

Lecture 3

Stationary processes

References: CC08 Chapter 2 & Chapter 4.1-4.3; BD16
Chapter 1.3-1.6; SS17 Chapter 1.2-1.6

MATH 8090 Time Series Analysis
Week 3

Mean and
Autocovariance
Functions

Stationarity

Some Examples of
Stationary Processes

Estimation of Mean
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Functions

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Review: The Additive Decomposition

- The additive model for a time series $\{Y_t\}$ is

$$Y_t = \mu_t + s_t + \eta_t,$$

where

- μ_t is the **trend** component
- s_t is the **seasonal** component
- η_t is the **random (noise)** component with $\mathbb{E}(\eta_t) = 0$
- Standard procedure:
 - (1) Estimate/remove the trend and seasonal components
 - (2) Analyze the remainder, the residuals $\hat{\eta}_t = y_t - \hat{\mu}_t - \hat{s}_t$
- We will focus on (2) for the next few weeks

- A **time series model** is a specification of the probabilistic distribution of a sequence of random variables (RVs) η_t

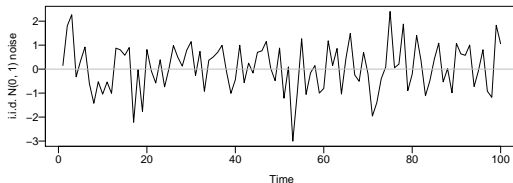
(The observed time series is a **realization** of such a sequence of random variables)

- The simplest time series is **i.i.d. (*independent and identically distributed*) noise**
 - $\{\eta_t\}$ is a sequence of independent and identically distributed zero-mean (i.e., $\mathbb{E}(\eta_t) = 0, \forall t$) random variables
 \Rightarrow **no temporal dependence**
 - It is of little value of using i.i.d. noise model to conduct **forecast** as there is no information from the past observations
 - **But**, we will use i.i.d. model as a building block to develop time series models that can accommodate time dependence

Example Realizations of i.i.d. Noise

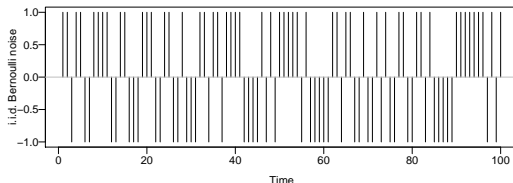
- Gaussian (normal) i.i.d. noise with mean 0 and variance $\sigma^2 > 0$

$$f(\eta_t) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\eta_t^2}{2\sigma^2}\right)$$



- Bernoulli i.i.d. noise with “success” probability

$$\mathbb{P}(\eta_t = 1) = p = 1 - \mathbb{P}(\eta_t = -1)$$



A time series model could also be a specification of the **means** and **autocovariances** of the RVs

- The **mean function** of $\{\eta_t\}$ is

$$\mu_t = \mathbb{E}(\eta_t).$$

- μ_t is the population mean at time t , which can be computed as:

$$\mu_t = \begin{cases} \int_{-\infty}^{\infty} \eta_t f(\eta_t) d\eta_t & \text{when } \eta_t \text{ is a continuous RV;} \\ \sum_{-\infty}^{\infty} \eta_t p(\eta_t), & \text{when } \eta_t \text{ is a discrete RV,} \end{cases}$$

where $f(\cdot)$ and $p(\cdot)$ are the probability density function and probability mass function of η_t , respectively

- **Example 1:** What is the mean function for $\{\eta_t\}$, an i.i.d. $N(0, \sigma^2)$ process?
- **Example 2:** For each time point, let $Y_t = \beta_0 + \beta_1 t + \eta_t$ with β_0 and β_1 some constants and η_t is defined above. What is $\mu_Y(t)$?

Review: The Covariance Between Two RVs

- The **covariance** between the RVs X and Y is

$$\begin{aligned}\mathbb{Cov}(X, Y) &= \mathbb{E}\{(X - \mu_X)(Y - \mu_Y)\} \\ &= \mathbb{E}(XY) - \mu_X \mu_Y.\end{aligned}$$

It is a measure of **linear dependence** between the two RVs. When $X = Y$ we have

$$\mathbb{Cov}(X, X) = \mathbb{Var}(X).$$

- For constants a, b, c , and RVs X, Y, Z :

$$\begin{aligned}\mathbb{Cov}(aX + bY + c, Z) &= \mathbb{Cov}(aX, Z) + \mathbb{Cov}(bY, Z) \\ &= a\mathbb{Cov}(X, Z) + b\mathbb{Cov}(Y, Z)\end{aligned}$$

\Rightarrow

$$\begin{aligned}\mathbb{Var}(X + Y) &= \mathbb{Cov}(X, X) + \mathbb{Cov}(X, Y) + \mathbb{Cov}(Y, X) + \mathbb{Cov}(Y, Y) \\ &= \mathbb{Var}(X) + \mathbb{Var}(Y) + 2\mathbb{Cov}(X, Y)\end{aligned}$$

- The autocovariance function of $\{\eta_t\}$ is

$$\gamma(s, t) = \text{Cov}(\eta_s, \eta_t) = \mathbb{E}[(\eta_s - \mu_s)(\eta_t - \mu_t)]$$

It measures the strength of linear dependence between two RVs η_s and η_t

- **Properties:**

- $\gamma(s, t) = \gamma(t, s)$ for each s and t
- When $s = t$ we have

$$\gamma(t, t) = \text{Cov}(\eta_t, \eta_t) = \text{Cov}(\eta_t) = \sigma_t^2$$

the value of the variance function at time t

- $\gamma(s, t)$ is a non-negative definite function (will come back to this later)

- The autocorrelation function of $\{\eta_t\}$ is

$$\rho(s, t) = \text{Corr}(\eta_s, \eta_t) = \frac{\gamma(s, t)}{\sqrt{\gamma(s, s)\gamma(t, t)}}$$

It measures the “scale invariant” linear association between η_s and η_t

- **Properties:**

- $-1 \leq \rho(s, t) \leq 1$ for each s and t
- $\rho(s, t) = \rho(t, s)$ for each s and t
- $\rho(t, t) = 1$ for each t
- $\rho(\cdot, \cdot)$ is a non-negative definite function

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- We typically need “replicates” to estimate population quantities. For example, we use

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

to be the estimate of μ_X , the population mean of the **single** RV, X

- However, in time series analysis, we have $n = 1$ (i.e., no replication) because we only have one realized value at each time point
- Stationarity means that some characteristic of $\{\eta_t\}$ does not depend on the time point, t , only on the “time lag” between time points **so that we can create “replicates”**

Next, we will talk about strict stationarity and weak stationarity

- A time series, $\{\eta_t\}$, is **strictly stationary** if

$$[\eta_1, \eta_2, \dots, \eta_T] \stackrel{d}{=} [\eta_{1+h}, \eta_{2+h}, \dots, \eta_{T+h}],$$

for all integers h and $T \geq 1 \Rightarrow$ the **joint distribution** are unaffected by time shifts

- Under such the strict stationarity
 - $\{\eta_t\}$ is **identically distributed** but not (necessarily) **independent**
 - When μ_t is finite, $\mu_t = \mu$ is independent of time t
 - When the variance function exists,

$$\gamma(s, t) = \gamma(s + h, t + h),$$

for any s, t , and h

- $\{\eta_t\}$ is **weakly stationary** if
 - $\mathbb{E}(\eta_t) = \mu_t = \mu$
 - $\text{Cov}(\eta_t, \eta_{t+h}) = \gamma(t, t+h) = \gamma(h)$, finite constant that can depend on h **but not on t**
- Other names for this type of stationarity include **second-order, covariance, wide sense**. The quantity h is called the **lag**
- Weak and strict stationarity
 - A strictly stationary process $\{\eta_t\}$ is also weakly stationary as long as μ is finite
 - **Weak stationarity does not imply strict stationarity!**

The autocovariance function (ACVF) of a stationary process $\{\eta_t\}$ is defined to be

$$\begin{aligned}\gamma(h) &= \text{Cov}(\eta_t, \eta_{t+h}) \\ &= \mathbb{E}[(\eta_t - \mu)(\eta_{t+h} - \mu)],\end{aligned}$$

which measures the lag- h time dependence

Properties of the ACVF:

- $\gamma(0) = \text{Var}(\eta_t)$
- $\gamma(-h) = \gamma(h)$ for each h
- $\gamma(s - t)$ as a function of $(s - t)$ is non-negative definite

The autocorrelation function (ACF) of a stationary process $\{\eta_t\}$ is defined to be

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)}$$

which measures the “scale invariant” lag- h time dependence

Properties of the ACF:

- $-1 \leq \rho(h) \leq 1$ and $\rho(0) = 1$ for each h
- $\rho(-h) = \rho(h)$ for each h
- $\rho(s-t)$ as a function of $(s-t)$ is non-negative definite

The White Noise Process

Let's assume $\mathbb{E}(\eta_t) = \mu$ and $\text{Var}(\eta_t) = \sigma^2 < \infty$. $\{\eta_t\}$ is a **white noise** or **WN**(μ, σ^2) process if

$$\gamma(h) = 0,$$

for $h \neq 0$

- $\{\eta_t\}$ is stationary
- However, distributions of η_t and η_{t+1} **can be different!**
- All i.i.d. noise with finite variance ($\sigma^2 < \infty$) is **white noise** but **the converse need not be true**

Examples Realizations of White Noise Processes

Stationary processes

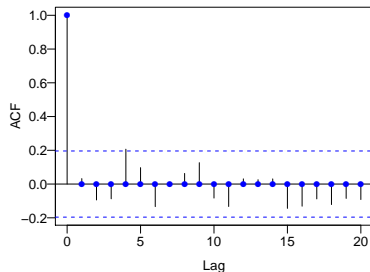
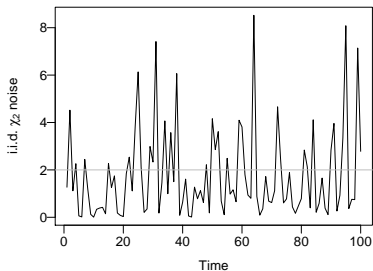
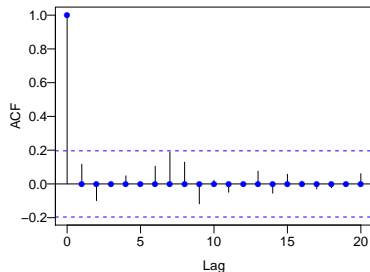
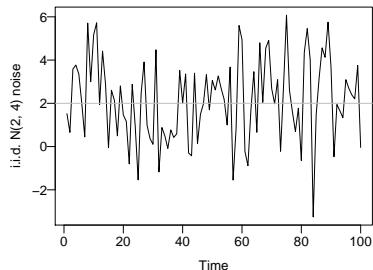
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The Moving Average Process of First Order (MA(1))

Let $\{Z_t\}$ be a $WN(0, \sigma^2)$ process and θ be some constant $\in \mathbb{R}$.
For each integer t , let

$$\eta_t = Z_t + \theta Z_{t-1}.$$

- The sequences of RVs $\{\eta_t\}$ is called the **moving average process of order 1** or MA(1) process
- One can show that the MA(1) process $\{\eta_t\}$ is **stationary**

First-Order Moving Average Process: Mean Function

Need to show the mean function is NOT a function of time t

$$\begin{aligned}\mathbb{E}[\eta_t] &= \mathbb{E}[Z_t + \theta Z_{t-1}] \\ &= \mathbb{E}[Z_t] + \theta \mathbb{E}[Z_{t-1}] \\ &= 0 + \theta \times 0 \\ &= 0, \quad \forall t\end{aligned}$$



First-Order Moving Average Process: Covariance Function

Need to show the autocovariance function $\gamma(\cdot, \cdot)$ is a function of time lag only

$$\begin{aligned}\gamma(t, t+h) &= \text{Cov}(\eta_t, \eta_{t+h}) \\ &= \text{Cov}(Z_t + \theta Z_{t-1}, Z_{t+h} + \theta Z_{t+h-1}) \\ &= \text{Cov}(Z_t, Z_{t+h}) + \text{Cov}(Z_t, \theta Z_{t+h-1}) \\ &\quad + \text{Cov}(\theta Z_{t-1}, Z_{t+h}) + \text{Cov}(\theta Z_{t-1}, \theta Z_{t+h-1})\end{aligned}$$

$$\begin{aligned}\text{if } h = 0, \text{ we have} \quad & \gamma(t, t+h) = \sigma^2 + \theta^2 \sigma^2 = \sigma^2(1 + \theta^2) \\ \text{if } h = \pm 1, \text{ we have} \quad & \gamma(t, t+h) = \theta \sigma^2 \\ \text{if } |h| \geq 2, \text{ we have} \quad & \gamma(t, t+h) = 0\end{aligned}$$

$\Rightarrow \gamma(t, t+h)$ only depends on h but not on t 😊

First-Order Moving Average Process: ACVF & ACF

ACVF:

$$\gamma(h) = \begin{cases} \sigma^2(1 + \theta^2) & h = 0; \\ \theta\sigma^2 & |h| = 1; \\ 0 & |h| \geq 2 \end{cases}$$

We can get **ACF** by dividing everything by $\gamma(0) = \sigma^2(1 + \theta^2)$

$$\rho(h) = \begin{cases} 1 & h = 0; \\ \frac{\theta}{1+\theta^2} & |h| = 1; \\ 0 & |h| \geq 2. \end{cases}$$

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First-order autoregressive process, AR(1)

Let $\{Z_t\}$ be a $WN(0, \sigma^2)$ process, and $-1 < \phi < 1$ be a constant.
Let's assume $\{\eta_t\}$ is a **stationary process** with

$$\eta_t = \phi\eta_{t-1} + Z_t,$$

for each integer t , where η_s and Z_t are **uncorrelated** for each
 $s < t \Rightarrow$ future noise is uncorrelated with the current time point)

We will see later there is only one unique solution to this
equation. Such a sequence $\{\eta_t\}$ of RVs is called an **AR(1)
process**

Properties of the AR(1) process

Want to find the mean value μ under the weakly stationarity assumption

$$\begin{aligned}\mathbb{E}[\eta_t] &= \mathbb{E}[\phi\eta_{t-1} + Z_t] \\ \mu &= \phi\mathbb{E}[\eta_{t-1}] + \mathbb{E}[Z_t] \\ \mu &= \phi\mu + 0 \\ \Rightarrow \mu &= 0, \quad \forall t\end{aligned}$$



Want to find $\gamma(h)$ under the weakly stationarity assumption

$$\begin{aligned}\text{Cov}(\eta_t, \eta_{t-h}) &= \text{Cov}(\phi\eta_{t-1} + Z_t, \eta_{t-h}) \\ \gamma(-h) &= \phi\text{Cov}(\eta_{t-1}, \eta_{t-h}) + \text{Cov}(Z_t, \eta_{t-h}) \\ \gamma(h) &= \phi\gamma(h-1) + 0 \\ \Rightarrow \gamma(h) &= \phi\gamma(h-1) = \dots = \phi^{|h|}\gamma(0)\end{aligned}$$

Next, need to figure out $\gamma(0)$

Properties of the AR(1) process Cont'd

$$\text{Var}(\eta_t) = \text{Var}(\phi\eta_{t-1} + Z_t)$$

$$\gamma(0) = \phi^2\gamma(0) + \sigma^2$$

$$\Rightarrow (1 - \phi^2)\gamma(0) = \sigma^2$$

$$\Rightarrow \gamma(0) = \frac{\sigma^2}{1 - \phi^2}$$



Therefore, we have

$$\gamma(h) = \begin{cases} \frac{\sigma^2}{1-\phi^2} & h = 0; \\ \phi^{|h|} \frac{\sigma^2}{1-\phi^2} & h \neq 0, \end{cases}$$

and

$$\rho(h) = \begin{cases} 1 & h = 0; \\ \phi^{|h|} & h \neq 0. \end{cases}$$

Examples Realizations of AR(1) Processes

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Let $\{Z_t\}$ be a $WN(0, \sigma^2)$ process and for $t \geq 1$ define

$$\eta_t = Z_1 + Z_2 + \cdots + Z_t = \sum_{s=1}^t Z_s.$$

- The sequence of RVs $\{\eta_t\}$ is called a **random walk process**
- **Special case:** If we have $\{Z_t\}$ such that for each t

$$\mathbb{P}(Z_t = z) = \begin{cases} \frac{1}{2}, & z = 1; \\ \frac{1}{2}, & z = -1, \end{cases}$$

then $\{\eta_t\}$ is a **simple symmetric random walk**

- **The random walk process is not stationary!**

Example Realizations of Random Walk Processes

Stationary processes



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$\{\eta_t\}$ is a **Gaussian process (GP)** if the joint distribution of any collection of the RVs has a multivariate normal (aka Gaussian) distribution

- The distribution of a GP is fully characterized by $\mu(\cdot)$, the mean function, and $\gamma(\cdot, \cdot)$, the autocovariance function. The joint probability density function of $\boldsymbol{\eta} = (\eta_1, \eta_2, \dots, \eta_T)^T$ is

$$f(\boldsymbol{\eta}) = \frac{1}{(2\pi)^{\frac{T}{2}} |\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\boldsymbol{\eta} - \boldsymbol{\mu})^T \Sigma^{-1}(\boldsymbol{\eta} - \boldsymbol{\mu})\right),$$

where $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_T)^T$ and the (i, j) element of the covariance matrix Σ is $\gamma(i, j)$

- If a GP $\{\eta_t\}$ is **weakly stationary** then the process is also **strictly stationary**

Estimating the Mean of Stationary Processes

Let $\{\eta_t\}$ be stationary with mean μ and ACVF $\gamma(s, t) = \gamma(s - t)$

- A natural estimator of μ is the sample mean

$$\bar{\eta} = \frac{1}{T} \sum_{t=1}^T \eta_t.$$

$\bar{\eta}$ is an unbiased estimator of μ , i.e.

- Since $\{\eta_t\}$ is stationary, we have

$$\begin{aligned} \text{Var}(\bar{\eta}) &= \frac{1}{T^2} \text{Var}\left(\sum_{i=1}^T \eta_t\right) \\ &= \frac{1}{T^2} \sum_{s=1}^T \sum_{t=1}^T \text{Cov}(\eta_s, \eta_t) \\ &= \frac{1}{T^2} \sum_{s=1}^T \sum_{t=1}^T \gamma(s - t) \end{aligned}$$

- **Exercise:** Show

$$\text{Var}(\bar{\eta}) = \frac{1}{T} \sum_{h=-(T-1)}^{T-1} \left(1 - \frac{|h|}{T}\right) \gamma(h)$$

AR(1) Example

Suppose $\{\eta_1, \eta_2, \eta_3\}$ is an AR(1) process with $|\phi| < 1$ and innovation variance σ^2 . Show that the variance of $\bar{\eta}$ is $\frac{\sigma^2}{9(1-\phi^2)}(3 + 4\phi + 2\phi^2)$

Solution:

Let

$$v_T = \sum_{h=-(T-1)}^{(T-1)} \left(1 - \frac{|h|}{T}\right) \gamma(h)$$

- If $\{\eta_t\}$ is **Gaussian** we have

$$\sqrt{T}(\bar{\eta} - \mu) \sim N(0, v_T)$$

- The result above is **approximate** for many **non-Gaussian** time series
- In practice we also need to **estimate** $\gamma(h)$ from the data

- If $\gamma(h) \rightarrow 0$ as $h \rightarrow \infty$ then

$$v = \lim_{T \rightarrow \infty} v_T = \sum_{h=-\infty}^{\infty} \gamma(h) \text{ exists.}$$

- Further, if $\{\eta_t\}$ is **Gaussian** and

$$\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty,$$

then an **approximate large-sample** 95% CI for μ is given by

$$\left[\bar{\eta} - 1.96\sqrt{\frac{v}{T}}, \bar{\eta} + 1.96\sqrt{\frac{v}{T}} \right]$$

- Parametric:

- Assume a parametric model $\gamma_{\theta}(\cdot)$, and calculate

$$\hat{v} = \sum_{h=-\infty}^{\infty} \gamma_{\hat{\theta}}(h)$$

based on the ACVF for that model

- The standard error, v , will depend on the parameters θ of the parametric model

- Nonparametric:

- Estimate v by

$$\hat{v} = \sum_{h=-\infty}^{\infty} \hat{\gamma}(h),$$

where $\hat{\gamma}(\cdot)$ is an nonparametric estimate of ACVF

Examples of Parametric Forms for v

- **i.i.d. Gaussian Noise:** $v = \gamma(0) = \sigma \Rightarrow$ CI reduces to the classical case:

$$\left[\bar{\eta} - 1.96\sqrt{\frac{\sigma^2}{T}}, \bar{\eta} + 1.96\sqrt{\frac{\sigma^2}{T}} \right]$$

- **MA(1) process:** We have

$$\begin{aligned} v &= \sum_{h=-\infty}^{\infty} \gamma(h) = \gamma(-1) + \gamma(0) + \gamma(1) \\ &= \gamma(0) + 2\gamma(1) \\ &= \sigma^2(1 + \theta^2 + 2\theta) = \sigma^2(1 + \theta)^2 \end{aligned}$$

- **Exercise:** Show for an **AR(1)** process we have

$$v = \frac{\sigma^2}{(1 - \phi)^2}$$

Goal: Want to estimate

$$\gamma(h) = \text{Cov}(\eta_t, \eta_{t+h}) = \mathbb{E}[(\eta_t - \mu)(\eta_{t+h} - \mu)]$$

using data $\{\eta_t\}_{t=1}^T$

- For $|h| < T$, consider $\hat{\gamma}(h) = \frac{1}{T} \sum_{t=1}^{T-|h|} (\eta_t - \bar{\eta})(\eta_{t+|h|} - \bar{\eta})$. We call $\hat{\gamma}(h)$ the **sample ACVF**
- The sample ACVF is a **biased** estimator of $\gamma(h)$, but, it is used as the **standard** estimate of $\gamma(h)$
- $\hat{\gamma}(h)$ are **even** and **non-negative definite**