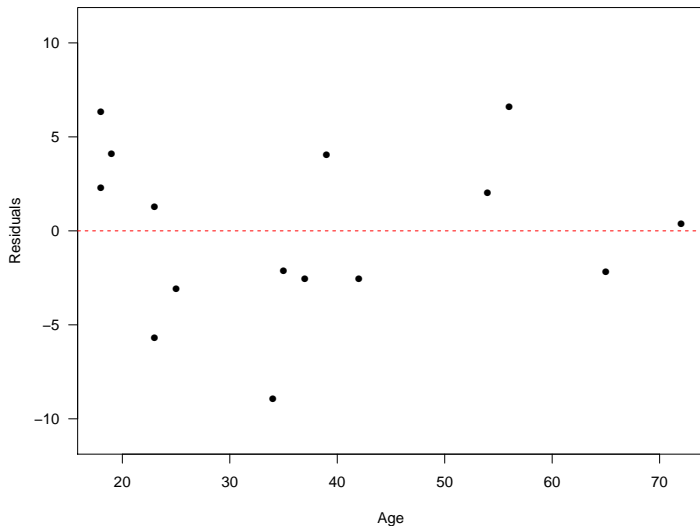
where $\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i$

- e_i is NOT the error term $\varepsilon_i = Y_i - E[Y_i]$
- Residuals are very useful in assessing the appropriateness of the assumptions on ε_i . Recall
 - $E[\varepsilon_i] = 0$
 - $\text{Var}[\varepsilon_i] = \sigma^2$
 - $\text{Cov}[\varepsilon_i, \varepsilon_j] = 0, \quad i \neq j$

Maximum Heart Rate vs. Age Residual Plot: ε vs. X

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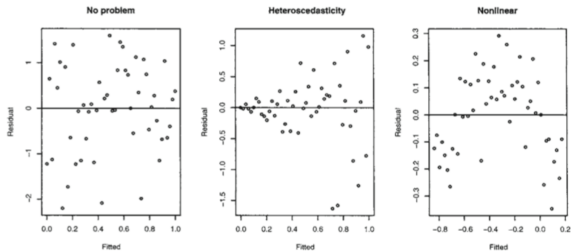


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Interpreting Residual Plots

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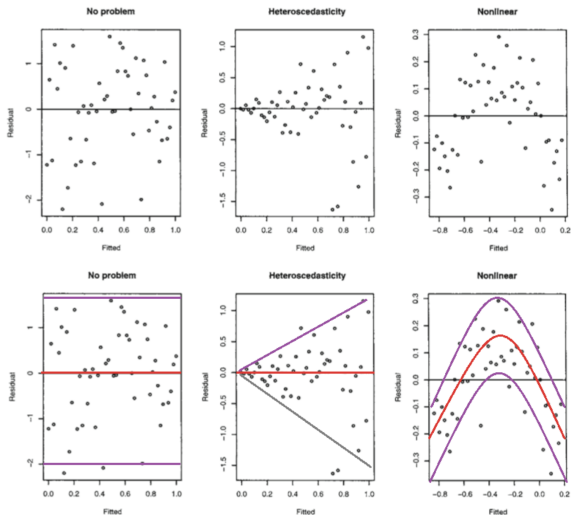


Figure: Figure courtesy of Faraway's Linear Models with R (2005, p. 59).