Lecture 3

Stationary Processes: Properties, Mean, and Covariance Functions

References: CC08 Chapter 2 & Chapter 4.1-4.3; BD16 Chapter 1.4, 1.5, 2.1, 2.2; SS17 Chapter 1.2-1.4

MATH 8090 Time Series Analysis
Week 3

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Agenda

Time Series Modeling Strategy

Additive Decompstion:

$$Y_t = \mu_t + s_t + \eta_t, \quad t = 1, 2, \cdots, T$$

1. Plot the data y_t to explore the form of μ_t and s_t, and check for non-constant variation in η_t

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5. Estimate parameters in μ_t, s_t, and η_t (ideally simultaneously in one step)
6. Check for fit of model (poor fit $\Rightarrow$ return to step 1)
7. Use model for inference: predicting future y_t's, describing changes in y_t over time, hypothesis testing, etc

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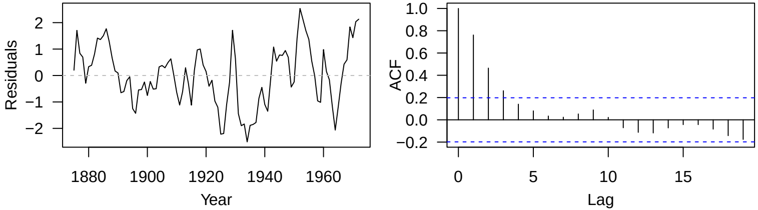
- Residuals typically suggest temporal dependence in $\{\eta_t\}$
- Time series models concern the modeling of temporal dependence in $\{\eta_t\}$
- Stationarity assumption typically employed to overcome the issue of "one sample"
- Weakly stationary: constant mean and variance over time, with covariance depending only on time lags

The Implications of Temporal Dependence



- There is a consistent relationship between consecutive residuals
- The usual regression assumptions are violated, and t- and F-tests are not valid ☹
- We can get better predictions of future values by modeling autocorrelation ☺

Time Series Models

- A time series model is a specification of the probabilistic distribution of a sequence of random variables (RVs) $\{\eta_t\}_{t=1}^{T}$

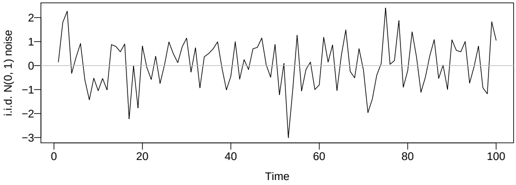
  (The observed time series is a realization of such a sequence of random variables)

- The simplest time series is i.i.d. (*independent and identically distributed*) noise
  - $\{\eta_t\}$ is a sequence of independent and identically distributed zero-mean (i.e., $\mathbb{E}(\eta_t) = 0, \forall t$) random variables $\Rightarrow$ no temporal dependence
  - It is of little value of using i.i.d. noise model to conduct forecast as there is no information from the past observations
  - **But**, we will use i.i.d. model as a building block to develop time series models that can accommodate time dependence

Example Realizations of i.i.d. Noise

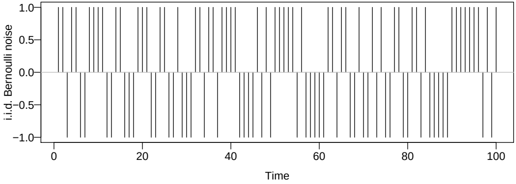
- Gaussian (normal) i.i.d. noise with mean 0 and variance $\sigma^2 > 0$

$$f(\eta_t) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\eta_t^2}{2\sigma^2}\right)$$



- Bernoulli i.i.d. noise with "success" probability

$$\mathbb{P}(\eta_t = 1) = p = 1 - \mathbb{P}(\eta_t = -1)$$



Means and Autocovaraınces

A time series model could also be a specification of the means and autocovariances of the RVs

- The mean function of $\{\eta_t\}$ is

$$\mu_t = \mathbb{E}(\eta_t).$$

- μ_t is the population mean at time t, which can be computed as:

$$\mu_t = \begin{cases} \int_{-\infty}^{\infty} \eta_t f(\eta_t)\, d\eta_t & \text{when } \eta_t \text{ is a continuous RV;} \\ \sum_{-\infty}^{\infty} \eta_t p(\eta_t), & \text{when } \eta_t \text{ is a discrete RV,} \end{cases}$$

where $f(\cdot)$ and $p(\cdot)$ are the probability density function and probability mass function of η_t, respectively

Examples of Mean Functions

- **Example 1**: What is the mean function for $\{\eta_t\}$, an i.i.d. $\mathrm{N}(0, \sigma^2)$ process?

- **Example 2**: For each time point, let $Y_t = \beta_0 + \beta_1 t + \eta_t$ with β_0 and β_1 some constants and η_t is defined above. What is $\mu_Y(t)$?

Review: The Covariance Between Two RVs

- The covariance between the RVs X and Y is

$$\begin{aligned}\mathrm{Cov}(X,Y) &= \mathbb{E}\{(X-\mu_X)(Y-\mu_Y)\}\\ &= \mathbb{E}(XY) - \mu_X\mu_Y.\end{aligned}$$

  It is a measure of linear dependence between the two RVs. When $X = Y$ we have

$$\mathrm{Cov}(X,X) = \mathrm{Var}(X).$$

- For constants a, b, c, and RVs X, Y, Z:

$$\begin{aligned}\mathrm{Cov}(aX+bY+c,Z) &= \mathrm{Cov}(aX,Z)+\mathrm{Cov}(bY,Z)\\ &= a\mathrm{Cov}(X,Z)+b\mathrm{Cov}(Y,Z)\end{aligned}$$

  $\Rightarrow$

$$\begin{aligned}\mathrm{Var}(X+Y) &= \mathrm{Cov}(X,X)+\mathrm{Cov}(X,Y)+\mathrm{Cov}(Y,X)+\mathrm{Cov}(Y,Y)\\ &= \mathrm{Var}(X)+\mathrm{Var}(Y)+2\mathrm{Cov}(X,Y)\end{aligned}$$

Autocovariance Function

- The autocovariance function of $\{\eta_t\}$ is

$$\gamma(s,t) = \mathrm{Cov}(\eta_s, \eta_t) = \mathbb{E}[(\eta_s - \mu_s)(\eta_t - \mu_t)]$$

It measures the strength of linear dependence between two RVs η_s and η_t

- **Properties**:
  - $\gamma(s,t) = \gamma(t,s)$ for each s and t
  - When $s = t$ we have

$$\gamma(t,t) = \mathrm{Cov}(\eta_t, \eta_t) = \mathrm{Cov}(\eta_t) = \sigma_t^2$$

  the value of the variance function at time t
  - $\gamma(s,t)$ is a non-negative definite function (will come back to this later)

Autocorrelation Function

- The autocorrelation function of $\{\eta_t\}$ is

$$\rho(s,t) = \mathrm{Corr}(\eta_s, \eta_t) = \frac{\gamma(s,t)}{\sqrt{\gamma(s,s)\gamma(t,t)}}$$

It measures the "scale invariant" linear association between η_s and η_t

- **Properties**:
  - $-1 \le \rho(s,t) \le 1$ for each s and t
  - $\rho(s,t) = \rho(t,s)$ for each s and t
  - $\rho(t,t) = 1$ for each t
  - $\rho(\cdot,\cdot)$ is a non-negative definite function

Why Stationarity Matters in Time Series

- We typically need "replicates" to estimate population quantities. For example, we use

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

as an estimate of μ_X, the population mean of the **single** RV, X

- However, in time series analysis, we have $n = 1$ (i.e., no replication), since we only observe one realized value at each time point
- Stationarity means that some characteristic of $\{\eta_t\}$ does not depend on the time point t, but only on the time lag between time points, **so that we can create "replicates."**

> Next, we will talk about strict stationarity and weak stationarity

Strictly Stationary Processes

- A time series, $\{\eta_t\}$, is strictly stationary if

$$[\eta_1, \eta_2, \cdots \eta_T] \stackrel{d}{=} [\eta_{1+h}, \eta_{2+h}, \cdots \eta_{T+h}],$$

  for all integers h and $T \geq 1 \Rightarrow$ the joint distribution are unaffected by time shifts

- Under such the strict stationarity
  - $\{\eta_t\}$ is identically distributed but not (necessarily) independent
  - When μ_t is finite, $\mu_t = \mu$ is independent of time t
  - When the variance function exists,

$$\gamma(s,t) = \gamma(s+h, t+h),$$

    for any s, t, and h

Weakly Stationary Processes

- $\{\eta_t\}$ is weakly stationary if
  - $\mathbb{E}(\eta_t) = \mu_t = \mu$
  - $\mathrm{Cov}(\eta_t, \eta_{t+h}) = \gamma(t, t+h) = \gamma(h)$, finite constant that can depend on h but not on t
- Other names for this type of stationarity include second-order, covariance, wide senese. The quantity h is called the lag
- Weak and strict stationarity
  - A strictly stationary process $\{\eta_t\}$ is also weakly stationary as long as μ is finite
  - Weak stationarity does not imply strict stationarity!

Autocovariance Function of Stationary Processes

The autocovariance function (ACVF) of a stationary process $\{\eta_t\}$ is defined to be

$$\begin{aligned}\gamma(h) &= \mathrm{Cov}(\eta_t, \eta_{t+h}) \\ &= \mathbb{E}[(\eta_t - \mu)(\eta_{t+h} - \mu)],\end{aligned}$$

which measures the lag-h time dependence

Properties of the ACVF:

- $\gamma(0) = \mathrm{Var}(\eta_t)$
- $\gamma(-h) = \gamma(h)$ for each h
- $\gamma(s-t)$ as a function of $(s-t)$ is non-negative definite

Autocorrelation Function of Stationary Processes

The autocorrelation function (ACF) of a stationary process $\{\eta_t\}$ is defined to be

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)}$$

which measures the "scale invariant" lag-h time dependence

Properties of the ACF:

- $-1 \le \rho(h) \le 1$ and $\rho(0) = 1$ for each h
- $\rho(-h) = \rho(h)$ for each h
- $\rho(s-t)$ as a function of $(s-t)$ is non-negative definite

The White Noise Process

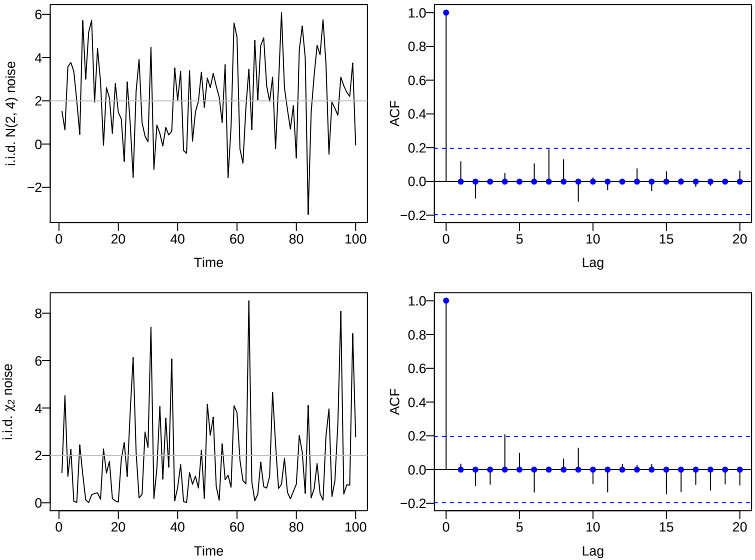
Let's assume $\mathbb{E}(\eta_t) = \mu$ and $\mathrm{Var}(\eta_t) = \sigma^2 < \infty$. $\{\eta_t\}$ is a white noise or $\mathrm{WN}(\mu, \sigma^2)$ process if

$$\gamma(h) = 0,$$

for $h \neq 0$

- $\{\eta_t\}$ is stationary
- However, distributions of η_t and η_{t+1} can be different!
- All i.i.d. noise with finite variance ($\sigma^2 < 0$) is white noise but the converse need not be true

Examples Realizations of White Noise Processes



The Moving Average Process of First Order (MA(1))

Let $\{Z_t\}$ be a $\mathrm{WN}(0, \sigma^2)$ process and θ be some constant $\in \mathbb{R}$. For each integer t, let

$$\eta_t = Z_t + \theta Z_{t-1}.$$

- The sequences of RVs $\{\eta_t\}$ is called the moving average process of order 1 or MA(1) process
- One can show that the MA(1) process $\{\eta_t\}$ is stationary

First-Order Moving Average Process: Mean Function

Need to show the mean function is NOT a function of time t

$$\begin{aligned}\mathbb{E}[\eta_t] &= \mathbb{E}[Z_t + \theta Z_{t-1}] \\ &= \mathbb{E}[Z_t] + \theta \mathbb{E}[Z_{t-1}] \\ &= 0 + \theta \times 0 \\ &= 0, \quad \forall t\end{aligned}$$

☺

First-Order Moving Average Process: Covariance Function

Need to show the autovariance function $\gamma(\cdot, \cdot)$ is a function of time lag only

$$
\begin{aligned}
\gamma(t, t+h) &= \mathrm{Cov}(\eta_t, \eta_{t+h}) \\
&= \mathrm{Cov}(Z_t + \theta Z_{t-1}, Z_{t+h} + \theta Z_{t+h-1}) \\
&= \mathrm{Cov}(Z_t, Z_{t+h}) + \mathrm{Cov}(Z_t, \theta Z_{t+h-1}) \\
&+ \mathrm{Cov}(\theta Z_{t-1}, Z_{t+h}) + \mathrm{Cov}(\theta Z_{t-1}, \theta Z_{t+h-1})
\end{aligned}
$$

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\end{aligned}
$$

if $h=0,$ we have $\quad \gamma(t,t+h) = \sigma^2+\theta^2\sigma^2 = \sigma^2(1+\theta^2)$
if $h=\pm 1,$ we have $\quad \gamma(t,t+h) = \theta\sigma^2$
if $|h|\geq 2,$ we have $\quad \gamma(t,t+h) = 0$

$\Rightarrow \gamma(t,t+h)$ only depends on h but not on t ☺

First-Order Moving Average Process: ACVF & ACF

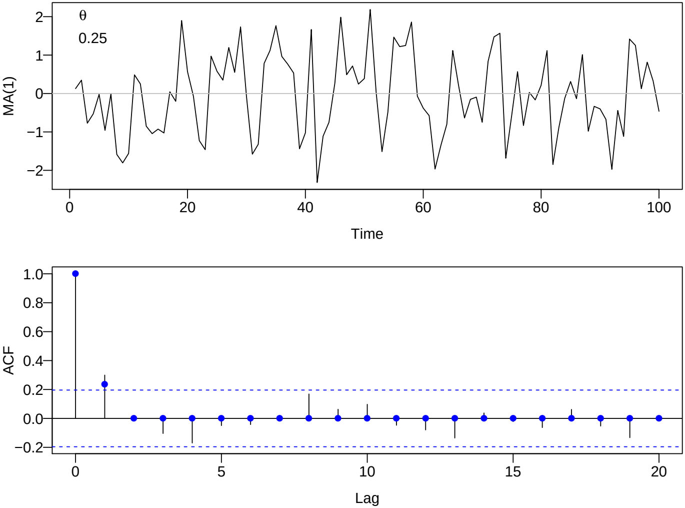
ACVF:

$$\gamma(h) = \begin{cases} \sigma^2(1+\theta^2) & h = 0; \\ \theta\sigma^2 & |h| = 1; \\ 0 & |h| \geq 2 \end{cases}$$

We can get **ACF** by dividing everything by $\gamma(0) = \sigma^2(1+\theta^2)$

$$\rho(h) = \begin{cases} 1 & h = 0; \\ \frac{\theta}{1+\theta^2} & |h| = 1; \\ 0 & |h| \geq 2. \end{cases}$$

Examples Realizations of MA(1) Processes



First-order Autoregressive Process, AR(1)

Let $\{Z_t\}$ be a $\mathrm{WN}(0, \sigma^2)$ process, and $-1 < \phi < 1$ be a constant. Let's assume $\{\eta_t\}$ is a stationary process with

$$\eta_t = \phi\eta_{t-1} + Z_t,$$

for each integer t, where η_s and Z_t are uncorrelated for each $s < t \Rightarrow$ future noise is uncorrelated with the current time point

We will see later there is only one unique solution to this equation. Such a sequence $\{\eta_t\}$ of RVs is called an AR(1) process

Properties of the AR(1) process

Want to find the mean value μ under the weakly stationarity assumption

$$\begin{aligned}
\mathbb{E}[\eta_t] &= \mathbb{E}[\phi\eta_{t-1} + Z_t] \\
\mu &= \phi\mathbb{E}[\eta_{t-1}] + \mathbb{E}[Z_t] \\
\mu &= \phi\mu + 0 \\
&\Rightarrow \mu = 0, \quad \forall t
\end{aligned}$$

☺

Want to find $\gamma(h)$ under the weakly stationarity assumption

$$\begin{aligned}
\mathrm{Cov}(\eta_t, \eta_{t-h}) &= \mathrm{Cov}(\phi\eta_{t-1} + Z_t, \eta_{t-h}) \\
\gamma(-h) &= \phi\mathrm{Cov}(\eta_{t-1}, \eta_{t-h}) + \mathrm{Cov}(Z_t, \eta_{t-h}) \\
\gamma(h) &= \phi\gamma(h-1) + 0 \\
\Rightarrow \gamma(h) &= \phi\gamma(h-1) = \cdots = \phi^{|h|}\gamma(0)
\end{aligned}$$

Next, need to figure out $\gamma(0)$

Properties of the AR(1) process Cont'd

$$\begin{aligned}
\mathrm{Var}(\eta_t) &= \mathrm{Var}(\phi\eta_{t-1} + Z_t) \\
\gamma(0) &= \phi^2\gamma(0) + \sigma^2 \\
\Rightarrow (1-\phi^2)\gamma(0) &= \sigma^2 \\
\Rightarrow \gamma(0) &= \frac{\sigma^2}{1-\phi^2}
\end{aligned}$$

☺

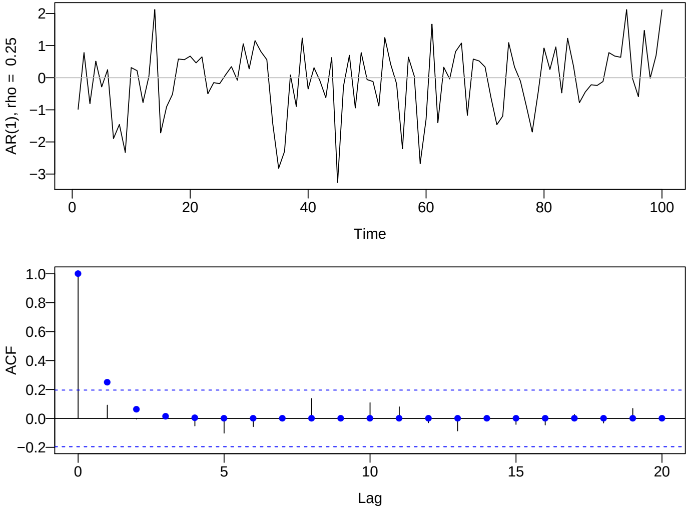
Therefore, we have

$$\gamma(h) = \begin{cases} \frac{\sigma^2}{1-\phi^2} & h = 0; \\ \frac{\phi^{|h|}\sigma^2}{1-\phi^2} & |h| \geq 1, \end{cases}$$

and

$$\rho(h) = \begin{cases} 1 & h = 0; \\ \phi^{|h|} & |h| \geq 1. \end{cases}$$

Examples Realizations of AR(1) Processes



The Random Walk Process

Let $\{Z_t\}$ be a $\mathrm{WN}(0, \sigma^2)$ process and for $t \geq 1$ definite

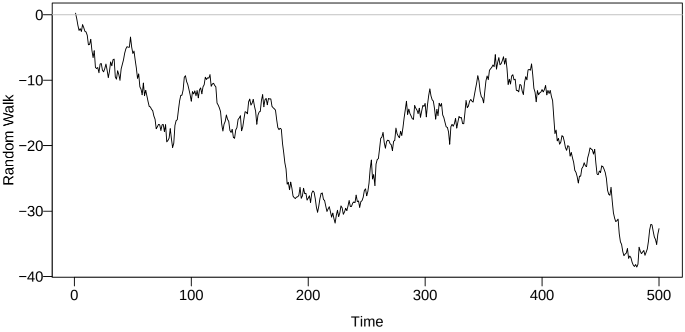
$$\eta_t = Z_1 + Z_2 + \cdots + Z_t = \sum_{s=1}^{t} Z_s.$$

- The sequence of RVs $\{\eta_t\}$ is called a random walk process
- **Special case**: If we have $\{Z_t\}$ such that for each t

$$\mathbb{P}(Z_t = z) = \begin{cases} \frac{1}{2}, & z = 1; \\ \frac{1}{2}, & z = -1, \end{cases}$$

  then $\{\eta_t\}$ is a simple symmetric random walk
- The random walk process is not stationary!

Example Realizations of Random Walk Processes



Gaussian Processes

$\{\eta_t\}$ is a Gaussian process (GP) if the joint distribution of any collection of the RVs has a multivariate normal (aka Gaussian) distribution

- The distribution of a GP is fully characterized by $\mu(\cdot)$, the mean function, and $\gamma(\cdot,\cdot)$, the autocovariance function. The joint probability density function of $\boldsymbol{\eta} = (\eta_1, \eta_2, \cdots, \eta_T)^T$ is

$$f(\boldsymbol{\eta}) = \frac{1}{(2\pi)^{\frac{T}{2}}|\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\boldsymbol{\eta}-\boldsymbol{\mu})^T\Sigma^{-1}(\boldsymbol{\eta}-\boldsymbol{\mu})\right),$$

  where $\boldsymbol{\mu} = (\mu_1, \mu_2, \cdots, \mu_T)^T$ and the (i,j) element of the covariance matrix Σ is $\gamma(i,j)$

- If a GP $\{\eta_t\}$ is weakly stationary then the process is also strictly stationary

Linear Processes

- A time series $\{\eta_t\}$ is a linear process with mean μ if we can write it as

$$\eta_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}, \quad \forall t,$$

where μ is a real-valued constant, $\{Z_t\}$ is a $\mathrm{WN}(0, \sigma^2)$ process and $\{\psi_j\}$ is a set of absolutely summable constants[1]

- Absolute summability of the constants guarantees that the infinite sum converges

[1] A set of real-valued constants $\{\psi_j : j \in \mathbb{Z}\}$ is absolutely summable if $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$

Example: Moving Average Process of Order q, MA(q)

Let $\{Z_t\}$ be a $\mathrm{WN}(0, \sigma^2)$ process. For an integer $q > 0$ and constants $\theta_1, \cdots, \theta_q$ with $\theta_q \neq 0$, define

$$
\begin{aligned}
\eta_t &= Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q} \\
&= \theta_0 Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q} \\
&= \sum_{j=0}^{q} \theta_j Z_{t-j},
\end{aligned}
$$

where we let $\theta_0 = 1$

$\{\eta_t\}$ is known as the moving average process of order q, or the MA(q) process, and, by definition, is a linear process

Defining Linear Processes with Backward Shifts

- Recall the backward shift operator, B, is defined by $B\eta_t = \eta_{t-1}$
- We can represent a linear process using the backward shift operator as $\eta_t = \mu + \psi(B)Z_t$, where we let $\psi(B) = \sum_{j=-\infty}^{\infty} \psi_j B^j$
- Example: we can write a mean zero MA(1) process as

$$\eta_t = \mu + \psi(B)Z_t,$$

where $\mu = 0$ and $\psi(B) = 1 + \theta B$

Linear Filtering Preserves Stationarity

- Let $\{Y_t\}$ be a time series and $\{\psi_j\}$ be a set of absolutely summable constants that does not depend on time

- **Definition**: A linear time invariant filtering of $\{Y_t\}$ with coefficients $\{\psi_j\}$ that do not depend on time is defined by

$$X_t = \psi(B)Y_t$$

- **Theorem**: Suppose $\{Y_t\}$ is a zero mean stationary series with ACVF $\gamma_Y(\cdot)$. Then $\{X_t\}$ is a zero mean stationary process with ACVF

$$\gamma_X(h) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_Y(j - k + h)$$

Example: The MA(q) Process is Stationary

By the filtering preserves stationarity result, the MA(q) process is a stationary process with mean zero and ACVF

$$\gamma(h) = \sigma^2 \sum_{j=0}^{q} \theta_j \theta_{j+h}$$

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$$\gamma(h) = \sigma^2 \sum_{j=0}^{q} \theta_j \theta_{j+h}$$

$$\begin{aligned}
\gamma(h) &= \sum_{j=0}^{q} \sum_{k=0}^{q} \theta_j \theta_k \gamma_Z(j-k+h) \\
&= \sigma^2 \sum_{j=0}^{q} \sum_{k=0}^{q} \theta_j \theta_k \mathbb{1}(k = j+h) \\
&= \sigma^2 \sum_{j=0}^{q} \theta_j \theta_{j+h}
\end{aligned}$$

Processes with a Correlation that Cuts Off

- A time series η_t is q-correlated if

$$\eta_t \text{ and } \eta_s \text{ are uncorrelated } \forall |t-s| > q,$$

i.e., $\mathrm{Cov}(\eta_t, \eta_s) = 0, \forall |t-s| > q$

- A time series $\{\eta_t\}$ is q-dependent if

$$\eta_t \text{ and } \eta_s \text{ are independent } \forall |t-s| > q.$$

- **Theorem**: if $\{\eta_t\}$ is a stationary q-correlated time series with zero mean, then it can be always be represented as an MA(q) process

The Autoregressive Process of Order p, AR(p)

- This process is attributed to George Udny Yule. The AR(1) process has also been called the Markov process
- Let $\{Z_t\}$ be a $\mathrm{WN}(0, \sigma^2)$ process and let $\{\phi_1, \cdots, \phi_p\}$ be a set of constants for some integer $p > 0$ with $\phi_p \neq 0$
- The AR(p) process is defined to be the solution to the equation

$$\eta_t = \sum_{j=1}^{p} \phi_j \eta_{t-j} + Z_t \Rightarrow \underbrace{\eta_t - \sum_{j=1}^{p} \phi_j \eta_{t-j}}_{\phi(B)\eta_t} = Z_t,$$

where we let $\phi(B) = 1 - \sum_{j=1}^{p} \phi_j B^j$

A Stationary Solution for AR(1)

- We want the solution to the AR equation to yield a stationary process. Let's first consider AR(1). We will demonstrate that **a stationary solution exists for** $|\phi_1| < 1$.

- We first write

$$
\begin{aligned}
\eta_t = \phi_1 \eta_{t-1} + Z_t &= \phi_1(\phi_1 \eta_{t-2} + Z_{t-1}) + Z_t \\
&= \phi_1^2 \eta_{t-2} + \phi_1 Z_{t-1} + Z_t \\
&\vdots \\
&= \phi_1^k \eta_{t-k} + \sum_{j=0}^{k-1} \phi_1^j Z_{t-j} \\
&\vdots \\
&= \sum_{j=0}^{\infty} \phi_1^j Z_{t-j}
\end{aligned}
$$

AR(1) Example Cont'd

- Now let $\psi_j = \phi_1^j$. We then have

$$\eta_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}.$$

Using the fact that, for $|a| < 1$, $\sum_{j=0}^{\infty} a^j = \frac{1}{1-a}$, the sequence $\{\psi_j\}$ is absolutely summable

- Thus, since $\{\eta_t\}$ is a linear process, it follows by the filtering preserves stationarity result that $\{\eta_t\}$ is a zero mean stationary process with ACVF

$$\begin{aligned}\gamma(h) &= \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h} \\ &= \sigma^2 \sum_{j=0}^{\infty} \phi_1^j \phi_1^{j+h} \\ &= \sigma^2 \phi^h \sum_{j=0}^{\infty} (\phi_1^2)^j\end{aligned}$$

AR(1) Example Cont'd

Now $|\phi_1| < 1$ implies that $|\phi_1^2| < 1$ and therefore we have

$$\gamma(h) = \frac{\sigma^2 \phi_1^h}{1 - \phi_1^2}$$

When $|\phi_1| \geq 1$

- No stationary solutions exist for $|\phi_1| = 1$
- When $|\phi_1| > 1$, dividing by ϕ_1 for both sides we get

$$\phi_1^{-1} \eta_t = \eta_{t-1} + \phi_1^{-1} Z_t$$
$$\Rightarrow \eta_{t-1} = \phi_1^{-1} \eta_t - \phi_1^{-1} Z_t$$

  A linear combination of **future** Z_t's $\Rightarrow$ we have a stationary solution, but, η_t depends on future $\{Z_t\}$'s–This process is said to be not causal

- If we assume that η_s and Z_t are uncorrelated for each $t > s$, $|\phi_1| < 1$ is the only stationary solution to the AR equation

The Autoregressive Operator

- AR(1) process

$$\eta_t = \phi_1 \eta_{t-1} + Z_t \Rightarrow (1 - \phi_1 B)\eta_t = Z_t \Rightarrow \eta_t = (1 - \phi_1 B)^{-1} Z_t$$

- Recall $\sum_{j=0}^{\infty} a^j = \frac{1}{1-a} = (1-a)^{-1}$. We have

$$\eta_t = \sum_{j=0}^{\infty} (\phi_1 B)^j Z_t = \sum_{j=0}^{\infty} \phi_1^j B^j Z_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}$$

$\Rightarrow$ **This is another way to show that AR(1) is a linear process**

- Here $1 - \phi_1 B$ is the AR characteristic polynomial