MATH 8090: Stationary processes and Linear Processes

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Contents

ACF

Population ACF

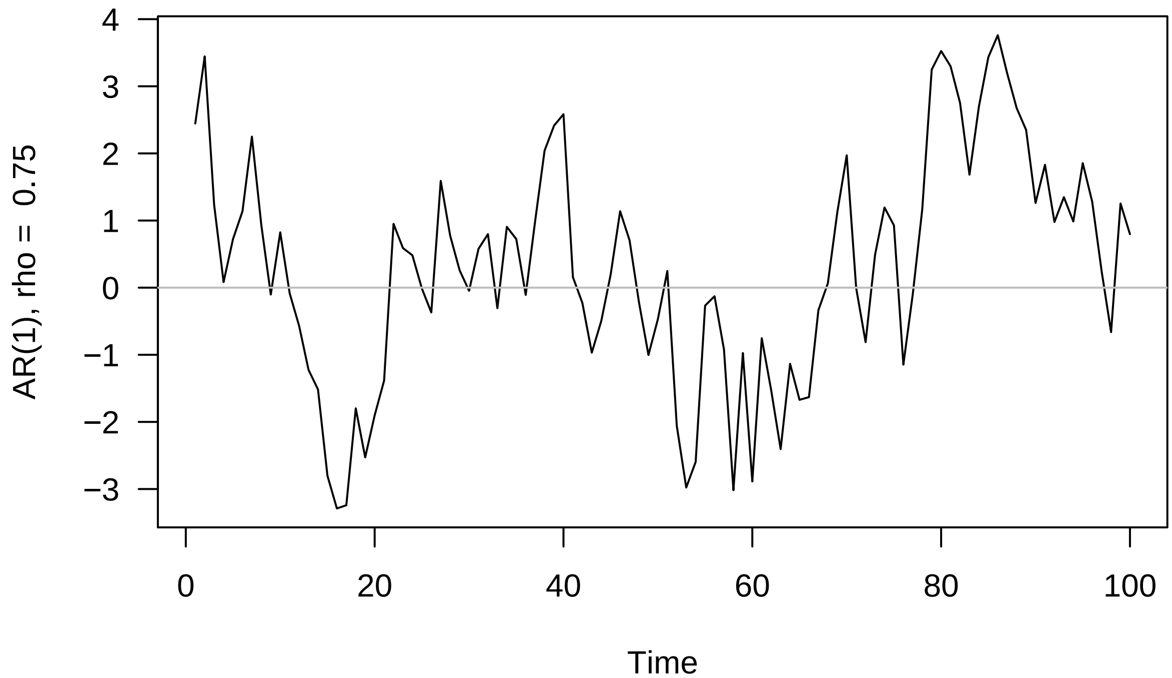
$$\rho(h) = \mathrm{Cor}(\eta_t, \eta_{t+h}) = \frac{\mathrm{E}\left[(\eta_t - \mu)(\eta_{t+h} - \mu)\right]}{\sqrt{\mathrm{Var}(\eta_t)\mathrm{Var}(\eta_{t+h})}}$$

Sample ACF

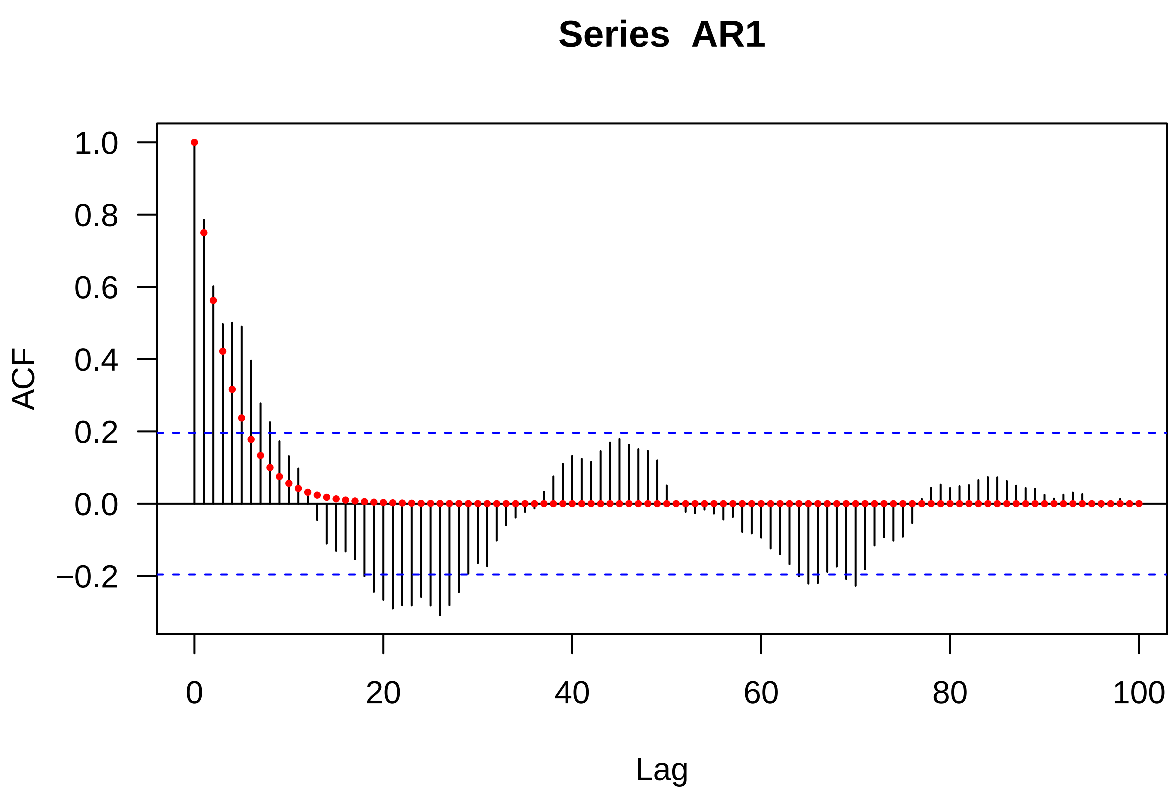
$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma(0)}},$$

where $\hat{\gamma}(h) = \frac{1}{T}\sum_{t=1}^{T-|h|}(\eta_t - \bar{\eta})(\eta_{t+h} - \bar{\eta})$.

```
AR1 <- arima.sim(n = 100, list(ar = c(0.75)))
par(las = 1)
plot(1:100, AR1, type = "l", xlab = "Time", ylab = paste("AR(1), rho = ", 0.75))
abline(h = 0, col = "gray")
```

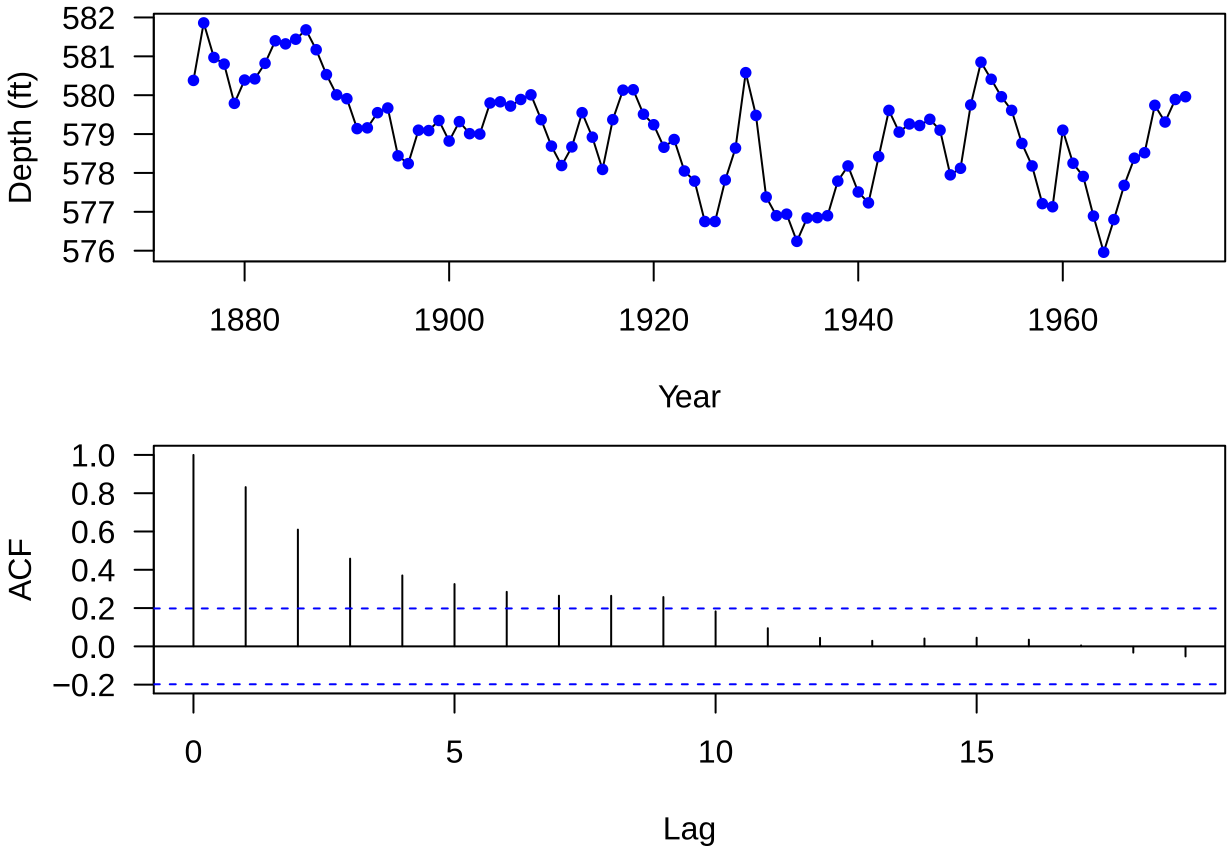



```
acf(AR1, lag = 100)
acf_true <- ARMAacf(ar = c(0.75), lag.max = 100)
points(0:100, acf_true, pch = 16, cex = 0.5, col = "red")
```

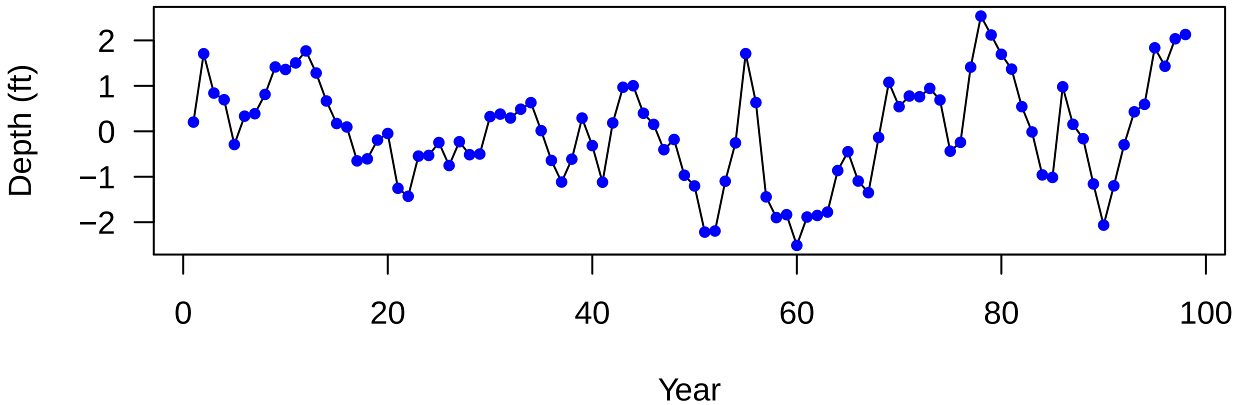


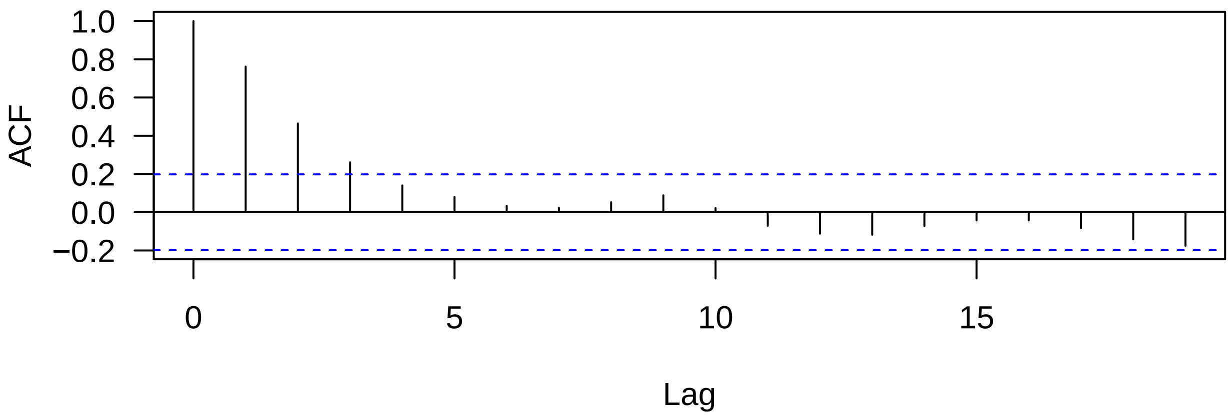
Lake Huron Example

```
data(LakeHuron)
par(las = 1, mfrow = c(2, 1), mar = c(4, 4, 0.8, 0.6))
plot(LakeHuron, ylab = "Depth (ft)", xlab = "Year")
points(LakeHuron, cex = 0.8, col = "blue", pch = 16)
acf(LakeHuron)
```

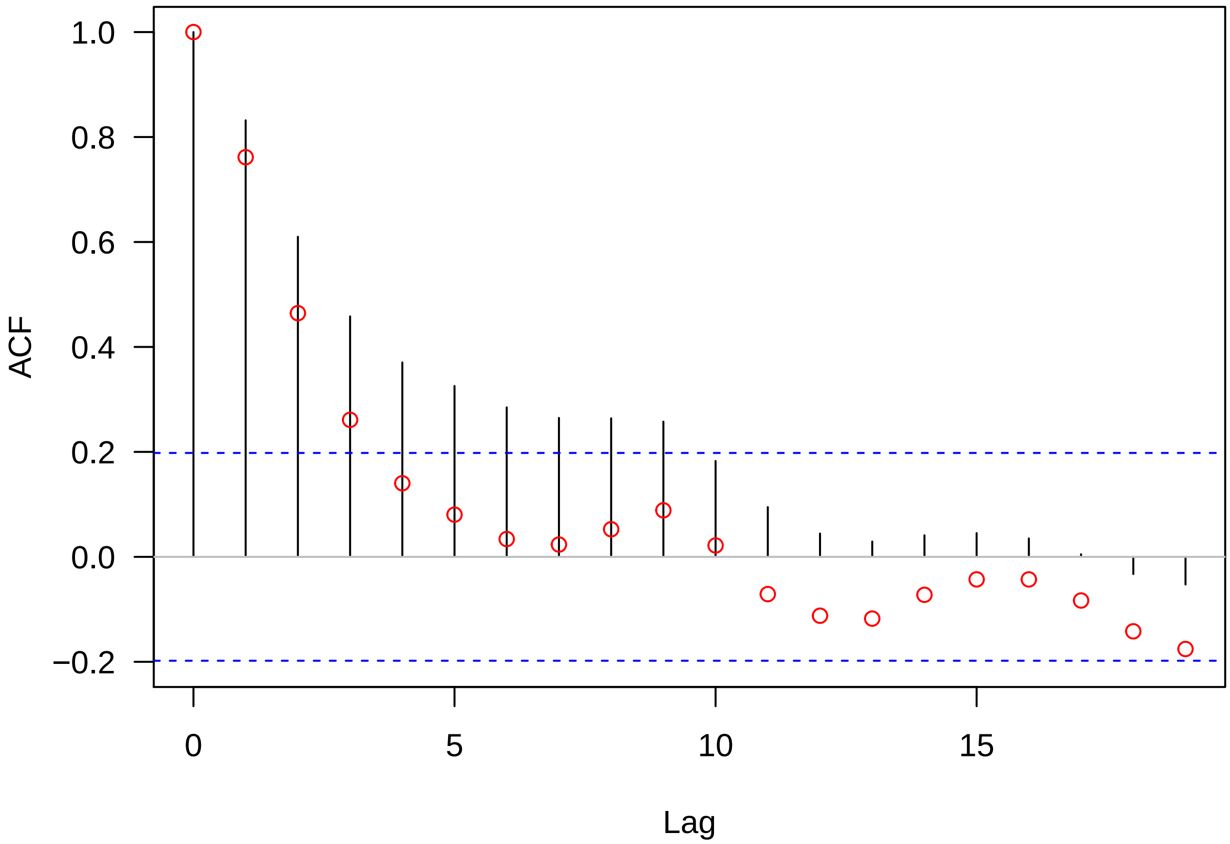



```
# Let's remove the (linear trend)
yr <- 1875:1972
lm <- lm(LakeHuron ~ yr)
plot(lm$residuals, ylab = "Depth (ft)", xlab = "Year", type = "l")
points(lm$residuals, cex = 0.8, col = "blue", pch = 16)
acf(lm$residuals)
```






```
par(mfrow = c(1, 1), las = 1)
plot(0:19, acf(LakeHuron, plot = F)$acf, type = "h", xlab = "Lag", ylab = "ACF", ylim = c(-0.2, 1))
abline(h = 0, col = "gray")
abline(h = c(-1, 1) * qnorm(0.975) / sqrt(length(LakeHuron)) , col = "blue", lty = 2)
acf_detrend <- acf(lm$residuals, plot = F)$acf
points(0:19, acf_detrend, col = "red")
```



Box test for temporal independence

Box and Pierce test Box and Pierce (1970)

We wish to test:

$$H_0 : \{\eta_1, \eta_2, \cdots, \eta_T\} \text{ is an i.i.d. noise sequence}$$
$$H_1 : H_0 \text{ is false}$$

1. Under H_0,
$$\hat{\rho}(h) \dot{\sim} \mathrm{N}(0, \frac{1}{T}) \stackrel{d}{=} \frac{1}{\sqrt{n}}\mathrm{N}(0, 1)$$
2. Hence
$$Q = T \sum_{i=1}^{k} \hat{\rho}^2(h) \dot{\sim} \chi^2_{df=k}$$
3. We reject H_0 if $Q > \chi^2_k(1 - \alpha)$, the $1 - \alpha$ quatile of the chi-squared distribution with k degrees of freedom

Ljung-Box Test

$$Q_{LB} = T(T - 2) \sum_{h=1}^{k} \frac{\hat{\rho^2}(h)}{n - h} \dot{\sim} \chi^2_k.$$

The Ljung-Box test Ljung and Box (1978) can be more powerful than the Box and Pierce test

```r
Box.test(rnorm(100), 20)
```

```
##
##  Box-Pierce test
##
## data:  rnorm(100)
## X-squared = 15.751, df = 20, p-value = 0.7319
```

```r
Box.test(LakeHuron, 20)
```

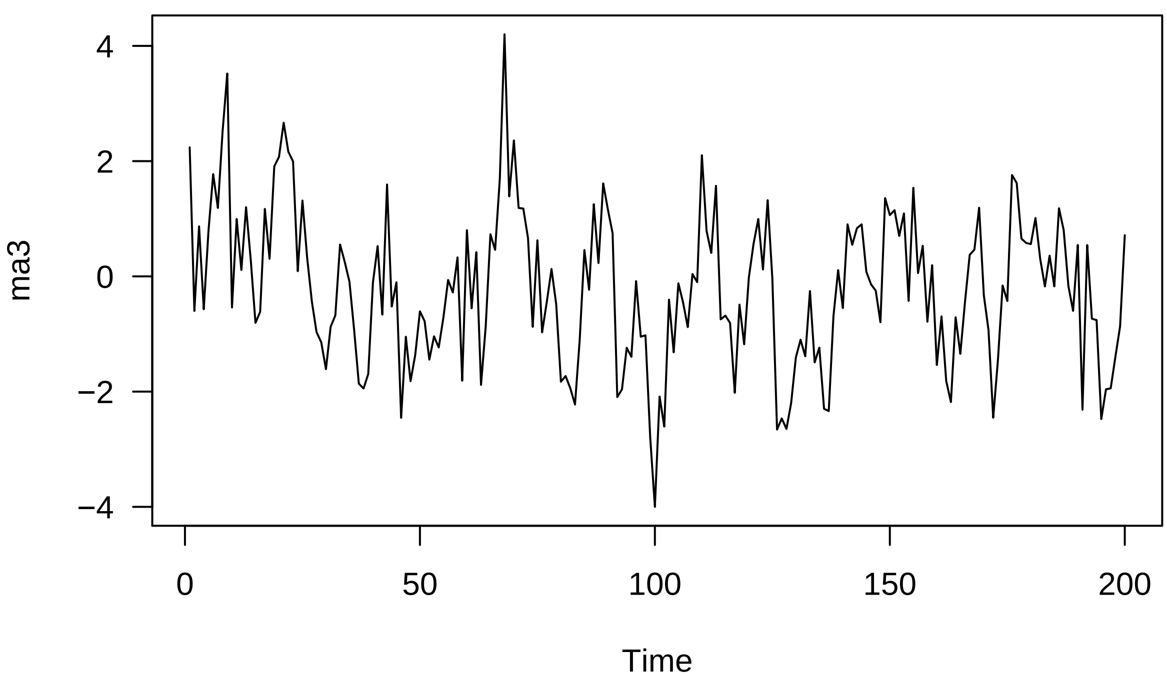
```
##
##  Box-Pierce test
##
## data:  LakeHuron
## X-squared = 182.43, df = 20, p-value < 2.2e-16
```

```r
Box.test(LakeHuron, 20, type = "Ljung")
```

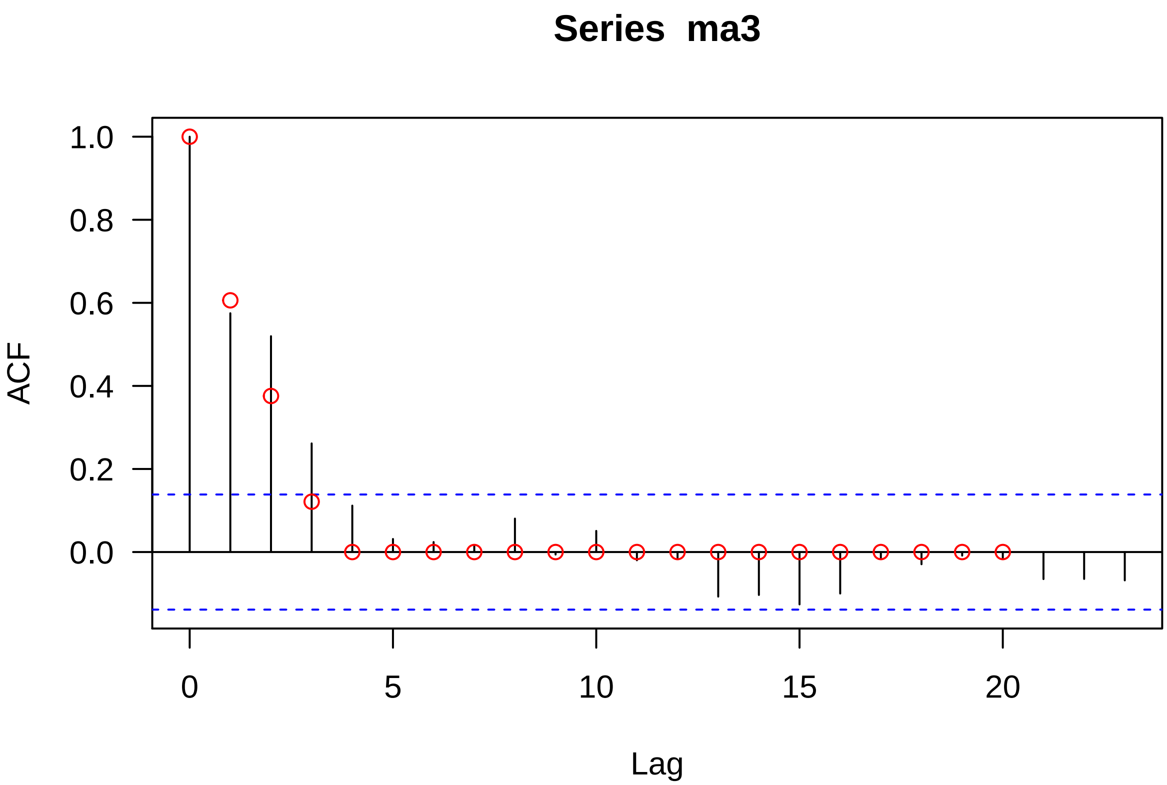
```
##
##  Box-Ljung test
##
## data:  LakeHuron
## X-squared = 192.6, df = 20, p-value < 2.2e-16
```

Examples of MA(q) processes

```r
ma3 <- arima.sim(n = 200, list(ma = c(0.6, 0.5, 0.2)))
par(las = 1)
ts.plot(ma3)
```



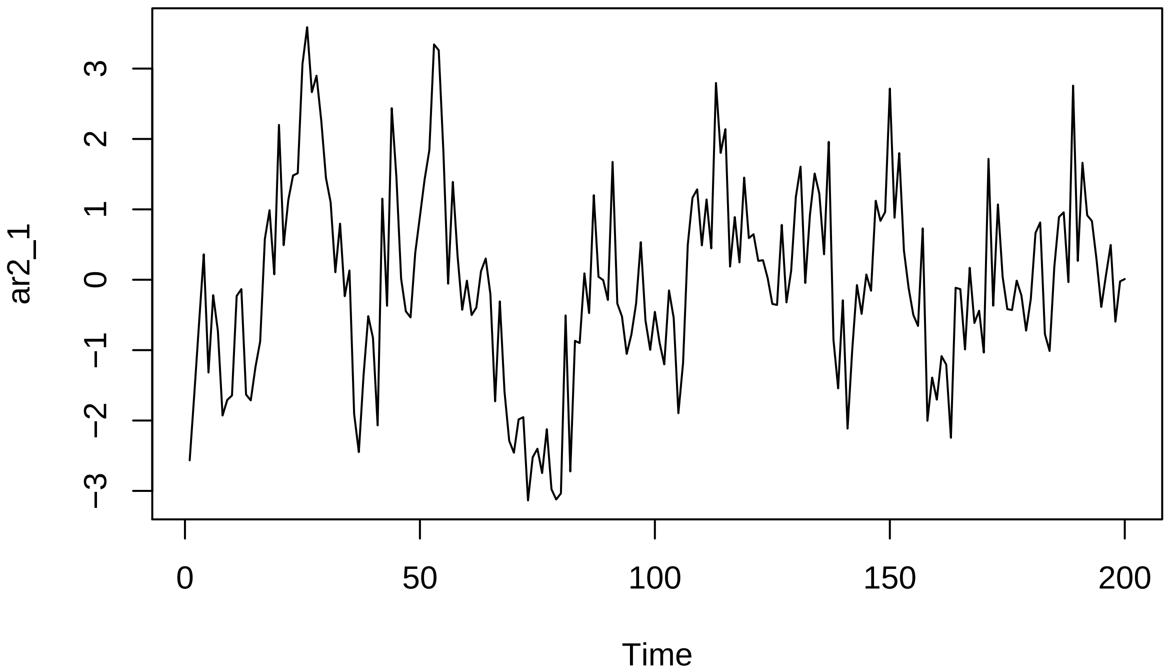
```r
acf(ma3)
acf_true <- ARMAacf(ma = c(0.6, 0.5, 0.2), lag.max = 20)
points(0:20, acf_true, col = "red")
```



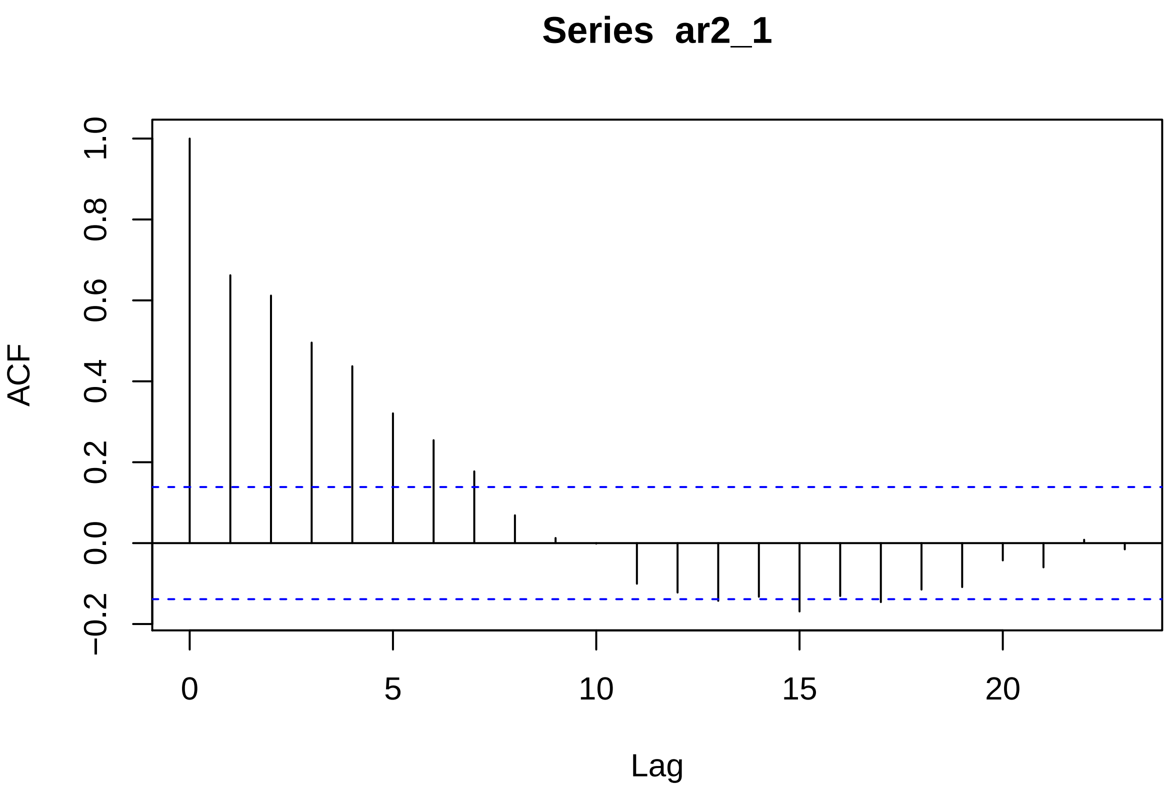
Examples of AR(2) processes

```r
ar2_1 <- arima.sim(n = 200, list(ar = c(0.5, 0.25)))

ts.plot(ar2_1)
```




```
acf(ar2_1)
```



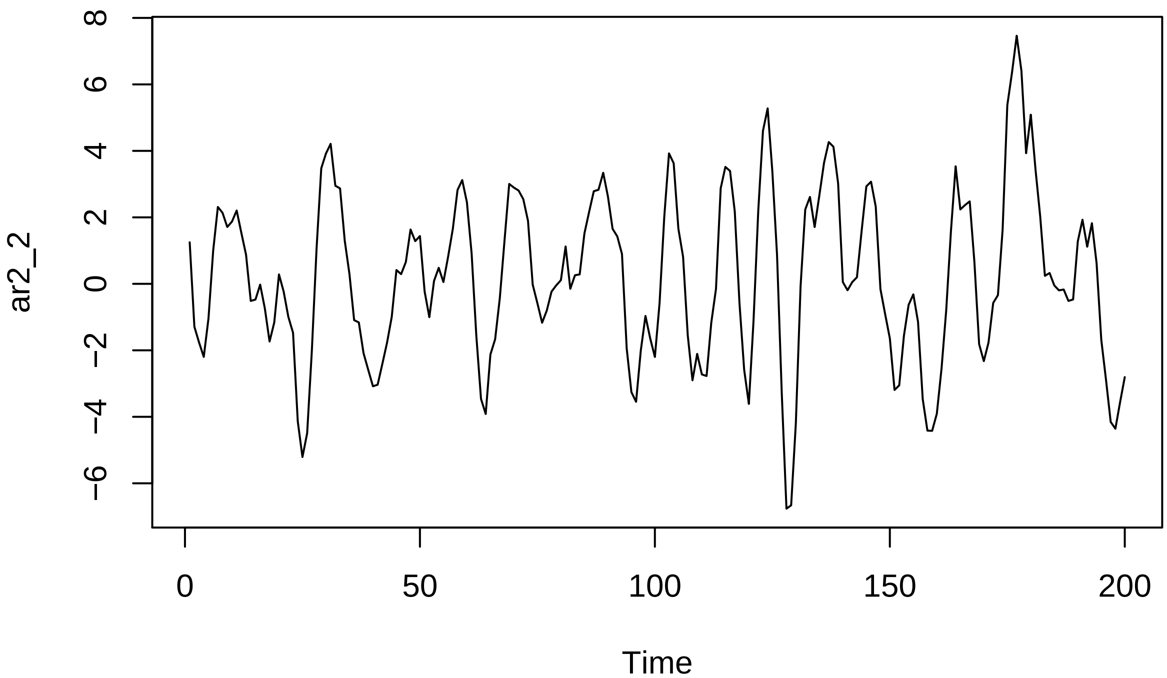

```
ar2_2 <- arima.sim(n = 200, list(ar = c(1.5, -0.75)))

ARMAacf(ar = c(0.5, 0.25), lag.max = 20)
```

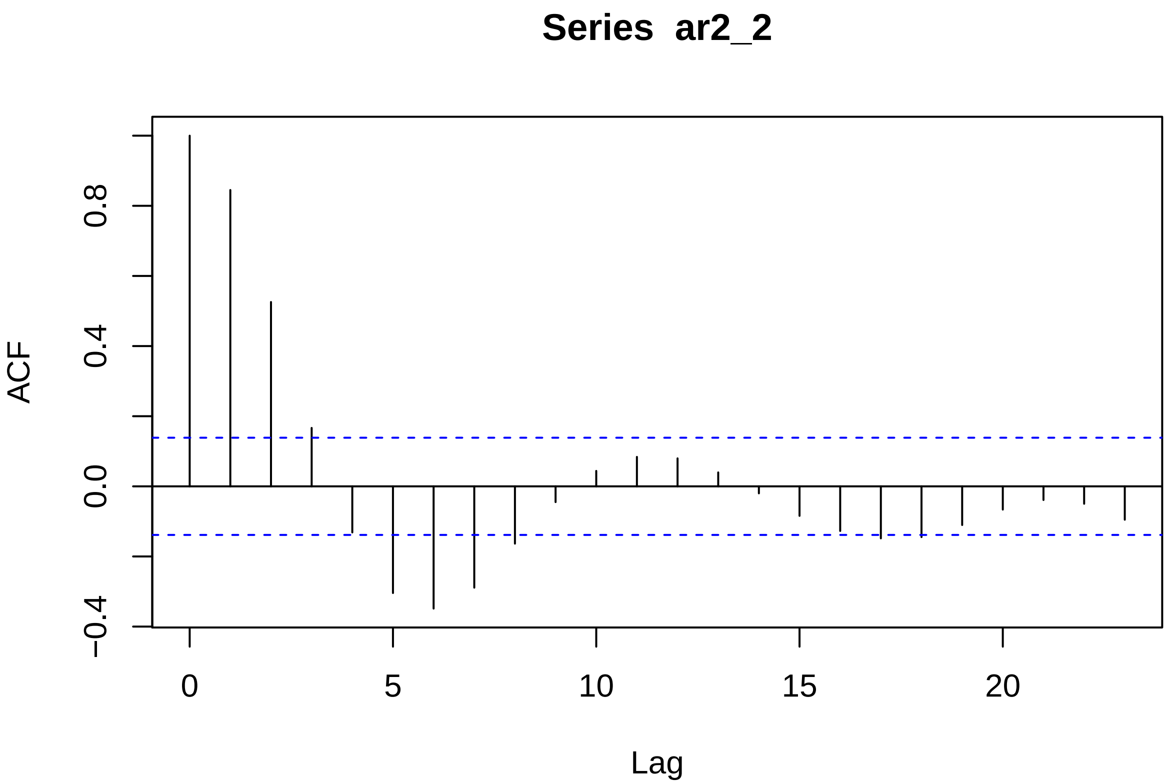
```
##             0           1           2           3           4           5           6
```

```
## 1.00000000 0.66666667 0.58333333 0.45833333 0.37500000 0.30208333 0.24479167
##          7          8          9         10         11         12         13
## 0.19791667 0.16015625 0.12955729 0.10481771 0.08479818 0.06860352 0.05550130
##         14         15         16         17         18         19         20
## 0.04490153 0.03632609 0.02938843 0.02377574 0.01923498 0.01556142 0.01258945
```

```
ts.plot(ar2_2)
```




```
acf(ar2_2)
```



References

Box, George EP, and David A Pierce. 1970. “Distribution of Residual Autocorrelations in Autoregressive-Integrated Moving Average Time Series Models.” *Journal of the American Statistical Association* 65 (332): 1509–26.

Ljung, Greta M, and George EP Box. 1978. “On a Measure of Lack of Fit in Time Series Models.” *Biometrika* 65 (2): 297–303.