# Lecture 5

## Time Series Regression

Reading: Forecasting, Time Series, and Regression (4th edition) by Bowerman, O'Connell, and Koehler: Chapter 6

*MATH 4070: Regression and Time-Series Analysis*

Whitney Huang
Clemson University

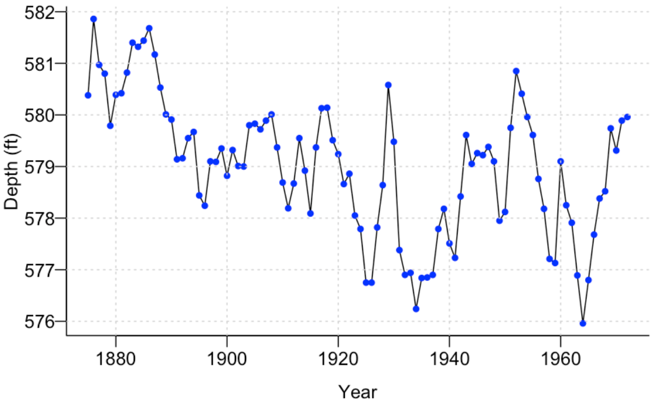
# Agenda

# Level of Lake Huron 1875–1972

Annual measurements of the level of Lake Huron in feet.

[Source: Brockwell & Davis, 1991]

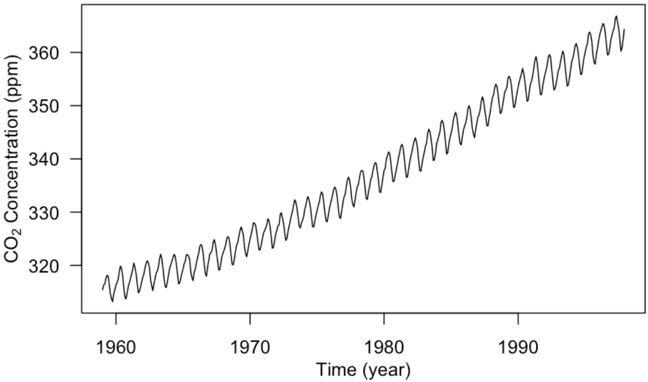
```{r}
par(mar = c(3.2, 3.2, 0.5, 0.5), mgp = c(2, 0.5, 0), bty = "L")
data(LakeHuron)
plot(LakeHuron, ylab = "Depth (ft)", xlab = "Year", las = 1)
points(LakeHuron, cex = 0.8, col = "blue", pch = 16)
grid()
```



# Mauna Loa Atmospheric $CO_2$ Concentration

Monthly atmospheric concentrations of $CO_2$ at the Mauna Loa Observatory [Source: Keeling & Whorf, Scripps Institution of Oceanography]

```{r}
data(co2)
par(mar = c(3.8, 4, 0.8, 0.6))
plot(co2, las = 1, xlab = "", ylab = "")
mtext("Time (year)", side = 1, line = 2)
mtext(expression(paste("CO"[2], " Concentration (ppm)")), side = 2, line = 2.5)
```



# Time Series Data

- A time series is a collection of observations $\{y_t, t \in T\}$ taken sequentially in time $(t)$ with the index set $T$
  - $T = \{0, 1, 2, \cdots, T\} \subset \mathbb{Z} \Rightarrow$ discrete-time time series
  - $T = [0, T] \subset \mathbb{R} \Rightarrow$ continuous-time time series
- A discrete-time time series might be intrinsically discrete or might arise from a underlying continuous-time time series via
  - sampling (e.g., instantaneous wind speed)
  - aggregation (e.g., daily accumulated precipitation amount)
  - extrema (e.g., daily maximum temperature)
- We will focus on dealing with discrete-time real-valued $(Y_t \in \mathbb{R})$ time series in this course

# Exploratory Time Series Analysis







- Start with a time series plot, i.e., to plot $y_t$ versus $t$
- Look at the following:
  - Are there abrupt changes?
  - Are there "outliers"?
  - Is there a need to transform the data?
- Examine the trend, seasonal components, and the "noise" term

## Features of Times Series

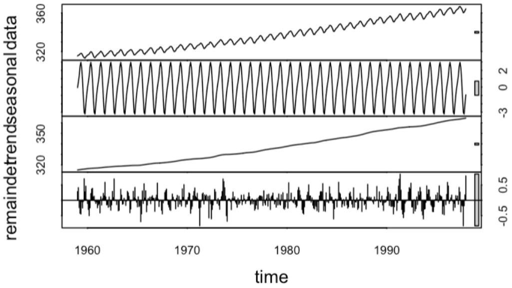
- Trends ($\mu_t$)
  - $\mu_t$ represents continuous changes, usually in the mean, over longer time scales. *"The essential idea of trend is that it shall be smooth."* - [Kendall, 1973]
  - The form of the trend is typically unknown and needs to be estimated. Removing the trend yields a detrended series
- Seasonal or Periodic Components ($s_t$)
  - $s_t$ repeats consistently over time, i.e., $s_t = s_{t+kd}$
  - The form and period $d$ of the seasonal component must be estimated to deseasonalize the series.
- The "Noise" Process ($\eta_t$)
  - $\eta_t$ represents the component that is neither trend nor seasonality
  - Focus on finding plausible statistical models for this process

## Combining Trend, Seasonality, and Noise Together

There are two commonly used approaches

- Additive model:

$$Y_t = \mu_t + s_t + \eta_t, \quad t = 1, \cdots, T$$




- Multiplicative model:

$$Y_t = \mu_t s_t \eta_t, \quad t = 1, \cdots, T$$

If all $\{y_t\}$ are positive then we obtain the additive model by taking logarithms:

$$\log Y_t = \log \mu_t + \log s_t + \log \eta_t, \quad t = 1, \cdots, T$$

# The (Additive) Decomposition Model





- The additive model for a time series $\{Y_t\}$ is

$$Y_t = \mu_t + s_t + \eta_t,$$

where

  - $\mu_t$ is the trend component
  - $s_t$ is the seasonal component
  - $\eta_t$ is the random (noise) component with $\mathbb{E}(\eta_t) = 0$

- Standard procedure:

  (1) Estimate/remove the trend and seasonal components

  (2) Analyze the remainder, the residuals $\hat{\eta}_t = y_t - \hat{\mu}_t - \hat{s}_t$

- We will focus on (1) for this week

# Estimating Trend for Nonseasonal Model

Assuming $s_t = 0$ (i.e., there is no “seasonal” variation), we have

$$Y_t = \mu_t + \eta_t,$$

with $\mathbb{E}(\eta_t) = 0$

Methods for estimating trends

- Least squares regression
- Smoothing

## Trend Estimation: Linear Regression



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- The additive nonseasonal time series model for $\{Y_t\}$ is

$$Y_t = \mu_t + \eta_t,$$

where the trend is assumed to be a linear combination of known covariate series $\{x_{it}\}_{i=1}^{p}$

$$\mu_t = \beta_0 + \sum_{i=1}^{p} \beta_i x_{it}.$$

- Here we want to **estimate** $\boldsymbol{\beta} = (\beta_0, \beta_1, \cdots, \beta_p)^T$ from the data $\{y_t, \{x_{it}\}_{i=1}^{p}\}_{t=1}^{T}$

- You're likely quite familiar with this formulation already ⇒ Regression Analysis

## Some Examples of Covariate Series $\{x_{it}\}$

- **Simple linear regression model:**

$$\mu_t = \beta_0 + \beta_1 x_t,$$

for example, the temperature trend at time $t$ could be a constant ($\beta_0$) plus a multiple ($\beta_1$) of the carbon dioxide level at time $t$ ($x_t$)

- **Polynomial regression model:**

$$\mu_t = \beta_0 + \sum_{i=1}^{p} \beta_i t^i$$

- **Change point model:**

$$\mu_t = \begin{cases} \beta_0 & \text{if } t \leq t^*; \\ \beta_0 + \beta_1 & \text{if } t \geq t^*. \end{cases}$$

# Parameter Estimation: Ordinary Least Squares



- Like in the linear regression setting, we can estimate the parameters via ordinary least squares (OLS)

- Specifically, we minimize the following objective function:

$$\ell_{ols} = \sum_{t=1}^{T}(y_t - \beta_0 - \sum_{k=1}^{p} x_{kt}\beta_k)^2.$$

- The estimates $\boldsymbol{\beta} = (\beta_0, \beta_1, \cdots, \beta_p)^T$ minimizing the above objective function are called the OLS estimates of $\boldsymbol{\beta}$ $\Rightarrow$ they are easiest to express in **matrix form**

# The Model and Parameter Estimates in Matrix Form

- Matrix representation:

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\eta},$$

where $\boldsymbol{Y} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_T \end{bmatrix}$, $\boldsymbol{X} = \begin{bmatrix} 1 & x_{11} & x_{12} & \cdots & x_{1p} \\ 1 & x_{21} & x_{22} & \cdots & x_{2p} \\ 1 & \vdots & \cdots & \cdots & \vdots \\ 1 & x_{T1} & x_{T2} & \cdots & x_{Tp} \end{bmatrix}$, and

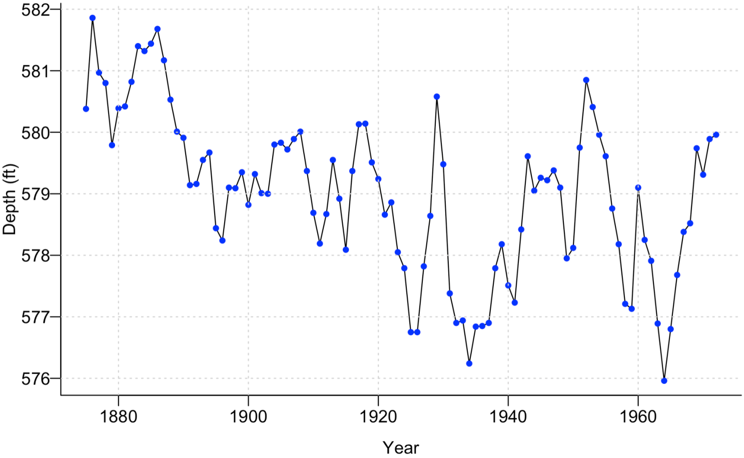
$\boldsymbol{\eta} = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_T \end{bmatrix}$

- Assuming $\boldsymbol{X}^T\boldsymbol{X}$ is **invertible**, the OLS estimate of $\beta$ can be shown to be

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y},$$

and the `lm` function in `R` calculates OLS estimates

# Lake Huron Example Revisited




Let's **assume** there is a linear trend in time $\Rightarrow$ we need to estimate the **intercept** $\beta_0$ and **slope** $\beta_1$

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# The R Output

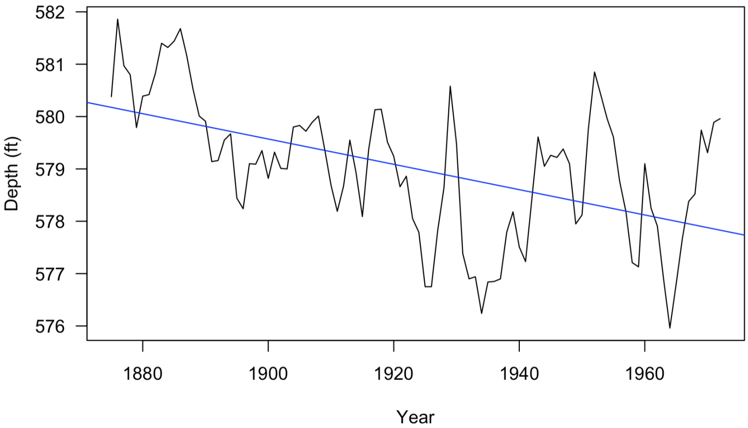
```
Call:
lm(formula = LakeHuron ~ yr)

Residuals:
     Min       1Q   Median       3Q      Max 
-2.50997 -0.72726  0.00083  0.74402  2.53565 

Coefficients:
              Estimate Std. Error t value Pr(>|t|)    
(Intercept) 625.554918   7.764293  80.568  < 2e-16 ***
yr           -0.024201   0.004036  -5.996 3.55e-08 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 1.13 on 96 degrees of freedom
Multiple R-squared:  0.2725,    Adjusted R-squared:  0.2649 
F-statistic: 35.95 on 1 and 96 DF,  p-value: 3.545e-08
```

# Plot the (Estimated) Trend $\hat{\mu}_t = \hat{\beta}_0 + \hat{\beta}_1 t$



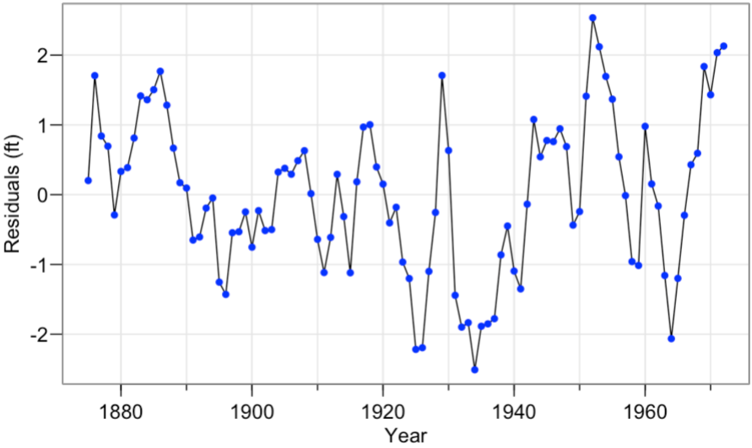

$\hat{\beta}_1 = -0.0242$ (ft/yr) $\Rightarrow$ there seems to be a decreasing trend

# Plot the Residuals $\{\hat{\eta}_t = y_t - \hat{\beta}_0 - \hat{\beta}_1 t\}$







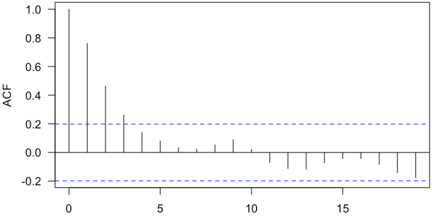



$\{\hat{\eta}_t\}$ seems to exhibit some temporal dependence structure, should we worry about the results we have (recall OLS makes an i.i.d. assumption)?

# Statistical Properties of the OLS Estimates with Correlated Errors

- Assume the components of $\boldsymbol{X}$ are not random, the OLS estimates $\hat{\boldsymbol{\beta}}$ are unbiased for $\boldsymbol{\beta}$
  **Proof:**

- Since $\{\eta_t\}$ is typically not an i.i.d. process (see the acf plot below), statistical inferences regarding $\boldsymbol{\beta}$ will be invalid



## Smoothing or Local Averaging

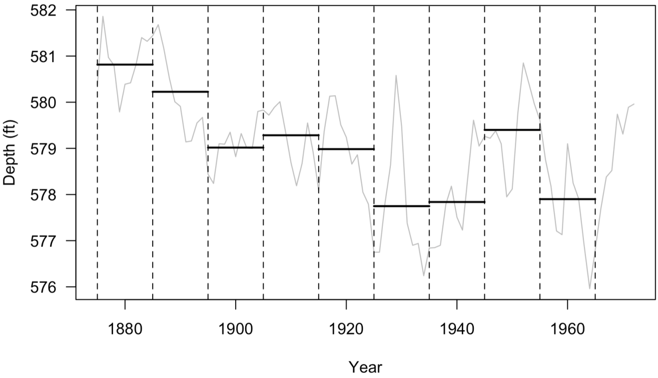






In some cases, we may relax the trend assumption using a 'non-parametric' approach.

We divide the time series into small blocks (each with 10 years of data) and average each block.

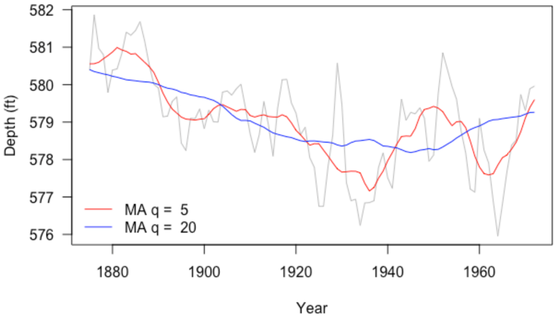



Doing this gives a very rough estimate of the trend. **Can we do better?**

## Moving Average Smoother

A moving average smoother estimates the trend at time $t$ by averaging the current observation and the $q$ nearest observations from either side. That is

$$\hat{\mu}_t = \frac{1}{2q+1} \sum_{j=-q}^{q} y_{t-j}$$



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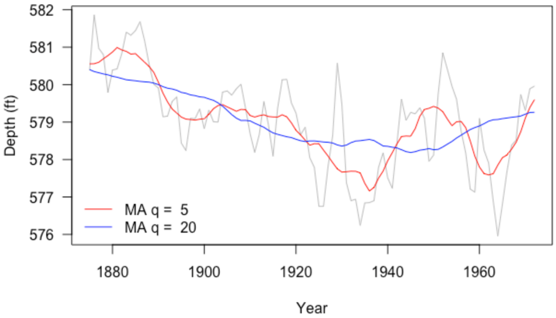



$q$ is the “smoothing” parameter, which controls the smoothness of the estimated trend $\hat{\mu}_t$

## Exponential Smoothing



- Let $\alpha \in [0, 1]$ be some fixed constant, defined

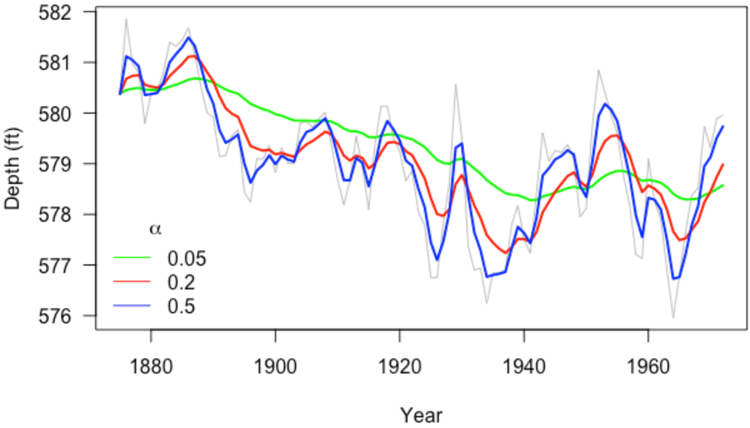
$$\hat{\mu}_t = \begin{cases} Y_1 & \text{if } t = 1; \\ \alpha Y_t + (1-\alpha)\hat{\mu}_{t-1} & t = 2, \cdots T. \end{cases}$$

- For $t = 2, \cdots, T$, we can rewrite $\hat{\mu}_t$ as

$$\sum_{j=0}^{t-2} \alpha(1-\alpha)^j Y_{t-j} + (1-\alpha)^{t-1} Y_1.$$

$\Rightarrow$ it is a one-sided moving average filter with exponentially decreasing weights. One can alter $\alpha$ to control the amounts of smoothing (see next slide for an example)

# $\alpha$ is the Smoothing Parameter for Exponential Smoothing




The smaller the $\alpha$, the smoother the resulting trend

# Seasonal Component Estimation





Let's consider a situation where a time series consists of only a seasonal component (assuming the trend has been estimated/removed). In this scenario,

$$Y_t = s_t + \eta_t,$$

with $\{s_t\}$ having period $d$ (i.e., $s_{t+jd} = s_t$ for all integers $j$ and $t$), $\sum_{t=1}^{d} s_t = 0$ and $\mathbb{E}(\eta_t) = 0$

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Two methods to estimate $\{s_t\}$

- Harmonic regression
- Seasonal mean model

## Harmonic Regression

- A harmonic regression model has the form

$$s_t = \sum_{j=1}^{k} A_j \cos(2\pi f_j + \phi_j).$$

For each $j = 1, \cdots, k$:

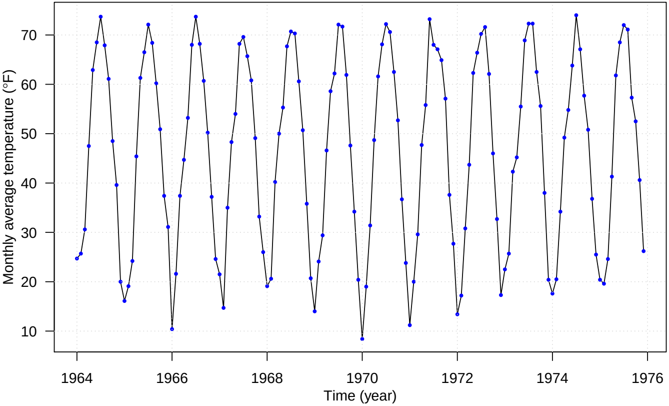
  - $A_j > 0$ is the amplitude of the $j$-th cosine wave
  - $f_j$ controls the the frequency of the $j$-th cosine wave (how often waves repeats)
  - $\phi_j \in [-\pi, \pi]$ is the phase of the $j$-th wave (where it starts)

- The above can be expressed as

$$\sum_{j=1}^{k} \left(\beta_{1j} \cos(2\pi f_j) + \beta_{2j} \sin(2\pi f_j)\right),$$

where $\beta_{1j} = A_j \cos(\phi_j)$ and $\beta_{2j} = A_j \sin(\phi_j) \Rightarrow$ if $\{f_j\}_{j=1}^{k}$ are known, **we can use regression techniques to estimate the parameters** $\{\beta_{1j}, \beta_{2j}\}_{j=1}^{k}$

# Monthly Average Temperature in Dubuque, IA [Cryer & Chan, 2008]




Let's assume that there is no trend in this time series. In this context, our goal is to estimate $s_t$, the seasonal component.

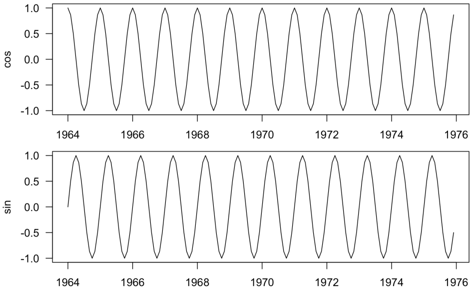
# Use a Harmonic Regression to Model Annual Cycles

**Model:** $s_t = \beta_0 + \beta_1 \cos(2\pi t) + \beta_2 \sin(2\pi t)$

$\Rightarrow$ annual cycles can be modeled by a linear combination of $\cos$ and $\sin$ with 1-year period.

In `R`, we can easily create these harmonics using the `harmonic` function in the `TSA` package

```
harmonics <- harmonic(tempdub, 1)
```



# R Code & Output

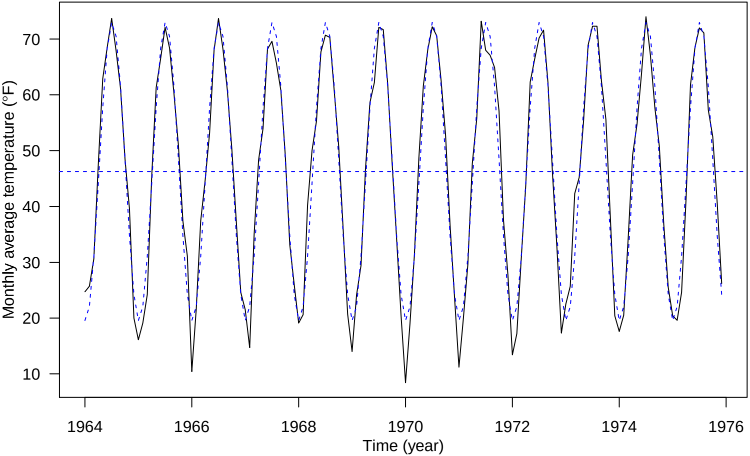
```{r}
harReg <- lm(tempdub ~ harmonics)
summary(harReg)
```

```
Call:
lm(formula = tempdub ~ harmonics)

Residuals:
     Min       1Q   Median       3Q      Max
-11.1580  -2.2756  -0.1457   2.3754  11.2671

Coefficients:
                    Estimate Std. Error t value Pr(>|t|)
(Intercept)          46.2660     0.3088 149.816  < 2e-16 ***
harmonicscos(2*pi*t) -26.7079     0.4367 -61.154  < 2e-16 ***
harmonicssin(2*pi*t)  -2.1697     0.4367  -4.968 1.93e-06 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# The Harmonic Regression Model Fit







# Seasonal Means Model





- **Harmonics regression** assumes the seasonal pattern has a regular shape, i.e., the height of the peaks is the same as the depth of the troughs

- A less restrictive approach is to model $\{s_t\}$ as

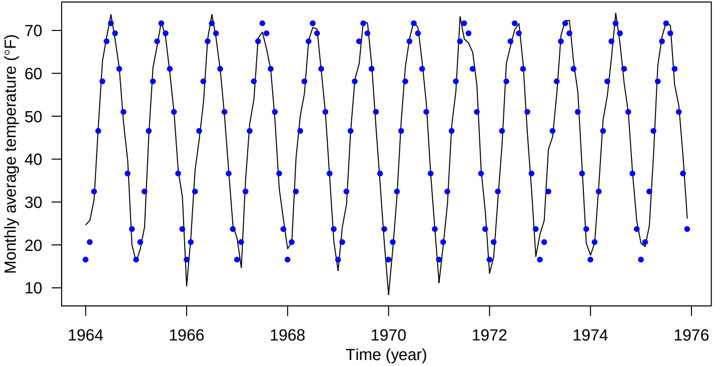
$$s_t = \begin{cases} \beta_1 & \text{for } t = 1, 1+d, 1+2d, \cdots \quad ; \\ \beta_2 & \text{for } t = 2, 2+d, 2+2d, \cdots \quad ; \\ \vdots & \vdots \quad ; \\ \beta_d & \text{for } t = d, 2d, 3d, \cdots \quad . \end{cases}$$

- This is the seasonal means model, the parameters $(\beta_1, \beta_2, \cdots, \beta_d)^T$ can be estimated under the linear model framework (think about ANOVA)

# R Output

```
Call:
lm(formula = tempdub ~ month - 1)

Residuals:
    Min      1Q  Median      3Q     Max
-8.2750 -2.2479  0.1125  1.8896  9.8250

Coefficients:
               Estimate Std. Error t value Pr(>|t|)
monthJanuary     16.608      0.987   16.83   <2e-16 ***
monthFebruary    20.650      0.987   20.92   <2e-16 ***
monthMarch       32.475      0.987   32.90   <2e-16 ***
monthApril       46.525      0.987   47.14   <2e-16 ***
monthMay         58.092      0.987   58.86   <2e-16 ***
monthJune        67.500      0.987   68.39   <2e-16 ***
monthJuly        71.717      0.987   72.66   <2e-16 ***
monthAugust      69.333      0.987   70.25   <2e-16 ***
monthSeptember   61.025      0.987   61.83   <2e-16 ***
monthOctober     50.975      0.987   51.65   <2e-16 ***
monthNovember    36.650      0.987   37.13   <2e-16 ***
monthDecember    23.642      0.987   23.95   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

# The Seasonal Means Model Fit

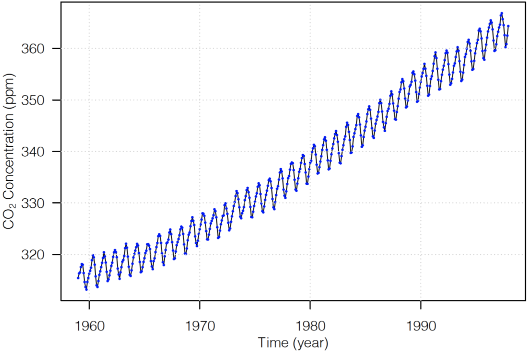


## Estimating the Trend and Seasonal variation Together







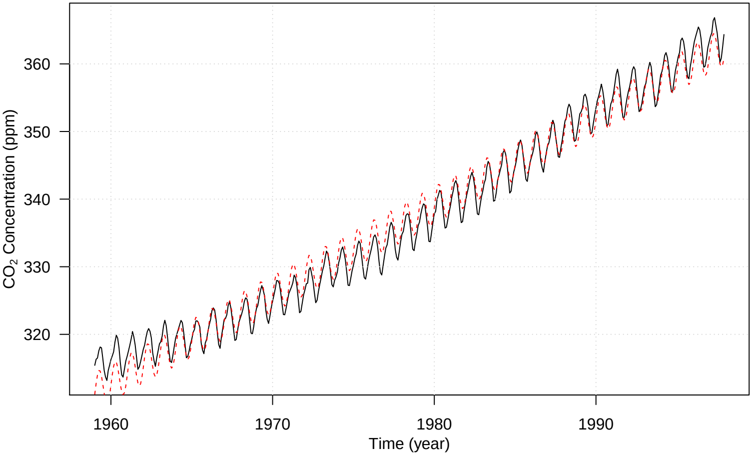



Let's perform a regression analysis to model both $\mu_t$ (assuming a linear time trend) and $s_t$ (using $\cos$ and $\sin$)

```{r}
time <- as.numeric(time(co2))
harmonics <- harmonic(co2, 1)

lm_trendSeason <- lm(co2 ~ time + harmonics)
summary(lm_trendSeason)
```

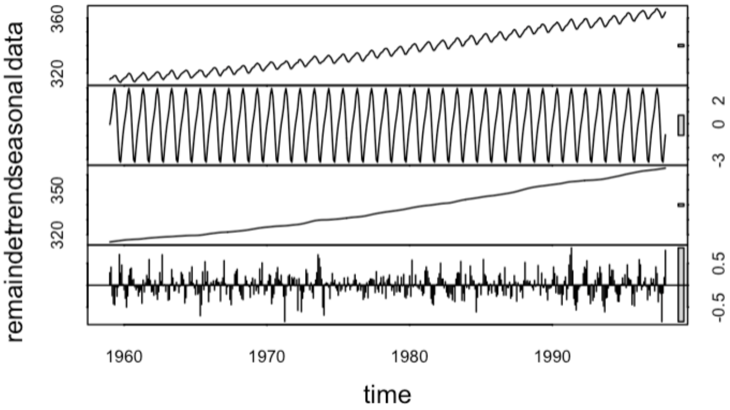
# The Regression Fit





# Seasonal and Trend decomposition using Loess [Cleveland, et. al., 1990]

```{r}
# Seasonal and Trend decomposition using Loess (STL)
par(mar = c(4, 3.6, 0.8, 0.6))
stl <- stl(co2, s.window = "periodic")
plot(stl, las = 1)
```



# Summary





These slides cover:

- Main features of a time series: trend, seasonality, and “noise”
- Estimating trends using multiple linear regression and “nonparametric” smoothing
- Estimating seasonality using harmonic regression and the seasonal mean model

## R Functions to Know

- Visualizing time series data: `plot` (for ts objects), `ts.plot`, `tsplot` (`astsa` package)
- Fitting time series regression: `lm`, `harmonic` (`TSA` package) for creating harmonic predictors, `filter` for smoothing
- Seasonal and trend decomposition: `stl`