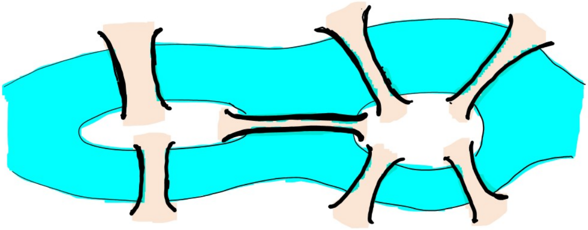
CS201

Mathematics for Computer Science I

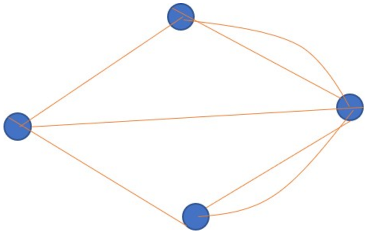
Lecture 17

Seven Bridges of Königsberg



Seven Bridges of Königsberg

- Is there a way to traverse all the bridges without repetition?
- Formulate as graph problem: graph is not simple as there are multiple edges between two vertices.
- Such graphs are called multigraphs.
- Find a path that passes through all edges exactly once.



Euler Walks

An Euler walk of a graph is a path that passes through every edge exactly once.

Theorem (Euler)

A multigraph $G = (V, E)$ has an Euler walk if and only if the graph is connected and at most two vertices have odd degree.

Proof

- Let $W = u_0, u_1, \ldots, u_k$ be an Euler walk of graph G.
- Every vertex of G of degree > 0 must be on the path since Euler walk visits all edges.
- Let $v \in V$ occur ℓ times in the walk.
- If $v \neq u_0, u_k$, then $\deg(v) = 2\ell$.
- If $v = u_0 = u_k$, then $\deg(v) = 2\ell - 2$.
- If $v \in \{u_0, u_k\}$ and $u_0 \neq u_k$, then $\deg(v) = 2\ell - 1$.

Proof

- Suppose G has two vertices of odd degree: v_1 and v_2.
- Start a walk from v_1 and keep extending it as long as an unvisited edge can be found.
- Suppose the walk is extended up to vertex u and that it cannot be extended any further.
- Then $\deg(u)$ must be odd and $u \neq v_1$.
- Hence, $u = v_2$.
- Denote the walk by W_1 and remove all the edges from G that are visited by W_1.
- The degree of all vertices becomes even now.

Proof

- If no edges left, we are done.
- Otherwise, pick any vertex with non-zero degree and remove one of the edges incident on it.
- We now have exactly two vertices with odd degree and so repeat the construction of a walk as before.
- This walk will end in the other odd degree vertex.
- Add to the walk the deleted edge, thus making it end at starting vertex.
- Denote the walk by W_2 and remove all edges from G visited by W_2, and repeat.
- Eventually, we get a sequence of walks W_1, W_2, $\ldots$, W_r such that (i) their edge sets are disjoint, (ii) union of their edge sets equals E.

Proof

- Moreover, all walks except W_1 start and end at the same vertex.
- Extend W_1 as follows: take a vertex in W_1 that occurs as first vertex of another walk W_i.
- Insert entire W_i in W_1 at this point.
- Repeat until no other walk is left – since G is connected, it can be done.
- The case when no vertex of G has odd degree is already addressed above – we get an Euler walk with same start and end vertex.

Characterization of Planar Graphs

- K_5 is not planar.
  - Assume that K_5 is planar.
  - Recall that for a connected planar graph $G = (V, E)$: $|V| - |E| + f = 1$ where f is the number of faces in a planar embedding of G.
  - K_5 has $|V| = 5$, $|E| = 10$.
  - We also showed in last lecture that $3f \leq 2|E|$.
  - This can be improved by observing that in a planar embedding of a connected graph, the outer boundary will have at least three edges and each of these will be an external edge.
  - Hence, $3f \leq 2|E| - 3$.
  - This implies that $f \leq 17/3$, implying $f \leq 5$.
  - Therefore, $|V| - |E| + f \leq 5 - 10 + 5 \leq 0 < 1$.

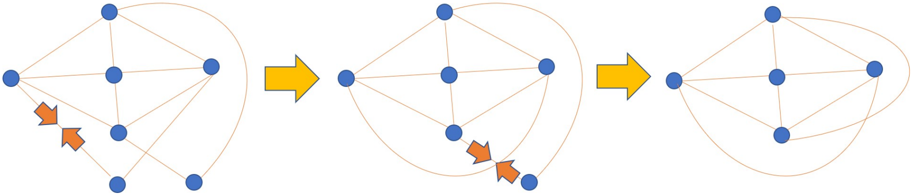
Characterization of Planar Graphs

- $K_{3,3}$ is not planar.
  - Assume that $K_{3,3}$ is planar.
  - $K_{3,3}$ has $|V| = 6$, $|E| = 9$.
  - Faces in a planar embedding of $K_{3,3}$ are bounded by at least four vertices.
  - Hence, $4f \leq 2|E| - 3$.
  - This implies that $f \leq 15/4$ implying $f \leq 3$.
  - Therefore, $|V| - |E| + f \leq 6 - 9 + 3 \leq 0 < 1$.

Characterization of Planar Graphs

Minors

H is a minor of *G* if *H* can be obtained from *G* by removing and contracting edges. Edge contraction is operation in which two endpoint of the edge are made to coincide.



Characterization of Planar Graphs

Kuratowski's Theorem

Graph G is planar if and only if neither K_5 nor $K_{3,3}$ are its minors.