An Introduction to Extreme Value Analysis

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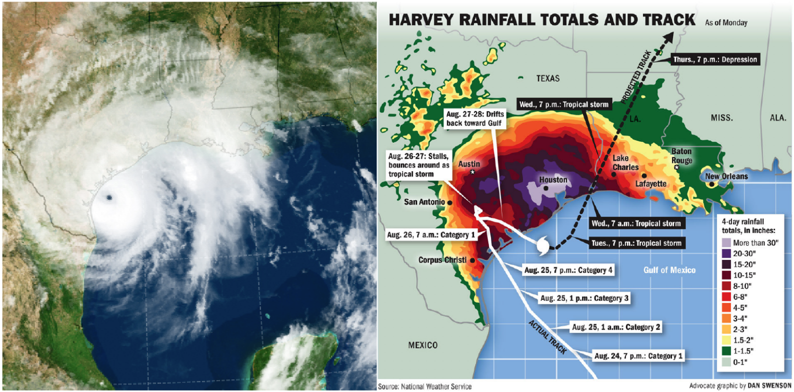
Outline

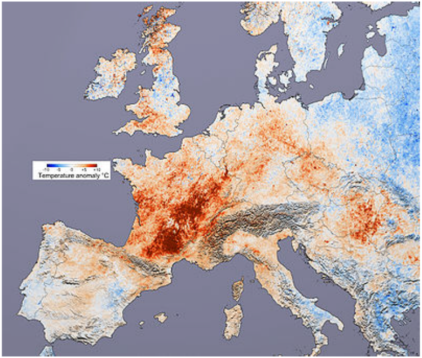
Extreme Rainfall During Hurricane Harvey



Source: NASA (Left); National Weather Service (Right)

- *"A storm forces Houston, the limitless city, to consider its limits"* – The New York Times (8.31.17)

Environmental Extremes: Heatwaves, Storm Surges, etc.



- **Heat wave**: The 2003 European heat wave led to the hottest summer on record in Europe since 1540 that resulted in at least 30,000 deaths
- **Storm Surge**: Hurricane Katrina produced the highest storm surge ever recorded (27.8 feet) on the U.S. coast

Scientific Questions

- How to estimate the magnitude of extreme events (e.g. 100-year rainfall)?
- How extremes vary in space?
- How extremes may change in future climate conditions?

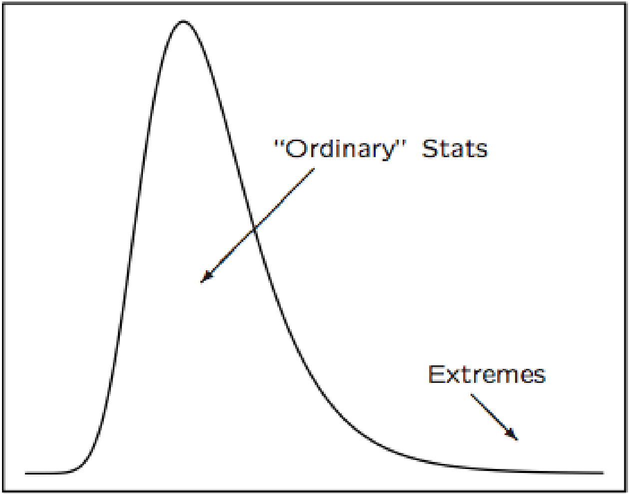
Outline

Motivation

Extreme Value Theorem & Block Maxima Method

Peaks–Over–Threshold (POT) Method

Usual vs Extremes



Probability Framework

Let $X_1, \cdots, X_n \overset{\text{iid}}{\sim} F$ and define $M_n = \max\{X_1, \cdots, X_n\}$
Then the distribution function of M_n is

$$\begin{aligned}\mathbb{P}(M_n \le x) &= \mathbb{P}(X_1 \le x, \cdots, X_n \le x)\\ &= \mathbb{P}(X_1 \le x) \times \cdots \times \mathbb{P}(X_n \le x) = F^n(x)\end{aligned}$$

Remark

$$F^n(x) \xlongequal{n\to\infty} \begin{cases} 0 & \text{if } F(x) < 1 \\ 1 & \text{if } F(x) = 1 \end{cases}$$

$\Rightarrow$ the limiting distribution is degenerate.

Asymptotic: Classical Limit Laws

Recall the Central Limit Theorem:

$$\frac{S_n - n\mu}{\sqrt{n}\sigma} \xrightarrow{d} N(0,1)$$

$\Rightarrow$ rescaling is the key to obtain a non-degenerate distribution

Question: Can we get the limiting distribution of

$$\frac{M_n - b_n}{a_n}$$

for suitable sequence $\{a_n\} > 0$ and $\{b_n\}$?

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Extremal Types Theorem (Fisher–Tippett 1928, Gnedenko 1943)

Define $M_n = \max\{X_1, \cdots, X_n\}$ where $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} F$. If $\exists\, a_n > 0$ and $b_n \in \mathbb{R}$ such that, as $n \to \infty$, if

$$\mathbb{P}\left(\frac{M_n - b_n}{a_n} \le x\right) \xrightarrow{d} \mathsf{G}(x)$$

then G must be the same type of the following form:

$$\mathsf{G}(x; \mu, \sigma, \xi) = \exp\left\{ - \left[1 + \xi(\frac{x-\mu}{\sigma})\right]_+^{\frac{-1}{\xi}} \right\}$$

where $x_+ = \max(x, 0)$ and $G(x)$ is the distribution function of the generalized extreme value distribution (GEV(μ, σ, ξ))

- μ and σ are location and scale parameters
- ξ is a shape parameter determining the rate of tail decay, with
  - $\xi > 0$ giving the heavy-tailed case (Fréchet)
  - $\xi = 0$ giving the light-tailed case (Gumbel)
  - $\xi < 0$ giving the bounded-tailed case (reversed Weibull)

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Max-Stability and GEV

Definition

A distribution G is said to be max-stable if

$$G^k(a_k x + b_k) = G(x), \qquad k \in \mathbb{N}$$

for some constants $a_k > 0$ and b_k

- Taking powers of a distribution function results only in a change of location and scale
- A distribution is max-stable $\iff$ it is a GEV distribution

Quantiles and Return Levels

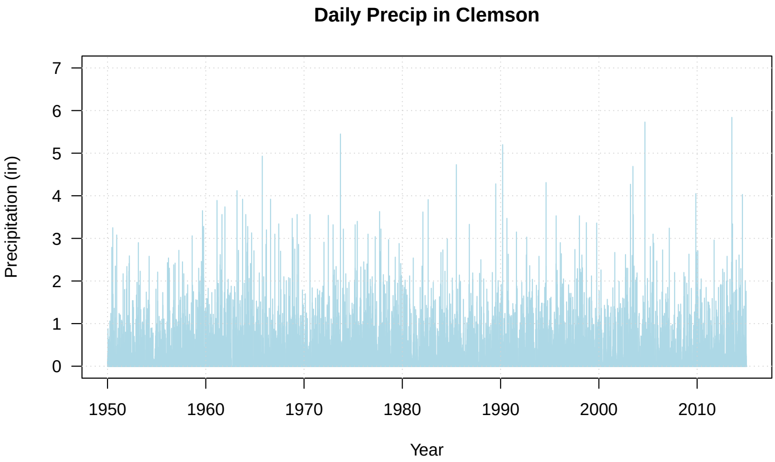
- Quantiles of GEV

$$G(x_p) = \exp\left\{ - \left[1 + \xi \left(\frac{x_p - \mu}{\sigma} \right) \right]_+^{\frac{-1}{\xi}} \right\} = 1 - p$$

$$\Rightarrow x_p = \mu - \frac{\sigma}{\xi} \left[1 - \{ - \log(1-p)^{-\xi} \} \right] \qquad 0 < p < 1$$

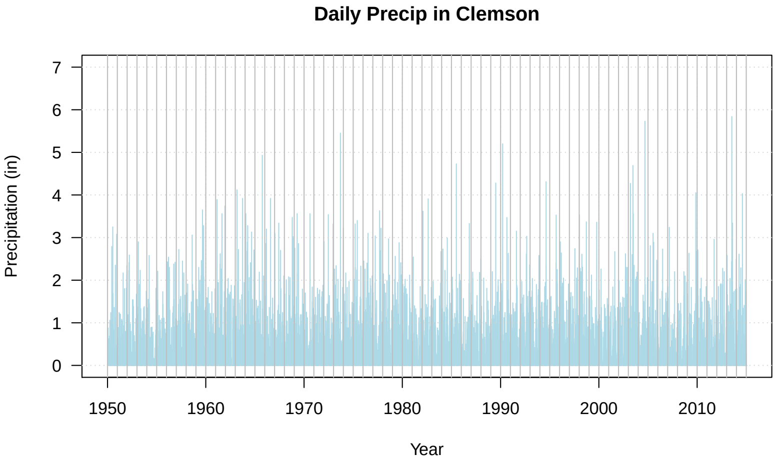
- In the extreme value terminology, x_p is the return level associated with the return period $\frac{1}{p}$

Clemson Daily Precipitation [Data Source: USHCN]



Block Maxima Method (Gumbel 1958)

1. Determine the block size and extract the block maxima



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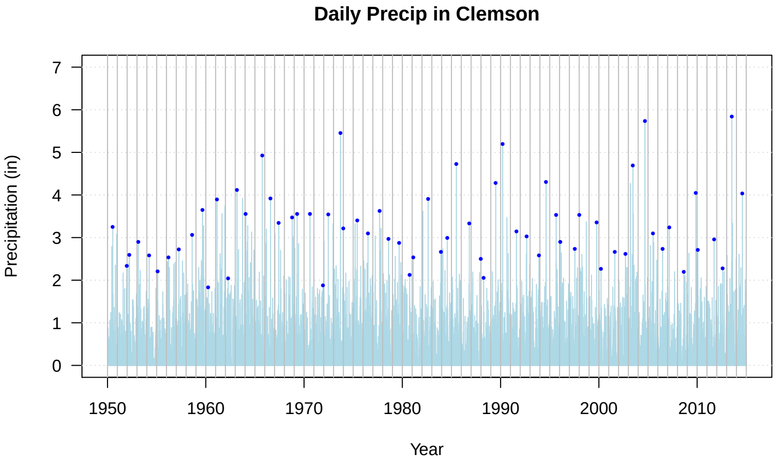
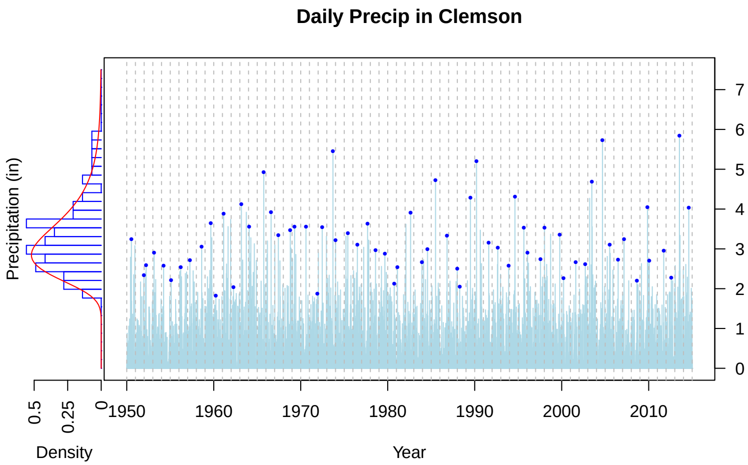


Block Maxima Method (Gumbel 1958)

2. Fit the GEV to the maximal and assess the fit



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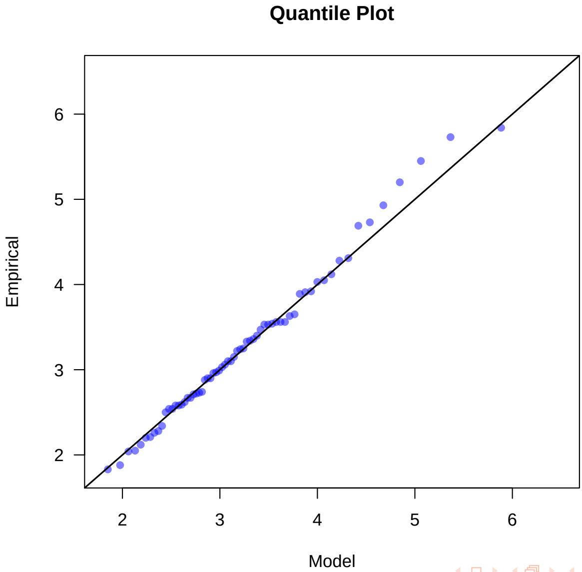
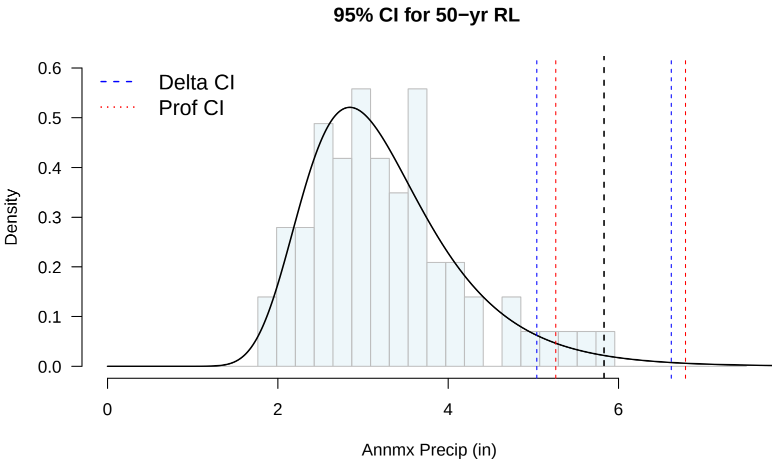


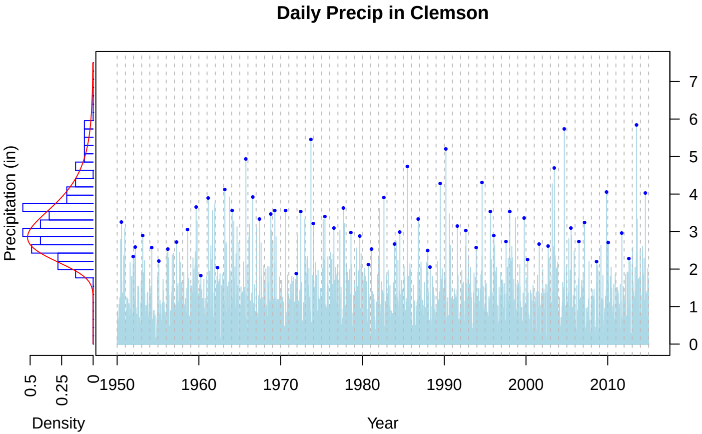
Block Maxima Method (Gumbel 1958)

3. Perform inference for return levels, probabilities, etc



Outline

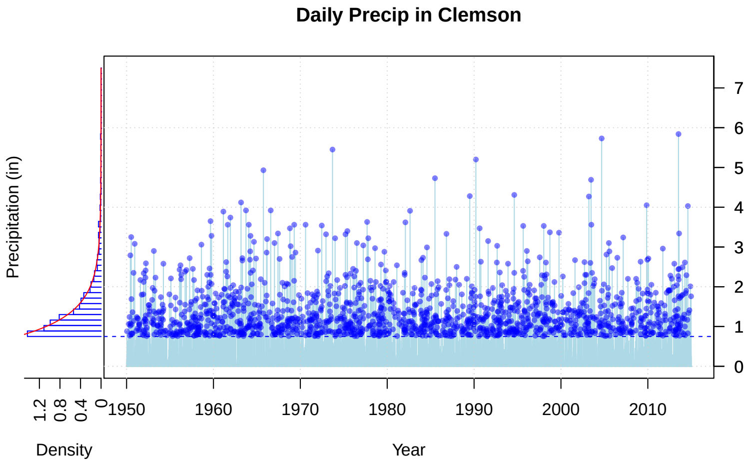
Recall the Block Maxima Method



Question: Can we use data more efficiently?

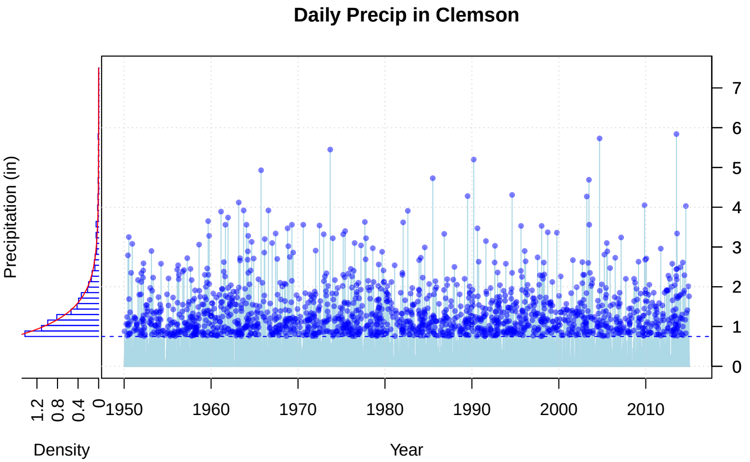
Peaks–over–threshold (POT) method [Davison & Smith 1990]

1. Select a "sufficiently large" threshold u, extract the exceedances



Peaks–over–threshold (POT) method [Davison & Smith 1990]

2. Fit an appropriate model to exceedances



GPD for Exceedances

If $M_n = \max_{i=1,\cdots,n} X_i$ (for a large n) can be apprximated by a GEV(μ, σ, ξ), then for sufficently large u,

$$
\begin{aligned}
\mathbb{P}(X_i > x + u | X_i > u) &= \frac{n\mathbb{P}(X_i > x + u)}{n\mathbb{P}(X_i > u)} \\
&\to \left(\frac{1 + \xi \frac{x+u-b_n}{a_n}}{1 + \xi \frac{u-b_n}{a_n}} \right)^{\frac{-1}{\xi}} \\
&= \left(1 + \frac{\xi x}{a_n + \xi(u - b_n)} \right)^{\frac{-1}{\xi}}
\end{aligned}
$$

$\Rightarrow$ Survival function of generalized Pareto distribution

Pickands–Balkema–de Haan Theorem (1974, 1975)

If $M_n = \max_{1 \le i \le n}\{X_i\} \approx \text{GEV}(\mu, \sigma, \xi)$, then, for a "large" u (i.e., $u \to x_F = \sup\{x : F(x) < 1\}$),

$$\mathbb{P}(X > u) \approx \frac{1}{n}\left[1 + \xi\left(\frac{u-\mu}{\sigma}\right)\right]^{\frac{-1}{\xi}}$$

$F_u = \mathbb{P}(X - u < y | X > u)$ is well approximated by the generalized Pareto distribution (GPD). That is:

$$F_u(y) \xrightarrow{d} H_{\tilde{\sigma},\xi}(y) \qquad u \to x_F$$

where

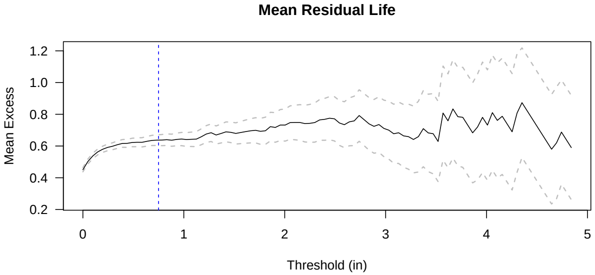
$$H_{\tilde{\sigma},\xi}(y) = \begin{cases} 1 - (1 + \xi y/\tilde{\sigma})^{-1/\xi} & \xi \neq 0; \\ 1 - \exp(-y/\tilde{\sigma}) & \xi = 0. \end{cases}$$

and $\tilde{\sigma} = \sigma + \xi(u - \mu)$

How to Choose the Threshold?

Bias–variance tradeoff:

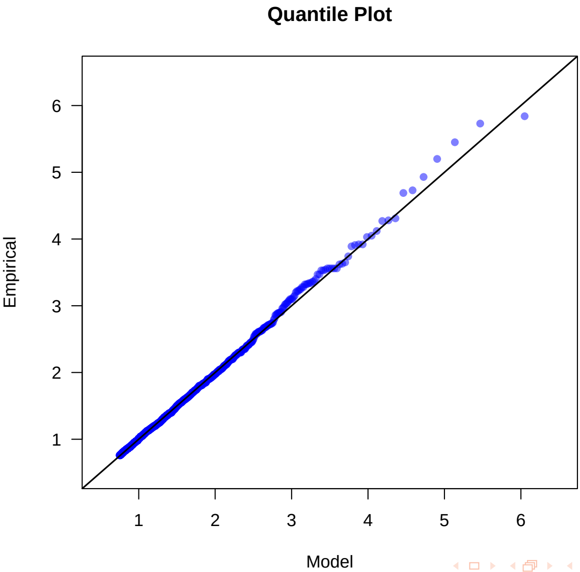
- Threshold too low $\Rightarrow$ bias because of the model asymptotics being invalid
- Threshold too high $\Rightarrow$ variance is large due to few data points



Task: To choose a u_0 s.t. the Mean Residual Life curve behaves linearly $\forall u > u_0$

Peaks–over–threshold (POT) method [Davison & Smith 1990]

2. Fit an appropriate model to exceedances and assess the fit



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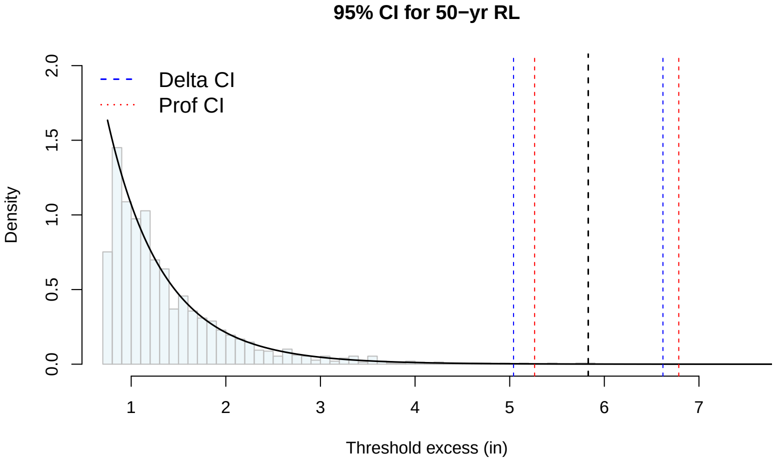

Summary & Discussion

- Extreme value theory provides a framework to model extreme values
  - GEV for fitting block maxima
  - GPD for fitting threshold exceedances
  - Return level for communicating risk
- Practical Issues: seasonality, temporal dependence, non-stationarity, ...

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For Further Reading

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