

Lecture 11

Canonical Correlation Analysis

DSA 8070 Multivariate Analysis
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Notes

Agenda

1 Canonical Correlations

2 Sales Data Example



Notes

Overview

Canonical correlation analysis (CCA) is a method for exploring the relationships between two sets of multivariate variables $\mathbf{X} = (X_1, X_2, \dots, X_p)^T$ and $\mathbf{Y} = (Y_1, Y_2, \dots, Y_q)^T$

For example, \mathbf{X} could be a vector of variables associated with **environmental health** such as species diversity, total biomass, productivity of the environment, etc while \mathbf{Y} could be concentrations of heavy metals, pesticides, dioxin that measure **environmental toxins**

CCA relates two sets of variables \mathbf{X} and \mathbf{Y} by finding linear combinations of variables that maximally correlated

Motivation: relates \mathbf{X} and \mathbf{Y} using a small number of linear combinations for ease of interpretation



Notes

Linear Combinations of Two Sets of Variables

Recall we have $\mathbf{X} = (X_1, X_2, \dots, X_p)^T$ and $\mathbf{Y} = (Y_1, Y_2, \dots, Y_q)^T$. Without loss of generality, let's assume $p \leq q$.

Similar to PCA, we define a set of linear combinations

$$U_1 = a_{11}X_1 + a_{12}X_2 + \dots + a_{1p}X_p$$

$$U_2 = a_{21}X_1 + a_{22}X_2 + \dots + a_{2p}X_p$$

$$\vdots = \dots$$

$$U_p = a_{p1}X_1 + a_{p2}X_2 + \dots + a_{pp}X_p$$

and

$$V_1 = b_{11}Y_1 + b_{12}Y_2 + \dots + b_{1q}Y_q$$

$$V_2 = b_{21}Y_1 + b_{22}Y_2 + \dots + b_{2q}Y_q$$

$$\vdots = \dots$$

$$V_p = b_{p1}Y_1 + b_{p2}Y_2 + \dots + b_{pq}Y_q$$

We want to find **linear combinations that maximize the correlation of** (U_i, V_i) , $i = 1, \dots, p$



Notes

Defining Canonical Variates

We call (U_i, V_i) be the i^{th} **canonical variate** pair. One can compute the variance of U_i with the following expression:

$$\text{Var}(U_i) = \sum_{k=1}^p \sum_{\ell=1}^p a_{ik}a_{i\ell} \text{Cov}(X_k, X_\ell), \quad i = 1, \dots, p.$$

Similarly, we compute the variance of V_j with the following expression:

$$\text{Var}(V_j) = \sum_{k=1}^q \sum_{\ell=1}^q b_{jk}b_{j\ell} \text{Cov}(Y_k, Y_\ell), \quad j = 1, \dots, p.$$

The covariance between U_i and V_j is:

$$\text{Cov}(U_i, V_j) = \sum_{k=1}^p \sum_{\ell=1}^q a_{ik}b_{j\ell} \text{Cov}(X_k, Y_\ell).$$

The **canonical correlation** for the i^{th} canonical variate pair is simply the correlation between U_i and V_i :

$$\rho_i^* = \frac{\text{Cov}(U_i, V_i)}{\sqrt{\text{Var}(U_i)\text{Var}(V_i)}}$$



Notes

Finding Canonical Variates

Let us look at each of the p canonical variates pair one by one.

First canonical variable pair (U_1, V_1) : The coefficients $a_{11}, a_{12}, \dots, a_{1p}$ and $b_{11}, b_{12}, \dots, b_{1q}$ are chosen to **maximize the canonical correlation** ρ_1^* . As in PCA, this is subject to the constraint that $\text{Var}(U_1) = \text{Var}(V_1) = 1$

Second canonical variable pair (U_2, V_2) : Similarly we want to find $a_{21}, a_{22}, \dots, a_{2p}$ and $b_{21}, b_{22}, \dots, b_{2q}$ that maximize ρ_2^* under the following constraints:

$$\text{Var}(U_2) = \text{Var}(V_2) = 1,$$

$$\text{Cov}(U_1, U_2) = \text{Cov}(V_1, V_2) = 0,$$

$$\text{Cov}(U_1, V_2) = \text{Cov}(U_2, V_1) = 0.$$

This procedure is repeated for each pair of canonical variates



Notes

Finding Canonical Variates Cont'd

Let $\text{Var}(\mathbf{X}) = \Sigma_X$ and $\text{Var}(\mathbf{Y}) = \Sigma_Y$ and let $\mathbf{Z}^T = (\mathbf{X}^T, \mathbf{Y}^T)$. Then the covariance matrix of \mathbf{Z} is

$$\begin{bmatrix} \text{Var}(\mathbf{X}) & \text{Cov}(\mathbf{X}, \mathbf{Y}) \\ \text{Cov}(\mathbf{Y}, \mathbf{X}) & \text{Var}(\mathbf{Y}) \end{bmatrix} = \begin{bmatrix} \underbrace{\Sigma_X}_{p \times p} & \underbrace{\Sigma_{XY}}_{p \times q} \\ \underbrace{\Sigma_{YX}}_{q \times p} & \underbrace{\Sigma_Y}_{q \times q} \end{bmatrix}$$

The i^{th} pair of canonical variates is given by

$$U_i = \underbrace{\mathbf{u}_i^T \Sigma_X^{-1/2}}_{a_i^*} \mathbf{X} \text{ and } V_i = \underbrace{\mathbf{v}_i^T \Sigma_Y^{-1/2}}_{b_i^*} \mathbf{Y},$$

where

- \mathbf{u}_i is the i^{th} eigenvector of $\Sigma_X^{-1/2} \Sigma_{XY} \Sigma_Y^{-1} \Sigma_{YX} \Sigma_X^{-1/2}$
- \mathbf{v}_i is the i^{th} eigenvector of $\Sigma_Y^{-1/2} \Sigma_{YX} \Sigma_X^{-1} \Sigma_{XY} \Sigma_Y^{-1/2}$
- The i^{th} canonical correlation is given by, $\text{Cor}(U_i, V_i) = \rho_i^*$, where ρ_i^{*2} is the i^{th} eigenvalue of $\Sigma_X^{-1/2} \Sigma_{XY} \Sigma_Y^{-1} \Sigma_{YX} \Sigma_X^{-1/2}$



Notes

Likelihood Ratio Test: Is CCA Worthwhile?

Note that if $\Sigma_{XY} = \mathbf{0}$, then $\text{Cov}(\mathbf{U}, \mathbf{V}) = \mathbf{a}^T \Sigma_{XY} \mathbf{b} = 0$ for all \mathbf{a} and $\mathbf{b} \Rightarrow$ all canonical correlations must be zero and there is no point in pursuing CCA.

For large n , we reject $H_0 : \Sigma_{XY} = \mathbf{0}$ in favor of $H_1 : \Sigma_{XY} \neq \mathbf{0}$ if

$$-2 \log(\Lambda) = n \log \left(\frac{|\mathbf{S}_X| |\mathbf{S}_Y|}{|\mathbf{S}|} \right) = -n \sum_{j=1}^p \log(1 - \hat{\rho}_j^{*2})$$

is larger than $\chi_{pq}^2(\alpha)$ For an improvement to the χ^2 approximation, Bartlett suggested using the following test statistic:

$$-2 \log(\Lambda) = -[n - 1 - \frac{1}{2}(p + q + 1)] \sum_{j=1}^p \log(1 - \hat{\rho}_j^{*2})$$



Notes

Example: Sales Data [Source: PSU STAT 505]

The example data comes from a firm that surveyed a random sample of $n = 50$ of its employees in an attempt to determine which factors influence sales performance. Two collections of variables were measured:

- **Sales Performance:** Sales Growth, Sales Profitability, New Account Sales
- **Test Scores as a Measure of Intelligence:** Creativity, Mechanical Reasoning, Abstract Reasoning, Mathematics

We are going to carry out a canonical correlation analysis using R



Notes

Likelihood Ratio Test: Is CCA Worthwhile?

Let's first determine if there is any relationship between the two sets of variables at all.

H_0	Approximate F value	p-value
$\rho_1^* = \rho_2^* = \rho_3^* = 0$	87.39	~ 0
$\rho_2^* = \rho_3^* = 0$	18.53	8.25×10^{-14}
$\rho_3^* = 0$	3.88	0.028

All three canonical variate pairs are significantly correlated and dependent on one another. This suggests that we may summarize all three pairs.

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11.10

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Estimates of Canonical Correlation

Since we rejected the hypotheses of independence, the next step is to obtain estimates of canonical correlation

i	Canonical Correlation (ρ_i^*)	ρ_i^{*2}
1	0.9945	0.9890
2	0.8781	0.7711
3	0.3836	0.1472

98.9% of the variation in U_1 is explained by the variation in V_1 , 77.11% of the variation in U_2 is explained by V_2 , only 14.72% of the variation in U_3 is explained by V_3

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11.11

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Obtain the Canonical Coefficients

	U_1	U_2	U_3
Growth	0.0624	-0.1741	-0.3772
Profit	0.0209	0.2422	0.1035
New	0.0783	-0.2383	0.3834

The first canonical variable for sales is

$$U_1 = 0.0624X_{growth} + 0.0209X_{profit} + 0.0783X_{new}$$

	V_1	V_2	V_3
Creativity	0.0697	-0.1924	0.2466
Mechanical	0.0307	0.2016	-0.1419
Abstract	0.08956	-0.4958	-0.2802
Math	0.0628	0.0683	-0.0113

The first canonical variable for test scores is

$$V_1 = 0.0697Y_{create} + 0.0307Y_{mech} + 0.0896Y_{abstract} + 0.0628Y_{math}$$

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11.12

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Correlations Between Each Variable and The Corresponding Canonical Variate

Correlations Between X 's and U 's

	U_1	U_2	U_3
Growth	0.9799	0.0006	-0.1996
Profit	0.9464	0.3229	0.0075
New	0.9519	-0.1863	0.2434

Correlations Between Y 's and V 's

	V_1	V_2	V_3
Creativity	0.6383	-0.2157	0.6514
Mechanical	0.7212	0.2376	-0.0677
Abstract	0.6472	-0.5013	-0.5742
Math	0.9441	0.1975	-0.0942

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11.13

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Correlations Between Each Set of Variables and The Opposite Group of Canonical Variates

Correlations Between X 's and V 's

	V_1	V_2	V_3
Growth	0.9745	0.0006	-0.0766
Profit	0.9412	0.2835	0.0029
New	0.9466	-0.1636	0.0934

Correlations Between Y 's and U 's

	U_1	U_2	U_3
Creativity	0.6348	-0.1894	0.2499
Mechanical	0.7172	0.2086	-0.0260
Abstract	0.6437	-0.4402	-0.2203
Math	0.9389	0.1735	-0.0361

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11.14

Notes

Notes
