

Lecture 30

Inference for Proportions I

STAT 8010 Statistical Methods I

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In the next few lectures we will focus on **categorical data analysis**:

- Inference for a single proportion p
- Comparison of two proportions p_1 and p_2
- χ^2 tests: Inference for Multi-category data and contingency tables

Inference for a single proportion: Motivated Example

Researchers in the development of new treatments for cancer patients often evaluate the effectiveness of new therapies by reporting the **proportion** of patients who survive for a specified period of time after completion of the treatment. A new genetic treatment of 870 patients with a particular type of cancer resulted in 330 patients surviving at least 5 years after treatment. **Estimate** the proportion of all patients with the specified type of cancer who would survive at least 5 years after being administered this treatment.

- Dichotomous (two-category) outcomes: “success” & “failure”
- Similar to the inferential problem for μ , the population mean, we would like to infer p , the population proportion of success \Rightarrow **point estimate, interval estimate, hypothesis testing**

- Point estimate:

$$\hat{p} = \frac{X(\text{\# of "successes"})}{n}$$

Recall the Binomial random variable, we have $\mathbb{E}[X] = np$
where $X \sim \text{Bin}(n, p) \Rightarrow \mathbb{E}[\hat{p}] = \mathbb{E}\left[\frac{X}{n}\right] = \frac{1}{n}\mathbb{E}[X] = p$

- $100(1 - \alpha)\%$ CI:

$$\hat{p} \pm z_{\alpha/2} \sqrt{\frac{(\hat{p})(1 - \hat{p})}{n}}$$

Why?

- CLT approximation: $\hat{p} \approx N(p, \sigma_{\hat{p}}^2)$ where n “sufficiently large”
 $\Rightarrow \min(np, n(1 - p)) \geq 5$
- $\sigma_{\hat{p}}^2 = \text{Var}\left(\frac{X}{n}\right) = \frac{1}{n^2} \text{Var}(X) = \frac{1}{n^2} n(p)(1 - p) = \frac{p(1-p)}{n}$

A new genetic treatment of 870 patients with a particular type of cancer resulted in 330 patients surviving at least 5 years after treatment.

- 1 Estimate the proportion of all patients who would survive at least 5 years after being administered this treatment.
- 2 Construct a 95% CI for p

Another Example

Among 900 randomly selected registered voters nationwide, 63% of them are somewhat or very concerned about the spread of bird flu in the United States.

- 1 What is the point estimate for p (Proportion of U.S. voters who are concerned about the spread of bird flu?)
- 2 Construct a 99% CI for p
- 3 Construct a 95% CI for p

Margin of error & Sample Size Calculation

- Margin of error:

$$z_{\alpha/2} \sqrt{\frac{n\hat{p}(1-\hat{p})}{n}}$$

\Rightarrow CI for $p = \hat{p} \pm$ margin of error

- Sample size determination:

$$n = \frac{\tilde{p}(1-\tilde{p}) \times z_{\alpha/2}^2}{\text{margin of error}^2},$$

What value of \tilde{p} to use?

- An educated guess
- A value from previous research
- Use a pilot study
- The “most conservative” choice is to use $\tilde{p} = 0.5$

Example

A researcher wants to estimate the proportion of voters who will vote for candidate A. She wants to estimate to within 0.05 with 90% confidence.

- 1 How large a sample does she need if she thinks the true proportion is about .9?
- 2 How large a sample does she need if she thinks the true proportion is about .6?
- 3 How large a sample does she need if she wants to use the most conservative estimate?

- 1 State the null and alternative hypotheses:

$$H_0 : p = p_0 \text{ vs. } H_a : p > \text{ or } \neq \text{ or } < p_0$$

- 2 Compute the test statistic:

$$z_{obs} = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

- 3 Make the decision of the test:

Rejection Region/ P-Value Methods

- 4 Draw the conclusion of the test:

We (do/do not) have enough statistical evidence to conclude that (H_a in words) at $\alpha\%$ significant level.

Among 900 randomly selected registered voters nationwide, 63% of them are somewhat or very concerned about the spread of bird flu in the United States. Conduct a hypothesis test at .01 level to assess the research hypothesis: $p < .667$.

Another CI for p : Wilson Score Confidence Interval

Idea: Solving $p = \hat{p} \pm z_{\alpha/2} \sqrt{\frac{p(1-p)}{n}} \Rightarrow (p - \hat{p})^2 = z_{\alpha/2}^2 \frac{p(1-p)}{n}$

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$100(1 - \alpha)\%$ Wilson Score Confidence Interval:

$$\frac{X + \frac{z_{\alpha/2}^2}{2}}{n + z_{\alpha/2}^2} \pm \frac{z_{\alpha/2}}{n + z_{\alpha/2}^2} \sqrt{\frac{X(n - X)}{n} + \frac{z_{\alpha/2}^2}{4}}$$

Rule of Three: An Approximate 95% CI for p When $\hat{p} = 0$ or 1

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Rule of Three: An Approximate 95% CI for p When $\hat{p} = 0$ or 1

When $\hat{p} = 0$, we have

$$\hat{p} \pm z_{\alpha/2} \sqrt{\frac{(\hat{p})(1 - \hat{p})}{n}} = 0 \pm z_{\alpha/2} \times 0 = (0, 0)$$

Similarly, when $\hat{p} = 1$, we have

$$\hat{p} \pm z_{\alpha/2} \sqrt{\frac{(\hat{p})(1 - \hat{p})}{n}} = 1 \pm z_{\alpha/2} \times 0 = (1, 1)$$

These CIs degenerate to a point, which do not reflect the estimation uncertainty. Here we could apply the **rule of three** to approximate 95% CI:

$$\begin{array}{ll} (0, 3/n), & \text{if } \hat{p} = 0 \\ (1 - 3/n, 1), & \text{if } \hat{p} = 1 \end{array}$$

In this lecture, we learned statistical inference for population proportion, p :

- Point estimate
- Interval estimate
- Hypothesis testing

In next lecture we will learn statistical inference for two proportions p_1 and p_2