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Stochastic Volatility Model

Lecture 10

Univariate Volatility Modeling

Reading: An introduction to analysis of financial data with R (2013) by Ruey Tsay

MATH 8090 Time Series Analysis
Week 10

Whitney Huang
Clemson University

Agenda

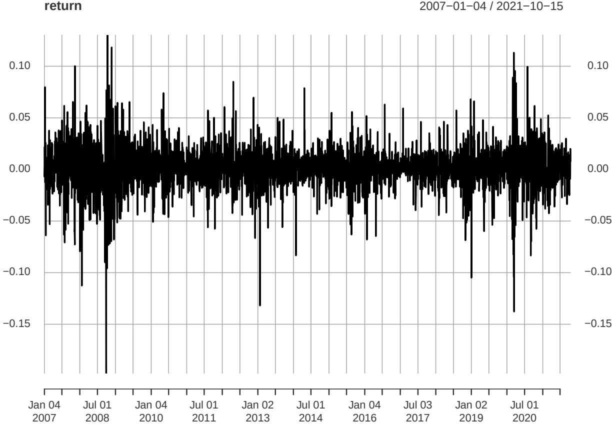
1. Background
2. ARCH Model
3. GARCH Model
4. IGARCH and EGARCH Models
5. Stochastic Volatility Model

Financial Time Series

Source: Google Finance

Log Returns of Apple Stock

$$r_t = \log\left(\frac{y_t}{y_{t-1}}\right), \quad \text{where } y_t \text{ is the price at time } t$$



Periods of high uncertainty or large price movements tend to cluster together ⇒ Volatility Clustering

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Modeling Volatility

Volatility measures the degree of variation in asset prices over time, typically quantified as the (conditional) standard deviation of log returns.

Why is volatility important?

- Option pricing - e.g., Black-Scholes model
- Risk management - e.g., Value-at-Risk (VaR)
- Asset allocation - e.g., minimum-variance portfolios
- Forecasting - e.g., interval prediction of returns

Key challenge: Volatility is not directly observable

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How to Model Volatility?

We will take an econometric approach by modeling the conditional standard deviation (σ_t) of daily or monthly returns

Basic structure

$$r_t = \mu_t + a_t, \quad \mu_t = \mathbb{E}(r_t | F_{t-1}) = \phi_0 + \sum_{i=1}^{p} \phi_i r_{t-i} + \sum_{j=1}^{q} \theta_j a_{t-j}$$

Volatility models are concerned with time-evolution of

$$\mathrm{Var}(r_t | F_{t-1}) = \mathrm{Var}(a_t | F_{t-1}) = \sigma_t^2,$$

the conditional variance of a return

Literature on Univariate Volatility Modeling

- Autoregressive conditional heteroscedastic (ARCH) model [Engle, 1982]
- Generalized Autoregressive Conditional Heteroskedasticity (GARCH) Model [Bollerslev, 1986]
- Integrated Generalized Autoregressive Conditional heteroskedasticity (IGARCH) model
- Exponential general autoregressive conditional heteroskedastic (EGARCH) model [Nelson, 1991]
- Asymmetric parametric ARCH models [Ding, Granger, and Engle, 1994]
- Stochastic volatility (SV) models [Melino and Turnbull, 1990; Harvey, Ruiz, and Shephard, 1994; Jacqier, Polson. and Rossi, 1994]

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Autoregressive Conditional Heteroscedastic (ARCH) Model

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ARCH(m) model:

$$a_t = \sigma_t \epsilon_t, \qquad \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \cdots + \alpha_m a_{t-m}^2, \qquad \alpha_i \geq 0 \text{ for } 1 \leq i \leq m$$

where $\{\epsilon_t\}$ is an i.i.d. sequence with:

- $\mathbb{E}[\epsilon_t] = 0$
- $\mathrm{Var}[\epsilon_t] = 1$
- Common choices: standard normal, standardized Student-t, or generalized error distribution (and their skewed forms)

ARMA Analogy: ARCH(m) is analogous to an AR(m) model, but for the squared innovations

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Properties of ARCH Models

ARCH(1) model

$$a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2, \alpha_0 > 0, \quad \alpha_1 \geq 0$$

Main Properties:

- $\mathbb{E}(a_t) = 0$, $\mathrm{Var}(a_t) = \dfrac{\alpha_0}{1-\alpha_1}$, valid if $0 < \alpha_1 < 1$
- Under normality:

$$\mathbb{E}[a_t^4] = \frac{3\alpha_0^2(1+\alpha_1)}{(1-\alpha_1)(1-3\alpha_1^2)}$$

Finite only if $0 < \alpha_1^2 < \frac{1}{3}$

Implications:

- Finite variance: $\alpha_1 < 1$. Finite kurtosis: $\alpha_1^2 < \frac{1}{3}$
- Large $\alpha_1 \Rightarrow$ heavy-tailed a_t

Even with Gaussian errors, ARCH(1) produces heavy-tailed returns

Building an ARCH Model

1. **Model the mean and test for ARCH effects**

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Building an ARCH Model

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   - Fit a mean model: $a_t = y_t - \hat{\mu}_t$

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$$H_0 : \text{no ARCH effects} \quad \text{vs.} \quad H_1 : \text{ARCH effects present}$$

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   - Use the Ljung-Box test on $\{a_t^2\}$ [McLeod and Li, 1983] or the Lagrange Multiplier test [Engle, 1982]

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2. **Determine the order** m: Examine the PACF of $\{a_t^2\}$ to identify significant lags

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   - Check Q-statistics for standardized residuals and their squares

Building an ARCH Model

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   - Check Q-statistics for standardized residuals and their squares
   - Assess skewness and kurtosis of residuals

Building an ARCH Model

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3. **Estimate parameters**: Use Conditional Maximum Likelihood Estimation
4. **Check model adequacy**
   - Check Q-statistics for standardized residuals and their squares
   - Assess skewness and kurtosis of residuals
   - Verify no remaining ARCH effects

Advantages and Limitations of ARCH Models

Advantages:

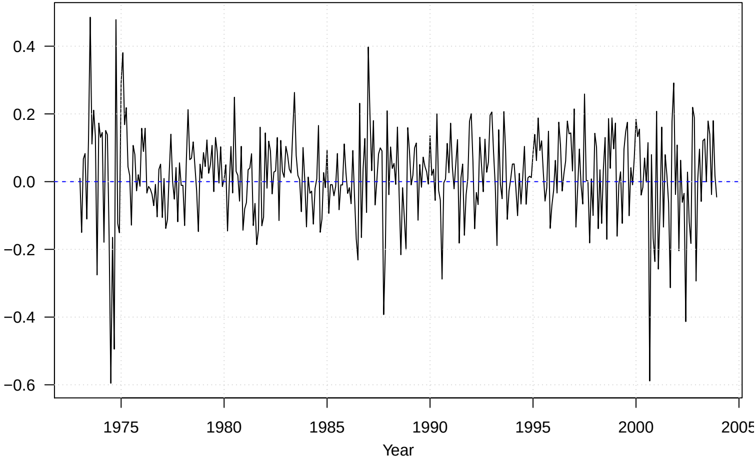
- **Simple and intuitive:** easy to understand and estimate
- **Captures volatility clustering:** periods of high and low variability
- **Produces heavy tails:** helps explain large market movements

Limitations:

- **Symmetric response:** treats positive and negative shocks equally
- **Parameter constraints:** e.g., ARCH(1) requires $\alpha_1^2 \in [0, 1/3]$
- **Limited flexibility:** volatility forecasts can change too slowly

Example: Monthly Log Returns of Intel Stock

Here we use the monthly log returns of Intel stock to illustrate ARCH modeling



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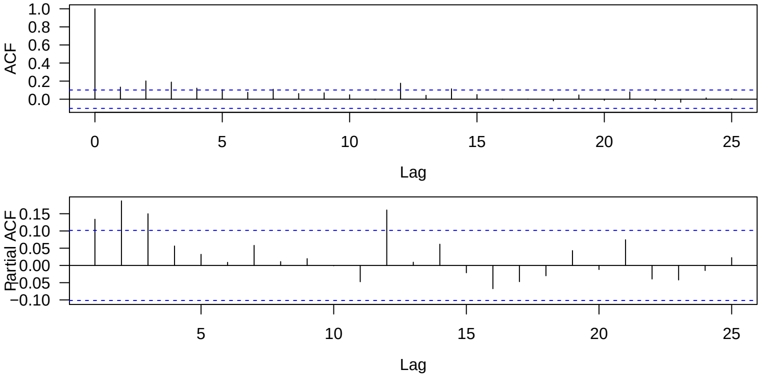
Testing ARCH Effect

Here we test and examine the temporal pattern of the squared residuals

```
> Box.test(y^2, lag = 12, type = 'Ljung')

        Box-Ljung test

data:  y^2
X-squared = 68.67, df = 12, p-value = 5.676e-10
```



ARCH Model Fitting

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Here we fit an ARCH(3) for the volatility:

$$r_t = \mu + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \sum_{i=1}^{3} \alpha_i a_{t-i}^2,$$

assuming $\epsilon_t \overset{i.i.d.}{\sim} \mathrm{N}(0, 1)$.

```
Error Analysis:
        Estimate  Std. Error  t value Pr(>|t|)
mu      0.016572    0.006423    2.580  0.00988 **
omega   0.012043    0.001579    7.627  2.4e-14 ***
alpha1  0.208649    0.129177    1.615  0.10626
alpha2  0.071837    0.048551    1.480  0.13897
alpha3  0.049045    0.048847    1.004  0.31536
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

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Here we fit an ARCH(3) for the volatility:

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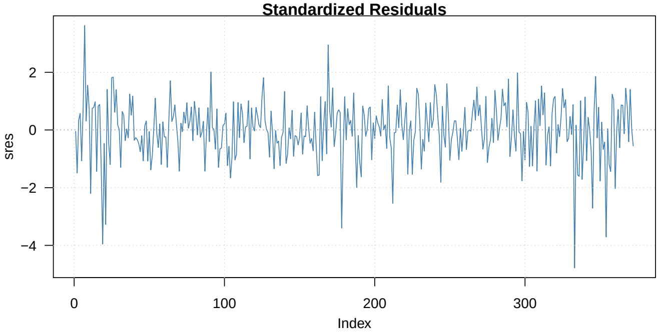
alpha3  0.049045    0.048847    1.004  0.31536
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

Let’s fit a simplified ARCH(1) model

ARCH(1) Model Fitting

```
Error Analysis:
        Estimate  Std. Error  t value Pr(>|t|)
mu      0.016570    0.006161    2.689  0.00716 **
omega   0.012490    0.001549    8.061 6.66e-16 ***
alpha1  0.363447    0.131598    2.762  0.00575 **
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```



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ARCH(1) Model Checking

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Standardised Residuals Tests:

			Statistic	p-Value
Jarque-Bera Test	R	Chi^2	122.404	0
Shapiro-Wilk Test	R	W	0.9647625	8.273101e-08
Ljung-Box Test	R	Q(10)	13.72604	0.1858587
Ljung-Box Test	R	Q(15)	22.31714	0.09975386
Ljung-Box Test	R	Q(20)	23.88257	0.2475594
Ljung-Box Test	R^2	Q(10)	12.50025	0.25297
Ljung-Box Test	R^2	Q(15)	30.11276	0.01152131
Ljung-Box Test	R^2	Q(20)	31.46404	0.04935483
LM Arch Test	R	TR^2	22.036	0.0371183

- Jarque-Berg & Shapiro-Wilk Tests: Normality
- Lagrange multiplier (LM) Test: ARCH Effects

ARCH(1) Model with Student-t Innovations


```
Error Analysis:
        Estimate  Std. Error  t value Pr(>|t|)
mu      0.021571    0.006054    3.563 0.000366 ***
omega   0.013424    0.001968    6.820 9.09e-12 ***
alpha1  0.259867    0.119901    2.167 0.030209 *
shape   5.985979    1.660030    3.606 0.000311 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Log Likelihood:
 242.9678    normalized:  0.6531391

Description:
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Standardised Residuals Tests:
                                Statistic p-Value
 Jarque-Bera Test   R    Chi^2  130.8931  0
 Shapiro-Wilk Test  R    W      0.9637533 5.744995e-08
 Ljung-Box Test     R    Q(10)  14.31288  0.1591926
 Ljung-Box Test     R    Q(15)  23.34043  0.07717449
 Ljung-Box Test     R    Q(20)  24.87286  0.2063387
 Ljung-Box Test     R^2  Q(10)  15.35917  0.1195054
 Ljung-Box Test     R^2  Q(15)  33.96318  0.003446127
 Ljung-Box Test     R^2  Q(20)  35.46828  0.01774746
 LM Arch Test       R    TR^2   24.11517  0.01961957
```

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GARCH: Generalized Autoregressive Conditional Heteroskedasticity

For a log-return series r_t, let $a_t = r_t - \mu_t$ denote the innovation at time t.

$$a_t = \sigma_t \varepsilon_t, \qquad \sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i a_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2$$

where $\varepsilon_t \sim$ i.i.d.$(0,1)$, $\alpha_0 > 0$, $\alpha_i, \beta_j \geq 0$, and $\sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i) < 1$ ensures stationarity

Interpretation: Past shocks (a_{t-i}^2) and past volatility (σ_{t-j}^2) drive current volatility

ARCH reacts only to past shocks, while GARCH also accounts for past volatility - giving smoother and more realistic dynamics in financial returns

GARCH(1,1) Model

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Model:

$$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

Key Properties:

- **Stationarity:** holds if $0 \le \alpha_1, \beta_1 \le 1$ and $\alpha_1 + \beta_1 < 1$
- **Volatility clustering:** large shocks tend to be followed by large shocks
- **Heavy tails:**

$$\frac{\mathbb{E}(a_t^4)}{[\mathbb{E}(a_t^2)]^2} = \frac{3[1-(\alpha_1+\beta_1)^2]}{1-(\alpha_1+\beta_1)^2 - 2\alpha_1^2} > 3$$

- **1-step forecast:**

$$\sigma_t^2(1) = \alpha_0 + \alpha_1 a_t^2 + \beta_1 \sigma_t^2$$

Multi-Step Ahead Forecasts

For multi-step ahead forecasts, use $a_t^2 = \sigma_t^2 \epsilon_t^2$ and rewrite the model as

$$\sigma_{t+1}^2 = \alpha_0 + (\alpha_1 + \beta_1)\sigma_t^2 + \alpha_1 \sigma_t^2(\epsilon_t^2 - 1)$$

We have 2-step ahead volatility forecast

$$\sigma_t^2(2) = \alpha_0 + (\alpha_1 + \beta_1)\sigma_t^2(1)$$

In general, we have

$$\begin{aligned}
\sigma_t^2(\ell) &= \alpha_0 + (\alpha_1 + \beta_1)\sigma_t^2(\ell - 1), \quad \ell > 1 \\
&= \frac{\alpha_0[1 - (\alpha_1 + \beta_1)^{\ell-1}]}{1 - \alpha_1 - \beta_1} + (\alpha_1 + \beta_1)^{\ell-1}\sigma_t^2(1)
\end{aligned}$$

Therefore

$$\sigma_t^2(\ell) \to \frac{\alpha_0}{1 - \alpha_1 - \beta_1}, \quad \text{as } \ell \to \infty$$

Intel Example Revisited

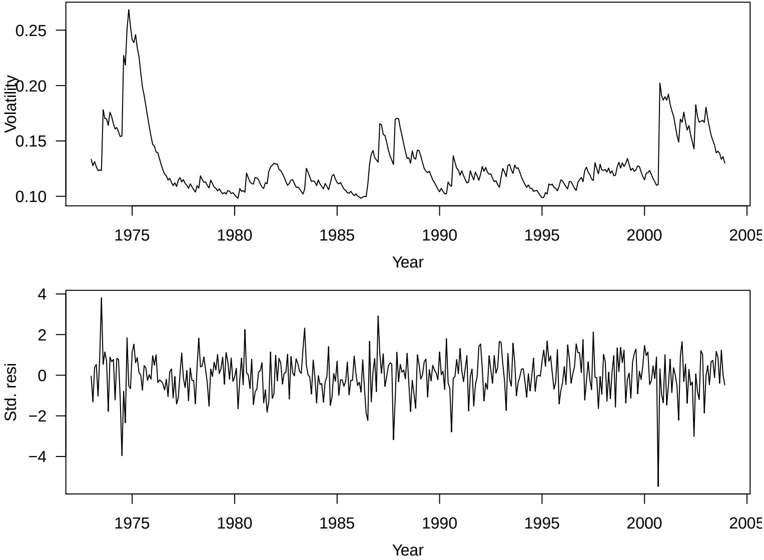
```
Error Analysis:
        Estimate  Std. Error  t value Pr(>|t|)
mu      0.0163276  0.0062624    2.607  0.00913 **
omega   0.0010918  0.0005291    2.063  0.03907 *
alpha1  0.0802716  0.0281162    2.855  0.00430 **
beta1   0.8553014  0.0461374   18.538  < 2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Log Likelihood:
 239.5189    normalized:  0.6438681

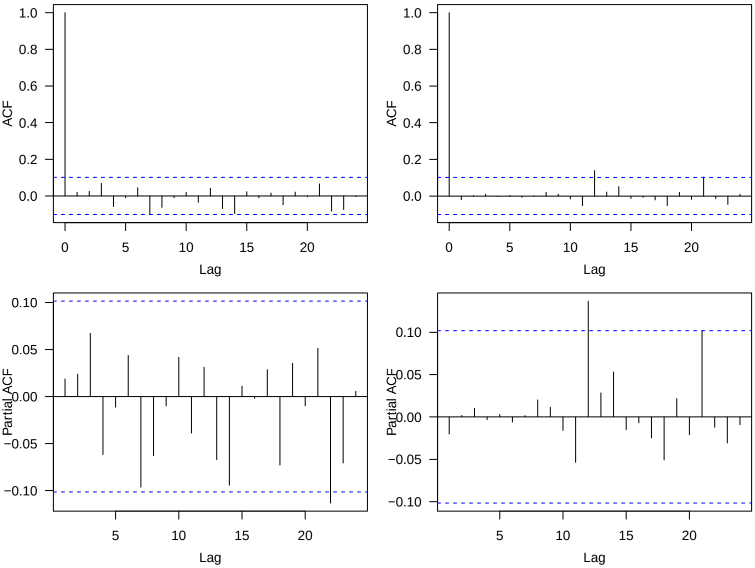
Description:
 Mon Oct 18 15:44:32 2021 by user:


Standardised Residuals Tests:
                                Statistic p-Value
 Jarque-Bera Test   R    Chi^2  156.5138  0
 Shapiro-Wilk Test  R    W      0.9676933 2.471139e-07
 Ljung-Box Test     R    Q(10)  9.805485  0.4577215
 Ljung-Box Test     R    Q(15)  16.54435  0.346824
 Ljung-Box Test     R    Q(20)  17.8005   0.6005484
 Ljung-Box Test     R^2  Q(10)  0.5130171 0.9999925
 Ljung-Box Test     R^2  Q(15)  10.24557  0.8040151
 Ljung-Box Test     R^2  Q(20)  11.77988  0.9234441
 LM Arch Test       R    TR^2   9.334459  0.6741288
```

Volatility Series and Standardized Residuals



GARCH Model Checking: ACF and PACF



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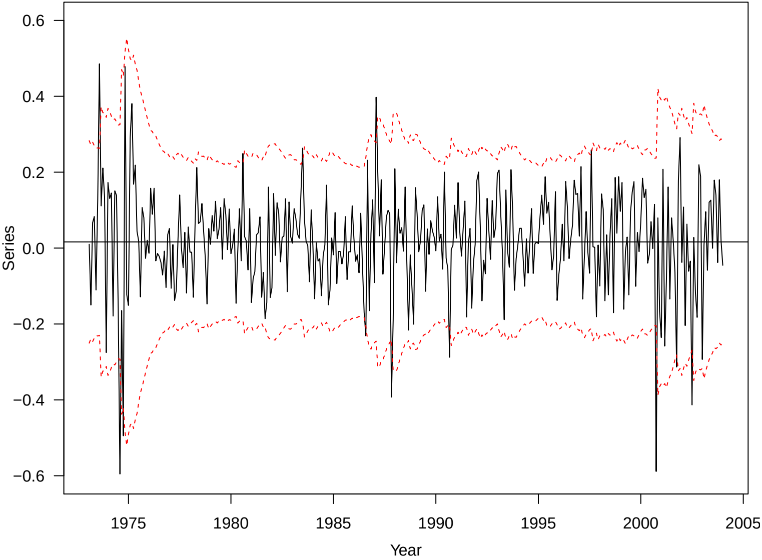
95% Pointwise Prediction Intervals



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The Integrated GARCH Model

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If the AR polynomial of the GARCH representation has unit root then we have an IGARCH model

An IGARCH(1, 1) model:

$$a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \beta_1 \sigma_{t-1}^2 + (1 - \beta_1) a_{t-1}^2$$

ℓ-step ahead forecasts

$$\sigma_t^2(\ell) = \sigma_t(1)^2 + (\ell - 1)\alpha_0, \quad \ell \geq 1$$

$\Rightarrow$ the effect of $\sigma_t^2(1)$ on future volatilities is persistent, and the volatility forecasts form a straight line with slope α_0

Exponential GARCH (EGARCH) Model [Nelson, 1991]

Motivation: EGARCH captures asymmetric volatility effects-negative shocks often increase volatility more than positive shocks of the same size.

Innovation function:

$$g(\varepsilon_t) = \theta\varepsilon_t + \gamma\big(|\varepsilon_t| - \mathbb{E}|\varepsilon_t|\big), \quad \text{where } \mathbb{E}[g(\varepsilon_t)] = 0$$

Interpretation:

- If $\gamma > 0$, negative returns ($\varepsilon_t < 0$) cause higher volatility than positive ones.
- The term $|\varepsilon_t| - \mathbb{E}|\varepsilon_t|$ ensures mean zero.

Asymmetric form:

$$g(\varepsilon_t) = \begin{cases} (\theta + \gamma)\varepsilon_t - \gamma\mathbb{E}|\varepsilon_t|, & \varepsilon_t \geq 0, \\ (\theta - \gamma)\varepsilon_t - \gamma\mathbb{E}|\varepsilon_t|, & \varepsilon_t < 0 \end{cases}$$

EGARCH(m, s) model:

$$a_t = \sigma_t\varepsilon_t, \qquad \log(\sigma_t^2) = \alpha_0 + \frac{1 + \beta_1 B + \cdots + \beta_{s-1}B^{s-1}}{1 - \alpha_1 B - \cdots - \alpha_m B^m} g(\varepsilon_{t-1})$$

EGARCH(1, 1) Model

Model:

$$a_t = \sigma_t \epsilon_t, \quad (1 - \alpha B) \log(\sigma_t^2) = (1 - \alpha)\alpha_0 + g(\epsilon_{t-1}),$$

where the ϵ_t are i.i.d. standard normal. In this case, $\mathbb{E}(|\epsilon_t|) = \sqrt{\frac{2}{\pi}}$ and the model for $\log(\sigma_t^2)$ becomes

$$(1-\alpha B) \log(\sigma_t^2) = \begin{cases} (1 - \alpha)\alpha_0 - \sqrt{\frac{2}{\pi}}\gamma + (\gamma + \theta)\epsilon_{t-1} & \text{if } \epsilon_{t-1} \geq 0, \\ (1 - \alpha)\alpha_0 - \sqrt{\frac{2}{\pi}}\gamma + (\gamma - \theta)(-\epsilon_{t-1}) & \text{if } \epsilon_{t-1} < 0 \end{cases}$$

Finally, we have

$$\sigma_t^2 = \sigma_{t-1}^{2\alpha} \exp\left((1 - \alpha)\alpha_0 - \sqrt{\frac{2}{\pi}}\gamma \right) \begin{cases} \exp\left[(\gamma + \theta) \frac{a_{t-1}}{\sigma_{t-1}} \right] & \text{if } a_{t-1} \geq 0, \\ \exp\left[(\gamma - \theta) \frac{|a_{t-1}|}{\sigma_{t-1}} \right] & \text{if } a_{t-1} < 0. \end{cases}$$

IBM Stock Example

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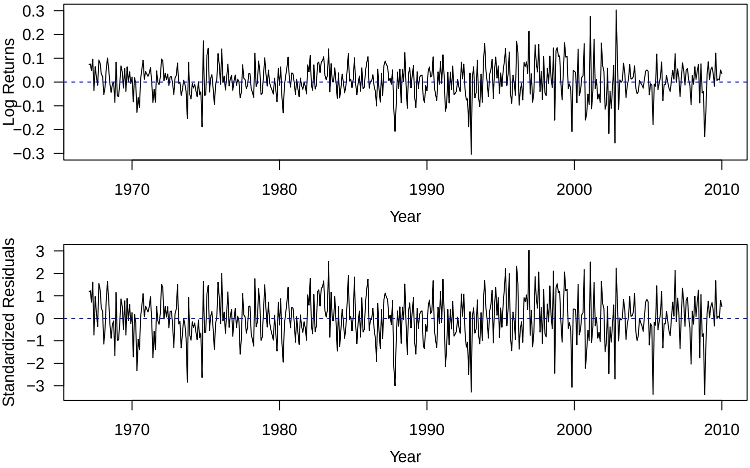
Stochastic Volatility Model

We consider the monthly log returns of IBM stock from January 1967 to December 2009



EGARCH(1, 1) Model Fit

$$r_t = 0.067 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim \mathrm{N}(0,1)$$

$$\log(\sigma_t^2) = -0.598 + 0.218(|\epsilon_{t-1}| - 0.423\epsilon_{t-1}) + 0.920\log(\sigma_{t-1}^2)$$

Therefore, we have

$$\sigma_t^2 = \sigma_{t-1}^{2\times 0.920} \exp(-0.598) \times \begin{cases} \exp(0.125) & \text{if } \epsilon_{t-1} \geq 0, \\ \exp(-0.310) & \text{if } \epsilon_{t-1} < 0. \end{cases}$$

For example, for a standardized shock with magnitude 2 (i.e., two standard deviations), we have

$$\frac{\sigma_t^2(\epsilon_{t-1} = -2)}{\sigma_t^2(\epsilon_{t-1} = 2)} = \frac{\exp\left(-0.31 \times (-2)\right)}{\exp\left(0.125 \times 2\right)} = e^{0.37} = 1.448$$

Therefore, the impact of a negative shock of size two standard deviations is about 44.8% higher than that of a positive shock of the same size

Background
ARCH Model
GARCH Model
IGARCH and EGARCH Models
Stochastic Volatility Model

Stochastic Volatility (SV) Model

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A simple stochastic volatility (SV) model assumes time-varying volatility driven by a latent process:

$$a_t = \sigma_t \varepsilon_t, \qquad (1 - \alpha_1 B - \cdots - \alpha_m B^m) \log(\sigma_t^2) = \alpha_0 + \nu_t$$

where:

- $\varepsilon_t \sim$ i.i.d. $N(0, 1)$ are shocks to returns
- $\nu_t \sim$ i.i.d. $N(0, \sigma_\nu^2)$ are shocks to volatility
- ε_t and ν_t are independent

Key idea: Volatility follows its own stochastic process rather than being a deterministic function of past shocks (as in GARCH)

Long-Memory Stochastic Volatility (LMSV) Model

Model:

$$a_t = \sigma_t \varepsilon_t, \quad \sigma_t = \sigma \exp(u_t/2), \quad (1-B)^d u_t = \eta_t, \quad 0 < d < 0.5$$

where $\varepsilon_t \sim N(0,1)$, $\eta_t \sim N(0,\sigma_\eta^2)$, and they are independent

By taking logs,

$$\begin{aligned}
\log(a_t^2) &= \log(\sigma^2) + u_t + \log(\varepsilon_t^2) \\
&= \left[\log(\sigma^2) + \mathbb{E}(\log(\varepsilon_t^2))\right] + u_t + \left[\log(\varepsilon_t^2) - \mathbb{E}(\log(\varepsilon_t^2))\right] \\
&= \mu + u_t + e_t
\end{aligned}$$

so $\log(a_t^2)$ is a Gaussian long-memory signal plus a non-Gaussian noise

Inference: Because σ_t (or u_t) is unobserved, the likelihood is intractable. Hence, estimation is typically done using **MCMC or particle filtering**

In the R session, the `stochvol` package uses MCMC sampling to approximate the posterior of volatility paths and parameters