CS201

Mathematics for Computer Science I

Lecture 9

Binomial Theorem

Binomial Theorem

$$(1+x)^m = \sum_{i=0}^{m} \binom{m}{i} x^i.$$

- Consequences:

$$\begin{aligned} 2^m &= \sum_{i=0}^{m} \binom{m}{i} \\ 0 &= \sum_{i=0}^{m} (-1)^i \binom{m}{i} \\ \binom{m}{k} &= \binom{m-1}{k} + \binom{m-1}{k-1} \end{aligned}$$

for $1 \leq k \leq m$.

Binomial Theorem

Proof.

- Multiply out the m operands and collect terms of the kind x^i.
- Since the product is a degree m polynomial in x, there will be exactly $m+1$ terms.
- For the term x^i, consider a sequence of m slots and fill i of them with x and rest with 1.
- Number of ways of choosing i slots for x's is exactly $\binom{m}{i}$.
- Hence,

$$(1+x)^m = \sum_{i=0}^{m} \binom{m}{i} x^i.$$

Counting size of union set

- Suppose A_1, A_2, ..., A_n are finite sets.
- We wish to count the size of the set $A_1 \cup A_2 \cup \cdots \cup A_n$.
- $|A_1| + |A_2| + \cdots + |A_n|$ is clearly an upper bound on the size.
- However, it may not be equal since sets may intersect.
- $|A_i \cap A_j|$ is the common elements between A_i and A_j.
- To avoid overcounting common elements, we can subtract $\sum_{1 \leq i < j \leq n} |A_i \cap A_j|$ from $\sum_{1 \leq i \leq n} |A_i|$.
- However, this may undercount since elements common to three sets will not be counted.

Inclusion-Exclusion Principle

Theorem

For finite sets A_1, A_2, ..., A_n:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|.$$

Inclusion-Exclusion Principle

Proof.

- Consider an element a that is contained in exactly m A_i's.
- It is counted $\binom{m}{1}$ times in $\sum_{1 \le i \le n} |A_i|$, $\binom{m}{2}$ times in $\sum_{1 \le i < j \le n} |A_i \cap A_j|$, $\binom{m}{3}$ times in $\sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k|$ etc.
- Hence its contribution to the sum is:

$$\binom{m}{1} - \binom{m}{2} + \binom{m}{3} - \cdots + (-1)^{m-1}\binom{m}{m} = \binom{m}{0} = 1,$$

by Binomial Theorem.

Counting Onto Mappings

- How many onto mappings exist from finite set A to finite set B?
  - If $|B| > |A|$ then no onto mapping exists.
  - Let $|B| = m \leq n = |A|$.
  - For $b \in B$, define F_b to be the set of mappings from A to B that map A to $B \backslash \{b\}$.
  - Then, the number of onto mappings equals $m^n - |\cup_{b \in B} F_b|$.
  - For any subset $C \subseteq B$, $\cap_{b \in C} F_b$ contains all mappings that map A to $B \backslash C$.
  - Hence,

$$|\cap_{b \in C} F_b| = (m - |C|)^n.$$

Counting Onto Mappings

- By Inclusion-Exclusion Principle:

$$
\begin{aligned}
|\cup_{b \in B} F_b| &= \sum_{b \in B} |F_b| - \sum_{b,c \in B, b \neq c} |F_b \cap F_c| + \\
&\quad \cdots + (-1)^{m-1} |\cap_{b \in B} F_b| \\
&= \binom{m}{1}(m-1)^n - \binom{m}{2}(m-2)^n + \\
&\quad \cdots + (-1)^{m-1}\binom{m}{m}(m-m)^n \\
&= \sum_{i=1}^{m} (-1)^{i-1}\binom{m}{i}(m-i)^n
\end{aligned}
$$

Counting Onto Mappings

- Therefore, number of onto mappings are:

$$
\begin{aligned}
m^n - |\cup_{b\in B} F_b| &= m^n - \sum_{i=1}^{m} (-1)^{i-1}\binom{m}{i}(m-i)^n \\
&= \sum_{i=0}^{m} (-1)^{i}\binom{m}{i}(m-i)^m.
\end{aligned}
$$

Derangements

- How many bijections are there from the set $[1, n] = \{1, 2, 3, \ldots, n\}$ to $[1, n]$ such that $f(i) \neq i$ for every i?
- Such a mapping is called a derangement.
- Define F_i to be the set of bijections that map i to i.
- The number of derangements equals

$$n! - | \cup_{1 \leq i \leq n} F_i|.$$

- For any subset $J \subseteq [1, n]$, $\cap_{i \in J} F_i$ is the set of bijections that map every $i \in J$ to itself.
- Hence,

$$| \cap_{i \in J} F_i| = (n - |J|)!.$$

Derangements

- By Inclusion-Exclusion Principle,

$$
\begin{aligned}
|\cup_{1\le i\le n} F_i| &= \sum_{1\le i\le n} |F_i| - \sum_{1\le i<j\le n} |F_i \cap F_j| + \\
&\quad \cdots + (-1)^{n-1} |\cap_{1\le i\le n} F_i| \\
&= \binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \\
&\quad \cdots + (-1)^{n-1}\binom{n}{n}(n-n)! \\
&= \sum_{i=1}^{n} (-1)^{i-1}\binom{n}{i}(n-i)! \\
&= \sum_{i=1}^{n} (-1)^{i-1}\frac{n!}{i!(n-i)!}(n-i)! \\
&= \sum_{i=1}^{n} (-1)^{i-1}\frac{n!}{i!}
\end{aligned}
$$

Derangements

- Therefore, total number of derangements are:

$$
\begin{aligned}
n! - |\cup_{1 \leq i \leq n} F_i| &= n! - \sum_{i=1}^{n} (-1)^{i-1} \frac{n!}{i!} \\
&= n! (\sum_{i=0}^{n} (-1)^{i} \frac{1}{i!}).
\end{aligned}
$$