# Lecture 6

## Probability II

Text: Chapter 4

*STAT 8010 Statistical Methods I*
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Notes

## Agenda

1. Union, Intersection, and Logical Relationships among Events
2. Complement Rule and General Addition Rule
3. Independence and Conditional Probability
4. Law of Total Probability
5. Bayes' Rule





Notes

## Probability Rules

1. Any probability must be between 0 and 1 inclusively
2. The sum of the probabilities for all the experimental outcomes must equal 1

If a probability model satisfies the two rules above, it is said to be legitimate





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### Example

An experiment with three outcomes has been repeated 50 times, and it was learned that outcome 1 occurred 20 times, outcome 2 occurred 13 times, and outcome 3 occurred 17 times. Assign probabilities to the outcomes. What method did you use?

**Solution.**





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### Example

A decision maker subjectively assigned the following probabilities to the four possible outcomes of an experiment:

$$P(E_1) = 0.1 \; P(E_2) = 0.15 \; P(E_3) = 0.4 \; P(E_4) = 0.2$$

Are these probability assignments legitimate? Explain.

**Solution.**





Notes

# Union, Intersection, and Logical Relationships among Events





Notes

## Intersection and Union

- Intersection: the intersection of two events $A$ and $B$, denoted by $A \cap B$, is the event that contains all outcomes of $A$ that also belong to $B \Rightarrow$ AND

  Example: Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 4, 5\}$, then $A \cap B = \{1, 2\}$
- Union: the union of two events $A$ and $B$, denoted by $A \cup B$, is the event of all outcomes that belong to either $A$ or $B \Rightarrow$ OR

  Example: Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 4, 5\}$, then $A \cup B = \{1, 2, 3, 4, 5\}$





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## Example

Suppose we flipped 3 fair coins. Let $A$ be the event of **exactly 2 tails**. Let $B$ be the event that the **first 2 tosses are tails**. Let $C$ be the event that **all 3 tosses are tails**. What are $A \cap B$, $A \cup C$, and $(A \cap B) \cup C$?

**Solution.**

$$A = \{(T, T, H), (T, H, T), (H, T, T)\}$$
$$B = \{(T, T, T), (T, T, H)\}$$
$$C = \{T, T, T\}$$

1. $A \cap B = \{(T, T, H)\}$
2. $A \cup C = \{(T, T, H), (T, H, T), (H, T, T), (T, T, T)\}$
3. $(A \cap B) \cup C =$ $\{T, T, H\} \cup \{T, T, T\} = \{(T, T, H), (T, T, T)\}$

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## Logical Relationships among Events

- Mutually exclusive: refers to two (or more) events that cannot both occur when the random experiment is formed.
$$A \cap B = \emptyset$$
- Exhaustive: refers to event(s) that comprise the sample space.
$$A \cup B = \Omega$$
- Partition: events that are both mutually exclusive and exhaustive.
$$A \cap B = \emptyset \quad \text{and} \quad A \cup B = \Omega$$

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# Complement Rule and General Addition Rule



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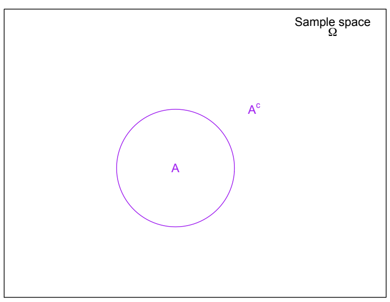
## Complement



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## Complement Rule

1. By the definition of complement

$$A \cup A^c = \Omega$$

2. Apply the probability operator

$$\mathbb{P}(A \cup A^c) = \mathbb{P}(\Omega) = 1$$

3. Since $A$ and $A^c$ are mutually exclusive

$$\mathbb{P}(A \cup A^c) = \mathbb{P}(A) + \mathbb{P}(A^c)$$

4. Hence we get $\boxed{\mathbb{P}(A) = 1 - \mathbb{P}(A^c)}$



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## Example

Suppose we rolled a fair, six–sided die 10 times. Let $T$ be the event that we roll at least 1 three. If one were to calculate $T$ you would need to find the probability of 1 three, 2 threes, $\cdots$, and 10 threes and add them all up. However, you can use the complement rule to calculate $\mathbb{P}(T)$

**Solution.**

Let $X$ be the times that we rolled a $3$, then
$\mathbb{P}(T) = \mathbb{P}(X \geq 1) =$
$\underbrace{\mathbb{P}(X=1) + \mathbb{P}(X=2) + \cdots + \mathbb{P}(X=10)}_{\text{need to compute 10 probabilities}}$
If we apply the complement rule
$\mathbb{P}(T) = 1 - \mathbb{P}(T^c) = 1 - \mathbb{P}(X=0)$




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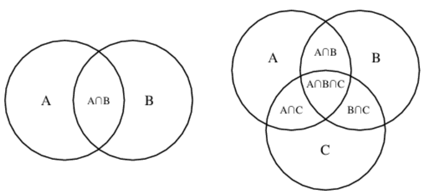
## Venn Diagram

A Venn diagram is a diagram that shows all possible logical relations between a finite collection of events.







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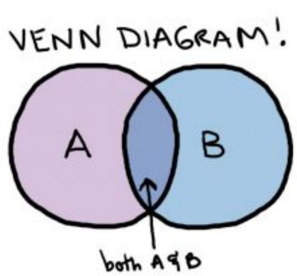
## General Addition Rule

The general addition rule is a way of finding the probability of a union of 2 events. It is
$\boxed{\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)}$







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## Example

Three of the major commercial computer operating systems are Windows, Mac OS, and Red Hat Linux Enterprise. A Computer Science professor selects 50 of her students and asks which of these three operating systems they use. The results for the 50 students are summarized below.

- 30 students use Windows
- 16 students use at least two of the operating systems
- 9 students use all three operating systems
- 18 students use Mac OS
- 46 students use at least one of the operating systems
- 11 students use both Windows and Linux
- 11 students use both Windows and Mac OS





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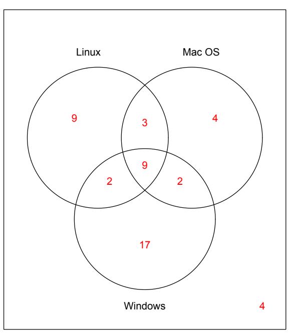
## Example cont'd





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# Independence and Conditional Probability





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## Independence: A Motivating Example



### Example

You toss a fair coin and it comes up “Heads" three times. What is the chance that the next toss will also be a “Head"?



Notes

## Independence and Conditional Probability



### Conditional Probability

Let $A$ and $B$ be events. The probability that event $B$ occurs given (knowing) that event $A$ occurs is called a conditional probability and is denoted by $P(B|A)$. The formula of conditional probability is

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

### Independent events

Suppose $P(A) > 0,\ P(B) > 0$. We say that event $B$ is independent of event $A$ if the occurrence of event $A$ does not affect the probability that event $B$ occurs.

$$\boxed{P(B|A) = P(B) \Rightarrow P(B \cap A) = P(B)P(A)}$$



Notes

## Law of Partitions & Multiplication Rule



### Law of partitions

Let $A_1, A_2, \cdots, A_k$ form a partition of $\Omega$. Then, for all events $B$,

$$\mathbb{P}(B) = \sum_{i=1}^{k} \mathbb{P}(A_i \cap B)$$

### Multiplication rule

- 2 events:

$$\mathbb{P}(B \cap A) = \mathbb{P}(A) \times \mathbb{P}(B|A) = \mathbb{P}(B) \times \mathbb{P}(A|B)$$

- More than 2 events:

$$\mathbb{P}(\cap_{i=1}^{n} A_i) = \mathbb{P}(A_1) \times \mathbb{P}(A_2|A_1) \times \mathbb{P}(A_3|A_1 \cap A_2) \\ \times \cdots \times \mathbb{P}(A_n|A_{n-1} \cap \cdots \cap A_1)$$



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## Law of Total Probability

Let $A_1, A_2, \cdots, A_k$ form a partition of $\Omega$. Then, for all events $B$,

$$\mathbb{P}(B) = \underbrace{\sum_{i=1}^{k} \mathbb{P}(A_i \cap B)}_{\text{Law of partitions}}$$

$$= \underbrace{\sum_{i=1}^{k} \mathbb{P}(B|A_i) \times \mathbb{P}(A_i)}_{\text{Multiplication rule}}$$





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## Example

Suppose that two factories supply light bulbs to the market. Factory X's bulbs work for over 5000 hours in 99% of cases, whereas factory Y's bulbs work for over 5000 hours in 95% of cases. It is known that factory X supplies 60% of the total bulbs available and Y supplies 40% of the total bulbs available. What is the chance that a purchased bulb will work for longer than 5000 hours?

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## Bayes' Rule: Motivating example

### The Monty Hall Problem

There was an old television show called Let's Make a Deal, whose original host was named Monty Hall. The set–up is as follows. You are on a game show and you are given the choice of three doors. Behind one door is a car, behind the others are goats. You pick a door, and the host, who knows what is behind the doors, opens another door (not your pick) which has a goat behind it. Then he asks you if you want to change your original pick. The question we ask you is, "Is it to your advantage to switch your choice?"

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## The Monty Hall Problem






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## The Monty Hall Problem Solution

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## Bayes' Rule

### General form

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}$$

Let $A_1, A_2, \cdots, A_k$ form a partition of the sample space. Then for every event $B$ in the sample space,

$$\boxed{\mathbb{P}(A_j|B) = \frac{\mathbb{P}(B|A_j) \times \mathbb{P}(A_j)}{\sum_{i=1}^{k} \mathbb{P}(B|A_i) \times \mathbb{P}(A_i)}}, \; j = 1, 2, \cdots, k$$

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## Example

Let us assume that a specific disease is only present in 5 out of every 1,000 people. Suppose that the test for the disease is accurate 99% of the time a person has the disease and 95% of the time that a person lacks the disease. What is the probability that the person has the disease given that they tested positive?

**Solution.**

$\mathbb{P}(D|+) = \frac{\mathbb{P}(D\cap+)}{\mathbb{P}(+)} = \frac{.005\times.99}{.005\times.99+.995\times.05} = \frac{.00495}{.0547} = .0905$

The reason we get such a surprising result is because the disease is so rare that the number of false positives greatly outnumbers the people who truly have the disease.





Notes

## Review of Probability (we learned so far)

**Basic Concepts:**

- Random Experiment, Sample Space, Outcome, Event
- Frequentist Interpretation of Probability and Equally Likely Framework
- Union and Intersection, Mutually Exclusive, Exhaustive, Partition
- Venn Diagram
- Independence and Conditional Probability

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## Review: Probability Rules

- $0 \leq \mathbb{P}(A) \leq 1$ for any event $A$, $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\Omega) = 1$
- Complement rule: $\mathbb{P}(A) = 1 - \mathbb{P}(A^c)$
- General addition rule: $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$
- Multiplication rule: $\mathbb{P}(A \cap B) = \mathbb{P}(A|B) \times \mathbb{P}(B) = \mathbb{P}(B|A) \times \mathbb{P}(A)$
- Conditional probability: $\mathbb{P}(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}$
- Law of total probability: $\mathbb{P}(B) = \sum_{i=1}^{k} \mathbb{P}(B \cap A_i) = \sum_{i=1}^{k} \mathbb{P}(B|A_i) \times \mathbb{P}(A_i)$
- Independence: if $A$ and $B$ are independent, then $\mathbb{P}(A|B) = \mathbb{P}(A)$, $\mathbb{P}(B|A) = \mathbb{P}(B)$, and $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$

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