Lecture 9

ARMA Models: Properties, Identification, and Estimation

Reading: Bowerman, O'Connell, and Koehler (2005): Chapter 9.2-9.4; Capter 10.1; Cryer and Chen (2008): Chapter 4.4-4.6; Chapter 6.1-6.3

MATH 4070: Regression and Time-Series Analysis

Whitney Huang
Clemson University

Properties of ARMA Models: Stationarity, Causality, and Invertibility

Tentative Model Identification Using ACF and PACF

Parameter Estimation

Agenda

1 Properties of ARMA Models: Stationarity, Causality, and Invertibility

2 Tentative Model Identification Using ACF and PACF

3 Parameter Estimation

Properties of ARMA Models: Stationarity, Causality, and Invertibility
Tentative Model Identification Using ACF and PACF
Parameter Estimation

ARMA(p, q) Processes

$\{\eta_t\}$ is an ARMA(p, q) process if it satisfies

$$\eta_t - \sum_{i=1}^{p} \phi_i \eta_{t-i} = Z_t + \sum_{j=1}^{q} \theta_j Z_{t-j},$$

where $\{Z_t\}$ is a $\mathrm{WN}(0, \sigma^2)$ process.

- Let $\phi(B) = 1 - \sum_{i=1}^{p} \phi_i B^i$ and $\theta(B) = 1 + \sum_{j=1}^{q} \theta_j B^j$. Then we can write it as

$$\phi(B)\eta_t = \theta(B)Z_t$$

ARMA(p, q) Processes

$\{\eta_t\}$ is an ARMA(p, q) process if it satisfies

$$\eta_t - \sum_{i=1}^{p} \phi_i \eta_{t-i} = Z_t + \sum_{j=1}^{q} \theta_j Z_{t-j},$$

where $\{Z_t\}$ is a $\mathrm{WN}(0, \sigma^2)$ process.

- Let $\phi(B) = 1 - \sum_{i=1}^{p} \phi_i B^i$ and $\theta(B) = 1 + \sum_{j=1}^{q} \theta_j B^j$. Then we can write it as

$$\phi(B)\eta_t = \theta(B)Z_t$$

- An ARMA(p, q) process $\{\tilde{\eta}_t\}$ with mean μ can be written as

$$\phi(B)(\tilde{\eta}_t - \mu) = \theta(B)Z_t$$

Properties of ARMA Models: Stationarity, Causality, and Invertibility
Tentative Model Identification Using ACF and PACF
Parameter Estimation

A Stationary Solution to the ARMA Equation

A zero-mean ARMA process is stationary if it can be written as a linear process, i.e., $\eta_t = \psi(B)Z_t$, where $\psi(B) = \sum_{j=-\infty}^{\infty} \psi_j B^j$ for an absolutely summable sequence $\{\psi_j\}$

- This only happens if one can "divide" by $\phi(B)$, i.e., it is stationary only if the following makes sense:

$$(\phi(B))^{-1}\,\phi(B)\eta_t = (\phi(B))^{-1}\,\theta(B)Z_t$$

$$\Rightarrow \eta_t = \underbrace{\frac{\theta(B)}{\phi(B)}}_{=\psi(B)} Z_t$$

A Stationary Solution to the ARMA Equation

A zero-mean ARMA process is stationary if it can be written as a linear process, i.e., $\eta_t = \psi(B)Z_t$, where $\psi(B) = \sum_{j=-\infty}^{\infty} \psi_j B^j$ for an absolutely summable sequence $\{\psi_j\}$

- This only happens if one can "divide" by $\phi(B)$, i.e., it is stationary only if the following makes sense:

$$\begin{aligned}(\phi(B))^{-1}\phi(B)\eta_t &= (\phi(B))^{-1}\theta(B)Z_t \\ \Rightarrow \eta_t &= \underbrace{\frac{\theta(B)}{\phi(B)}}_{=\psi(B)} Z_t\end{aligned}$$

- Let's forget about B is the backshift operator and replace it with z. Now consider whether we can divide $\theta(z)$ by $\phi(z)$

Roots of the AR Characteristic Polynomial and Stationarity

- A root of the polynomial $f(z) = \sum_{j=0}^{p} a_j z^j$ is a value ξ such that $f(\xi) = 0 \Rightarrow$ it can be real-valued $\mathbb{R}$ or complex-valued $\mathbb{C}$

Roots of the AR Characteristic Polynomial and Stationarity

- A root of the polynomial $f(z) = \sum_{j=0}^{p} a_j z^j$ is a value ξ such that $f(\xi) = 0 \Rightarrow$ it can be real-valued $\mathbb{R}$ or complex-valued $\mathbb{C}$
- For example, a root can take the form $\xi = a + b\,i$ for real number a and b. The modulus of a complex number $|\xi|$ is defined by

$$|\xi| = \sqrt{a^2 + b^2}$$

Roots of the AR Characteristic Polynomial and Stationarity

- A root of the polynomial $f(z) = \sum_{j=0}^{p} a_j z^j$ is a value ξ such that $f(\xi) = 0 \Rightarrow$ it can be real-valued $\mathbb{R}$ or complex-valued $\mathbb{C}$
- For example, a root can take the form $\xi = a + b\,i$ for real number a and b. The modulus of a complex number $|\xi|$ is defined by

$$|\xi| = \sqrt{a^2 + b^2}$$

- For any ARMA(p,q) process, a stationary and unique solution exists if and only if

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0,$$

for all $|z| = 1 \Rightarrow$ None of the roots of the AR characteristic equation have a modulus of exactly 1

Roots of the AR Characteristic Polynomial and Stationarity

- A root of the polynomial $f(z) = \sum_{j=0}^{p} a_j z^j$ is a value ξ such that $f(\xi) = 0 \Rightarrow$ it can be real-valued $\mathbb{R}$ or complex-valued $\mathbb{C}$
- For example, a root can take the form $\xi = a + b\,i$ for real number a and b. The modulus of a complex number $|\xi|$ is defined by

$$|\xi| = \sqrt{a^2 + b^2}$$

- For any ARMA(p,q) process, a stationary and unique solution exists if and only if

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0,$$

for all $|z| = 1 \Rightarrow$ None of the roots of the AR characteristic equation have a modulus of exactly 1

Roots of the AR Characteristic Polynomial and Stationarity

- A root of the polynomial $f(z) = \sum_{j=0}^{p} a_j z^j$ is a value ξ such that $f(\xi) = 0 \Rightarrow$ it can be real-valued $\mathbb{R}$ or complex-valued $\mathbb{C}$
- For example, a root can take the form $\xi = a + b\,i$ for real number a and b. The modulus of a complex number $|\xi|$ is defined by

$$|\xi| = \sqrt{a^2 + b^2}$$

- For any ARMA(p,q) process, a stationary and unique solution exists if and only if

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0,$$

for all $|z| = 1 \Rightarrow$ None of the roots of the AR characteristic equation have a modulus of exactly 1

Note: Stationarity of the ARMA process has nothing to do with the MA polynomial!

Properties of ARMA Models: Stationarity, Causality, and Invertibility

Tentative Model Identification Using ACF and PACF

Parameter Estimation

AR(4) Example

Consider the following AR(4) process

$$\eta_t = 2.7607\eta_{t-1} - 3.8106\eta_{t-2} + 2.6535\eta_{t-3} - 0.9238\eta_{t-4} + Z_t,$$

the AR characteristic polynomial is

$$\phi(z) = 1 - 2.7607z + 3.8106z^2 - 2.6535z^3 + 0.9238z^4$$

- Hard to find the roots of $\phi(z)$ –we use the `polyroot` function in `R`:
- Use `Mod` in `R` to calculate the modulus of the roots
- **Conclusion:**

Causal ARMA Processes

An ARMA process is causal if there exists constants $\{\psi_j\}$ with $\sum_{j=0}^{\infty}|\psi_j| < 0$ and $\eta_t = \sum_{j=0}^{\infty}\psi_j Z_{t-j}$, that is, we can write $\{\eta_t\}$ as an MA(∞) process depending only on the current and past values of $\{Z_t\}$

- Equivalently, an ARMA process is causal if and only if

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0,$$

for all $|z| \leq 1 \Rightarrow$ None of the roots of the AR characteristic equation have a modulus less than 1

Properties of ARMA Models: Stationarity, Causality, and Invertibility
Tentative Model Identification Using ACF and PACF
Parameter Estimation

Causal ARMA Processes

An ARMA process is causal if there exists constants $\{\psi_j\}$ with $\sum_{j=0}^{\infty} |\psi_j| < 0$ and $\eta_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$, that is, we can write $\{\eta_t\}$ as an MA(∞) process depending only on the current and past values of $\{Z_t\}$

- Equivalently, an ARMA process is causal if and only if

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0,$$

  for all $|z| \leq 1 \Rightarrow$ None of the roots of the AR characteristic equation have a modulus less than 1

- The previous AR(4) example is causal since each zero, ξ, of $\phi(\cdot)$ is such that $|\xi| > 1$

Properties of ARMA Models: Stationarity, Causality, and Invertibility

Tentative Model Identification Using ACF and PACF

Parameter Estimation

Causal ARMA Processes

An ARMA process is causal if there exists constants $\{\psi_j\}$ with $\sum_{j=0}^{\infty} |\psi_j| < 0$ and $\eta_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$, that is, we can write $\{\eta_t\}$ as an MA(∞) process depending only on the current and past values of $\{Z_t\}$

- Equivalently, an ARMA process is causal if and only if

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0,$$

  for all $|z| \leq 1 \Rightarrow$ None of the roots of the AR characteristic equation have a modulus less than 1

- The previous AR(4) example is causal since each zero, ξ, of $\phi(\cdot)$ is such that $|\xi| > 1$

Properties of ARMA Models: Stationarity, Causality, and Invertibility

Tentative Model Identification Using ACF and PACF

Parameter Estimation

Causal ARMA Processes

An ARMA process is causal if there exists constants $\{\psi_j\}$ with $\sum_{j=0}^{\infty} |\psi_j| < 0$ and $\eta_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$, that is, we can write $\{\eta_t\}$ as an MA(∞) process depending only on the current and past values of $\{Z_t\}$

- Equivalently, an ARMA process is causal if and only if

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0,$$

  for all $|z| \leq 1 \Rightarrow$ None of the roots of the AR characteristic equation have a modulus less than 1

- The previous AR(4) example is causal since each zero, ξ, of $\phi(\cdot)$ is such that $|\xi| > 1$

> **Note**: The causality of the ARMA process depends only on the AR polynomial!

Invertible ARMA Processes

An ARMA process is invertible if there exists constants $\{\pi_j\}$ with $\sum_{j=0}^{\infty} |\pi_j| < \infty$ and

$$Z_t = \sum_{j=0}^{\infty} \pi_j \eta_{t-j},$$

that is, we can write $\{Z_t\}$ as an AR(∞) process depending only on the current and past values of $\{\eta_t\}$

- A process is invertible if and only if

$$\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q \neq 0,$$

for all $|z| \leq 1 \Rightarrow$ None of the roots of the MA characteristic equation have a modulus less than 1

Properties of ARMA Models: Stationarity, Causality, and Invertibility
Tentative Model Identification Using ACF and PACF
Parameter Estimation

Properties of ARMA Models: Stationarity, Causality, and Invertibility

Tentative Model Identification Using ACF and PACF

Parameter Estimation

Invertible ARMA Processes

An ARMA process is invertible if there exists constants $\{\pi_j\}$ with $\sum_{j=0}^{\infty} |\pi_j| < \infty$ and

$$Z_t = \sum_{j=0}^{\infty} \pi_j \eta_{t-j},$$

that is, we can write $\{Z_t\}$ as an AR(∞) process depending only on the current and past values of $\{\eta_t\}$

- A process is invertible if and only if

  $$\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q \neq 0,$$

  for all $|z| \leq 1 \Rightarrow$ None of the roots of the MA characteristic equation have a modulus less than 1

- An ARMA process

  $$\eta_t - 0.5\eta_{t-1} = Z_t + 0.4Z_{t-1},$$

Invertible ARMA Processes

An ARMA process is invertible if there exists constants $\{\pi_j\}$ with $\sum_{j=0}^{\infty} |\pi_j| < \infty$ and

$$Z_t = \sum_{j=0}^{\infty} \pi_j \eta_{t-j},$$

that is, we can write $\{Z_t\}$ as an AR(∞) process depending only on the current and past values of $\{\eta_t\}$

- A process is invertible if and only if

  $$\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q \neq 0,$$

  for all $|z| \leq 1 \Rightarrow$ None of the roots of the MA characteristic equation have a modulus less than 1

- An ARMA process

  $$\eta_t - 0.5\eta_{t-1} = Z_t + 0.4Z_{t-1},$$

Properties of ARMA Models: Stationarity, Causality, and Invertibility

Tentative Model Identification Using ACF and PACF

Parameter Estimation

Properties of ARMA Models: Stationarity, Causality, and Invertibility
Tentative Model Identification Using ACF and PACF
Parameter Estimation

Invertible ARMA Processes

An ARMA process is invertible if there exists constants $\{\pi_j\}$ with $\sum_{j=0}^{\infty} |\pi_j| < \infty$ and

$$Z_t = \sum_{j=0}^{\infty} \pi_j \eta_{t-j},$$

that is, we can write $\{Z_t\}$ as an AR(∞) process depending only on the current and past values of $\{\eta_t\}$

- A process is invertible if and only if

  $$\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q \neq 0,$$

  for all $|z| \leq 1 \Rightarrow$ None of the roots of the MA characteristic equation have a modulus less than 1

- An ARMA process

  $$\eta_t - 0.5\eta_{t-1} = Z_t + 0.4Z_{t-1},$$

  with $\phi(z) = 1 - 0.5z$ and $\theta(z) = 1 + 0.4z$ has a root of the MA characteristic polynomial at $z = \frac{-1}{0.4} = -2.5$

Review of the Autocorrelation Function (ACF)

The autocorrelation function (ACF) measures the correlation of a stationary time series η_t with its own lagged values

- The theoretical ACF for MA processes can be computed as $\rho(h) = \frac{\sum_{j=0}^{q} \theta_j \theta_{j+h}}{\sum_{j=0}^{q} \theta_j^2}$, and via the Yule-Walker equation for AR processes

Review of the Autocorrelation Function (ACF)

The autocorrelation function (ACF) measures the correlation of a stationary time series η_t with its own lagged values

- The theoretical ACF for MA processes can be computed as $\rho(h) = \frac{\sum_{j=0}^{q} \theta_j \theta_{j+h}}{\sum_{j=0}^{q} \theta_j^2}$, and via the Yule-Walker equation for AR processes
- The ACF is useful in identifying the MA(q) order, as it cuts off after lag q

Partial Autocorrelation Functions (PACF)

The partial autocorrelation function (PACF) represents the partial correlation of a stationary time series $\{\eta_t\}$ with its own lagged values, while regressing out the effects of the time series at all shorter lags

- The PACF at lag h is the autocorrelation between η_t and η_{t+h} with the linear dependence between η_t and $\eta_{t+1}, \ldots, \eta_{t+h-1}$ removed

Properties of ARMA Models: Stationarity, Causality, and Invertibility

Tentative Model Identification Using ACF and PACF

Parameter Estimation

Partial Autocorrelation Functions (PACF)

The partial autocorrelation function (PACF) represents the partial correlation of a stationary time series $\{\eta_t\}$ with its own lagged values, while regressing out the effects of the time series at all shorter lags

- The PACF at lag h is the autocorrelation between η_t and η_{t+h} with the linear dependence between η_t and $\eta_{t+1}, \ldots, \eta_{t+h-1}$ removed
- PACF plots are a commonly used tool for identifying the order of an AR model, as the theoretical PACF "shuts off" past the order of the model (see an example on the next slide)

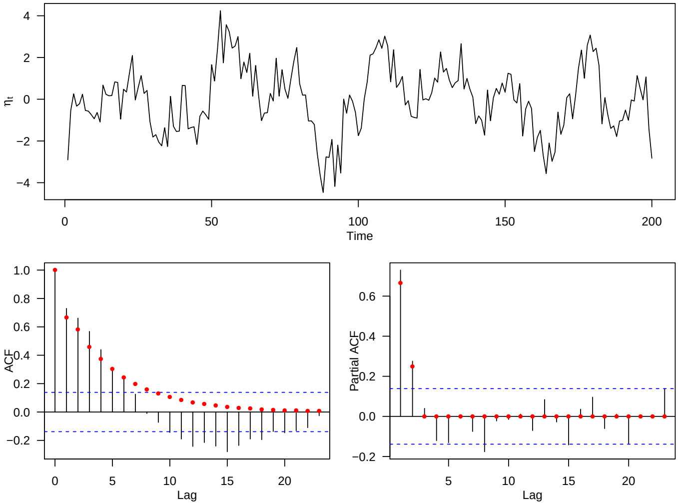
Partial Autocorrelation Functions (PACF)

The partial autocorrelation function (PACF) represents the partial correlation of a stationary time series $\{\eta_t\}$ with its own lagged values, while regressing out the effects of the time series at all shorter lags

- The PACF at lag h is the autocorrelation between η_t and η_{t+h} with the linear dependence between η_t and $\eta_{t+1}, \ldots, \eta_{t+h-1}$ removed
- PACF plots are a commonly used tool for identifying the order of an AR model, as the theoretical PACF “shuts off” past the order of the model (see an example on the next slide)
- One can use the function `pacf` in `R` to plot the PACF

An Example of PACF Plot

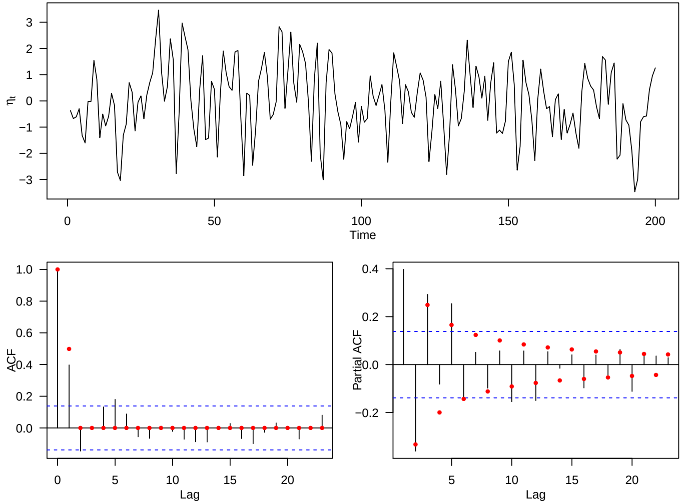
$$\eta_t - 0.5\eta_{t-1} - 0.25\eta_{t-2} = Z_t$$



The theoretical ACF decays exponentially, while the PACF cuts off at lag 2

PACF Plot for a MA Process

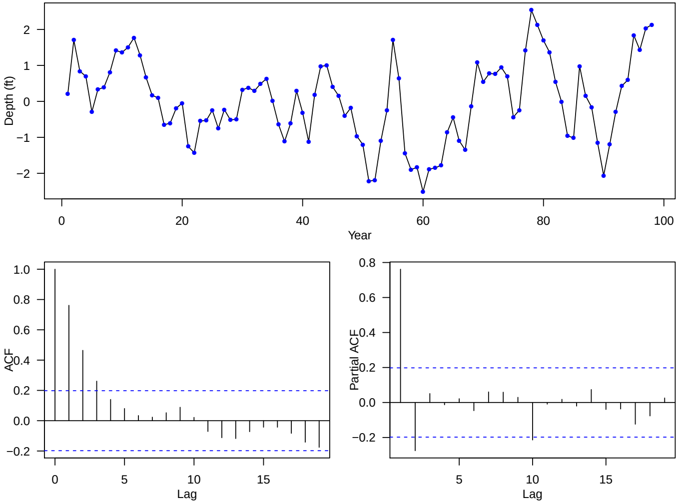
$$\eta_t = Z_t + Z_{t-1}$$



The theoretical ACF cuts off at lag 1, while the PACF decays exponentially

Properties of ARMA Models: Stationarity, Causality, and Invertibility
Tentative Model Identification Using ACF and PACF
Parameter Estimation

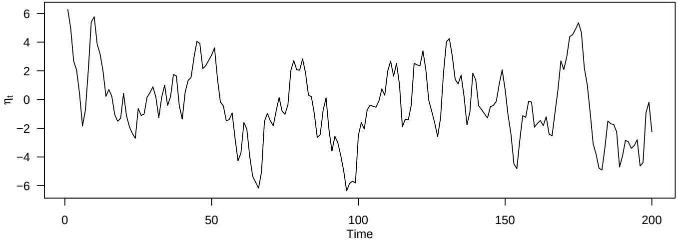
Lake Huron Series PACF Plot

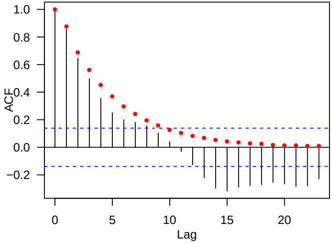


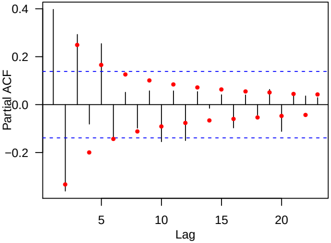
We can use both ACF and PACF plots to identify the potential ARMA model order

PACF Plot for a ARMA Process

$$\eta_t - 0.5\eta_{t-1} - 0.25\eta_{t-2} = Z_t + Z_{t-1}$$







Both the theoretical ACF and PACF decay exponentially

Identifying Plausible Stationary ARMA Models

We can use the sample ACF and PACF to help identify plausible models:

Model	ACF	PACF
MA(q)	cuts off after lag q	tails off exponentially
AR(p)	tails off exponentially	cuts off after lag p

Properties of ARMA Models: Stationarity, Causality, and Invertibility

Tentative Model Identification Using ACF and PACF

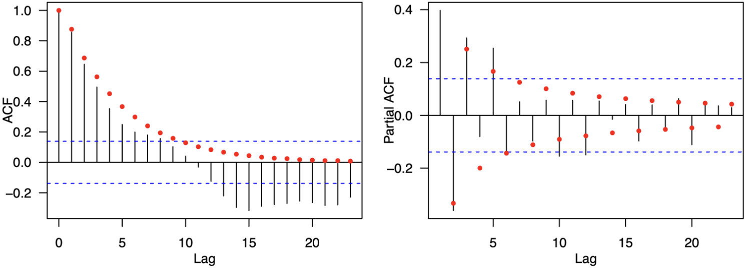
Parameter Estimation

Identifying Plausible Stationary ARMA Models

We can use the sample ACF and PACF to help identify plausible models:

Model	ACF	PACF
MA(q)	cuts off after lag q	tails off exponentially
AR(p)	tails off exponentially	cuts off after lag p

For ARMA(p, q) we will see a combination of the above



Estimation of the ARMA Process Parameters

Suppose we choose a ARMA(p, q) model for $\{\eta_t\}$

- Need to estimate the $p + q + 1$ parameters:
  - AR component $\{\phi_1, \cdots, \phi_p\}$
  - MA component $\{\theta_1, \cdots, \theta_q\}$
  - $\mathrm{Var}(Z_t) = \sigma^2$
- One strategy:
  - Do some preliminary estimation of the model parameters (e.g., via Yule-Walker estimates)
  - Follow-up with maximum likelihood estimation with Gaussian assumption

Properties of ARMA Models: Stationarity, Causality, and Invertibility

Tentative Model Identification Using ACF and PACF

Parameter Estimation

The Yule-Walker Method

Properties of ARMA Models: Stationarity, Causality, and Invertibility

Tentative Model Identification Using ACF and PACF

Parameter Estimation

Suppose η_t is a causal AR(p) process

$$\eta_t - \phi_1 \eta_{t-1} - \cdots - \phi_p \eta_{t-p} = Z_t$$

To estimate the parameters $\{\phi_1, \cdots, \phi_p\}$, we use a method of moments estimation scheme:

- Let $h = 0, 1, \cdots, p$. We multiply η_{t-h} to both sides

$$\eta_t \eta_{t-h} - \phi_1 \eta_{t-1} \eta_{t-h} - \cdots - \phi_p \eta_{t-p} \eta_{t-h} = Z_t \eta_{t-h}$$

- Taking expectations:

$$\mathbb{E}(\eta_t \eta_{t-h}) - \phi_1 \mathbb{E}(\eta_{t-1} \eta_{t-h}) - \cdots - \phi_p \mathbb{E}(\eta_{t-p} \eta_{t-h}) = \mathbb{E}(Z_t \eta_{t-h}),$$

we get $\boxed{\gamma(h) - \phi_1 \gamma(h-1) - \cdots - \phi_p \gamma(h-p) = \mathbb{E}(Z_t \eta_{t-h})}$

The Yule-Walker Equations

- When $h = 0$, $\mathbb{E}(Z_t \eta_{t-h}) = \mathrm{Cov}(Z_t, \eta_t) = \sigma^2$ (Why?) Therefore, we have

$$\gamma(0) - \sum_{j=1}^{p} \phi_j \gamma(j) = \sigma^2$$

- When $h > 0$, Z_t is uncorrelated with η_{t-h} (because the assumption of causality), thus $\mathbb{E}(Z_t \eta_{t-h}) = 0$ and we have

$$\gamma(h) - \sum_{j=1}^{p} \phi_j \gamma(h-j) = 0, \quad h = 1, 2, \cdots, p$$

- The Yule-Walker estimates are the solution of these equations when we replace $\gamma(h)$ by $\hat{\gamma}(h)$

The Yule-Walker Equations in Matrix Form

Let $\hat{\boldsymbol{\phi}} = (\hat{\phi}_1, \cdots, \hat{\phi}_p)^T$ be an estimate for $\boldsymbol{\phi} = (\phi_1, \cdots, \phi_p)^T$ and let

$$\hat{\boldsymbol{\Gamma}} = \begin{bmatrix} \hat{\gamma}(0) & \hat{\gamma}(1) & \cdots & \hat{\gamma}(p-1) \\ \hat{\gamma}(1) & \hat{\gamma}(0) & \cdots & \hat{\gamma}(p-2) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\gamma}(p-1) & \hat{\gamma}(p-2) & \cdots & \hat{\gamma}(0) \end{bmatrix}.$$

Then the Yule-Walker estimates of ϕ and σ^2 are

$$\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\Gamma}}^{-1}\hat{\boldsymbol{\gamma}},$$

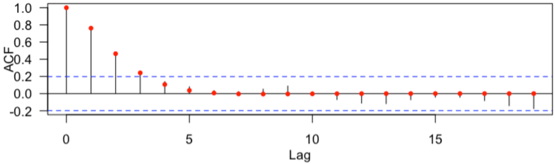
and

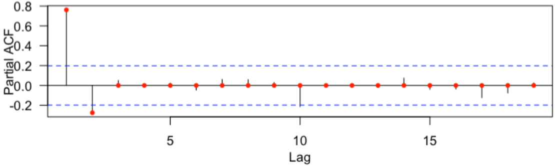
$$\hat{\sigma}^2 = \hat{\gamma}(0) - \hat{\boldsymbol{\phi}}^T\hat{\boldsymbol{\gamma}},$$

where $\hat{\boldsymbol{\gamma}} = (\hat{\gamma}(1), \cdots, \hat{\gamma}(p))^T$

Lake Huron Example in R

```
```{r}
YW_est <- ar(lm$residuals, aic = F, order.max = 2, method = "yw")
# plot sample and estimated acf/pacf
par(las = 1, mgp = c(2.2, 1, 0), mar = c(3.6, 3.6, 0.6, 0.6), mfrow = c(2, 1))
acf(lm$residuals)
acf_YWest <- ARMAacf(ar = YW_est$ar, lag.max = 23)
points(0:23, acf_YWest, col = "red", pch = 16, cex = 0.8)
pacf(lm$residuals)
pacf_YWest <- ARMAacf(ar = YW_est$ar, lag.max = 23, pacf = T)
points(1:23, pacf_YWest, col = "red", pch = 16, cex = 0.8)
```
```





Remarks on the Yule-Walker Method

- For large sample size, Yule-Walker estimator have (approximately) the same sampling distribution as maximum likelihood estimator (MLE), but with small sample size Yule-Walker estimator can be far less efficient than the MLE

[1] See **Least Squares Estimation** in Chapter 7.2 of Cryer and Chan (2008).

Remarks on the Yule-Walker Method

- For large sample size, Yule-Walker estimator have (approximately) the same sampling distribution as maximum likelihood estimator (MLE), but with small sample size Yule-Walker estimator can be far less efficient than the MLE

- The Yule-Walker method is a poor procedure for MA(q) and ARMA(p,q) processes with $q > 0$ (see Cryer Chan 2008, p. 150-151)

[1] See **Least Squares Estimation** in Chapter 7.2 of Cryer and Chan (2008).

Remarks on the Yule-Walker Method

- For large sample size, Yule-Walker estimator have (approximately) the same sampling distribution as maximum likelihood estimator (MLE), but with small sample size Yule-Walker estimator can be far less efficient than the MLE

- The Yule-Walker method is a poor procedure for MA(q) and ARMA(p,q) processes with $q > 0$ (see Cryer Chan 2008, p. 150-151)

- We move on the more versatile and popular method for estimating ARMA(p,q) parameters–maximum likelihood estimation[1]

[1] See **Least Squares Estimation** in Chapter 7.2 of Cryer and Chan (2008).

Maximum Likelihood Estimation

- The setup:

Properties of ARMA Models: Stationarity, Causality, and Invertibility

Tentative Model Identification Using ACF and PACF

Parameter Estimation

Maximum Likelihood Estimation

- The setup:
  - Model: $\boldsymbol{X} = (X_1, X_2, \cdots, X_n)$ has joint probability density function $f(\boldsymbol{x}|\boldsymbol{\omega})$ where $\boldsymbol{\omega} = (\omega_1, \omega_2, \cdots, \omega_p)$ is a vector of p parameters

Maximum Likelihood Estimation

- The setup:
  - Model: $\boldsymbol{X} = (X_1, X_2, \cdots, X_n)$ has joint probability density function $f(\boldsymbol{x}|\boldsymbol{\omega})$ where $\boldsymbol{\omega} = (\omega_1, \omega_2, \cdots, \omega_p)$ is a vector of p parameters
  - Data: $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$

Properties of ARMA Models: Stationarity, Causality, and Invertibility
Tentative Model Identification Using ACF and PACF
Parameter Estimation

Maximum Likelihood Estimation

- The setup:
  - Model: $\boldsymbol{X} = (X_1, X_2, \cdots, X_n)$ has joint probability density function $f(\boldsymbol{x}|\boldsymbol{\omega})$ where $\boldsymbol{\omega} = (\omega_1, \omega_2, \cdots, \omega_p)$ is a vector of p parameters
  - Data: $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$
- The likelihood function is defined as the the "likelihood" of the data, $\boldsymbol{x}$, given the parameters, $\boldsymbol{\omega}$

$$L_n(\boldsymbol{\omega}) = f(\boldsymbol{x}|\boldsymbol{\omega})$$

Maximum Likelihood Estimation

- The setup:
  - Model: $\boldsymbol{X} = (X_1, X_2, \cdots, X_n)$ has joint probability density function $f(\boldsymbol{x}|\boldsymbol{\omega})$ where $\boldsymbol{\omega} = (\omega_1, \omega_2, \cdots, \omega_p)$ is a vector of p parameters
  - Data: $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$
- The likelihood function is defined as the the "likelihood" of the data, $\boldsymbol{x}$, given the parameters, $\boldsymbol{\omega}$

$$L_n(\boldsymbol{\omega}) = f(\boldsymbol{x}|\boldsymbol{\omega})$$

- The maximum likelihood estimate (MLE) is the value of $\boldsymbol{\omega}$ which maximizes the likelihood, $L_n(\boldsymbol{\omega})$, of the data $\boldsymbol{x}$:

$$\hat{\boldsymbol{\omega}} = \underset{\boldsymbol{\omega}}{\operatorname{argmax}}\, L_n(\boldsymbol{\omega}).$$

  It is equivalent (and often easier) to maximize the log likelihood,

$$\ell_n(\boldsymbol{\omega}) = \log L_n(\boldsymbol{\omega})$$

The MLE for an i.i.d. Gaussian Process

Suppose $\{X_t\}$ be a Gaussian i.i.d. process with mean μ and variance σ^2. We observe a time series $\boldsymbol{x} = (x_1, \cdots, x_n)^T$.

- The likelihood function is

$$
\begin{aligned}
L_n(\mu, \sigma^2) &= f(\boldsymbol{x}|\mu, \sigma^2) \\
&= \prod_{t=1}^{n} f(x_t|\mu, \sigma) \\
&= \prod_{t=1}^{n} \left\{ \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x_t - \mu)^2}{2\sigma^2} \right] \right\} \\
&= (2\pi)^{-n/2} (\sigma^2)^{-n/2} \exp\left[-\frac{\sum_{t=1}^{n} (x_t - \mu)^2}{2\sigma^2} \right]
\end{aligned}
$$

The MLE for an i.i.d. Gaussian Process

Suppose $\{X_t\}$ be a Gaussian i.i.d. process with mean μ and variance σ^2. We observe a time series $\boldsymbol{x} = (x_1, \cdots, x_n)^T$.

- The likelihood function is

$$\begin{aligned} L_n(\mu, \sigma^2) &= f(\boldsymbol{x}|\mu, \sigma^2) \\ &= \prod_{t=1}^{n} f(x_t|\mu, \sigma) \\ &= \prod_{t=1}^{n} \left\{ \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x_t - \mu)^2}{2\sigma^2} \right] \right\} \\ &= (2\pi)^{-n/2} (\sigma^2)^{-n/2} \exp\left[-\frac{\sum_{t=1}^{n} (x_t - \mu)^2}{2\sigma^2} \right] \end{aligned}$$

- The log-likelihood function is

$$\begin{aligned} \ell_n(\mu, \sigma^2) &= \log L_n(\mu, \sigma^2) \\ &= -\frac{n}{2} \log(2\pi) - \frac{n}{2} \log(\sigma^2) - \frac{\sum_{t=1}^{n} (x_t - \mu)^2}{2\sigma^2} \end{aligned}$$

The MLE for an i.i.d. Gaussian Process

Suppose $\{X_t\}$ be a Gaussian i.i.d. process with mean μ and variance σ^2. We observe a time series $\boldsymbol{x} = (x_1, \cdots, x_n)^T$.

- The likelihood function is

$$
\begin{aligned}
L_n(\mu, \sigma^2) &= f(\boldsymbol{x}|\mu, \sigma^2) \\
&= \prod_{t=1}^{n} f(x_t|\mu, \sigma) \\
&= \prod_{t=1}^{n} \left\{ \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x_t - \mu)^2}{2\sigma^2} \right] \right\} \\
&= (2\pi)^{-n/2} (\sigma^2)^{-n/2} \exp\left[-\frac{\sum_{t=1}^{n} (x_t - \mu)^2}{2\sigma^2} \right]
\end{aligned}
$$

- The log-likelihood function is

$$
\begin{aligned}
\ell_n(\mu, \sigma^2) &= \log L_n(\mu, \sigma^2) \\
&= -\frac{n}{2} \log(2\pi) - \frac{n}{2} \log(\sigma^2) - \frac{\sum_{t=1}^{n} (x_t - \mu)^2}{2\sigma^2}
\end{aligned}
$$

The MLE for an i.i.d. Gaussian Process

Suppose $\{X_t\}$ be a Gaussian i.i.d. process with mean μ and variance σ^2. We observe a time series $\boldsymbol{x} = (x_1, \cdots, x_n)^T$.

- The likelihood function is

$$\begin{aligned} L_n(\mu, \sigma^2) &= f(\boldsymbol{x}|\mu, \sigma^2) \\ &= \prod_{t=1}^{n} f(x_t|\mu, \sigma) \\ &= \prod_{t=1}^{n} \left\{ \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x_t - \mu)^2}{2\sigma^2} \right] \right\} \\ &= (2\pi)^{-n/2} (\sigma^2)^{-n/2} \exp\left[-\frac{\sum_{t=1}^{n} (x_t - \mu)^2}{2\sigma^2} \right] \end{aligned}$$

- The log-likelihood function is

$$\begin{aligned} \ell_n(\mu, \sigma^2) &= \log L_n(\mu, \sigma^2) \\ &= -\frac{n}{2} \log(2\pi) - \frac{n}{2} \log(\sigma^2) - \frac{\sum_{t=1}^{n} (x_t - \mu)^2}{2\sigma^2} \end{aligned}$$

$$\Rightarrow \hat{\mu}_{\mathrm{MLE}} = \frac{\sum_{t=1}^{n} X_t}{n} = \bar{X}, \qquad \hat{\sigma}^2_{\mathrm{MLE}} = \frac{\sum_{t=1}^{n} (X_t - \bar{X})^2}{n}$$

Likelihood for Stationary Gaussian Time Series Models

Suppose $\{X_t\}$ be a mean zero stationary Gaussian time series with ACVF $\gamma(h)$. If $\gamma(h)$ depends on p parameters, $\boldsymbol{\omega} = (\omega_1, \cdots, \omega_p)$

- The likelihood of the data $\boldsymbol{x} = (x_1, \cdots, x_n)$ given the parameters $\boldsymbol{\omega}$ is

$$L_n(\boldsymbol{\omega}) = (2\pi)^{-n/2} |\boldsymbol{\Gamma}|^{-1/2} \exp\left(-\frac{1}{2}\boldsymbol{x}^T \boldsymbol{\Gamma}^{-1} \boldsymbol{x}\right),$$

  where $\boldsymbol{\Gamma}$ is the covariance matrix of $\boldsymbol{X} = (X_1, \cdots, X_n)^T$, $|\boldsymbol{\Gamma}|$ is the determinant of the matrix $\boldsymbol{\Gamma}$, and $\boldsymbol{\Gamma}^{-1}$ is the inverse of the matrix $\boldsymbol{\Gamma}$

- The log-likelihood is

$$\ell_n(\boldsymbol{\theta}) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\log|\boldsymbol{\Gamma}| - \frac{1}{2}\boldsymbol{x}^T \boldsymbol{\Gamma}^{-1} \boldsymbol{x}$$

Decomposing Joint Density into Conditional Densities

A joint distribution can be represented as the product of conditionals and a marginal distribution

- The simple version for $n = 2$ is:

$$f(x_1, x_2) = f(x_2|x_1)f(x_1)$$

Decomposing Joint Density into Conditional Densities

A joint distribution can be represented as the product of conditionals and a marginal distribution

- The simple version for $n = 2$ is:

$$f(x_1, x_2) = f(x_2|x_1)f(x_1)$$

- Extending for general n we get the following expression for the likelihood:

$$L_n(\boldsymbol{\theta}) = f(\boldsymbol{x};\boldsymbol{\theta}) = f(x_1)\prod_{t=2}^{n} f(x_t|x_{t-1},\cdots,x_1;\boldsymbol{\theta}),$$

and the log-likelihood is

$$\ell_n(\boldsymbol{\theta}) = \log f(\boldsymbol{x};\boldsymbol{\theta}) = \log(f(x_1)) + \sum_{t=2}^{n}\log f(x_t|x_{t-1},\cdots,x_1;\boldsymbol{\theta}).$$

AR(1) Log-likelihood

Let $\{\eta_1, \eta_2, \cdots, \eta_n\}$ be a realization of a zero-mean stationary AR(1) Gaussian time series. Let $\boldsymbol{\theta} = (\phi, \sigma^2)$

$$\ell_n(\boldsymbol{\theta}) = \underbrace{\log(f(\eta_1))}_{\ell_{n,1}} + \underbrace{\sum_{t=2}^{n} \log f(\eta_t|\eta_{t-1}, \cdots, \eta_1; \boldsymbol{\theta})}_{\ell_{n,2}}.$$

AR(1) Log-likelihood

Let $\{\eta_1, \eta_2, \cdots, \eta_n\}$ be a realization of a zero-mean stationary AR(1) Gaussian time series. Let $\boldsymbol{\theta} = (\phi, \sigma^2)$

$$\ell_n(\boldsymbol{\theta}) = \underbrace{\log(f(\eta_1))}_{\ell_{n,1}} + \underbrace{\sum_{t=2}^{n} \log f(\eta_t|\eta_{t-1}, \cdots, \eta_1; \boldsymbol{\theta})}_{\ell_{n,2}}.$$

Note that for $t \geq 2$, $f(\eta_t|\eta_{t-1}, \cdots, \eta_1) = f(\eta_t|\eta_{t-1})$, where $[\eta_t|\eta_{t-1}] \sim \mathrm{N}(\phi\eta_{t-1}, \sigma^2) \Rightarrow \ell_{n,2} =$

$$-\frac{(n-1)}{2}\log 2\pi - \frac{(n-1)}{2}\log \sigma^2 - \frac{\sum_{t=2}^{n}(\eta_t - \phi\eta_{t-1})^2}{2\sigma^2}$$

AR(1) Log-likelihood

Let $\{\eta_1, \eta_2, \cdots, \eta_n\}$ be a realization of a zero-mean stationary AR(1) Gaussian time series. Let $\boldsymbol{\theta} = (\phi, \sigma^2)$

$$\ell_n(\boldsymbol{\theta}) = \underbrace{\log(f(\eta_1))}_{\ell_{n,1}} + \underbrace{\sum_{t=2}^{n} \log f(\eta_t | \eta_{t-1}, \cdots, \eta_1; \boldsymbol{\theta})}_{\ell_{n,2}}.$$

Note that for $t \geq 2$, $f(\eta_t|\eta_{t-1}, \cdots, \eta_1) = f(\eta_t|\eta_{t-1})$, where $[\eta_t|\eta_{t-1}] \sim \mathrm{N}(\phi\eta_{t-1}, \sigma^2) \Rightarrow \ell_{n,2} =$

$$-\frac{(n-1)}{2}\log 2\pi - \frac{(n-1)}{2}\log\sigma^2 - \frac{\sum_{t=2}^{n}(\eta_t - \phi\eta_{t-1})^2}{2\sigma^2}$$

Also, we know $[\eta_1] \sim \mathrm{N}\left(0, \frac{\sigma^2}{(1-\phi^2)}\right) \Rightarrow \ell_{1,n} =$

$$\frac{-\log 2\pi}{2} - \frac{\log\sigma^2}{2} + \frac{\log(1-\phi^2)}{2} - \frac{(1-\phi^2)\eta_1^2}{2\sigma^2}$$

AR(1) Log-likelihood

Let $\{\eta_1, \eta_2, \cdots, \eta_n\}$ be a realization of a zero-mean stationary AR(1) Gaussian time series. Let $\boldsymbol{\theta} = (\phi, \sigma^2)$

$$\ell_n(\boldsymbol{\theta}) = \underbrace{\log(f(\eta_1))}_{\ell_{n,1}} + \underbrace{\sum_{t=2}^{n} \log f(\eta_t|\eta_{t-1}, \cdots, \eta_1; \boldsymbol{\theta})}_{\ell_{n,2}}.$$

Note that for $t \geq 2$, $f(\eta_t|\eta_{t-1}, \cdots, \eta_1) = f(\eta_t|\eta_{t-1})$, where $[\eta_t|\eta_{t-1}] \sim \mathrm{N}(\phi\eta_{t-1}, \sigma^2) \Rightarrow \ell_{n,2} =$

$$-\frac{(n-1)}{2}\log 2\pi - \frac{(n-1)}{2}\log\sigma^2 - \frac{\sum_{t=2}^{n}(\eta_t - \phi\eta_{t-1})^2}{2\sigma^2}$$

Also, we know $[\eta_1] \sim \mathrm{N}\left(0, \frac{\sigma^2}{(1-\phi^2)}\right) \Rightarrow \ell_{1,n} =$

$$\frac{-\log 2\pi}{2} - \frac{\log\sigma^2}{2} + \frac{\log(1-\phi^2)}{2} - \frac{(1-\phi^2)\eta_1^2}{2\sigma^2}$$

$$\Rightarrow \ell_n(\boldsymbol{\theta}) = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\sigma^2 - \frac{\sum_{t=2}^{n}(\eta_t - \phi\eta_{t-1})^2}{2\sigma^2} + \frac{\log(1-\phi^2)}{2} - \frac{(1-\phi^2)\eta_1^2}{2\sigma^2}$$

AR(1) Log-likelihood Cont'd

Properties of ARMA Models: Stationarity, Causality, and Invertibility

Tentative Model Identification Using ACF and PACF

Parameter Estimation

$$\ell_n(\boldsymbol{\theta}) = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\sigma^2 + \frac{\log(1-\phi^2)}{2} - \frac{S(\phi)}{2\sigma^2},$$

where $S(\phi) = \sum_{t=2}^{n}(\eta_t - \phi\eta_{t-1})^2 + (1-\phi^2)\eta_1^2$

- For given value of ϕ, $\ell_n(\phi, \sigma^2)$ can be maximized analytically with respect to σ^2

$$\hat{\sigma}^2 = \frac{S(\hat{\phi})}{n}$$

AR(1) Log-likelihood Cont'd

$$\ell_n(\boldsymbol{\theta}) = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\sigma^2 + \frac{\log(1-\phi^2)}{2} - \frac{S(\phi)}{2\sigma^2},$$

where $S(\phi) = \sum_{t=2}^{n}(\eta_t - \phi\eta_{t-1})^2 + (1-\phi^2)\eta_1^2$

- For given value of ϕ, $\ell_n(\phi, \sigma^2)$ can be maximized analytically with respect to σ^2

$$\hat{\sigma}^2 = \frac{S(\hat{\phi})}{n}$$

- Estimation of ϕ can be simplified by maximizing the **conditional sum-of-squares** $(\sum_{t=2}^{n}(\eta_t - \phi\eta_{t-1})^2)$

Likelihood Calculation via Innovations

Let the best linear one-step predictor of X_t be

$$\hat{X}_t = \begin{cases} 0, & t = 1; \\ P_{t-1} X_t, & t = 2, \cdots, n \end{cases}$$

- The one-step prediction errors or innovations are defined

  $$U_t = X_t - \hat{X}_t, \quad t = 1, \cdots, n,$$

  and the associated mean squared error is

  $$\nu_{t-1} = \mathbb{E}\left[(X_t - \hat{X}_t)^2\right] = \mathbb{E}(U_t^2), \quad t = 1, \cdots, n.$$

- For a causal ARMA process we can write $\nu_{t-1} = \sigma^2 r_{t-1}$, where r_t only depends on the AR and MA parameters $\boldsymbol{\phi}$ and $\boldsymbol{\theta}$, but not σ^2

Working with the Innovations

- Result I: $\{U_t\}$ is an independent set of RVs with

$$U_t \sim \mathrm{N}(0, \nu_{t-1}), t = 1, \cdots, n$$

$\Rightarrow$ the one-step prediction errors are uncorrelated with one another, and each each a normal distribution

Properties of ARMA Models: Stationarity, Causality, and Invertibility
Tentative Model Identification Using ACF and PACF
Parameter Estimation

Working with the Innovations

- Result I: $\{U_t\}$ is an independent set of RVs with

$$U_t \sim \mathrm{N}(0, \nu_{t-1}), t = 1, \cdots, n$$

$\Rightarrow$ the one-step prediction errors are uncorrelated with one another, and each each a normal distribution

- Result II: The likelihoods are the same if we use a model based on realizations of $\{X_t\}$ or a model based on realizations of $\{U_t\}$

Properties of ARMA Models: Stationarity, Causality, and Invertibility

Tentative Model Identification Using ACF and PACF

Parameter Estimation

Working with the Innovations

- Result I: $\{U_t\}$ is an independent set of RVs with

$$U_t \sim \mathrm{N}(0, \nu_{t-1}), t = 1, \cdots, n$$

  $\Rightarrow$ the one-step prediction errors are uncorrelated with one another, and each each a normal distribution

- Result II: The likelihoods are the same if we use a model based on realizations of $\{X_t\}$ or a model based on realizations of $\{U_t\}$

- Therefore

$$\ell_n(\boldsymbol{\omega}) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{n}\log(\nu_{t-1}) - \frac{1}{2}\sum_{t=1}^{n}\left(\frac{u_t^2}{\nu_{t-1}}\right).$$

  For a causal ARMA process this becomes

$$\ell_n(\boldsymbol{\phi}, \boldsymbol{\theta}, \sigma^2) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2}\sum_{t=1}^{n}\log(r_{t-1}) - \frac{1}{2\sigma^2}\sum_{t=1}^{n}\left(\frac{u_t^2}{r_{t-1}}\right)$$

The MLEs of σ^2, ϕ, and θ

- Now take the derivative of ℓ_n with respect to σ^2, setting the derivative equal to zero and solving for $\sigma^2 \Rightarrow$

$$\hat{\sigma}^2 = \frac{S(\boldsymbol{\phi}, \boldsymbol{\theta})}{n},$$

  where

$$S(\boldsymbol{\phi}, \boldsymbol{\theta}) = \sum_{t=1}^{n} \left(\frac{u_t^2}{r_{t-1}} \right).$$

- Substituting $\hat{\sigma}^2$ into ℓ_n, the MLE estimates of ϕ and $\boldsymbol{\theta}$, denoted by $\hat{\phi}$ and $\hat{\boldsymbol{\theta}}$, respectively, are those values which **maximize**

$$\tilde{\ell}_n(\boldsymbol{\phi}, \boldsymbol{\theta}, \hat{\sigma}^2) = -\frac{n}{2} \log(\hat{\sigma}^2) - \frac{1}{2} \sum_{t=1}^{n} \log(r_{t-1}) - \frac{1}{2\hat{\sigma}^2} \sum_{t=1}^{n} \left(\frac{u_t^2}{r_{t-1}} \right)$$

Properties of ARMA Models: Stationarity, Causality, and Invertibility

Tentative Model Identification Using ACF and PACF

Parameter Estimation