Lecture 5

Time Series Regression

Reading: Forecasting, Time Series, and Regression (4th edition) by Bowerman, O'Connell, and Koehler: Chapter 6

MATH 4070: Regression and Time-Series Analysis

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Agenda

1. Time Series Data

2. Trend Estimation

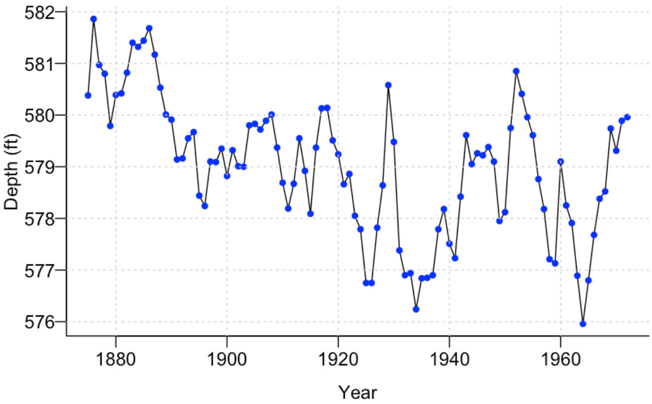
3. Estimating Seasonality

Level of Lake Huron 1875–1972

Annual measurements of the level of Lake Huron in feet.

[Source: Brockwell & Davis, 1991]

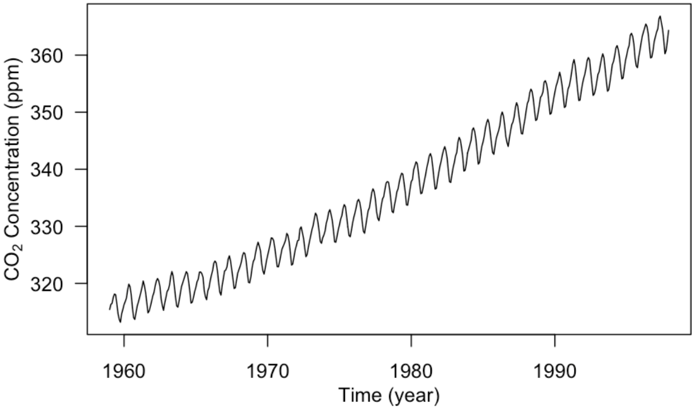
```{r}
par(mar = c(3.2, 3.2, 0.5, 0.5), mgp = c(2, 0.5, 0), bty = "L")
data(LakeHuron)
plot(LakeHuron, ylab = "Depth (ft)", xlab = "Year", las = 1)
points(LakeHuron, cex = 0.8, col = "blue", pch = 16)
grid()
```



Mauna Loa Monthly Atmospheric CO_2 Concentration

[Source: Keeling & Whorf, Scripps Institution of Oceanography]

````
```{r}
data(co2)
par(mar = c(3.8, 4, 0.8, 0.6))
plot(co2, las = 1, xlab = "", ylab = "")
mtext("Time (year)", side = 1, line = 2)
mtext(expression(paste("CO"[2], " Concentration (ppm)")), side = 2, line = 2.5)
```
````



Time Series Data

- A time series is a collection of observations $\{y_t, t \in T\}$ taken sequentially in time (t) with the index set T
  - $T = \{0, 1, 2, \cdots, T\} \subset \mathbb{Z} \Rightarrow$ discrete-time time series
  - $T = [0, T] \subset \mathbb{R} \Rightarrow$ continuous-time time series
- A discrete-time time series might be intrinsically discrete or might arise from a underlying continuous-time time series via
  - sampling (e.g., instantaneous wind speed)
  - aggregation (e.g., daily accumulated precipitation amount)
  - extrema (e.g., daily maximum temperature)
- We will focus on dealing with discrete-time real-valued $(Y_t \in \mathbb{R})$ time series in this course

Exploratory Time Series Analysis

- Start with a time series plot, i.e., to plot y_t versus t
- Look at the following:
  - Are there abrupt changes?
  - Are there "outliers"?
  - Is there a need to transform the data?
- Examine the trend, seasonal components, and the "noise" term

Features of Times Series

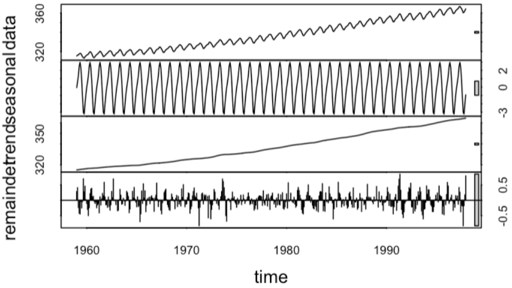
- Trends (μ_t)
  - μ_t represents continuous changes, usually in the mean, over longer time scales. *"The essential idea of trend is that it shall be smooth."* - [Kendall, 1973]
  - The form of the trend is typically unknown and needs to be estimated. Removing the trend yields a detrended series
- Seasonal or Periodic Components (s_t)
  - s_t repeats consistently over time, i.e., $s_t = s_{t+kd}$
  - The form and period d of the seasonal component must be estimated to deseasonalize the series.
- The "Noise" Process (η_t)
  - η_t represents the component that is neither trend nor seasonality
  - Focus on finding plausible statistical models for this process

Combining Trend, Seasonality, and Noise Together

There are two commonly used approaches

- Additive model:

$$Y_t = \mu_t + s_t + \eta_t, \quad t = 1, \cdots, T$$



- Multiplicative model:

$$Y_t = \mu_t s_t \eta_t, \quad t = 1, \cdots, T$$

If all $\{y_t\}$ are positive then we obtain the additive model by taking logarithms:

$$\log Y_t = \log \mu_t + \log s_t + \log \eta_t, \quad t = 1, \cdots, T$$

The (Additive) Decomposition Model

- The additive model for a time series $\{Y_t\}$ is

$$Y_t = \mu_t + s_t + \eta_t,$$

  where

  - μ_t is the trend component
  - s_t is the seasonal component
  - η_t is the random (noise) component with $\mathbb{E}(\eta_t) = 0$

- Standard procedure:

  (1) Estimate/remove the trend and seasonal components

  (2) Analyze the remainder, the residuals $\hat{\eta}_t = y_t - \hat{\mu}_t - \hat{s}_t$

- We will focus on (1) for this week

Estimating Trend for Nonseasonal Model

Assuming $s_t = 0$ (i.e., there is no "seasonal" variation), we have

$$Y_t = \mu_t + \eta_t,$$

with $\mathbb{E}(\eta_t) = 0$

Methods for estimating trends

- Least squares regression
- Smoothing

Trend Estimation: Linear Regression

- The additive nonseasonal time series model for $\{Y_t\}$ is

$$Y_t = \mu_t + \eta_t,$$

where the trend is assumed to be a linear combination of known covariate series $\{x_{it}\}_{i=1}^{p}$

$$\mu_t = \beta_0 + \sum_{i=1}^{p} \beta_i x_{it}.$$

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- Here we want to **estimate** $\boldsymbol{\beta} = (\beta_0, \beta_1, \cdots, \beta_p)^T$ from the data $\{y_t, \{x_{it}\}_{i=1}^p\}_{t=1}^T$

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- You're likely quite familiar with this formulation already ⇒ Regression Analysis

Some Examples of Covariate Series $\{x_{it}\}$

- **Simple linear regression model:**

$$\mu_t = \beta_0 + \beta_1 x_t,$$

for example, the temperature trend at time t could be a constant (β_0) plus a multiple (β_1) of the carbon dioxide level at time t (x_t)

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$$\mu_t = \beta_0 + \sum_{i=1}^{p} \beta_i t^i$$

- **Change point model:**

$$\mu_t = \begin{cases} \beta_0 & \text{if } t \le t^*; \\ \beta_0 + \beta_1 & \text{if } t \ge t^*. \end{cases}$$

Parameter Estimation: Ordinary Least Squares

- Like in the linear regression setting, we can estimate the parameters via ordinary least squares (OLS)

- Specifically, we minimize the following objective function:

$$\ell_{ols} = \sum_{t=1}^{T}(y_t - \beta_0 - \sum_{k=1}^{p} x_{kt}\beta_k)^2.$$

- The estimates $\boldsymbol{\beta} = (\beta_0, \beta_1, \cdots, \beta_p)^T$ minimizing the above objective function are called the OLS estimates of $\boldsymbol{\beta}$ $\Rightarrow$ they are easiest to express in **matrix form**

The Model and Parameter Estimates in Matrix Form

- Matrix representation:

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\eta},$$

where $\boldsymbol{Y} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_T \end{bmatrix}$, $\boldsymbol{X} = \begin{bmatrix} 1 & x_{11} & x_{12} & \cdots & x_{1p} \\ 1 & x_{21} & x_{22} & \cdots & x_{2p} \\ 1 & \vdots & \cdots & \cdots & \vdots \\ 1 & x_{T1} & x_{T2} & \cdots & x_{Tp} \end{bmatrix}$, and

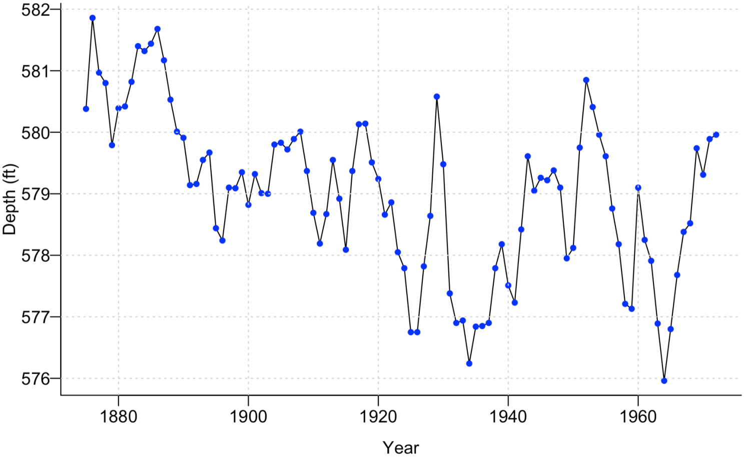
$\boldsymbol{\eta} = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_T \end{bmatrix}$

- Assuming $\boldsymbol{X}^T\boldsymbol{X}$ is **invertible**, the OLS estimate of β can be shown to be

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y},$$

and the `lm` function in `R` calculates OLS estimates

Lake Huron Example Revisited



Let's **assume** there is a linear trend in time ⇒ we need to estimate the **intercept** β_0 and **slope** β_1

The R Output

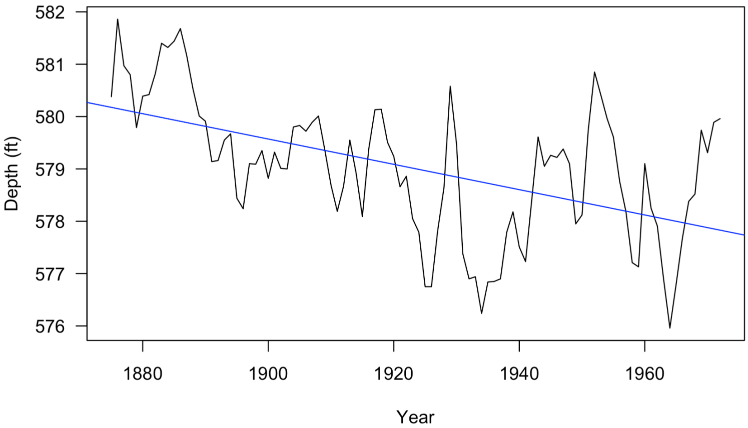
```
Call:
lm(formula = LakeHuron ~ yr)

Residuals:
     Min       1Q   Median       3Q      Max
-2.50997 -0.72726  0.00083  0.74402  2.53565

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept) 625.554918   7.764293  80.568  < 2e-16 ***
yr           -0.024201   0.004036  -5.996 3.55e-08 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

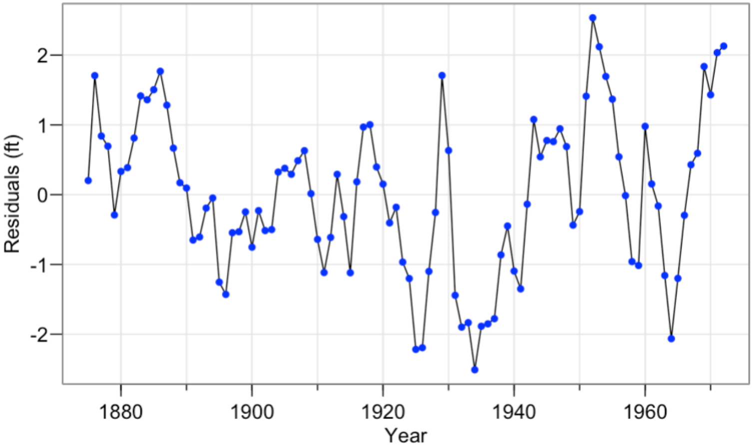
Residual standard error: 1.13 on 96 degrees of freedom
Multiple R-squared:  0.2725,    Adjusted R-squared:  0.2649
F-statistic: 35.95 on 1 and 96 DF,  p-value: 3.545e-08
```

Plot the (Estimated) Trend $\hat{\mu}_t = \hat{\beta}_0 + \hat{\beta}_1 t$



$\hat{\beta}_1 = -0.0242$ (ft/yr) $\Rightarrow$ there seems to be a decreasing trend

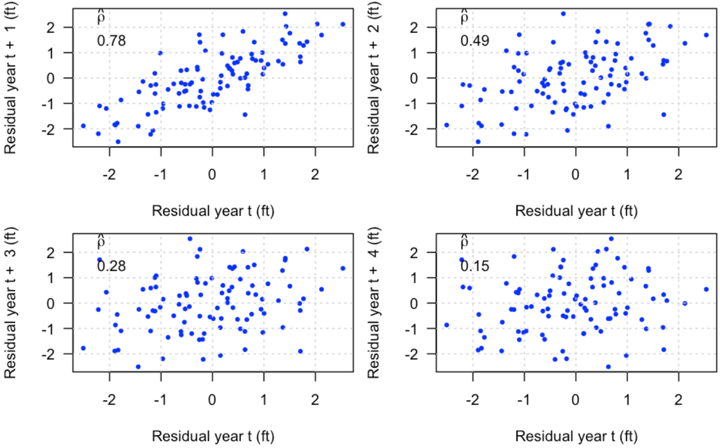
Plot the Residuals $\{\hat{\eta}_t = y_t - \hat{\beta}_0 - \hat{\beta}_1 t\}$



$\{\hat{\eta}_t\}$ seems to exhibit some temporal dependence structure, should we worry about the results we have (recall OLS makes an i.i.d. assumption)?

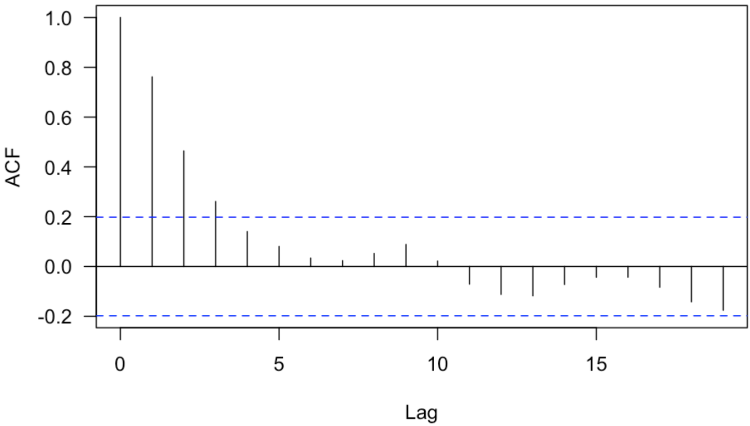
Exploring the Dependence Structure of “Noise” $\{\eta_t\}$

$\{\eta_t\}$ exhibit a temporal dependence structure, meaning that the nearby (in time) values tend to be more alike than those that are far part. To observe this, let’s create a few time lag plots



Further Exploration of the Temporal Dependence Structure

Let's plot the correlation as a function of the time lag

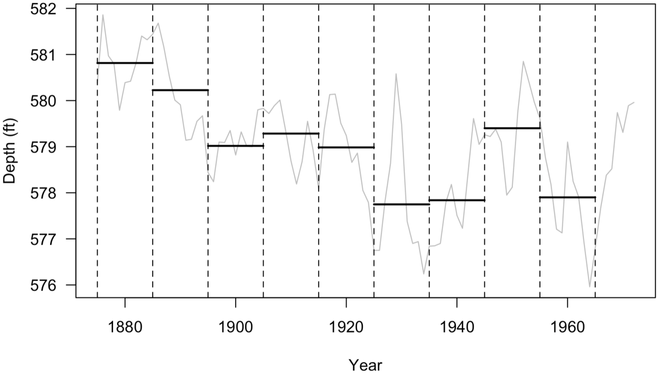


We will learn how to use this information to suggest an appropriate model

Smoothing or Local Averaging

In some cases, we may relax the trend assumption using a 'non-parametric' approach.

We divide the time series into small blocks (each with 10 years of data) and average each block.

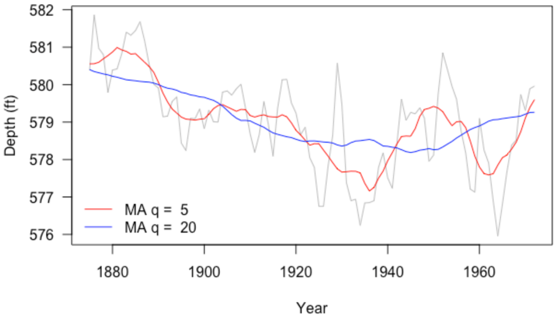


Doing this gives a very rough estimate of the trend. **Can we do better?**

Moving Average Smoother

A moving average smoother estimates the trend at time t by averaging the current observation and the q nearest observations from either side. That is

$$\hat{\mu}_t = \frac{1}{2q+1} \sum_{j=-q}^{q} y_{t-j}$$



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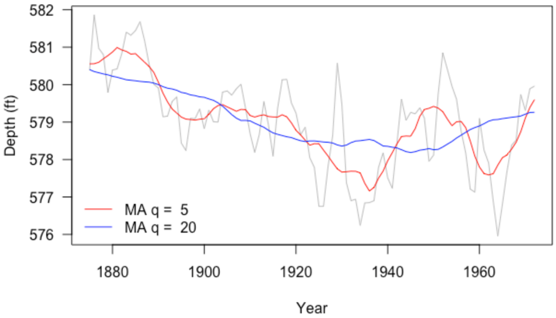


q is the “smoothing” parameter, which controls the smoothness of the estimated trend $\hat{\mu}_t$

Exponential Smoothing

- Let $\alpha \in [0,1]$ be some fixed constant, defined

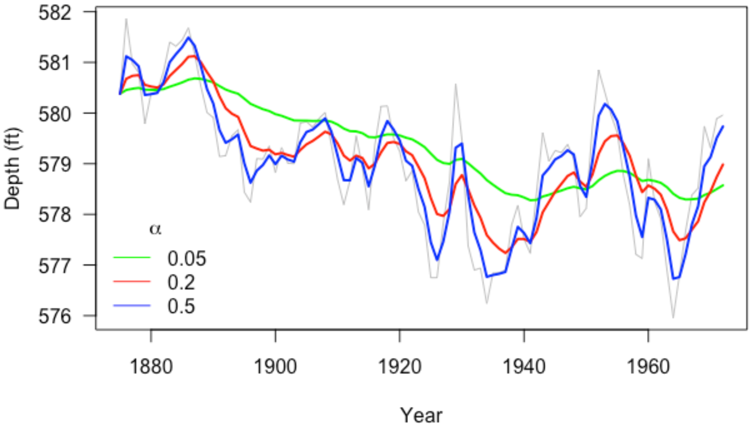
$$\hat{\mu}_t = \begin{cases} Y_1 & \text{if } t = 1; \\ \alpha Y_t + (1-\alpha)\hat{\mu}_{t-1} & t = 2, \cdots T. \end{cases}$$

- For $t = 2, \cdots, T$, we can rewrite $\hat{\mu}_t$ as

$$\sum_{j=0}^{t-2} \alpha(1-\alpha)^j Y_{t-j} + (1-\alpha)^{t-1} Y_1.$$

$\Rightarrow$ it is a one-sided moving average filter with exponentially decreasing weights. One can alter α to control the amounts of smoothing (see next slide for an example)

α is the Smoothing Parameter for Exponential Smoothing



The smaller the α, the smoother the resulting trend

Seasonal Component Estimation

Let's consider a situation where a time series consists of only a seasonal component (assuming the trend has been estimated/removed). In this scenario,

$$Y_t = s_t + \eta_t,$$

with $\{s_t\}$ having period d (i.e., $s_{t+jd} = s_t$ for all integers j and t), $\sum_{t=1}^{d} s_t = 0$ and $\mathbb{E}(\eta_t) = 0$

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Two methods to estimate $\{s_t\}$

- Harmonic regression
- Seasonal mean model

Describing Periodic/Cyclical Behavior

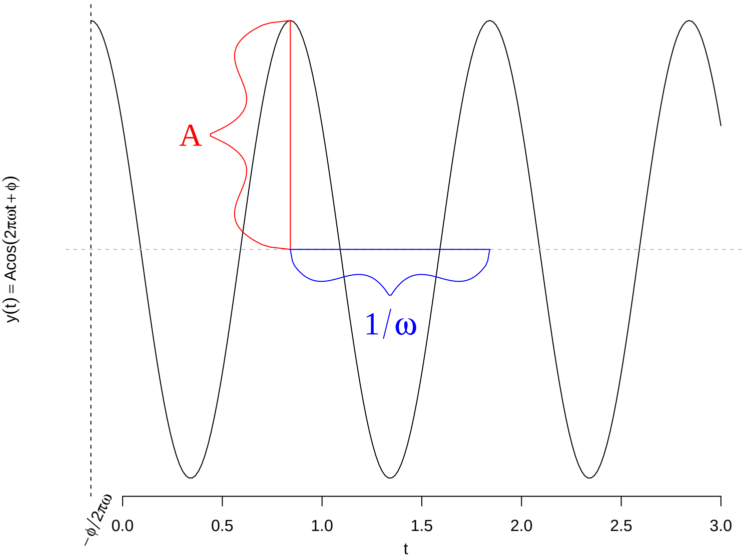
The simplest case is the cosine wave

$$\begin{aligned} s_t &= A\cos(2\pi\omega t + \phi) \\ &= \alpha_1 \cos(2\pi\omega t) + \alpha_2 \sin(2\pi\omega t), \end{aligned}$$

where

- A is amplitude
- ω is frequency, in cycles per time unit
- ϕ is phase, determining the start point of the cosine function
- $\alpha_1 = A\cos(\phi)$, $\alpha_2 = -A\sin(\phi)$, $A = \sqrt{\alpha_1^2 + \alpha_2^2}$, $\phi = \tan^{-1}\frac{-\alpha_2}{\alpha_1}$

Graphical Illustration of the Cosine Wave



Harmonic Regression

- A harmonic regression model has the form

$$s_t = \sum_{j=1}^{k} A_j \cos(2\pi\omega_j + \phi_j).$$

For each $j = 1, \cdots, k$:

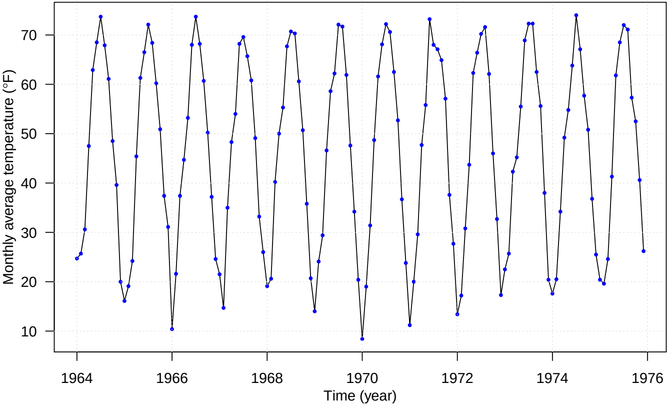
- $A_j > 0$ is the amplitude of the j-th cosine wave
- ω_j controls the the frequency of the j-th cosine wave (how often waves repeats)
- $\phi_j \in [-\pi, \pi]$ is the phase of the j-th wave (where it starts)

- The above can be expressed as

$$\sum_{j=1}^{k} \left(\beta_{1j} \cos(2\pi f_j) + \beta_{2j} \sin(2\pi f_j)\right),$$

where $\beta_{1j} = A_j \cos(\phi_j)$ and $\beta_{2j} = A_j \sin(\phi_j) \Rightarrow$ if $\{\omega_j\}_{j=1}^{k}$ are known, **we can use regression techniques to estimate the parameters** $\{\beta_{1j}, \beta_{2j}\}_{j=1}^{k}$

Monthly Average Temperature in Dubuque, IA [Cryer & Chan, 2008]



Let's assume that there is no trend in this time series. In this context, our goal is to estimate s_t, the seasonal component.

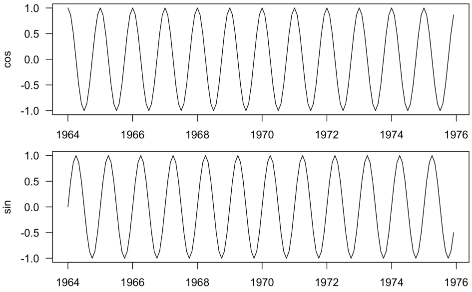
Use a Harmonic Regression to Model Annual Cycles

Model: $s_t = \beta_0 + \beta_1 \cos(2\pi t) + \beta_2 \sin(2\pi t)$

$\Rightarrow$ annual cycles can be modeled by a linear combination of $\cos$ and $\sin$ with 1-year period.

In `R`, we can easily create these harmonics using the `harmonic` function in the `TSA` package

```
harmonics <- harmonic(tempdub, 1)
```



R Code & Output

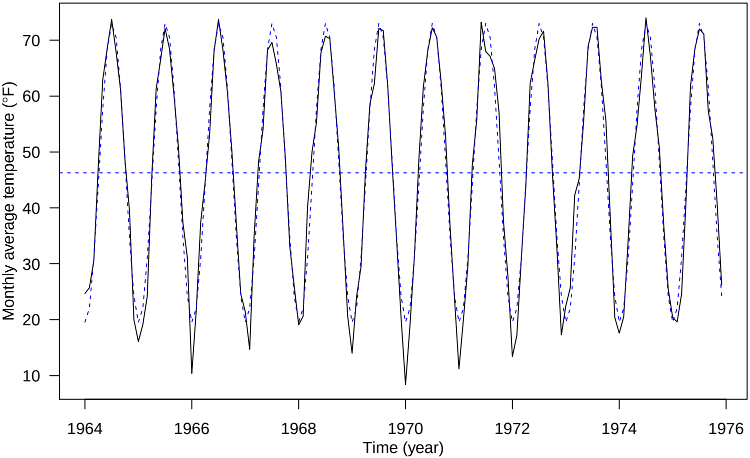
```
```{r}
harReg <- lm(tempdub ~ harmonics)
summary(harReg)
```

```
Call:
lm(formula = tempdub ~ harmonics)

Residuals:
     Min       1Q   Median       3Q      Max
-11.1580  -2.2756  -0.1457   2.3754  11.2671

Coefficients:
                     Estimate Std. Error t value Pr(>|t|)
(Intercept)           46.2660     0.3088 149.816  < 2e-16 ***
harmonicscos(2*pi*t) -26.7079     0.4367 -61.154  < 2e-16 ***
harmonicssin(2*pi*t)  -2.1697     0.4367  -4.968 1.93e-06 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```
```

The Harmonic Regression Model Fit



Seasonal Means Model

- **Harmonics regression** assumes the seasonal pattern has a regular shape, i.e., the height of the peaks is the same as the depth of the troughs

- A less restrictive approach is to model $\{s_t\}$ as

$$s_t = \begin{cases} \beta_1 & \text{for } t = 1, 1+d, 1+2d, \cdots \quad ; \\ \beta_2 & \text{for } t = 2, 2+d, 2+2d, \cdots \quad ; \\ \vdots & \vdots \quad ; \\ \beta_d & \text{for } t = d, 2d, 3d, \cdots \quad . \end{cases}$$

- This is the seasonal means model, the parameters $(\beta_1, \beta_2, \cdots, \beta_d)^T$ can be estimated under the linear model framework (think about ANOVA)

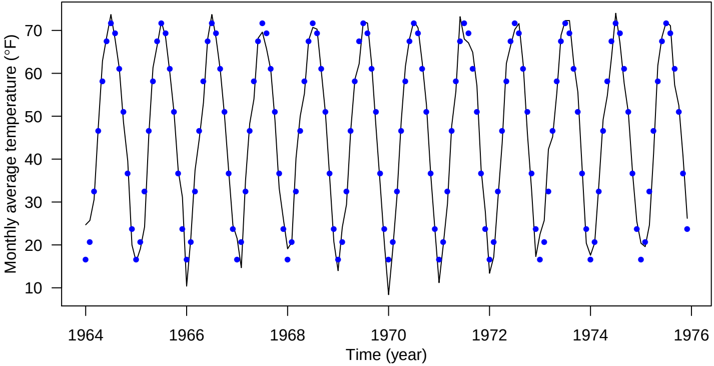
R Output

```
Call:
lm(formula = tempdub ~ month - 1)

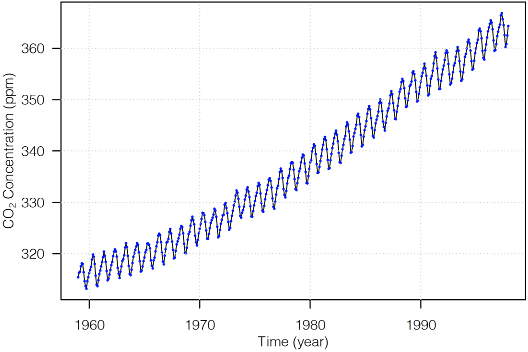
Residuals:
    Min      1Q  Median      3Q     Max
-8.2750 -2.2479  0.1125  1.8896  9.8250

Coefficients:
               Estimate Std. Error t value Pr(>|t|)
monthJanuary     16.608      0.987   16.83   <2e-16 ***
monthFebruary    20.650      0.987   20.92   <2e-16 ***
monthMarch       32.475      0.987   32.90   <2e-16 ***
monthApril       46.525      0.987   47.14   <2e-16 ***
monthMay         58.092      0.987   58.86   <2e-16 ***
monthJune        67.500      0.987   68.39   <2e-16 ***
monthJuly        71.717      0.987   72.66   <2e-16 ***
monthAugust      69.333      0.987   70.25   <2e-16 ***
monthSeptember   61.025      0.987   61.83   <2e-16 ***
monthOctober     50.975      0.987   51.65   <2e-16 ***
monthNovember    36.650      0.987   37.13   <2e-16 ***
monthDecember    23.642      0.987   23.95   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

The Seasonal Means Model Fit



Estimating the Trend and Seasonal variation Together

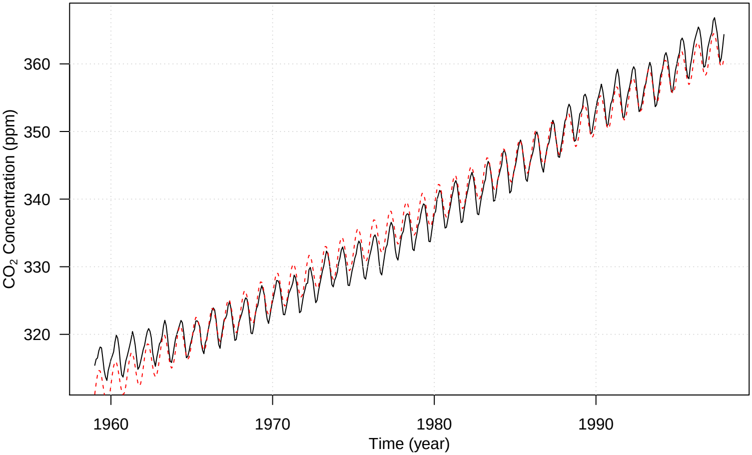


Let's perform a regression analysis to model both μ_t (assuming a linear time trend) and s_t (using $\cos$ and $\sin$)

```{r}
time <- as.numeric(time(co2))
harmonics <- harmonic(co2, 1)

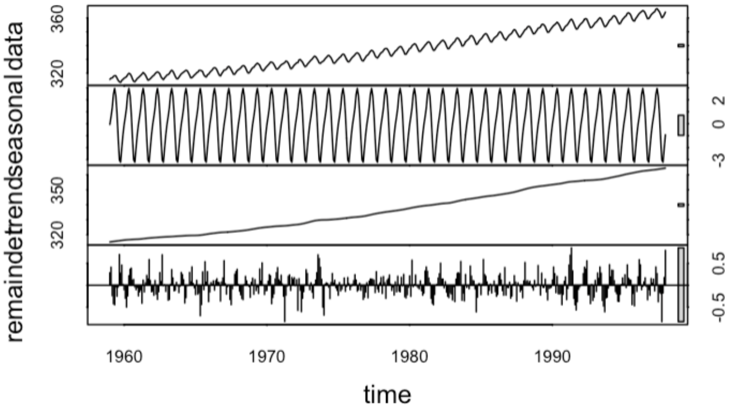
lm_trendSeason <- lm(co2 ~ time + harmonics)
summary(lm_trendSeason)
```

The Regression Fit



Seasonal and Trend decomposition using Loess [Cleveland, et. al., 1990]

```
```{r}
# Seasonal and Trend decomposition using Loess (STL)
par(mar = c(4, 3.6, 0.8, 0.6))
stl <- stl(co2, s.window = "periodic")
plot(stl, las = 1)
```
```



Summary

These slides cover:

- Main features of a time series: trend, seasonality, and "noise"
- Estimating trends using multiple linear regression and "nonparametric" smoothing
- Estimating seasonality using harmonic regression and the seasonal mean model

R Functions to Know

- Visualizing time series data: `plot` (for ts objects), `ts.plot`, `tsplot` (`astsa` package)
- Fitting time series regression: `lm`, `harmonic` (`TSA` package) for creating harmonic predictors, `filter` for smoothing
- Seasonal and trend decomposition: `stl`