Lecture 8

Multiple Linear Regression II

Reading: Chapter 12

STAT 8020 Statistical Methods II

September 6, 2019

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Agenda

1. **Coefficient of Determination**
2. **General Linear Test**
3. **Multicollinearity**

Coefficient of Determination

- Coefficient of Determination R^2 describes proportional of the variance in the response variable that is predictable from the predictors

$$R^2 = \frac{\text{SSR}}{\text{SST}} = 1 - \frac{\text{SSE}}{\text{SSR}}, \quad 0 \leq R^2 \leq 1$$

- R^2 usually increases with the increasing p, the number of the predictors
  - Adjusted R^2, denoted by $R^2_{\text{adj}} = \frac{\text{SSR}/(n-p)}{\text{SST}/(n-1)}$ attempts to account for p

Example

Suppose the true relationship between response Y and predictors (X_1, X_2) is

$$Y = 5 + 2X_1 + \varepsilon,$$

where $\varepsilon \sim$ N(0, 1) and X_1 and X_2 are independent to each other. Let's fit the following two models to the “data"

Model 1: $Y = \beta_0 + \beta_1 X_1 + \varepsilon^1$

Model 2: $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \varepsilon^2$

Question: Which model will “win” in terms of R^2?

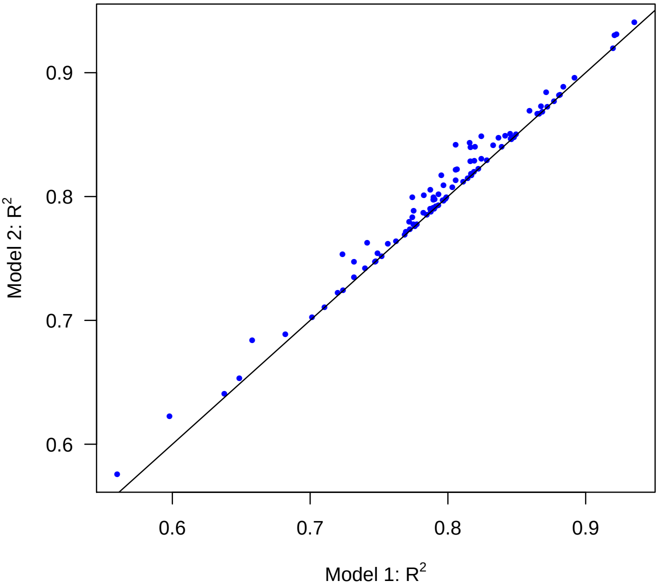
Model 1 Fit

```
> summary(fit1)

Call:
lm(formula = y ~ x1)

Residuals:
    Min      1Q  Median      3Q     Max
-1.6085 -0.5056 -0.2152  0.6932  2.0118

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   5.1720     0.1534   33.71  < 2e-16 ***
x1            1.8660     0.1589   11.74 2.47e-12 ***
---
Signif. codes:
0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.8393 on 28 degrees of freedom
Multiple R-squared:  0.8313,    Adjusted R-squared:  0.8253
F-statistic:   138 on 1 and 28 DF,  p-value: 2.467e-12
```

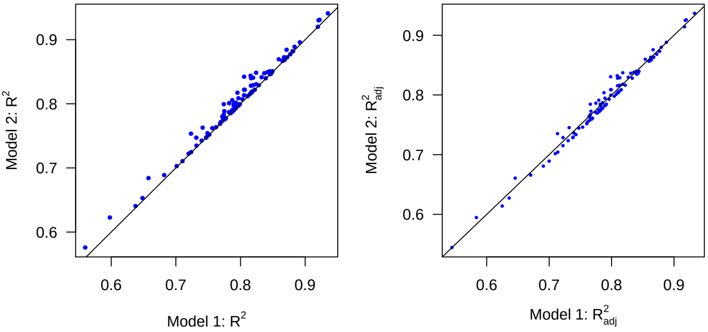
Model 2 Fit

```
> summary(fit2)

Call:
lm(formula = y ~ x1 + x2)

Residuals:
    Min      1Q  Median      3Q     Max 
-1.3926 -0.5775 -0.1383  0.5229  1.8385 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)   5.1792     0.1518  34.109  < 2e-16 ***
x1            1.8994     0.1593  11.923 2.88e-12 ***
x2           -0.2289     0.1797  -1.274    0.213    
---
Signif. codes:  
0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.8301 on 27 degrees of freedom
Multiple R-squared:  0.8408,	Adjusted R-squared:  0.8291 
F-statistic: 71.32 on 2 and 27 DF,  p-value: 1.677e-11
```

R^2: Model 1 vs. Model 2



R^2_{adj}: Model 1 vs. Model 2



General Linear Test

- Comparison of a “full model” and “reduced model” that involves a subset of full model predictors
- Consider a full model with k predictors and reduced model with l predictors ($l < k$)
- Test statistic: $F^* = \frac{\text{SSE(R)} - \text{SSE}(F)/(k-1)}{\text{SSE}(F)/(n-k-1)}$ $\Rightarrow$ Testing H_0 that the regression coefficients for the extra variables are all zero

Species Diversity on the Galapagos Islands Revisited: Reduce Model

```
> summary(gala_fit1)

Call:
lm(formula = Species ~ Elevation)

Residuals:
     Min       1Q   Median       3Q      Max 
-218.319  -30.721  -14.690    4.634  259.180 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) 11.33511   19.20529   0.590     0.56    
Elevation    0.20079    0.03465   5.795 3.18e-06 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 78.66 on 28 degrees of freedom
Multiple R-squared:  0.5454,	Adjusted R-squared:  0.5291 
F-statistic: 33.59 on 1 and 28 DF,  p-value: 3.177e-06
```

Species Diversity on the Galapagos Islands Revisited: Full Model

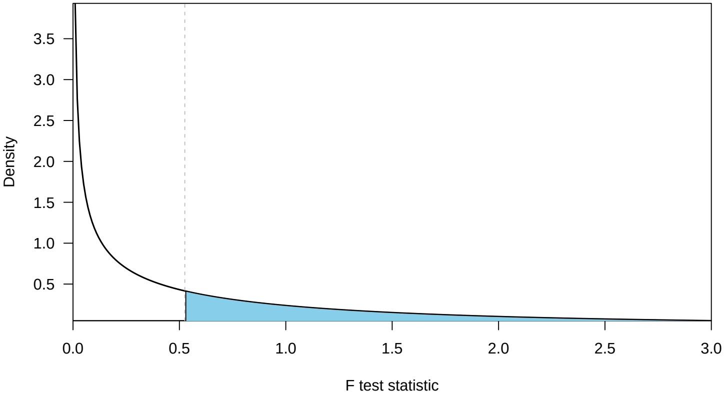
```
> summary(gala_fit2)

Call:
lm(formula = Species ~ Elevation + Area)

Residuals:
     Min       1Q   Median       3Q      Max
-192.619  -33.534  -19.199    7.541  261.514

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 17.10519   20.94211   0.817  0.42120
Elevation    0.17174    0.05317   3.230  0.00325 **
Area         0.01880    0.02594   0.725  0.47478
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 79.34 on 27 degrees of freedom
Multiple R-squared:  0.554,     Adjusted R-squared:  0.521
F-statistic: 16.77 on 2 and 27 DF,  p-value: 1.843e-05
```

Perform a General Linear Test

- $H_0 : \beta_{\texttt{Area}} = 0$ vs. $H_a : \beta_{\texttt{Area}} \neq 0$
- $F^* = \frac{(173254-169947)/(2-1)}{169947/(30-2-1)} = 0.5254$
- P-value: $\mathrm{P}[F > 0.5254] = 0.4748$, where $F \sim \mathrm{F}(1, 27)$

```
> anova(gala_fit1, gala_fit2)
Analysis of Variance Table

Model 1: Species ~ Elevation
Model 2: Species ~ Elevation + Area
  Res.Df    RSS Df Sum of Sq      F Pr(>F)
1     28 173254
2     27 169947  1      3307 0.5254 0.4748
```

P-value Calculation



P-value is the shaped area under the under the density curve

Multicollinearity

Another Simulated Example: Suppose the true relationship between response Y and predictors (X_1, X_2) is

$$Y = 4 + 0.8X_1 + 0.6X_2 + \varepsilon,$$

where $\varepsilon \sim \mathrm{N}(0, 1)$ and X_1 and X_2 are positively correlated with $\rho = 0.9$. Let's fit the following model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \varepsilon$$

```
Call:
lm(formula = Y ~ X1 + X2)

Residuals:
     Min       1Q   Median       3Q      Max 
-1.63912 -0.59978  0.01897  0.58691  1.74518 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)   4.0154     0.1646  24.390  < 2e-16 ***
X1           -0.1032     0.3426  -0.301    0.766    
X2            1.7471     0.3654   4.781 5.48e-05 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.8601 on 27 degrees of freedom
Multiple R-squared:  0.8166,	Adjusted R-squared:  0.803 
F-statistic: 60.12 on 2 and 27 DF,  p-value: 1.135e-10
```

Multicollinearity cont'd

- Numerical issue $\Rightarrow$ the matrix $X^T X$ is nearly singular
- Statistical issue
  - β's are not well estimated
  - β's may be meaningless
  - R^2 and predicted values are usually OK