Lecture 8

Normal Random Variables

Text: Chapter 4

STAT 8010 Statistical Methods I
February 3, 2020

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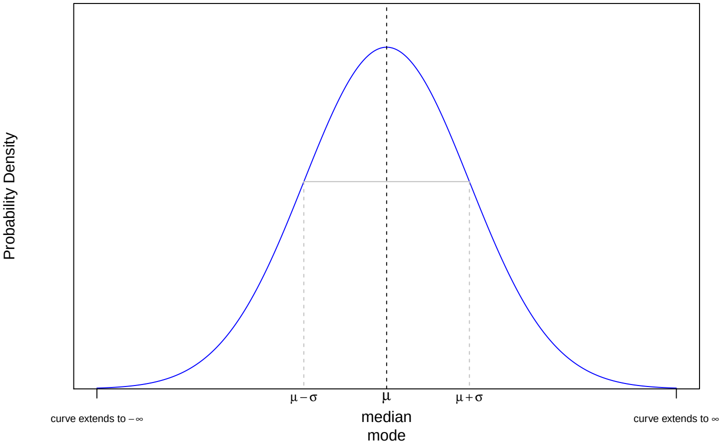
Clemson University

Agenda

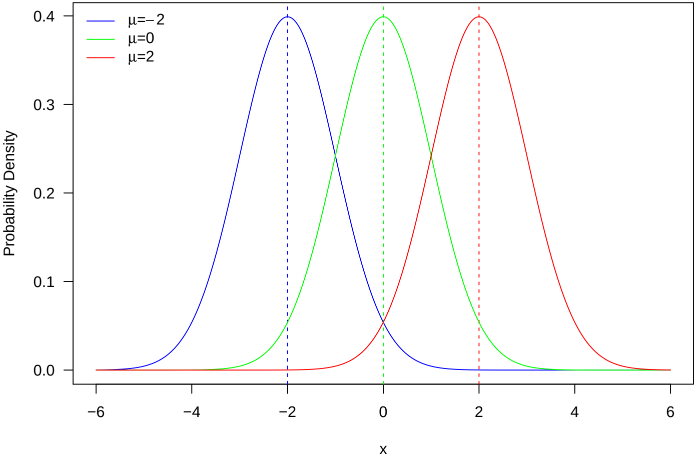
1. **Normal Density Curves**
2. **Standard Normal**
3. **Sums of Normal Random Variables**

Probability Density Curve for Normal Random Variable



Normal Density Curves

Different μ but same σ^2



Normal Density Curves Cont'd

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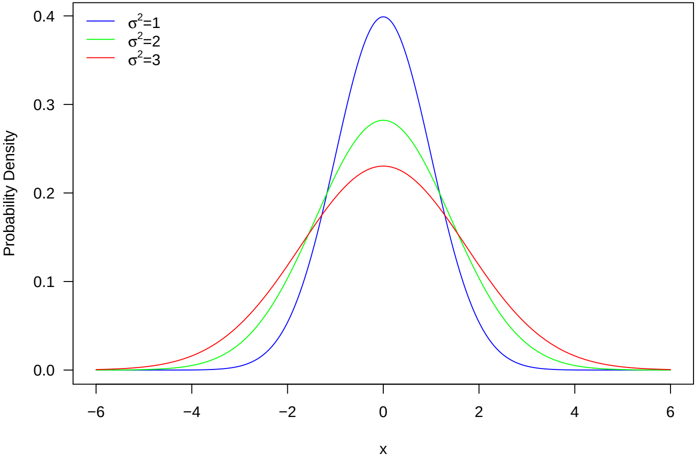

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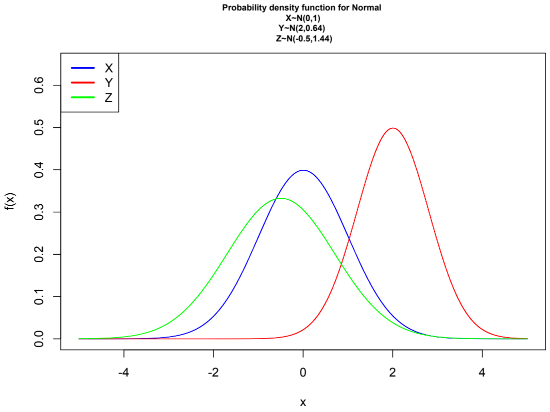

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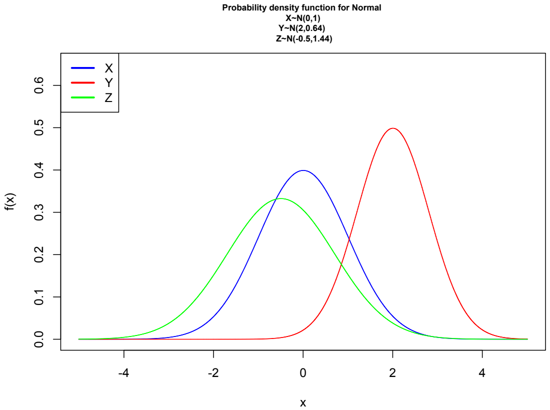

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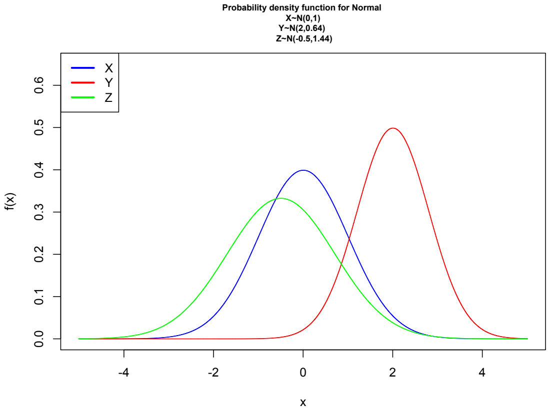

- The parameter μ determines the center of the distribution
- The parameter σ^2 determines the spread of the distribution
- Also called bell-shaped distribution

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Let X be a Normal r.v.

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- The variance: $Var(X) = \sigma^2$

Standard Normal $Z \sim N(\mu = 0, \sigma^2 = 1)$

- Normal random variable X with mean μ and standard deviation σ can convert to standard normal Z by the following :

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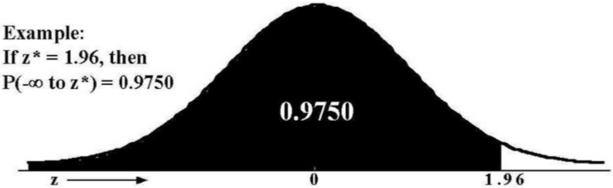
- The probability $\mathbb{P}(a \le X \le b)$ where $X \sim N(\mu, \sigma^2)$ can be compute

$$\mathbb{P}(a \le X \le b) = \mathbb{P}(\frac{a - \mu}{\sigma} \le Z \le \frac{b - \mu}{\sigma})$$

$$= \Phi(\frac{b - \mu}{\sigma}) - \Phi(\frac{a - \mu}{\sigma})$$

Standard Normal (Z) Table



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	**0.9750**	0.9756	0.9761	0.9767

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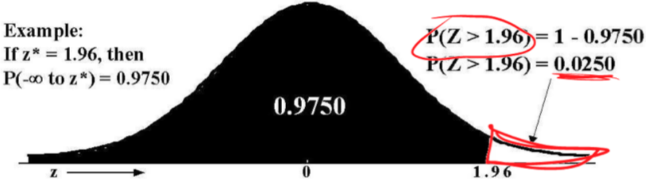

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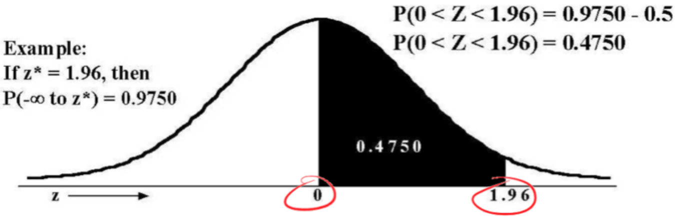

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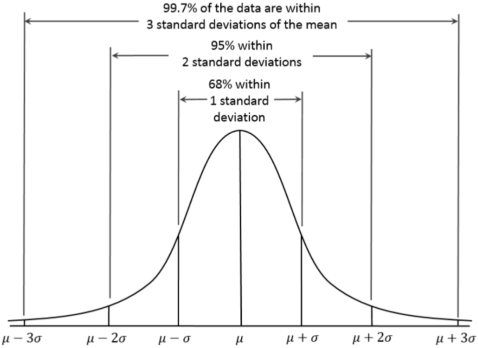
- $\Phi(-z) = 1 - \Phi(z)$
- $\mathbb{P}(Z > z) = 1 - \Phi(z) = \Phi(-z)$

The Empirical Rules

The Empirical Rules provide a quick way to approximate certain probabilities for the Normal Distribution as the following table:

Interval	Percentage with interval
$\mu \pm \sigma$	68%
$\mu \pm 2\sigma$	95%
$\mu \pm 3\sigma$	99.7%



Example

Let us examine Z. Find the following probabilities with respect to Z:

1. Z is at most -1.75
2. Z is between -2 and 2 inclusive
3. Z is less than .5

Example Cont'd

Solution.

1. $\mathbb{P}(Z \leq -1.75) = \Phi(-1.75) = .0401$

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1. $\mathbb{P}(Z \leq -1.75) = \Phi(-1.75) = .0401$
2. $\mathbb{P}(-2 \leq Z \leq 2) = \Phi(2) - \Phi(-2) = .9772 - .0228 = .9544$
3. $\mathbb{P}(Z < .5) = \Phi(.5) = .6915$

Example

Suppose a STAT-8020 exam score follows a normal distribution with mean 78 and variance 36. Let X to denote the exam score, answer the following questions:

1. What is the probability that a randomly chosen test taker got a score greater than 84?
2. Suppose the passing score for this exam is 75. What is the probability that a randomly chosen test taker got a score greater than 84 given that she/he pass the exam?
3. Using the empirical rule to find the 84_{th} percentile.

Example

Find the following percentile with respect to Z

1. 10_{th} percentile
2. 55_{th} percentile
3. 90_{th} percentile

Example Cont'd

Solution.

1. $Z_{10} = -1.28$
2. $Z_{55} = 0.13$
3. $Z_{90} = 1.28$

Example

Let X be Normal with a mean of 20 and a variance of 49. Find the following probabilities and percentile:

1. X is between 15 and 23
2. X is more than 30
3. X is more than 12 knowing it is less than 20
4. What is the value that is smaller than 20% of the distribution?

Example Cont'd

Solution.

1. $\mathbb{P}(15 \le X \le 23) = \Phi(\frac{15-20}{7}) - \Phi(\frac{23-20}{7}) = \Phi(0.43) - \Phi(-0.71) = .6664 - .2389 = .4275$
2. $\mathbb{P}(X > 30) = 1 - \mathbb{P}(X \le 30) = 1 - \Phi(\frac{30-20}{7}) = 1 - .9236 = .0764$
3. $\mathbb{P}(X > 12 | X < 20) = \frac{\mathbb{P}(12<X<20)}{\mathbb{P}(X<20)} = \frac{\Phi(0)-\Phi(-1.14)}{\Phi(0)} = .7458$
4. $Z_{80} = 0.84 \Rightarrow X_{80} = \mu + Z_{80} \times \sigma = 20 + 0.84 \times \sqrt{49} = 25.88$

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- This can be applied for any integer n

Example

Let X_1, X_2, and X_3 be mutually independent, Normal random variables. Let their means and standard deviations be $3k$ and k for $k = 1$, 2, and 3 respectively. Find the following distributions:

1. $\sum_{i=1}^{3} X_i$
2. $X_1 + 2X_2 - 3X_3$
3. $X_1 + 5X_3$

Example Cont'd

Solution.

1. $\sum_{i=1}^{3} X_i \sim N(\mu = 3+6+9 = 15, \sigma^2 = 1^2 + 2^2 + 3^2 = 14)$
2. $X_1 + 2X_2 - 3X_3 \sim N(\mu = 3 + 12 - 27 = -12, \sigma^2 = 1^2 + 4 \times 2^2 + 9 \times 3^2 = 98)$
3. $X_1 + 5X_3 \sim N(\mu = 3 + 45 = 48, \sigma^2 = 1^2 + 25 \times 3^2 = 226)$