# Lecture 3
## Stationary processes
Readings:

*MATH 8090 Time Series Analysis*
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# Agenda

# Review: The Additive Decomposition

- The additive model for a time series $\{Y_t\}$ is

$$Y_t = \mu_t + s_t + \eta_t,$$

where

  - $\mu_t$ is the trend component
  - $s_t$ is the seasonal component
  - $\eta_t$ is the random (noise) component with $\mathbb{E}(\eta_t) = 0$

- Standard procedure:

  (1) Estimate/remove the trend and seasonal components

  (2) Analyze the remainder, the residuals $\hat{\eta}_t = y_t - \hat{\mu}_t - \hat{s}_t$

- We will focus on (2) for the next few weeks

# Time Series Models

- A time series model is a specification of the probabilistic distribution of a sequence of random variables (RVs) $\eta_t$

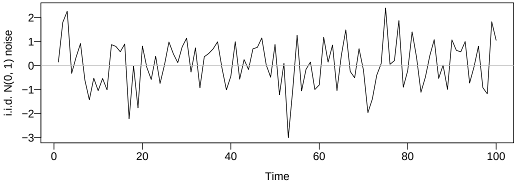
  (The observed time series is a realization of such a sequence of random variables)

- The simplest time series is i.i.d. (*independent and identically distributed*) noise
  - $\{\eta_t\}$ is a sequence of independent and identically distributed zero-mean (i.e., $\mathbb{E}(\eta_t) = 0, \forall t$) random variables $\Rightarrow$ no temporal dependence
  - It is of little value of using i.i.d. noise model to conduct forecast as there is no information from the past observations
  - **But**, we will use i.i.d. model as a building block to develop time series models that can accommodate time dependence

# Example Realizations of i.i.d. Noise

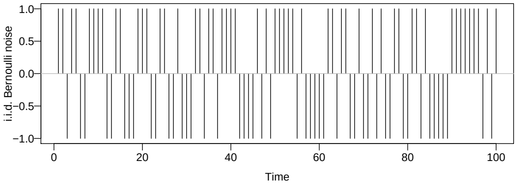
- Gaussian (normal) i.i.d. noise with mean $0$ and variance $\sigma^2 > 0$

$$f(\eta_t) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\eta_t^2}{2\sigma^2}\right)$$




- Bernoulli i.i.d. noise with "success" probability

$$\mathbb{P}(\eta_t = 1) = p = 1 - \mathbb{P}(\eta_t = -1)$$



## Means and Autocovaraínces

A time series model could also be a specification of the means and autocovariances of the RVs

- The mean function of $\{\eta_t\}$ is

$$\mu_t = \mathbb{E}(\eta_t).$$

- $\mu_t$ is the population mean at time $t$, which can be computed as:

$$\mu_t = \begin{cases} \int_{-\infty}^{\infty} \eta_t f(\eta_t)\, d\eta_t & \text{when } \eta_t \text{ is a continuous RV;} \\ \sum_{-\infty}^{\infty} \eta_t p(\eta_t), & \text{when } \eta_t \text{ is a discrete RV,} \end{cases}$$

where $f(\cdot)$ and $p(\cdot)$ are the probability density function and probability mass function of $\eta_t$, respectively

# Examples of Mean Functions





- **Example 1**: What is the mean function for $\{\eta_t\}$, an i.i.d. $\mathrm{N}(0, \sigma^2)$ process?

- **Example 2**: For each time point, let $Y_t = \beta_0 + \beta_1 t + \eta_t$ with $\beta_0$ and $\beta_1$ some constants and $\eta_t$ is defined above. What is $\mu_Y(t)$?

## Review: The Covariance Between Two RVs

- The covariance between the RVs $X$ and $Y$ is

$$\begin{aligned}\mathbb{C}\mathrm{ov}(X,Y) &= \mathbb{E}\{(X-\mu_X)(Y-\mu_Y)\}\\ &= \mathbb{E}(XY)-\mu_X\mu_Y.\end{aligned}$$

  It is a measure of linear dependence between the two RVs. When $X = Y$ we have

$$\mathbb{C}\mathrm{ov}(X,X) = \mathbb{V}\mathrm{ar}(X).$$

- For constants $a, b, c$, and RVs $X$, $Y$, $Z$:

$$\begin{aligned}\mathbb{C}\mathrm{ov}(aX+bY+c,Z) &= \mathbb{C}\mathrm{ov}(aX,Z)+\mathbb{C}\mathrm{ov}(bY,Z)\\ &= a\mathbb{C}\mathrm{ov}(X,Z)+b\mathbb{C}\mathrm{ov}(Y,Z)\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}\mathbb{V}\mathrm{ar}(X+Y) &= \mathbb{C}\mathrm{ov}(X,X)+\mathbb{C}\mathrm{ov}(X,Y)+\mathbb{C}\mathrm{ov}(Y,X)+\mathbb{C}\mathrm{ov}(Y,Y)\\ &= \mathbb{V}\mathrm{ar}(X)+\mathbb{V}\mathrm{ar}(Y)+2\mathbb{C}\mathrm{ov}(X,Y)\end{aligned}$$

## Autocovariance Function

- The autocovariance function of $\{\eta_t\}$ is

$$\gamma(s,t) = \mathbb{C}\mathrm{ov}(\eta_s, \eta_t) = \mathbb{E}[(\eta_s - \mu_s)(\eta_t - \mu_t)]$$

  It measures the strength of linear dependence between two RVs $\eta_s$ and $\eta_t$

- **Properties**:
  - $\gamma(s,t) = \gamma(t,s)$ for each $s$ and $t$
  - When $s = t$ we have

$$\gamma(t,t) = \mathbb{C}\mathrm{ov}(\eta_t, \eta_t) = \mathbb{C}\mathrm{ov}(\eta_t) = \sigma_t^2$$

    the value of the variance function at time $t$
  - $\gamma(s,t)$ is a non-negative definite function (will come back to this later)

## Autocorrelation Function

- The autocorrelation function of $\{\eta_t\}$ is

$$\rho(s,t) = \mathbb{Corr}(\eta_s, \eta_t) = \frac{\gamma(s,t)}{\sqrt{\gamma(s,s)\gamma(t,t)}}$$

  It measures the “scale invariant” linear association between $\eta_s$ and $\eta_t$

- **Properties**:
  - $-1 \le \rho(s,t) \le 1$ for each $s$ and $t$
  - $\rho(s,t) = \rho(t,s)$ for each $s$ and $t$
  - $\rho(t,t) = 1$ for each $t$
  - $\rho(\cdot,\cdot)$ is a non-negative definite function

# Stationarity





- We typically need "replicates" to estimate population quantities. For example, we use

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

to be the estimate of $\mu_X$, the population mean of the **single** RV, $X$

- However, in time series analysis, we have $n = 1$ (i.e., no replication) because we only have one realized value at each time point
- Stationarity means that some characteristic of $\{\eta_t\}$ does not depend on the time point, $t$, only on the "time lag" between time points **so that we can create "replicates"**

Next, we will talk about strict stationarity and weak stationarity

# Strictly Stationary Processes

- A time series, $\{\eta_t\}$, is strictly stationary if

$$[\eta_1, \eta_2, \cdots \eta_T] \stackrel{d}{=} [\eta_{1+h}, \eta_{2+h}, \cdots \eta_{T+h}],$$

for all integers $h$ and $T \geq 1 \Rightarrow$ the joint distribution are unaffected by time shifts

- Under such the strict stationarity
  - $\{\eta_t\}$ is identically distributed but not (necessarily) independent
  - When $\mu_t$ is finite, $\mu_t = \mu$ is independent of time $t$
  - When the variance function exists,

$$\gamma(s, t) = \gamma(s + h, t + h),$$

for any $s$, $t$, and $h$

# Weakly Stationary Processes





- $\{\eta_t\}$ is weakly stationary if
  - $\mathbb{E}(\eta_t) = \mu_t = \mu$
  - $\mathbb{Cov}(\eta_t, \eta_{t+h}) = \gamma(t, t+h) = \gamma(h)$, finite constant that can depend on $h$ but not on $t$
- Other names for this type of stationarity include second-order, covariance, wide senese. The quantity $h$ is called the lag
- Weak and strict stationarity
  - A strictly stationary process $\{\eta_t\}$ is also weakly stationary as long as $\mu$ is finite
  - Weak stationarity does not imply strict stationarity!

# Autocovariance Function of Stationary Processes

The autocovariance function (ACVF) of a stationary process $\{\eta_t\}$ is defined to be

$$\begin{aligned}\gamma(h) &= \mathbb{C}\mathrm{ov}(\eta_t, \eta_{t+h}) \\ &= \mathbb{E}[(\eta_t - \mu)(\eta_{t+h} - \mu)],\end{aligned}$$

which measures the lag-h time dependence

**Properties of the ACVF**:

- $\gamma(0) = \mathbb{V}\mathrm{ar}(\eta_t)$
- $\gamma(-h) = \gamma(h)$ for each $h$
- $\gamma(s-t)$ as a function of $(s-t)$ is non-negative definite

# Autocorrelation Function of Stationary Processes



The autocorrelation function (ACF) of a stationary process $\{\eta_t\}$ is defined to be

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)}$$

which measures the "scale invariant" lag-h time dependence

**Properties of the ACF**:

- $-1 \le \rho(h) \le 1$ and $\rho(0) = 1$ for each h
- $\rho(-h) = \rho(h)$ for each $h$
- $\rho(s-t)$ as a function of $(s-t)$ is non-negative definite

## The White Noise Process

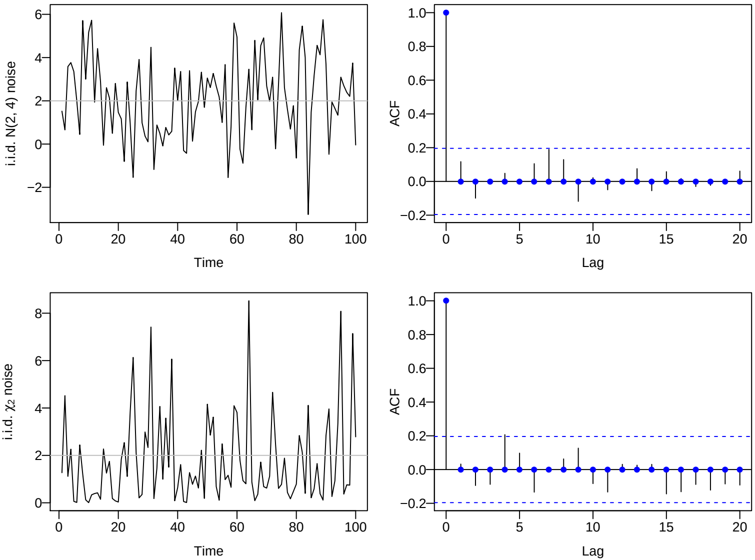
Let's assume $\mathbb{E}(\eta_t) = \mu$ and $\mathbb{Var}(\eta_t) = \sigma^2 < \infty$. $\{\eta_t\}$ is a white noise or $\mathrm{WN}(\mu, \sigma^2)$ process if

$$\gamma(h) = 0,$$

for $h \neq 0$

- $\{\eta_t\}$ is stationary
- However, distributions of $\eta_t$ and $\eta_{t+1}$ can be different!
- All i.i.d. noise with finite variance ($\sigma^2 < 0$) is white noise but the converse need not be true

# Examples Realizations of White Noise Processes



## The Moving Average Process of First Order (MA(1))





Let $\{Z_t\}$ be a $\mathrm{WN}(0, \sigma^2)$ process and $\theta$ be some constant $\in \mathbb{R}$. For each integer $t$, let

$$\eta_t = Z_t + \theta Z_{t-1}.$$

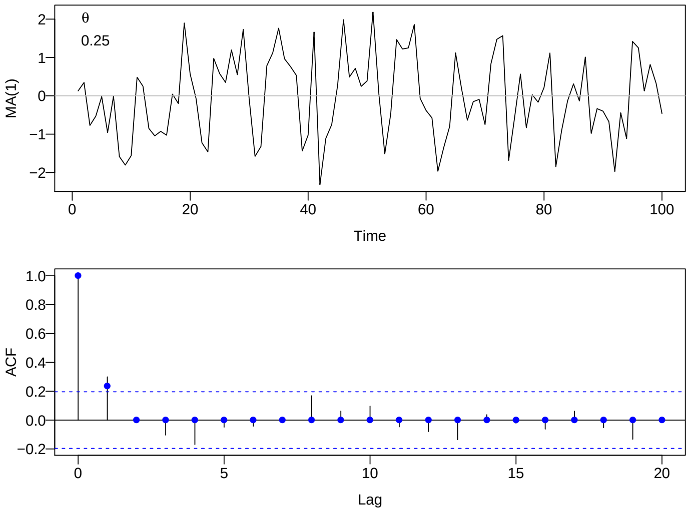
- The sequences of RVs $\{\eta_t\}$ is called the moving average process of order 1 or MA(1) process
- One can show that the MA(1) process $\{\eta_t\}$ is stationary

# First-Order Moving Average Process: Mean Function

# First-Order Moving Average Process: Covariance Function

# First-Order Moving Average Process: ACVF & ACF

# Examples Realizations of MA(1) Processes



# First-order autoregressive process, AR(1)





Let $\{Z_t\}$ be a $\mathrm{WN}(0, \sigma^2)$ process, and $-1 < \phi < 1$ be a constant. Let's assume $\{\eta_t\}$ is a stationary process with

$$\eta_t = \phi\eta_{t-1} + Z_t,$$

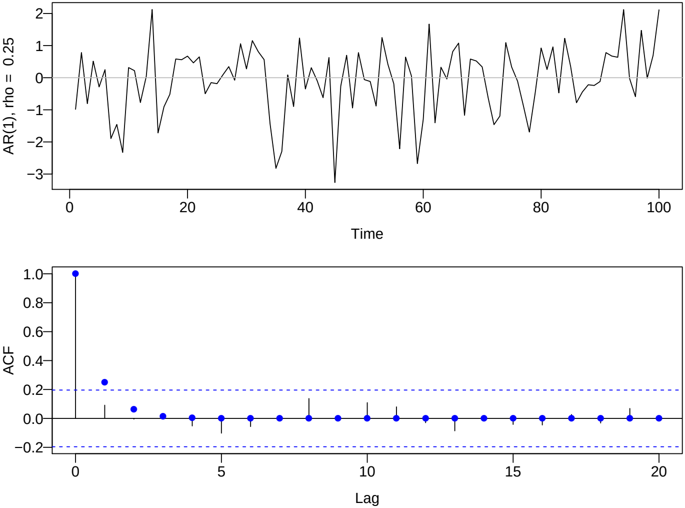
for each integer $t$, where $\eta_s$ and $Z_t$ are uncorrelated for each $s < t \Rightarrow$ future noise is uncorrelated with the current time point)

We will see later there is only one unique solution to this equation. Such a sequence $\{\eta_t\}$ of RVs is called an AR(1) process

# Properties of the AR(1) process

# Properties of the AR(1) process Cont'd

# Examples Realizations of AR(1) Processes



# The Random Walk Process

Let $\{Z_t\}$ be a $\mathrm{WN}(0, \sigma^2)$ process and for $t \geq 1$ definite

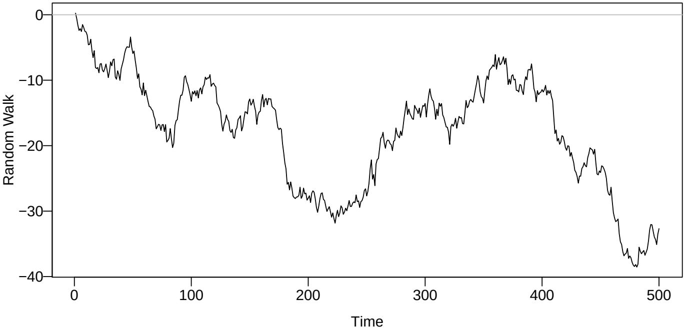
$$\eta_t = Z_1 + Z_2 + \cdots + Z_t = \sum_{s=1}^{t} Z_s.$$

- The sequence of RVs $\{\eta_t\}$ is called a random walk process
- **Special case**: If we have $\{Z_t\}$ such that for each $t$

$$\mathbb{P}(Z_t = z) = \begin{cases} \frac{1}{2}, & z = 1; \\ \frac{1}{2}, & z = -1, \end{cases}$$

  then $\{\eta_t\}$ is a simple symmetric random walk
- The random walk process is not stationary!

# Example Realizations of Random Walk Processes



## Gaussian Processes





$\{\eta_t\}$ is a Gaussian process (GP) if the joint distribution of any collection of the RVs has a multivariate normal (aka Gaussian) distribution

- The distribution of a GP is fully characterized by $\mu(\cdot)$, the mean function, and $\gamma(\cdot,\cdot)$, the autocovariance function. The joint probability density function of $\boldsymbol{\eta} = (\eta_1, \eta_2, \cdots, \eta_T)^T$ is

$$f(\boldsymbol{\eta}) = \frac{1}{(2\pi)^{\frac{n}{2}}|\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\boldsymbol{\eta}-\boldsymbol{\mu})^T\Sigma^{-1}(\boldsymbol{\eta}-\boldsymbol{\mu})\right),$$

where $\boldsymbol{\mu} = (\mu_1, \mu_2, \cdots, \mu_T)^T$ and the $(i,j)$ element of the covariance matrix $\Sigma$ is $\gamma(i,j)$

- If a GP $\{\eta_t\}$ is weakly stationary then the process is also strictly stationary

## Estimating the Mean of Stationary Processes

Let $\{\eta_t\}$ be stationary with mean $\mu$ and ACVF $\gamma(s,t)$

- A natural estimator of $\mu$ is the sample mean

$$\bar{\eta} = \frac{1}{T}\sum_{t=1}^{T}\eta_t.$$

$\bar{\eta}$ is an unbiased estimator of $\mu$, i.e.

- Since $\{\eta_t\}$ is stationary, we have

$$\begin{aligned}\mathbb{Var}(\bar{\eta}) &= \frac{1}{T^2}\mathbb{Var}\left(\sum_{i=1}^{T}\eta_t\right)\\ &= \frac{1}{T^2}\sum_{s=1}^{T}\sum_{t=1}^{T}\mathbb{Cov}(\eta_s,\eta_t)\\ &= \frac{1}{T^2}\sum_{s=1}^{T}\sum_{t=1}^{T}\gamma(s-t)\end{aligned}$$

- **Exercise**: Show

$$\mathbb{Var}(\bar{\eta}) = \frac{1}{T}\sum_{h=-(T-1)}^{T-1}\left(1-\frac{|h|}{T}\right)\gamma(h)$$

## AR(1) Example





Suppose $\{\eta_1, \eta_2, \eta_3\}$ is an AR(1) process with $|\phi| < 1$ and innovation variance $\sigma$. Show that the variance of $\bar{\eta}$ is $\frac{\sigma^2}{9}(3 + 4\phi + 2\phi^2)$

Solution:

# The Sampling Distribution of $\bar{\eta}$

Let

$$v_T = \sum_{h=-(T-1)}^{(T-1)} \left(1 - \frac{|h|}{T}\right) \gamma(h)$$

- If $\{\eta_t\}$ is Gaussian we have

$$\sqrt{T}(\bar{\eta} - \mu) \sim \mathrm{N}(0, v_T)$$

- The result above is approximate for many non-Gaussian time series

- In practice we also need to estimate $\gamma(h)$ from the data

## Confidence Intervals for $\mu$





- If $\gamma(h) \to 0$ as $h \to \infty$ then

$$v = \lim_{T\to\infty} v_n = \sum_{h=-\infty}^{\infty} \gamma(h) \text{ exists.}$$

- Further, if $\{\eta_t\}$ is Gaussian and

$$\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty,$$

then an approximate large-sample 95% CI for $\mu$ is given by

$$\left[\bar{\eta} - 1.96\sqrt{\frac{v}{T}}, \bar{\eta} + 1.96\sqrt{\frac{v}{T}}\right]$$

## Strategies for Estimating $v$

1. Parametric:

- Assume a parametric model $\gamma_{\boldsymbol{\theta}}(\cdot)$, and calculate $v = \sum_{h=-(T-1)}^{T-1}\left(1 - \frac{|h|}{T}\right)\gamma_{\hat{\boldsymbol{\theta}}}(h)$ based on the ACVF for that model
- The standard error, $v$, will depend on the parameters $\boldsymbol{\theta}$ of the parametric model

- Nonparametric:

  - Estimate $v$ by

$$\hat{v} = \sum\left(1 - \frac{|h|}{T}\right)\hat{\gamma}(h),$$

where $\hat{\gamma}(\cdot)$ is an nonparametric estimate of ACVF

## Examples of Parametric Forms for $v$

- i.i.d. Gaussian Noise: $v = \gamma(0) = \sigma \Rightarrow$ CI reduces to the classical case:

$$\left[\bar{\eta} - 1.96\sqrt{\frac{\sigma}{T}}, \bar{\eta} + 1.96\sqrt{\frac{\sigma}{T}}\right]$$

- MA(1) process: We have

$$\begin{aligned} v = \sum_{h=-\infty}^{\infty} \gamma(h) &= \gamma(-1) + \gamma(0) + \gamma(1) \\ &= \gamma(0) + 2\gamma(1) \\ &= \sigma^2(1 + \theta^2 + 2\theta) = \sigma^2(1 + \theta)^2 \end{aligned}$$

- **Exercise**: Show for an AR(1) process we have

$$v = \frac{\sigma^2}{(1 - \phi)^2}$$

# An Estimate of $\gamma(\cdot)$

**Goal**: Want to estimate

$$\gamma(h) = \mathbb{C}\mathrm{ov}(\eta_t, \eta_{t+h}) = \mathbb{E}\left[(\eta_t - \mu)(\eta_{t+h} - \mu)\right]$$

using data $\{\eta_t\}_{t=1}^T$

- For $|h| < T$, consider $\hat{\gamma}(h) = \frac{1}{T}\sum_{i=1}^{n-|h|}(\eta_t - \bar{\eta})(\eta_{t+|h|} - \bar{\eta})$. We call $\hat{\gamma}(h)$ the sample ACVF
- The sample ACVF is a biased estimator of $\gamma(h)$, but, it is used as the **standard** estimate of $\gamma(h)$
- $\hat{\gamma}(h)$ are even and non-negative definite

# The Sample Autocorrelation Function

- The sample autocorrelation function (ACF) is defined for $|h| < T$ by
$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}.$$

- **Rule of thumb**: Box and Jenkins (1976) recommend using $\hat{\rho}(h)$ and $\hat{\gamma}(h)$ only for $\frac{|h|}{T} \leq \frac{1}{4}$ and $T \geq 50$

- This is because estimates $\hat{\rho}(h)$ and $\hat{\gamma}(h)$ are unstable for large $|h|$ as there will be no enough data points going into the estimator

# Calculating the Sample ACF in R

# Asymptotic Distribution of the Sample ACF Bartlett, 1946

Let $\{\eta_t\}$ be a stationary process we suppose that the ACF

$$\boldsymbol{\rho} = (\rho(1), \rho(2), \cdots, \rho(k))^T$$

is estimated by

$$\hat{\boldsymbol{\rho}} = (\hat{\rho}(1), \hat{\rho}(2), \cdots, \hat{\rho}(k))^T$$

- For large $T$

$$\hat{\boldsymbol{\rho}} \dot{\sim} \mathrm{N}_k(\boldsymbol{\rho}, \frac{1}{T} W),$$

where $\mathrm{N}_k$ is the K-variate normal distribution and $W$ is an $k \times k$ covariance matrix with $(i, j)$ element defined by

$$w_{ij} = \sum_{k=1}^{\infty} a_{ik} a_{jk},$$

where $a_{ik} = \rho(k+i) + \rho(k-i) - 2\rho(k)\rho(i)$

# Using the ACF as a Test for i.i.d. Noise



When $\{\eta_t\}$ is an i.i.d. process with finite variance, Bartlett's result simplifies for each $h \neq 0$

$$\hat{\rho}(h) \dot{\sim} \mathrm{N}(0, \frac{1}{h}).$$

This suggests a diagnostic for i.i.d. noise:

1. Plot the lag $h$ versus the sample ACF $\hat{\rho}(h)$
2. Draw two horizontal lines at $\pm\frac{1.96}{T}$ (blue dashed lines in R)
3. About 95% of the $\{\hat{\rho}(h) : h = 1, 2, 3, \cdots\}$ should be within the lines if we have i.i.d. noise

## The Portmanteau Test [Box and Pierce, 1970]for i.i.d. Noise

Suppose we wish to test:

$H_0 : \{\eta_1, \eta_2, \cdots, \eta_T\}$ is an i.i.d. noise sequence

$H_1 : H_0$ is false

- Under $H_0$,

$$\hat{\rho}(h) \dot{\sim} \mathrm{N}(0, \frac{1}{T}) \overset{d}{=} \frac{1}{\sqrt{n}} \mathrm{N}(0,1)$$

- Hence

$$Q = T \sum_{i=1}^{k} \hat{\rho}^2(h) \dot{\sim} \chi^2_{df=k}$$

- We reject $H_0$ if $Q > \chi^2_k(1-\alpha)$, the $1-\alpha$ quatile of the chi-squared distribution with $k$ degrees of freedom

# Ljung-Box Test [Ljung and Box, 1978]

Ljung and Box [1978] showed that

$$Q_{LB} = T(T-2)\sum_{h=1}^{k}\frac{\hat{\rho}^2(h)}{n-h} \;\dot{\sim}\; \chi^2_k.$$

The Ljung-Box test can be more powerful than the Portmanteau Test

Both the Portmanteau Test (aka Box-Pierce test) and Ljung-Box test can be carried out in `R` using the function `Box.text`

# Examples in R