Lecture 18

Inference for Proportions

Text: Chapter 10

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Whitney Huang
Clemson University

Inference for Categorical Data

In the next few lectures we will focus on categorical data analysis, i.e, statistical inference for categorical data

- Inference for a single proportion p
- Comparison of two proportions p_1 and p_2
- Inference for multi-category data and multivariate category data

Inference for a single proportion: Motivated Example

> Researchers in the development of new treatments for cancer patients often evaluate the effectiveness of new therapies by reporting the proportion of patients who survive for a specified period of time after completion of the treatment. A new genetic treatment of 870 patients with a particular type of cancer resulted in 330 patients surviving at least 5 years after treatment. Estimate the proportion of all patients with the specified type of cancer who would survive at least 5 years after being administered this treatment.

- Binary (two-category) outcomes: “success” & “failure”
- Similar to the inferential problem for μ, we would like to infer p, the population proportion of success $\Rightarrow$ point estimate, interval estimate, hypothesis testing

Point/Interval Estimation

- Point estimate:

$$\hat{p} = \frac{X(\text{\# of "successes"})}{n}$$

Recall: $X \sim \text{Bin}(n,p) \Rightarrow \mathbb{E}[\hat{p}] = \mathbb{E}[\frac{X}{n}] = \frac{1}{n}\mathbb{E}[X] = \frac{np}{n} = p$

- $100(1-\alpha)\%$ CI for p:

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{(\hat{p})(1-\hat{p})}{n}}$$

Why?

  - CLT approximation: $\hat{p} \approx \text{N}(p, \sigma^2_{\hat{p}})$ where n "sufficiently large" $\Rightarrow \min(np, n(1-p)) \geq 5$
  - $\sigma^2_{\hat{p}} = Var(\frac{X}{n}) = \frac{1}{n^2}Var(X) = \frac{1}{n^2}n(p)(1-p) = \frac{p(1-p)}{n}$

Motivated Example Revisited

A new genetic treatment of 870 patients with a particular type of cancer resulted in 330 patients surviving at least 5 years after treatment.

1. Estimate the proportion of all patients who would survive at least 5 years after being administered this treatment.
2. Construct a 95% CI for p

Another Example

Among 900 randomly selected registered voters nationwide, 63% of them are somewhat or very concerned about the spread of bird flu in the United States.

1. What is the point estimate for p, the proportion of U.S. voters who are concerned about the spread of bird flu?
2. Construct a 95% CI for p

Margin of Error & Sample Size Calculation

- Margin of error (ME):

$$z_{\alpha/2}\sqrt{\frac{n\hat{p}(1-\hat{p})}{n}}$$

$\Rightarrow$ CI for $p = \hat{p} \pm$ ME

- Sample size determination:

$$n = \frac{\tilde{p}(1-\tilde{p}) \times z_{\alpha/2}^2}{\text{ME}^2},$$

What value of $\tilde{p}$ to use?

  - An educated guess
  - A value from previous research
  - Use a pilot study
  - The "most conservative" choice is to use $\tilde{p} = 0.5$

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Example

A researcher wants to estimate the proportion of voters who will vote for candidate A. She wants to estimate to within 0.05 with 90% confidence.

1. How large a sample does she need if she thinks the true proportion is about .9?
2. How large a sample does she need if she thinks the true proportion is about .6?
3. How large a sample does she need if she wants to use the most conservative estimate?

Hypothesis Testing for p

1. State the null and alternative hypotheses:

$$H_0 : p = p_0 \text{ vs. } H_a : p > \text{ or } \neq \text{ or } < p_0$$

2. Compute the test statistic:

$$z_{obs} = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

3. Make the decision of the test:

   Rejection Region/ P-Value Methods

4. Draw the conclusion of the test:

   We (do/do not) have enough statistical evidence to conclude that (H_a in words) at α significant level.

Bird Flu Example Revisited

Among 900 randomly selected registered voters nationwide, 63% of them are somewhat or very concerned about the spread of bird flu in the United States. Conduct a hypothesis test at .01 level to assess the research hypothesis: $p > .6$.

Recap: Inference for p

- Point estimate:

$$\hat{p} = \frac{x}{n}$$

where x is the number of "successes" in a sample with sample size n, and the probability of success, p, is the parameter of interest

- $100(1-\alpha)\%$ confidence interval:

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{(\hat{p})(1-\hat{p})}{n}}$$

- Hypothesis Testing: $H_0 : p = p_0$ vs. $H_a : p >$ or $\neq$ or $< p_0$

$$z^* = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

Under $H_0 : p = p_0, \quad z^* \sim \mathrm{N}(0, 1)$

Another CI for p: Wilson Score Confidence Interval

- The actual coverage probability of $100(1-\alpha)\%$ CI $\hat{p} \pm z_{\alpha/2}\sqrt{\frac{(\hat{p})(1-\hat{p})}{n}}$ is usually falls below $(1-\alpha)$ ☹

- E.B. Wilson proposed one solution in 1927
  Idea: Solving $\frac{p-\hat{p}}{\sqrt{\frac{p(1-p)}{n}}} = \pm z_{\alpha/2}$ for p

$$\Rightarrow (p-\hat{p})^2 = z_{\alpha/2}^2 \frac{p(1-p)}{n}$$

Another CI for p: Wilson Score Confidence Interval

- The actual coverage probability of $100(1-\alpha)\%$ CI $\hat{p} \pm z_{\alpha/2}\sqrt{\frac{(\hat{p})(1-\hat{p})}{n}}$ is usually falls below $(1-\alpha)$ ☹

- E.B. Wilson proposed one solution in 1927
  Idea: Solving $\frac{p-\hat{p}}{\sqrt{\frac{p(1-p)}{n}}} = \pm z_{\alpha/2}$ for p

$$\Rightarrow (p-\hat{p})^2 = z^2_{\alpha/2}\frac{p(1-p)}{n}$$

$100(1-\alpha)\%$ Wilson Score Confidence Interval:

$$\frac{X + \frac{z^2_{\alpha/2}}{2}}{n + z^2_{\alpha/2}} \pm \frac{z_{\alpha/2}}{n + z^2_{\alpha/2}}\sqrt{\frac{X(n-X)}{n} + \frac{z^2_{\alpha/2}}{4}}$$

Example

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Suppose we would like to estimate p, the probability of being vegetarian (for all the CU student). We take a sample with sample size $n = 25$ and none of them are vegetarian (i.e., $x = 0$). Construct a 95% CI for p.

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Rule of Three: An Approximate 95% CI for p When $\hat{p} = 0$ or 1

When $\hat{p} = 0$, we have

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{(\hat{p})(1-\hat{p})}{n}} = 0 \pm z_{\alpha/2} \times 0 = (0, 0)$$

Similarly, when $\hat{p} = 1$, we have

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{(\hat{p})(1-\hat{p})}{n}} = 1 \pm z_{\alpha/2} \times 0 = (1, 1)$$

These Wald CIs degenerate to a point , which do not reflect the estimation uncertainty. Here we could apply the rule of three to approximate 95% CI:

$$\begin{array}{ll} (0, 3/n), & \text{if } \hat{p} = 0 \\ (1 - 3/n, 1), & \text{if } \hat{p} = 1 \end{array}$$

Comparing Two Population Proportions p_1 and p_2

- We often interested in comparing two groups, e.g., does a particular treatment increase the survival probability for cancer patients ?
- We would like to infer $p_1 - p_2$, the difference between two population proportions $\Rightarrow$ point estimate, interval estimate, hypothesis testing

Notation

- Parameters
  - p_1, p_2: population proportions
  - $p_1 - p_2$: the difference between two population proportions
- Sample Statistics
  - n_1, n_2: sample sizes
  - $\hat{p}_1 = \frac{x_1}{n_1}, \hat{p}_2 = \frac{x_2}{n_2}$: sample proportions

$$\Rightarrow \hat{p}_1 - \hat{p}_2 = \frac{x_1}{n_1} - \frac{x_2}{n_2}$$

$$se(\hat{p}_1 - \hat{p}_2) = \sqrt{\frac{(\hat{p}_1)(1 - \hat{p}_1)}{n_1} + \frac{(\hat{p}_2)(1 - \hat{p}_2)}{n_2}}$$

Point/Interval Estimation for $p_1 - p_2$

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- Point estimate:

$$\hat{p}_1 - \hat{p}_2 = \frac{X_1}{n_1} - \frac{X_2}{n_2}$$

- $100(1-\alpha)\%$ CI based on CLT:

$$\hat{p}_1 - \hat{p}_2 \pm z_{\alpha/2}\sqrt{\frac{(\hat{p}_1)(1-\hat{p}_1)}{n_1} + \frac{(\hat{p}_2)(1-\hat{p}_2)}{n_2}}$$

Hypothesis Testing for $p_1 - p_2$

1. State the null and alternative hypotheses:

$$H_0 : p_1 - p_2 = 0 \text{ vs. } H_a : p_1 - p_2 > \text{ or } \neq \text{ or } < 0$$

2. Compute the test statistic:

$$z_{obs} = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\frac{\bar{p}(1-\bar{p})}{n_1} + \frac{\bar{p}(1-\bar{p})}{n_2}}},$$

where $\bar{p} = \frac{x_1+x_2}{n_1+n_2}$

3. Make the decision of the test:

Rejection Region/ P-Value Methods

4. Draw the conclusion of the test:

We (do/do not) have enough statistical evidence to conclude that (H_a in words) at α% significant level.

Example

A Simple Random Simple of 100 CU graduate students is taken and it is found that 79 “strongly agree” that they would recommend their current graduate program. A Simple Random Simple of 85 USC graduate students is taken and it is found that 52 “strongly agree” that they would recommend their current graduate program. At 5 % level, can we conclude that the proportion of “strongly agree” is higher at CU?

Summary

In this lecture, we learned statistical inference for population proportion p:

- Point estimate
- Interval estimate
- Hypothesis testing

In next lecture we will learn statistical inference for multi-category data and bivariate categorical data