Lecture 7

Nonstationary Time Series Models

Readings: Cryer & Chan Ch 5; Brockwell & Davis Ch 6.1-6.4; Shumway & Stoffer Ch 3.6-3.7

MATH 8090 Time Series Analysis
September 28 & September 30, 2021

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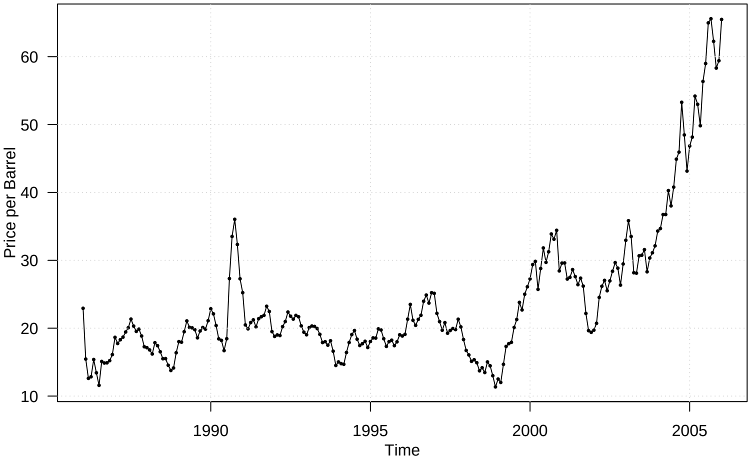
Monthly Price of Oil: January 1986–January 2006



Random Walks Revisited

Recall the random walk process

$$X_t = Z_1 + Z_2 + \cdots + Z_t = \sum_{j=1}^{t} Z_j,$$

where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$

$\{X_t\}$ is a nonstationary process

- We can obtain a stationary process by differencing

$$\Delta X_t = X_t - X_{t-1} = (1 - B)X_t = Z_t$$

- $\{X_t\}$ is an example of an autoregressive integrated moving average (ARIMA) process– ARIMA(0, 1, 0) process

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ARIMA Models

An ARIMA model is an ARMA process after differencing

- Let d be a non-negative integer. Then X_t is an ARIMA(p, d, q) process if

$$Y_t = \Delta^d X_t = (1 - B)^d X_t$$

  is a causal ARMA process

- Let $\phi(B)$ be the AR polynomial and $\theta(B)$ be the MA polynomial. Then for $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$

$$\phi(B) Y_t = \theta(B) Z_t,$$

  and since $Y_t = (1 - B)^d X_t$

$$\phi(B)(1 - B)^d X_t = \theta(B) Z_t$$

Example: ARIMA(1, 1, 0)

Let $\phi(z) = 1 - \phi_1 z$, $\theta(z) = 1$ and $d = 1$. For a causal stationary solution (after differencing) we need to assume $|\phi_1| < 1$. Then $\{X_t\}$ is an ARIMA (1, 1, 0) process,

$$(1 - \phi_1 B)(1 - B)X_t = Z_t,$$

where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$

Now let $Y_t = (1 - B)X_t = X_t - X_{t-1}$, afer some rearrangements we we

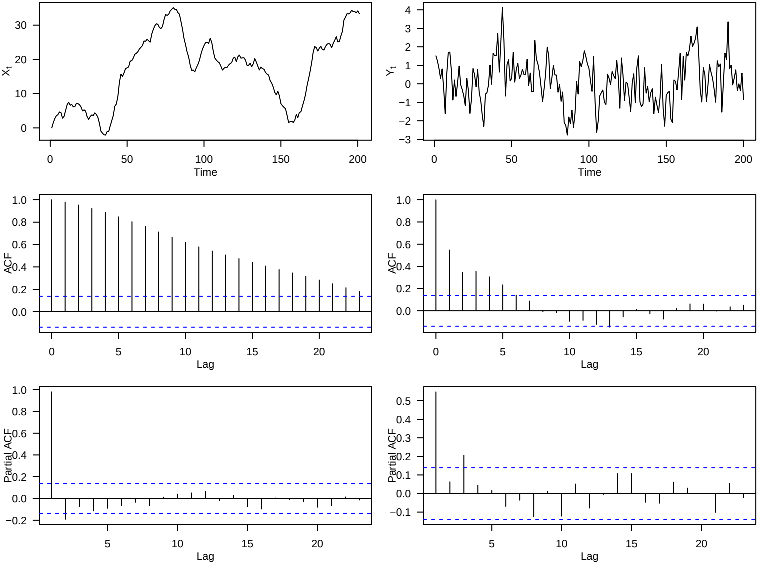
$$\begin{aligned} X_t &= X_{t-1} + Y_t \\ &= (X_{t-2} + Y_{t-1}) + Y_t \\ &\vdots \\ &= X_0 + \sum_{j=1}^{t} Y_j \end{aligned}$$

Thus $\{X_t\}$ is a “sort of random walk”–we cumulatively sum an AR(1) process, $\{Y_t\}$

Simulated ARIMA and Differenced ARMA Process

We simulate an ARIMA$(1,1,0)$:

$$(1-0.5B)(1-B)X_t = Z_t, \quad \{Z_t\} \sim \mathrm{N}(0,1)$$



Adding a Polynomial Trend

For $d \geq 1$, let $\{X_t\}$ be an ARIMA(p, d, q) process. Then $\{X_t\}$ satisfies the equation

$$\phi(B)(1-B)^d X_t = \theta(B) Z_t$$

- Let μ_t be a polynomial of degree $(d-1)$, i.e., $\mu_t = \sum_{j=0}^{d-1} a_j t^j$ for constants $\{a_j\}$
- Now let $V_t = \mu_t + X_t$, then

$$\begin{aligned}
\phi(B)(1-B)^d V_t &= \phi(B)(1-B)^d(\mu_t + X_t) \\
&= \phi(B)(1-B)^d \mu_t + \phi(B)(1-B)^d X_t \\
&= 0 + \phi(B)(1-B)^d X_t \\
&= \theta(B) Z_t
\end{aligned}$$

- Takeaway: ARIMA(p, d, q) are useful for modeling data with polynomial trends, due to the inherent differencing that can be used to remove trends

Typical Steps for Modeling ARIMA Processes: Exploratory Data Analysis

- Plot the data, ACF, PACF and Q-Q plots
  - Check for unusual features of the data
  - Check for stationarity
  - Do we need to transform the data?
- Eliminate trend
  - Estimating the trend and removing it from the series
  - Or, differencing the series (i.e., select d in the ARIMA model)
- Plot the sample ACF/PACF for the stationary component
  - Identify candidate values of p and q

Typical Steps for Modeling ARIMA Processes: Model Estimation

- Estimate the ARMA process parameters for the candidate models
- Check the goodness of fit: Are the time series residuals, $\{r_t\}$ a sample of *i.i.d.* noise?
- Model selection:
  - Using information criteria such as AIC and AICC
  - Test model parameters to compare between the "full" model and the "subset" model

Forecasting ARIMA Processes

We need more assumptions to forecast ARIMA(p, d, q) processes. Let us start with the case of $d = 1$, i.e.,

$$\phi(B)(1 - B)X_t = \theta(B)Z_t,$$

where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$

- **Note:** $Y_t = (1 - B)X_t = X_t - X_{t-1}$ is an ARMA(p, q) process
- We want to find the best linear predictor (BLP) of X_{n+1} based on $X_0, X_1, \cdots, X_n$
  - We konw that $X_{n+1} = X_n + Y_{n+1} \Rightarrow$ only need to figure out the BLP of Y_{n+1} based on $\{X_0, Y_1, \cdots, Y_n\}$
  - We need to know $\mathbb{E}(X_0^2)$ and $\mathbb{E}(X_0 Y_j)$ for $j = 1, \cdots, n + 1$

Forecasting ARIMA(p, 1, q) Processes (Cont'd)

Problem: What is $\mathbb{E}(X_0 Y_j)$?

- We assume that X_0 is uncorrelated with $Y_1, Y_2, \cdots$
- Then the BLP of X_{n+1} based on $\{X_0, X_1, \cdots, X_n\}$ is the same as the BLP of X_{n+1} based on $\{Y_1, Y_2, \cdots, Y_n\}$
- This extends to ARIMA(p, d, q) processes:

  If we assume that $\{X_{1-d}, \cdots, X_0\}$ is uncorrelated with $Y_1, Y_2, \cdots$, then the BLP of Y_{n+1} based on $\{X_{1-d}, \cdots, X_0, \cdots, X_n\}$ is the same as the BLP based on $\{Y_1, Y_2, \cdots, Y_n\}$

Percentage Changes and Logarithms

Suppose X_t tends to have relatively stable percentage changes from one time period to the next. Specifically, assume that

$$X_t = (1 + Y_t)X_{t-1},$$

where $100Y_t$ is the percentage change from X_{t-1} to X_t. Then

$$\log(X_t) - \log(X_{t-1}) = \log\left(\frac{X_t}{X_{t-1}}\right) = \log(1 + Y_t).$$

If Y_t is restricted to, say, $|Y_t| < 0.2$ (ie., the percentage changes are at most ± 20%), then, to a good approximation, $\log(1 + Y_t) \approx Y_t$. Consequently

$$\Delta[\log(X_t)] \approx Y_t$$

will be relatively stable and perhaps well-modeled by a stationary process.

In financial literature, the differences of the (natural) logarithms are usually called returns

Time Series Plots of Monthly US Electricity Production