Lecture 12

Spectral Analysis of Time Series I

Readings: Cryer & Chan Ch 13-14; Brockwell & Davis Ch 4; Shumway & Stoffer Ch 4.1-4.4

MATH 8090 Time Series Analysis
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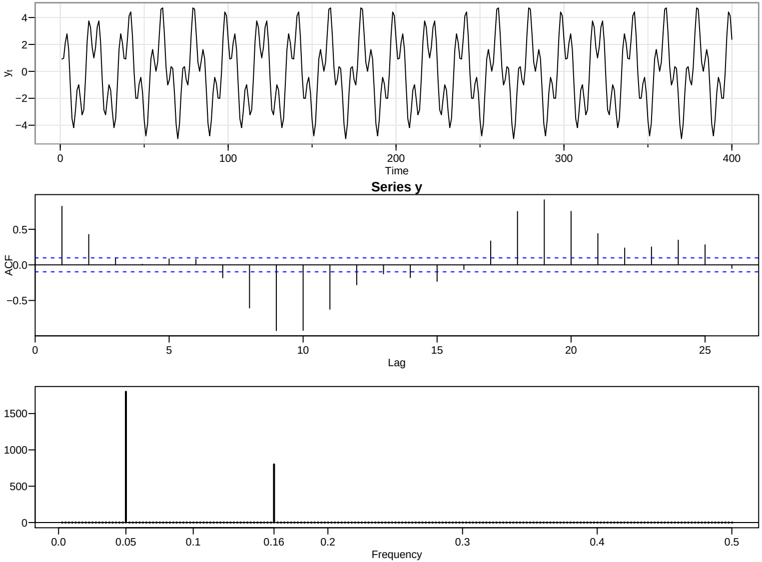
Agenda

1. **Overview**
2. **The Spectral Density and Periodogram**
3. **Spectral Estimation**

Searching Hidden Periodicities



Time Domain Analysis vs Frequency Domain Analysis

- Time domain methods [Box and Jenkins, 1970]:
  - Regress present on past

    Example: $Y_t = \phi Y_{t-1} + Z_t, \quad |\phi| < 1, \{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$
  - Capture dynamics in terms of "velocity", "acceleration", etc
- Frequency domain methods [Priestley, 1981]:
  - Regress present on periodic sines and cosines

    Example: $Y_t = \alpha_0 + \sum_{j=1}^{p} \left[\alpha_{1j} \cos(2\pi\omega_j t) + \alpha_{2j} \sin(2\pi\omega_j t)\right]$
  - Capture dynamics in terms of resonant frequencies

Describing Cyclical Behavior

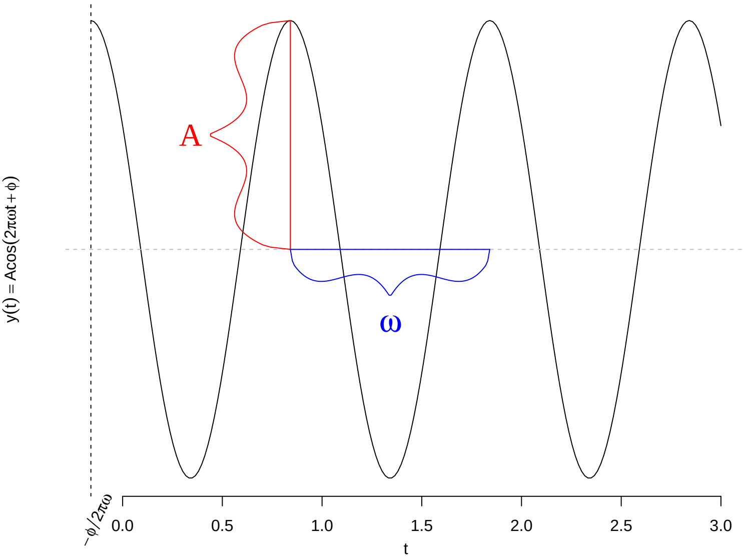
The simplest case is the periodic process

$$\begin{aligned} Y_t &= A\cos(2\pi\omega t + \phi) \\ &= \alpha_1 \cos(2\pi\omega t) + \alpha_2 \sin(2\pi\omega t), \end{aligned}$$

where

- A is amplitude
- ω is frequency, in cycles per sample
- ϕ is phase, determining the start point of the cosine function
- $\alpha_1 = A\cos(\phi)$, $\alpha_2 = -A\sin(\phi)$, $A = \sqrt{\alpha_1^2 + \alpha_2^2}$, $\phi = \tan^{-1} \frac{-\alpha_2}{\alpha_1}$

Graphical Illustration of the Periodic Process



A Periodic Time Series

If

$$\begin{aligned} Y_t &= A\cos(2\pi\omega t + \phi) \\ &= \alpha_1 \cos(2\pi\omega t) + \alpha_2 \sin(2\pi\omega t), \end{aligned}$$

and ϕ is random, uniformly distribiuted on $[-\pi, \pi)$, then:

$$\begin{aligned} \mathbb{E}(Y_t) &= 0 \\ \mathbb{E}(Y_{t+h}Y_t) &= \frac{1}{2}A^2 \cos(2\pi\omega h) \end{aligned}$$

$\Rightarrow Y_t$ is weakly stationary

A Generalization of Periodic Time Series

Also

$$\mathbb{E}(\alpha_1) = \mathbb{E}(\alpha_2) = 0,$$
$$\mathbb{E}(\alpha_1^2) = \mathbb{E}(\alpha_2^2) = \frac{1}{2}A^2,$$

and $\mathbb{E}(\alpha_1\alpha_2) = 0.$

Alternatively, if the α's have these properties, then Y_t is stationary with the same mean and autocovariances:

$$\mathbb{E}(Y_t) = 0,$$
$$\mathbb{E}(Y_{t+h}Y_t) = \frac{1}{2}A^2\cos(2\pi\omega h).$$

A Periodic Time Series (Cont'd)

More generally, if

$$Y_t = \sum_{k=1}^{q} \left[\alpha_{k,1} \cos(2\pi\omega_k t) + \alpha_{k,2} \sin(2\pi\omega_k t)\right],$$

where:

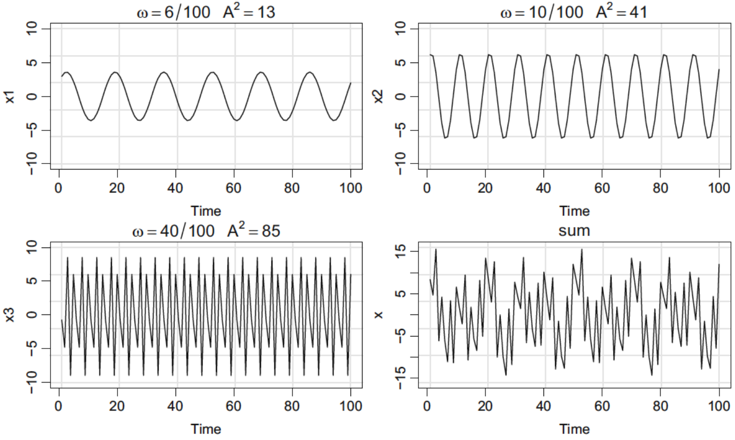
- The α's are uncorrelated with zero mean;
- $\mathrm{Var}(\alpha_{k,1}) = \mathrm{Var}(\alpha_{k,2}) = \sigma_k^2$;

then Y_t is stationary with zero mean and autocovariances

$$\gamma(h) = \sum_{k=1}^{q} \sigma_k^2 \cos(2\pi\omega_k h)$$

$$\Rightarrow \gamma(0) = \mathbb{V}\mathrm{ar}(Y_t) = \sum_{k=1}^{q} \sigma_k^2$$

An Example of Periodic Time Series



Source: Fig. 4.1. of Shumway and Stoffer, 2017

Folding Frequency and Aliasing

- If $\omega = 0$, $Y_t = A\cos(\phi)$
- If $\omega = 1$, $Y(0) = Y(1) = Y(2) = \cdots = A\cos(\phi)$

  $\Rightarrow \omega = 0$ is an alias of $\omega = 1$

- All frequencies higher than $\omega = \frac{1}{2}$ have an alias in $0 \leq \omega \leq \frac{1}{2}$

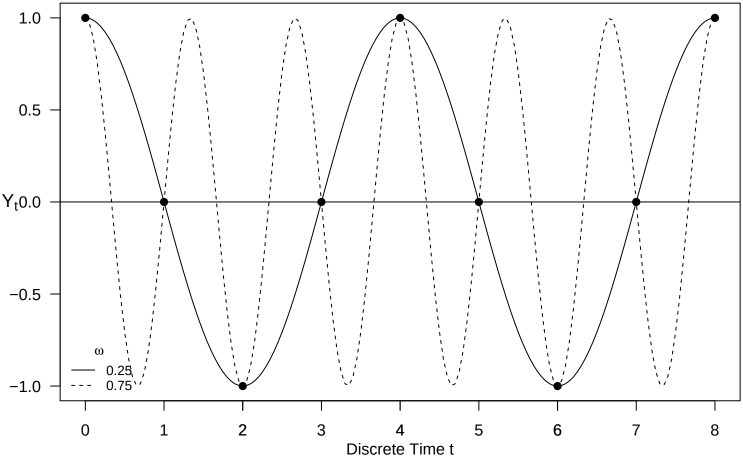
$$\cos(2\pi\omega t + \phi) = \cos(2\pi(1-\omega)t + \phi)$$

- $\omega = \frac{1}{2}$ is the folding frequency (aka Nyquist frequency)

Takeaway: It suffices to limit attention to $\omega \in [0, \frac{1}{2}]$

Illustration of Aliasing

$\omega = 0.25$ and $\omega = 0.75$ are aliased with one another



Representing Periodic Functions by Fourier Series

Any time series sample $y_1, y_2, \cdots, y_n$ can be written

$$y_t = \alpha_0 + \sum_{j=1}^{(n-1)/2} \left[\alpha_j \cos(2\pi jt/n) + \beta_j \sin(2\pi jt/n)\right],$$

if n is odd; if n is even, an extra term is needed

- The (sclaed) periodogram is

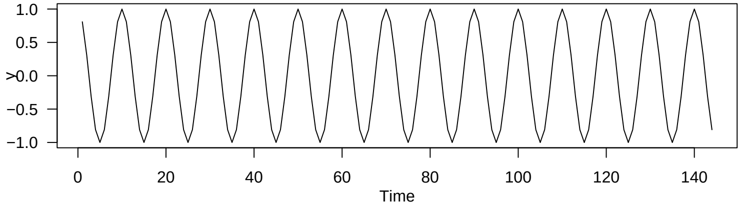
  $$P(j/n) = \alpha_j^2 + \beta_j^2$$

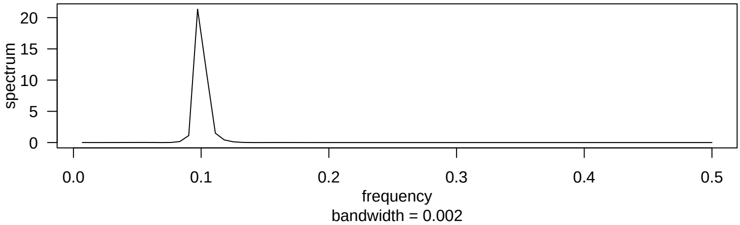
  the sample variance at each frequency component

- The R function `spectrum` can calculate and plot the periodogram

An Example: $Y_t = \cos(2\pi 0.1 t)$

```
y = cos(2 * pi * (0.1) * (1:144))
ts.plot(y); spectrum(x, log = "no", main = "")
```





The Spectral Density

- The periodogram shows which frequencies are strong in a finite sample $\{y_1, y_2, \cdots, y_n\}$
- What about a population model for Y_t, such as a stationary time series?
- The spectral density plays the corresponding role

The Discrete Fourier Transform (DFT)

- Given data $y_1, y_2, \cdots, y_n$, the discrete Fourier transform is

$$d(\omega_j) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} y_t e^{-2\pi\omega_j t}, \quad j = 0, 1, \cdots, n-1.$$

- The frequencies $\omega_j = j/n$ are the Fourier or fundamental frequencies
- Like any other Fourier transform, it has an inverse transform:

$$y_t = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} d(\omega_j) e^{2\pi\omega_j t}, \quad t = 1, 2, \cdots, n$$

The Periodogram

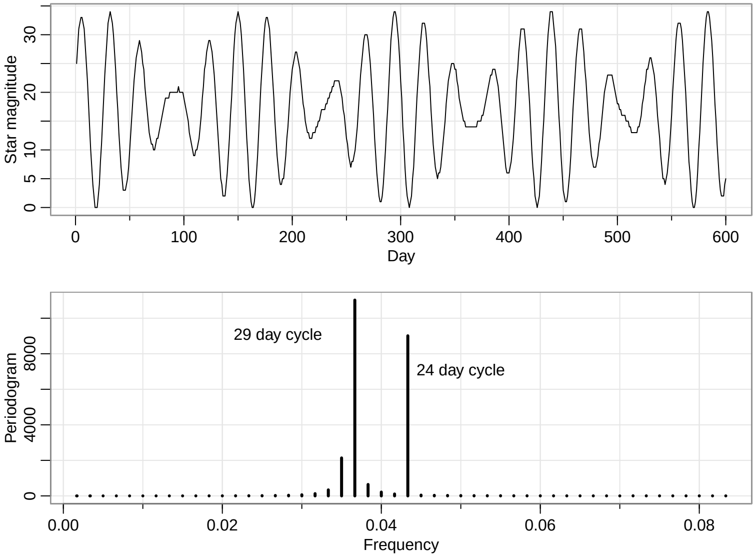
- The periodogram is $I(\omega_j) = |d(w_j)|^2, \quad j = 0, 1, \cdots, n-1$
- The sclaed periodogram we used earlier is

$$P(\omega_j) = (4/n) I(\omega_j)$$

- In terms of sample autocovariances: $I(0) = n\bar{y}^2$, and for $j \neq 0$,

$$\begin{aligned} I(\omega_j) &= \sum_{h=-(n-1)}^{n-1} \hat{\gamma}(h) e^{-2\pi i \omega_j h} \\ &= \hat{\gamma}(0) + 2 \sum_{h=1}^{n-1} \hat{\gamma}(h) \cos(2\pi\omega_j h). \end{aligned}$$

Star Magnitude Example [Example 4.3, Shumway & Stoffer, 2017]



The Mathematics of the Spectrum

Every weakly stationary time series Y_t with autocovariances $\gamma(h)$ has a non-decreasing spectrum or spectral distribution function $F(\omega)$ for which

$$\gamma(h) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i\omega h} dF(\omega) = 2\int_{0}^{\frac{1}{2}} \cos(2\pi\omega h)\, dF(\omega).$$

If $F(\omega)$ is *absolutely continuous*, it has a spectral density function $f(\omega) = F^{'}(\omega)$, and

$$\gamma(h) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i\omega h} f(\omega)\, d\omega = 2\int_{0}^{\frac{1}{2}} \cos(2\pi\omega h) f(\omega)\, d\omega$$

> The autocovariance and the spectral distribution function contain the same information

The Mathematics of the Spectrum (Cont'd)

Under various conditions on $\gamma(h)$, such as

$$\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$$

$f(\omega)$ can be written as the sum

$$f(\omega) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \omega_j h} = \gamma(0) + 2 \sum_{h=1}^{\infty} \gamma(h) \cos(2\pi\omega_j h)$$

Properties of the spectral density:

- $f(\omega) \geq 0;$
- $f(-\omega) = f(\omega);$
- $\int_{-\frac{1}{2}}^{\frac{1}{2}} f(\omega)\, d\omega = \gamma(0) < \infty$

Example: White Noise

For white noise $\{Z_t\}$, we have seen that $\gamma(0) = \sigma_Z^2$ and $\gamma(h) = 0$ for $h \neq 0$. Thus,

$$\begin{aligned} f(\omega) &= \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \omega h} \\ &= \gamma(0) = \sigma_Z^2 \end{aligned}$$

That is, the spectral density is constant across all frequencies: each frequency in the spectrum contributes equally to the variance. This is the origin of the name *white noise*: it is like white light, which is a uniform mixture of all frequencies in the visible spectrum

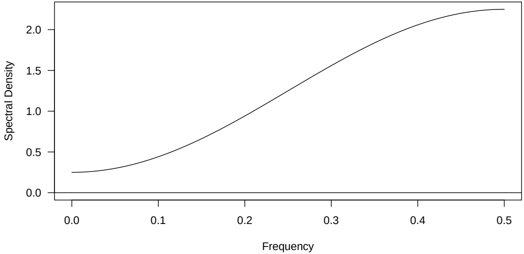
Examples: MA(1)

An MA(1) process $Y_t = \theta Z_{t-1} + Z_t$ is a simple filtering of white noise. Therefore, we have the (power) transfer function of the MA filter is:

$$\begin{aligned} \left|1 - \theta e^{2\pi i\omega}\right|^2 &= (1 - \theta e^{2\pi i\omega})(1 - \theta e^{-2\pi i\omega}) \\ &= 1 + \theta^2 - \theta(e^{2\pi i\omega} + e^{-2\pi i\omega}) \\ &= 1 + \theta^2 - 2\theta\cos(2\pi\omega). \end{aligned}$$

Thus, we have

$$f(\omega) = \left[1 + \theta^2 - 2\theta\cos(2\pi\omega)\right]\sigma_Z^2$$



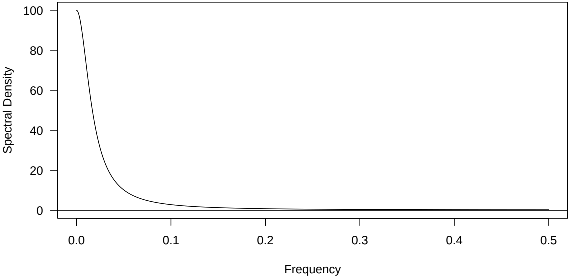
Example: AR(1)

For an AR(1) $Y_t = \phi Y_{t-1} + Z_t$, we have

$$\left[1 + \phi^2 - 2\phi\cos(2\pi\omega)\right] f(\omega) = \sigma_Z^2$$

Thus, we have

$$f(\omega) = \frac{\sigma_Z^2}{1 + \phi^2 - 2\phi\cos(2\pi\omega)}$$



Examples: ARMA(p, q)

- ARMA: using results about linear filtering, we shall show that the spectral density of the ARMA(p, q) process

$$\phi(B)Y_t = \theta(B)Z_t$$

is

$$f(\omega) = \sigma_Z^2 \frac{|\theta(e^{-2\pi i\omega})|^2}{|\phi(e^{-2\pi i\omega})|^2}$$

- Note that this gives the polynomials $\phi(\cdot)$ and $\theta(\cdot)$ a concrete meaning: they determine how strongly the series tends to fluctuate at each frequency

Estimating Spectral Density Using Periodogram

If n is large

$$
\begin{aligned}
\mathbb{E}\left[I(\omega_j)\right] &\approx \sum_{h=-(n-1)}^{n-1} \gamma(h) e^{-2\pi i \omega_j h} \\
&\approx \sum_{-\infty}^{\infty} \gamma(h) e^{-2\pi \omega_j h} = \gamma(0) + 2\sum_{h=1}^{\infty} \gamma(h)\cos(2\pi\omega_j h) \\
&= f(\omega_j) \text{ ☺}.
\end{aligned}
$$

- Heuristically, the spectral density is the approximate expected value of the periodogram
- Conversely, the periodogram can be used as an estimator of the spectral density
- But the periodogram values have only two degrees of freedom each, which makes it a poor estimate ☹

The Periodogram

Recall: the discrete Fourier transform

$$d(\omega_j) = n^{-\frac{1}{2}} \sum_{t=1}^{n} y_t e^{-2\pi i \omega_j t}, \quad j = 0, 1, \cdots, n-1,$$

and the periodogram

$$I(\omega_j) = |d(\omega_j)|^2, \quad j = 0, 1, \cdots, n-1,$$

where ω_j is one of the Fourier frequencies

$$\omega_j = \frac{j}{n}.$$

Periodogram is the squared modulus of the DFT

Sine and Cosine Transforms

For $j = 0, 1, \cdots, n-1$

$$
\begin{aligned}
d(\omega_j) &= n^{-\frac{1}{2}} \sum_{t=1}^{n} y_t e^{-2\pi i \omega_j t} \\
&= n^{-\frac{1}{2}} \sum_{t=1}^{n} y_t \cos(2\pi\omega_j t) - i \times n^{-\frac{1}{2}} y_t \sin(2\pi\omega_j t) \\
&= d_{\cos}(\omega_j) - i \times d_{\sin}(\omega_j).
\end{aligned}
$$

- $d_{\cos}(\omega_j)$ and $d_{\cos}(\omega_j)$ are the cosine transform and sine transform, respectively, of $y_1, y_2, \cdots, y_n$
- The periodogram is $I(\omega_j) = d_{\cos}(\omega_j)^2 + d_{\sin}(\omega_j)^2$

Sampling Properties of the Periodogram

For convenience, suppose that n is odd: $n = 2m + 1$

- White noise: orthogonality properties of sines and cosines mean that $d_{\cos}(\omega_1), d_{\sin}(\omega_1), d_{\cos}(\omega_2), d_{\sin}(\omega_2), \cdots, d_{\cos}(\omega_m), d_{\sin}(\omega_m)$ have zero mean, variance $\frac{\sigma_Z^2}{2}$, and uncorrelated
- Gaussian white noise: $d_{\cos}(\omega_1), d_{\sin}(\omega_1), d_{\cos}(\omega_2), d_{\sin}(\omega_2), \cdots, d_{\cos}(\omega_m), d_{\sin}(\omega_m)$ are $\text{i.i.d.}\, \mathrm{N}(0, \frac{\sigma_Z^2}{2})$
- So for Gaussian white noise

$$I(\omega_j) \sim \frac{\sigma_Z^2}{2} \times \chi_2^2$$

The periodogram is not a consistent estimator of the spectral density (why?)

Sampling Distributions: General Case

General case:

$d_{\cos}(\omega_1), d_{\sin}(\omega_1), d_{\cos}(\omega_2), d_{\sin}(\omega_2), \cdots, d_{\cos}(\omega_m), d_{\sin}(\omega_m)$, have zero mean and are approximately uncorrelated, and

$$\mathrm{Var}\left[d_{\cos}(\omega_j)\right] \approx \mathrm{Var}\left[d_{\sin}(\omega_j)\right] \approx \frac{1}{2} f(\omega_j),$$

where $f(\omega_j)$ is the spectral density function

If Y_t is Gaussian,

$$\frac{I(\omega_j)}{\frac{1}{2} f(\omega_j)} = \frac{d_{\cos}(\omega_j)^2 + d_{\sin}(\omega_j)^2}{\frac{1}{2} f(\omega_j)} \approx \chi_2^2,$$

and $I(\omega_1), I(\omega_2), \cdots, I(\omega_m)$ are approximately independent

The periodogram is not a consistent estimator!

Spectral ANOVA

- For odd $n = 2m + 1$, the inverse transform can be written

$$y_t - \bar{y} = \frac{2}{\sqrt{n}} \sum_{j=1}^{m} \left[d_{\cos}(\omega_j) \cos(2\pi\omega_j)t + d_{\sin}(\omega_j) \sin(2\pi\omega_j t) \right].$$

- Square and sum over t; orthogonality of sines and cosines implies that

$$\begin{aligned} \sum_{t=1}^{n} (y_t - \bar{y})^2 &= 2 \sum_{j=1}^{m} \left[d_{\cos}(\omega_j)^2 + d_{\sin}(\omega_j)^2 \right] \\ &= 2 \sum_{j=1}^{m} I(\omega_j) \end{aligned}$$

ANOVA Table and Hypothesis Testing

Source	df	SS	MS
ω_1	2	$2I(\omega_1)$	$I(\omega_1)$
ω_2	2	$2I(\omega_2)$	$I(\omega_2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
ω_m	2	$2I(\omega_m)$	$I(\omega_m)$
Total	$2m = n - 1$	$\sum(y_t - \bar{y})^2$	

Consider the model:

$$Y_t = A\cos(2\pi\omega_j t + \phi) + Z_t.$$

Hypotheses:

- $H_0 : A = 0 \Rightarrow Y_t = Z_t$ white noise
- $H_1 : A > 0$, white noise plus a sine wave

Hypothesis Testing

Two cases:

- ω_j known: use

$$F_j = \frac{I(\omega_j)}{(m-1)^{-1}\sum_{j'\neq j} I(\omega_{j'})},$$

which is $F_{2,2(m-1)}$ under H_0

- ω_j unknown: use

$$\kappa = \frac{\max\{I(\omega_j), j = 1, , 2, \cdots, n\}}{m^{-1}\sum_j I(\omega_j)}$$

and

$$\mathbb{P}(\kappa > \xi) \approx 1 - \exp\left\{-\exp\left[-\xi\frac{(m-1-\log(m))}{m-\xi}\right]\right\}.$$

Nonparametric Spectrum Estimates

Recall:

$$\frac{I(\omega_j)}{\frac{1}{2}f(\omega_j)} = \frac{d_{\cos}(\omega_j)^2 + d_{\sin}(\omega_j)^2}{\frac{1}{2}f(\omega_j)} \approx \chi_2^2,$$

and $I(\omega_1), I(\omega_2), \cdots, I(\omega_m)$ are approximately independent

Problem: $I(\omega_j)$ is an approximately unbiased estimator of $f(\omega_j)$ but with too few degrees of freedom (df = 2) to be useful. Specifically, $I(\omega) \dot{\sim} \frac{1}{2}f(\omega)\chi_2^2$, which implies

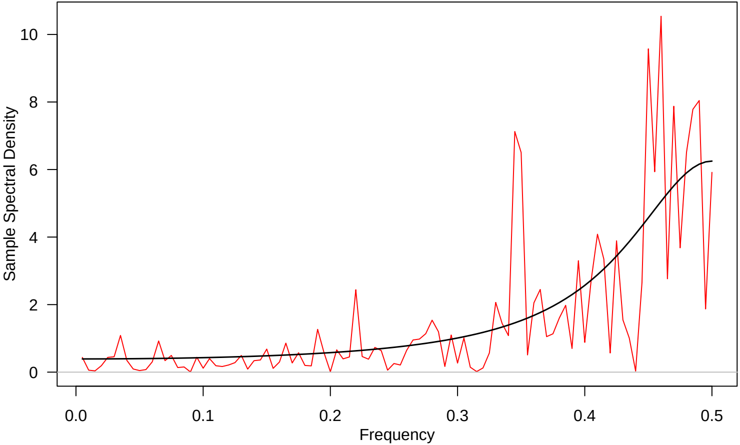
$$\mathbb{E}[I(\omega)] \approx f(\omega)$$

and

$$\mathbb{Var}[I(\omega)] \approx f^2(\omega)$$

Consequently, $\mathbb{Var}[I(\omega)] \overset{n\to\infty}{\neq} 0$ and thus the periodogram is not a consistent estimator of the spectral density

Smoothing the Periodogram

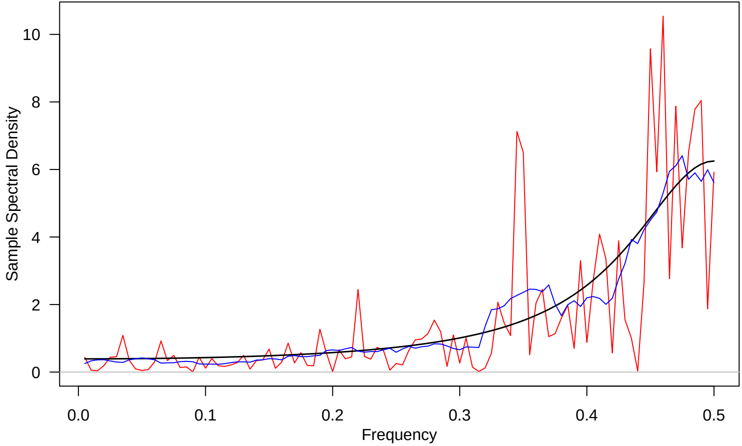


Main idea: "average" the values of the periodogram over "small" intervals of frequencies to reduce the estimation variability

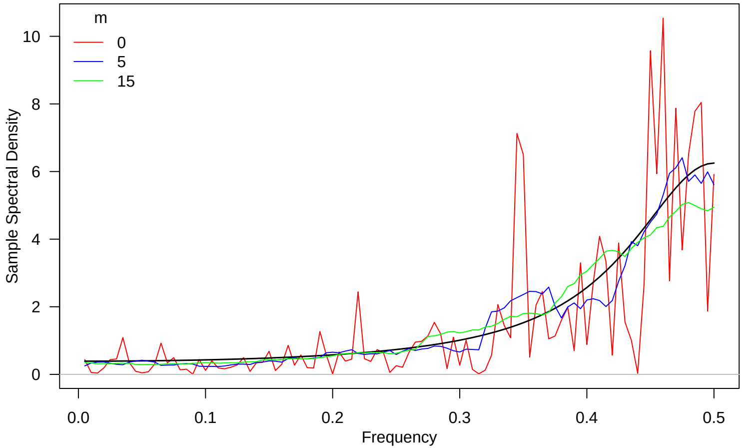
Averaged Periodogram [Daniell Spectral Window]

Use the band $[\omega_{j-m}, \omega_{j+m}]$ containing $L = 2m + 1$ Fourier frequencies:

$$\bar{f}(\omega_j) = \frac{1}{L} \sum_{k=-m}^{m} I(\omega_{j+k})$$



Tuning Parameter: m



A large m can effectively reduce the estimation variability but can also introduce bias

Bias and Variance

Let's assume the true spectral density does not change much locally, then a short Taylor expansion produces

$$
\begin{aligned}
\mathbb{E}[\bar{f}(\omega)] &\approx \sum_{k=-m}^{m} W_m(k) f(\omega + \frac{k}{n}) \\
&\approx \sum_{k=-m}^{m} W_m(k) \left[f(\omega) + \frac{k}{n} f'(\omega) + \frac{1}{2} (\frac{k}{n})^2 f''(\omega) \right] \\
&\approx f(\omega) + \frac{1}{n^2} \frac{f''(\omega)}{2} \sum_{k=-m}^{m} k^2 W_m(k)
\end{aligned}
$$

$$
\text{Bias} \approx \frac{1}{n^2} \frac{f''(\omega)}{2} \sum_{k=-m}^{m} k^2 W_m(k)
$$

$$
\text{Variance} \approx f^2(\omega) \sum_{k=-m}^{m} W_m^2(k)
$$

Example: for Daniell rectangular spectral window, we have bias = $\frac{2}{n^2(2m+1)} \left(\frac{m^3}{3} + \frac{m^2}{2} + \frac{m}{6} \right)$ and variance $\frac{1}{2m+1}$

Pointwise Confidence Intervals for $f(\omega)$

The distribution of $\frac{\nu \bar{f}(\omega)}{f(\omega)}$ can be approximated by $\chi^2_{df=\nu}$, where

$$\nu = \frac{2}{\sum_{k=-m}^{m} W_m^2(k)}$$

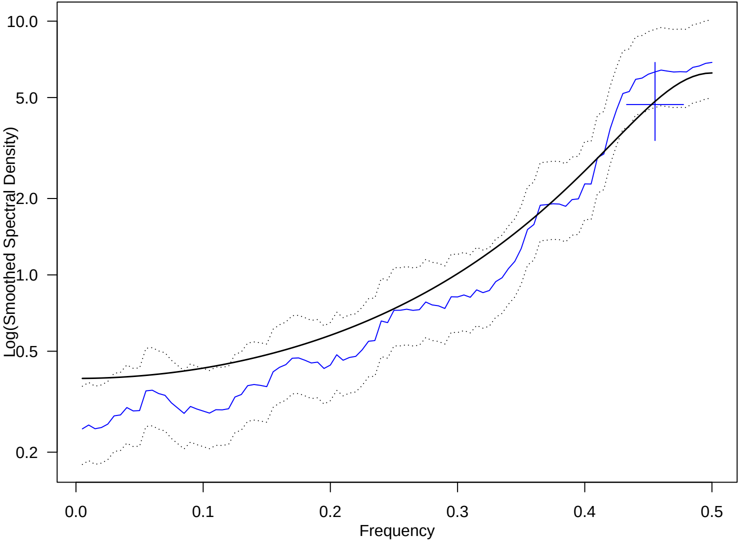
$\Rightarrow 100(1-\alpha)\%$ CI for $f(\omega)$

$$\frac{\nu \bar{f}(\omega)}{\chi^2_{df=\nu,\,1-\frac{\alpha}{2}}} < f(\omega) < \frac{\nu \bar{f}(\omega)}{\chi^2_{df=\nu,\,\frac{\alpha}{2}}}$$

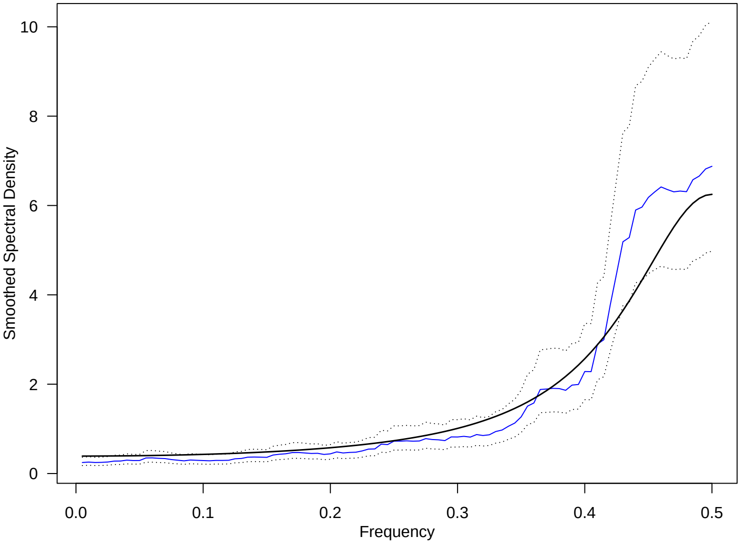
Taking logs we obtain an interval for the logged spectrum:

$$\log[\bar{f}(\omega)] + \log\left[\frac{\nu}{\chi^2_{\nu,1-\frac{\alpha}{2}}}\right] < \log[f(\omega)] < \log[\bar{f}(\omega)] + \log\left[\frac{\nu}{\chi^2_{\nu,\frac{\alpha}{2}}}\right]$$

Pointwise Confidence Intervals for $f(\omega)$: Log-Scale



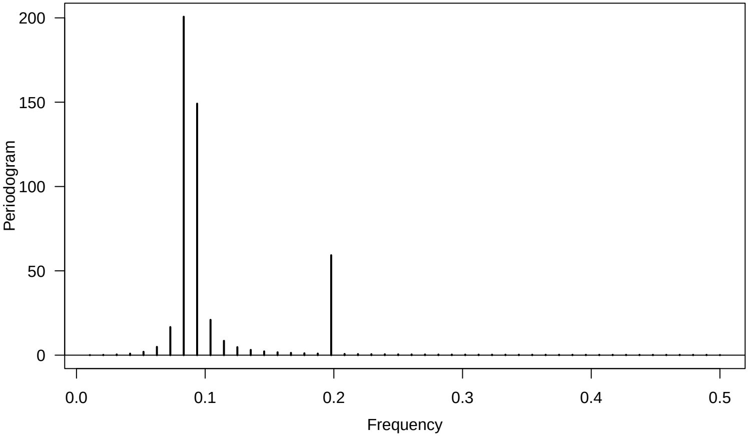
Pointwise Confidence Intervals for $f(\omega)$: Original Scale



Spectral Leakage

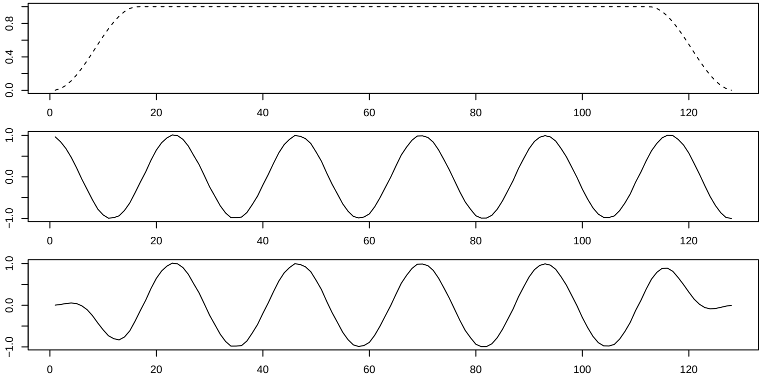
Much of the previous discussion has assumed that the frequencies of interest are the Fourier frequencies, i.e., $\omega_j = \frac{j}{n}$. What happens if that is not the case?

Example: $Y_t = 3\cos(2\pi(0088)t) + \sin(2\pi(\frac{19}{96})t), \quad t = 1, \cdots, 96$



Tapering

Tapering is one method used to improve the issue of spectral leakage, where power at non-Fourier frequencies leak into the nearby Fourier frequencies



Tapering (Cont'd)