CS201 Assignment 1: The Concept of Numbers

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Maximum Marks: $20 \times 5 = 100$

Before we start discussion on numbers, let us examine the axioms of set theory and why they are required. Define U to be the collection of all sets.

- Show that U is not a set as per the Zermalo Fraenkel Axioms.

> **Solution:**
> Consider, U, which is a collection of all sets,
> By the *Axiom of Paring*, we know, $\{U, U\}$ is also a set, which implies that $\{U\}$ is also a set, because, $\{A, A\}$ is the same as $\{A\}$
> Since, the only element in $\{U\}$, is U itself,
>
> $$U \in \{U\} \tag{1}$$
>
> The *Axiom of Regularity* states that if A is a set then $\exists\ x$ in A such that, $x \cap A = \phi$
> Equation (1) clearly contradicts *Axiom of Regularity*, hence, it is not a set as per the Zermalo Fraenkel Axioms.

The motivation to define these axioms was a paradox discovered by Bertrand Russell: Suppose we allow U to be a set. Then $U \in U$ by definition. Define:

$$V = \{A \mid A \notin A\}.$$

- Derive a contradiction using the question " is $V \in V$?".

> **Solution:**
> Assume, V to be a set.
>
> - Let $V \in V$, then by definition of V, $V \notin V$.
> - Let $V \notin V$, then by definition of V, $V \in V$.
>
> Thus, we arrive at a contradiction for both the cases.
> Therefore , V can not be a set.

This is the reason that circularity in definition of sets was explicitly not permitted by the axioms.

Let us now move to numbers. In the class, we discussed the definition of natural numbers through Peano's Axioms. How does one define numbers in general? One possible way is to define numbers as any set that admits four arithmetic operations: addition, subtraction, multiplication, and division. But to define arithmetic operations, we need numbers! This is resolved by defining both together. Let us develop axioms for this. Consider addition and subtraction first.

Define set of *numbers with addition* $(N, +)$ as:

1. $+ : N \times N \mapsto N$. We will write $+(a, b)$ as $a + b$.
2. $(a + b) + c = a + (b + c)$ for all $a, b, c \in N$.
3. There is an element $0 \in N$ such that $a + 0 = 0 + a = a$ for all $a \in N$.
4. For all $a \in N$, there is an element $b \in N$ such that $a + b = 0$.
5. $a + b = b + a$ for all $a, b \in N$.

With above definition, subtraction can be defined as: $a - b = a + c$ where c is such that $b + c = 0$. Does this capture the addition and subtraction properly? Show that:

- There is a unique number 0 satisfying third axiom.

Solution:
Let, $0_1, 0_2 \in N$ and $0_1 \neq 0_2$ are two such numbers which satisfy the third axiom.
Then, for any $a \in N$,

$$a + 0_1 = 0_1 + a = a \tag{2}$$

$$a + 0_2 = 0_2 + a = a \tag{3}$$

From (1) and (2) we have,

$$a + 0_1 = a + 0_2 \tag{4}$$

which implies,

$$0_1 = 0_2 \tag{5}$$

which is a contradiction,
$\therefore$ 0 must be a unique number which satisfies the third axiom.

- For every $a \in N$, there is a unique b satisfying fourth axiom.

Solution:
Let, $b_1, b_2 \in N$ and $b_1 \neq b_2$ are two such numbers which satisfy the fourth axiom.
Then, for any $a \in N$,

$$a + b_1 = 0 \tag{1}$$

$$a + b_2 = a \tag{2}$$

From (1) and (2) we have,

$$a + b_1 = a + b_2 \tag{3}$$

which implies,

$$b_1 = b_2 \tag{4}$$

which is a contradiction,
$\therefore b$ must be a unique number which satisfies the fourth axiom.

- Define $-a$ to be the number such that $a + (-a) = 0$. For every $a, b \in N$, $a - b = -(b - a)$.

Solution:
Let,

$$b - a = \alpha$$

We know,

$$\alpha + (-\alpha) = 0 \tag{1}$$

Also,

$$= b + (-a) + a + (-b) = b + (-b) + a + (-a) = 0 \tag{2}$$

$$(b - a) + = \alpha$$

$$= \begin{array}{c} (b + (-a)) + (a + (-b)) \\ b + (-a) + a + (-b) \end{array} \tag{2}$$

From (1) and (2) we have,

$$(b - a) + (a - b) = (b - a) + (-(b - a)) \tag{3}$$

$$a - b = -(b - a) \tag{4}$$

Now let us add multiplication and division. Define set of *numbers with multiplication* $(N, *)$ as:

1. $* : N \times N \mapsto N$. We will write $*(a, b)$ as $a * b$.

2. $(a * b) * c = a * (b * c)$ for all $a, b, c \in N$.
3. There is an element $1 \in N$ such that $a * 1 = 1 * a = a$ for all $a \in N$.
4. For all $a \in N$, there is an element $b \in N$ such that $a * b = 1$.
5. $a * b = b * a$ for all $a, b \in N$.

These axioms are identical to first ones except for the name of operation and replacement of 0 by 1. Division operation is defined analogously to subtraction. It is easy to see that the definition of '$-$' and '/' is entirely determined by the definition of $+$ and $*$ respectively.

Finally define set of *numbers with addition and multiplication* $(N, +, *)$ as:

1. $(N, +)$ is a set of numbers with addition.
2. $(N \backslash \{0\}, *)$ is a set of numbers with multiplication.
3. For all $a, b, c \in N$, $a * (b + c) = a * b + a * c$.

Why is the number '0' excluded from N in second axiom above? It is to avoid division by zero. Show that:

- If 0 is included in N for the second axiom, then $1 = 0$.

Solution:
If 0 is included in the *second axiom for a set of numbers with addition and multiplication*, then from the *fourth axiom of multiplication* there must exist an element $k \in N$ such that,

$$0 * k = 1 \tag{1}$$

Therefore,

$$(0 - 0) * k = 1 \tag{2}$$

$$0 * k - 0 * k = 1$$

$$0 = 1$$

The addition and multiplication operations can be different for different sets of numbers:

- Give two examples of sets of numbers with different addition and multiplication operations.

Solution:

1. The set $(N_2, +_2, *_2)$

$$a +_2 b = (a + b) \mod 5$$
$$a *_2 b = (a * b) \mod 5$$

This set therefore contains, $\{0, 1, 2, 3, 4\}$

The additive identity for this set is, $0 \in N_2$ and the multiplicative identity for this set is $1 \in N_2$.

The additive inverse for element $k \in N_2$ is $(5 - k) \in N_2$ and the multiplicative inverse pairs are, $(1, 1)$, $(2, 3)$, $(3, 2)$, $(4, 4)$

2. The set $(N, +_3, *_3)$

$$a +_3 b = (a + b)$$
$$a *_3 b = (a * b * 2)$$

The additive identity for this set is, $0 \in N_3$ and the multiplicative identity for this set is $\frac{1}{2} \in N_3$.

The additive inverse for element $k \in N_3$ is $(-k) \in N_3$ and the multiplicative inverse for element $k \in N_3$ is $\frac{1}{2k} \in N_3$

Does a set of numbers defined as above contains natural numbers? Show that:

- There is a set of numbers $(N, +, *)$ such that N is finite.

Solution:
The *first example* mentioned as a solution to the above is a finite set of numbers.
The set $(N_2, +_2, *_2)$

$$a +_2 b = (a + b) \mod 5$$
$$a *_2 b = (a * b) \mod 5$$

This set therefore contains, $\{0, 1, 2, 3, 4\}$
The additive identity for this set is, $0 \in N_2$ and the multiplicative identity for this set is $1 \in N_2$.
The additive inverse for element $k \in N_2$ is $(5 - k) \in N_2$ and the multiplicative inverse pairs are, $(1, 1)$, $(2, 3)$, $(3, 2)$, $(4, 4)$

Does this mean that we have not been able to capture the notion of numbers properly? Later in the course, we will show that it is not so. A set of numbers *can* be finite, and such numbers are extremely useful!

In order to identify set of numbers that contain $\mathbb{N}$, define *multiplicity* of set $(N, +, *)$ to be the smallest k for which $\underbrace{1+1+\cdots+1}_{k \text{ times}} = 0$. When there is no such k, then we set multiplicity of $(N, +, *)$ to 0. Show that:

- Multiplicity of $(N, +, *)$ is either 0 or a prime number.

Solution:
If $k = 1$, then $1 = 0$, which is clearly wrong.

Suppose the multiplicity is k, which is a composite number and can be written as, $k = a * b$,

$$\underbrace{1+1+\cdots+1}_{k \text{ times}} = 0 \tag{1}$$

This can be split into b groups of a one's.

$$\underbrace{\underbrace{1+\cdots+1}_{a \text{ times}} + \cdots + \underbrace{1+\cdots+1}_{a \text{ times}}}_{b \text{ times}} = 0 \tag{2}$$

Let, $\underbrace{1+\cdots+1}_{a \text{ times}} = p$ and, $\underbrace{1+\cdots+1}_{b \text{ times}} = q$

Claim:

$$a * (\underbrace{1+\cdots+1}_{k \text{ times}}) = \underbrace{a+\cdots+a}_{k \text{ times}}$$

Proof:
For $k = 1$, it is trivially true.
Assume it to be true for some $k - 1$

$$a * (\underbrace{1+\cdots+1}_{(k-1) \text{ times}}) = \underbrace{a+\cdots+a}_{k \text{ times}} \tag{3}$$

For k,

$$a * (\underbrace{1+\cdots+1}_{(k-1) \text{ times}}) + a * 1 = \underbrace{a+\cdots+a}_{k \text{ times}} + a$$

Using the *third axiom for additions*, we can write,

$$a * (\underbrace{1+\cdots+1}_{(k) \text{ times}}) = \underbrace{a+\cdots+a}_{k \text{ times}}$$

Hence, by the principle of induction the Claim is valid for all $kin\mathbb{N}$

Now from (2), $\underbrace{p+\cdots+p}_{(b)\text{ times}} = c*(\underbrace{1+\cdots+1}_{(b)\text{ times}})$

Which implies,

$$c*d=0$$

We know that according to the *second axiom for a set of numbers*, 0 is not included in the Domain for multiplication,
We also know neither of c, d is zero, since k is the minimum such value, when a sum of ones, yield 0.
This, $c*d \neq 0$,
Which implies that our assumption that k is composite was wrong.
k is either 0 or prime

- Any set of numbers $(N, +, *)$ of multiplicity 0 contains $\mathbb{N}$.

Solution:
Since, multiplicity of the set $(N, \times, +)$ is 0 and addition and multiplication operations are defined on this set.
Which implies that the set contains the elements, $0, 1$, at least. and $0 \neq 1$, ...
Since, 1 is in the set and addition is defined there must be an element $-(1)$ such that, $1+(-1)=0$
Therefore our set now contains, $-1, 0, 1$.
We also know, since multiplicity is 0, $1+1 \neq 0$, which implies that there must be a number, say 2, such that, $1+1=2$.
Therefore our set now contains, $-1, 0, 1, 2$.
Since, 2 is in the set and addition is defined there must be an element $-(2)$ such that, $2+(-2)=0$
Therefore our set now contains, $-2, -1, 0, 1, 2$.
Continuing this argument inductively, and infinitely many times, we can create the set,
$..., -2, -1, 0, 1, 2....,$
which clearly contains N in it.

- For any set of numbers $(N, +, *)$ of multiplicity 0, for any $k \in \mathbb{N} \subseteq N$, for any $a \in N$, $k*a = \underbrace{a+a+\cdots+a}_{k\text{ times}}$.

Solution:
Since, multiplicity of the set $(N, \times, +)$ is 0 and addition and multiplication operations are defined on this set.

$$k = \underbrace{1 + 1 + \cdots + 1}_{k \text{ times}} \neq 0 \tag{1}$$

Therefore, for any $a \in N$,

$$k * a = (\underbrace{1 + 1 + \cdots + 1}_{k \text{ times}}) * a \tag{2}$$

Using the third axiom,

$$k * a = (\underbrace{a + a + \cdots + a}_{k \text{ times}}) \tag{3}$$

As was done in the class with $\mathbb{N}$, is there way to identify a unique set of numbers using equivalence classes? The answer is no, as there can be finite as well as infinite set of numbers. Moreover, there are binary operations defined on numbers and any equivalence between two sets of numbers must equate the operations as well. Define an *isomorphism* h between two sets of numbers $(N_1, +_1, *_1)$ and $(N_2, +_2, *_2)$ as:

1. $h : N_1 \mapsto N_2$ is a bijection,
2. For all $a, b \in N_1$, $h(a +_1 b) = h(a) +_2 h(b)$,
3. For all $a, b \in N_1$, $h(a *_1 b) = h(a) *_2 h(b)$.

Show that:

- The relation defined by isomorphism between two sets of numbers is an equivalence relation on the set of all sets of numbers.

Solution:
In order to prove that an isomorphism is an equivalence relation, we need to prove that it is a reflexive, symmetric, and transitive relation.

Reflexive
This is trivially true, as there will always exist an isomorphism from $(N, +, *)$ to $(N, +, *)$, which will be a bijection as well.

Symmetric
Consider a bijection, $h : N_1 \mapsto N_2$, since h is an bijection, the inverse function $h^{-1} : N_2 \mapsto N_1$ must exist as well.

- Let, $c = h(a)$ and $d = h(b)$ for some $c, d \in N_2$ and $a, b \in N_1$, thus, $a = h^{-1}(c)$ and $b = h^{-1}(d)$

  Consider, $h(a +_1 b) = h(a) +_2 h(b)$

Operating h^{-1} on both sides,

$$h^{-1}(h(a +_1 b)) = h^{-1}(h(a) +_2 h(b)) \tag{1}$$

$$h^{-1}(c) +_1 h^{-1}(d) = h^{-1}(c +_2 d))$$

- Let, $c = h(a)$ and $d = h(b)$ for some $c, d \in N_2$ and $a, b \in N_1$, thus, $a = h^{-1}(c)$ and $b = h^{-1}(d)$

  Consider, $h(a *_1 b) = h(a) *_2 h(b)$

  Operating h^{-1} on both sides,

$$h^{-1}(h(a *_1 b)) = h^{-1}(h(a) *_2 h(b)) \tag{2}$$

$$h^{-1}(c) *_1 h^{-1}(d) = h^{-1}(c *_2 d))$$

Transitive
Consider the three sets, $(N_1, +_1, *_1)$, $(N_2, +_2, *_2)$ and $(N_3, +_3, *_3)$, with bijections, $f : N_1 \mapsto N_2$ and $g : N_2 \mapsto N_3$.
Now, consider a map, $h : N_1 \mapsto N_3$, and $h(x) = g(f(x))$, which is a bijection since g, f are also bijections.

- For $a, b \in N_1$, $f(a +_1 b) = f(a) +_2 f(b)$ and for $c, d \in N_2$, $g(c +_2 d) = g(c) +_3 g(b)$

  Thus, $h(a +_1 b) = g(f(a +_1 b) = g(f(a)) +_3 g(f(b)) = h(a) +_3 h(b)$

- For $a, b \in N_1$, $f(a *_1 b) = f(a) *_2 f(b)$ and for $c, d \in N_2$, $g(c *_2 d) = g(c) *_3 g(b)$

  Thus, $h(a *_1 b) = g(f(a *_1 b) = g(f(a)) *_3 g(f(b)) = h(a) *_3 h(b)$

Since the isomorphism is reflexive, symmetric, and transitive relation, it is an equivalance relation.

- If h is an isomorphism from $(N_1, +_1, *_1)$ to $(N_2, +_2, *_2)$ then $h(0_1) = 0_2$ and $h(1_1) = h(1_2)$.

Solution:
Since, $0_1 +_1 0_1 = 0_1$

$$h(0_1) +_2 h(0_1) = h(0_1) \tag{1}$$

Since, $h(0_1) \in N_2$, there must exist a numbers, $-h(0_1) \in N_2$ such

that $h(0_1) +_2 (-h(0_1)) = 0_2$.
Adding it to both sides of the equality in (1),

$$h(0_1) +_2 h(0_1) +_2 (-h(0_1)) = h(0_1) +_2 (-h(0_1)) \tag{2}$$

$$h(0_1) = 0_2 \tag{3}$$

Since, $1_1 *_1 1_1 = 1_1$

$$h(1_1) *_2 h(1_1) = h(1_1) \tag{4}$$

Since, $h(1_1) \in N_2$, there must exist a numbers, $h(1_1)' \neq 0_2 \in N_2$ such that $h(1_1) *_2 h(1_1)' = 1_2$.
Multiplying it to both sides of the equality in (4),

$$h(1_1) *_2 h(1_1) *_2 *h(1_1)' = h(1_1) *_2 h(1_1)' \tag{5}$$

$$h(1_1) = 1_2 \tag{6}$$

- If h is an isomorphism from $(N_1, +_1, *_1)$ to $(N_2, +_2, *_2)$ then $h(a -_1 b) = h(a) -_2 h(b)$ and $h(a/_1 b) = h(a)/_2 h(b)$.

Solution:
Since, $h(0_1) = 0_2$,

$$h(a +_1 -a) = h(0_1) = 0_2$$

$$h(a) +_1 h(-a) = 0_2$$

$$h(-a) = -h(a)$$

Now,

$$h(a +_1 (-b)) = h(a -_1 b) \tag{1}$$

$$h(a +_1 (-b)) = h(a) +_2 h(-b) = h(a) -_2 h(b) \tag{2}$$

From (1) and (2), we have,

$$h(a -_1 b) = h(a) -_2 h(b) \tag{3}$$

Similarly,
Since, $h(1_1) = 1_2$,

$$h(a *_1 (\frac{1}{a}) = h(1_1) = 1_2$$

$$h(a) *_1 h(\frac{1}{a}) = 1_2$$

$$h(\frac{1}{a}) = \frac{1}{h(a)}$$

Now,

$$h(a *_1 \frac{1}{b})) = h(a/_1 b) \tag{4}$$

$$h(a/_1 b) = h(a *_1 \frac{1}{b})) = h(a) *_2 h(\frac{1}{b})$$

$$h(a) *_2 h(\frac{1}{b}) = h(a) *_2 \frac{1}{h(\frac{1}{b})} = h(a)/_2 h(b) \tag{5}$$

From (1) and (2), we have,

$$h(a/_1 b) = h(a)/_2 h(b) \tag{6}$$

Do two sets of numbers of same cardinality always have isomorphism between them? The answer is no. Define a 0-1 polynomial to be $\sum_{i=0}^{k} c_i x^i$ with $c_i = 0, 1$. Define addition of these polynomials as $x^i + x^i = 0$ for every i.

- Prove that the set of 0-1 polynomials with addition defined as above and usual multiplication of polynomials is a set of numbers. It is represented as $F_2(x)$.

The additive identity for the set is 0 itself, as for any polynomial $P(x) + 0 = 0 + P(x) = P(x)$
The multiplicative identity for the set is 1 itself as multiplication is defined as on normal polynomials, $1 * P(x) = P(x) * 1 = P(x)$

For Rational Functions, addition can be defined as :

$$\frac{s_1}{t_1} + \frac{s_2}{t_2} = \frac{s_1 * t_2 + s_2 * t_1}{t_1 * t_2}$$

where s_1, s_2, t_1, t_2 are $0 - 1$ polynomials.

Addition Axioms

- The addition of two elements from this set belongs to this set.
- If s, t are 0-1 polynomials, then $s+t = t+s$, as $s = \sum_{i=0}^{k_1} a_i x^i$ and $t = \sum_{i=0}^{k_2} b_i x^i$.

$$s + t = \sum_{i=0}^{k_2} (a_i + b_i) x^i + \sum_{i=k_2}^{k_1} a_i x^i$$

$$t + s = \sum_{i=0}^{k_2}(b_i + a_i)x^i + \sum_{i=k_2}^{k_1} a_i x^i$$

(assuming $k_1 > k_2$).

$$a_i + b_i = b_i + a_i$$

Similar proof can be constructed for the case when $k_2 > k_1$.

- For associativity,

$$\frac{s_1}{t_1} + (\frac{s_2}{t_2} + \frac{s_3}{t_3}) = \frac{s_1}{t_1} + \frac{s_2 * t_3 + s_3 * t_2}{t_2 * t_3}$$

$$= \frac{s_1 * t_2 * t_3 + s_2 * t_1 * t_3 + s_3 * t_1 * t_2}{t_1 * t_2 * t_3}$$

- The additive inverse for $\frac{s}{t}$ is $-\frac{s_2}{t_2}$

Hence, all axioms of addition are satisfied.
Multiplication is defined as normal polynomial multiplication hence it will by default follow all of the axioms of multiplication.

- Show that there is a bijection between rational numbers $\mathbb{Q}$ and $F_2(x)$.

To prove the existence of a bijection ,we need to show the existence of two one-one functions, one from $\mathbb{Q}$ to $F_2(x)$ and one from $F_2(x)$ to $\mathbb{Q}$.

- Function from $F_2(x)$ to $\mathbb{Q}$:

  Writing down the coefficients of the numerator and the denominator in decreasing orders of power, ie. if the degree of the polynomial is n write down the coefficients as $c_n c_{n-1} ... c_1 c_0$

  Since, $c_i \in 0, 1$, the above written number is a binary number which can be converted into a number in base 10.
  Once we have this representation for both the numerator and the denominator, it can be expresses as a number, $x = 2^a.3^b$

  Thus, a one-one map exists.

- Function from $\mathbb{Q}$ to $F_2(x)$:

  Express the given rational number in the $\frac{a}{b}$ form where

> $a \in N$ and $b \in Z$ and gcd(a,b)=1.
>
> Now, convert a and b into their binary forms and assign the coefficients of a to the numerator function in ascending order, till it exists and repeat the same for b.
>
> If the sign of b is positive, keep the constant value of numerator to be 1 and denominator to be 0, else keep the constant of both numerator and denominator to be 1.
> This gives a one-one function.

- Show that there is no isomorphism between $\mathbb{Q}$ and $F_2(x)$.

> Define h to be an isomorphism to $F_2(x)$ from $\mathbb{Q}$, and the polynomial addition to be $+_1$
>
> - h(0) = 0, which is the zero polynomial.
> - h(1) = 1, which is the identity polynomial.
>
> $$h(1+1) = h(1) +_1 (1)$$
>
> According to the defintion, $x^i + x^i = 0$ for every i.
>
> $$h(2) = 0$$
>
> Thus, h is not a bijection and hence no isomorphism is possible.

As per the definition above, the set of integers $\mathbb{Z}$ is not a set of numbers. This is unsatisfactory. The problem is that division is generally not possible in $\mathbb{Z}$. To address this, define a set of *numbers without division* $(N, +, *)$ to be a set of numbers in which the fourth axiom for $(N, *)$ is removed. Show that:

- $(\mathbb{Z}, +, *)$ is a set of numbers without division.

> **Solution:**
> Given, $(Z, +, *)$ is a set of *numbers without division* if it satisfies the *four axioms of addition* along with *the axioms for multiplication*, except the fourth axiom.
>
> - For $(Z, +)$ all *axioms for addition* are trivially satisfied, by the definition itself.
>
>   The 0 element is 0 and additive inverse for $n \in Z$ is $-n$
> - For $(Z, +)$ axioms except the fourth axiom are trivially sat-

isfied.

For the *fourth axiom for multiplication* which states that,
For every element $a \in Z$, there must exist an element $b \in Z$ such that, $a * b = 1$.
For, any $a, b \in Z$ and $a, b \neq \{1, -1\}$, $a * b = 1$ is never possible.
Also for all $a, b, c \in Z$

$$a * (b + c) = a * b + a * c$$

Holds, true as there is no use of the fourth axiom for the above.
Hence, Z is a set of numbers without division.

Such set of numbers can also have unexpected properties. Show that:

- There is a set of numbers without division $(N, +, *)$ such that there are $a, b \in N$, $a \neq 0$, $b \neq 0$, but $a * b = 0$.

Solution:
Consider the set $(N_6, +_6, *_6)$, with

$$+_6(a, b) = (a + b) \mod 6$$

$$*_6(a, b) = (a * b) \mod 6$$

Thus, $N_6 = \{0, 1, 2, 3, 4, 5\}$
For N_6, all the *four axioms for addition* as well as the *axioms for multiplication*, apart from the *fourth axiom* are valid.
Consider the elements $2 \neq 0$ and $3 \neq 0$ and $2, 3 \in N_6$, we have,

$$(2 *_6 3) \mod 6 = 0 \tag{1}$$

Therefore, there exists a set of numbers

- There is a set of numbers without division $(N, +, *)$ such that there is $a \in N$, $a \neq 0$, but $a^3 = a * a * a = 0$.

Solution:
Consider the set $(N_8, +_8, *_8)$, with

$$+_8(a, b) = (a + b) \mod 8$$

$$*_8(a, b) = (a * b) \mod 8$$

Thus, $N_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$
For N_8, all the *four axioms for addition* as well as the *axioms for multiplication*, apart from the *fourth axiom* are valid.

Consider the element $2 \neq 0$ and $2 \in N_8$, we have,

$$2^3 = (2 *_8 2 *_8 2) \mod 8 = 0 \tag{1}$$

Therefore, there exists such a set of numbers as well.

Later in the course, we will see utility of these types of numbers as well.