# CS201: Midsem Examination

September 21, 2023

**Duration**: Two Hours **Maximum Marks**: 50

**Question 1. (10 marks)** Consider $\mathbb{R}^2$, the usual two-dimensional space. A *triangle* $T \subset \mathbb{R}^2$ is the set of all points inside the boundary defined by lines joining three given points. In other words,

$$T = \{(x,y) \in \mathbb{R}^2 \mid (x,y) = \alpha(x_1,y_1) + \beta(x_2,y_2) + \gamma(x_3,y_3), 0 \leq \alpha, \beta, \gamma \leq 1, \alpha+\beta+\gamma = 1\}.$$

Let $\mathbb{T}$ be the set of all triangles in $\mathbb{R}^2$. Prove that $|\mathbb{T}| = |\mathbb{R}|$.

**Answer.** A triangle is completely specified by its three apex points $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$. Therefore, there exists a one-to-one mapping from $\mathbb{T}$ to $\mathbb{R}^6$. Define a mapping from $\mathbb{R}^6$ to $\mathbb{R}$ as follows. Given $(x_1, x_2 \ldots, x_6) \in \mathbb{R}^6$, let $f(x_i) = \frac{1}{1+e^{x_i}}$. As discussed in the class, $f$ is a one-to-one map from $\mathbb{R}$ to the interval $(0, 1)$. Let $y_i = 0.d_{i,1}d_{i,2}\cdots$ where $d_{i,j} \in \{0, 1, 2, \ldots, 9\}$. Define $g(x_1, x_2, \ldots, x_6) = 0.d_{1,1}d_{2,1}d_{3,1}d_{4,1}d_{5,1}d_{6,1}d_{1,2}d_{2,2}d_{3,2}d_{4,2}d_{5,2}d_{6,2}\cdots$. It is clear that $g$ is a one-to-one map from $\mathbb{R}^6$ to the interval $(0, 1)$.

**Question 2. (10 marks)** Recall the definition of set of numbers with addition and multiplication $(N, +, *)$ given in the assignment. Consider such a set of numbers $N$ that is finite. Let $m$ be multiplicity of $(N, +, *)$. Prove that $m$ divides $|N|$.

**Answer.** Since $m$ is the multiplicity of $N$, we have $m = \underbrace{1+1+\cdots+1}_{m\ times} = 0$. Therefore, for any $a \in N$, $a*m = a*(\underbrace{1+1+\cdots+1}_{m\ times}) = \underbrace{a+a+\cdots+a}_{m\ times} = 0$. The numbers $0, 1, 2, \ldots, m-1 \in N$ must all be distinct since if $i = j$ for $0 \leq i < j < m$, we have $j - i = 0$ violating the minimality of $m$. Define a binary relation $R$ on elements of $N$ as: $aRb$ iff $a = b + j$ for some $0 \leq j < m$. $R$ is clearly reflexive, transitive, and symmetric. Hence an equivalence relation. Therefore, $N$ is split into equivalence classes by $R$. Consider any equivalence class. By definition, there exists an $a \in N$ such that every element of this equivalence class can be written as $a + j$ for $0 \leq j < m$. Further, $a + i \neq a + j$ for $0 \leq i \neq j < m$ since otherwise $j - i = 0$. Hence, each equivalence class has exactly $m$ elements proving that size of $N$ is divisible by $m$.

**Question 3. (10+10 marks)** Define polynomial $Q_k(y) = 0^k + 1^k y + 2^k y^2 + 3^k y^3 + \cdots + n^k y^n = \sum_{i=0}^{n} i^k y^i$, for $k \geq 0$. Generating function for polynomials $Q_k(y)$ is $G(x) = \sum_{k \geq 0} Q_k(y) x^k$. Derive a formula for $G(x)$.
Let $Q(y) = Q_\infty(y) = \sum_{i \geq 0} i^k y^i$. Derive a formula for generating function $Q(y)$.

**Answer.** We have

$$\begin{aligned}
G(x) &= \sum_{k\geq 0} Q_k(y)x^k \\
&= \sum_{k\geq 0}\sum_{0\leq j\leq n} j^k y^j x^k \\
&= \sum_{0\leq j\leq n} (\sum_{k\geq 0} j^k x^k) y^j \\
&= \sum_{0\leq j\leq n} \frac{1}{1-jx} y^j
\end{aligned}$$

Define $P_k(y) = \sum_{i\geq 0} i^k y^i$. Then, $P_0(y) = \frac{1}{1-y}$. Moreover, $\frac{d}{dy}P_k(y) = \sum_{i\geq 0} i^{k+1}y^{i-1}$. Therefore, $P_{k+1}(y) = y\frac{d}{dy}P_k(y)$. [Showing this gets 8/10 marks] This gives, $P_1(y) = y\frac{1}{(1-y)^2} = \frac{1}{(1-y)^2} - \frac{1}{1-y}$. In general, suppose that $P_k(y) = \sum_{1\leq j\leq k} \frac{a_j[k]}{(1-y)^j}$ for $a_j[k] \in \mathbb{Z}$. Then,

$$\begin{aligned}
P_{k+1}(y) &= y \cdot \left( \sum_{1\leq j\leq k} \frac{ja_j[k]}{(1-y)^{j+1}} \right) \\
&= \sum_{1\leq j\leq k} \frac{ja_j[k]}{(1-y)^{j+1}} - \sum_{1\leq j\leq k} \frac{ja_j[k]}{(1-y)^j} \\
&= \sum_{1\leq j\leq k+1} \frac{(j-1)a_{j-1}[k]}{(1-y)^j} - \sum_{1\leq j\leq k} \frac{ja_j[k]}{(1-y)^j} \\
&= \sum_{1\leq j\leq k+1} \frac{(j-1)a_{j-1}[k] - ja_j[k]}{(1-y)^j}
\end{aligned}$$

setting $a_j[k] = 0$ for $j > k$. Therefore, $a_j[k+1] = (j-1)a_{j-1}[k] - ja_j[k]$. [Showing this recurrence gets 10/10 marks]

Using matrices, we can derive an explicit formula for $a_j[k]$. For $1 \leq \ell < k$, using the recurrence relations, we can write:

$$\begin{bmatrix} a_1[\ell+1] & a_2[\ell+1] & \cdots & a_k[\ell+1] \end{bmatrix} = \begin{bmatrix} a_1[\ell] & a_2[\ell] & \cdots & a_k[\ell] \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -2 & 2 & \cdots & 0 & 0 \\ 0 & 0 & -3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -(k-1) & k-1 \\ 0 & 0 & 0 & \cdots & 0 & -k \end{bmatrix}$$

Let

$$a[\ell] = \begin{bmatrix} a_1[\ell] & a_2[\ell] & \cdots & a_k[\ell] \end{bmatrix}$$

and

$$M = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -2 & 2 & \cdots & 0 & 0 \\ 0 & 0 & -3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -(k-1) & k-1 \\ 0 & 0 & 0 & \cdots & 0 & -k \end{bmatrix}.$$

Then,

$$a[\ell+1] = a[\ell]\cdot M = a[\ell-1]\cdot M^2 = \cdots = a[1]\cdot M^\ell$$

with $a[1] = \begin{bmatrix}1 & 0\cdots & 0\end{bmatrix}$. To compute $M^\ell$, we note that eigenvalues of $M$ are $-1$, $-2$, …, $-k$. The corresponding *eigenvectors* are

$$v_j = \begin{bmatrix}0 & \cdots & 0 & \binom{j-1}{0} & \binom{j}{1} & \binom{j+1}{2} & \cdots & \binom{k-1}{k-j}\end{bmatrix}.$$

This follows since

$$\begin{aligned} v_j \cdot M &= \begin{bmatrix}0 & \cdots & 0 & -j\binom{j-1}{0} & j\binom{j-1}{0} - (j+1)\binom{j}{j-1} & \cdots & (k-1)\binom{k-2}{k-j-1} - k\binom{k-1}{k-j}\end{bmatrix} \\ &= -jv_j \end{aligned}$$

as

$$(i-1)\binom{i-2}{i-j-1} - i\binom{i-1}{i-j} = \frac{(i-1)!}{(i-j-1)!(j-1)!} - i\frac{(i-1)!}{(i-j)!(j-1)!} = -j\binom{i-1}{i-j}$$

for $j < i \le k$.

Letting matrix $V = \begin{bmatrix}v_1 \\ v_2 \\ \vdots \\ v_k\end{bmatrix}$, we have:

$$V\cdot M = D\cdot V$$

where

$$D = \begin{bmatrix}-1 & 0 & \cdots & 0 & 0 \\ 0 & -2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -(k-1) & 0 \\ 0 & 0 & \cdots & 0 & -k\end{bmatrix}.$$

Matrix $V$ is an upper triangular matrix with non-zero diagonals, and so is invertible. This gives us a formula for $M$:

$$M = V^{-1}DV.$$

Further,

$$M^\ell = V^{-1}D^\ell V = V^{-1}\begin{bmatrix}(-1)^\ell & 0 & \cdots & 0 & 0 \\ 0 & (-2)^\ell & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & (-k+1)^\ell & 0 \\ 0 & 0 & \cdots & 0 & (-k)^\ell\end{bmatrix}V.$$

What is $V^{-1}$? Consider vector

$$u_i = \begin{bmatrix}(-1)^{i-1}\binom{i-1}{i-1} & (-1)^{i-2}\binom{i-1}{i-2} & \cdots & \binom{i-1}{0} & 0 & \cdots & 0\end{bmatrix}.$$

For $i < j$, it is straightforward to see that $v_j \cdot u_i^T = 0$. For $i \geq j$:

$$\begin{aligned}
v_j \cdot u_i^T &= \sum_{r=j}^{i} (-1)^{i-r} \binom{i-1}{i-r} \binom{r-1}{r-j} \\
&= \sum_{r=j}^{i} (-1)^{i-r} \frac{(i-1)!}{(i-r)!(r-1)!} \frac{(r-1)!}{(r-j)!(j-1)!} \\
&= \sum_{r=j}^{i} (-1)^{i-r} \frac{(i-1)!}{(i-r)!(r-j)!(j-1)!} \\
&= \sum_{r=0}^{i-j} (-1)^{i-j-r} \frac{(i-1)!}{(j-1)!} \frac{1}{(i-j-r)!r!} \\
&= (-1)^{i-j} \frac{(i-1)!}{(i-j)!(j-1)!} \sum_{r=0}^{i-j} (-1)^{r} \frac{(i-j)!}{(i-j-r)!r!} \\
&= (-1)^{i-j} \binom{i-1}{j-1} (1-1)^{i-j}
\end{aligned}$$

Therefore, $v_j \cdot u_i^T = 1$ if $i = j$ and 0 if $i \neq j$. Hence,

$$V^{-1} = \begin{bmatrix} u_1^T & u_2^T & \cdots & u_k^T \end{bmatrix}.$$

Combining all this, we have

$$\begin{aligned}
a[\ell+1] &= a[1] \cdot M^{\ell} \\
&= a[1] \cdot V^{-1} D^{\ell} V \\
&= \begin{bmatrix} \binom{0}{0} & -\binom{1}{1} & \binom{2}{2} & \cdots & (-1)^{k-1}\binom{k-1}{k-1} \end{bmatrix} D^{\ell} V \\
&= \begin{bmatrix} (-1)^{\ell} & -(-2)^{\ell} & (-3)^{\ell} & \cdots & (-1)^{k-1}(-k)^{\ell} \end{bmatrix} V
\end{aligned}$$

This gives

$$\begin{aligned}
a_j[\ell+1] &= (-1)^{\ell} \sum_{r=1}^{j} (-1)^{r-1} r^{\ell} \binom{j-1}{j-r} \\
&= (-1)^{\ell} \sum_{r=0}^{j-1} (-1)^{r} \binom{j-1}{j-1-r} (r+1)^{\ell} \\
&= (-1)^{\ell} \sum_{r=0}^{j-1} (-1)^{r} \binom{j-1}{r} (r+1)^{\ell}
\end{aligned}$$

for any $0 \leq \ell < k$.

**Question 4. (10 marks)** Consider the following C function:

```
int f(int *A, int n)  {
    int i;
    int value;

    for (i = 1, value = A[0]; i < n; i++) {
```

```
        value += f(A+i, n-i);
    }
    return value;
}
```

Derive a formula for time complexity of function `f`.

**Answer.** Let $T(n)$ be the time complexity of function `f(A,n)`. Then we have:

$$T(n) \leq T(n-1) + T(n-2) + \cdots + T(0) + c \cdot n$$

and $T(0) \leq c$ for some constant $c > 0$. Define $U(0) = c$ and $U(n) = U(n-1) + U(n-2) + \cdots + U(0) + c \cdot n$. Then $T(n) \leq U(n)$ for all $n \geq 0$. We have $U(n) - U(n-1) = U(n-1) + c$ giving us:

$$\begin{aligned} U(n) &= 2U(n-1) + c \\ &= 4U(n-2) + 3c \\ &= 8U(n-3) + 7c \\ &\vdots \quad \vdots \\ &= 2^n U(0) + (2^n - 1)c \\ &= (2^{n+1} - 1)c. \end{aligned}$$

Therefore, $T(n) = O(2^n)$.