Lecture 5

Law of Total Probability, Bayes' Rule, and Random Variables

Text: Chapter 4

STAT 8010 Statistical Methods I
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Agenda

1. **Law of Total Probability**
2. **Bayes' Rule**
3. **Random Variables**

Law of Partitions & Multiplication Rule

Law of partitions

Let $A_1, A_2, \cdots, A_k$ form a partition of Ω. Then, for all events B,

$$\mathbb{P}(B) = \sum_{i=1}^{k} \mathbb{P}(A_i \cap B)$$

Multiplication rule

- 2 events:

$$\mathbb{P}(B \cap A) = \mathbb{P}(A) \times \mathbb{P}(B|A) = \mathbb{P}(B) \times \mathbb{P}(A|B)$$

- More than 2 events:

$$\mathbb{P}(\cap_{i=1}^{n} A_i) = \mathbb{P}(A_1) \times \mathbb{P}(A_2|A_1) \times \mathbb{P}(A_3|A_1 \cap A_2) \times \cdots \times \mathbb{P}(A_n|A_{n-1} \cap \cdots \cap A_1)$$

Law of Total Probability

Let $A_1, A_2, \cdots, A_k$ form a partition of Ω. Then, for all events B,

$$\mathbb{P}(B) = \underbrace{\sum_{i=1}^{k} \mathbb{P}(A_i \cap B)}_{\text{Law of partitions}}$$

$$= \underbrace{\sum_{i=1}^{k} \mathbb{P}(B|A_i) \times \mathbb{P}(A_i)}_{\text{Multiplication rule}}$$

Example

Suppose that two factories supply light bulbs to the market. Factory X's bulbs work for over 5000 hours in 99% of cases, whereas factory Y's bulbs work for over 5000 hours in 95% of cases. It is known that factory X supplies 60% of the total bulbs available and Y supplies 40% of the total bulbs available. What is the chance that a purchased bulb will work for longer than 5000 hours?

Bayes' Rule: Motivating example

The Monty Hall Problem

There was an old television show called Let's Make a Deal, whose original host was named Monty Hall. The set–up is as follows. You are on a game show and you are given the choice of three doors. Behind one door is a car, behind the others are goats. You pick a door, and the host, who knows what is behind the doors, opens another door (not your pick) which has a goat behind it. Then he asks you if you want to change your original pick. The question we ask you is, "Is it to your advantage to switch your choice?"

The Monty Hall Problem

The Monty Hall Problem Solution

Bayes' Rule

General form

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}$$

Let $A_1, A_2, \cdots, A_k$ form a partition of the sample space. Then for every event B in the sample space,

$$\boxed{\mathbb{P}(A_j|B) = \frac{\mathbb{P}(B|A_j) \times \mathbb{P}(A_j)}{\sum_{i=1}^{k} \mathbb{P}(B|A_i) \times \mathbb{P}(A_i)}}, \; j = 1, 2, \cdots, k$$

Example

Let us assume that a specific disease is only present in 5 out of every 1,000 people. Suppose that the test for the disease is accurate 99% of the time a person has the disease and 95% of the time that a person lacks the disease. What is the probability that the person has the disease given that they tested positive?

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Solution.

$$\mathbb{P}(D|+) = \frac{\mathbb{P}(D\cap+)}{\mathbb{P}(+)} = \frac{.005\times.99}{.005\times.99+.995\times.05} = \frac{.00495}{.0547} = .0905$$

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The reason we get such a surprising result is because the disease is so rare that the number of false positives greatly outnumbers the people who truly have the disease.

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- Independence and Conditional Probability

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- Independence: if A and B are independent, then
  $\mathbb{P}(A|B) = \mathbb{P}(A)$, $\mathbb{P}(B|A) = \mathbb{P}(B)$, and $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$

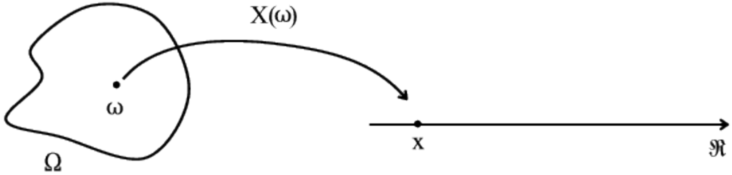
Random Variables

Random Variables

A random variable is a real–valued function whose domain is the sample space of a random experiment. In other words, a random variable is a function

$$X : \Omega \mapsto \mathbb{R}$$

where Ω is the sample space of the random experiment under consideration and $\mathbb{R}$ represents the set of all real numbers.



Example

The following is a chart describing the number of siblings each student in a particular class has.

Siblings (X)	Frequency	Relative Frequency
0	8	.200
1	17	.425
2	11	.275
3	3	.075
4	1	.025
Total	40	1

Example cont'd

Let's define the event A as the event that a randomly chosen student has 2 or more siblings. What is $\mathbb{P}(X \in A)$?

Solution.

$$\begin{aligned}\mathbb{P}(X \in A) = \mathbb{P}(X \geq 2) &= \mathbb{P}(X = 2) + \mathbb{P}(X = 3) + \mathbb{P}(X = 4) \\ &= .275 + 0.75 + 0.25 = .375\end{aligned}$$

Types of Random Variables

There are two main types of quantitative random variables: discrete and continuous. A discrete random variable often involves a count of something. Examples may include number of cars per household, etc.

Discrete random variable

A random variable X is called a discrete random variable if the outcome of the random variable is limited to a countable set of real numbers (usually integers).

Probability Mass Function

Let X be a discrete random variable. Then the probability mass function (pmf) of X is the real–valued function defined on $\mathbb{R}$ by

$$p_X(x) = \mathbb{P}(X = x)$$

The capital letter, X, is used to denote random variable. Lowercase letter, x, is used to denote possible values of the random variable.

Example

Flip a fair coin 3 times. Let X denote the number of heads tossed in the 3 flips. Create a pmf for X

Solution.

The random variable X maps any outcome to an integer (e.g. $X\left(\left(\text{T, T, T}\right)\right) = 0$)

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X	0	1	2	3
$p_X(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Properties of a PMF

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- $\{x \in \mathbb{R} : p_X(x) \neq 0\}$ is countable
- $\sum_x p_X(x) = 1$

Example

Let X be a random variable with pmf defined as follows:

$$p_X(x) = \begin{cases} k(5-x) & \text{if} \quad x = 0, 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}$$

1. Find the value of k that makes $p_X(x)$ a legitimate pmf.
2. What is the probability that X is between 1 and 3 inclusive?
3. If X is not 0, what is the probability that X is less than 3?

Mean of Discrete Random Variables

The mean of a discrete r.v. X, denoted by $\mathbb{E}[X]$, is defined by

$$\mathbb{E}[X] = \sum_x x \times p_X(x)$$

Remark:

The mean of a discrete r.v. is a weighted average of its possible values, and the weight used is its probability. Sometimes we refer to the expected value as the expectation (expected value), or the first moment.

For any function, say $g(X)$, we can also find an expectation of that function. It is

$$\mathbb{E}[g(X)] = \sum_x g(x) \times p_X(x)$$

Example

$$\mathbb{E}[X^2] = \sum_x x^2 \times p_X(x)$$

Properties of Mean

Let X and Y be discrete r.v.s defined on the same sample space and having finite expectation (i.e. $\mathbb{E}[X], \mathbb{E}[Y] < \infty$). Let a and b be constants. Then the following hold:

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- $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$
- $\mathbb{E}[aX + b] = a \times \mathbb{E}[X] + b$

Number of Siblings Example Revisited

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$$\mathbb{E}[X] = \sum_x x p_X(x) = 0\times.200+1\times.425+2\times.275+3\times.075+4\times.025 = 1.3$$

Variance/Standard Deviation of Discrete r.v.'s

The **variance** of a (discrete) r.v., denoted by $Var(X)$, is a measure of the spread, or variability, in the r.v. $Var(X)$ is defined by

$$\boxed{Var(X) = \mathbb{E}[(X - \mathbb{E}[x])^2]}$$

or

$$\boxed{Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2}$$

The **standard deviation**, denoted by $Sd(X)$, is the square root of its variance

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4. $\mathbb{E}[X^2 - 4]$
5. $\mathbb{E}[(X - 4)^2]$
6. $Var(2X - 4)$