

Lecture 23

Analysis of Variance (ANOVA)

STAT 8010 Statistical Methods I
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Notes

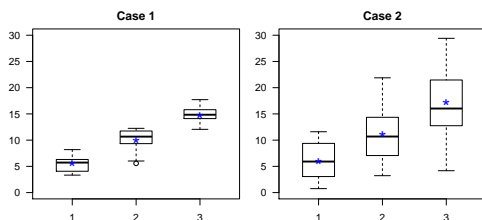
Testing for a Difference in More Than Two Means

- In the last few lectures we have seen how to test a difference in two means, using **two sample t-test**
- **Question:** what if we want to test if there are differences in a set of **more than two means**?
- The statistical tool for doing this is called **analysis of variance (ANOVA)**

Notes

A Quick Quiz: To Detect Differences in Means

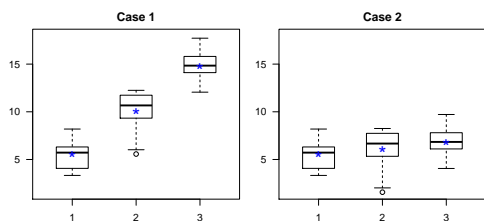
Question: Are group 1, 2, 3 for each case come from the same population?



Notes

Another Quiz: To Detect Differences in Means

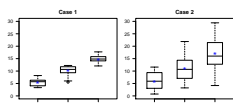
Question: Are group 1, 2, 3 for each case come from the same population?



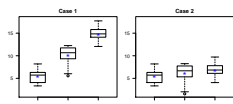
Notes

Decomposing Variance to Test for a Difference in Means

- In the first quiz, the data within each group is not very spread out for Case 1, while in Case 2 it is



- In the second quiz, the group means are quite different for Case 1, while they are not in Case 2



- In ANOVA, we compare **between group variance** ("signal") to **within group variance** ("noise") to detect a difference in means

Notes

Notation

$$X_{ij} = \mu_j + \varepsilon_{ij}, \varepsilon_{ij} \stackrel{i.i.d.}{\sim} N(0, \sigma^2), i = 1, \dots, n_j, 1 \leq j \leq J$$

- J : number of groups
- $\mu_j, j = 1, \dots, J$: population mean for j_{th} group
- $\bar{X}_j, j = 1, \dots, J$: sample mean for j_{th} group
- $s_j^2, j = 1, \dots, J$: sample variance for j_{th} group
- $N = \sum_{j=1}^J n_j$: overall sample size
- $\bar{X} = \frac{\sum_{j=1}^J \sum_{i=1}^{n_j} X_{ij}}{N}$: overall sample mean

Notes

Partition of Sums of Squares

“Sums of squares” refers to sums of squared deviations from some mean. ANOVA decomposes the **total sum of squares** into **treatment sum of squares** and **error sum of squares**:

- **Total sum of square:** $SSTo = \sum_{j=1}^J \sum_{i=1}^{n_j} (X_{ij} - \bar{X})^2$
- **Treatment sum of square:** $SSTr = \sum_{j=1}^J n_j (\bar{X}_j - \bar{X})^2$
- **Error sum of square:** $SSE = \sum_{j=1}^J (n_j - 1) s_j^2$

We can show that $SSTo = SSTr + SSE$



Notes

Mean squares

A mean square is a sum of squares divided by its associated degrees of freedom

- **Mean square of treatments:** $MSTr = \frac{SSTr}{J-1}$
- **Mean square of error:** $MSE = \frac{SSE}{N-J}$

Think of MSTr as the “signal”, and MSE as the “noise” when detecting a difference in means (μ_1, \dots, μ_J) . A nature test statistic is the signal-to-noise ratio i.e.,

$$F^* = \frac{MSTr}{MSE}$$



Notes

ANOVA Table and F Test

Source	df	SS	MS	F statistic
Treatment	$J - 1$	$SSTr$	$MSTr = \frac{SSTr}{J-1}$	$F = \frac{MSTr}{MSE}$
Error	$N - J$	SSE	$MSE = \frac{SSE}{N-J}$	
Total	$N - 1$	$SSTo$		

F-Test

- $H_0 : \mu_1 = \mu_2 = \dots = \mu_J$
 $H_a : \text{At least one mean is different}$
- Test Statistic: $F^* = \frac{MSTr}{MSE}$. Under H_0 , $F^* \sim F_{df_1=J-1, df_2=N-J}$
- **Assumptions:**
 - The distribution of each group is normal with equal variance (i.e. $\sigma_1^2 = \sigma_2^2 = \dots = \sigma_J^2$)
 - Responses for a given group are independent to each other



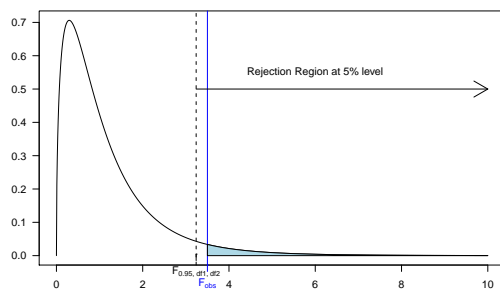
Notes

F Distribution and the Overall F-Test

Consider the observed F test statistic: $F_{obs} = \frac{MSTR}{MSE}$

- Should be “near” 1 if the means are equal
- Should be “larger than” 1 if means are not equal

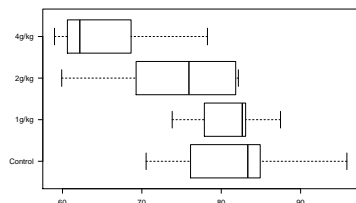
⇒ We use the null distribution of $F^* \sim F_{df_1=J-1, df_2=N-J}$ to quantify if F_{obs} is large enough to reject H_0



Notes

Example

A researcher who studies sleep is interested in the effects of ethanol on sleep time. She gets a sample of 20 rats and gives each an injection having a particular concentration of ethanol per body weight. There are 4 treatment groups, with 5 rats per treatment. She records Rapid eye movement (REM) sleep time for each rat over a 24-period. The results are plotted below:



Notes

Set Up Hypotheses and Compute Sums of Squares

- $H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4$ vs.
 $H_a : \text{At least one mean is different}$

- Sample statistics:

Treatment	Control	1g/kg	2g/kg	4g/kg
Mean	82.2	81.0	73.8	65.7
Std	9.6	5.3	9.4	7.9

- Overall Mean $\bar{X} = \frac{\sum_{j=1}^4 \sum_{i=1}^5 X_{ij}}{20} = 75.67$
- $SSTo = \sum_{j=1}^4 \sum_{i=1}^5 (X_{ij} - \bar{X})^2 = 1940.69$
- $SSTr = \sum_{j=1}^4 5 \times (\bar{X}_j - \bar{X})^2 = 861.13$
- $SSE = \sum_{j=1}^4 (5 - 1) \times s_j^2 = 1079.56$

Notes

ANOVA Table and F-Test

Source	df	SS	MS	F statistic
Treatment	$4 - 1 = 3$	861.13	$\frac{861.13}{3} = 287.04$	$\frac{287.04}{67.47} = 4.25$
Error	$20 - 4 = 16$	1079.56	$\frac{1079.56}{16} = 67.47$	
Total	19	1940.69		

Suppose we use $\alpha = 0.05$

- **Rejection Region Method:**

$$F_{obs} = 4.25 > F_{0.95, df_1=3, df_2=16} = 3.24$$

- **P-value Method:**

$$\mathbb{P}(F^* > F_{obs}) = \mathbb{P}(F^* > 4.25) = 0.022 < 0.05$$

Reject $H_0 \Rightarrow$ We do have enough evidence that not all of population means are equal at 5% level.

Notes

R Output

Analysis of Variance Table

Response: Response

	Df	Sum Sq	Mean Sq
Treatment	3	861.13	287.044
Residuals	16	1079.56	67.472

	F value	Pr(>F)
Treatment	4.2542	0.02173 *
Residuals		

Signif. codes:

0 '***' 0.001 '**' 0.01 '*'

0.05 '.' 0.1 ' ' 1

Notes

Summary

In this lecture, we learned

- **Analysis of Variance (ANOVA)**

- Between group variance vs. within group variance

- ANOVA Table

- Overall F-Test

If we reject H_0 , we'll want to know which group means are different. Therefore, in next lecture we will learn

- Multiple Comparisons

Notes