Lecture 13
Time Series Analysis II

DSA 8020 Statistical Methods II

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Modeling Trend, Seasonality, and Noise

Recall the trend, seasonality, noise decomposition mentioned last week:

$$Y_t = \mu_t + s_t + \eta_t,$$

where

- μ_t: trend component is a long-term pattern or directionality observed over time;
- s_t: seasonal component is a pattern that repeats at regular intervals within a specific time period;
- η_t: random noise represents the irregular fluctuations that may be correlated in time.

We are going to learn two approaches for estimating s_t, the seasonal component

Seasonal Component Estimation

- Let's consider the situation that a time series consists of seasonal component only (assuming the trend has been estimated/removed), that is,

$$Y_t = s_t + \eta_t,$$

with $\{s_t\}$ having period d (i.e., $s_{t+jd} = s_t$ for all integers j and t), $\sum_{t=1}^{d} s_t = 0$ and $\mathbb{E}(\eta_t) = 0$

- Two regression methods to estimate $\{s_t\}$
  - Harmonic regression
  - Seasonal mean model

Harmonic Regression

- A harmonic regression model has the form

$$s_t = \sum_{j=1}^{k} A_j \cos(2\pi f_j t + \phi_j).$$

For each $j = 1, \cdots, k$:

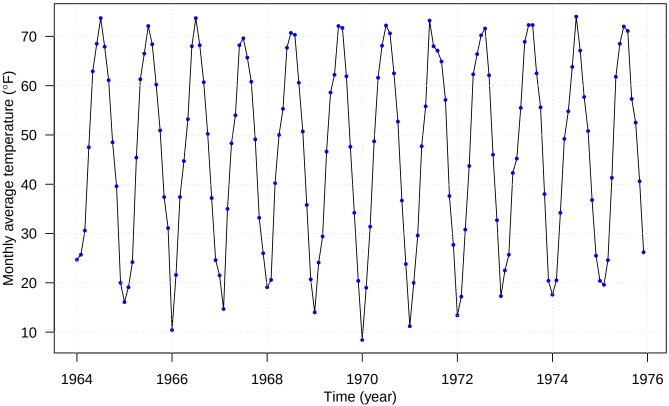
  - $A_j > 0$ is the amplitude of the j-th cosine wave
  - f_j controls the frequency of the j-th cosine wave (how often waves repeats)
  - $\phi_j \in [-\pi, \pi]$ is the phase of the j-th wave (where it starts)

- The above can be expressed as

$$\sum_{j=1}^{k} \{\beta_{1j} \cos(2\pi f_j t) + \beta_{2j} \sin(2\pi f_j t)\},$$

where $\beta_{1j} = A_j \cos(\phi_j)$ and $\beta_{2j} = -A_j \sin(\phi_j) \Rightarrow$ if $\{f_j\}_{j=1}^{k}$ are known, **we can use regression techniques to estimate the parameters** $\{\beta_{1j}, \beta_{2j}\}_{j=1}^{k}$

Monthly Temperature in Dubuque, IA [Cryer & Chan, 2008]



Let's assume there is no trend in this time series. Here we want to estimate s_t, the seasonal component

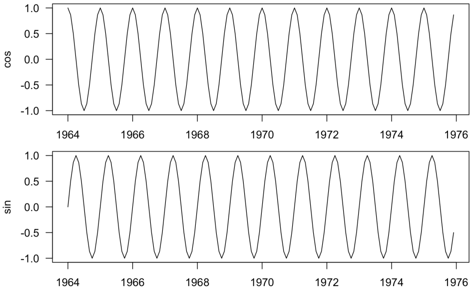
Modeling Annual Cycle via Harmonic Regression

Model: $s_t = \beta_0 + \beta_1 \cos(2\pi t) + \beta_2 \sin(2\pi t)$

$\Rightarrow$ **annual cycles can be modeled by a linear combination of** $\cos$ **and** $\sin$ **with 1-year period** $\Rightarrow d = 12$.

In `R`, we can easily create these harmonics using the `harmonic` function in the `TSA` package

```
harmonics <- harmonic(tempdub, 1)
```



R Code & Output

```
```{r}
harReg <- lm(tempdub ~ harmonics)
summary(harReg)
```

```
Call:
lm(formula = tempdub ~ harmonics)

Residuals:
     Min       1Q   Median       3Q      Max
-11.1580  -2.2756  -0.1457   2.3754  11.2671

Coefficients:
                    Estimate Std. Error t value Pr(>|t|)
(Intercept)          46.2660     0.3088 149.816  < 2e-16 ***
harmonicscos(2*pi*t) -26.7079     0.4367 -61.154  < 2e-16 ***
harmonicssin(2*pi*t)  -2.1697     0.4367  -4.968 1.93e-06 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```
```

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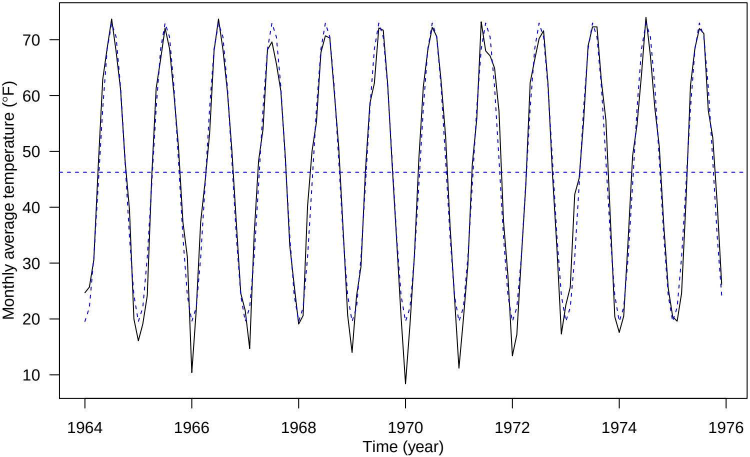
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Question: What assumptions are we making here?

The Harmonic Regression Model Fit



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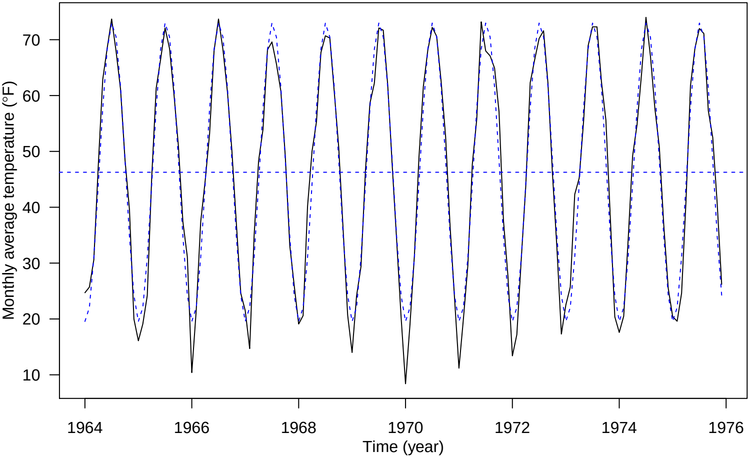


Question: What can be the model limitations?

Seasonal Means Model

- **Harmonics regression** assumes the seasonal pattern has a regular shape, i.e., the height of the peaks is the same as the depth of the troughs

- A less restrictive approach is to model $\{s_t\}$ as

$$s_t = \begin{cases} \beta_1 & \text{for } t = 1, 1+d, 1+2d, \cdots \quad ; \\ \beta_2 & \text{for } t = 2, 2+d, 2+2d, \cdots \quad ; \\ \vdots & \vdots \quad ; \\ \beta_d & \text{for } t = d, 2d, 3d, \cdots \quad . \end{cases}$$

- This is the seasonal means model, the parameters $(\beta_1, \beta_2, \cdots, \beta_d)^T$ can be estimated under the linear model framework (think about ANOVA)

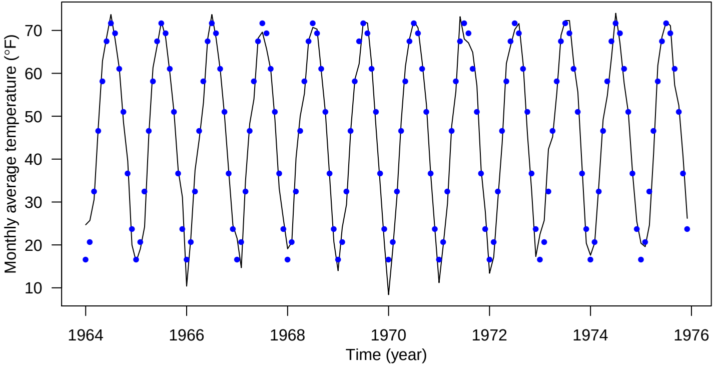
R Output

```
Call:
lm(formula = tempdub ~ month - 1)

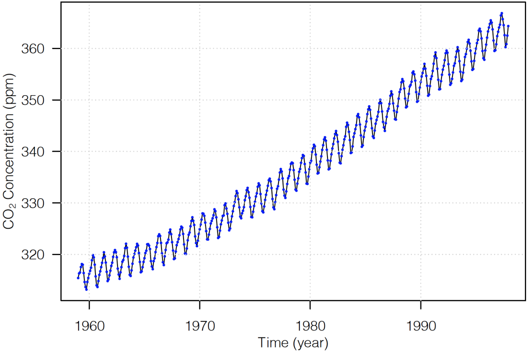
Residuals:
    Min      1Q  Median      3Q     Max 
-8.2750 -2.2479  0.1125  1.8896  9.8250 

Coefficients:
               Estimate Std. Error t value Pr(>|t|)    
monthJanuary     16.608      0.987   16.83   <2e-16 ***
monthFebruary    20.650      0.987   20.92   <2e-16 ***
monthMarch       32.475      0.987   32.90   <2e-16 ***
monthApril       46.525      0.987   47.14   <2e-16 ***
monthMay         58.092      0.987   58.86   <2e-16 ***
monthJune        67.500      0.987   68.39   <2e-16 ***
monthJuly        71.717      0.987   72.66   <2e-16 ***
monthAugust      69.333      0.987   70.25   <2e-16 ***
monthSeptember   61.025      0.987   61.83   <2e-16 ***
monthOctober     50.975      0.987   51.65   <2e-16 ***
monthNovember    36.650      0.987   37.13   <2e-16 ***
monthDecember    23.642      0.987   23.95   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

The Seasonal Means Model Fit



Estimating the Trend and Seasonal Components Together

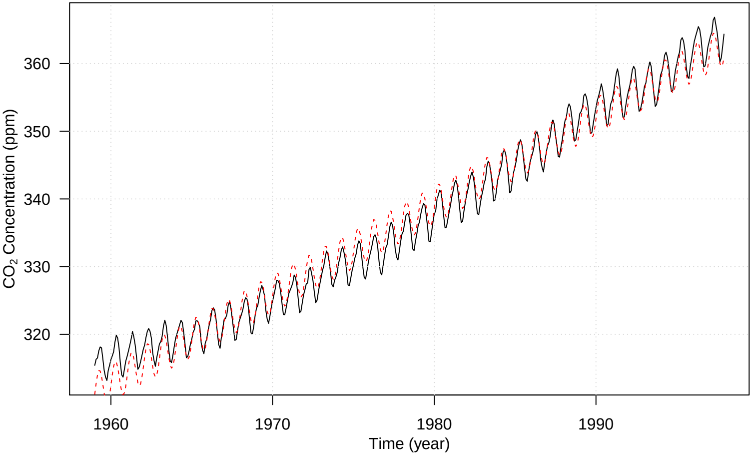


Let's perform a regression analysis to model both μ_t (assuming a linear time trend) and s_t (using $\cos$ and $\sin$)

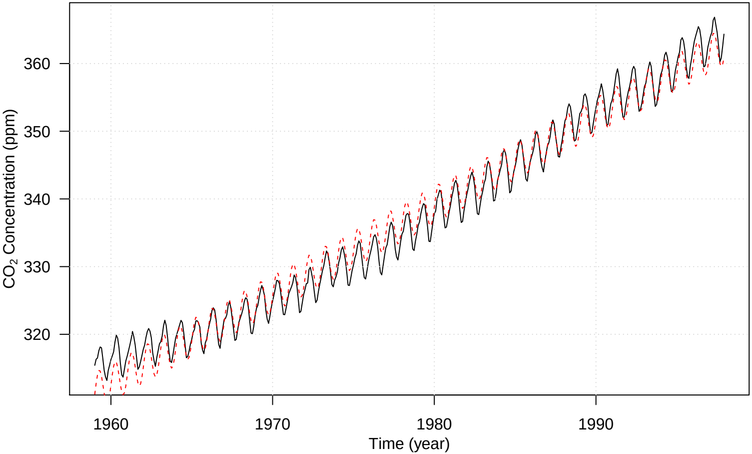
```
```{r}
time <- as.numeric(time(co2))
harmonics <- harmonic(co2, 1)

lm_trendSeason <- lm(co2 ~ time + harmonics)
summary(lm_trendSeason)
```
```

The Regression Fit



The Regression Fit



Question: How well the model fits the data?

Backshift Operator in Time Series

- We define the first order difference operator ∇ as

$$\nabla Y_t = Y_t - Y_{t-1} = (1 - B)Y_t,$$

where B is the **backshift operator** and is defined as $BY_t = Y_{t-1}$.

- Similarly the general order difference operator $\nabla^q Y_t$ is **defined recursively** as $\nabla[\nabla^{q-1} Y_t]$

- The backshift operator of power q is defined as $B^q Y_t = Y_{t-q}$

- A seasonal difference is the difference between an observation and the previous observation from the same season:

$$Y_t - Y_{t-s} = Y_t - B^s Y_t = (1 - B^s)Y_t$$

The Seasonal ARIMA (SARIMA) Model

Let d and D be non-negative integers. Then $\{X_t\}$ is a seasonal ARIMA$(p, d, q) \times (P, D, Q)$ process with period s if

$$Y_t = \nabla^d \nabla_s^D X_t = (1 - B)^d (1 - B^s)^D X_t,$$

is a casual ARMA process define by

$$\phi(B)\Phi(B^s)Y_t = \theta(B)\Theta(B^s)Z_t,$$

where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$.

An Illustration of Seasonal Model

Consider a monthly time series $\{X_t\}$ with both a trend, and a seasonal component of period $s = 12$.

- Suppose we know the values of d and D such that $Y_t = (1 - B)^d (1 - B^{12})^D X_t$ is stationary
- We can arrange the data this way:

	Month 1	Month 2	$\cdots$	Month 12
Year 1	Y_1	Y_2	$\cdots$	Y_{12}
Year 2	Y_{13}	Y_{14}	$\cdots$	Y_{24}
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
Year r	$Y_{1+12(r-1)}$	$Y_{2+12(r-1)}$	$\cdots$	$Y_{12+12(r-1)}$

The Inter-annual Model

Here we view each column (month) of the data table from the previous slide as a separate time series

- For each month m, we assume the same ARMA(P, Q) model. We have

$$Y_{m+12s} - \sum_{i=1}^{P} \Phi_i Y_{m+12(s-i)}$$
$$= U_{m+12s} + \sum_{j=1}^{Q} \Phi_j U_{m+12(s-j)},$$

  for each $s = 0, \cdots, r-1$, where $\{U_{m+12s:s=0,\cdots,r-1}\} \sim \mathrm{WN}(0, \sigma_U^2)$ for each m

- We can write this as

$$\Phi(B^{12}) Y_t = \Theta(B^{12}) U_t,$$

  and this defines the inter-annual model

The Intra-Annual Model

We induce correlation between the months by letting the process $\{U_t\}$ follow an ARMA(p, q) model,

$$\phi(B)U_t = \theta(B)Z_t,$$

where $Z_t \sim \mathrm{WN}(0, \sigma^2)$

- This is the intra-annual model
- The combination of the inter-annual and intra-annual models for the differenced stationary series,

  $$Y_t = (1 - B)^d(1 - B^{12})^D X_t,$$

  yields a SARIMA model for $\{X_t\}$

Steps for Modeling SARIMA Processes

1. Transform data if necessary

2. Find d and D so that

$$Y_t = (1 - B)^d (1 - B^s)^D X_t$$

is stationary

3. Examine the sample ACF/PACF of $\{Y_t\}$ at lags that are multiples of s for plausible values for P and Q

4. Examine the sample ACF/PACF at lags $\{1, 2, \cdots, s - 1\}$, to identify possible values for p and q

Modeling SARIMA Processes (Cont'd)

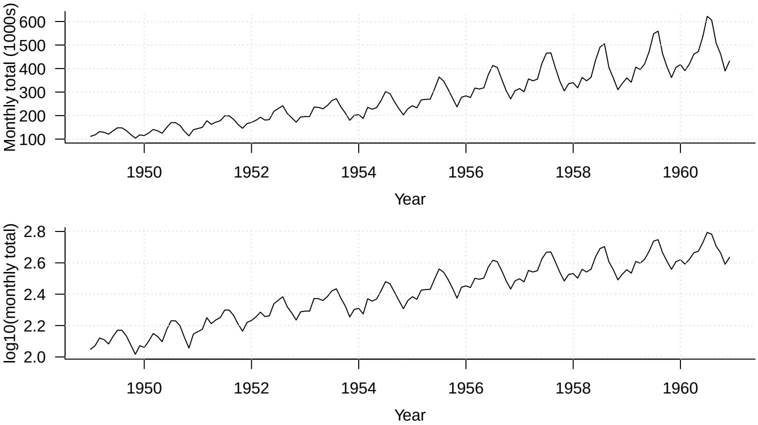
5. Use maximum likelihood method to fit the models

6. Use model summaries, diagnostics, AIC (AICC) to determine the best SARIMA model

7. Conduct forecast

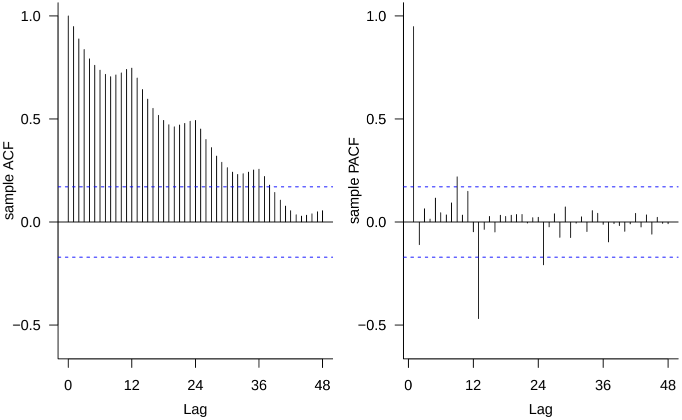
Airline Passengers Example

We consider the data set `airpassengers`, which are the monthly totals of international airline passengers from 1949 to 1960, taken from Box and Jenkins [1970]



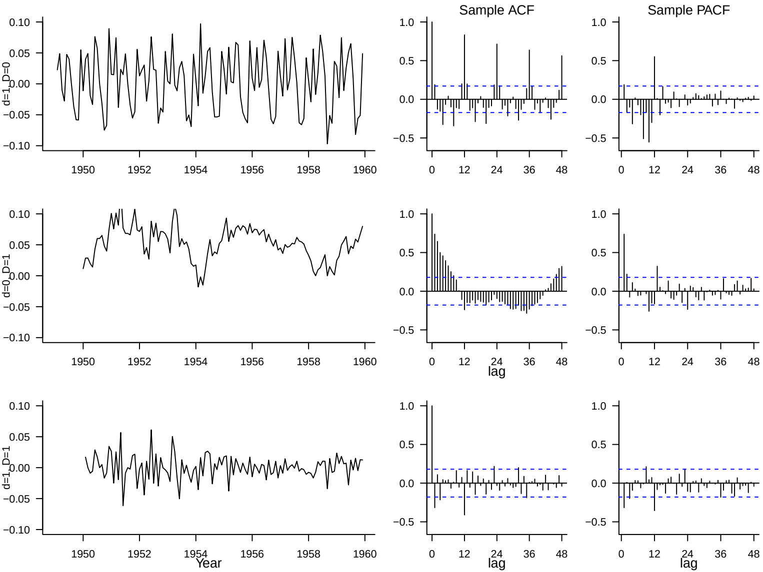
Here we stabilize the variance with a $\log_{10}$ transformation

Sample ACF/PACF Plots



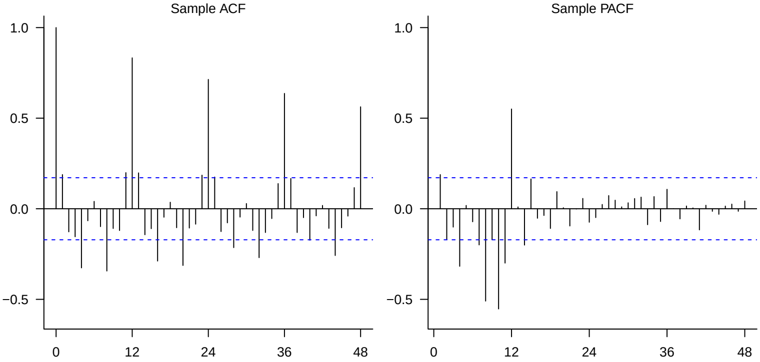
- The sample ACF decays slowly with a wave structure ⇒ seasonality
- The lag one PACF is close to one, indicating that differencing the data would be reasonable

Trying Different Orders of Differencing



Choosing Candidate SARIMA Models

We choose a SARIMA$(p, 1, q) \times (P, 0, Q)$ model. Next we examine the sample ACF/PACF of the process $Y_t = (1 - B)X_t$



Now we need to choose P , Q, p, and q

Fitting a SARIMA$(1,1,0)\times(1,0,0)$ model


```
> fit1 <- arima(diff.1.0, order = c(1, 0, 0), seasonal = list(order = c(1, 0, 0), period = 12))
> fit1

Call:
arima(x = diff.1.0, order = c(1, 0, 0), seasonal = list(order = c(1, 0, 0),
    period = 12))

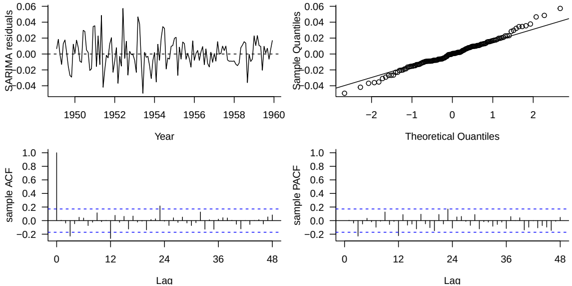
Coefficients:
          ar1    sar1  intercept
      -0.2667  0.9291     0.0039
s.e.   0.0865  0.0235     0.0096

sigma^2 estimated as 0.0003298:  log likelihood = 327.27,  aic = -646.54
```

```
> Box.test(fit1$residuals, lag = 48, type = "Ljung-Box")

        Box-Ljung test

data:  fit1$residuals
X-squared = 55.372, df = 48, p-value = 0.2164
```



A Discussion of the Model Fit

- The spread of the residuals is larger in 1949-1955 compared to the later years and the residual distribution has heavy tails
- The Ljung-Box test result indicates the fitted SARIMA $(1,1,0) \times (1,0,0)$ has sufficiently account for the temporal dependence
- 95% CI for ϕ_1 and Φ_1 do not contain zero $\Rightarrow$ no need to go with simpler model

Our estimated model is

$$(1 + 0.2667B)(1 - 0.9291B^{12})(X_t - 0.0039) = Z_t,$$

where $\{Z_t\} \overset{i.i.d.}{\sim} \mathrm{N}(0, \sigma^2)$ with $\hat{\sigma}^2 = 0.00033$

Comparing with a SARIMA$(0,1,0) \times (1,0,0)$ Model

```
> (fit2 <- arima(diff.1.0, seasonal = list(order = c(1, 0, 0), period = 12)))

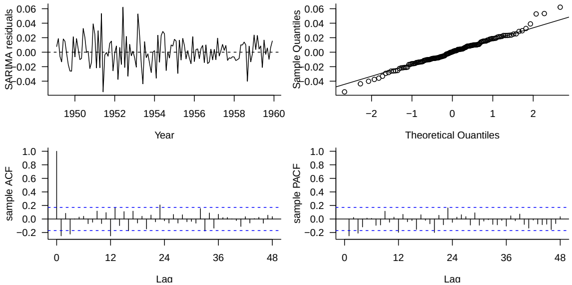
Call:
arima(x = diff.1.0, seasonal = list(order = c(1, 0, 0), period = 12))

Coefficients:
        sar1  intercept
      0.9081     0.0040
s.e.  0.0278     0.0108

sigma^2 estimated as 0.0003616:  log likelihood = 322.75,  aic = -639.51
> Box.test(fit2$residuals, lag = 48, type = "Ljung-Box")

        Box-Ljung test

data:  fit2$residuals
X-squared = 80.641, df = 48, p-value = 0.002209
```



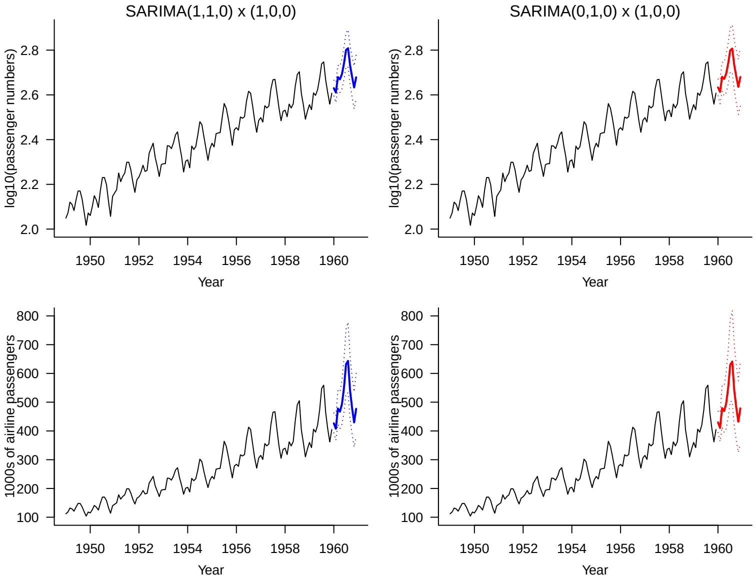
A Discussion of Model Fit2

Here we drop the AR(1) term

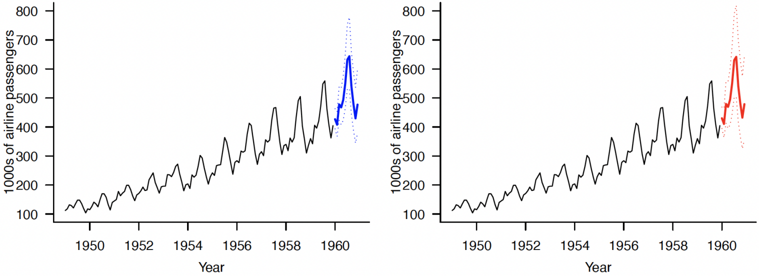
- The residual plots looks quite similar to before: The spread of the residuals is larger in 1949-1955 compared to the later years and the residual distribution has heavy tails
- Both $\hat{\sigma}^2$ and AIC increase (compared with model fit1)
- The lag 1 of ACF and PACF now lies outside the IID noise bounds. The Ljung-Box P-value of 0.0022, leads us to reject the IID residual assumption

In conclusion, the SARIMA$(1,1,0) \times (1,0,0)$ model fits better than SARIMA$(0,1,0) \times (1,0,0)$

Forecasting the 1960 Data



Evaluating Forecast Performance



Metrics	Model Fit1	Model Fit2
Root Mean Square Error	30.36	31.32
Mean Relative Error	0.057	0.060
Empirical Coverage	0.917	1.000

Summary

These slides cover two methods for estimating seasonality:

- Harmonic regression models
- Seasonal ARIMA Models
- Ways to evaluate forecasting performance

`R` functions to know:

- `harmonic` (under the package `TSA`)for constructing harmonic functions
- Incorporating `seasonal = list(order = c(P, D, Q), period = d)` in `arima` for SARIMA modeling