CS201

Mathematics for Computer Science I

Lecture 2

Set Notations

- A set is denoted by curly braces with elements inside.
  - Examples: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, $\{(x, y) \mid x \in \mathbb{R} \;\&\; y = x^2\}$
- If a is an element of A, it is denoted by $a \in A$.
- If B is a subset of A, it is denoted by $B \subseteq A$.
- If B is a subset of A and $B \neq A$, it is denoted by $B \subset A$. Such B is called a proper subset of A.

Set Operations

- Union of two sets A and B is the set containing all elements belonging to either A or B. It is denoted by $A \cup B$.
- Intersection of two sets A and B is the set containing all elements belonging to both A and B. It is denoted by $A \cap B$.
- Subtraction of set B from A is the set containing all elements of A not belonging to B. It is denoted by $A \backslash B$.
- Symmetric difference of sets A and B is the set containing all elements of A and B that do not belong to both. It is denoted by $A \Delta B$.
- It is easy to see that

$$A \Delta B = (A \cup B) \backslash (A \cap B) = (A \backslash B) \cup (B \backslash A).$$

Set Operations

- Cardinality of a finite set A denotes the number of elements in it. It is denoted by $|A|$.
- This notion, when extended to infinite sets, leads to many interesting conclusions.
- Let us define the cardinality of set of integers, $\mathbb{Z}$, as $\aleph_0$.
- Is the cardinality of set of even integers, denoted as $2\mathbb{Z}$, different?
  - Since $2\mathbb{Z} \subset \mathbb{Z}$, $|2\mathbb{Z}|$ cannot be more than $\aleph_0$.
- How does one compare cardinalities of infinite sets?

Comparing Cardinalities

- If A and B are both finite, then their cardinalities can be easily compared since they are positive numbers.
- If exactly one of them is infinite, then the infinite set has higher cardinality than finite set.
- To compare cardinalities when both are infinite, we define the notion of bijection.
- Mapping $f : A \mapsto B$ is a bijection if the following properties are satisfied:
  - f is one-to-one: for any $a \neq a' \in A$, $f(a) \neq f(a')$.
  - f is onto: for any $b \in B$, there exists an $a \in A$ such that $f(a) = b$.
- If f is a bijection mapping A to B, f^{-1} is a bijection mapping B to A.

Comparing Cardinalities

Definition

Set A and B, finite or infinite, have the same cardinality if there exists a bijection between them.

- For finite sets, this definition matches with the standard one.
- For infinite sets, it allows us to compare the cardinalities of two sets.

Comparing Cardinalities

- $\mathbb{Z}$ and $2\mathbb{Z}$:
  - $f(n) = 2n$ is a bijection between them.
  - Hence, they have the same cardinality!
- $\mathbb{Q}$ and $\mathbb{Z}$:
  - $f(\frac{m}{n}) = 2^m \cdot (2n+1)$ is a one-to-one map from $\mathbb{Q}$ to $\mathbb{Z}$.
  - $g(n) = n$ is a one-to-one map from $\mathbb{Z}$ to $\mathbb{Q}$.
  - We now prove Cantor-Bernstein-Schroeder theorem to show that their cardinalities are the same.

Cantor-Bernstein-Schroeder Theorem

Theorem

If there exist one-to-one maps from A to B and vice-versa, then there is a bijection between A and B.

Proof.

- Let $f : A \mapsto B$ and $g : B \mapsto A$ be one-to-one maps.
- Then, f^{-1} is a map from a subset of B to A and g^{-1} is a map from a subset of A to B.
- Since f and g are one-to-one, f^{-1} and g^{-1} are well-defined as maps.

Cantor-Bernstein-Schroeder Theorem

- Define a chain as a sequence of alternating elements from A and B,

$$\ldots, a_{i-1}, b_{i-1}, a_i, b_i, a_{i+1}, b_{i+1}, \ldots$$

  such that $g^{-1}(a_j) = b_j$ and $f^{-1}(b_j) = a_{j+1}$ for every j.
- A chain will be of infinite length (shown later), and there may be multiple, possibly infinite, chains are present.
- A key property is: any element of $A \cup B$ is present in one and only one chain:
  - It cannot be present in more than one chain since f^{-1} and g^{-1} are one-to-one
  - An element is trivially present in at least one chain.

Cantor-Bernstein-Schroeder Theorem

- For element $a \in A$, define length of chain at a to be the number of elements in the chain after a. The number can be infinite.
- Define mapping $h : A \mapsto B$ as:

$$h(a) = \begin{cases} f(a) & \text{length of chain at } a \text{ is even or infinite} \\ g^{-1}(a) & \text{otherwise} \end{cases}$$

- We claim that h is the desired bijection.

Cantor-Bernstein-Schroeder Theorem

- h is one-to-one:
  - Suppose $h(a) = h(a')$.
  - If h equals f or g^{-1} on both a and a' then $a = a'$ since both f and g^{-1} are one-to-one.
  - Suppose $h(a) = f(a)$ and $h(a') = g^{-1}(a')$.
  - Then length of chain at a' is odd and length of chain at a is two less and so also odd.
  - Hence $h(a) = g^{-1}(a)$ by definition which contradicts the assumption.

Cantor-Bernstein-Schroeder Theorem

- h is onto:
  - Any element $b \in B$ lies on an infinite length chain:

$$\ldots, f(g(f(g(b)))), g(f(g(b))), f(g(b)), g(b), b, \ldots.$$

  - If the length of the chain at $g(b) \in A$ is odd, $h(g(b)) = b$.
  - If the length of the chain at $g(b)$ is even or infinite, then $f^{-1}(b)$ exists and $h(f^{-1}(b)) = b$.

Comparing Cardinalities

- $\mathbb{Q}$ and $\mathbb{Z}$:
  - $f(\frac{m}{n}) = 2^m \cdot (2n+1)$ is a one-to-one map from $\mathbb{Q}$ to $\mathbb{Z}$.
  - $g(n) = n$ is a one-to-one map from $\mathbb{Z}$ to $\mathbb{Q}$.
  - By Cantor-Bernstein-Schroeder Theorem, there is a bijection between $\mathbb{Z}$ and $\mathbb{Q}$ and hence they have the same cardinality.
- $\mathbb{R}$ and $\mathbb{Z}$:
  - Their cardinalities are not the same!
  - We prove this in next lecture.