CS201: Endsem Examination

November 20, 2023

Duration: Twenty Four Hours **Maximum Marks**: 52

Question 1. (8+8 marks) Consider a finite field F. Define a bipartite graph $G = (U, V, E)$ where vertices of U and V both correspond to elements of F, and the edges $(a, a^2), (a, a^3) \in E$ for all $a \in U$. Show that G has a perfect matching if $F = F_{71}$ and no perfect matching if $F = F_{73}$.

Question 2. (4+8+4 marks) Let $\phi : [1, n] \mapsto [1, n]$ be a permutation (a 1-1, onto mapping). Consider a set of n balls, numbered 1 to n. Each ball is to be colored by one of three colors. Let $C : [1, n] \mapsto \{1, 2, 3\}$ denote a coloring of the balls. Two colorings C_1 and C_2 are said to be ϕ-related if $\phi^j(C_1) = C_2$ for some $j \geq 0$. Prove that "ϕ-related" is an equivalence relation.

The number of unrelated colorings equals the number of equivalence classes induced by the ϕ-related relation. This number will depend on the structure of ϕ. Consider the following specific ϕ for $n = 15$:

$$\phi(1) = 3, \phi(2) = 8, \phi(3) = 7, \phi(4) = 1, \phi(5) = 13$$
$$\phi(6) = 15, \phi(7) = 11, \phi(8) = 5, \phi(9) = 2. \phi(10) = 12$$
$$\phi(11) = 4, \phi(12) = 14, \phi(13) = 6, \phi(14) = 10, \phi(15) = 9$$

Compute the number of unrelated colorings for this value of ϕ. Which value of ϕ results in largest possible number of unrelated colorings?

Question 3. (4+8+4+4 marks) Let F be a field and $F_0 \subset F$ be a subfield (F_0 is a subset of F and a field under same operations as F). For any $\alpha \in F \backslash F_0$, define *minimum polynomial of α wrt F_0* to be the smallest degree polynomial $p(x)$ with coefficients from F_0 such that $p(\alpha) = 0$. In case there is no such polynomial, α is called *transcendental* wrt F_0. In case there exists such a polynomial, α is called *algebraic* wrt F_0. Give an example, with proof, of $F \supset \mathbb{Q}$ and $\alpha \in F$ such that α wrt $\mathbb{Q}$ is transcendental.

For any field F and α algebraic wrt F, define

$$F[\alpha] = \{a_0 + a_1\alpha + \cdots + a_{d-1}\alpha^{d-1} \mid a_0, a_1, \cdots, a_{d-1} \in F\},$$

where d is the degree of minimum polynomial of α wrt F. Prove that $F[\alpha]$ is a field.

For every $j \geq 1$, let $F_j = \mathbb{Q}[5^{1/2^j}]$ with $F_0 = \mathbb{Q}$. Let ring $R_j = F_{j-1}[x]$. Prove that polynomial $x^2 - 5^{1/2^{j-1}}$ is irreducible in the ring R_j. In other words, show that the polynomial does not have roots in the field F_{j-1}. Hence, $I = (x^2 - 5^{1/2^{j-1}})$ is a maximal ideal in R_j, and therefore, $\hat{F} = R_j/I$ is a field. Prove that $\hat{F}$ is isomorphic to F_j (the isomorphism must respect both ring operations).