CS201

Mathematics for Computer Science I

Lecture 10

Binomial Theorem Revisited

Binomial Theorem

For integer m:

$$(1+x)^m = \sum_{i=0}^{m} \binom{m}{i} x^i.$$

- What if m is not an integer?
- $m = \frac{1}{2}$ or $m = -\pi$?
- We use α instead of m when m is not integer.

Binomial Theorem Revisited

- We can use Taylor series expansion to compute expansion of $(1+x)^\alpha$ in general.

Taylor series expansion

For any function $f(x)$ that converges in the neighborhood of 0:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots.$$

- Using it for $f(x) = (1+x)^\alpha$, we get

$$f^{(n)}(0) = \left(\prod_{j=0}^{n-1}(\alpha - j)\right).$$

Binomial Theorem Revisited

Generalized Binomial Coefficient

Define
$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}$$
for any α and any integer $n \geq 0$.

Generalized Binomial Theorem

For any α:
$$(1+x)^\alpha = \sum_{n\geq 0} \binom{\alpha}{n} x^n.$$

Examples

- When α is a positive integer, then this coincides with the binomial theorem since
$$\prod_{j=0}^{n-1}(\alpha - j) = 0$$
for $n > \alpha$.
- When α is negative integer, then
$$\binom{\alpha}{n} = (-1)^n \frac{(n + |\alpha| - 1)!}{(|\alpha| - 1)!n!} = (-1)^n \binom{|\alpha| + n - 1}{n}.$$

Generating Functions

Generating Function

Given a possibly infinite sequence of numbers a_0, a_1, a_2, $\ldots$, function

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

is called generating function for the sequence.

- Captures a sequence of related numbers in a single formula.
- Handy in deriving properties of the sequences.

Examples

- Generating function for sequence $\{\binom{n+1}{n} \mid n \geq 0\}$
  - We saw that $(1+x)^{-2} = \sum_{n\geq 0}(-1)^n\binom{n+1}{n}x^n$.
  - This gives $(1-x)^{-2}$ as generating function.
- Generating function for sequence $\{\binom{2n}{n} \mid n \geq 0\}$
  - We have $\binom{-1/2}{n} = (-1)^n\frac{1}{4^n}\binom{2n}{n}$ (Verify!)
  - This gives $(1-4x)^{-\frac{1}{2}}$ as generating function.

Identities

Identity-I

$$\sum_{r=0}^{n} (-1)^r \binom{r+\alpha-1}{r} \binom{\alpha}{n-r} = 0$$

for $\alpha \geq 0$ and integer $n \geq 1$.

Proof.

- We know that

$$\begin{aligned}
(1+x)^{\alpha} &= \sum_{s \geq 0} \binom{\alpha}{s} x^s \\
(1+x)^{-\alpha} &= \sum_{r \geq 0} (-1)^r \binom{r+\alpha-1}{r} x^r
\end{aligned}$$

Identities

- Multiplying the two, we get

$$\begin{aligned} 1 &= \sum_{n\geq 0}\left(\sum_{r+s=n}(-1)^r\binom{r+\alpha-1}{r}\binom{\alpha}{s}\right)x^n \\ &= \sum_{n\geq 0}\left(\sum_{r=0}^{n}(-1)^r\binom{r+\alpha-1}{r}\binom{\alpha}{n-r}\right)x^n \end{aligned}$$

- Therefore, for every $n \geq 1$:

$$\sum_{r=0}^{n}(-1)^r\binom{r+\alpha-1}{r}\binom{\alpha}{n-r} = 0.$$

Identities

Identity-II

$$\sum_{r=0}^{k} \binom{\alpha}{r} \binom{\beta}{k-r} = \binom{\alpha+\beta}{k}$$

for $k \geq 0$.

- We give three proofs of this identify!
- Combinatorial proof, inductive proof, and proof by generating functions.

Combinatorial proof

- This works when α and β are non-negative integers.
- Suppose we have $\alpha + \beta$ distinct objects and wish to choose k of them.
- The number of ways of choosing is $\binom{\alpha+\beta}{k}$.
- Counting another way, divide the $\alpha + \beta$ objects into two groups of α and β each.
- We can pick r objects from first group and $k - r$ objects from second group to choose k objects.
- This number is $\binom{\alpha}{r}\binom{\beta}{k-r}$.
- Hence,

$$\sum_{r=0}^{k} \binom{\alpha}{r}\binom{\beta}{k-r} = \binom{\alpha+\beta}{k}.$$

Proof by induction

- We do induction on β, this works only when β is non-negative integer.
- Base case is $\beta = 0$. Then we have:

$$\sum_{r=0}^{k} \binom{\alpha}{r} \binom{0}{k-r} = \binom{\alpha}{k}.$$

- Assume for β and consider $\beta + 1$:

$$\begin{aligned}
\sum_{r=0}^{k} \binom{\alpha}{r} \binom{\beta+1}{k-r} &= \sum_{r=0}^{k} \binom{\alpha}{r} \left\{ \binom{\beta}{k-r} + \binom{\beta}{k-r-1} \right\} \\
&= \sum_{r=0}^{k} \binom{\alpha}{r} \binom{\beta}{k-r} + \sum_{r=0}^{k-1} \binom{\alpha}{r} \binom{\beta}{k-r-1} \\
&= \binom{\alpha+\beta}{k} + \binom{\alpha+\beta}{k-1} \\
&= \binom{\alpha+\beta+1}{k}.
\end{aligned}$$

Proof by generating functions

- Generating function for $\{\binom{\alpha+\beta}{k} \mid k \geq 0\}$ is $(1+x)^{\alpha+\beta}$.
- We have:

$$
\begin{aligned}
(1+x)^{\alpha+\beta} &= (1+x)^{\alpha} \cdot (1+x)^{\beta} \\
&= \sum_{r \geq 0} \binom{\alpha}{r} x^r \cdot \sum_{s \geq 0} \binom{\beta}{s} x^s \\
&= \sum_{k \geq 0} \left\{ \sum_{r+s=k} \binom{\alpha}{r} \binom{\beta}{s} \right\} x^k \\
&= \sum_{k \geq 0} \left\{ \sum_{r=0}^{k} \binom{\alpha}{r} \binom{\beta}{k-r} \right\} x^k
\end{aligned}
$$

- Equating coefficients of x^k on both sides gives:

$$\binom{\alpha+\beta}{k} = \sum_{r=0}^{k} \binom{\alpha}{r} \binom{\beta}{k-r}.$$