Lecture 16

Multiple Comparisons and Linear Contrasts

Text: Chapter 9

STAT 8010 Statistical Methods I
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One-Way ANOVA & Overall F-Test

- We use one-way ANOVA to compare means of **J (≥ 3) groups/conditions**

$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_J$$
$$H_a : \text{at least a pair } \mu\text{'s differ}$$

- If H_0 is rejected, ANOVA just states that there is a significant difference between the groups **but not where those differences occur**

- We need to perform additional post hoc tests, multiple comparisons, to determine where the group differences are

Pairwise T-Tests

- Suppose we have 4 groups, i.e. $J = 4$, then we need to perform $\binom{4}{2} = 6$ two-sample tests to locate where the group differences are

$$H_0 : \mu_1 = \mu_2 \text{ vs. } H_a : \mu_1 \neq \mu_2$$
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- What if we simply perform these tests using, say, $\alpha = 0.05$ for each test?

$$\mathbb{P}(\text{making a least one type I error}) = 1 - (1 - 0.05)^6 = 0.265$$

if each test was independent

Family-Wise Error Rate (FWER)

Family-Wise Error Rate (FWER) $\bar{\alpha}$: the probability of making 1 or more type I errors in a set of hypothesis tests

For m independent tests, each with individual type I error rate α, then we have

$$\bar{\alpha} = 1 - (1 - \alpha)^m$$

	α		
m	0.1	0.05	0.01
1	0.100	0.050	0.010
3	0.271	0.143	0.030
6	0.469	0.265	0.059
10	0.651	0.401	0.096
15	0.794	0.537	0.140
21	0.891	0.659	0.190

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The Bonferroni Correction

If we would like to control the FWER to be α, then we adjust the significant level for each of the m tests to be $\frac{\alpha}{m}$

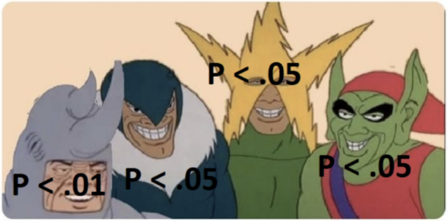
$$FWER = \mathbb{P}(\cup_{i=1}^{m} p_i \leq \frac{\alpha}{m}) \leq \sum_{i=1}^{m} \mathbb{P}(p_i \leq \frac{\alpha}{m}) = m\frac{\alpha}{m} = \alpha$$

where p_i is the p-value for the i_{th} test

If we have 4 treatment groups, then we need to perform 6 tests ($m = 6$) $\Rightarrow$ will need to set the significant level for each individual pairwise t-test to be $0.05/6 = 0.0083$ to ensure that FWER is less than 0.05

Remark: Bonferroni procedure can be very conservative but gives guaranteed control over FWER at the risk of reducing statistical power. Does not assume independence of the comparisons.

Me and the significant boys



Me and the significant boys after Bonferroni correction

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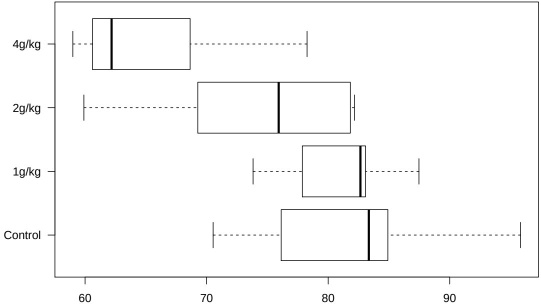
Example

A researcher who studies sleep is interested in the effects of ethanol on sleep time. She gets a sample of 20 rats and gives each an injection having a particular concentration of ethanol per body weight. There are 4 treatment groups, with 5 rats per treatment. She records Rapid eye movement (REM) sleep time for each rat over a 24-period.

Treatment	Control	1g/kg	2g/kg	4g/kg
Mean	82.2	81.0	73.8	65.7
Std	9.6	5.3	9.4	7.9

Recall in last lecture we reject $H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4$ at 0.05 level. But where these differences are?

Example: Multiple Testing with Bonferroni Correction



P-value

Test	μ_1, μ_2	μ_1, μ_3	μ_1, μ_4	μ_2, μ_3	μ_2, μ_4	μ_3, μ_4
Pooled	0.816	0.202	0.018	0.175	0.007	0.179
Non-pooled	0.818	0.202	0.019	0.185	0.009	0.180

Fisher's Protected Least Significant Difference (LSD) Procedure

- We conclude that μ_i and μ_j differ at α significance level if $|\bar{X}_i - \bar{X}_j| > LSD$, where

$$LSD = t_{\alpha/2, df=N-J} \sqrt{\text{MSE}\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}$$

- This procedure builds on the equal variances t-test of the difference between two means
- The test statistic is improved by using MSE rather than s_p^2

Tukey's Honest Significance Difference (HSD) Test

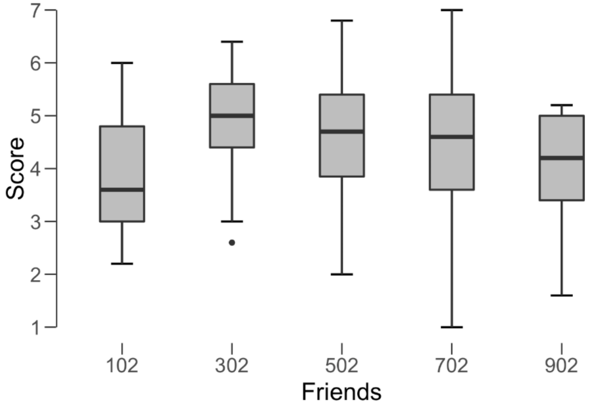
- The test procedure:
  - Requires equal sample size n per populations
  - Find a critical value ω as follows:

$$\omega = q_\alpha(J, N-J)\sqrt{\frac{\text{MSE}}{n}}$$

    where $q_\alpha(J, N-J)$ can be obtained from the studentized range table
- If $\bar{X}_{max} - \bar{X}_{min} > \omega \Rightarrow$ there is sufficient evidence to conclude that $\mu_{max} > \mu_{min}$
- Repeat this procedure for each pair of samples. Rank the means if possible

Facebook Friends Example

A researcher would like to investigate the relationship between Facebook social attractiveness and the number of Facebook friends. An experiment was conducted where five groups of participant judge the same Facebook profiles, except for the one aspect that was manipulated: the number of friends for that profile.



Example: Descriptive Statistics

	Score				
	102	302	502	702	902
Valid	24	33	26	30	21
Missing	0	0	0	0	0
Mean	3.817	4.879	4.562	4.407	3.990
Std. Deviation	0.999	0.851	1.070	1.428	1.023
Minimum	2.200	2.600	2.000	1.000	1.600
Maximum	6.000	6.400	6.800	7.000	5.200

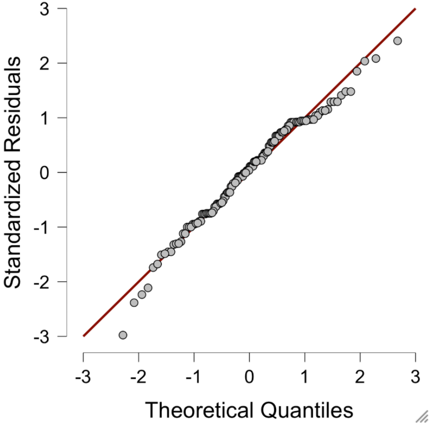
Example: Checking Model Assumptions

Assumption Checks ▼

Test for Equality of Variances (Levene's)

F	df1	df2	p
2.607	4.000	129.000	0.039

Q-Q Plot ▼



Facebook Friends: Overall F-Test

Question: Are Facebook attractiveness affected by # of friends?

$H_0: \mu_1 = \mu_2 = \cdots = \mu_5$
H_a: At least one group mean is different from others

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```
Analysis of Variance Table

Response: Score
           Df Sum Sq Mean Sq F value
Friends     4  19.89  4.9726   4.142
Residuals 129 154.87  1.2005
           Pr(>F)
Friends   0.00344 **
Residuals
```

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Residuals
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Next, we need to figure out where these differences occur

Facebook Example: Fisher's LSD

We conclude that μ_i and μ_j differ at α level if $|\bar{X}_i - \bar{X}_j| > LSD$, where

$$LSD = t_{\alpha/2, df=N-J}\sqrt{\text{MSE}\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}$$

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```
> LSD_none$groups
       Score groups
302 4.878788      a
502 4.561538     ab
702 4.406667    abc
902 3.990476     bc
102 3.816667      c
```

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Facebook Example: Tukey's HSD Test

Yet there is another method to deal with multiple testing: Tukey's Honest Significant Difference (HSD) test. We conclude that μ_i and μ_j differ at α familywise level if $|\bar{X}_i - \bar{X}_j| > \omega$, where

$$\omega = q_\alpha(J, N-J)\sqrt{\frac{\text{MSE}}{n}},$$

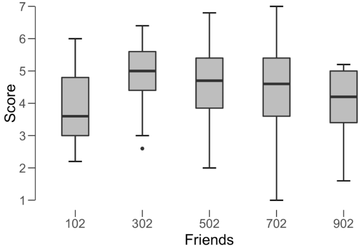
$q_\alpha(J, N-J)$ can be obtained from the studentized range table

Critical Values of Studentized Range Distribution(q) for Familywise ALPHA = .05.

Denominator DF	Number of Groups (a.k.a. Treatments) 3	4	5	6	7	8	9	10
51	3.414	3.756	3.999	4.187	4.340	4.469	4.580	4.677
52	3.412	3.753	3.996	4.184	4.337	4.465	4.576	4.673
53	3.410	3.751	3.994	4.181	4.334	4.462	4.572	4.669
54	3.408	3.749	3.991	4.178	4.331	4.459	4.569	4.666
55	3.406	3.747	3.989	4.176	4.328	4.455	4.566	4.662
56	3.405	3.745	3.986	4.173	4.325	4.452	4.562	4.659
57	3.403	3.743	3.984	4.170	4.322	4.449	4.559	4.656
58	3.402	3.741	3.982	4.168	4.319	4.447	4.556	4.652
59	3.400	3.739	3.979	4.165	4.317	4.444	4.553	4.649
60	3.399	3.737	3.977	4.163	4.314	4.441	4.550	4.646

Facebook Example: Tukey's HSD Test

	diff	lwr	upr	p adj
302-102	1.0621212	0.2488644	1.87537798	0.003889635
502-102	0.7448718	-0.1132433	1.60298691	0.121456224
702-102	0.5900000	-0.2402014	1.42020143	0.288431585
902-102	0.1738095	-0.7320145	1.07963355	0.984016816
502-302	-0.3172494	-1.1121910	0.47769215	0.804080046
702-302	-0.4721212	-1.2368466	0.29260420	0.432633745
902-302	-0.8883117	-1.7345313	-0.04209203	0.034535577
702-502	-0.1548718	-0.9671402	0.65739661	0.984391504
902-502	-0.5710623	-1.4604793	0.31835479	0.391768065
902-702	-0.4161905	-1.2787075	0.44632652	0.669927748



Linear Contrasts

Suppose we have J populations (e.g. response for J different treatments) of interest. We have seen how to perform multiple comparisons. For example, the comparison between μ_1 and μ_2 can be conducted using the test: $H_0 : \mu_1 - \mu_2 = 0$ vs. $H_a : \mu_1 - \mu_2 \neq 0$. This comparison is actually a special case of linear contrasts

Linear Contrasts

Let $c_1, c_2, \cdots, c_J$ are constants where $\sum_{j=1}^{J} c_j = 0$, then $\sum_{j=1}^{J} c_j \mu_j$ is called a **linear contrast** of the population means.

Example: Suppose $J = 4$

1. $\mu_1 - \mu_3 : c_1 = 1, c_2 = 0, c_3 = -1, c_4 = 0$
2. $\mu_2 - \mu_4 : c_1 = 0, c_2 = 1, c_3 = 0, c_4 = -1$
3. $\mu_1 - \frac{1}{3}\mu_2 - \frac{1}{3}\mu_3 - \frac{1}{3}\mu_4 : c_1 = 1, c_2 = c_3 = c_4 = -\frac{1}{3}$

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Inferences for Linear Contrasts

If we want to make a inference about $L = \sum_{i=1}^{J} c_j\mu_j$. Then we use

$$\hat{L} = \sum_{j=1}^{J} c_j\bar{X}_j$$

as the point estimate. Furthermore, we can construct a $100(1-\alpha)\%$ CI for L:

$$(\hat{L} - t_{(\alpha/2, df=N-J)}\hat{se}_{\hat{L}}, \hat{L} + t_{(\alpha/2, df=N-J)}\hat{se}_{\hat{L}}),$$

where $\hat{se}_{\hat{L}} = \sqrt{\text{MSE}\left(\frac{c_1^2}{n_1} + \cdots + \frac{c_J^2}{n_J}\right)}$

To test whether L is significantly different from 0, we can conduct the following test:

$$H_0 : \sum_{j=1}^{J} c_j\mu_j = 0 \text{ vs. } H_a : \sum_{j=1}^{J} c_j\mu_j \neq 0$$

Hypothesis Testing for Linear Contrasts

1. Null and Alternative Hypotheses:

$$H_0 : \sum_{j=1}^{J} c_j\mu_j = 0 \text{ vs. } H_a : \sum_{j=1}^{J} c_j\mu_j \neq 0$$

2. Test Statistic:

$$t_{obs} = \frac{\hat{L} - 0}{\hat{se}_{\hat{L}}} = \frac{\sum_{j=1}^{J} c_j\bar{X}_j}{\sqrt{\text{MSE}\left(\frac{c_1^2}{n_1} + \cdots + \frac{c_J^2}{n_J}\right)}}$$

3. Decision:

Reject H_0 if $|t_{obs}| > t_{\alpha/2, df=N-J}$ (or p-value $< \alpha$)

Facebook Example: Linear Contrast

Suppose we'd like to compare μ_1 vs. $\frac{\mu_3+\mu_4}{2}$. Let $L = 1\mu_1 - \frac{1}{2}\mu_3 - \frac{1}{2}\mu_4$. Then the above comparison is equivalent to test whether L is different from 0

1. $H_0 : L = 0$ vs. $H_a : L \neq 0$
2. $t_{obs} = \frac{\hat{L}}{\hat{se}_{\hat{L}}} = \frac{1\times 3.817-0.5\times 4.562-0.5\times 4.407}{\sqrt{1.2005\times(\frac{1^2}{24}+\frac{0.5^2}{26}+\frac{0.5^2}{30})}} = \frac{-0.6674}{0.2675} = -2.495$
3. Since $|t_{obs}| = |-2.495| = 2.495 > t_{0.025,df=129} = 1.9785$. We reject H_0 at 0.05 level

Note: If we are performing several tests for different linear contrasts simultaneously, we'll need to adjust α level accordingly to control the FWER