

# Lecture 14

## Paired T-Tests; Analysis of Variance (ANOVA)

Text: Chapters 6, 8

STAT 8010 Statistical Methods I  
March 3, 2020

Whitney Huang  
Clemson University



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### Another Example

A simple random sample with sample size 37 is taken and are subjected to a treatment ( $\bar{X}_1 = 19.45, s_1 = 4.3$ ). A Simple random sample with sample size 31 is taken and given a placebo ( $\bar{X}_2 = 18.2, s_2 = 2.2$ ). At the 10% level can we say that the means are different between the two groups?



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## Paired T-Tests



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Paired T-Test: Motivating Example

Insurance handlers are concerned about the high estimates they are receiving for auto repairs from garage I compared to garage II. To verify their suspicions, each of 15 cars recently involved in an accident was taken to both garages for separate estimates of repair costs. The estimates from the two garages are given in the following table

| Garage I | Garage II | Garage I | Garage II | Garage I | Garage II |
|----------|-----------|----------|-----------|----------|-----------|
| 17.6     | 17.3      | 20.2     | 19.1      | 19.5     | 18.4      |
| 11.3     | 11.5      | 13.0     | 12.7      | 16.3     | 15.8      |
| 15.3     | 14.9      | 16.2     | 15.3      | 12.2     | 12.0      |
| 14.8     | 14.2      | 21.3     | 21.0      | 22.1     | 21.0      |
| 16.9     | 16.1      | 17.6     | 16.7      | 18.4     | 17.5      |

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Example Cont'd

Suppose we perform a two-sample test

Sample statistics:  
 $\bar{X}_1 = 16.85, \bar{X}_2 = 16.23, s_1 = 3.20, s_2 = 2.94$

- $H_0 : \mu_1 - \mu_2 = 0$  vs.  $H_a : \mu_1 - \mu_2 > 0$
- $t_{obs} = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{16.85 - 16.23}{\sqrt{\frac{3.2^2}{15} + \frac{2.94^2}{15}}} = \frac{0.62}{1.12} = 0.55$
- Critical value for rejection region:  $t_{0.05, df=27} = 1.70$
- Since  $t_{obs}$  is not in the rejection region. We fail to reject  $H_0$  at 0.05 level.

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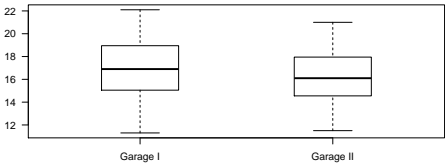
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Boxplots and R Output



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Welch Two Sample t-test

data: GarageI and GarageII
t = 0.54616, df = 27.797, p-value = 0.2947
alternative hypothesis: true difference in means is greater than 0
95 percent confidence interval:
-1.29749      Inf
sample estimates:
mean of x mean of y
16.84667 16.23333
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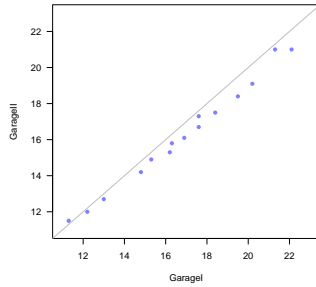
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For all but one of the 15 cars, the estimates from garage I were higher than that from garage II.

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## Analyzing Matched Pairs

- Matched pairs are **dependent samples** where each unit in the first sample is directly linked with a unit in the second sample
- This could occur in several situations, for example, before/after study, study on twins, pairing subjects based on similar characteristics
- We need different strategy for testing two dependent samples  $\Rightarrow$  **Paired T-Tests**

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## Paired T-Tests

- $H_0 : \mu_{diff} = 0$  vs.  $H_a : \mu_{diff} > 0$  (Upper-tailed);  $\mu_{diff} < 0$  (Lower-tailed);  $\mu_{diff} \neq 0$  (Two-tailed)
- Test statistic:  $t^* = \frac{\bar{x}_{diff} - 0}{\frac{s_{diff}}{\sqrt{n}}}$ . If  $\mu_{diff} = 0$ , then  $t^* \sim t_{df=n-1}$
- Use rejection region method or P-value method to make a decision

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Car Repair Example Revisited

Paired T-Tests;  
Analysis of  
Variance (ANOVA)

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Paired T-Test

| Garage I - Garage II | Garage I - Garage II | Garage I - Garage II |
|----------------------|----------------------|----------------------|
| 17.6 - 17.3 = 0.3    | 20.2 - 19.1 = 1.1    | 19.5 - 18.4 = 1.1    |
| 11.3 - 11.5 = -0.2   | 13.0 - 12.7 = 0.3    | 16.3 - 15.8 = 0.5    |
| 15.3 - 14.9 = 0.4    | 16.2 - 15.3 = 0.9    | 12.2 - 12.0 = 0.2    |
| 14.8 - 14.2 = 0.6    | 21.3 - 21.0 = 0.3    | 22.1 - 21.0 = 1.1    |
| 16.9 - 16.1 = 0.8    | 17.6 - 16.7 = 0.9    | 18.4 - 17.5 = 0.9    |

1

First, compute the difference in paired samples

2

Compute the sample mean and standard deviation for the differences

3

Then perform a one sample t-test

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Car Repair Example Cont'd

Paired T-Tests;  
Analysis of  
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Paired T-Test

$\bar{X}_{diff} = 0.61, s_{diff} = 0.39$

1

$H_0 : \mu_{diff} = 0$  vs.  $H_a : \mu_{diff} > 0$

2

$t_{obs} = \frac{0.61}{\frac{0.39}{\sqrt{15}}} = 6.03$

3

Critical value for rejection region:  $t_{0.05,df=14} = 1.76 \Rightarrow$  reject  $H_0$

4

We do have enough evidence that the true mean repair cost difference for the garage I and II is greater than 0

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Boxplot and R Output

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Paired T-Test

Garage I - Garage II

Paired t-test

data: GarageI and GarageII  
t = 6.0234, df = 14, p-value = 1.563e-05  
alternative hypothesis: true difference in means is greater than 0  
95 percent confidence interval:  
0.4339886 Inf  
sample estimates:  
mean of the differences  
0.6133333

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# Analysis of Variance (ANOVA)

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## Testing for a Difference in More Than Two Means

- In the last few lectures we have seen how to test a difference in two means, using **two sample t-test**
- **Question:** what if we want to test if there are differences in a set of **more than two means**?
- The statistical tool for doing this is called **analysis of variance (ANOVA)**

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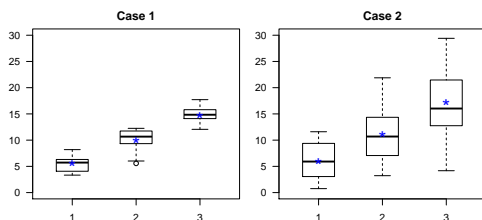
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## A Quick Quiz: To Detect Differences in Means

**Question:** Are group 1, 2, 3 for each case come from the same population?



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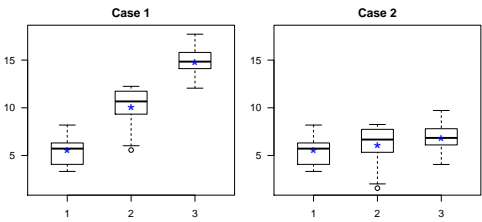
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Another Quiz: To Detect Differences in Means

**Question:** Are group 1, 2, 3 for each case come from the same population?



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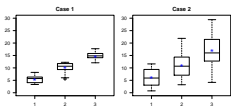
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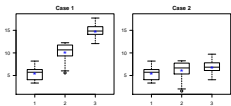
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Decomposing Variance to Test for a Difference in Means

- In the first quiz, the data within each group is not very spread out for Case 1, while in Case 2 it is



- In the second quiz, the group means are quite different for Case 1, while they are not in Case 2



- In ANOVA, we compare **between group variance** ("signal") to **within group variance** ("noise") to detect a difference in means

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Notation

$X_{ij} = \mu_j + \varepsilon_{ij}, \varepsilon_{ij} \stackrel{i.i.d.}{\sim} N(0, \sigma^2), i = 1, \dots, n_j, 1 \leq j \leq J$

- $J$ : number of groups
- $\mu_j, j = 1, \dots, J$ : population mean for  $j_{th}$  group
- $\bar{X}_j, j = 1, \dots, J$ : sample mean for  $j_{th}$  group
- $s_j^2, j = 1, \dots, J$ : sample variance for  $j_{th}$  group
- $N = \sum_{j=1}^J n_j$ : overall sample size
- $\bar{X} = \frac{\sum_{j=1}^J \sum_{i=1}^{n_j} X_{ij}}{N}$ : overall sample mean

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Partition of Sums of Squares

“Sums of squares” refers to sums of squared deviations from some mean. ANOVA decomposes the **total sum of squares** into **treatment sum of squares** and **error sum of squares**:

- **Total sum of square:**  $SSTo = \sum_{j=1}^J \sum_{i=1}^{n_j} (X_{ij} - \bar{X})^2$
- **Treatment sum of square:**  $SSTr = \sum_{j=1}^J n_j (\bar{X}_j - \bar{X})^2$
- **Error sum of square:**  $SSE = \sum_{j=1}^J (n_j - 1) s_j^2$

We can show that  $SSTo = SSTr + SSE$

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Mean squares

A mean square is a sum of squares divided by its associated degrees of freedom

- **Mean square of treatments:**  $MSTr = \frac{SSTr}{J-1}$
- **Mean square of error:**  $MSE = \frac{SSE}{N-J}$

Think of MSTr as the “signal”, and MSE as the “noise” when detecting a difference in means  $(\mu_1, \dots, \mu_J)$ . A nature test statistic is the signal-to-noise ratio i.e.,

$$F^* = \frac{MSTr}{MSE}$$

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ANOVA Table and F Test

| Source    | df      | SS     | MS                        | F statistic            |
|-----------|---------|--------|---------------------------|------------------------|
| Treatment | $J - 1$ | $SSTr$ | $MSTr = \frac{SSTr}{J-1}$ | $F = \frac{MSTr}{MSE}$ |
| Error     | $N - J$ | $SSE$  | $MSE = \frac{SSE}{N-J}$   |                        |
| Total     | $N - 1$ | $SSTo$ |                           |                        |

F-Test

- $H_0 : \mu_1 = \mu_2 = \dots = \mu_J$   
 $H_a : \text{At least one mean is different}$
- Test Statistic:  $F^* = \frac{MSTr}{MSE}$ . Under  $H_0$ ,  
 $F^* \sim F_{df_1=J-1, df_2=N-J}$
- **Assumptions:**
  - The distribution of each group is normal with equal variance (i.e.  $\sigma_1^2 = \sigma_2^2 = \dots = \sigma_J^2$ )
  - Responses for a given group are independent to each other

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Paired T-Test

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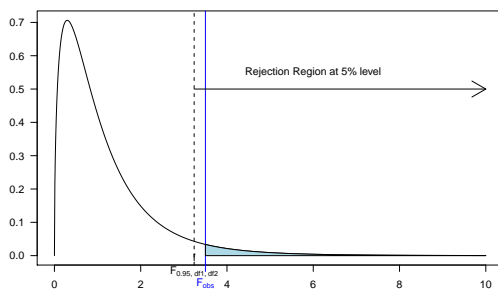
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## F Distribution and the Overall F-Test

Consider the observed F test statistic:  $F_{obs} = \frac{MSTr}{MSE}$

- Should be "near" 1 if the means are equal
- Should be "larger than" 1 if means are not equal

⇒ We use the null distribution of  $F^* \sim F_{df_1=J-1, df_2=N-J}$  to quantify if  $F_{obs}$  is large enough to reject  $H_0$



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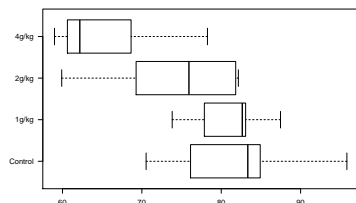
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## Example

A researcher who studies sleep is interested in the effects of ethanol on sleep time. She gets a sample of 20 rats and gives each an injection having a particular concentration of ethanol per body weight. There are 4 treatment groups, with 5 rats per treatment. She records Rapid eye movement (REM) sleep time for each rat over a 24-period. The results are plotted below:



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## Set Up Hypotheses and Compute Sums of Squares

- $H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4$  vs.  
 $H_a$  : At least one mean is different

- Sample statistics:

| Treatment | Control | 1g/kg | 2g/kg | 4g/kg |
|-----------|---------|-------|-------|-------|
| Mean      | 82.2    | 81.0  | 73.8  | 65.7  |
| Std       | 9.6     | 5.3   | 9.4   | 7.9   |

- Overall Mean  $\bar{X} = \frac{\sum_{j=1}^4 \sum_{i=1}^5 X_{ij}}{20} = 75.67$

- $SSTo = \sum_{j=1}^4 \sum_{i=1}^5 (X_{ij} - \bar{X})^2 = 1940.69$

- $SSTr = \sum_{j=1}^4 5 \times (\bar{X}_j - \bar{X})^2 = 861.13$

- $SSE = \sum_{j=1}^4 (5 - 1) \times s_j^2 = 1079.56$

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ANOVA Table and F-Test

| Source    | df            | SS      | MS                           | F statistic                   |
|-----------|---------------|---------|------------------------------|-------------------------------|
| Treatment | $4 - 1 = 3$   | 861.13  | $\frac{861.13}{3} = 287.04$  | $\frac{287.04}{67.47} = 4.25$ |
| Error     | $20 - 4 = 16$ | 1079.56 | $\frac{1079.56}{16} = 67.47$ |                               |
| Total     | 19            | 1940.69 |                              |                               |

Suppose we use  $\alpha = 0.05$

- **Rejection Region Method:**  
 $F_{obs} = 4.25 > F_{0.95, df_1=3, df_2=16} = 3.24$
- **P-value Method:**  
 $\mathbb{P}(F^* > F_{obs}) = \mathbb{P}(F^* > 4.25) = 0.022 < 0.05$

Reject  $H_0 \Rightarrow$  We do have enough evidence that not all of population means are equal at 5% level.

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Paired T-Test

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R Output

Analysis of Variance Table

Response: Response

|           |    |         |         |
|-----------|----|---------|---------|
|           | Df | Sum Sq  | Mean Sq |
| Treatment | 3  | 861.13  | 287.044 |
| Residuals | 16 | 1079.56 | 67.472  |

|           |         |           |
|-----------|---------|-----------|
|           | F value | Pr(>F)    |
| Treatment | 4.2542  | 0.02173 * |
| Residuals |         |           |

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Signif. codes:

|      |       |       |      |      |     |
|------|-------|-------|------|------|-----|
| 0    | ‘***’ | 0.001 | ‘**’ | 0.01 | ‘*’ |
| 0.05 | ‘.’   | 0.1   | ‘ ’  | 1    |     |

Paired T-Tests;  
Analysis of  
Variance (ANOVA)

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Paired T-Test

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