CS201

Mathematics for Computer Science I

Lecture 8

Counting Onto Mappings

- How many onto mappings exist from finite set A to finite set B?
  - If $|B| > |A|$ then no onto mapping exists.
  - Let $|B| = m \leq n = |A|$.
  - Total number of mappings from A to B are m^n, but not all are onto.
  - A mapping that is not onto maps A within a subset of size $m-1$ of B.
  - There are $\binom{m}{m-1}$ ways of selecting a subset of size $m-1$ of B.
  - Number of mappings to such a subset of B from A are $(m-1)^n$.
  - Are number of onto mappings $m^n - \binom{m}{m-1}(m-1)^n$?
  - No! Because a mapping can belong to more than one such collection.

Counting Number of Ways

- Given 15 Red, 20 Blue, and 12 Green balls, how many sets can be formed by picking 3 Red, 10 Blue, and 5 Green balls?
  - When balls of a color are identical then there is only one way of forming the set.
  - When balls of a color are all distinct, then Red balls can be chosen in $\binom{15}{3}$ ways, Blue balls can be chosen in $\binom{20}{10}$ ways, and Green balls can be chosen in $\binom{12}{5}$ ways.
  - Thus the total number of sets is

$$\binom{15}{3} \cdot \binom{20}{10} \cdot \binom{12}{5}.$$

Counting Number of Ways

- How many sets can be formed containing n balls from an unlimited number of identical Red, Blue, and Green balls?
  - Represent any choice by lining up Red balls, followed by Blue and then Green balls.
  - Put two markers, one between Red and Blue balls, and other between Blue and Green balls.
  - View the sequence as $n + 2$ slots occupied by two markers and Red balls before first marker, Blue balls between two markers, and Green balls after second marker.
  - Once the location of two markers are fixed, there is only one way to fill up remaining slots.
  - Hence, the total number of such sequences is exactly $\binom{n+2}{2}$.

Counting Number of Ways

- Given an $m \times n$ grid, how many paths are there from bottom-left to top-right when one can move either right or up?
  - Represent a path by a sequence of right and up moves.
  - There will be exactly n right and m up moves, so $n+m$ moves overall.
  - Once location of right moves is fixed, remaining locations will have up moves.
  - Each sequence is a valid path and every valid path corresponds to such a sequence.
  - Hence, total number of paths is exactly $\binom{n+m}{n}$.

Counting Number of Ways

- Given 20 Red and 20 Blue beads, how many distinct necklaces can be made from them?
  - View the necklace as a sequence with 40 slots, twenty of which have Blue beads and remaining Red beads.
  - Once the location of Blue beads is fixed, there is only one way to fill the remaining slots.
  - There are exactly $\binom{40}{20}$ ways of picking slots for Blue beads.
  - However, it does not give the number of necklaces due to rotational symmetries.
  - Even dividing the number by 40 does not count properly.

Counting Number of Ways

- In how many ways positive integer n can partitioned as a sum of smaller numbers?
  - A partition can have different number of numbers.
  - We can allow 0 and fix the number of numbers in the partition to be exactly n.
  - Also, by ordering the numbers in the partition, we can view it as a sequence with $2n-1$ slots containing $n-1$ markers denoting transition, slots before first marker containing 0, slots between marker no i and $i+1$ containing number i, and slots after marker no $n-1$ containing number $n-1$.
  - Number of such sequences is $\binom{2n-1}{n-1}$.
  - However, not all of them add up to n.