CS201

Mathematics for Computer Science I

Lecture 16

Map Coloring Problem

- Given a map of countries, every country is to be colored such that no two neighboring countries get the same color. What is the minimum number of colors needed?
- It can be formulated as a graph problem:
  - Represent every country as a vertex.
  - Join two vertices with an edge if corresponding countries are neighboring.
  - Minimum number of colors required to color every vertex so that no two adjacent vertices get same color.

Planar Graphs

- Graph K_n requires n colors.
- However, a map cannot give rise to K_n for any $n \geq 5$.
- Follows from the special nature of map graphs: they can be drawn on a plane.

Planar Graphs

Graph $G = (V, E)$ is a planar graph if G can be drawn on a plane such that no two edges cross. A drawing of G on a plane is called planar embedding of G.

Faces of Planar Graphs

Given a planar embedding of G, a face of G is a region in the embedding bounded on all sides by edges and vertices of G.

- An edge of G bounds at most two faces.
- An edge that bounds less than two faces is called an external edge of G.

Euler's Theorem

Theorem

Let f be the number of faces in a planar embedding of a connected graph $G = (V, E)$. Then: $|V| - |E| + f = 1$.

Proof.

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- Proof is by induction on $|E|$.
- Base case of $|E| = 0$ is trivial.
- Assume for $|E| = m - 1$ and consider $|E| = m$.

Euler's Theorem

- Let e be an external edge of G.
- Suppose e bounds one face of G.
- Then, on removing e, G remains connected but loses one face.
- Then, $H = (V, E \setminus \{e\})$ is a connected planar graph with $m - 1$ edges.
- By induction hypothesis, $|V| - (|E| - 1) + (f - 1) = 1$.
- This gives: $|V| - |E| + f = 1$.

Euler's Theorem

- Suppose e does not bound any face of G.
- Then, on removing e, G gets disconnected but does not lose any face.
- Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be the two disjoint components with f_1 and f_2 faces respectively.
- Since $|E_1|, |E_2| < m$, by induction hypothesis:

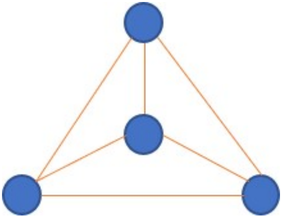
$$|V_1| - |E_1| + f_1 = 1 = |V_2| - |E_2| + f_2.$$

- Hence,

$$|V| - |E| + f = |V_1| + |V_2| - |E_1| - |E_2| - 1 + f_1 + f_2 = 1 + 1 - 1 = 1.$$

Back to Map Coloring Problem

- Coloring of a graph is assignment of colors to vertices of the graph so that no two adjacent vertices get the same color.
- What is the minimum number of colors required to color a planar graph?
- K_4 is a planar graph, hence minimum number of colors required is ≥ 4:



Five Color Theorem

Theorem

Every planar graph can be colored using five colors.

Proof.

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- Let $G = (V, E)$ be a planar graph with $|V| = n$ and f faces.
- Proof is by induction on n.
- Base case of $n \leq 5$ is trivially true.
- Assume for $n - 1$ and consider G.

Five Color Theorem

- Since G is planar, $|E| \leq 3n - 3$:
  - Let r be the number of pairs (e, F) for $e \in E$ and F a face of G such that e bounds it.
  - Since each edge bounds at most two faces, $r \leq 2|E|$.
  - Since each face is bounded by at least three edges, $r \geq 3f$.
  - Therefore, $|E| \leq |E| + (2|E| - 3f) \leq 3(|E| - f) = 3n - 3$.
- If all vertices of G have degree ≥ 6, then $6n \leq 2|E|$ or $|E| \geq 3n$, not possible.

Five Color Theorem

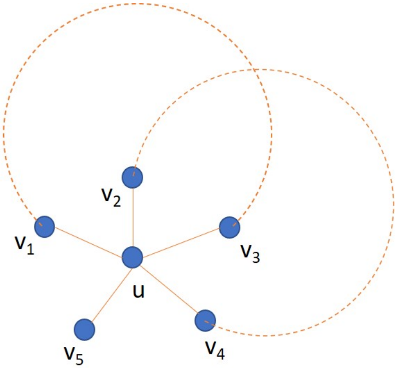
- Let u be a vertex of G with $\deg(u) \leq 5$.
- Remove u from G and let H be resulting planar graph.
- By induction hypothesis, H can be colored with five colors.
- Color all vertices of G except u using coloring of H.
- If $\deg(u) \leq 4$, u can be colored properly and thus G is five colorable.

Five Color Theorem

- Let $\deg(u) = 5$.
- If five adjacent vertices to u do not have distinct colors, we are done.
- Let the neighbors of u be v_1, ..., v_5 ordered clockwise and let i denote color of v_i.
- An (a, b)-path, for $1 \leq a < b \leq 5$, starts from v_a, ends in v_b, and only passes through vertices of color either a or b.
- In an (a, b)-path, colors of vertices alternate between a and b.
- If there is no (a, b)-path in G for some $1 \leq a < b \leq 5$, then color of v_a can be changed to b by flipping colors of vertices on all paths starting from v_a and passing through vertices of colors only a or b.
- Then, u can be colored with a and we are done.

Five Color Theorem

- Now assume that G has both $(1,3)$-path and $(2,4)$-path.
- This is not possible since these two paths must intersect due to planarity of G.



Four Color Theorem

Appel-Haken, 1976

Every planar graph can be colored using four colors.

- Proved after a lot of efforts.
- Required computer simulation to rule out certain configurations.