

Lecture 2

Characterizing and Displaying Multivariate Data

DSA 8070 Multivariate Analysis
August 23-27, 2021

Whitney Huang
Clemson University

1 Descriptive Statistics

2 Graphs and Visualization

- We will use n to denote the number of individuals or units in our sample and use p to denote the number of variables measured on each unit.
- If $p = 1$, then we are back in the usual univariate setting.
- x_{ik} is the value of the k -th measurement on the i -th unit. For the i -th unit we have measurements

$$(x_{i1}, x_{i2}, \dots, x_{ip})$$

- We often display measurements from a sample of n units in matrix form:

$$X_{n \times p} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{np} \end{bmatrix}$$

is a matrix with n rows (one for each unit) and p columns (one for each measured trait or variable).

- The sample mean of the k th variable ($k = 1, \dots, p$) is computed as

$$\bar{x}_k = \frac{1}{n} \sum_{i=1}^n x_{ik}$$

- The sample variance of the k th variable is usually computed as

$$s_k^2 = \frac{1}{n-1} \sum_{i=1}^n (x_{ik} - \bar{x}_k)^2$$

and the sample standard deviation is given by

$$s_k = \sqrt{s_k^2}$$

Descriptive Statistics: Sample Covariance

- We often use s_{kk} to denote the sample variance for the k -th variable. Thus,

$$s_k^2 = \frac{1}{n-1} \sum_{i=1}^n (x_{ik} - \bar{x}_k)^2 = s_{kk}$$

- The **sample covariance** between variable k and variable j is computed as

$$s_{jk} = \frac{1}{n-1} \sum_{i=1}^n (x_{ij} - \bar{x}_j)(x_{ik} - \bar{x}_k)$$

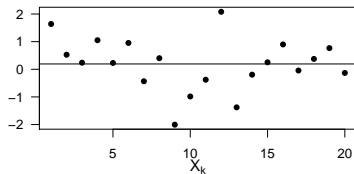
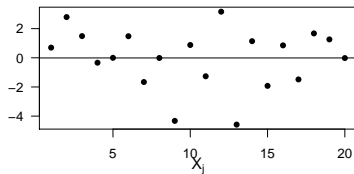
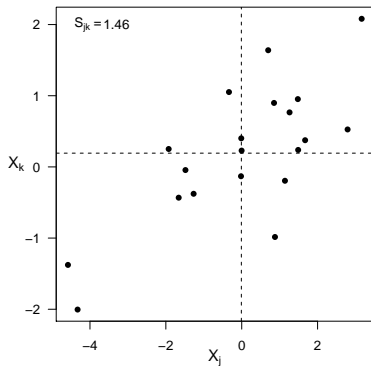


- If variables k and j are independent, the population covariance will be exactly zero, but the sample covariance will vary about zero

```
***{r}  
dat <- mvrnorm(n = 50, mu = c(0, 0), Sigma = matrix(c(1, 0, 0, 1), 2))  
cov(dat[, 1], dat[, 2])  
***
```

[1] -0.1508848

Sample Covariance



- The sample correlation between variables k and j is defined as

$$r_{jk} = \frac{s_{jk}}{\sqrt{s_{jj}}\sqrt{s_{kk}}}$$

- r_{jk} is between -1 and 1
- $r_{jk} = r_{kj}$

- The sample correlation is equal to the sample covariance if measurements are standardized (i.e., $s_{kk} = s_{jj} = 1$)
- Covariance and correlation measure linear association. Other non-linear dependencies may exist among variables even if $r_{jk} = 0$
- The sample correlation (r_{ij}) will vary about the value of the population correlation (ρ_{ij})

Sample statistics of a p -dimensional multivariate data can be organized as vectors and matrices:

- $\bar{\mathbf{x}} = [\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p]^T$ is the $p \times 1$ vector of sample means

- $\mathbf{S} = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1p} \\ s_{21} & s_{22} & \cdots & s_{2p} \\ \vdots & \cdots & \cdots & \cdots \\ s_{p1} & s_{p2} & \cdots & s_{pp} \end{bmatrix}$ is the $p \times p$ symmetric matrix of variance (on the diagonal) and covariances (the off-diagonal elements)

- $\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1p} \\ r_{21} & r_{22} & \cdots & r_{2p} \\ \vdots & \cdots & \cdots & \cdots \\ r_{p1} & r_{p2} & \cdots & r_{pp} \end{bmatrix}$ is the $p \times p$ symmetric matrix of sample correlations. Diagonal elements are all equal to 1

Example: Bivariate Data

- Data consist of $n = 5$ receipts from a bookstore. On each receipt we observe the total amount of the sale (\$) and the number of books sold ($p = 2$). Then

$$X_{5 \times 2} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \\ x_{41} & x_{42} \\ x_{51} & x_{52} \end{bmatrix} = \begin{bmatrix} 42 & 2 \\ 52 & 5 \\ 88 & 7 \\ 58 & 4 \\ 60 & 5 \end{bmatrix}$$

- Sample mean vector is:

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 60 \\ 5 \end{bmatrix}$$

- Sample covariance matrix is

$$\mathbf{S} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} = \begin{bmatrix} 294.0 & 19.0 \\ 19.0 & 1.5 \end{bmatrix}$$

- Sample correlation matrix is

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0.90476 \\ 0.90476 & 1 \end{bmatrix}$$

- The generalized variance is a scalar value which generalizes variance for multivariate random variables
- The generalized variance is defined as the determinant of the (sample) covariance matrix S , $\det(S)$
- **Example:**

```
```{r}
data(mtcars)
vars <- which(names(mtcars) %in% c("mpg", "disp", "hp", "drat", "wt"))
car <- mtcars[, vars]; S <- cov(car)
(genVar <- det(S))
```
```

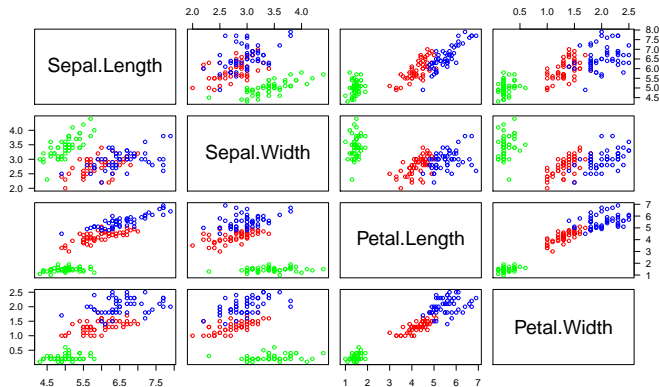
```
[1] 3951786
```

Graphs and Visualization

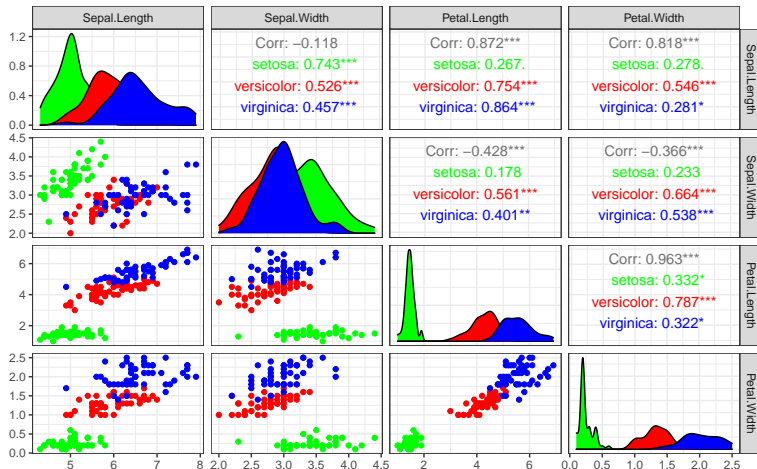
- Graphs convey information about associations between variables and also about unusual observations
- One difficulty with multivariate data is their visualization, in particular when $p > 3$.
- At the very least, we can construct pairwise scatter plots of variables

Example: Fisher's Iris Data

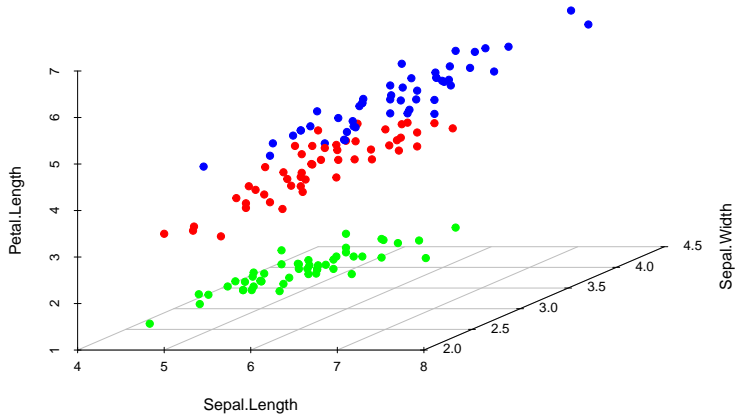
5 variables (sepal length and width, petal length and width, species (**setosa**, **versicolor**, and **virginica**), 50 flowers from each of 3 species $\Rightarrow p = 4, n = 50 \times 3 = 150$



Plotting Iris Data using *ggpairs*



3D Scatter Plot



Chernoff Faces

> head(mtcars)

| | mpg | cyl | dis | hp | drat | wt | qsec | vs | am | gear | carb |
|-------------------|------|-----|-----|-----|------|-------|-------|----|----|------|------|
| Mazda RX4 | 21.0 | 6 | 160 | 110 | 3.90 | 2.620 | 16.46 | 0 | 1 | 4 | 4 |
| Mazda RX4 Wag | 21.0 | 6 | 160 | 110 | 3.90 | 2.875 | 17.02 | 0 | 1 | 4 | 4 |
| Datsun 710 | 22.8 | 4 | 108 | 93 | 3.85 | 2.320 | 18.61 | 1 | 1 | 4 | 1 |
| Hornet 4 Drive | 21.4 | 6 | 258 | 110 | 3.08 | 3.215 | 19.44 | 1 | 0 | 3 | 1 |
| Hornet Sportabout | 18.7 | 8 | 360 | 175 | 3.15 | 3.440 | 17.02 | 0 | 0 | 3 | 2 |
| Valiant | 18.1 | 6 | 225 | 105 | 2.76 | 3.460 | 20.22 | 1 | 0 | 3 | 1 |

Mazda RX4



Mazda RX4 Wag



Datsun 710



Hornet 4 Drive



Hornet Sportabout



Valiant



Duster 360



Merc 240D



Merc 230



Merc 280



Merc 280C



Merc 450SE



Merc 450GL



Merc 450SLC



Cadillac Fleetwood



Lincoln Continental



Chrysler Imperial



Fiat 128



Honda Civic



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Toyota Corona



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Camaro Z28



Pontiac Firebird



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Porsche 914-2



Lotus Europa



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Ferrari Dino



Maserati Bora



Volvo 142E



Visualizing Summary Statistics

