Extreme Value Analysis for Climate Research

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Clemson Math Club, January 17, 2020

Who am I?

- **First year** Assistant Professor of Applied Statistics
- Born in Laramie, Wyoming, grew up in Taiwan

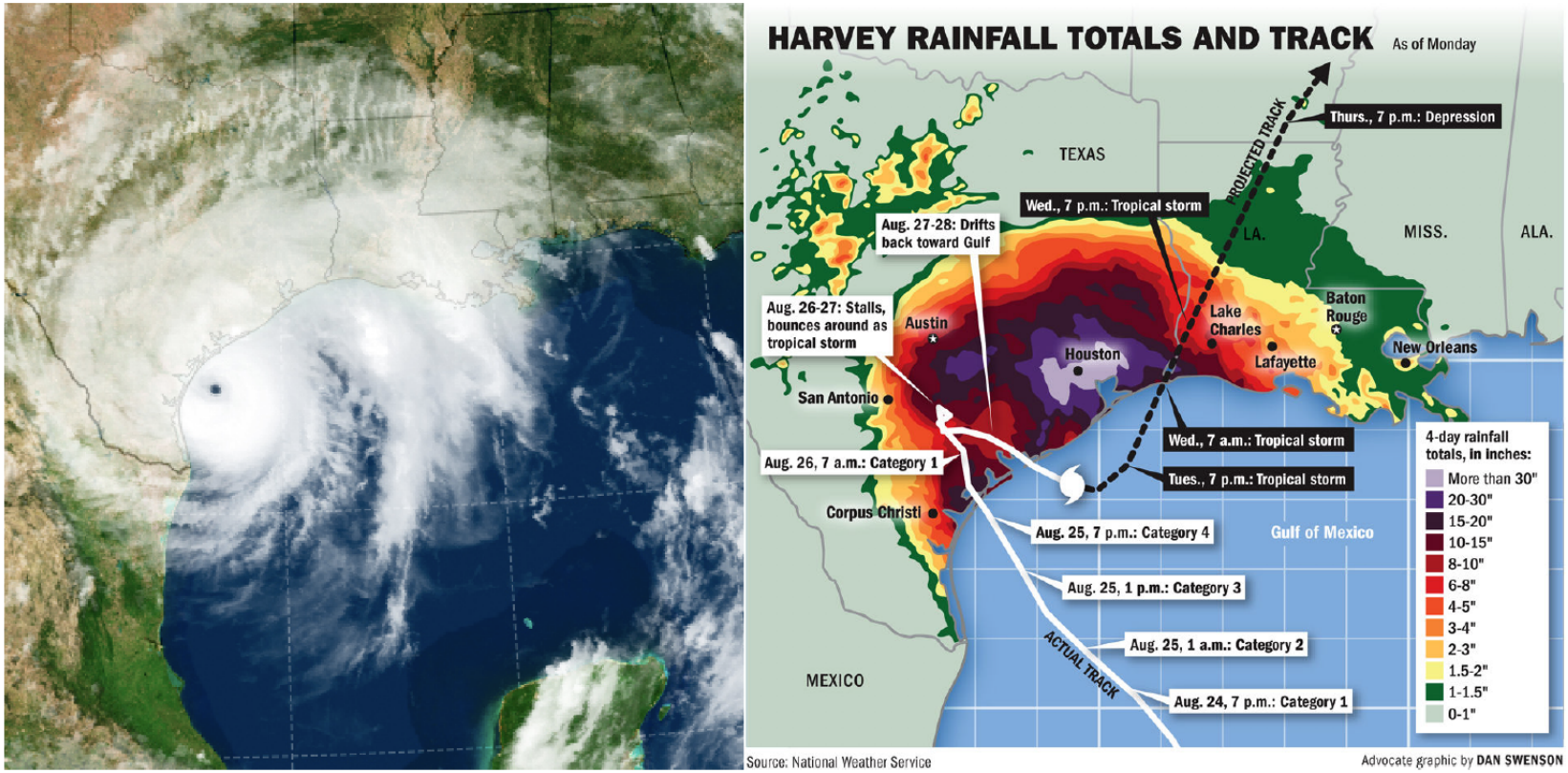
- Got a B.S. in Mechanical Engineering, switched to Statistics in graduate school
- Ph.D. in Statistics, 2017, Purdue; SAMSI/CANSSI Postdoc 2017-2019

PURDUE

samsi
NSF Duke NCSU UNC

University of Victoria

Extreme Rainfall During Hurricane Harvey



Source: NASA (Left); National Weather Service (Right)

- *"A storm forces Houston, the limitless city, to consider its limits"* – The New York Times (8.31.17)

Scientific Questions

- How to estimate the magnitude of extreme events (e.g. 100-year rainfall)?
- How extremes vary in space?
- How extremes may change in future climate conditions?

Digression: Central Limit Theorem

What is Central Limit Theorem (CLT)?

The **sampling distribution** of the **mean** will become approximately **normally distributed** as the **sample size becomes larger**, **irrespective of the shape of the population distribution!**

Let $X_1, X_2, \cdots, X_n \overset{i.i.d.}{\sim} F$ with $\mu = \mathbb{E}[X_i]$ and $\sigma^2 = \text{Var}[X_i] < \infty$. Then $\bar{X}_n = \frac{\sum_{i=1}^n X_i}{n} \xrightarrow{d} \text{N}(\mu, \frac{\sigma^2}{n})$ as $n \to \infty$.

▸ Proof

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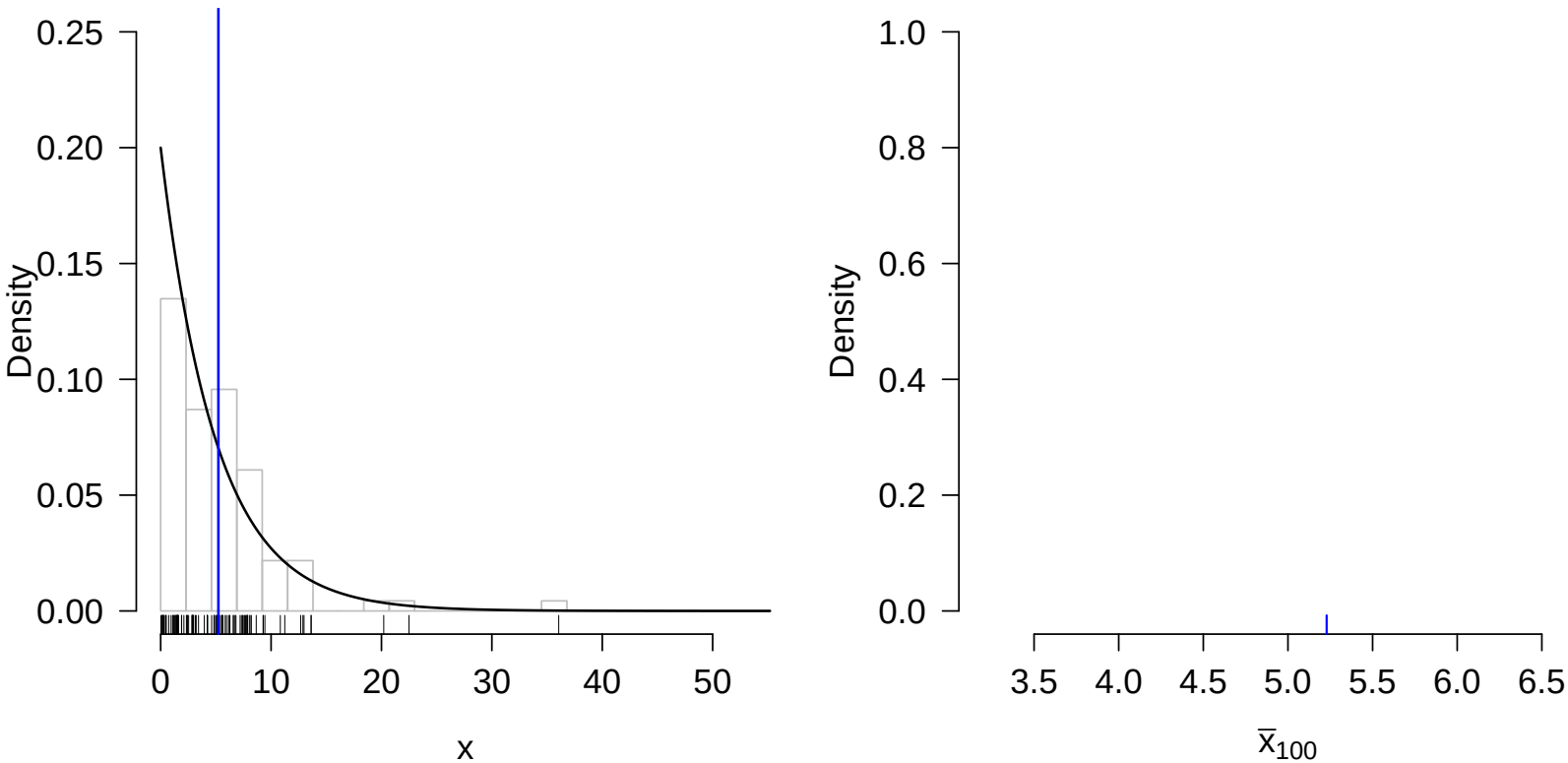
CLT in Action

1. Generate 100 random numbers $(n = 100)$ from an Exponential distribution
2. Compute the sample mean of these 100 random numbers
3. Repeat this process 120 times



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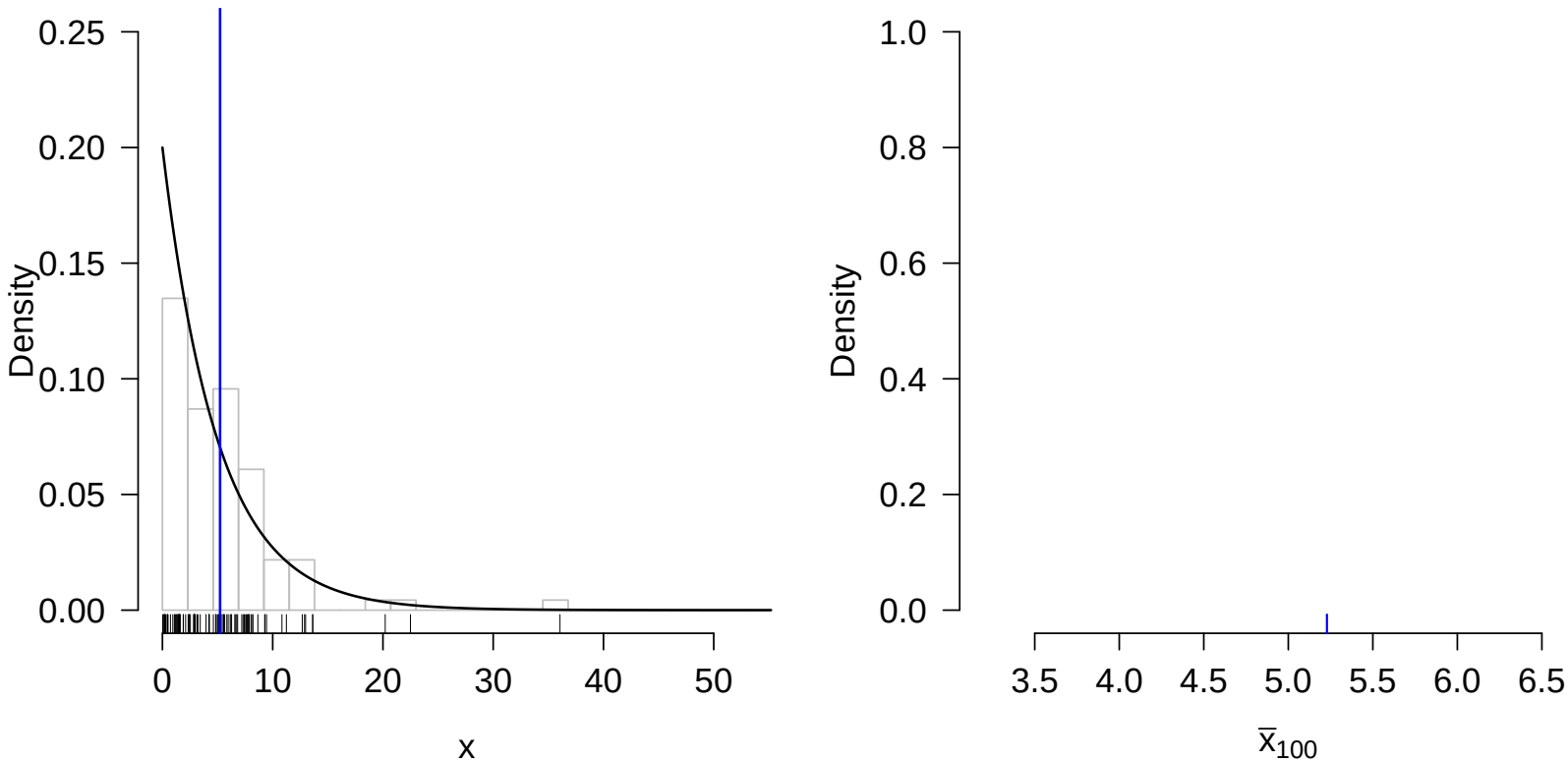

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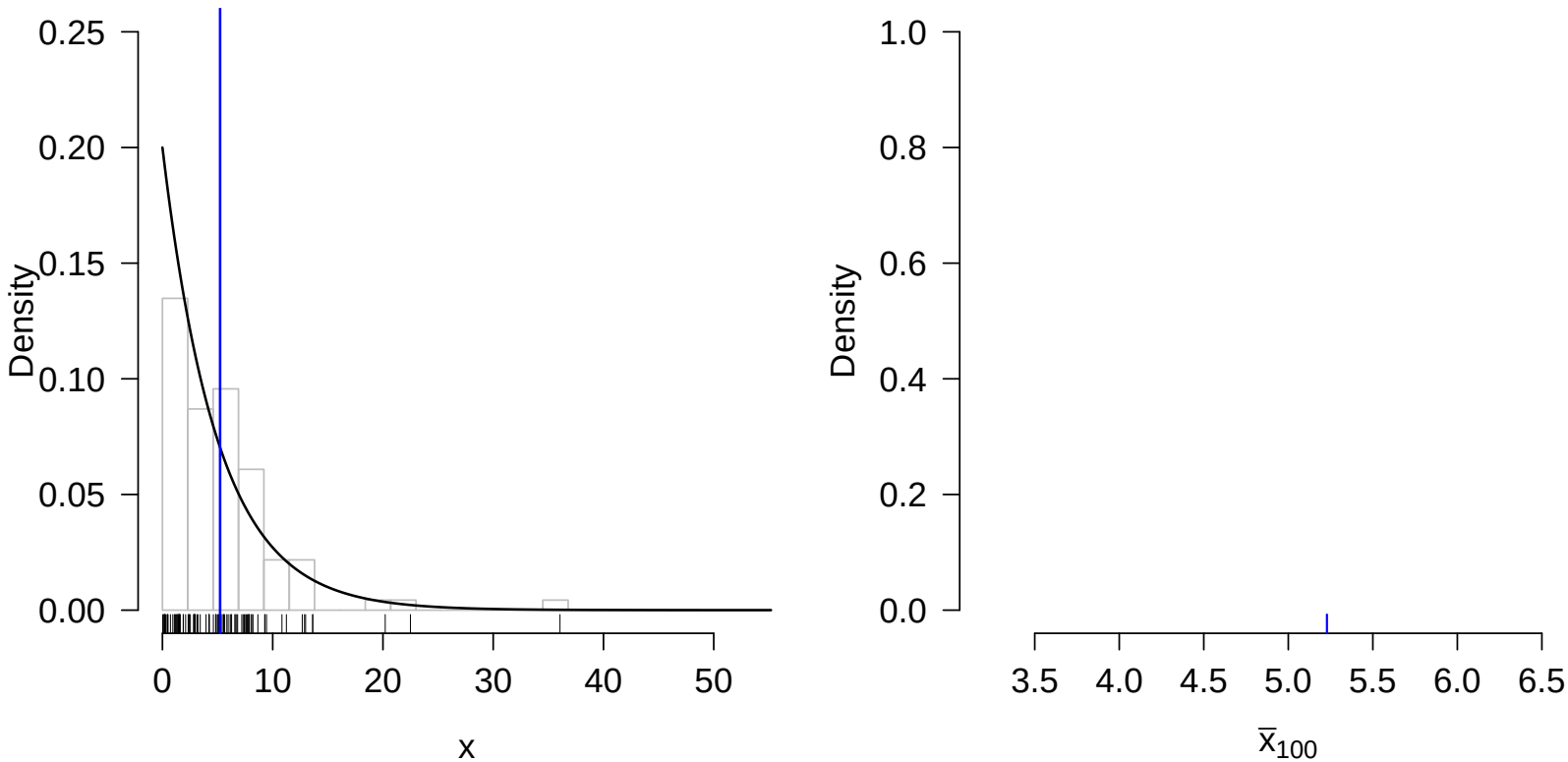

Can we find an analog of CLT for the unusually large values?

Distribution of the Sample Maximum

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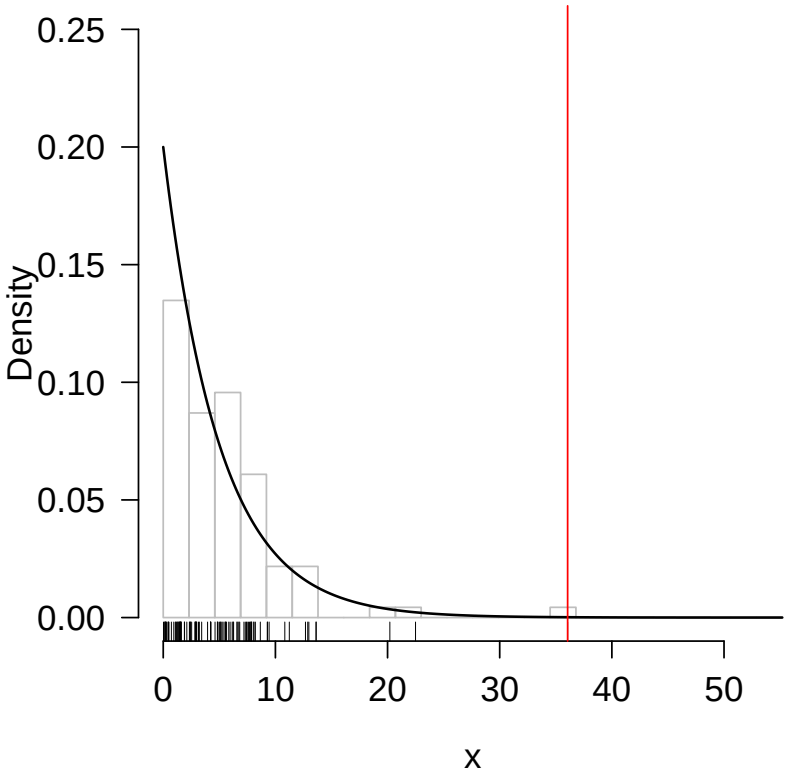
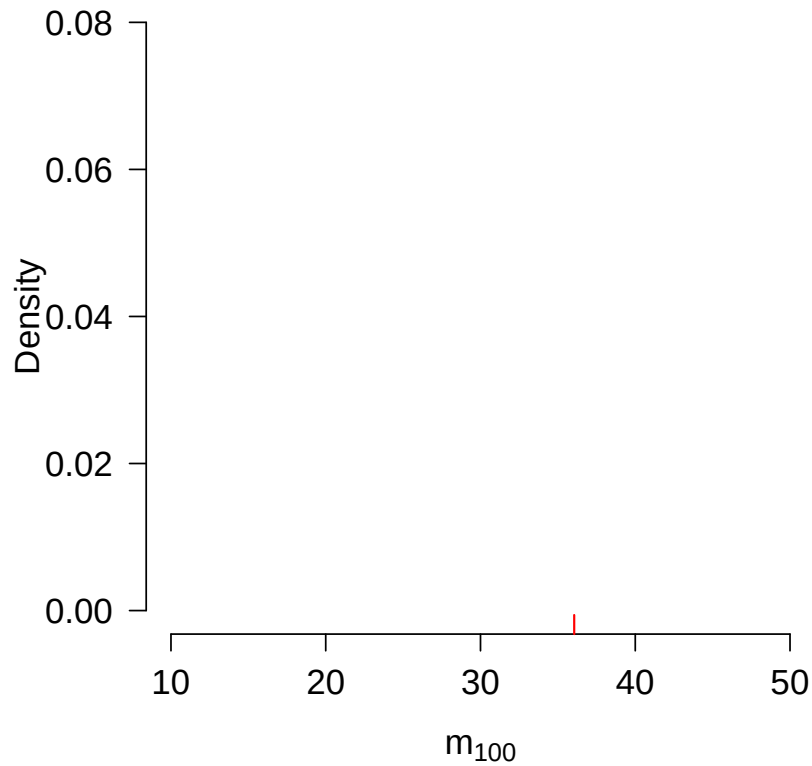
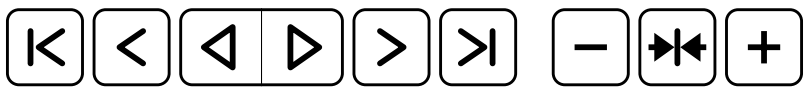






Extremal Types Theorem (Fisher–Tippett 1928, Gnedenko 1943)

Define $M_n = \max\{X_1, \cdots, X_n\}$ where $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} F$. If $\exists\, a_n > 0$ and $b_n \in \mathbb{R}$ such that, as $n \to \infty$, if

$$\mathbb{P}\left(\frac{M_n - b_n}{a_n} \leq x\right) \xrightarrow{d} \mathsf{G}(x)$$

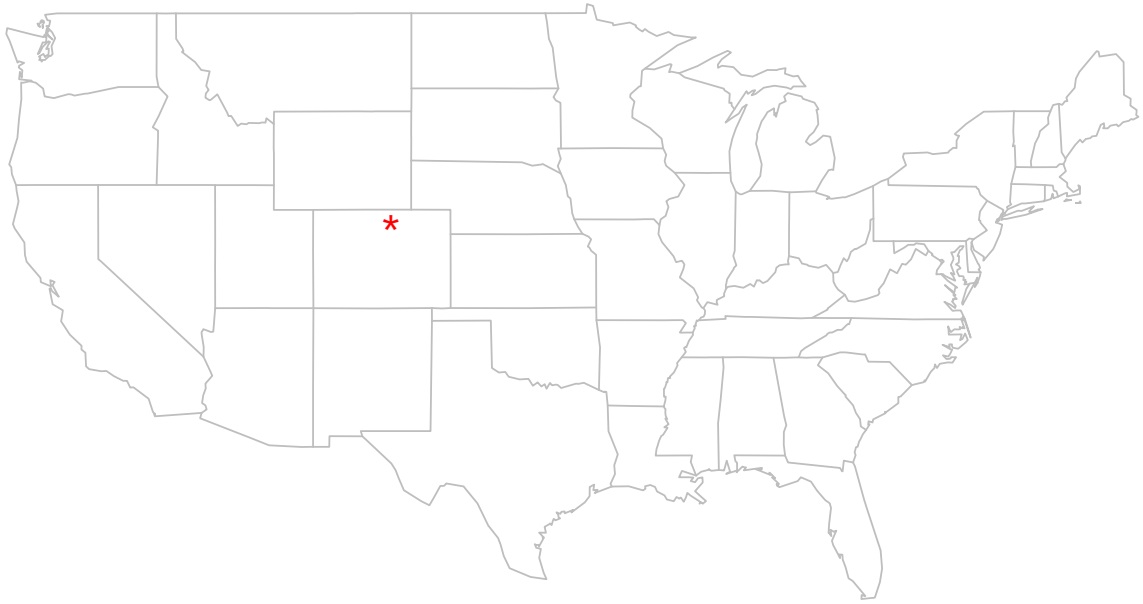
then G must be the same type of the following form:

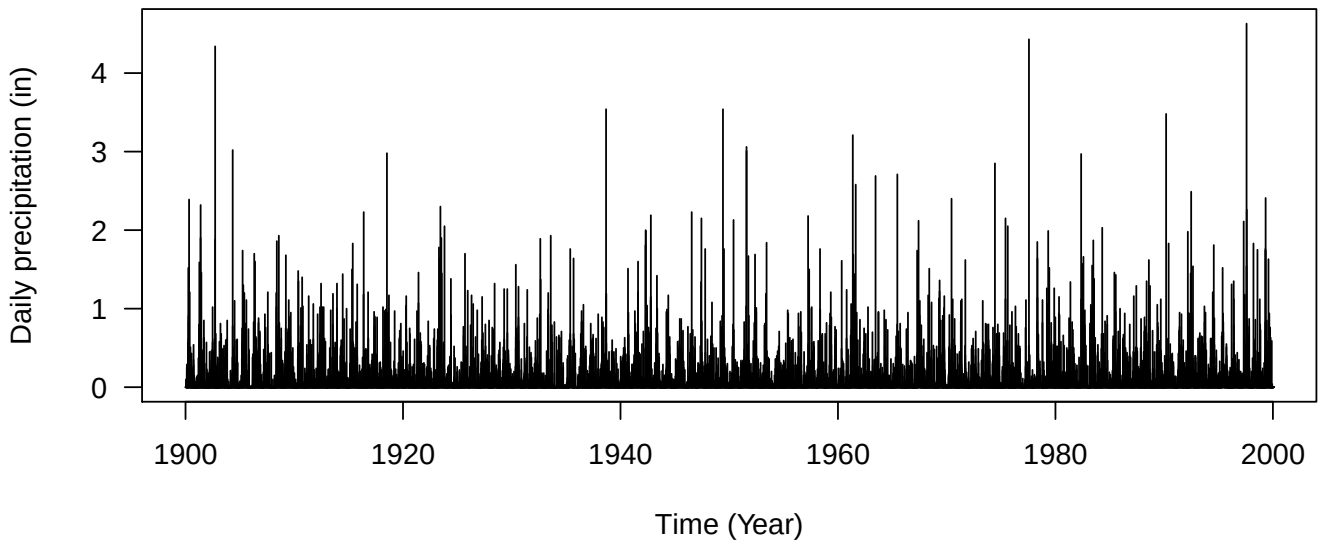
$$\mathsf{G}(x; \mu, \sigma, \xi) = \exp\left\{ -\left[1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right]_+^{\frac{-1}{\xi}} \right\}$$

where $x_+ = \max(x, 0)$ and $G(x)$ is the distribution function of the generalized extreme value distribution (GEV)

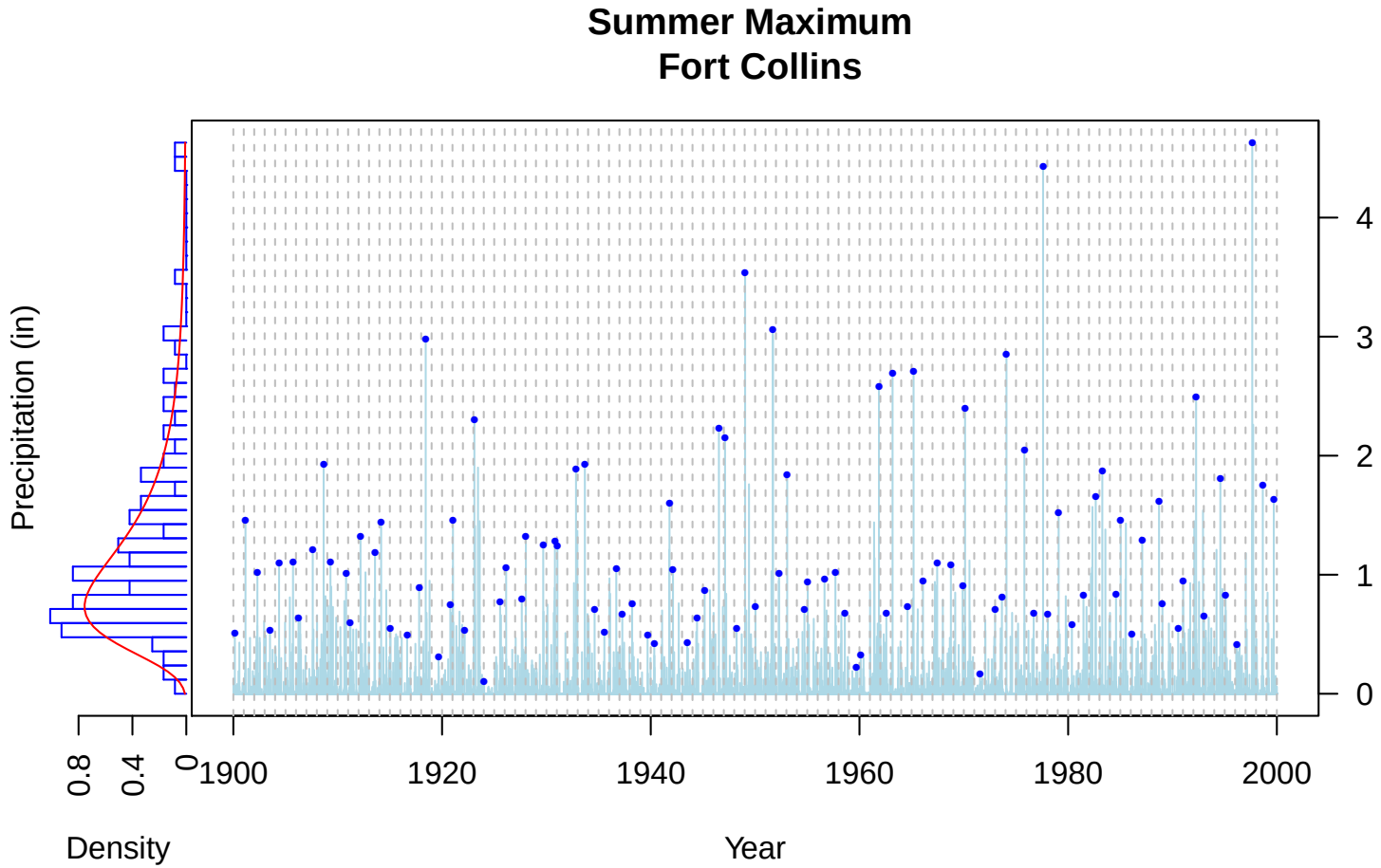
- μ and σ are location and scale parameters
- ξ is a shape parameter determining the rate of tail decay, with
  - $\xi > 0$ giving the heavy-tailed case (Fréchet)
  - $\xi = 0$ giving the light-tailed case (Gumbel)
  - $\xi < 0$ giving the bounded-tailed case (reversed Weibull)

Fort Collins Daily Precipitation





Estimating Extremes Using Block Maxima (Gumbel 1958)



Which distribution to use for annual maxima?

$\Rightarrow$ generalized extreme value distribution (GEV(μ, σ, ξ))

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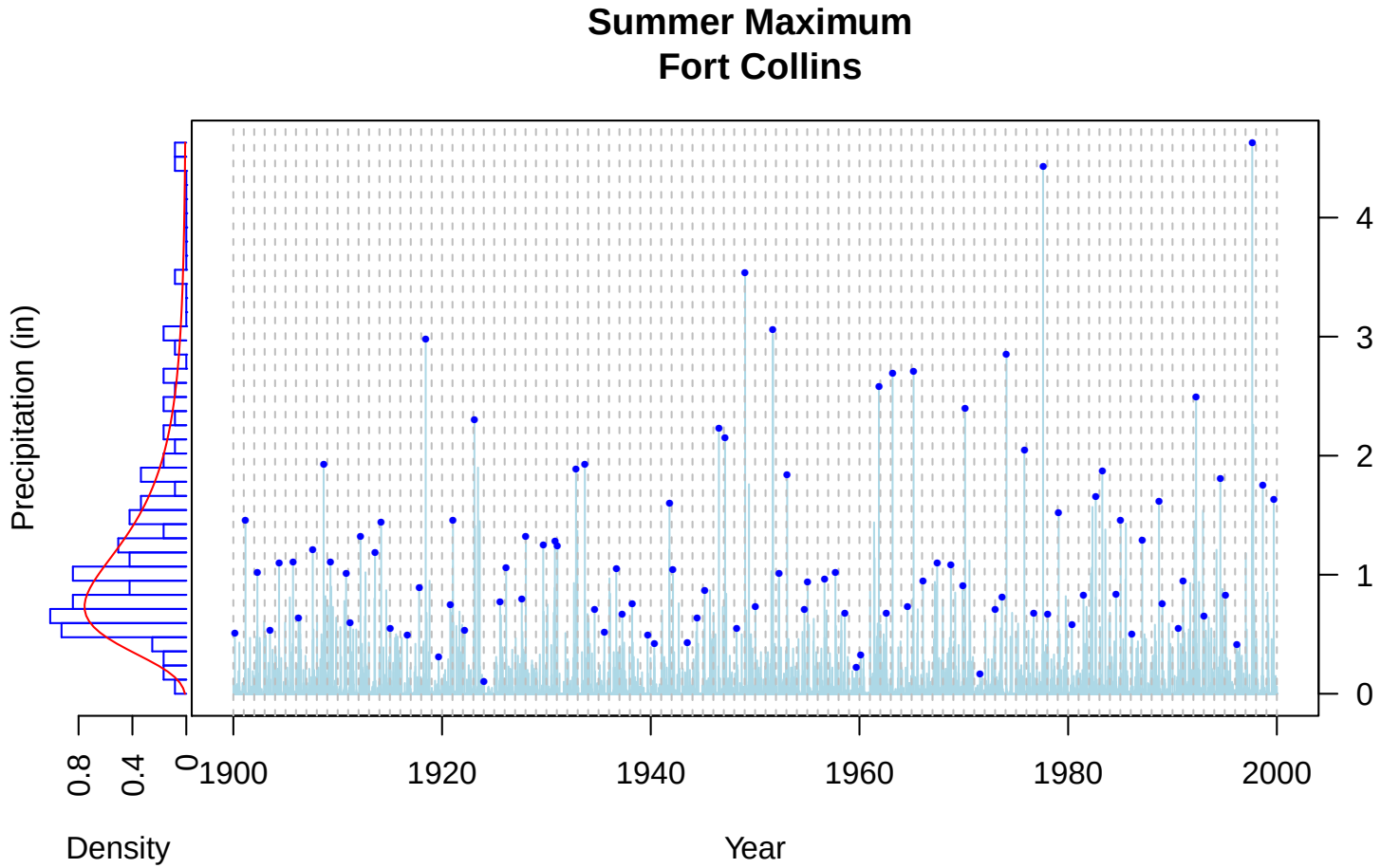

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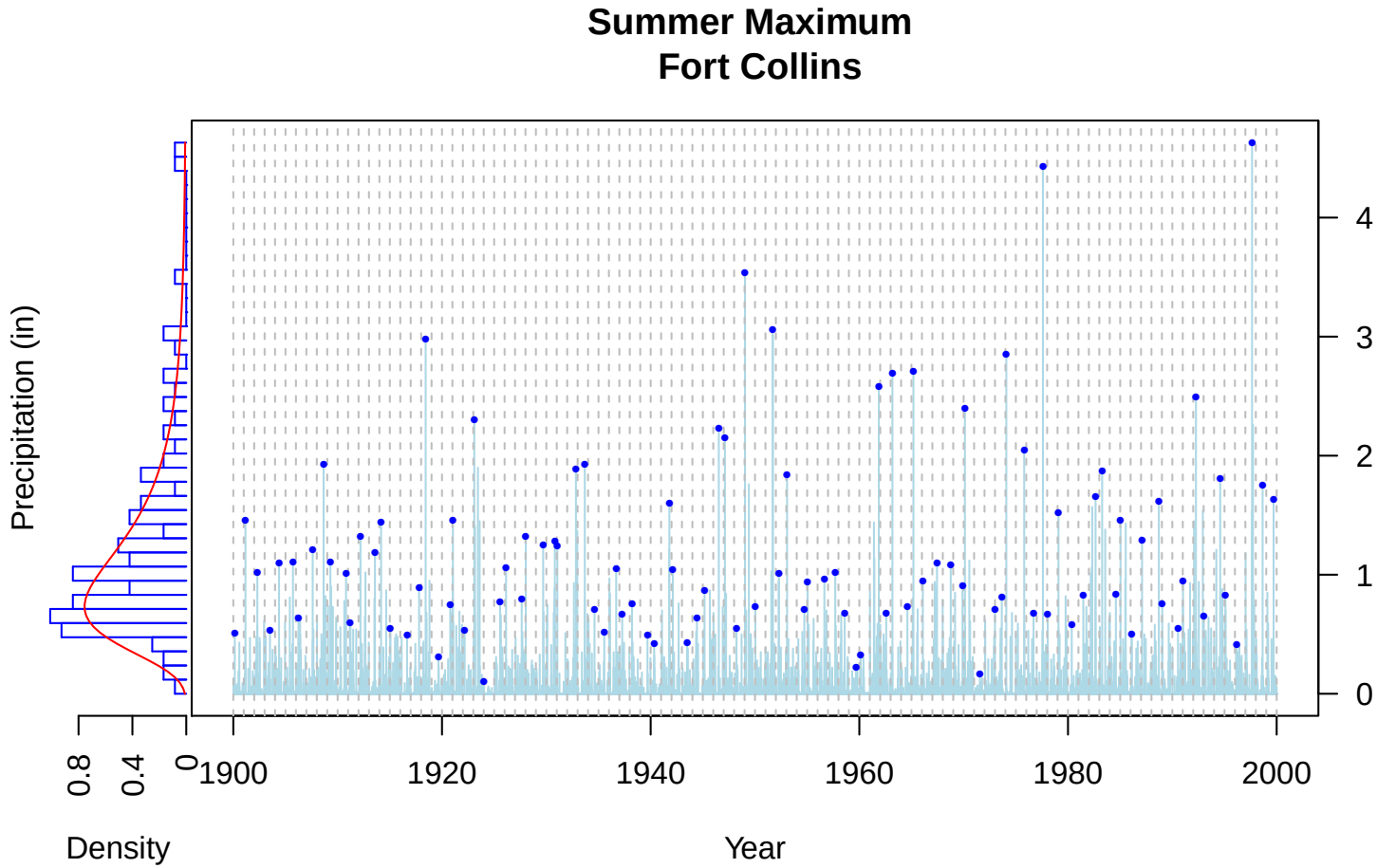

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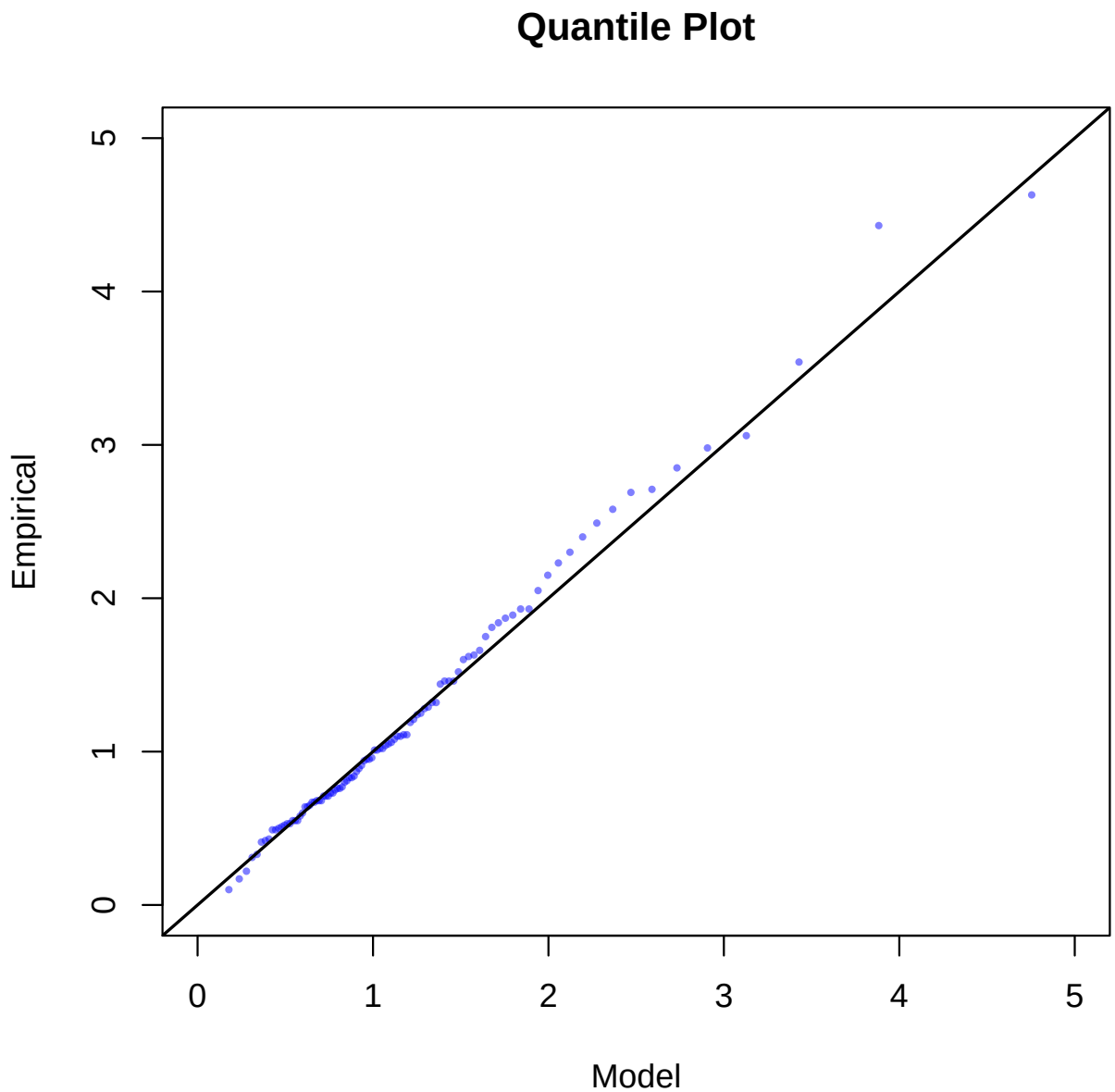
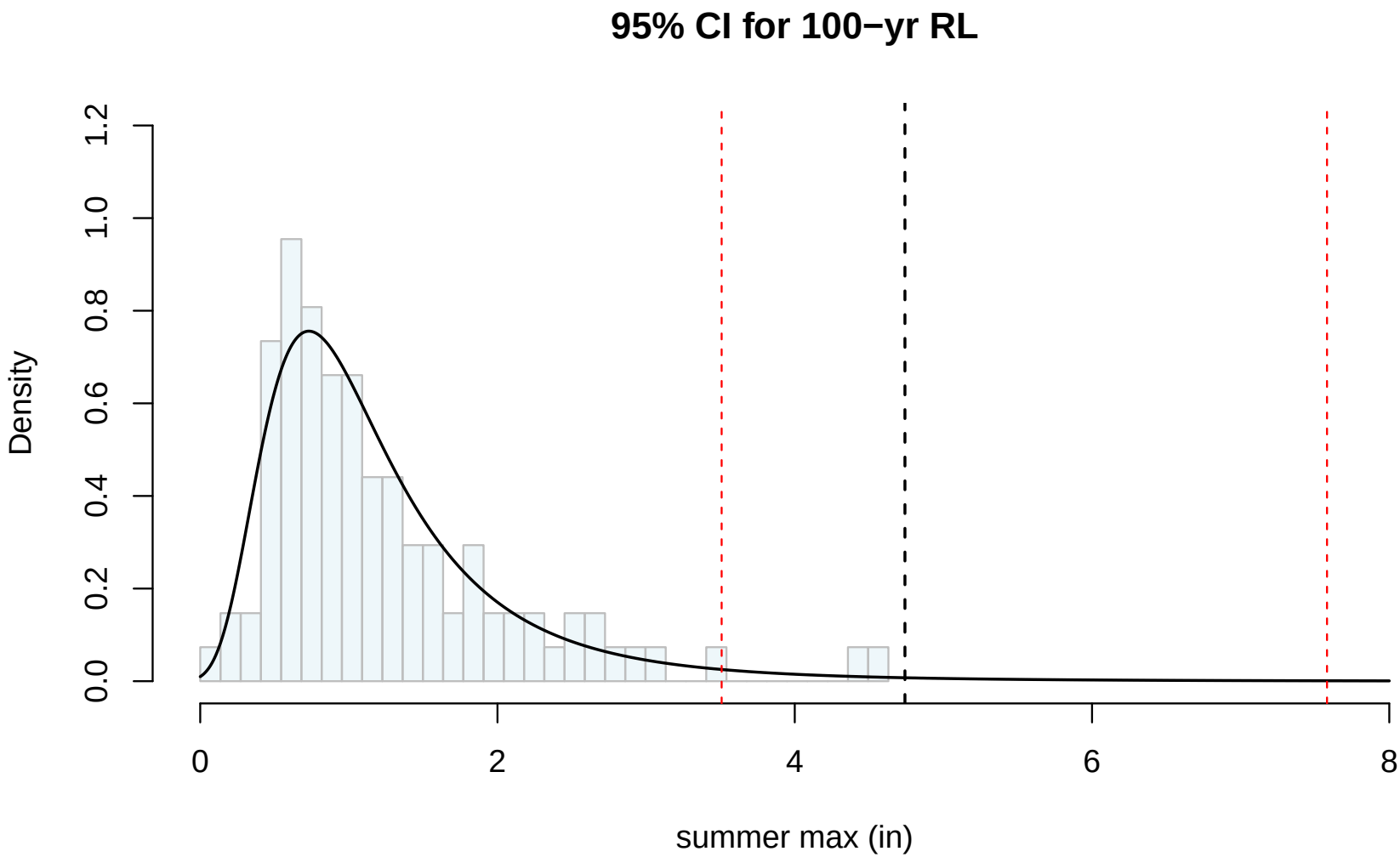
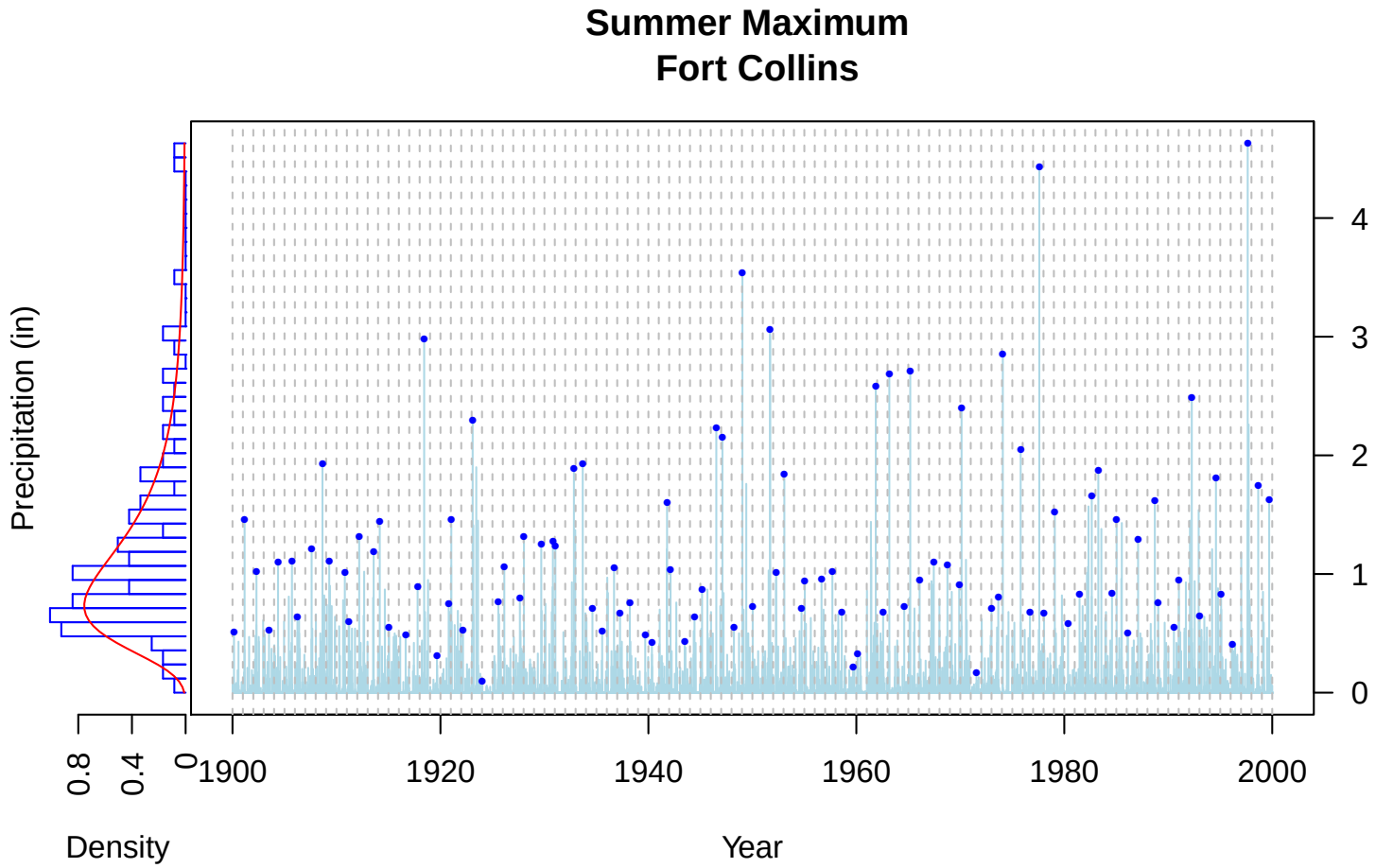
GEV Fit Diagnostics



Inference for 100-Year Event



Recall the Block Maxima Method

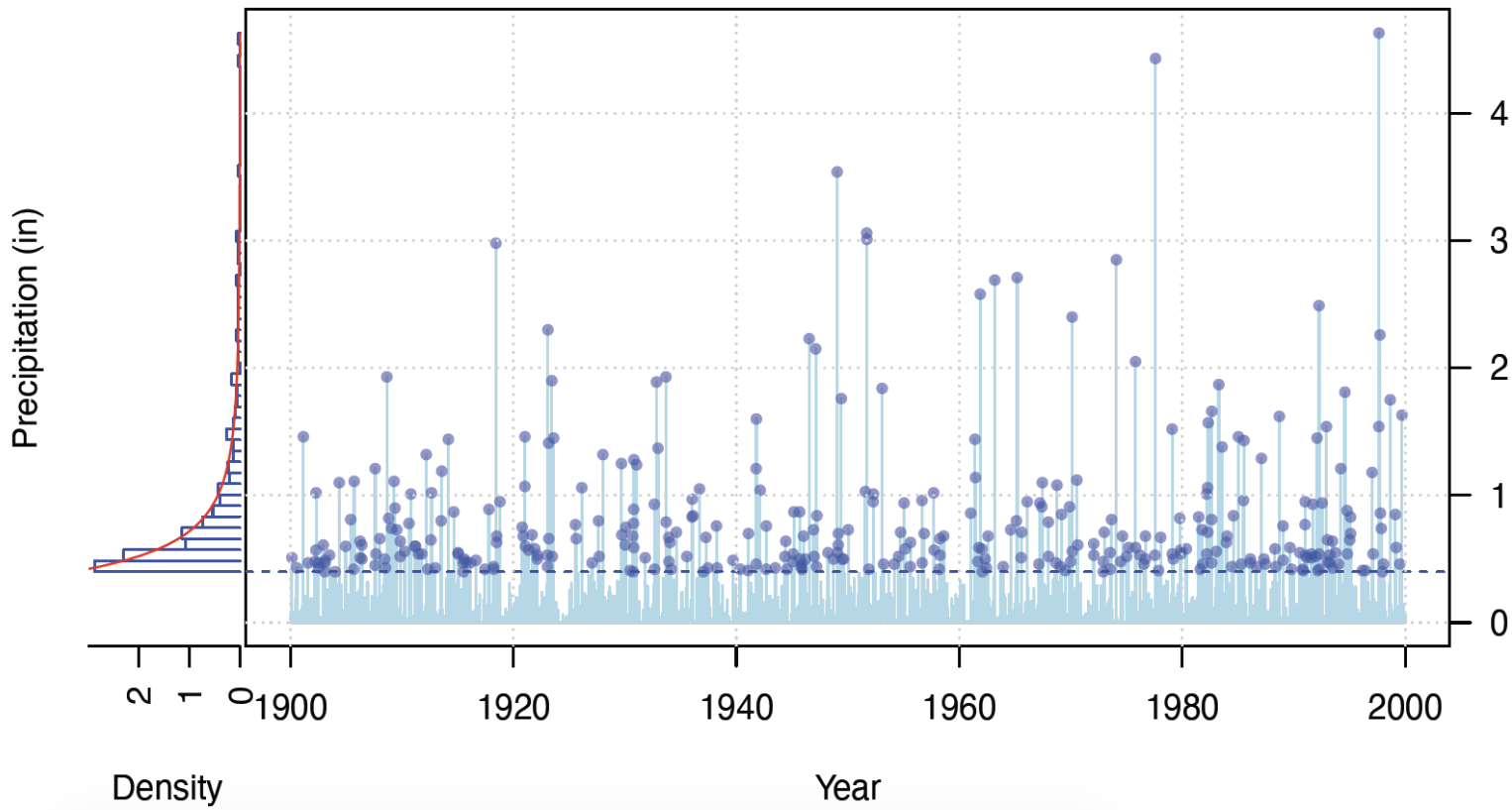


Question: Can we use data more efficiently?

Peaks–over–threshold (POT) method [Davison & Smith 1990]

Theorem: GEV for block maxima $\Rightarrow$ generalized Pareto distribution (GPD) for excesses over high threshold

▸ Pickands–Balkema–de Haan Theorem



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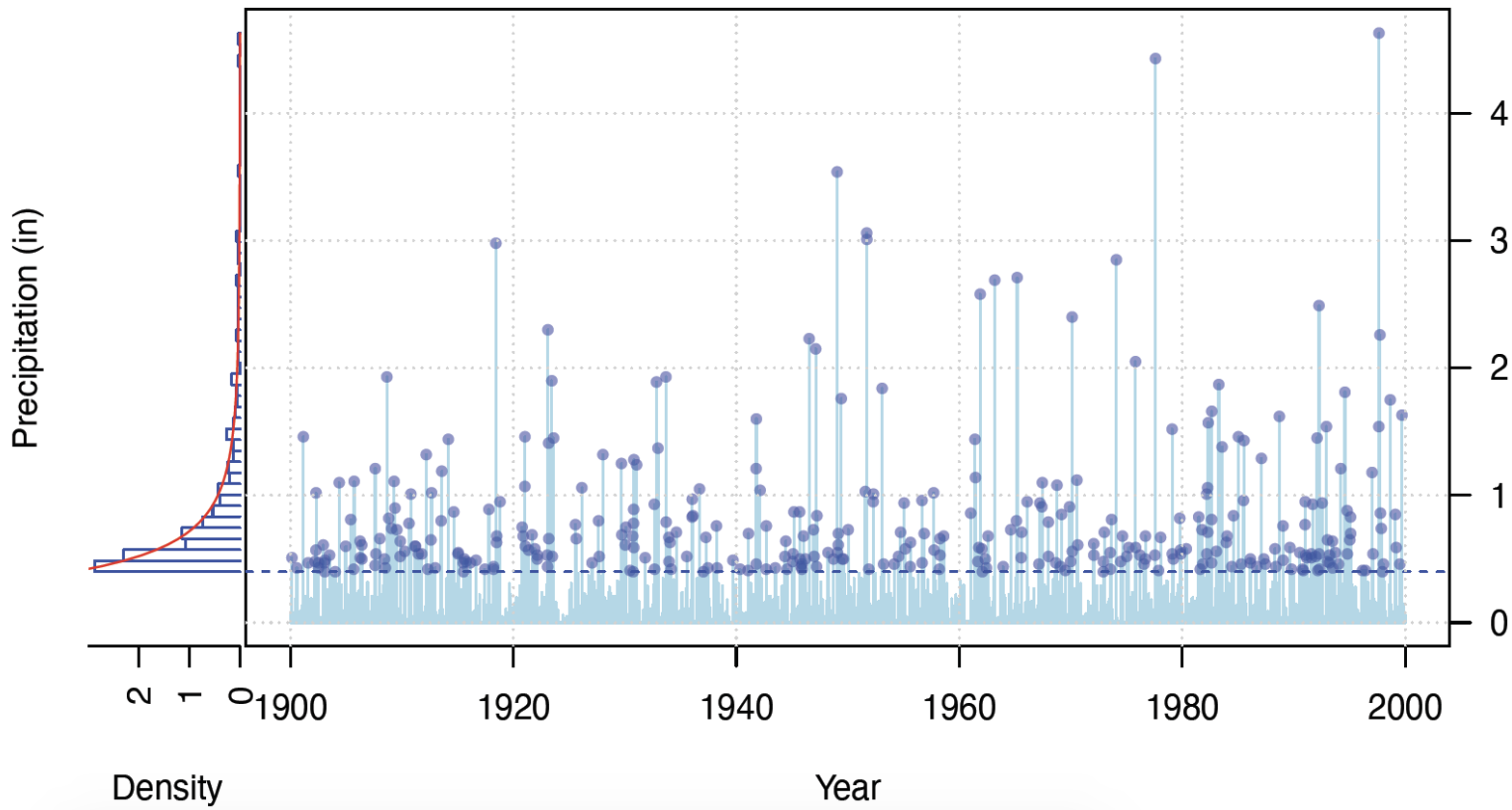

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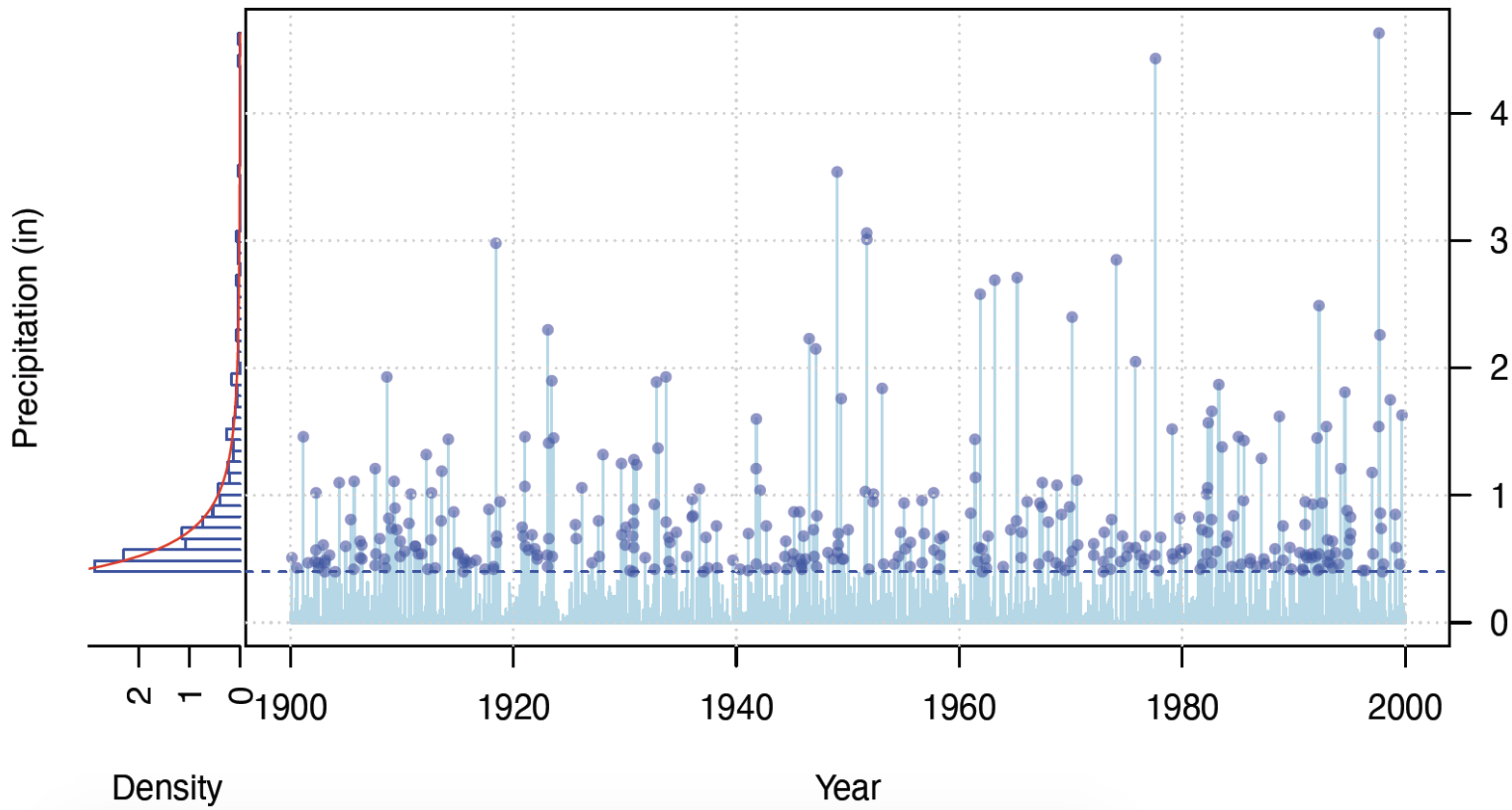

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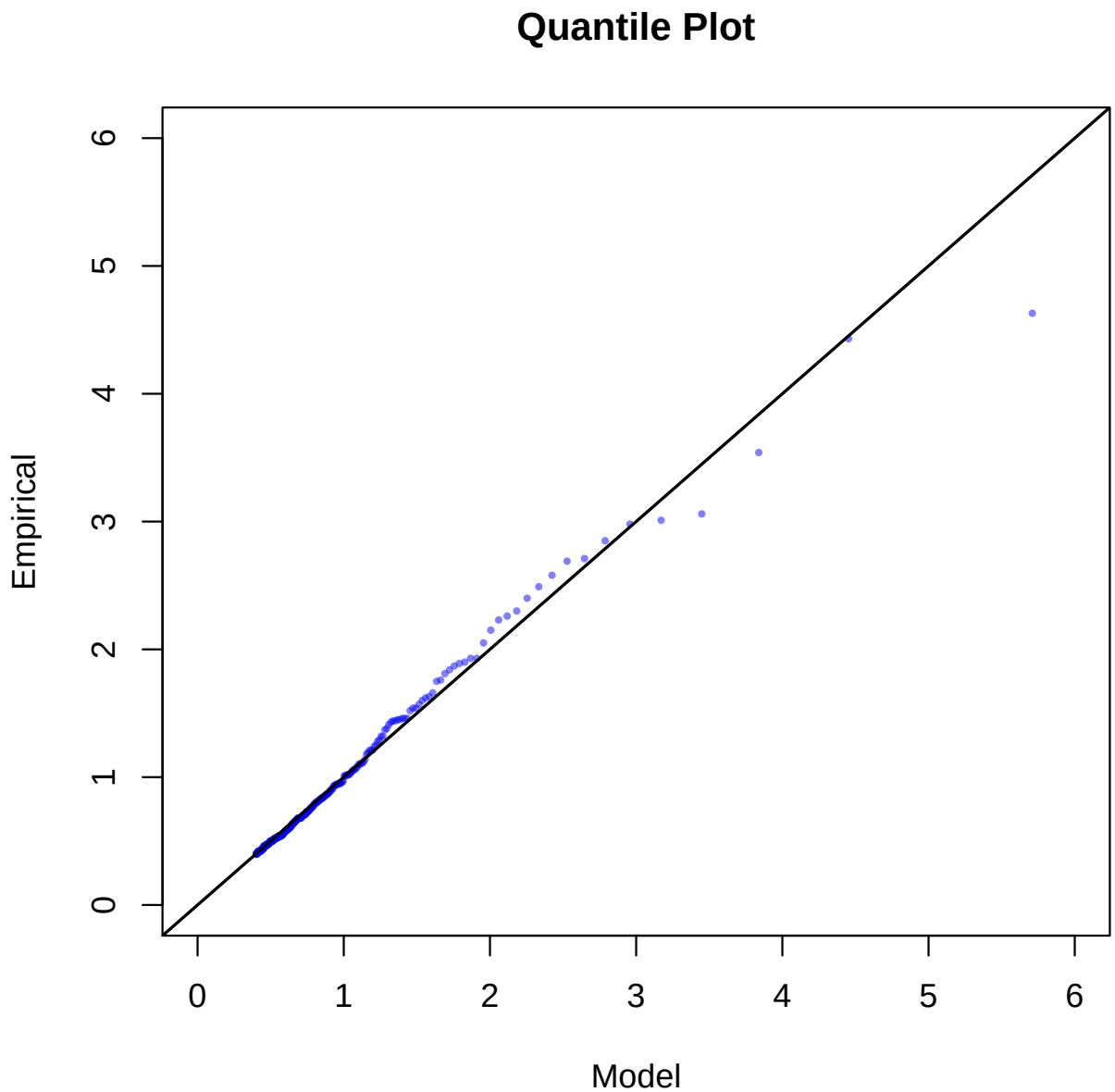
GPD Fit Diagnostics



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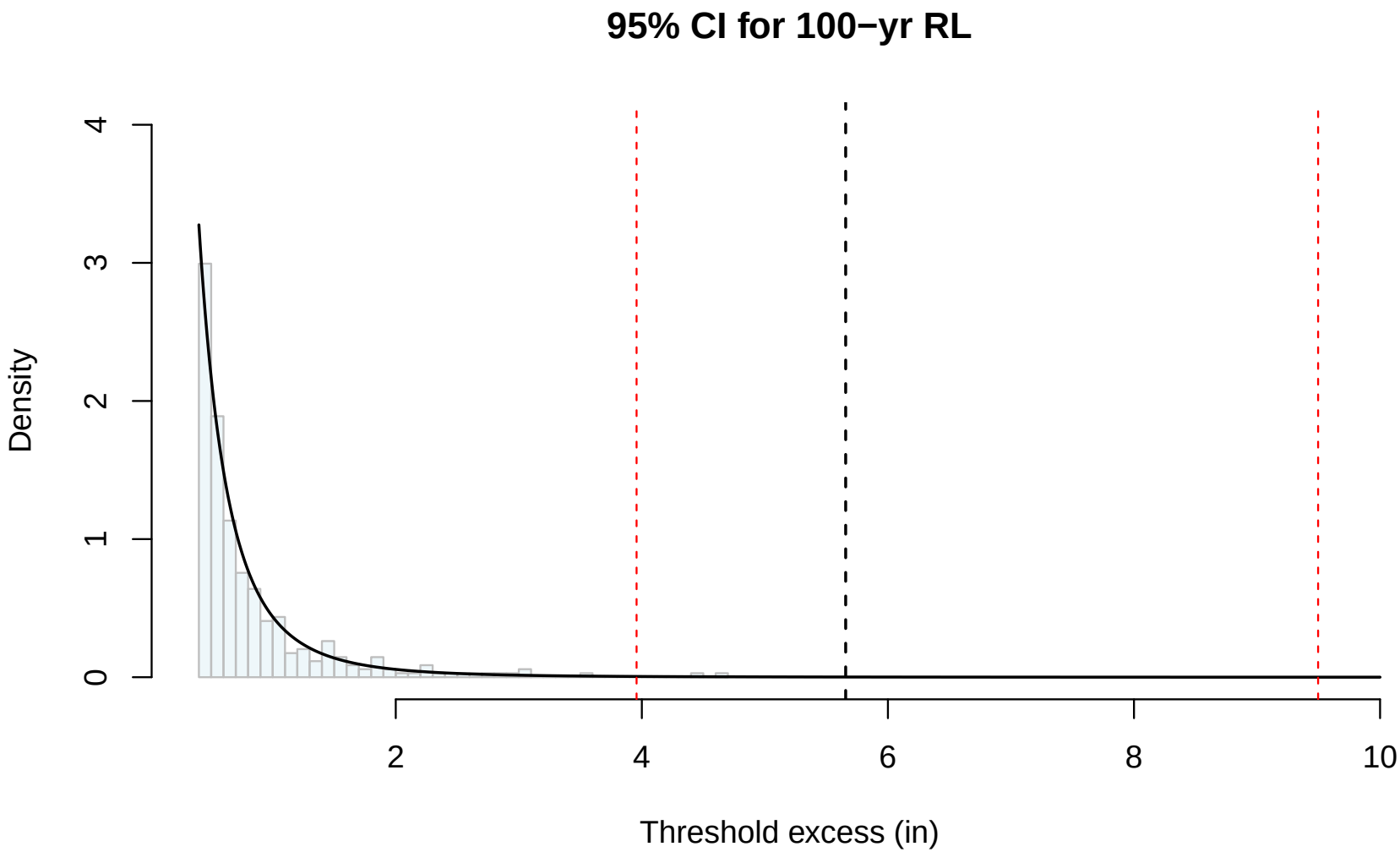

Summary & Discussion

- Climate extremes can have large impacts on both human society and environmental systems
- Extreme value theory provides a framework to model extreme values
  - GEV for fitting block maxima
  - GPD for fitting threshold exceedances
  - Return level for communicating risk
- Practical Issues: seasonality, temporal dependence, non-stationarity, ...

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How to React Me?

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- **Office**: O-221 Martin Hall

Go Tigers!

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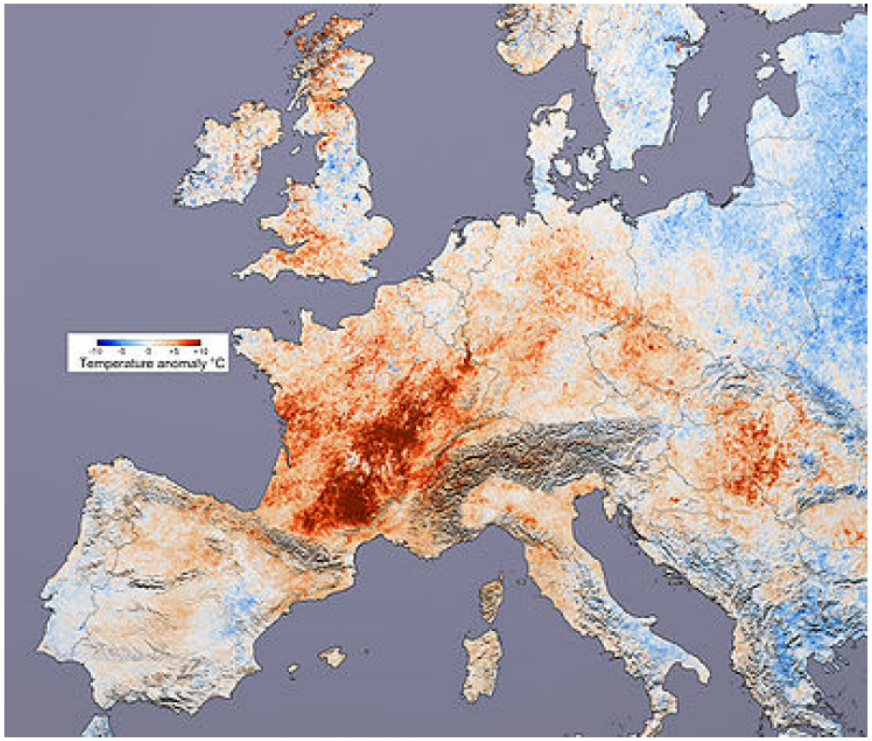
Backup Slides

Environmental Extremes: Heatwaves, storm surges, etc.



- **Heat wave**: The 2003 European heat wave led to the hottest summer on record in Europe since 1540 that resulted in at least 30,000 deaths

- **Storm Surge**: Hurricane Katrina produced the highest storm surge ever recorded (27.8 feet) on the U.S. coast

Sketch of a Proof of CLT

Let's consider the case where $\mu = 0$ and $\sigma^2 = 1$. When will use the **characteristics function** $\varphi_X(t) = \mathbb{E}[e^{itX}]$ for this proof. Let $Z_n = \frac{\sum_{i=1}^{n} X_i}{\sqrt{n}}$. What we need to show is that $\varphi_{Z_n}(t) \overset{n\to\infty}{\longrightarrow} e^{-\frac{t^2}{2}}$.

$$\varphi_{Z_n}(t) = \mathbb{E}[e^{it\{(\sum_{i=1}^{n} X_i)/\sqrt{n}\}}] = \mathbb{E}[\prod_{i=1}^{n} e^{it(X_i/\sqrt{n})}]$$

$$= \left[\mathbb{E}[e^{it(X_i/\sqrt{n})}]\right]^n = \left[\mathbb{E}[1 + it(\frac{X_i}{\sqrt{n}}) + 1/2(\frac{itX_i}{\sqrt{n}})^2 + o[(\frac{itX_i}{\sqrt{n}})^2]\right]^n$$

$$= \left[1 - \frac{t^2}{2n} + o(\frac{-t^2}{2n})\right]^n \Rightarrow \lim_{n\to\infty} \varphi_{Z_n}(t) \to e^{\frac{-t^2}{2}} \quad \text{☺}$$

Back

Pickands–Balkema–de Haan Theorem (1974, 1975)

If $M_n = \max_{1 \le i \le n}\{X_i\} \approx \text{GEV}(\mu, \sigma, \xi)$, then, for a "large" u (i.e., $u \to x_F = \sup\{x : F(x) < 1\}$), $F_u = \mathbb{P}(X - u < y | X > u)$ is well approximated by the generalized Pareto distribution (GPD). That is:

$$F_u(y) \xrightarrow{d} H_{\tilde{\sigma},\xi}(y) \qquad u \to x_F$$

where

$$H_{\tilde{\sigma},\xi}(y) = \begin{cases} 1 - (1 + \xi y/\tilde{\sigma})^{-1/\xi} & \xi \neq 0; \\ \\ 1 - \exp(-y/\tilde{\sigma}) & \xi = 0. \end{cases}$$

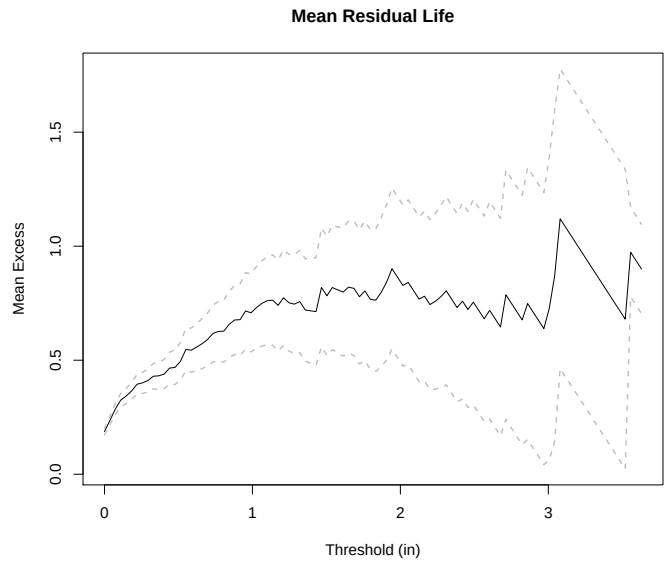
and $\tilde{\sigma} = \sigma + \xi(u - \mu)$

Back

How to Choose the Threshold?

Bias–variance tradeoff:

- Threshold too low $\Rightarrow$ bias because of the model asymptotics being invalid
- Threshold too high $\Rightarrow$ variance is large due to few data points



Task: To choose a u_0 s.t. the Mean Residual Life curve behaves linearly $\forall u > u_0$

Back