Lecture 24

Simple Linear Regression: Confidence/Prediction Intervals and Hypothesis Testing

Text: Chapter 11

STAT 8010 Statistical Methods I
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Agenda

1. **Review of Last Class**
2. **Confidence/Prediction Intervals**
3. **Hypothesis Testing**

Simple Linear Regression (SLR)

Y: dependent (response) variable; X: independent (predictor) variable

- In SLR we **assume** there is a **linear relationship** between X and Y:

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i,$$

  where $\mathrm{E}(\varepsilon_i) = 0$, and $\mathrm{Var}(\varepsilon_i) = \sigma^2, \forall i$. Furthermore, $\mathrm{Cov}(\varepsilon_i, \varepsilon_j) = 0, \forall i \neq j$

- **Least Squares Estimation:** $\mathrm{argmin}_{\beta_0,\beta_1} \sum_{i=1}^{n} (Y_i - (\beta_0 + \beta_1 X_i))^2 \Rightarrow$
  - $\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y})}{\sum_{i=1}^{n}(X_i-\bar{X})^2}$
  - $\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$
  - $\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(Y_i-\hat{Y}_i)^2}{n-2}$

- **Residuals**: $e_i = Y_i - \hat{Y}_i$, where $\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i$

Residual Analysis

- **Residual Analysis:** To check the appropriateness of SLR model
  - Is the regression function linear?
  - Do ε_i's have constant variance σ^2?
  - Are ε_i's indepdent to each other?

We plot residuals e_i's against X_i's (or $\hat{Y}_i$'s) to assess these aspects

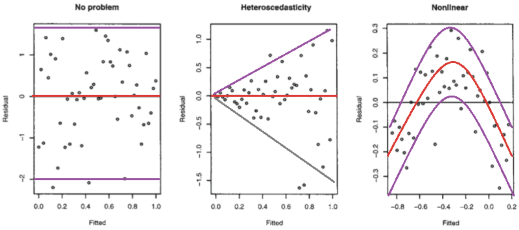
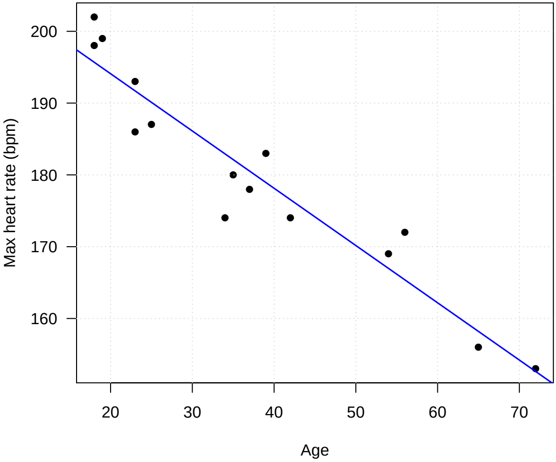


Figure: Figure courtesy of Faraway's Linear Models with R (2005, p. 59).

How (Un)certain We Are?



Can we formally quantify our estimation uncertainty? $\Rightarrow$

We need additional (distributional) assumption on ε

Normal Error Regression Model

Recall

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$$

- Further assume $\varepsilon_i \sim \mathrm{N}(0, \sigma^2) \Rightarrow Y_i \sim \mathrm{N}(\beta_0 + \beta_1 X_i, \sigma^2)$
- With normality assumption, we can derive the **sampling distribution** of $\hat{\beta}_1$ and $\hat{\beta}_0 \Rightarrow$
  - $\frac{\hat{\beta}_1 - \beta_1}{\hat{\sigma}_{\hat{\beta}_1}} \sim t_{n-2}, \quad \hat{\sigma}_{\hat{\beta}_1} = \frac{\hat{\sigma}}{\sqrt{\sum_{i=1}^n (X_i - \bar{X})^2}}$
  - $\frac{\hat{\beta}_0 - \beta_0}{\hat{\sigma}_{\hat{\beta}_0}} \sim t_{n-2}, \quad \hat{\sigma}_{\hat{\beta}_0} = \hat{\sigma}\sqrt{\left(\frac{1}{n} + \frac{\bar{X}^2}{\sum_{i=1}^n (X_i - \bar{X})^2}\right)}$

  where t_{n-2} denotes the Student's t distribution with $n-2$ degrees of freedom

Confidence Intervals

- Recall $\frac{\hat{\beta}_1 - \beta_1}{\hat{\sigma}_{\hat{\beta}_1}} \sim t_{n-2}$, we use this fact to construct **confidence intervals (CIs)** for β_1:

$$\left[\hat{\beta}_1 - t_{\alpha/2,n-2}\hat{\sigma}_{\hat{\beta}_1}, \hat{\beta}_1 + t_{\alpha/2,n-2}\hat{\sigma}_{\hat{\beta}_1}\right],$$

  where α is the **confidence level** and $t_{\alpha/2,n-2}$ denotes the $1 - \alpha/2$ percentile of a student's t distribution with $n - 2$ degrees of freedom

- Similarly, we can construct CIs for β_0:

$$\left[\hat{\beta}_0 - t_{\alpha/2,n-2}\hat{\sigma}_{\hat{\beta}_0}, \hat{\beta}_0 + t_{\alpha/2,n-2}\hat{\sigma}_{\hat{\beta}_0}\right]$$

Interval Estimation of $\mathrm{E}(Y_h)$

- We often interested in estimating the **mean** response for a particular value of predictor, say, X_h. Therefore we would like to construct CI for $\mathrm{E}[Y_h]$
- We need sampling distribution of $\hat{Y}_h$ to form CI:
  - $\frac{\hat{Y}_h - Y_h}{\hat{\sigma}_{\hat{Y}_h}} \sim t_{n-2}, \quad \hat{\sigma}_{\hat{Y}_h} = \hat{\sigma}\sqrt{\left(\frac{1}{n} + \frac{(X_h - \bar{X})^2}{\sum_{i=1}^n (X_i - \bar{X})^2}\right)}$
  - CI:

$$\left[\hat{Y}_h - t_{\alpha/2,n-2}\hat{\sigma}_{\hat{Y}_h}, \hat{Y}_h + t_{\alpha/2,n-2}\hat{\sigma}_{\hat{Y}_h}\right]$$

- **Quiz:** Use this formula to construct CI for β_0

Prediction Intervals

- Suppose we want to predict the response of a future observation given $X = X_h$
- We need to account for added variability as a new observation does not fall directly on the regression line (i.e., $Y_{\text{h(new)}} = \mathrm{E}[Y_h] + \varepsilon_h$)
- Replace $\hat{\sigma}_{\hat{Y}_h}$ by $\hat{\sigma}_{\hat{Y}_{\text{h(new)}}} = \hat{\sigma}\sqrt{\left(1 + \frac{1}{n} + \frac{(X_h - \bar{X})^2}{\sum_{i=1}^{n}(X_i - \bar{X})^2}\right)}$ to construct CIs for $Y_{\text{h(new)}}$

Maximum Heart Rate vs. Age Revisited

The maximum heart rate `MaxHeartRate` (HR_{max}) of a person is often said to be related to age `Age` by the equation:

$$HR_{max} = 220 - \text{Age}.$$

Suppose we have 15 people of varying ages are tested for their maximum heart rate (bpm)

Age	18	23	25	35	65	54	34	56	72	19	23	42	18	39	37
HR_{max}	202	186	187	180	156	169	174	172	153	199	193	174	198	183	178

- Construct the 95% CI for β_1
- Compute the estimate for mean `MaxHeartRate` given `Age` $= 40$ and construct the associated 90% CI
- Construct the prediction interval for a new observation given `Age` $= 40$

Maximum Heart Rate vs. Age: Hypothesis Test for Slope

1. $H_0 : \beta_1 = 0$ vs. $H_a : \beta_1 \neq 0$
2. Compute the **test statistic**: $t^* = \frac{\hat{\beta}_1 - 0}{\hat{\sigma}_{\hat{\beta}_1}} = \frac{-0.7977}{0.06996} = -11.40$
3. Compute **P-value**: $\mathrm{P}(|t^*| \geq |t_{obs}|) = 3.85 \times 10^{-8}$
4. Compare to α and draw conclusion:

> Reject H_0 at $\alpha = .05$ level, evidence suggests a negative linear relationship between `MaxHeartRate` and `Age`

Maximum Heart Rate vs. Age: Hypothesis Test for Intercept

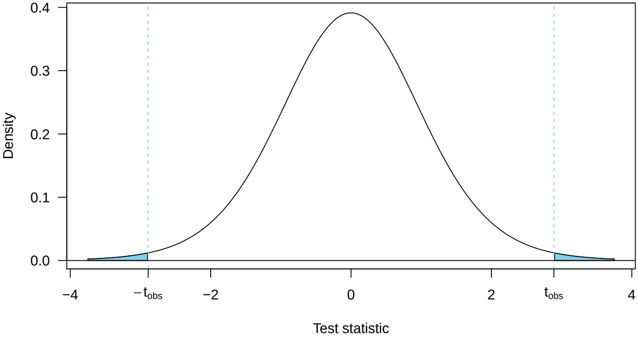
1. $H_0 : \beta_0 = 0$ vs. $H_a : \beta_0 \neq 0$
2. Compute the **test statistic**: $t^* = \frac{\hat{\beta}_0 - 0}{\hat{\sigma}_{\beta_0}} = \frac{210.0485}{2.86694} = 73.27$
3. Compute **P-value**: $\mathrm{P}(|t^*| \geq |t_{obs}|) \simeq 0$
4. Compare to α and draw conclusion:

> Reject H_0 at $\alpha = .05$ level, evidence suggests evidence suggests the intercept (the expected `MaxHeartRate` at age 0) is different from 0

Hypothesis Tests for $\beta_{\text{age}} = -1$

$H_0 : \beta_{\text{age}} = -1$ vs. $H_a : \beta_{\text{age}} \neq -1$

Test Statistic: $\frac{\hat{\beta}_{\text{age}} - (-1)}{\hat{\sigma}_{\hat{\beta}_{\text{age}}}} = \frac{-0.79773 - (-1)}{0.06996} = 2.8912$



P-value: $2 \times \mathbb{P}(t^* > 2.8912) = 0.013$, where $t^* \sim t_{df=13}$

Summary

In this lecture, we learned

- **Normal Error Regression Model** and statistical inference for β_0 and β_1
- **Confidence/Prediction Intervals**
- **Hypothesis Testing**

Next time we will talk about

1. Analysis of Variance (ANOVA) Approach to Regression
2. Correlation (r) & Coefficient of Determination (R^2)