CS201

Mathematics for Computer Science I

Lecture 7

Counting Problems

- How many subsets does a finite set have?
- How many mappings exist from finite set A to finite set B?
  - One-to-one mappings?
  - Onto mappings?
  - Bijections?
- How many bijections h exist from finite set A to A such that $h(a) \neq a$ for all $a \in A$?

Counting Problems

- Given 15 Red, 20 Blue, and 12 Green balls, how many sets can be formed by picking 3 Red, 10 Blue, and 5 Green balls:
  - When balls of a color are identical?
  - When balls of a color are all distinct?
- Given 10 Red and 20 Blue beads, how many distinct necklaces can be made from them?

Counting Problems

- Assume we are given unlimited numbers of Red, Blue, and Green balls. Balls of a color are identical.
- How many different sets can be formed containing n balls?
- In how many ways positive integer n can partitioned as a sum of smaller numbers?
  - Six ways of partitioning 5 are:

$$5 = 1+1+1+1+1 = 1+1+1+2 = 1+2+2 = 1+1+3 = 2+3 = 1+4$$

Counting Problems

- Consider the following algorithm for sorting n numbers:

```
merge-sort(Array A, size n) {
    if (|A| <= 1)
      return;
    Let m = |A|/2;
    merge-sort(A, m);
    merge-sort(A+m, n-m);
    merge(A, A+m);
}
```

  - merge takes two sorted arrays of numbers and merges then in a single sorted array
  - It takes cn steps when c is a constant and n is the sum of the sizes of input arrays.

Independent Events

- Suppose there are n variables: x_1, x_2, …, x_n.
- Variable x_i can take m_i possible values.
- A configuration is $(v_1, v_2, \ldots, v_n)$ where v_i is the value taken by x_i.
- We wish to count the total number of configurations.
- Further suppose that the event of x_i taking value v_i is independent of x_j taking value v_j for every $i \neq j$ and all values v_i and v_j.
- Viewing each x_i as a random variable, we can state above property as:

$$\Pr[x_i = v_i \ \& \ x_j = v_j] = \Pr[x_i = v_i] \cdot \Pr[x_j = v_j]$$

  for all $i \neq j$, v_i and v_j.
- In this settings, the total number of configurations equals the product of numbers of values x_i's can take, that is $\prod_{i=1}^{n} m_i$.

Independent Events: Examples

- How many subsets does a finite set have?
  - Let A be a finite set with $|A| = n$.
  - Define n variables x_1, x_2, $\ldots$, x_n with each x_i taking value 0 or 1.
  - Variable x_i represents ith element of A.
  - A configuration of these variables represents a subset of A: $x_i = 0$ means ith element is not in the subset, otherwise it is.
  - The total number of configurations equals the number of subsets of A.
  - Since the events are independent, this number is 2^n.

Independent Events: Examples

- How many mappings exist from finite set A to finite set B?
  - Let $|A| = n$ and $|B| = m$.
  - Define n variables x_1, x_2, ..., x_n with each x_i taking values from set B.
  - Variable x_i represents mapping of ith element of A.
  - A configuration of these variables represents a mapping of A to B: x_i denotes the element of B that the ith element of A is mapped to.
  - The total number of configurations equals the number of mappings of A to B.
  - Since the events are independent, this number is m^n.

Number of one-to-one mappings

- How many one-to-one mappings exist from finite set A to finite set B?
  - Let $|A| = n$ and $|B| = m$.
  - Define n variables x_1, x_2, $\ldots$, x_n with each x_i taking values from set B.
  - Variable x_i represents mapping of ith element of A.
  - A configuration of these variables represents a mapping of A to B: x_i denotes the element of B that the ith element of A is mapped to.
  - The total number of configurations representing one-to-one mappings equals the number of mappings of A to B.
  - The events are now not independent: if $x_i = c \in B$, then no other x_j can be equal to c.

Permutations

- How many total orderings exist on a set of m elements?
  - First element of ordering can be any of the m elements of the set.
  - Second element can be all elements except the one that is already at first place, so $m - 1$ possibilities.
  - Third element has $m - 2$ possibilities, and so on.
  - The total number of orderings is, therefore, $m!$.

Combinations

- How many ways exist to select n elements from a set A of m elements?
  - If $n > m$, the number is obviously 0.
  - Otherwise, order elements of set A and pick first n elements.
  - This gives $m!$ ways, however, there are many repetitions.
  - Ordering within first n and last $m - n$ elements is irrelevant.
  - Hence, dividing $m!$ by $n!$ and $(m-n)!$ gives the number of ways:

$$\binom{m}{n} = \frac{m!}{n!(m-n)!}.$$

  - $\binom{m}{n}$ is also written as ${}^mC_n$.

Example: Number of one-to-one mappings

- How many one-to-one mappings exist from finite set A to finite set B?
  - Let $|A| = n$ and $|B| = m$.
  - If $n > m$, there cannot be any one-to-one mapping from A to B.
  - If $n \leq m$, a one-to-one mapping maps n elements of A to some n elements of B.
  - The number of ways of selecting n elements of B equals $\binom{m}{n}$.
  - After selection, order these elements (in $n!$ ways) and map ith element of A to ith element of this ordering.
  - Thus, total number of one-to-one mappings from A to B equals

$$\binom{m}{n} \cdot n! = \frac{m!}{(m-n)!}.$$

  - This number also written as ${}^{m}P_{n}$.

Example: Number of bijections

- How many bijections exist between finite sets A and B?
  - If $|A| \neq |B|$, there is no bijection.
  - Otherwise, let $|A| = |B| = n$, and order elements of B.
  - Map ith element of A to ith element of B in the ordering.
  - This shows the number of bijections to be $n!$.