CS201

Mathematics for Computer Science I

Lecture 11

Identities

Identity-III

$$\sum_{r=0}^{n} \binom{n}{r}^2 = \binom{2n}{n}$$

for $n \geq 0$.

- Set $\alpha = \beta = k = n$ in Identity-II.

Identities

Identity-IV

$$\sum_{r=0}^{n} r\binom{n}{r} = n2^{n-1}$$

for $n \geq 0$.

- We know that $(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$.
- Differentiating wrt x:

$$n(1+x)^{n-1} = \sum_{r=0}^{n} r\binom{n}{r} x^{r-1}.$$

- Set $x = 1$.

Convergence of Generating Functions

- When setting x to a number in a generating function, one needs to be careful.
- Consider
$$\frac{1}{(1-x)^2} = \sum_{n \geq 0} (n+1) x^n,$$
and set $x = -1$. We get:
$$\frac{1}{4} = 1 - 2 + 3 - 4 + 5 - 6 + \cdots$$
which is absurd.
- This happens because the infinite sum $\sum_{n \geq 0} (n+1) x^n$ does not converge at $x = 1$.

Convergence of Generating Functions

- The sum does converge for any x between -1 and 1, or equivalently $|x| < 1$.
- This is called radius of convergence for the sum $\sum_{n\geq 0}(n+1)x^n$.
- The name comes from the fact that x can be a complex number too.
- Within the radius of convergence, the equality

$$\frac{1}{(1-x)^2} = \sum_{n\geq 0}(n+1)x^n$$

  holds.
- For example, setting $x = \frac{1}{2}$:

$$4 = \sum_{n\geq 0}(n+1)\frac{1}{2^n}.$$

Partition Numbers

- Recall the problem of counting number of ways to write n as a sum of smaller or equal positive numbers.
- It is called partition number and denoted as $p(n)$.
- Euler defined and studied the following function called Euler function:

$$\phi(z) = \prod_{r \geq 1} (1 - z^r).$$

- We have:

$$\begin{aligned} \frac{1}{\phi(z)} &= \frac{1}{\prod_{r \geq 1}(1 - z^r)} \\ &= \prod_{r \geq 1} (1 + z^r + z^{2r} + z^{3r} + \cdots) \\ &= \sum_{n \geq 0} p(n) z^n \end{aligned}$$

taking $p(0) = 1$.

Partition Numbers

- Therefore, $\frac{1}{\phi(z)}$ is generating function for partition numbers.
- Taking natural logarithm:

$$\begin{aligned}
\ln \phi(z) &= \sum_{r \geq 1} \ln(1 - z^r) \\
&= -\sum_{r \geq 1} \sum_{m \geq 1} \frac{z^{mr}}{m} \\
&= -\sum_{n \geq 1} \Big(\sum_{m, m|n} \frac{1}{m} \Big) z^n \\
&= -\sum_{n \geq 1} \Big(\sum_{m, m|n} m \Big) \frac{z^n}{n}
\end{aligned}$$

- Therefore, $\ln \phi(z)$ is generating function for sum-of-divisors $\sigma(n) = \sum_{m, m|n} m$.

Dedekind function and Partition Numbers

- Define

$$\eta(\tau) = z^{1/24}\phi(z),$$

  by setting $z = e^{2\pi i \tau}$.

- It is called Dedekind eta function.
- The function has many remarkable properties:

$$\begin{aligned} \eta(\tau+1) &= e^{\frac{\pi i}{12}}\eta(\tau) \\ \eta(-\frac{1}{\tau}) &= \sqrt{-i\tau}\,\eta(\tau) \end{aligned}$$

- Using these properties, Hardy-Ramanujan showed that

$$p(n) \sim \frac{1}{\sqrt{48}n} e^{2\pi\sqrt{n/6}}$$

  denoting asymptomatic convergence.

Recurrence Relations

- For many sequences of numbers, a number in the sequence can be written in terms of previous numbers in the sequence.
- For example, consider Pingala sequence of numbers (also known as Fibonacci sequence):

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

- Each number in the sequence is the sum of previous two numbers and first two numbers are 1.
- Let $P(n)$ be the $(n+1)$st number in the sequence.
- Can we derive a formula for $P(n)$?

Pingala Numbers

- We use generating functions to derive a formula for $P(n)$.
- Let
$$f(x) = \sum_{n \geq 0} P(n) x^n.$$
- By definition,
$$P(n) = \begin{cases} P(n-1) + P(n-2) & n \geq 2 \\ 1 & n = 0, 1 \end{cases}$$

Pingala Numbers

- Therefore,

$$\begin{aligned} f(x) &= 1 + x + \sum_{n\geq 2}(P(n-1) + P(n-2))x^n \\ &= 1 + x + x\sum_{n\geq 2} P(n-1)x^{n-1} + x^2\sum_{n\geq 2} P(n-2)x^{n-2} \\ &= 1 + x\sum_{n\geq 1} P(n-1)x^{n-1} + x^2\sum_{n\geq 0} P(n)x^n \\ &= 1 + xf(x) + x^2 f(x) \end{aligned}$$

- This gives:

$$f(x) = \frac{1}{1-x-x^2}.$$

Pingala Numbers

- $1 - x - x^2$ can be factored as:

$$1 - x - x^2 = (\frac{\sqrt{5}-1}{2} - x)(\frac{\sqrt{5}+1}{2} + x).$$

- Using this, we can derive:

$$\begin{aligned} f(x) &= \frac{1}{(\frac{\sqrt{5}-1}{2} - x)(\frac{\sqrt{5}+1}{2} + x)} \\ &= \frac{1}{\sqrt{5}}\left(\frac{1}{\frac{\sqrt{5}-1}{2} - x} + \frac{1}{\frac{\sqrt{5}+1}{2} + x}\right) \\ &= \frac{1}{\sqrt{5}}\left(\frac{\frac{\sqrt{5}+1}{2}}{1 - \frac{\sqrt{5}+1}{2}x} + \frac{\frac{\sqrt{5}-1}{2}}{1 + \frac{\sqrt{5}-1}{2}x}\right) \end{aligned}$$

Pingala Numbers

- Continuing...

$$= \frac{\sqrt{5}+1}{2\sqrt{5}} \sum_{n \geq 0} \left(\frac{\sqrt{5}+1}{2}\right)^n x^n + \frac{\sqrt{5}-1}{2\sqrt{5}} \sum_{n \geq 0} \left(\frac{1-\sqrt{5}}{2}\right)^n x^n$$

$$= \sum_{n \geq 0} \frac{1}{\sqrt{5}} \left(\left(\frac{\sqrt{5}+1}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \right) x^n$$

- Therefore,

$$P(n) = \frac{1}{\sqrt{5}} \left(\left(\frac{\sqrt{5}+1}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \right).$$