Lecture 5

Introduction to Probability

Text: Chapter 4

STAT 8010 Statistical Methods I

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Agenda

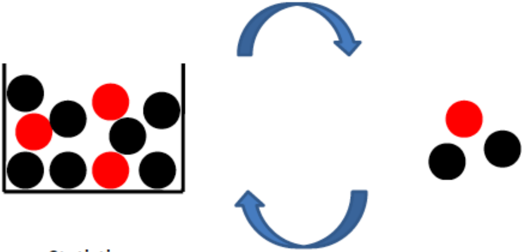
1. **Probability and Statistics**
2. **Terminology/Concepts**
3. **Union, Intersection, and Logical Relationships among Events**
4. **Complement Rule and General Addition Rule**
5. **Independence and Conditional Probability**

Probability & Statistics

Probability and Statistics

<u>Probability</u>:

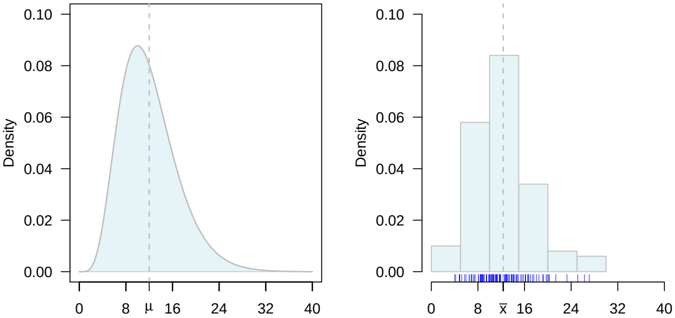
What is the probability to get 1 red and 2 black balls?



<u>Statistics:</u>

What percentage of balls in the box are red?

Probability and Statistics



Terminology/Concepts

Definitions

The framework of Probability is based on the paradigm of a random experiment, i.e., an action whose outcome cannot be predicted beforehand.

- Outcome: A particular result of an (random) experiment. (e.g. rolling a 3 on a die roll)
- Event: A collection of one or more outcomes of an experiment. (e.g. rolling an odd number on a die roll)
- Sample space: the set of all possible outcomes for an experiment. We will use Ω to denote it
- Probability: A number between 0 and 1 that reflects the likelihood of occurrence of some events.

Example

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Solution.

Example

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Example

Let us examine what happens in the flip of 3 fair coins. In this case $\Omega = \{(T,T,T),(T,T,H),(T,H,T),(H,T,T),(T,H,H),(H,T,H),(H,H,T),(H,H,H)\}$. Let A be the event of exactly 2 tails. Let B be the event that the first 2 tosses are tails. Let C be the event that all 3 tosses are tails. Write out the possible outcomes for each of these 3 events

Solution.

Example

Suppose a fair six–sided die is rolled twice. Determine the number of possible outcomes

1. For this experiment
2. The sum of the two rolls is 5
3. The two rolls are the same
4. The sum of the two rolls is an even number

Solution.

Finding the Probability of an Event

Frequentist Interpretation of Probability

The probability of an event is the long–run proportion of times that the event occurs in independent repetitions of the random experiment. This is referred to as an empirical probability and can be written as

$$P(event) = \frac{\text{number of times that event occurs}}{\text{number of random experiment}}$$

Equally Likely Framework

$$P(event) = \frac{\text{number of outcomes for the event}}{\text{number of all possible outcomes}}$$

Remark:

- Any individual outcome of the sample space is equally likely as any other outcome in the sample space.
- In an equally likely framework, the probability of any event is the number of ways the event occurs divided by the number of total events possible.

Dice Roll Example

Find the probabilities associated with parts 2–4 of the previous example

Solution.

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- The probability that the sum of the two rolls is 5: $\frac{4}{36} = \frac{1}{9}$
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- The probability that the two rolls are the same:
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- The probability that the sum of the two rolls is an even number:
  $\frac{18}{36} = \frac{1}{2}$

Probability Rules

1. Any probability must be between 0 and 1 inclusively
2. The sum of the probabilities for all the experimental outcomes must equal 1

If a probability model satisfies the two rules above, it is said to be legitimate

Example

An experiment with three outcomes has been repeated 50 times, and it was learned that outcome 1 occurred 20 times, outcome 2 occurred 13 times, and outcome 3 occurred 17 times. Assign probabilities to the outcomes. What method did you use?

Solution.

Example

A decision maker subjectively assigned the following probabilities to the four possible outcomes of an experiment:

$$P(E_1) = 0.1 \; P(E_2) = 0.15 \; P(E_3) = 0.4 \; P(E_4) = 0.2$$

Are these probability assignments legitimate? Explain.

Solution.

Union, Intersection, and Logical Relationships among Events

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Suppose we flipped 3 fair coins. Let A be the event of **exactly 2 tails**. Let B be the event that the **first 2 tosses are tails**. Let C be the event that **all 3 tosses are tails**. What are $A \cap B$, $A \cup C$, and $(A \cap B) \cup C$?

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$$A = \{(T,T,H),(T,H,T),(H,T,T)\}$$
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2. $A \cup C = \{(T,T,H),(T,H,T),(H,T,T),(T,T,T)\}$
3. $(A \cap B) \cup C = \{T,T,H\} \cup \{T,T,T\} = \{(T,T,H),(T,T,T)\}$

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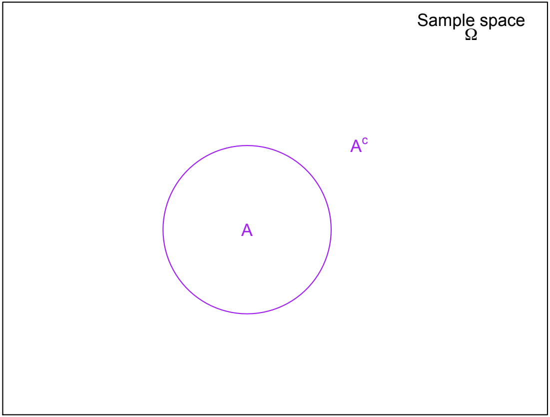
Complement Rule and General Addition Rule

Complement



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4. Hence we get $\boxed{\mathbb{P}(A) = 1 - \mathbb{P}(A^c)}$

Example

Suppose we rolled a fair, six–sided die 10 times. Let T be the event that we roll at least 1 three. If one were to calculate T you would need to find the probability of 1 three, 2 threes, $\cdots$, and 10 threes and add them all up. However, you can use the complement rule to calculate $\mathbb{P}(T)$

Solution.

Let X be the times that we rolled a 3, then

$$\mathbb{P}(T) = \mathbb{P}(X \geq 1) = \underbrace{\mathbb{P}(X = 1) + \mathbb{P}(X = 2) + \cdots + \mathbb{P}(X = 10)}_{\text{need to compute 10 probabilities}}$$

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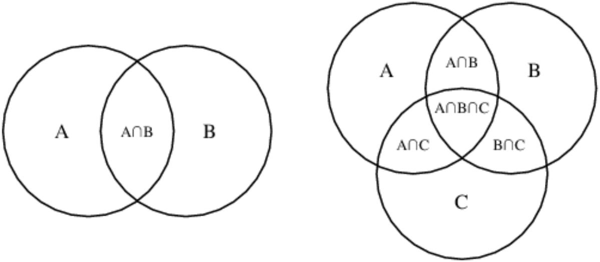
If we apply the complement rule

$$\mathbb{P}(T) = 1 - \mathbb{P}(T^c) = 1 - \mathbb{P}(X = 0)$$

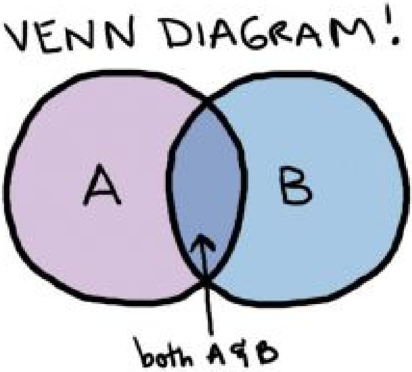
Venn Diagram

A Venn diagram is a diagram that shows all possible logical relations between a finite collection of events.



General Addition Rule

The general addition rule is a way of finding the probability of a union of 2 events. It is $\boxed{\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)}$

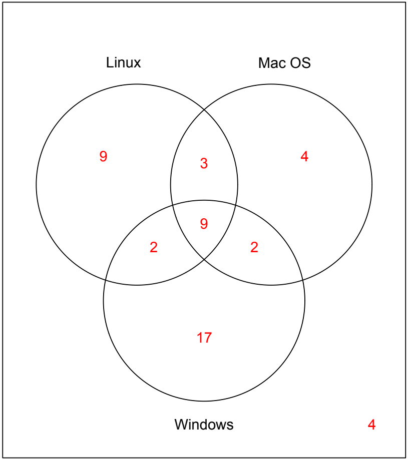


Example

Three of the major commercial computer operating systems are Windows, Mac OS, and Red Hat Linux Enterprise. A Computer Science professor selects 50 of her students and asks which of these three operating systems they use. The results for the 50 students are summarized below.

- 30 students use Windows
- 16 students use at least two of the operating systems
- 9 students use all three operating systems
- 18 students use Mac OS
- 46 students use at least one of the operating systems
- 11 students use both Windows and Linux
- 11 students use both Windows and Mac OS

Example cont'd



Independence and Conditional Probability

Independence: A Motivating Example

Example

You toss a fair coin and it comes up “Heads" three times. What is the chance that the next toss will also be a “Head"?

Independence and Conditional Probability

Conditional Probability

Let A and B be events. The probability that event B occurs given (knowing) that event A occurs is called a conditional probability and is denoted by $P(B|A)$. The formula of conditional probability is

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

Independent events

Suppose $P(A) > 0,\ P(B) > 0$. We say that event B is independent of event A if the occurrence of event A does not affect the probability that event B occurs.

$$\boxed{P(B|A) = P(B) \Rightarrow P(B \cap A) = P(B)P(A)}$$

Summary

In this lecture, we learned

- Some definitions: Outcome, Event, Sample Space
- The Frequentist Interpretation of Probability, the Equally Likely Framework, and the Probability Rules
- Union, Intersection, Mutually Exclusive, Exhaustive, Partition
- Complement Rule and General Addition Rule
- Independence and Conditional Probability