CS201

Mathematics for Computer Science I

Lecture 15

Motivation

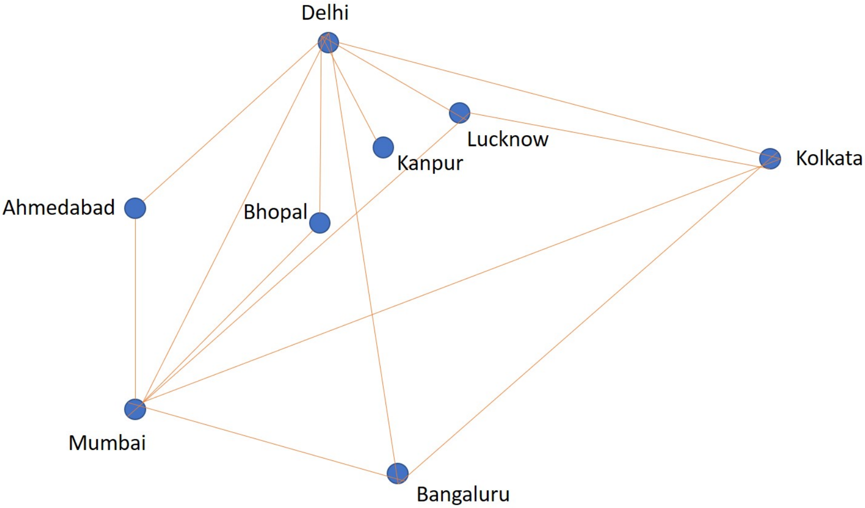
- Given a set of elements, many relationships between them are binary:
  - Friend relationship between a set of people
  - Order relationship between a set of numbers
  - Subset relationship between sets
  - Neighborhood relationship between a set of points
  - Flight connectivity between a set of cities
- Such relationships are best captured by graphs.

Definition

A graph $G = (V, E)$ consists of a set V of vertices and a set E of edges with $E \subseteq V \times V$.

- E represents a binary relationship between vertices of V.

Example: Air Connectivity between cities



Types of Graphs

Directed and Undirected

Graph $G = (V, E)$ is undirected if E is symmetric, otherwise directed.

Graph without self-loops

Graph $G = (V, E)$ is simple if E is irreflexive.

Unless otherwise stated, a graph is simple, undirected, and with finite number of vertices.

Subgraphs

Graph $H = (V_H, E_H)$ is a subgraph of $G = (V_G, E_G)$ if $V_H \subseteq V_G$ and $E_H \subseteq E_G$.

Induced Subgraphs

Graph $H = (V_H, E_H)$ is an induced subgraph of $G = (V_G, E_G)$ if $V_H \subseteq V_G$ and

$$E_H = \{(u, v) \mid (u, v) \in E_G \ \& \ u, v \in V_H\}.$$

Paths

Let $u, v \in V$ be two vertices of graph $G = (V, E)$. A path from u to v is a sequence of vertices u_0, u_1, $\ldots$, u_k with $u_0 = u$, $u_k = v$, and $(u_i, u_{i+1}) \in E$ for every i, $0 \leq i < k$. Length of a path is the number of edges in it $(= k)$.

Cycles

A cycle of the graph G is a path from vertex u to u of length at least one and with no repeated vertex.

Connected Graphs and Components

Graph $G = (V, E)$ is connected if there exists a path between any two vertices of the graph. A connected component of G is an induced subgraph that is connected.

Transitive Closure

Graph $G^* = (V, E^*)$ is a transitive closure of graph $G = (V, E)$ if

$$E^* = \{(u, v) \mid \text{there is a path from } u \text{ to } v \text{ in } G\}.$$

- E^* is an equivalence relation on V with connected components of G as equivalence classes.

Special Types of Graphs

Let $G = (V, E)$ be a graph. Following are special types of G:

- Complete Graph: $E = V \times V$. Denoted as $K_{|V|}$.
- Independent Set: $E = \emptyset$.
- Bipartite Graph: $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ and $E \subseteq V_1 \times V_2$.
- Complete Bipartite Graph: Bipartite graph with $E = V_1 \times V_2$. Denoted as $K_{|V_1|,|V_2|}$.
- Forest: G has no cycles.
- Tree: G is a forest and connected.

Properties of Forests

Theorem

If $G = (V, E)$ is a forest then $|E| \leq |V| - 1$. Forest G is a tree iff $|E| = |V| - 1$.

Proof.

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- Proof is by induction on $m = |E|$.
- Trivially true for $m = 0$. Assume for $m - 1$.

Properties of Forests

- Let G have G_1, G_2, $\ldots$, G_k, $G_i = (V_i, E_i)$, as connected components.
- Each G_i is a tree.
- If more than one connected component contains an edge then each G_i is a tree with $|E_i| < m$.
- By induction hypothesis, $|E| = \sum_{i=1}^{k} |E_i| = \sum_{i=1}^{k} (|V_i| - 1) < |V| - 1$.
- If only G_1 has edges, deleting one edge from G_1 makes it disconnected into two components: say H_1 and H_2.
- Both are trees with fewer edges than m and so by induction hypothesis:

$$|E| = \#\text{Edges in } H_1 + \#\text{Edges in } H_2 + 1 = |V_1| - 1 \leq |V| - 1.$$

- If G is a tree then $G_1 = G$ and hence $|E| = |V_1| - 1 = |V| - 1$.

Degree of Vertices

Incidence, Adjacent, and Endpoint

For edge $e \in E$, if $e = (u, v)$ then vertices u and v are called endpoints of e. Vertices u and v are called adjacent. Edge $e \in E$ is incident on vertex $v \in V$ if v is an endpoint of e.

Degree

Degree of vertex $v \in V$ is the number of edges incident on it. It is denoted as $\deg(v)$.

Degrees and Edges

Theorem

$\sum_{v \in V} \deg(v) = 2|E|$.

- Every edge $e \in E$ is counted exactly twice in the sum $\sum_{v \in V} \deg v$: once each for the two endpoints of e.