CS201

Mathematics for Computer Science I

Lecture 4

Why Sets in a Discrete Mathematics Course?

- Study of sets captures all of mathematics, whether discrete or continuous.
- Arguments used are, therefore, applicable everywhere.
- Discrete mathematics is study of discrete sets: all finite sets, $\mathbb{Z}$ etc.
- Arguments from set theory find many applications.

Relations

- Structure of a set is captured by relationships between its elements.
- A k-ary relation on set A is typically viewed as a predicate taking true or false value on any collection of k-elements of A.
- Examples:
  - $\subseteq$ is a 2-ary relation (also called binary) on $\mathcal{P}(A)$ for any set A: $\subseteq(X, Y)$ is true iff $X \subseteq Y$ for $X, Y \subseteq A$.
  - $\leq$ is a binary relation on set of numbers: $\leq(a, b)$ is true iff $a \leq b$.
  - For any function $f : A \mapsto A$, f defines a binary relation on A: $R_f(a, b)$ is true iff $f(a) = b$.
  - $+$ is a 3-ary relation on set of numbers: $+(a, b, c)$ is true iff $a + b = c$.
  - Set membership is a unary relation: given $A \subseteq B$, $R_A(b)$ is true for $b \in B$ iff $b \in A$.

Relations

- A relation can be represented as a set: if R is a k-ary relation on set A, then the set

$$\{(a_1, a_2, \ldots, a_k) \mid a_1, \ldots, a_k \in A \text{ and } R(a_1, \ldots, a_k) \text{ is true}\}$$

  is set representation of R. It is also denoted by R.
- We will focus on special kind binary relations as these are easier to analyze and still capture interesting structures.

Transitive Relations

- Let R be a binary relation on set A.
- Instead of $R(a,b)$, we write aRb.
- R is a transitive relation if the following condition is satisfied:
  For every $a, b, c \in A$, if aRb and bRc then aRc.
- Examples:
  - $\subseteq$ is transitive
  - $\leq$ is transitive
  - Relation induced by function f is not transitive

Transitive Relations

- Transitive relations give an ordering to elements of the set:
  - Say a is "below" or "less than equal to" b iff aRb for a transitive relation R.
  - transitivity of R ensures that the intuitive expectation from an ordering is satisfied.
- There are two special classes of transitive relations. These are defined using reflexive and symmetric properties.

Reflexive Relations

- Relation R on set A is reflexive if aRa for every $a \in A$.
- Examples:
  - $\subseteq$ and $\leq$ are reflexive.
  - If elements of A are sets, define Equal cardinality relation $\equiv$ on A as: $a \equiv b$ iff $|a| = |b|$.
  - $\equiv$ is reflexive.
  - Relation $<$ is not reflexive.

Symmetric and Antisymmetric Relations

- Relation R on set A is symmetric if aRb implies bRa for every $a, b \in A$.
- Relation R on set A is antisymmetric if aRb and bRa implies $a = b$ for every $a, b \in A$.
- Examples:
  - $\subseteq$ and $\leq$ are antisymmetric.
  - $\equiv$ is symmetric.

Equivalence Relations

Definition

Relation R on set A in an equivalence relation if R is transitive, reflexive and symmetric.

- $\equiv$ is an equivalence relation.

Structure Induced by Equivalence Relations

Definition

If R is an equivalence relation on A then an equivalence class of (A, R) is a subset X of A such that aRb for every $a, b \in X$, and aRb is false for every $a \in X$ and $b \in A \backslash X$.

- All subsets of cardinality 1 of $\mathcal{P}(\mathbb{Z})$ form an equivalence class of $(\mathcal{P}(\mathbb{Z}), \equiv)$.

Structure Induced by Equivalence Relations

Theorem

An equivalence relation R on set A splits it into disjoint equivalent classes.

Proof.

- Let $a \in A$. Consider subset X_a of A such that

$$X_a = \{b \mid b \in A \text{ and } aRb\}.$$

- X_a is an equivalence class of (A, R).

Partial Orders

Definition

Relation R on set A is a partial order if R is transitive, reflexive, and anti-symmetric.

- $\subseteq$ and $\leq$ are partial orders.

Total Orders

Definition

Relation R on set A is a total order if R is a partial order on A and for every $a, b \in A$, either aRb or bRa.

- $\leq$ is a total order but $\subseteq$ is not.

Structure Induced by Partial Orders

- Let R be a partial order on set A.
- A chain of (A, R) is a subset C of A such that aRb or bRa for any two $a, b \in C$.
- Clearly, R is a total order on C since R remains a partial order on any subset of A.
- A maximal element of (A, R) is an element $a \in A$ such that there is no $b \in A$ with aRb.
- A minimal element of (A, R) is an element $a \in A$ such that there is no $b \in A$ with bRa.

Total Orders on Sets

- Every set admits a partial order trivially: define an empty relation on the set.
- Is it possible to define a total ordering on all sets?
- Yes if set is finite:
    - Let
    $$A = \{a_1, a_2, \ldots, a_k\}.$$
    - Define relation R such that $a_i R a_j$ for $i \leq j$.
- What about infinite sets?
- Leads to Axiom of Choice, Well-ordering Principle, and Zorn's Lemma!