Lecture 12

Inference for One Population Mean

Readings: IntroStat Chapter ; OpenIntro Chapter 7.1

STAT 8010 Statistical Methods I
June 1, 2023

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Agenda

1. Statistical Inferences
2. Point/Interval Estimation
3. Confidence Intervals

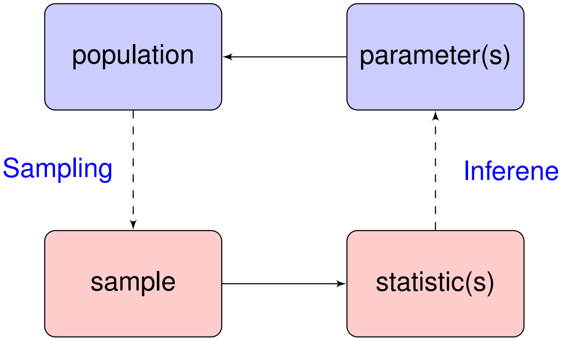
Statistical Inference

For the rest of the semester, we will focus on conducting statistical inferences for the following tasks:

- Estimating one population mean
- Comparing two population means
- Comparing more than two population means
- Estimating population proportions
- Estimating relationship between two quantitative variables

Statistical Inference Cont'd

- We use parameters to describe the population
  Example: population mean (μ_X); population variance (σ_X^2)



- We use statistics of a sample to infer the population
  Example: sample mean ($\bar{X}$); sample variance (s_X^2)

Estimating Population Mean μ

Goal: To estimate the population mean using a (representative) sample:

- The sample mean, $\bar{X}_n = \frac{\sum_i^n X_i}{n}$, is a reasonable point estimate of the population mean μ_X

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- Need to quantify the level of uncertainty of the point estimate $\Rightarrow$ Interval estimation
- Need to figure out the sampling distribution of $\bar{X}_n$ in order to construct interval estimates $\Rightarrow$ Central Limit Theorem (CLT)

Central Limit Theorem (CLT)

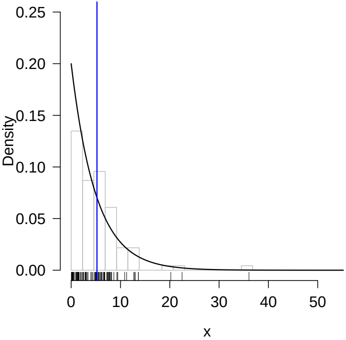
CLT

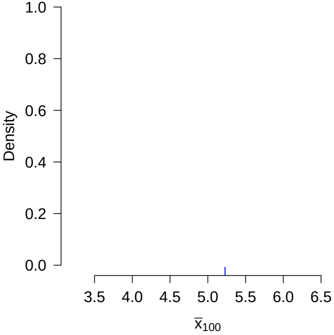
The **sampling distribution** of $\bar{X}_n$ will become approximately **normally distributed** as the **sample size (n) becomes "large"**, **regardless of the shape of the population distribution!**

Let $X_1, X_2, \cdots, X_n$ be a random sample from a population X with $\mu = \mathbb{E}[X]$ and $\sigma^2 = \text{Var}[X]$. Then $\bar{X}_n = \frac{\sum_{i=1}^n X_i}{n} \xrightarrow{d} \text{N}(\mu, \frac{\sigma^2}{n})$ as $n \to \infty$.

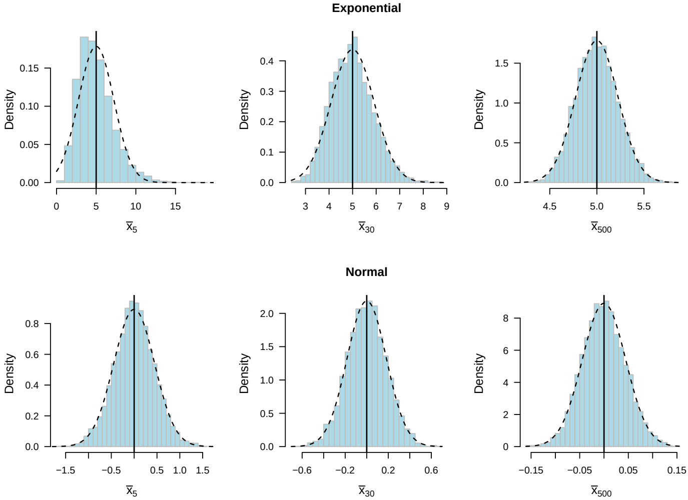
CLT In Action

1. Generate 100 (n) random numbers from an Exponential distribution (population distribution)
2. Compute the sample mean of these 100 random numbers
3. Repeat this process 120 times





CLT: Sample Size (n) and the Normal Approximation



Why CLT is important?

- CLT tells us the **distribution** of our estimator

$$\bar{X}_n \approx \mathrm{N}(\mu, \frac{\sigma^2}{n})$$

  - The distribution of $\bar{X}_n$ is center around the true mean μ
  - The variance of $\bar{X}_n$ is decrease with n
  - With normality approximation of the sampling distribution of $\bar{X}_n$, we can perform interval estimation about μ

- Applications: Confidence Interval, Hypothesis testing

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$$\left[\bar{X}_n - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}, \bar{X}_n + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right],$$

where $z_{\frac{\alpha}{2}}$ is the $1-\frac{\alpha}{2}$ percentile of $Z \sim \mathrm{N}(0,1)$

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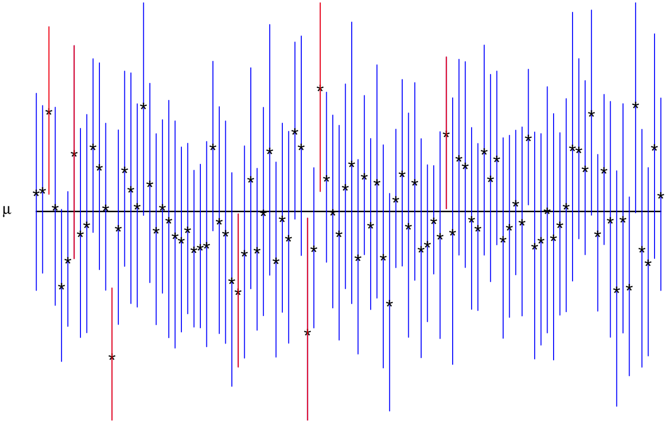
- $\frac{\sigma}{\sqrt{n}}$ is the standard error of $\bar{X}_n$, that is, the standard deviation of its sampling distribution

Making Sense of Confidence Intervals

For any $\alpha \in (0, 1)$:

$$
\begin{aligned}
&\mathrm{P}\left(\bar{X}_n - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X}_n + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right)\\
=&\mathrm{P}\left(-z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} \le \bar{X}_n - \mu \le z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right)\\
=&\mathrm{P}\left(-z_{\frac{\alpha}{2}} \le \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \le z_{\frac{\alpha}{2}}\right)\\
=&\mathrm{P}\left(-z_{\frac{\alpha}{2}} \le Z \le z_{\frac{\alpha}{2}}\right)\\
=&\Phi(z_{\frac{\alpha}{2}}) - \Phi(-z_{\frac{\alpha}{2}})\\
=&1 - \frac{\alpha}{2} - \frac{\alpha}{2} = 1 - \alpha
\end{aligned}
$$

Making Sense of Confidence Intervals Cont'd



Example: Average Height

We measure the heights of 40 randomly chosen men, and get a mean height of 5’9” (≈175cm). Suppose we know the standard deviation of men’s heights is 4” (≈10cm). Find the 95% confidence interval of the true mean height of ALL men.

Average Height Example Cont'd

1. Point estimate: $\bar{X}_{40} = \frac{\sum_{i=1}^{40} X_i}{40} = 69$ inches

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4. 95%CI: Need to find $z_{0.05/2} = 1.96$ from the Z-table
5. 95% CI for μ_X is:

$$[69 - 1.96 \times 0.63, 69 + 1.96 \times 0.63]$$
$$= [67.77, 70.23]$$

Properties of Confidence Intervals

- In contrast with the point estimate, $\bar{X}_n$, a $(1-\alpha)\%$ CI is an interval estimate, where the **length of CI** reflects our estimation uncertainty

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- The length of a CI depends on
  - Population Standard Deviation: σ
  - Confidence Level: $1-\alpha$
  - Sample Size: n

Sample Size Calculation

- We may want to estimate μ with a confidence interval with a predetermined margin of error (i.e. $z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$)
- For example, in estimating the true mean height of All men we may want our CI to be just 0.5 inches in width
- The question is then, "**how many observations do we need to take** so that we have the desired margin of error?"

Sample Size Calculation Cont'd

To compute the sample size needed to get a CI for μ with a specified margin of error, we use the formula below

$$n = \left(\frac{z_{\alpha/2} \times \sigma}{\text{Margin of error}} \right)^2$$

Exercise: Derive this formula using margin of error $= z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$

Average Height Example Revisited

Compute the sample size needed in order to estimate the true mean height of All men such that the 95% CI to be 0.5 inches in width

Average Height Example Revisited

Compute the sample size needed in order to estimate the true mean height of All men such that the 95% CI to be 0.5 inches in width

1. Length of CI: $2 \times z_{0.025} \frac{\sigma}{\sqrt{n}} = 2\times$ margin of error
2. Want to find n s.t. $z_{0.025} \frac{\sigma}{\sqrt{n}} = 0.25$
3. We have $n = \left(\frac{1.96\times 4}{0.25}\right)^2 = 983.4496$

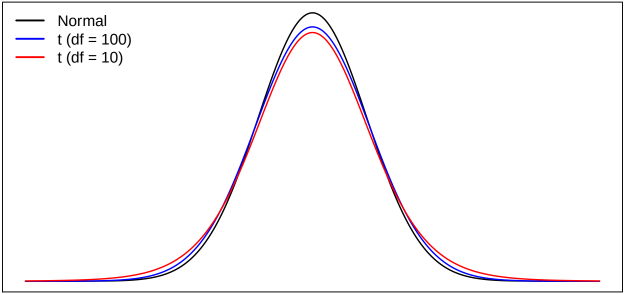
Therefore, the required sample size is 984

Confidence Intervals When σ Unknown

- In practice, it is unlikely that σ is available to us
- One reasonable option is to replace σ with s, the sample standard deviation
- We need to account for this added uncertainty with a (slightly) different sampling distribution that has fatter tails

$\Rightarrow$ Student t Distribution (William Gosset, 1908)

Student t Distribution



- Recall the standardize sampling distribution $\frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \sim \mathrm{N}(0, 1)$
- Similarly , the studentized sampling distribution $\frac{\bar{X}_n - \mu}{\frac{s}{\sqrt{n}}} \sim \mathrm{t}_{df=n-1}$

Confidence Intervals (CIs) for μ When σ is Unknown

- $(1-\alpha)\times 100\%$ CI for μ:

$$\left[\bar{X}_n - t_{\frac{\alpha}{2},n-1}\frac{s}{\sqrt{n}}, \bar{X}_n + t_{\frac{\alpha}{2},n-1}\frac{s}{\sqrt{n}}\right],$$

  where $t_{\frac{\alpha}{2},n-1}$ is the $1-\frac{\alpha}{2}$ percentile of a student t distribution with the degrees of freedom $= n-1$

- $\frac{s}{\sqrt{n}}$ is an estimate of the standard error of $\bar{X}_n$

Average Height Example Revisited

We measure the heights of 40 randomly chosen men, and get a mean height of 5'9" ($\approx$175cm), and a standard deviation of 4.5" ($\approx$11.4cm). Find the 95% confidence interval of the true mean height of ALL men.

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5. 95% CI for μ_X is:

$$[69 - 2.02 \times 0.71, 69 + 2.02 \times 0.71]$$
$$= [67.57, 70.43]$$

Summary

In this lecture, we learned

- Statistical Inferences
- Point and interval estimation
- Confidence Intervals