Lecture 30

Inference for Proportions

STAT 8010 Statistical Methods I
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Inference for Categorical Data

In the next few lectures we will focus on categorical data analysis:

- Inference for a single proportion p
- Comparison of two proportions p_1 and p_2
- χ^2 tests: Inference for Multi-category data and contingency tables

Inference for a single proportion: Motivated Example

> Researchers in the development of new treatments for cancer patients often evaluate the effectiveness of new therapies by reporting the proportion of patients who survive for a specified period of time after completion of the treatment. A new genetic treatment of 870 patients with a particular type of cancer resulted in 330 patients surviving at least 5 years after treatment. Estimate the proportion of all patients with the specified type of cancer who would survive at least 5 years after being administered this treatment.

- Dichotomous (two-category) outcomes: “success” & “failure”
- Similar to the inferential problem for μ, the population mean, we would like to infer p, the population proportion of success $\Rightarrow$ point estimate, interval estimate, hypothesis testing

Point/Interval Estimation

- Point estimate:

$$\hat{p} = \frac{X(\text{\# of "successes"})}{n}$$

Recall the Binomial random variable, we have $\mathbb{E}[X] = np$ where $X \sim \text{Bin}(n,p) \Rightarrow \mathbb{E}[\frac{X}{n}] = \mathbb{E}[\hat{p}] = p$

- $100(1-\alpha)\%$ CI:

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{(\hat{p})(1-\hat{p})}{n}}$$

Why?

  - CLT approximation: $\hat{p} \approx \mathrm{N}(p, \sigma^2_{\hat{p}})$ where n "sufficiently large" $\Rightarrow \min(np, n(1-p)) \geq 5$
  - $\sigma^2_{\hat{p}} = Var(\frac{X}{n}) = \frac{1}{n^2}Var(X) = \frac{1}{n^2}n(p)(1-p) = \frac{p(1-p)}{n}$

Motivated Example Revisited

A new genetic treatment of 870 patients with a particular type of cancer resulted in 330 patients surviving at least 5 years after treatment.

1. Estimate the proportion of all patients who would survive at least 5 years after being administered this treatment.
2. Construct a 95% CI for p

Another Example

Among 900 randomly selected registered voters nationwide, 63% of them are somewhat or very concerned about the spread of bird flu in the United States.

1. What is the point estimate for p (Proportion of U.S. voters who are concerned about the spread of bird flu?
2. Construct a 99% CI for p
3. Is it reasonable to conclude that p is .600? in the United States)

Margin of error & Sample Size Calculation

- Margin of error:

$$z_{\alpha/2}\sqrt{\frac{n\hat{p}(1-\hat{p})}{n}}$$

$\Rightarrow$ CI for $p = \hat{p} \pm$ margin of error

- Sample size determination:

$$n = \tilde{p}(1-\tilde{p})\left(\frac{z_{\alpha/2}}{\text{margin of error}}\right)^2,$$

What value of $\tilde{p}$ to use?

  - An educated guess
  - A value from previous research
  - Use a pilot study
  - The “most conservative” choice is to use $\tilde{p} = 0.5$

Example

A researcher wants to estimate the proportion of voters who will vote for candidate A. She wants to estimate to within 0.05 with 90% confidence.

1. How large a sample does she need if she thinks the true proportion is about .9?
2. How large a sample does she need if she thinks the true proportion is about .6?
3. How large a sample does she need if she wants to use the most conservative estimate?

Hypothesis Testing for p

1. State the null and alternative hypotheses:

$$H_0 : p = p_0 \text{ vs. } H_a : p > \text{ or } \neq \text{ or } < p_0$$

2. Compute the test statistic:

$$z_{obs} = \frac{\hat{p} - p_0}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}}$$

3. Make the decision of the test:

Rejection Region/ P-Value Methods

4. Draw the conclusion of the test:

We (do/do not) have enough statistical evidence to conclude that (H_a in words) at α% significant level.

Bird Flu Example Revisited

Among 900 randomly selected registered voters nationwide, 63% of them are somewhat or very concerned about the spread of bird flu in the United States. Conduct a hypothesis test at .01 level to assess if $p > .667$.

Another CI for p: Wilson Score Confidence Interval

Idea: Solving $p = \hat{p} \pm z_{\alpha/2}\sqrt{\frac{p(1-p)}{n}} \Rightarrow (p-\hat{p})^2 = z_{\alpha/2}^2 \frac{p(1-p)}{n}$

$100(1-\alpha)\%$ Wilson Score Confidence Interval:

$$\frac{X + \frac{z_{\alpha/2}^2}{2}}{n + z_{\alpha/2}^2} \pm \frac{z_{\alpha/2}}{n + z_{\alpha/2}^2}\sqrt{\frac{X(n-X)}{n} + \frac{z_{\alpha/2}^2}{4}}$$

Rule of Three: An Approximate 95% CI for p When $\hat{p} = 0$ or 1

When $\hat{p} = 0$, we have

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{(\hat{p})(1-\hat{p})}{n}} = 0 \pm z_{\alpha/2} \times 0 = (0, 0)$$

Similarly, when $\hat{p} = 1$, we have

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{(\hat{p})(1-\hat{p})}{n}} = 1 \pm z_{\alpha/2} \times 0 = (1, 1)$$

These CIs degenerate to a point , which do not reflect the estimation uncertainty. Here we could apply the rule of three to approximate 95% CI:

$$\begin{array}{ll} (0, 3/n), & \text{if } \hat{p} = 0 \\ (1 - 3/n, 1), & \text{if } \hat{p} = 1 \end{array}$$