Lecture 11

Spectral Analysis of Time Series I

Readings: CC08 Chapter 13-14; BD16 Ch 4; SS17 Chapter 4.1-4.4

MATH 8090 Time Series Analysis
Week 11

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Agenda

Time vs. Frequency: Two Views of the Same Process

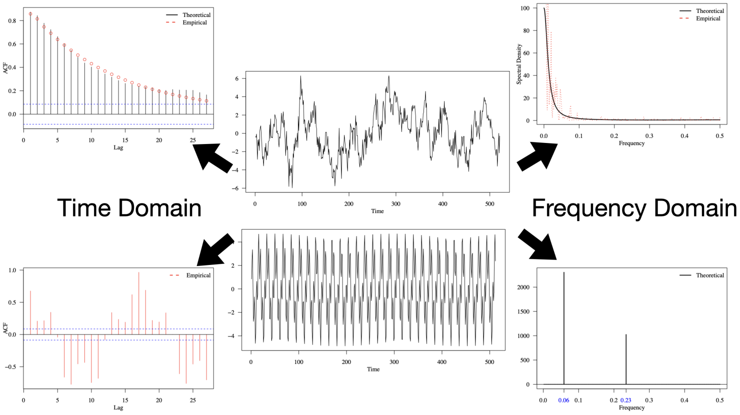
Time domain (Box & Jenkins, 1970)

- **Example (AR(1)):** $Y_t = \phi Y_{t-1} + Z_t, \quad |\phi| < 1,$ $Z_t \sim \mathrm{WN}(0, \sigma^2)$
- **Idea:** Predict the present from the past
- **Physical view:** Dynamics in *state variables*: position → velocity → acceleration; cause → effect in time
- **Good for:** Shocks, transients, local dependencies
- **Asks:** "How does the system evolve through time?"

Frequency domain (Priestley, 1981)

- **Example (har-regression):** $Y_t = \alpha_0 +$ $\sum_{j=1}^{p} \left[\alpha_{1j} \cos(\omega_j^* t) + \alpha_{2j} \sin(\omega_j^* t)\right]$
- **Idea:** Decompose into sines/cosines (*frequency*)
- **Physical view:** *Energy at resonant frequencies* (oscillators ringing at certain rates)
- **Good for:** Cycles, seasonality, filtering
- **Asks:** "Which time scales dominate the variability?"

From Autocorrelation to Spectrum: Dual Representations of Time Series



Waves as Building Blocks of Cyclic Time Series

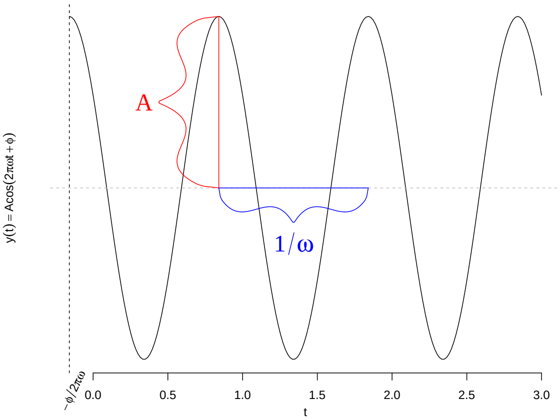
The simplest case is the cosine wave

$$\begin{aligned} Y_t &= A\cos(2\pi\omega t + \phi) \\ &= \alpha_1 \cos(2\pi\omega t) + \alpha_2 \sin(2\pi\omega t), \end{aligned}$$

$\alpha_1 = A\cos(\phi)$, $\alpha_2 = -A\sin(\phi)$, $A = \sqrt{\alpha_1^2 + \alpha_2^2}$, $\phi = \tan^{-1}\frac{-\alpha_2}{\alpha_1}$
where

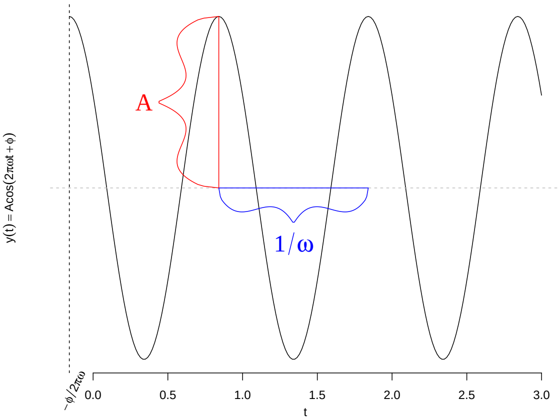
- A is amplitude: the height of the wave
- ω is frequency: how many cycles per unit time
- ϕ is phase: the start point of the cosine function

Graphical Illustration of the Cosine Wave



- Changing amplitude stretches vertically

Graphical Illustration of the Cosine Wave



- Changing amplitude stretches vertically
- Changing frequency compresses horizontally

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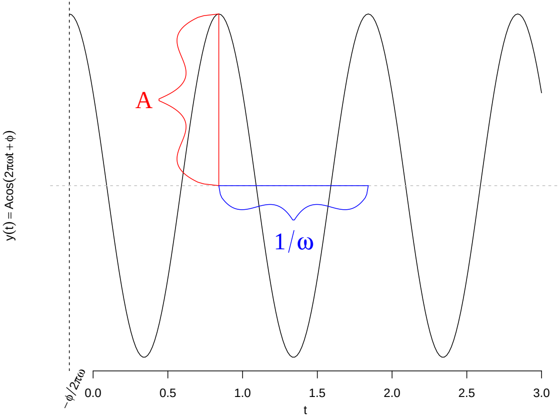
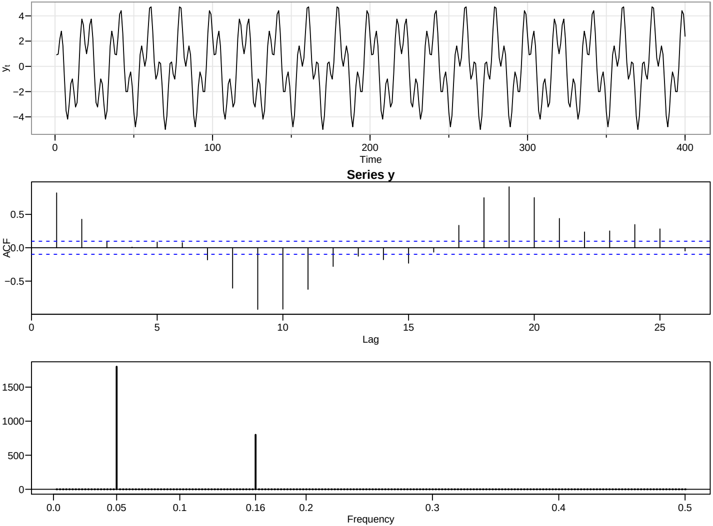
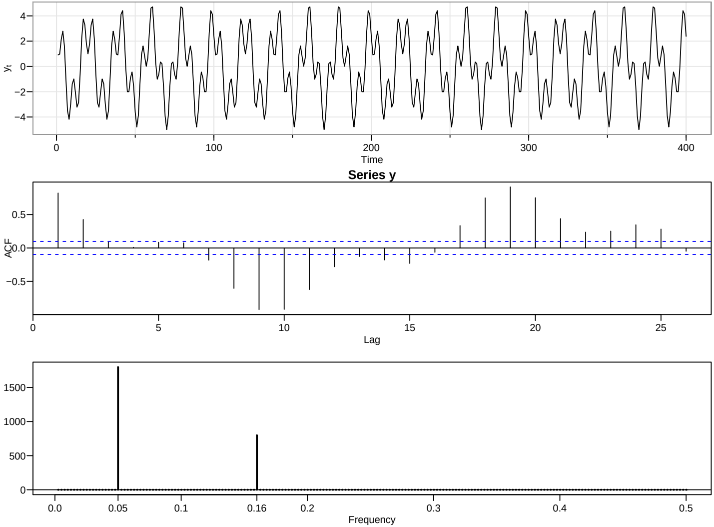


- Changing amplitude stretches vertically
- Changing frequency compresses horizontally
- Changing phase shifts horizontally

Searching Hidden Periodicities

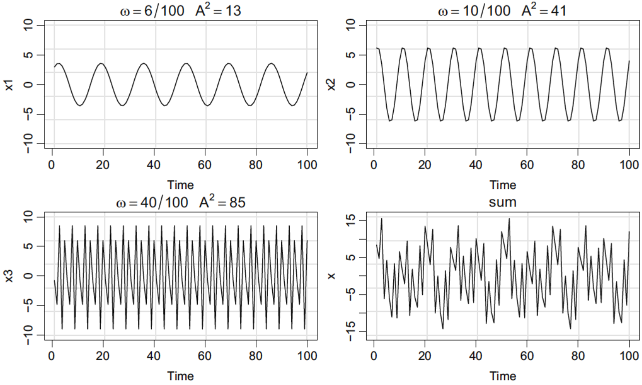


Searching Hidden Periodicities



$$y_t = 3\cos\left(2\pi(\frac{10}{200})t\right) + 2\cos\left(2\pi(\frac{32}{200}t + 0.3)\right)$$

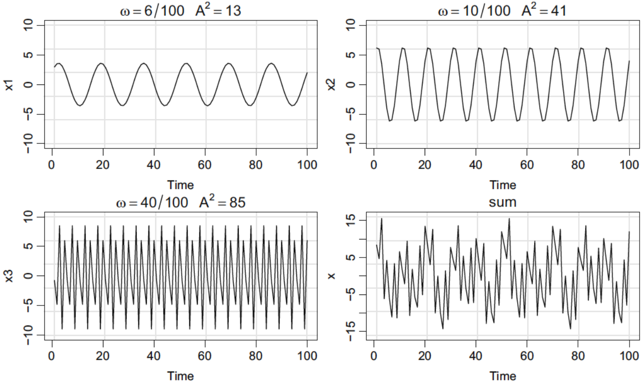
Examples of Periodic Time Series



Source: Fig. 4.1. of Shumway and Stoffer, 2017

- Combining sine and cosine waves with different frequencies and amplitudes can produce complex periodic series

Examples of Periodic Time Series



Source: Fig. 4.1. of Shumway and Stoffer, 2017

- Combining sine and cosine waves with different frequencies and amplitudes can produce complex periodic series
- With a continuous spectrum (infinitely many frequencies), we can represent any stationary time series

Folding Frequency and Aliasing

Let's consider $Y_{1,t} = \cos(2\pi(0.2)t)$ and $Y_{2,t} = \cos(2\pi(1.2)t)$

- At $t = 1$, $Y_{1,t} = \cos(0.4\pi t)$,
  $Y_{2,t} = \cos(2.4\pi t) = \cos(2\pi t + 0.4\pi t) = \cos(0.4\pi t) = Y_{1,t}$
- This is true for all integer values of t

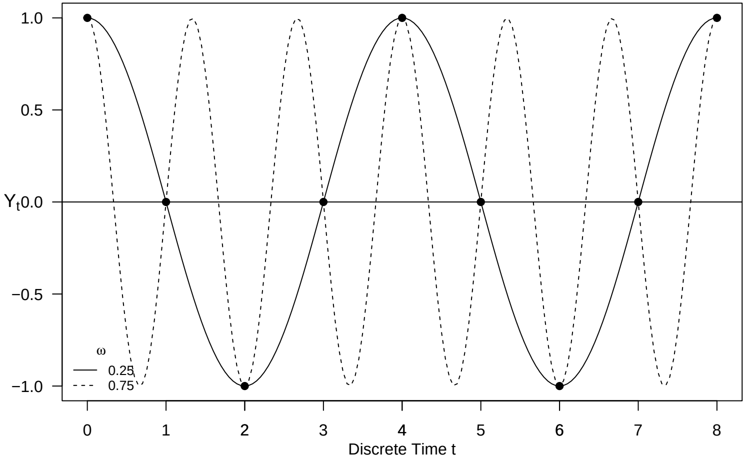
  $\Rightarrow \omega = 1.2$ is an alias of $\omega = 0.2$.

  In general, all frequencies higher than $\omega = \frac{1}{2}$ have an alias in $0 \leq \omega \leq \frac{1}{2}$
- $\omega = \frac{1}{2}$ is the folding frequency (aka Nyquist frequency), because the shortest period that can be observed is $\frac{1}{\omega} = 2$.

Takeaway: It suffices to limit attention to $\omega \in [0, \frac{1}{2}]$

Illustration of Aliasing

$\omega = 0.25$ and $\omega = 0.75$ are aliased with one another



A Cosine Wave With Random Phase

If

$$\begin{aligned} Y_t &= A\cos(2\pi\omega t + \phi) \\ &= \alpha_1 \cos(2\pi\omega t) + \alpha_2 \sin(2\pi\omega t), \end{aligned}$$

and ϕ is random, uniformly distribiuted on $[-\pi, \pi)$, then:

$$\begin{aligned} \mathbb{E}(Y_t) &= 0 \\ \mathbb{E}(Y_{t+h}Y_t) &= \frac{1}{2}A^2\cos(2\pi\omega h) \end{aligned}$$

$\Rightarrow Y_t$ is weakly stationary

A Cosine Wave With Random Phase (Continued)

Also

$$\mathbb{E}(\alpha_1) = \mathbb{E}(\alpha_2) = 0,$$
$$\mathbb{E}(\alpha_1^2) = \mathbb{E}(\alpha_2^2) = \frac{1}{2}A^2,$$

and $\mathbb{E}(\alpha_1\alpha_2) = 0.$

Alternatively, if the α's have these properties, then Y_t is stationary with the same mean and autocovariances:

$$\mathbb{E}(Y_t) = 0,$$
$$\mathbb{E}(Y_{t+h}Y_t) = \frac{1}{2}A^2\cos(2\pi\omega h).$$

Representing a Periodic Process as Multiple Sines/Cosines

More generally, if $Y_t = \sum_{k=1}^{K} \left[\alpha_{k,1} \cos(2\pi\omega_k t) + \alpha_{k,2} \sin(2\pi\omega_k t)\right]$, where:

- The α's are uncorrelated with zero mean;
- $\mathrm{Var}(\alpha_{k,1}) = \mathrm{Var}(\alpha_{k,2}) = \sigma_k^2$;

then Y_t is stationary with zero mean and autocovariances

$$\gamma(h) = \sum_{k=1}^{K} \sigma_k^2 \cos(2\pi\omega_k h)$$

$$\Rightarrow \gamma(0) = \mathrm{Var}(Y_t) = \sum_{k=1}^{K} \sigma_k^2$$

As $K \to \infty$, the discrete sum turns into a process with variance continuously distributed across frequencies $\Rightarrow$ spectral density function

From Autocovariance to Spectrum

Every weakly stationary time series Y_t with $\gamma(h)$ has a way to describe how its variance is distributed over frequencies

$$\gamma(h) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i\omega h} dF(\omega) = 2\int_{0}^{\frac{1}{2}} \cos(2\pi\omega h)\, dF(\omega),$$

where $F(\omega)$ analogous to a CDF, describing the accumulated variance up to frequency ω. If $F(\omega)$ is *absolutely continuous*, it admits a spectral density function $f(\omega) = F^{'}(\omega)$, and

$$\gamma(h) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i\omega h} f(\omega)\, d\omega = 2\int_{0}^{\frac{1}{2}} \cos(2\pi\omega h) f(\omega)\, d\omega$$

> The autocovariance and the spectral distribution function contain the same information

From Autocovariance to Spectrum (Cont'd)

Under the absolute-summability condition $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$ $f(\omega)$ exists and can be written as the (uniformly convergent) Fourier series

$$f(\omega) = \sum_{h=-\infty}^{\infty} \gamma(h)\, e^{-2\pi i \omega h} = \gamma(0) + 2\sum_{h=1}^{\infty} \gamma(h)\, \cos(2\pi\omega h), \quad \omega \in \mathbb{R}.$$

Properties of the spectral density:

- $f(\omega) \geq 0$ for all ω (nonnegativity)
- $f(-\omega) = f(\omega)$ (evenness)
- $f(\omega + 1) = f(\omega)$ (1-periodicity in frequency)
- $\int_{-1/2}^{1/2} f(\omega)\, d\omega = \gamma(0) < \infty$ (total variance)

Example: White Noise

For white noise $\{Z_t\}$, we have seen that $\gamma(0) = \sigma_Z^2$ and $\gamma(h) = 0$ for $h \neq 0$. Thus,

$$\begin{aligned} f(\omega) &= \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \omega h} \\ &= \gamma(0) = \sigma_Z^2 \end{aligned}$$

That is, the spectral density is constant across all frequencies: each frequency in the spectrum contributes equally to the variance.

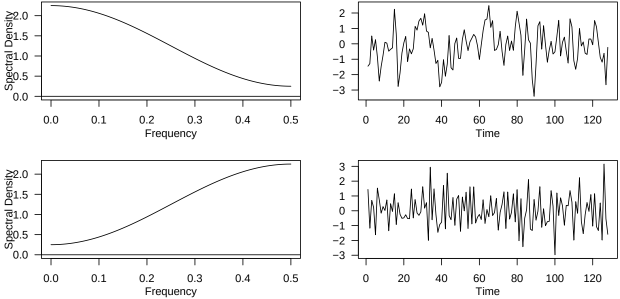
This is the origin of the name *white noise*: it is like white light, which is a uniform mixture of all frequencies in the visible spectrum

Examples: MA(1)

An MA(1) process $Y_t = \theta Z_{t-1} + Z_t$ is a simple filtering of white noise. Therefore, the (power) transfer function of the MA filter ($Y_t = \psi(B)Z_t$ with $\psi(z) = 1 + \theta z$) is given by:

$$\begin{aligned} |1 + \theta e^{-2\pi i\omega}|^2 &= (1 + \theta e^{-2\pi i\omega})(1 + \theta e^{2\pi i\omega}) \\ &= 1 + \theta^2 + \theta(e^{2\pi i\omega} + e^{-2\pi i\omega}) \\ &= 1 + \theta^2 + 2\theta\cos(2\pi\omega). \end{aligned}$$

Thus, we have: $f(\omega) = \left[1 + \theta^2 + 2\theta\cos(2\pi\omega)\right]\sigma_Z^2$



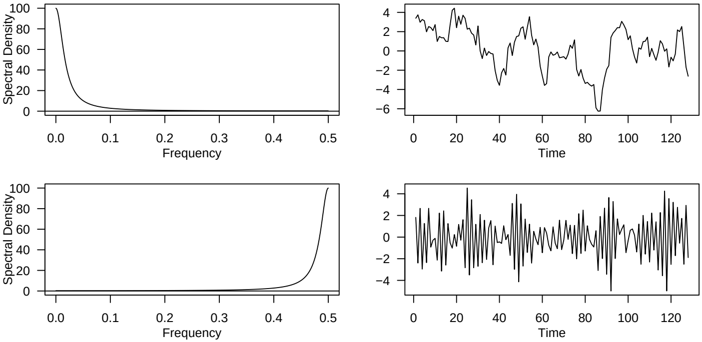
Example: AR(1)

For an AR(1) process $Y_t = \phi Y_{t-1} + Z_t$, we can write $(1 - \phi B) Y_t = Z_t \Rightarrow \psi(B) = \frac{1}{1-\phi B}$. Then we have

$$\left[1 + \phi^2 - 2\phi \cos(2\pi\omega)\right] f(\omega) = \sigma_Z^2$$

Thus,

$$f(\omega) = \frac{\sigma_Z^2}{1 + \phi^2 - 2\phi \cos(2\pi\omega)}$$



Examples: ARMA(p, q)

- For an ARMA(p, q) process:

$$\phi(B)Y_t = \theta(B)Z_t,$$

where $\phi(B)$ and $\theta(B)$ are the AR and MA polynomials, respectively.

Using the *linear filtering* result, the spectral density is:

$$f(\omega) = \sigma_Z^2 \frac{|\theta(e^{-2\pi i\omega})|^2}{|\phi(e^{-2\pi i\omega})|^2}.$$

- **Interpretation:** The spectral density shows how the variance of Y_t is distributed across frequencies. The AR and MA polynomials determine this distribution:
  - $\phi(\cdot)$: controls how strongly the series smooths or amplifies low vs. high frequencies (persistence)
  - $\theta(\cdot)$: determines short-term fluctuations or fine-scale noise

From Spectral Density to the Periodogram

Any finite time series sample $y_1, y_2, \cdots, y_n$ can be expressed as

$$y_t = \alpha_0 + \sum_{j=1}^{(n-1)/2} \left[\alpha_j \cos\left(\frac{2\pi j t}{n}\right) + \beta_j \sin\left(\frac{2\pi j t}{n}\right) \right],$$

(with one extra term if n is even)

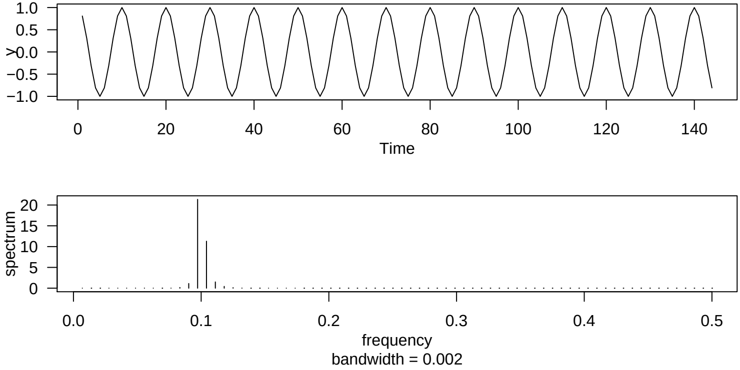
- The coefficients α_j and β_j measure how much of each frequency is present in the data
- The (scaled) periodogram

  $$P(j/n) = \alpha_j^2 + \beta_j^2$$

  estimates how variance is distributed across frequencies - a sample analogue of the spectral density
- In `R`: use `spectrum()` to compute and plot $P(j/n)$

An Example: $Y_t = \cos(2\pi(0.1)t)$

```
y = cos(2 * pi * (0.1) * (1:144))
ts.plot(y)
spectrum(y, log = "no", main = "", type = "h")
```



Power around 0.1 shows the true frequency, while nearby spikes reflect spectral leakage-we'll revisit this later

Discrete Fourier Transform (DFT) and Periodogram

- The DFT transforms a time series $y_0, \ldots, y_{n-1}$ into its frequency-domain representation:

$$d(\omega_j) = \frac{1}{\sqrt{n}} \sum_{t=0}^{n-1} y_t e^{-2\pi i \omega_j t}, \quad \omega_j = \frac{j}{n},\ j = 0, \ldots, n-1.$$

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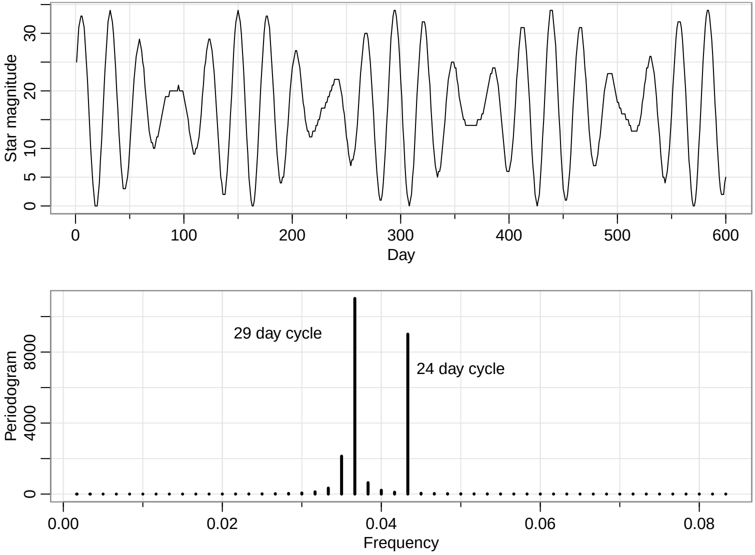
$$I(\omega_j) \propto |d(\omega_j)|^2.$$

- Alternatively, in the time domain:

$$I(\omega_j) = \hat{\gamma}(0) + 2 \sum_{h=1}^{n-1} \hat{\gamma}(h) \cos(2\pi\omega_j h),$$

where $\hat{\gamma}(h)$ is the sample autocovariance at lag h

Example: Detecting Periodicities in Star Brightness [Example 4.3, SS17]



Estimating Spectral Density Using Periodogram

If n is large

$$
\begin{aligned}
\mathbb{E}\left[I(\omega_j)\right] &\approx \sum_{h=-(n-1)}^{n-1} \gamma(h) e^{-2\pi i \omega_j h} \\
&\approx \sum_{-\infty}^{\infty} \gamma(h) e^{-2\pi \omega_j h} = \gamma(0) + 2\sum_{h=1}^{\infty} \gamma(h)\cos(2\pi\omega_j h) \\
&= f(\omega_j) \quad ☺
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- Conversely, the periodogram can be used as an estimator of the spectral density

Estimating Spectral Density Using Periodogram

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\end{aligned}
$$

- Heuristically, the spectral density is the approximate expected value of the periodogram
- Conversely, the periodogram can be used as an estimator of the spectral density
- But the periodogram values have only two degrees of freedom each, which makes it a poor estimate ☹

DFT and Two Degrees of Freedom

For $j = 0, 1, \cdots, n-1$

$$\begin{aligned} d(\omega_j) &= n^{-\frac{1}{2}} \sum_{t=1}^{n} y_t e^{-2\pi i \omega_j t} \\ &= n^{-\frac{1}{2}} \sum_{t=1}^{n} y_t \cos(2\pi\omega_j t) - i \times n^{-\frac{1}{2}} \sum_{t=1}^{n} y_t \sin(2\pi\omega_j t) \\ &= d_{\cos}(\omega_j) - i \times d_{\sin}(\omega_j). \end{aligned}$$

- $d_{\cos}(\omega_j)$ and $d_{\cos}(\omega_j)$ are the cosine transform and sine transform, respectively, of $y_1, y_2, \cdots, y_n$
- The periodogram is $I(\omega_j) = d_{\cos}(\omega_j)^2 + d_{\sin}(\omega_j)^2$

Sampling Properties of the Periodogram

For convenience, suppose that n is odd: $n = 2m + 1$

- White noise: orthogonality properties of sines and cosines mean that $d_{\cos}(\omega_1), d_{\sin}(\omega_1), d_{\cos}(\omega_2), d_{\sin}(\omega_2), \cdots, d_{\cos}(\omega_m), d_{\sin}(\omega_m)$ have zero mean, variance $\frac{\sigma_Z^2}{2}$, and uncorrelated

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- Gaussian white noise:
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  are $\text{i.i.d.}\, \text{N}(0, \frac{\sigma_Z^2}{2})$

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- Gaussian white noise: $d_{\cos}(\omega_1), d_{\sin}(\omega_1), d_{\cos}(\omega_2), d_{\sin}(\omega_2), \cdots, d_{\cos}(\omega_m), d_{\sin}(\omega_m)$ are $\text{i.i.d.}\, \text{N}(0, \frac{\sigma_Z^2}{2})$
- So for Gaussian white noise

$$I(\omega_j) \sim \frac{\sigma_Z^2}{2} \times \chi_2^2$$

Sampling Distributions: General Case

General case:

$d_{\cos}(\omega_1), d_{\sin}(\omega_1), d_{\cos}(\omega_2), d_{\sin}(\omega_2), \cdots, d_{\cos}(\omega_m), d_{\sin}(\omega_m)$, have zero mean and are approximately uncorrelated, and

$$\mathrm{Var}\left[d_{\cos}(\omega_j)\right] \approx \mathrm{Var}\left[d_{\sin}(\omega_j)\right] \approx \frac{1}{2} f(\omega_j),$$

where $f(\omega_j)$ is the spectral density function

If Y_t is Gaussian,

$$\frac{I(\omega_j)}{\frac{1}{2} f(\omega_j)} = \frac{d_{\cos}(\omega_j)^2 + d_{\sin}(\omega_j)^2}{\frac{1}{2} f(\omega_j)} \approx \chi_2^2,$$

and $I(\omega_1), I(\omega_2), \cdots, I(\omega_m)$ are approximately independent

The periodogram is not a consistent estimator!

Nonparametric Spectrum Estimates

From previous slide:

$$\frac{I(\omega_j)}{\frac{1}{2}f(\omega_j)} = \frac{d_{\cos}(\omega_j)^2 + d_{\sin}(\omega_j)^2}{\frac{1}{2}f(\omega_j)} \approx \chi_2^2,$$

and $I(\omega_1), I(\omega_2), \cdots, I(\omega_m)$ are approximately independent

Problem: $I(\omega_j)$ is an approximately unbiased estimator of $f(\omega_j)$ but with too few degrees of freedom (df = 2) to be useful. Specifically, $I(\omega) \dot{\sim} \frac{1}{2}f(\omega)\chi_2^2$, which implies

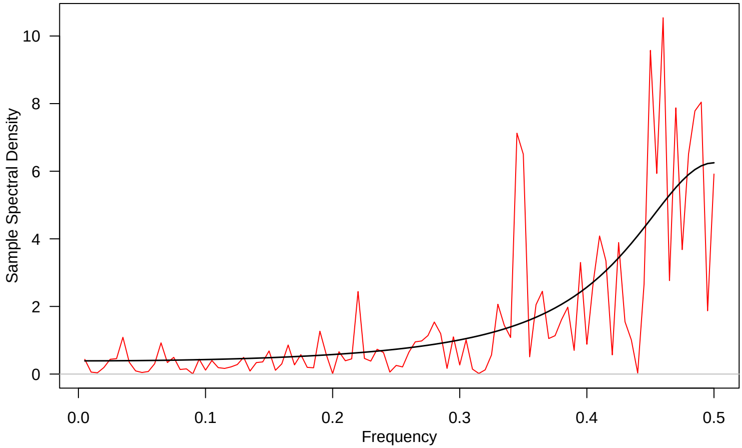
$$\mathbb{E}[I(\omega)] \approx f(\omega)$$

and

$$\mathrm{Var}[I(\omega)] \approx f^2(\omega)$$

Consequently, $\mathrm{Var}[I(\omega)] \overset{n\to\infty}{\neq} 0$ and thus the periodogram is not a consistent estimator of the spectral density

Smoothing the Periodogram

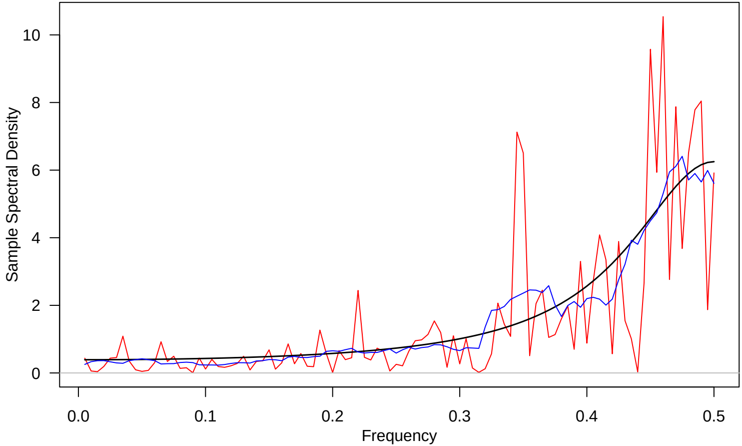


Main idea: “average” the values of the periodogram over “small” intervals of frequencies to reduce the estimation variability

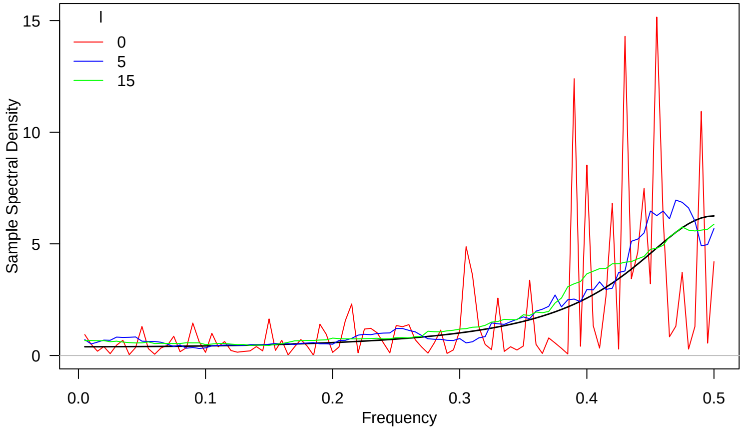
Averaged Periodogram [Daniell Spectral Window]

Use the band $[\omega_{j-l}, \omega_{j+l}]$ containing $L = 2l + 1$ Fourier frequencies:

$$\bar{f}(\omega_j) = \frac{1}{L} \sum_{k=-l}^{l} I(\omega_{j+k})$$



Tuning Parameter: l



A large l can effectively reduce the estimation variability but can also introduce bias

Bias and Variance

Let's assume the true spectral density does not change much locally, then a Taylor expansion produces

$$
\begin{aligned}
\mathbb{E}[\bar{f}(\omega)] &\approx \sum_{k=-l}^{l} W_l(k) f(\omega + \frac{k}{n}) \\
&\approx \sum_{k=-l}^{l} W_l(k) \left[f(\omega) + \frac{k}{n} f^{'}(\omega) + \frac{1}{2} (\frac{k}{n})^2 f^{''}(\omega) \right] \\
&\approx f(\omega) + \frac{1}{n^2} \frac{f^{''}(\omega)}{2} \sum_{k=-l}^{l} k^2 W_l(k)
\end{aligned}
$$

$$
\text{Bias} \approx \frac{1}{n^2} \frac{f^{''}(\omega)}{2} \sum_{k=-l}^{l} k^2 W_l(k)
$$

$$
\text{Variance} \approx f^2(\omega) \sum_{k=-l}^{l} W_l^2(k)
$$

Example: for Daniell rectangular spectral window, we have bias $= \frac{f^{''}(\omega)}{2n^2} \frac{l(l+1)}{3}$ and variance $\frac{f^2(\omega)}{2l+1}$

Pointwise Confidence Intervals for $f(\omega)$

The distribution of $\frac{\nu \bar{f}(\omega)}{f(\omega)}$ can be approximated by $\chi^2_{df=\nu}$, where

$$\nu = \frac{2}{\sum_{k=-l}^{l} W_l^2(k)}$$

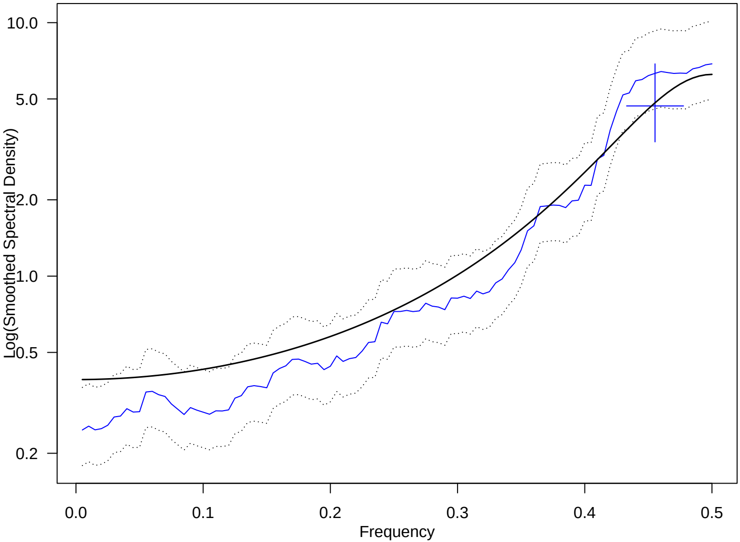
$\Rightarrow 100(1-\alpha)\%$ CI for $f(\omega)$

$$\frac{\nu \bar{f}(\omega)}{\chi^2_{df=\nu,1-\frac{\alpha}{2}}} < f(\omega) < \frac{\nu \bar{f}(\omega)}{\chi^2_{df=\nu,\frac{\alpha}{2}}}$$

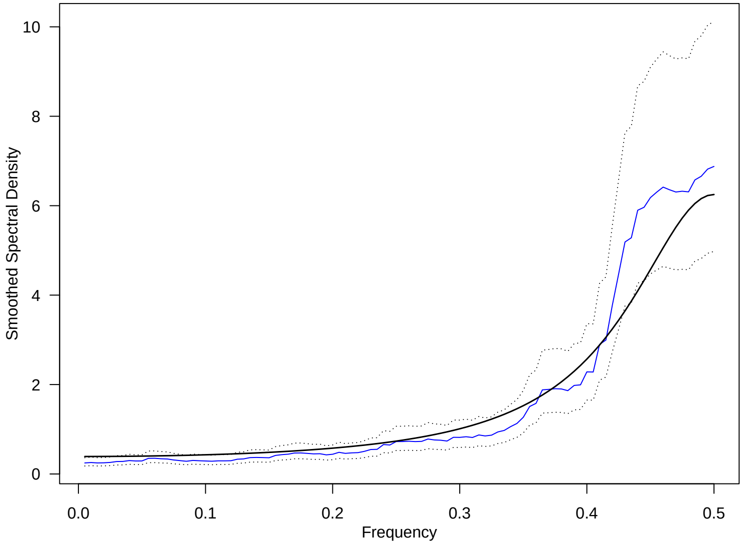
Taking logs we obtain an interval for the logged spectrum:

$$\log[\bar{f}(\omega)] + \log\left[\frac{\nu}{\chi^2_{\nu,1-\frac{\alpha}{2}}}\right] < \log[f(\omega)] < \log[\bar{f}(\omega)] + \log\left[\frac{\nu}{\chi^2_{\nu,\frac{\alpha}{2}}}\right]$$

Pointwise Confidence Intervals for $f(\omega)$: Log-Scale



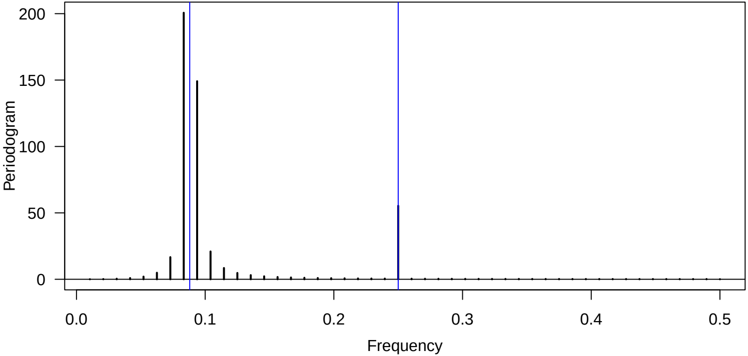
Pointwise Confidence Intervals for $f(\omega)$: Original Scale



Spectral Leakage

Much of the previous discussion has assumed that the frequencies of interest are the Fourier frequencies, i.e., $\omega_j = \frac{j}{n}$. What happens if that is not the case?

Example: $Y_t = 3\cos(2\pi(0.088)t) + \sin(2\pi(\frac{24}{96})t), \quad t = 1, \cdots, 96$

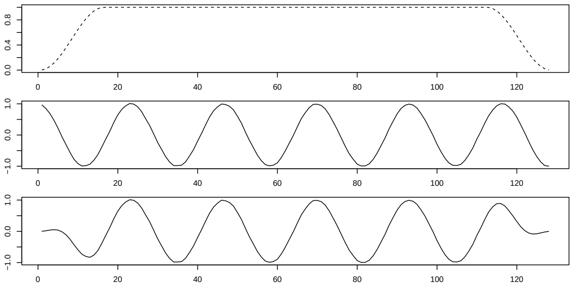


Power at non-Fourier frequencies will leak into the nearby Fourier frequencies

Tapering

Tapering is one method used to alleviate the issue of spectral leakage, where power at non-Fourier frequencies leak into the nearby Fourier frequencies

Main idea: replace the original series by the tapered series, i.e., $\tilde{y}_t = h_t y_t$. Tapers h_t's generally have a shape that enhances the center of the data relative to the extremities to reduce the end effects of computing a Fourier transform on a series of finite length



Effect of Tapering