DSA 8070 R Session 2: Matrix Algebra

Whitney Huang, Clemson University

Contents

Motor Trend Car Road Tests Data

```
data(mtcars)
vars <- which(names(mtcars) %in% c("mpg", "disp", "hp", "drat", "wt"))
cars <- mtcars[, vars]
```

Mean Vector and Covariance Matrix

Mean vector: $\frac{1}{n}\mathbf{X}^T\mathbf{1}$

Covariance Matrix: $\frac{1}{n-1}\mathbf{X}^T(\mathbf{I}-\frac{1}{n}\mathbf{1}\mathbf{1}^T)\mathbf{X} = \frac{1}{n-1}\left(\mathbf{X}-\mathbf{1}\,\bar{\mathbf{x}}^T\right)^T\left(\mathbf{X}-\mathbf{1}\,\bar{\mathbf{x}}^T\right)$

```
(mean <- apply(cars, 2, mean))
```

```
##        mpg       disp         hp       drat         wt
##  20.090625 230.721875 146.687500   3.596563   3.217250
```

```
n <- dim(cars)[1]; p <- dim(cars)[2]
X <- as.matrix(cars)
ones <- rep(1, n)
(meanCal <- (1 / n) * t(X) %*% ones)
```

```
##           [,1]
## mpg   20.090625
## disp 230.721875
## hp   146.687500
## drat   3.596563
## wt     3.217250
```

```r
(S <- cov(cars))
```

```
##             mpg        disp         hp        drat         wt
## mpg    36.324103  -633.09721 -320.73206   2.1950635  -5.1166847
## disp -633.097208 15360.79983 6721.15867 -47.0640192 107.6842040
## hp   -320.732056  6721.15867 4700.86694 -16.4511089  44.1926613
## drat    2.195064   -47.06402  -16.45111   0.2858814  -0.3727207
## wt     -5.116685   107.68420   44.19266  -0.3727207   0.9573790
```

```r
(Scal <- (1 / (n - 1)) * t(X) %*% (diag(n) - (1 / n) * ones %*% t(ones)) %*% X)
```

```
##             mpg        disp         hp        drat         wt
## mpg    36.324103  -633.09721 -320.73206   2.1950635  -5.1166847
## disp -633.097208 15360.79983 6721.15867 -47.0640192 107.6842040
## hp   -320.732056  6721.15867 4700.86694 -16.4511089  44.1926613
## drat    2.195064   -47.06402  -16.45111   0.2858814  -0.3727207
## wt     -5.116685   107.68420   44.19266  -0.3727207   0.9573790
```

```r
(Scal <- (1 / (n - 1)) * t(X - ones %*% t(meanCal))  %*% (X - ones %*% t(meanCal)))
```

```
##             mpg        disp         hp        drat         wt
## mpg    36.324103  -633.09721 -320.73206   2.1950635  -5.1166847
## disp -633.097208 15360.79983 6721.15867 -47.0640192 107.6842040
## hp   -320.732056  6721.15867 4700.86694 -16.4511089  44.1926613
## drat    2.195064   -47.06402  -16.45111   0.2858814  -0.3727207
## wt     -5.116685   107.68420   44.19266  -0.3727207   0.9573790
```

Inverse Matrix

```r
S_inv <- solve(S)
(S_inv %*% S)
```

```
##                mpg          disp            hp          drat            wt
## mpg   1.000000e+00 -6.712237e-14 -3.425998e-14  3.039963e-16 -5.763283e-16
## disp  8.465125e-17  1.000000e+00 -1.228264e-15  6.314000e-18 -1.950304e-17
## hp    3.496914e-17  4.144814e-17  1.000000e+00  1.061775e-18 -3.317849e-18
## drat -7.709220e-15  1.013106e-13  5.672211e-14  1.000000e+00  5.891006e-16
## wt   -4.844043e-15  1.174255e-13  4.141715e-14 -5.652220e-16  1.000000e+00
```

Orthogonal Matrix Example

```r
Q <- matrix(c(2, 1, 2, -2, 2, 1, 1, 2, -2), ncol = 3) / 3
#check
(Q %*% t(Q))
```

```
##              [,1]         [,2]         [,3]
## [1,] 1.000000e+00 1.541976e-17 1.233581e-17
## [2,] 1.541976e-17 1.000000e+00 2.467162e-17
## [3,] 1.233581e-17 2.467162e-17 1.000000e+00
```

Eigenvalues and Eigenvectors

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$$

```r
eigen <- eigen(S)

(S %*% eigen$vectors[, 1] / eigen$vectors[, 1])
```

```
##          [,1]
## mpg  18636.79
## disp 18636.79
## hp   18636.79
## drat 18636.79
## wt   18636.79
```

```r
eigen$values[1]
```

```
## [1] 18636.79
```

```r
t(eigen$vectors[, 1]) %*% eigen$vectors[, 1]
```

```
##      [,1]
## [1,]    1
```

Spectral Decomposition

$\mathbf{A} = \lambda_1 \mathbf{e}_1 \mathbf{e}_1^T + \lambda_2 \mathbf{e}_2 \mathbf{e}_2^T + \cdots + \lambda_p \mathbf{e}_p \mathbf{e}_p^T$

```r
temp <- array(dim = c(5, 5, 5))

for (i in 1:5){
  temp[i,,] <- eigen$values[i] * eigen$vectors[, i] %*% t(eigen$vectors[, i])
}
# Check the spectral decomposition
(out <- apply(temp, 2:3, sum))
```

```
##             [,1]        [,2]       [,3]        [,4]        [,5]
## [1,]   36.324103  -633.09721 -320.73206   2.1950635  -5.1166847
## [2,] -633.097208 15360.79983 6721.15867 -47.0640192 107.6842040
## [3,] -320.732056  6721.15867 4700.86694 -16.4511089  44.1926613
## [4,]    2.195064   -47.06402  -16.45111   0.2858814  -0.3727207
## [5,]   -5.116685   107.68420   44.19266  -0.3727207   0.9573790
```

```r
S
```

```
##             mpg        disp         hp        drat          wt
## mpg    36.324103  -633.09721 -320.73206   2.1950635  -5.1166847
## disp -633.097208 15360.79983 6721.15867 -47.0640192 107.6842040
## hp   -320.732056  6721.15867 4700.86694 -16.4511089  44.1926613
## drat    2.195064   -47.06402  -16.45111   0.2858814  -0.3727207
## wt     -5.116685   107.68420   44.19266  -0.3727207   0.9573790
```

Determinant and Trace

```r
# Trace
(trace <- sum(diag(S)))
```

```
## [1] 20099.23
```

```r
sum(eigen$values)
```

```
## [1] 20099.23
```

```r
# Determinant
det(S)
```

```
## [1] 3951786
```

```r
prod(eigen$values)
```

```
## [1] 3951786
```

Square-Root Matrices

$$\mathbf{A}^{\frac{1}{2}} = \mathbf{P}\Lambda^{\frac{1}{2}}\mathbf{P}^T = \sum_{j=1}^{p} \sqrt{\lambda_j}\mathbf{e}_j\mathbf{e}_j^T$$

```r
temp1 <- array(dim = c(5, 5, 5))

for (i in 1:5){
  temp1[i,,] <- (1 / eigen$values[i]) * eigen$vectors[, i] %*% t(eigen$vectors[, i])
}
# Check the spectral decomposition
(out1 <- apply(temp1, 2:3, sum))
```

```
##               [,1]          [,2]          [,3]        [,4]        [,5]
## [1,]  0.1695494031 -0.0006468718  0.0058975274 -0.29977161  0.58997555
## [2,] -0.0006468718  0.0005369064 -0.0003801427  0.02257595 -0.03751089
## [3,]  0.0058975274 -0.0003801427  0.0008208474 -0.02678451  0.02595898
## [4,] -0.2997716134  0.0225759526 -0.0267845083  8.50376340  0.40558365
## [5,]  0.5899755523 -0.0375108878  0.0259589804  0.40558365  7.37641228
```

```
S_inv
```

```
##                mpg          disp            hp        drat          wt
## mpg   0.1695494031 -0.0006468718  0.0058975274 -0.29977161  0.58997555
## disp -0.0006468718  0.0005369064 -0.0003801427  0.02257595 -0.03751089
## hp    0.0058975274 -0.0003801427  0.0008208474 -0.02678451  0.02595898
## drat -0.2997716134  0.0225759526 -0.0267845083  8.50376340  0.40558365
## wt    0.5899755523 -0.0375108878  0.0259589804  0.40558365  7.37641228
```

```
temp2 <- array(dim = c(5, 5, 5))

for (i in 1:5){
  temp2[i,,] <- sqrt(eigen$values[i]) * eigen$vectors[, i] %*% t(eigen$vectors[, i])
}

out2 <- apply(temp2, 2:3, sum)

(out2 %*% out2)
```

```
##             [,1]        [,2]       [,3]        [,4]        [,5]
## [1,]   36.324103  -633.09721 -320.73206   2.1950635  -5.1166847
## [2,] -633.097208 15360.79983 6721.15867 -47.0640192 107.6842040
## [3,] -320.732056  6721.15867 4700.86694 -16.4511089  44.1926613
## [4,]    2.195064   -47.06402  -16.45111   0.2858814  -0.3727207
## [5,]   -5.116685   107.68420   44.19266  -0.3727207   0.9573790
```

```
S
```

```
##              mpg        disp         hp        drat          wt
## mpg    36.324103  -633.09721 -320.73206   2.1950635  -5.1166847
## disp -633.097208 15360.79983 6721.15867 -47.0640192 107.6842040
## hp   -320.732056  6721.15867 4700.86694 -16.4511089  44.1926613
## drat    2.195064   -47.06402  -16.45111   0.2858814  -0.3727207
## wt     -5.116685   107.68420   44.19266  -0.3727207   0.9573790
```

Partitioning Random vectors

Let's partitioning the variables into two groups

1. *disp*, *hp*, *wt*
2. *mpg*, *drat*

```
vars1 <- which(names(mtcars) %in% c("disp", "hp", "wt"))
vars2 <- which(names(mtcars) %in% c("mpg", "drat"))

carPar <- mtcars[, c(vars1, vars2)]

(Sigma11 <- cov(carPar[1:3, 1:3]))
```

```
##          disp        hp       wt
## disp 901.3333 294.66667 7.410000
## hp   294.6667  96.33333 2.422500
## wt     7.4100   2.42250 0.077175
```

```r
(Sigma22 <- cov(carPar[4:5, 4:5]))
```

```
##          mpg     drat
## mpg   3.6450 -0.09450
## drat -0.0945  0.00245
```

```r
(Sigma12 <- cov(carPar)[1:3, 4:5])
```

```
##              mpg        drat
## disp -633.097208 -47.0640192
## hp   -320.732056 -16.4511089
## wt     -5.116685  -0.3727207
```