Lecture 3

Stationary processes

References: CC08 Chapter 2 & Chapter 4.1-4.3; BD16 Chapter 1.3-1.6; SS17 Chapter 1.2-1.6

MATH 8090 Time Series Analysis
Week 3

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Agenda

1. Mean and Autocovaraince Functions
2. Stationarity
3. Some Examples of Stationary Processes
4. Estimation of Mean and Autocovariance Functions

Review: The Additive Decomposition

- The additive model for a time series $\{Y_t\}$ is

$$Y_t = \mu_t + s_t + \eta_t,$$

where

  - μ_t is the trend component
  - s_t is the seasonal component
  - η_t is the random (noise) component with $\mathbb{E}(\eta_t) = 0$

- Standard procedure:

  (1) Estimate/remove the trend and seasonal components

  (2) Analyze the remainder, the residuals $\hat{\eta}_t = y_t - \hat{\mu}_t - \hat{s}_t$

- We will focus on (2) for the next few weeks

Time Series Models

- A time series model is a specification of the probabilistic distribution of a sequence of random variables (RVs) η_t

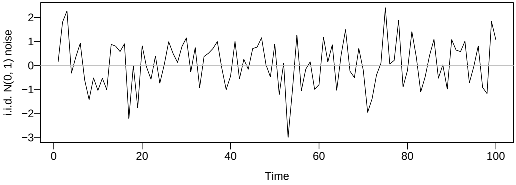
  (The observed time series is a realization of such a sequence of random variables)

- The simplest time series is i.i.d. (*independent and identically distributed*) noise
  - $\{\eta_t\}$ is a sequence of independent and identically distributed zero-mean (i.e., $\mathbb{E}(\eta_t) = 0, \forall t$) random variables $\Rightarrow$ no temporal dependence
  - It is of little value of using i.i.d. noise model to conduct forecast as there is no information from the past observations
  - **But**, we will use i.i.d. model as a building block to develop time series models that can accommodate time dependence

Example Realizations of i.i.d. Noise

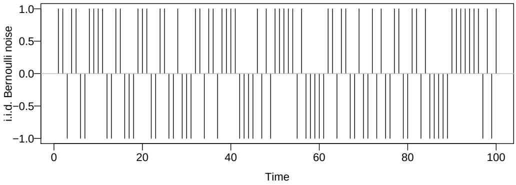
- Gaussian (normal) i.i.d. noise with mean 0 and variance $\sigma^2 > 0$

$$f(\eta_t) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{\eta_t^2}{2\sigma^2})$$



- Bernoulli i.i.d. noise with "success" probability

$$\mathbb{P}(\eta_t = 1) = p = 1 - \mathbb{P}(\eta_t = -1)$$



Means and Autocovaraince Functions

A time series model could also be a specification of the means and autocovariances of the RVs

- The mean function of $\{\eta_t\}$ is

$$\mu_t = \mathbb{E}(\eta_t).$$

- μ_t is the population mean at time t, which can be computed as:

$$\mu_t = \begin{cases} \int_{-\infty}^{\infty} \eta_t f(\eta_t)\, d\eta_t & \text{when } \eta_t \text{ is a continuous RV;} \\ \sum_{-\infty}^{\infty} \eta_t p(\eta_t), & \text{when } \eta_t \text{ is a discrete RV,} \end{cases}$$

where $f(\cdot)$ and $p(\cdot)$ are the probability density function and probability mass function of η_t, respectively

Examples of Mean Functions

- **Example 1**: What is the mean function for $\{\eta_t\}$, an i.i.d. $\mathrm{N}(0, \sigma^2)$ process?

- **Example 2**: For each time point, let $Y_t = \beta_0 + \beta_1 t + \eta_t$ with β_0 and β_1 some constants and η_t is defined above. What is $\mu_Y(t)$?

Review: The Covariance Between Two RVs

- The covariance between the RVs X and Y is

$$\begin{aligned}\mathbb{Cov}(X,Y) &= \mathbb{E}\{(X-\mu_X)(Y-\mu_Y)\}\\ &= \mathbb{E}(XY)-\mu_X\mu_Y.\end{aligned}$$

  It is a measure of linear dependence between the two RVs. When $X=Y$ we have

$$\mathbb{Cov}(X,X)=\mathbb{Var}(X).$$

- For constants a, b, c, and RVs X, Y, Z:

$$\begin{aligned}\mathbb{Cov}(aX+bY+c,Z) &= \mathbb{Cov}(aX,Z)+\mathbb{Cov}(bY,Z)\\ &= a\mathbb{Cov}(X,Z)+b\mathbb{Cov}(Y,Z)\end{aligned}$$

  $\Rightarrow$

$$\begin{aligned}\mathbb{Var}(X+Y) &= \mathbb{Cov}(X,X)+\mathbb{Cov}(X,Y)+\mathbb{Cov}(Y,X)+\mathbb{Cov}(Y,Y)\\ &= \mathbb{Var}(X)+\mathbb{Var}(Y)+2\mathbb{Cov}(X,Y)\end{aligned}$$

Autocovariance Function

- The autocovariance function of $\{\eta_t\}$ is

$$\gamma(s,t) = \mathbb{C}\mathrm{ov}(\eta_s, \eta_t) = \mathbb{E}[(\eta_s - \mu_s)(\eta_t - \mu_t)]$$

  It measures the strength of linear dependence between two RVs η_s and η_t

- **Properties**:
  - $\gamma(s,t) = \gamma(t,s)$ for each s and t
  - When $s = t$ we have

$$\gamma(t,t) = \mathbb{C}\mathrm{ov}(\eta_t, \eta_t) = \mathbb{C}\mathrm{ov}(\eta_t) = \sigma_t^2$$

    the value of the variance function at time t
  - $\gamma(s,t)$ is a non-negative definite function (will come back to this later)

Autocorrelation Function

- The autocorrelation function of $\{\eta_t\}$ is

$$\rho(s,t) = \mathbb{C}\mathrm{orr}(\eta_s, \eta_t) = \frac{\gamma(s,t)}{\sqrt{\gamma(s,s)\gamma(t,t)}}$$

  It measures the "scale invariant" linear association between η_s and η_t

- **Properties**:
  - $-1 \le \rho(s,t) \le 1$ for each s and t
  - $\rho(s,t) = \rho(t,s)$ for each s and t
  - $\rho(t,t) = 1$ for each t
  - $\rho(\cdot,\cdot)$ is a non-negative definite function

Stationarity

- We typically need "replicates" to estimate population quantities. For example, we use

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

  to be the estimate of μ_X, the population mean of the **single** RV, X

- However, in time series analysis, we have $n = 1$ (i.e., no replication) because we only have one realized value at each time point
- Stationarity means that some characteristic of $\{\eta_t\}$ does not depend on the time point, t, only on the "time lag" between time points **so that we can create "replicates"**

Next, we will talk about strict stationarity and weak stationarity

Strictly Stationary Processes

- A time series, $\{\eta_t\}$, is strictly stationary if

$$[\eta_1, \eta_2, \cdots \eta_T] \stackrel{d}{=} [\eta_{1+h}, \eta_{2+h}, \cdots \eta_{T+h}],$$

  for all integers h and $T \geq 1 \Rightarrow$ the joint distribution are unaffected by time shifts

- Under such the strict stationarity
  - $\{\eta_t\}$ is identically distributed but not (necessarily) independent
  - When μ_t is finite, $\mu_t = \mu$ is independent of time t
  - When the variance function exists,

$$\gamma(s,t) = \gamma(s+h, t+h),$$

    for any s, t, and h

Weakly Stationary Processes

- $\{\eta_t\}$ is weakly stationary if
  - $\mathbb{E}(\eta_t) = \mu_t = \mu$
  - $\mathbb{Cov}(\eta_t, \eta_{t+h}) = \gamma(t, t+h) = \gamma(h)$, finite constant that can depend on h but not on t
- Other names for this type of stationarity include second-order, covariance, wide senese. The quantity h is called the lag
- Weak and strict stationarity
  - A strictly stationary process $\{\eta_t\}$ is also weakly stationary as long as μ is finite
  - Weak stationarity does not imply strict stationarity!

Autocovariance Function of Stationary Processes

The autocovariance function (ACVF) of a stationary process $\{\eta_t\}$ is defined to be

$$\begin{aligned}\gamma(h) &= \mathbb{Cov}(\eta_t, \eta_{t+h}) \\ &= \mathbb{E}[(\eta_t - \mu)(\eta_{t+h} - \mu)],\end{aligned}$$

which measures the lag-h time dependence

Properties of the ACVF:

- $\gamma(0) = \mathbb{Var}(\eta_t)$
- $\gamma(-h) = \gamma(h)$ for each h
- $\gamma(s - t)$ as a function of $(s - t)$ is non-negative definite

Autocorrelation Function of Stationary Processes

The autocorrelation function (ACF) of a stationary process $\{\eta_t\}$ is defined to be

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)}$$

which measures the “scale invariant” lag-h time dependence

Properties of the ACF:

- $-1 \leq \rho(h) \leq 1$ and $\rho(0) = 1$ for each h
- $\rho(-h) = \rho(h)$ for each h
- $\rho(s-t)$ as a function of $(s-t)$ is non-negative definite

The White Noise Process

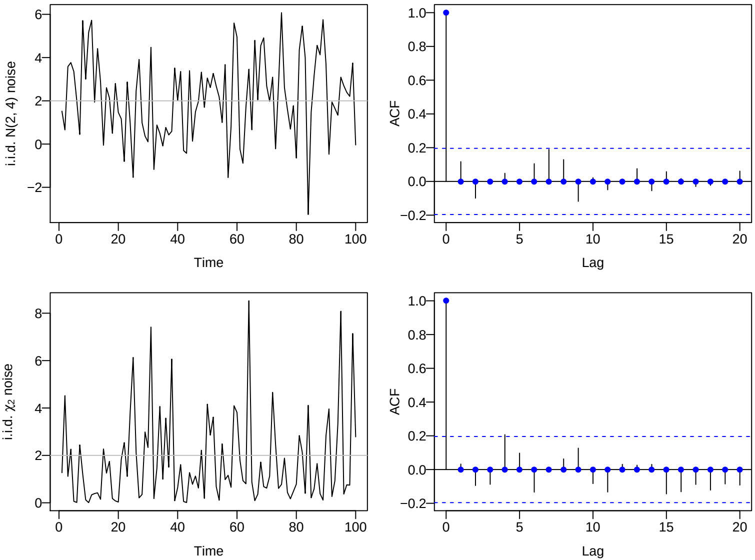
Let's assume $\mathbb{E}(\eta_t) = \mu$ and $\mathbb{Var}(\eta_t) = \sigma^2 < \infty$. $\{\eta_t\}$ is a white noise or $\mathrm{WN}(\mu, \sigma^2)$ process if

$$\gamma(h) = 0,$$

for $h \neq 0$

- $\{\eta_t\}$ is stationary
- However, distributions of η_t and η_{t+1} can be different!
- All i.i.d. noise with finite variance ($\sigma^2 < 0$) is white noise but the converse need not be true

Examples Realizations of White Noise Processes



The Moving Average Process of First Order (MA(1))

Let $\{Z_t\}$ be a $\mathrm{WN}(0, \sigma^2)$ process and θ be some constant $\in \mathbb{R}$. For each integer t, let

$$\eta_t = Z_t + \theta Z_{t-1}.$$

- The sequences of RVs $\{\eta_t\}$ is called the moving average process of order 1 or MA(1) process
- One can show that the MA(1) process $\{\eta_t\}$ is stationary

First-Order Moving Average Process: Mean Function

Need to show the mean function is NOT a function of time t

$$
\begin{aligned}
\mathbb{E}[\eta_t] &= \mathbb{E}[Z_t + \theta Z_{t-1}] \\
&= \mathbb{E}[Z_t] + \theta \mathbb{E}[Z_{t-1}] \\
&= 0 + \theta \times 0 \\
&= 0, \quad \forall t
\end{aligned}
$$

☺

First-Order Moving Average Process: Covariance Function

Need to show the autovariance function $\gamma(\cdot,\cdot)$ is a function of time lag only

$$\begin{aligned}
\gamma(t,t+h) &= \mathbb{Cov}(\eta_t,\eta_{t+h}) \\
&= \mathbb{Cov}(Z_t+\theta Z_{t-1}, Z_{t+h}+\theta Z_{t+h-1}) \\
&= \mathbb{Cov}(Z_t, Z_{t+h}) + \mathbb{Cov}(Z_t, \theta Z_{t+h-1}) \\
&\quad + \mathbb{Cov}(\theta Z_{t-1}, Z_{t+h}) + \mathbb{Cov}(\theta Z_{t-1}, \theta Z_{t+h-1})
\end{aligned}$$

if $h=0,$ we have $\quad \gamma(t,t+h) = \sigma^2+\theta^2\sigma^2 = \sigma^2(1+\theta^2)$
if $h=\pm 1,$ we have $\quad \gamma(t,t+h) = \theta\sigma^2$
if $|h|\geq 2,$ we have $\quad \gamma(t,t+h) = 0$

$\Rightarrow \gamma(t,t+h)$ only depends on h but not on t ☺

First-Order Moving Average Process: ACVF & ACF

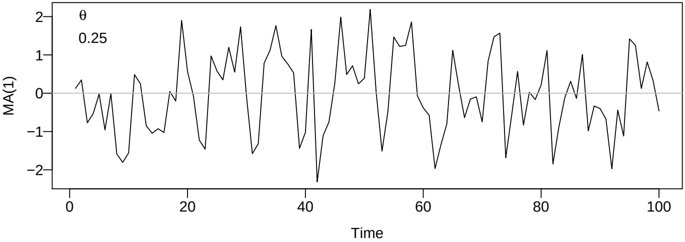
ACVF:

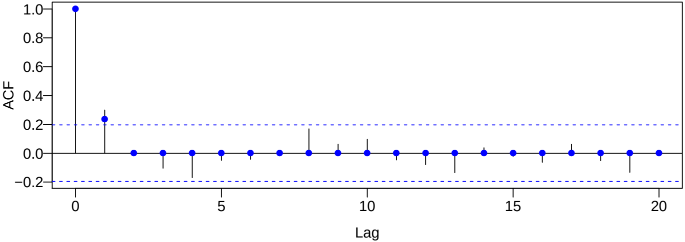
$$\gamma(h) = \begin{cases} \sigma^2(1+\theta^2) & h = 0; \\ \theta\sigma^2 & |h| = 1; \\ 0 & |h| \geq 2 \end{cases}$$

We can get **ACF** by dividing everything by $\gamma(0) = \sigma^2(1+\theta^2)$

$$\rho(h) = \begin{cases} 1 & h = 0; \\ \frac{\theta}{1+\theta^2} & |h| = 1; \\ 0 & |h| \geq 2. \end{cases}$$

Examples Realizations of MA(1) Processes





First-order autoregressive process, AR(1)

Let $\{Z_t\}$ be a $\mathrm{WN}(0, \sigma^2)$ process, and $-1 < \phi < 1$ be a constant. Let's assume $\{\eta_t\}$ is a stationary process with

$$\eta_t = \phi\eta_{t-1} + Z_t,$$

for each integer t, where η_s and Z_t are uncorrelated for each $s < t \Rightarrow$ future noise is uncorrelated with the current time point)

We will see later there is only one unique solution to this equation. Such a sequence $\{\eta_t\}$ of RVs is called an AR(1) process

Properties of the AR(1) process

Want to find the mean value μ under the weakly stationarity assumption

$$
\begin{aligned}
\mathbb{E}[\eta_t] &= \mathbb{E}[\phi\eta_{t-1} + Z_t] \\
\mu &= \phi\mathbb{E}[\eta_{t-1}] + \mathbb{E}[Z_t] \\
\mu &= \phi\mu + 0 \\
&\Rightarrow \mu = 0, \quad \forall t
\end{aligned}
$$

☺

Want to find $\gamma(h)$ under the weakly stationarity assumption

$$
\begin{aligned}
\mathbb{Cov}(\eta_t, \eta_{t-h}) &= \mathbb{Cov}(\phi\eta_{t-1} + Z_t, \eta_{t-h}) \\
\gamma(-h) &= \phi\mathbb{Cov}(\eta_{t-1}, \eta_{t-h}) + \mathbb{Cov}(Z_t, \eta_{t-h}) \\
\gamma(h) &= \phi\gamma(h-1) + 0 \\
\Rightarrow \gamma(h) &= \phi\gamma(h-1) = \cdots = \phi^{|h|}\gamma(0)
\end{aligned}
$$

Next, need to figure out $\gamma(0)$

Properties of the AR(1) process Cont'd

$$
\begin{aligned}
\mathbb{V}\mathrm{ar}(\eta_t) &= \mathbb{V}\mathrm{ar}(\phi\eta_{t-1} + Z_t) \\
\gamma(0) &= \phi^2\gamma(0) + \sigma^2 \\
\Rightarrow (1+\phi^2)\gamma(0) &= \sigma^2 \\
\Rightarrow \gamma(0) &= \frac{\sigma^2}{1-\phi^2}
\end{aligned}
$$

☺

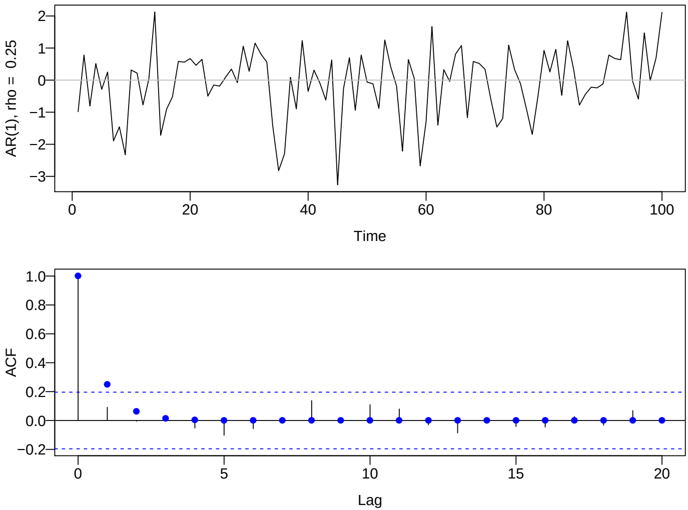
Therefore, we have

$$
\gamma(h) = \begin{cases} \frac{\sigma^2}{1-\phi^2} & h = 0; \\ \frac{\phi^{|h|}\sigma^2}{1-\phi^2} & h \neq 1, \end{cases}
$$

and

$$
\rho(h) = \begin{cases} 1 & h = 0; \\ \phi^{|h|} & h \neq 1. \end{cases}
$$

Examples Realizations of AR(1) Processes



The Random Walk Process

Let $\{Z_t\}$ be a $\mathrm{WN}(0, \sigma^2)$ process and for $t \geq 1$ definite

$$\eta_t = Z_1 + Z_2 + \cdots + Z_t = \sum_{s=1}^{t} Z_s.$$

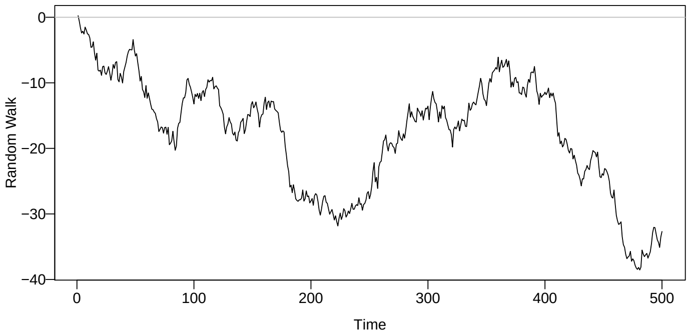
- The sequence of RVs $\{\eta_t\}$ is called a random walk process
- **Special case**: If we have $\{Z_t\}$ such that for each t

$$\mathbb{P}(Z_t = z) = \begin{cases} \frac{1}{2}, & z = 1; \\ \frac{1}{2}, & z = -1, \end{cases}$$

  then $\{\eta_t\}$ is a simple symmetric random walk

- The random walk process is not stationary!

Example Realizations of Random Walk Processes



Gaussian Processes

$\{\eta_t\}$ is a Gaussian process (GP) if the joint distribution of any collection of the RVs has a multivariate normal (aka Gaussian) distribution

- The distribution of a GP is fully characterized by $\mu(\cdot)$, the mean function, and $\gamma(\cdot, \cdot)$, the autocovariance function. The joint probability density function of $\boldsymbol{\eta} = (\eta_1, \eta_2, \cdots, \eta_T)^T$ is

$$f(\boldsymbol{\eta}) = \frac{1}{(2\pi)^{\frac{T}{2}} |\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\boldsymbol{\eta} - \boldsymbol{\mu})^T \Sigma^{-1} (\boldsymbol{\eta} - \boldsymbol{\mu})\right),$$

where $\boldsymbol{\mu} = (\mu_1, \mu_2, \cdots, \mu_T)^T$ and the (i, j) element of the covariance matrix Σ is $\gamma(i, j)$

- If a GP $\{\eta_t\}$ is weakly stationary then the process is also strictly stationary

Estimating the Mean of Stationary Processes

Let $\{\eta_t\}$ be stationary with mean μ and ACVF $\gamma(s,t)$

- A natural estimator of μ is the sample mean

$$\bar{\eta} = \frac{1}{T}\sum_{t=1}^{T}\eta_t.$$

  $\bar{\eta}$ is an unbiased estimator of μ, i.e.

- Since $\{\eta_t\}$ is stationary, we have

$$\begin{aligned}\mathbb{Var}(\bar{\eta}) &= \frac{1}{T^2}\mathbb{Var}\left(\sum_{i=1}^{T}\eta_t\right)\\ &= \frac{1}{T^2}\sum_{s=1}^{T}\sum_{t=1}^{T}\mathbb{Cov}(\eta_s,\eta_t)\\ &= \frac{1}{T^2}\sum_{s=1}^{T}\sum_{t=1}^{T}\gamma(s-t)\end{aligned}$$

- **Exercise**: Show

$$\mathbb{Var}(\bar{\eta}) = \frac{1}{T}\sum_{h=-(T-1)}^{T-1}\left(1-\frac{|h|}{T}\right)\gamma(h)$$

AR(1) Example

Suppose $\{\eta_1, \eta_2, \eta_3\}$ is an AR(1) process with $|\phi| < 1$ and innovation variance σ. Show that the variance of $\bar{\eta}$ is $\frac{\sigma^2}{9(1-\phi^2)}(3 + 4\phi + 2\phi^2)$

Solution:

The Sampling Distribution of $\bar{\eta}$

Let

$$v_T = \sum_{h=-(T-1)}^{(T-1)} \left(1 - \frac{|h|}{T}\right) \gamma(h)$$

- If $\{\eta_t\}$ is Gaussian we have

$$\sqrt{T}(\bar{\eta} - \mu) \sim \mathrm{N}(0, v_T)$$

- The result above is approximate for many non-Gaussian time series
- In practice we also need to estimate $\gamma(h)$ from the data

Confidence Intervals for μ

- If $\gamma(h) \to 0$ as $h \to \infty$ then

$$v = \lim_{T \to \infty} v_T = \sum_{h=-\infty}^{\infty} \gamma(h) \text{ exists.}$$

- Further, if $\{\eta_t\}$ is Gaussian and

$$\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty,$$

then an approximate large-sample 95% CI for μ is given by

$$\left[\bar{\eta} - 1.96\sqrt{\frac{v}{T}}, \bar{\eta} + 1.96\sqrt{\frac{v}{T}}\right]$$

Strategies for Estimating v

- Parametric:
  - Assume a parametric model $\gamma_{\boldsymbol{\theta}}(\cdot)$, and calculate $v = \sum_{h=-(T-1)}^{T-1} \left(1 - \frac{|h|}{T}\right) \gamma_{\hat{\boldsymbol{\theta}}}(h)$ based on the ACVF for that model
  - The standard error, v, will depend on the parameters $\boldsymbol{\theta}$ of the parametric model
- Nonparametric:
  - Estimate v by

$$\hat{v} = \sum \left(1 - \frac{|h|}{T}\right) \hat{\gamma}(h),$$

where $\hat{\gamma}(\cdot)$ is an nonparametric estimate of ACVF

Examples of Parametric Forms for v

- i.i.d. Gaussian Noise: $v = \gamma(0) = \sigma \Rightarrow$ CI reduces to the classical case:

$$\left[\bar{\eta} - 1.96\sqrt{\frac{\sigma}{T}}, \bar{\eta} + 1.96\sqrt{\frac{\sigma}{T}}\right]$$

- MA(1) process: We have

$$\begin{aligned} v = \sum_{h=-\infty}^{\infty} \gamma(h) &= \gamma(-1) + \gamma(0) + \gamma(1) \\ &= \gamma(0) + 2\gamma(1) \\ &= \sigma^2(1 + \theta^2 + 2\theta) = \sigma^2(1+\theta)^2 \end{aligned}$$

- **Exercise**: Show for an AR(1) process we have

$$v = \frac{\sigma^2}{(1-\phi)^2}$$

An Estimate of $\gamma(\cdot)$

Goal: Want to estimate

$$\gamma(h) = \mathbb{C}\text{ov}(\eta_t, \eta_{t+h}) = \mathbb{E}\left[(\eta_t - \mu)(\eta_{t+h} - \mu)\right]$$

using data $\{\eta_t\}_{t=1}^T$

- For $|h| < T$, consider $\hat{\gamma}(h) = \frac{1}{T}\sum_{t=1}^{T-|h|}(\eta_t - \bar{\eta})(\eta_{t+|h|} - \bar{\eta})$. We call $\hat{\gamma}(h)$ the sample ACVF
- The sample ACVF is a biased estimator of $\gamma(h)$, but, it is used as the **standard** estimate of $\gamma(h)$
- $\hat{\gamma}(h)$ are even and non-negative definite