Lecture 13

Continuous Random Variables

Text: Chapter 4

STAT 8010 Statistical Methods I
September 18, 2019

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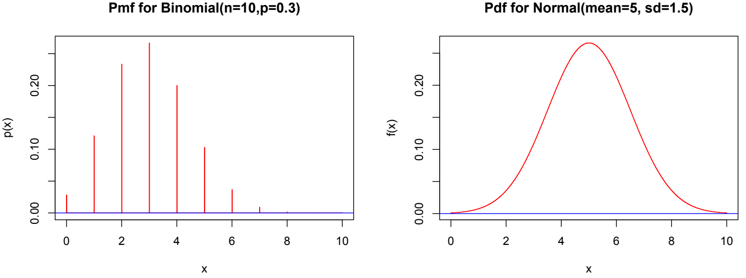
Agenda

1. From Discrete to Continuous Random Variables
2. Cumulative Distribution Functions
3. Expected Value and Variance
4. Normal Distributions

Probability Mass Functions vs. Probability Density Functions



Remarks:

- pmf assigns probabilities to each possible values of a discrete distribution
- pdf describes the relative likelihood for this random variable to take on a given interval

Probability Mass Functions v.s. Probability Density Functions cont'd

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Cumulative Distribution Functions (cdfs) for Continuous Distribution

- The cdf $F_X(x)$ is defined as $F_X(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f_X(x)\,dx$
- we use cdf to calculate probabilities of a continuous random variable within an interval, i.e. $\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x)\,dx = \int_{-\infty}^{b} f_X(x)\,dx - \int_{-\infty}^{a} f_X(x)\,dx = \boxed{F_X(b) - F_X(a)}$

Remark: $\mathbb{P}(X = x) = \int_x^x f_X(x)\,dx = 0$ for all possible values of x

Expected Value and Variance

Recall the expected value formula for the discrete random variable: $\mathbb{E}[X] = \sum_x x p_X(x)$
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Let a, b, and c are constant real numbers

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4. Find $F_X(x)$

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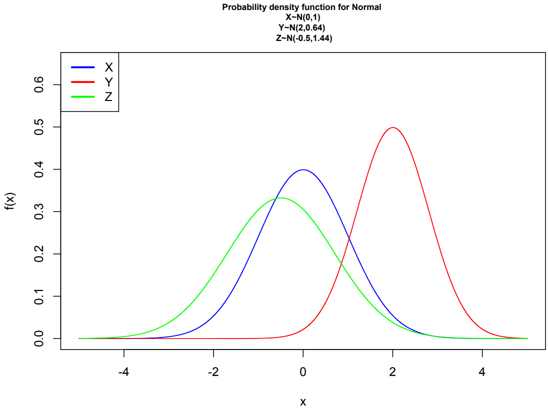
- The expected value: $\mathbb{E}[X] = \mu$
- The variance: $Var(X) = \sigma^2$

Normal Density Curves



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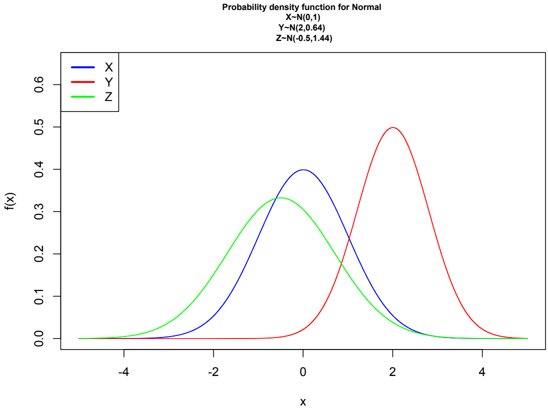

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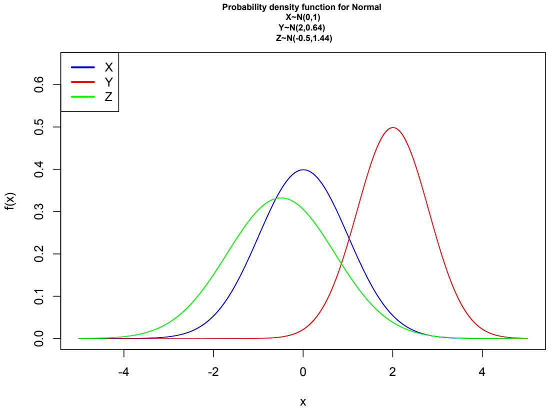

- The parameter μ determines the center of the distribution
- The parameter σ^2 determines the spread of the distribution
- Also called bell-shaped distribution

Standard Normal $Z \sim N(\mu = 0, \sigma^2 = 1)$

- Normal random variable X with mean μ and standard deviation σ can convert to standard normal Z by the following :

$$Z = \frac{X - \mu}{\sigma}$$

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- The probability $\mathbb{P}(a \leq X \leq b)$ where $X \sim N(\mu, \sigma^2)$ can be compute

$$\mathbb{P}(a \leq X \leq b) = \mathbb{P}(\frac{a - \mu}{\sigma} \leq Z \leq \frac{b - \mu}{\sigma})$$
$$= \Phi(\frac{b - \mu}{\sigma}) - \Phi(\frac{a - \mu}{\sigma})$$

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- $\mathbb{P}(Z > z) = 1 - \Phi(z) = \Phi(-z)$

Example

Let us examine Z. Find the following probabilities with respect to Z:

1. Z is at most -1.75
2. Z is between -2 and 2 inclusive
3. Z is less than .5

Example Cont'd

Solution.

1. $\mathbb{P}(Z \leq -1.75) = \Phi(-1.75) = .0401$

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Example Cont'd

Solution.

1. $\mathbb{P}(Z \leq -1.75) = \Phi(-1.75) = .0401$
2. $\mathbb{P}(-2 \leq Z \leq 2) = \Phi(2) - \Phi(-2) = .9772 - .0228 = .9544$
3. $\mathbb{P}(Z < .5) = \Phi(.5) = .6915$

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- This can be applied for any integer n

Example

Let X_1, X_2, and X_3 be mutually independent, Normal random variables. Let their means and standard deviations be $3k$ and k for $k = 1$, 2, and 3 respectively. Find the following distributions:

1. $\sum_{i=1}^{3} X_i$
2. $X_1 + 2X_2 - 3X_3$
3. $X_1 + 5X_3$

Example Cont'd

Solution.

1. $\sum_{i=1}^{3} X_i \sim N(\mu = 3+6+9 = 15, \sigma^2 = 1^2+2^2+3^2 = 14)$
2. $X_1 + 2X_2 - 3X_3 \sim N(\mu = 3+12-27 = -12, \sigma^2 = 1^2 + 4 \times 2^2 + 9 \times 3^2 = 98)$
3. $X_1 + 5X_3 \sim N(\mu = 3+45 = 48, \sigma^2 = 1^2 + 25 \times 3^2 = 226)$