# Lecture 38

## Simple Linear Regression: Confidence/Prediction Intervals & ANOVA Approach to Regression

*STAT 8010 Statistical Methods I*
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Notes

### Last Class

- **Residual Analysis:** To check the appropriateness of SLR model
  - Is the regression function linear?
  - Do $\varepsilon_i$'s have constant variance $\sigma^2$?
  - Are $\varepsilon_i$'s indepdent to each other?

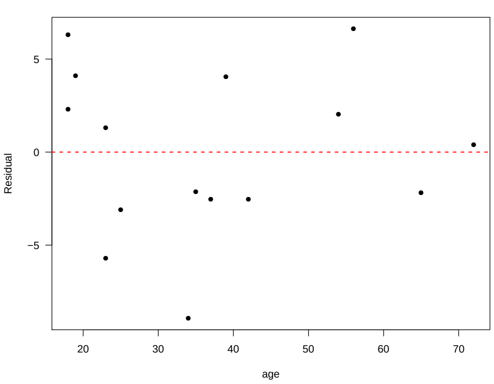
  We plot residuals $e_i$'s against $X_i$'s (or $\hat{Y}_i$'s) to assess these aspects
- **Hypothesis Tests for $\beta_1$ and $\beta_0$**
  - With additional normality assumption on $\varepsilon$, we obtained the **sampling distribution** for $\hat{\beta}_1$ and $\hat{\beta}_0$
  - Test statistic $\left(\hat{\beta}_1 - \beta_1\right)/\hat{\sigma}_{\hat{\beta}_1} \sim t_{n-2}$. With hypothesized value $\beta_1^*$ (i.e., $H_0: \beta_1 = \beta_1^*$), $H_a$ and significant level $\alpha$, we can compute the **P-value** to perform a test





Notes

### Residual Plot: $e_i$'s vs. $X_i$'s

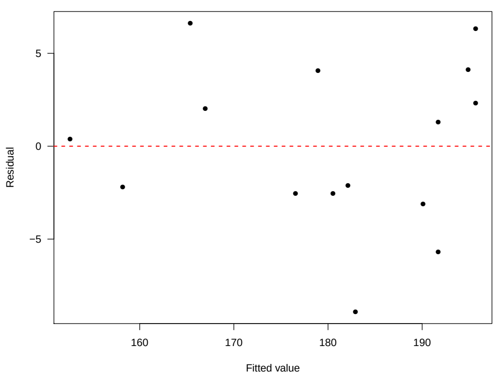







Notes

## Residual Plot: $e_i$'s vs. $\hat{Y}_i$'s





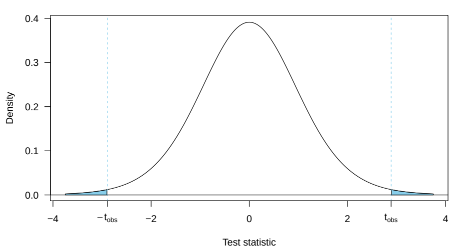



Notes

## Hypothesis Tests for $\beta_{\text{age}} = -1$

$H_0 : \beta_{\text{age}} = -1$ vs. $H_a : \beta_{\text{age}} \neq -1$

Test Statistic: $\frac{\hat{\beta}_{\text{age}} - (-1)}{\hat{\sigma}_{\hat{\beta}_{\text{age}}}} = \frac{-0.79773 - (-1)}{0.06996} = 2.8912$




P-value: $2 \times \mathbb{P}(t^* > 2.8912) = 0.013$, where $t^* \sim t_{df=13}$





Notes

## Confidence Intervals

- Recall $\frac{\hat{\beta}_1 - \beta_1}{\hat{\sigma}_{\hat{\beta}_1}} \sim t_{n-2}$, we use this fact to construct **confidence intervals (CIs)** for $\beta_1$:

$$\left[\hat{\beta}_1 - t_{\alpha/2,n-2}\hat{\sigma}_{\hat{\beta}_1}, \hat{\beta}_1 + t_{\alpha/2,n-2}\hat{\sigma}_{\hat{\beta}_1}\right],$$

  where $\alpha$ is the **confidence level** and $t_{\alpha/2,n-2}$ denotes the $1-\alpha/2$ percentile of a student's t distribution with $n-2$ degrees of freedom

- Similarly, we can construct CIs for $\beta_0$:

$$\left[\hat{\beta}_0 - t_{\alpha/2,n-2}\hat{\sigma}_{\hat{\beta}_0}, \hat{\beta}_0 + t_{\alpha/2,n-2}\hat{\sigma}_{\hat{\beta}_0}\right]$$

- Interpretation?





Notes

## Interval Estimation of $\mathrm{E}(Y_h)$



- We often interested in estimating the **mean** response for a particular value of predictor, say, $X_h$. Therefore we would like to construct CI for $\mathrm{E}[Y_h]$
- We need sampling distribution of $\hat{Y}_h$ to form CI:
  - $\frac{\hat{Y}_h - Y_h}{\hat{\sigma}_{\hat{Y}_h}} \sim t_{n-2}, \quad \hat{\sigma}_{\hat{Y}_h} = \hat{\sigma}\sqrt{\left(\frac{1}{n} + \frac{(X_h - \bar{X})^2}{\sum_{i=1}^n (X_i - \bar{X})^2}\right)}$
  - CI:
  $$\left[\hat{Y}_h - t_{\alpha/2,n-2}\hat{\sigma}_{\hat{Y}_h}, \hat{Y}_h + t_{\alpha/2,n-2}\hat{\sigma}_{\hat{Y}_h}\right]$$
- **Quiz:** Use this formula to construct CI for $\beta_0$



Notes

## Prediction Intervals

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Review of Last Class
Confidence/Prediction Intervals
Analysis of Variance (ANOVA) Approach to Regression

- Suppose we want to predict the response of a future observation given $X = X_h$
- We need to account for added variability as a new observation does not fall directly on the regression line (i.e., $Y_{\text{h(new)}} = \mathrm{E}[Y_h] + \varepsilon_h$)
- Replace $\hat{\sigma}_{\hat{Y}_h}$ by $\hat{\sigma}_{\hat{Y}_{\text{h(new)}}} = \hat{\sigma}\sqrt{\left(1 + \frac{1}{n} + \frac{(X_h - \bar{X})^2}{\sum_{i=1}^n (X_i - \bar{X})^2}\right)}$ to construct CIs for $Y_{\text{h(new)}}$



Notes

## Maximum Heart Rate vs. Age Revisited

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The maximum heart rate `MaxHeartRate` ($\text{HR}_{max}$) of a person is often said to be related to age `Age` by the equation:

$$\text{HR}_{max} = 220 - \text{Age}.$$

Suppose we have 15 people of varying ages are tested for their maximum heart rate (bpm)

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Age | 18 | 23 | 25 | 35 | 65 | 54 | 34 | 56 | 72 | 19 | 23 | 42 | 18 | 39 | 37 |
| $\text{HR}_{max}$ | 202 | 186 | 187 | 180 | 156 | 169 | 174 | 172 | 153 | 199 | 193 | 174 | 198 | 183 | 178 |

- Construct the 95% CI for $\beta_1$
- Compute the estimate for mean `MaxHeartRate` given `Age` = 40 and construct the associated 90% CI
- Construct the prediction interval for a new observation given `Age` = 40



Notes

## Analysis of Variance (ANOVA) Approach to Regression

### Partitioning Sums of Squares

- Total sums of squares in response

$$\text{SST} = \sum_{i=1}^{n}(Y_i - \bar{Y})^2$$

- We can rewrite SST as

$$\sum_{i=1}^{n}(Y_i - \bar{Y})^2 = \sum_{i=1}^{n}(Y_i - \hat{Y}_i + \hat{Y}_i - \bar{Y})^2$$
$$= \underbrace{\sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2}_{\text{Error}} + \underbrace{\sum_{i=1}^{n}(\hat{Y}_i - \bar{Y})^2}_{\text{Model}}$$





Notes

## Total Sum of Squares: SST

- If we ignored the predictor $X$, the $\bar{Y}$ would be the best (linear unbiased) predictor

$$Y_i = \beta_0 + \varepsilon_i \qquad (1)$$

- SST is the sum of squared deviations for this predictor (i.e., $\bar{Y}$)
- The **total mean square** is $\text{SST}/(n-1)$ and represents an unbiased estimate of $\sigma^2$ under the model (1).





Notes

## Regression Sum of Squares: SSR

- SSR: $\sum_{i=1}^{n}(\hat{Y}_i - \bar{Y})^2$
- Degrees of freedom is 1 due to the inclusion of the slope, i.e.,

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i \qquad (2)$$

- "Large" MSR = SSR/1 suggests a linear trend, because

$$\text{E}[MSE] = \sigma^2 + \beta_1^2 \sum_{i=1}^{n}(X_i - \bar{X})^2$$





Notes

## Error Sum of Squares: SSE

- SSE is simply the sum of squared residuals

$$\text{SSE} = \sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2$$

- Degrees of freedom is $n - 2$ (Why?)
- SSE large when |residuals| are "large" $\Rightarrow$ $Y_i$'s vary substantially around fitted regression line
- MSE = SSE$/(n - 2)$ and represents an unbiased estimate of $\sigma^2$ **when taking $X$ into account**





Notes

## ANOVA Table and F test

| **Source** | df | SS | MS |
|---|---|---|---|
| Model | 1 | $\text{SSR} = \sum_{i=1}^{n}(\hat{Y}_i - \bar{Y})^2$ | MSR = SSR/1 |
| Error | $n-2$ | $\text{SSE} = \sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2$ | MSE = SSE/(n-2) |
| Total | $n-1$ | $\text{SST} = \sum_{i=1}^{n}(Y_i - \bar{Y})^2$ | |

- **Goal:** To test $H_0 : \beta_1 = 0$
- Test statistics $F^* = \frac{\text{MSR}}{\text{MSE}}$
- If $\beta_1 = 0$ then $F^*$ should be near one $\Rightarrow$ reject $H_0$ when $F^*$ "large"
- We need sampling distribution of $F^*$ under $H_0 \Rightarrow$ $F_{1,n-2}$, where $F_{d_1,d_2}$ denotes a F distribution with degrees of freedom $d_1$ and $d_2$





Notes

## Summary

In this lecture, we learned

- Confidence/Prediction Intervals
- ANOVA Approach to Regression

Next time we will talk about

1. ANOVA Table and F test
2. Correlation $(r)$ & Coefficient of Determination $(R^2)$





Notes