Lecture 13

Continuous Random Variables

Text: Chapter 4

STAT 8010 Statistical Methods I
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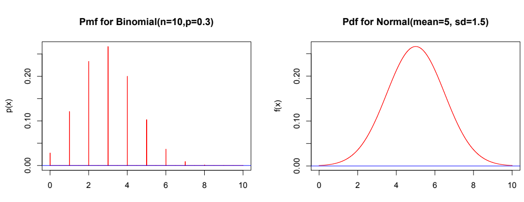
Notes

Agenda

Notes

Probability Mass Functions vs. Probability Density Functions



Remarks:

- pmf assigns probabilities to each possible values of a discrete distribution
- pdf describes the relative likelihood for this random variable to take on a given interval

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Probability Mass Functions v.s. Probability Density Functions cont'd

Recall the properties of discrete probability mass functions (Pmfs):

- $0 \leq p_X(x) \leq 1$ for all possible values of x
- $\sum_x p_X(x) = 1$
- $\mathbb{P}(a \leq X \leq b) = \sum_{x=a}^{x=b} p_X(x)$

For continuous distributions, the properties for probability density functions (Pdfs) are similar:

- $f_X(x) \geq 0$ for all possible values of x
- $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$
- $\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x)\, dx$

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Cumulative Distribution Functions (cdfs) for Continuous Distribution

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Expected Value and Variance
Normal Distributions

- The cdf $F_X(x)$ is defined as $F_X(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f_X(x)\, dx$
- we use cdf to calculate probabilities of a continuous random variable within an interval, i.e. $\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x)\, dx = \int_{-\infty}^{b} f_X(x)\, dx - \int_{-\infty}^{a} f_X(x)\, dx = \boxed{F_X(b) - F_X(a)}$

Remark: $\mathbb{P}(X = x) = \int_x^x f_X(x)\, dx = 0$ for all possible values of x

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Expected Value and Variance

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Recall the expected value formula for the discrete random variable: $\mathbb{E}[X] = \sum_x x p_X(x)$

For continuous random variables, we have similar formulas:

Let a, b, and c are constant real numbers

- $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$
- $\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$
- $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$
- $\mathbb{E}[cX] = c\mathbb{E}[X]$
- $\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$
- $\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx - \left(\int_{-\infty}^{\infty} x f_X(x)\, dx\right)^2$
- $\mathrm{Var}(cX) = c^2 \mathrm{Var}(X)$
- $\mathrm{Var}(X - c) = \mathrm{Var}(X)$

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Example

Let X represent the diameter in inches of a circular disk cut by a machine. Let $f_X(x) = c(4x - x^2)$ for $1 \leq x \leq 4$ and be 0 otherwise. Answer the following questions:

1. Find the value of c that makes this a valid pdf
2. Find the expected value and variance of X
3. What is the probability that X is within .5 inches of the expected diameter?
4. Find $F_X(x)$

Notes

Normal Distribution

Characteristics of the Normal random variable:
Let X be a Normal r.v.

- The support for X: $(-\infty, \infty)$
- Its parameter(s) and definition(s): μ : mean and σ^2 : variance
- The probability density function (pdf): $\frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$ for $-\infty < x < \infty$
- The cumulative distribution function (cdf): No explicit form, look at the value $\Phi(\frac{x-\mu}{\sigma})$ for $-\infty < x < \infty$ from standard normal table
- The expected value: $\mathbb{E}[X] = \mu$
- The variance: $Var(X) = \sigma^2$

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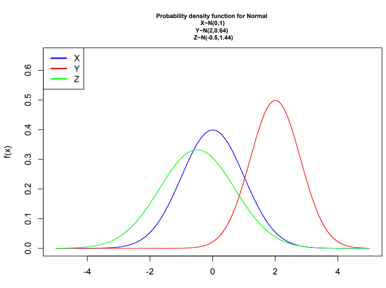
Normal Density Curves



- The parameter μ determines the center of the distribution
- The parameter σ^2 determines the spread of the distribution
- Also called bell-shaped distribution

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Standard Normal $Z \sim N(\mu = 0, \sigma^2 = 1)$

- Normal random variable X with mean μ and standard deviation σ can convert to standard normal Z by the following :

$$Z = \frac{X - \mu}{\sigma}$$

- The cdf of the standard normal, denoted by $\Phi(z)$, can be found from the standard normal table
- The probability $\mathbb{P}(a \leq X \leq b)$ where $X \sim N(\mu, \sigma^2)$ can be compute

$$\mathbb{P}(a \leq X \leq b) = \mathbb{P}(\frac{a-\mu}{\sigma} \leq Z \leq \frac{b-\mu}{\sigma})$$
$$= \Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})$$

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Properties of Φ

- $\Phi(0) = .50 \Rightarrow$ Mean and Median (50_{th} percentile) for standard normal are both 0
- $\Phi(-z) = 1 - \Phi(z)$
- $\mathbb{P}(Z > z) = 1 - \Phi(z) = \Phi(-z)$

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Example

Let us examine Z. Find the following probabilities with respect to Z:

1. Z is at most -1.75
2. Z is between -2 and 2 inclusive
3. Z is less than .5

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Example Cont'd

Solution.

1. $\mathbb{P}(Z \leq -1.75) = \Phi(-1.75) = .0401$
2. $\mathbb{P}(-2 \leq Z \leq 2) = \Phi(2) - \Phi(-2) = .9772 - .0228 = .9544$
3. $\mathbb{P}(Z < .5) = \Phi(.5) = .6915$

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Sums of Normal Random Variables

If X_i $1 \leq i \leq n$ are independent normal random variables with mean μ_i are variance σ_i^2, respectively.

- Let $S_n = \sum_{i=1}^n X_i$ then $S_n \sim N(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2)$
- This can be applied for any integer n

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Example

Let X_1, X_2, and X_3 be mutually independent, Normal random variables. Let their means and standard deviations be $3k$ and k for $k = 1, 2$, and 3 respectively. Find the following distributions:

1. $\sum_{i=1}^3 X_i$
2. $X_1 + 2X_2 - 3X_3$
3. $X_1 + 5X_3$

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Example Cont'd

Solution.

1. $\sum_{i=1}^{3} X_i \sim N(\mu = 3 + 6 + 9 = 15, \sigma^2 = 1^2 + 2^2 + 3^2 = 14)$
2. $X_1 + 2X_2 - 3X_3 \sim N(\mu = 3 + 12 - 27 = -12, \sigma^2 = 1^2 + 4 \times 2^2 + 9 \times 3^2 = 98)$
3. $X_1 + 5X_3 \sim N(\mu = 3 + 45 = 48, \sigma^2 = 1^2 + 25 \times 3^2 = 226)$

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