CS201

Mathematics for Computer Science I

Lecture 5

Infinite Sets of Cardinality $\aleph_0$

- Suppose A is infinite with $|A| = \aleph_0$.
- Then there is a bijection between A and $\mathbb{N}$.
- Let this be given by mapping f, $f : A \mapsto \mathbb{N}$.
- Define relation R on A as:

$$aRb \text{ iff } f(a) \leq f(b).$$

- R is a total order on A.

Infinite Sets of Cardinality $\aleph_1$

- If $|A| = \aleph_1$, there is a bijection between A and $\mathbb{R}$.
- Use this bijection in the same way as above to define a total order on A.
- But this cannot be done for all infinite cardinalities.
- Axiom of Choice ensures that it is possible.

Axiom of Choice

Axiom of Choice

Let A be a set whose elements are non-empty subsets of set U. Then there exists a mapping f, $f : A \mapsto U$ such that $f(X) \in X$ for all $X \in A$.

- The axiom says that it is possible to 'pick' one element from every set in A, no matter what A is.
- This axiom has two alternative forms which are also very useful.

Well-Ordering

Definition

Relation R on A is a well-ordering if for every $B \subseteq A$, (B, R) has a minimal element.

- $\leq$ is a well-ordering on $\mathbb{N}$ but not on $\mathbb{Z}$.
- A well-ordering is a total order also:
  - Let R be a well-ordering on A, and consider $a, b \in A$.
  - There is a minimal element of $(\{a, b\}, R)$ since R is a well-ordering.
  - Hence, either aRb or bRa.

Well-Ordering Principle

Well-Ordering Principle

There exists a well-ordering on every set.

- Well-Ordering Principle immediately implies that there exists a total order on every set.

Zorn's Lemma

Definition (Upper Bound)

Let R be a partial order on A and C be a chain of (A, R). An upper bound of C is an element $u \in A$ such that for every $a \in C$, aRu.

Zorn's Lemma

Let R be a partial order on A such that every chain C of (A, R) has an upper bound. Then (A, R) has a maximal element.

Three Equivalences

Axiom of Choice

$\Downarrow$

Zorn's Lemma

$\Downarrow$

Well-Ordering Principle

$\Downarrow$

Axiom of Choice

Axiom of Choice ⇒ Zorn's Lemma

- Let R be a partial order on A such that every chain of (A, R) has an upper bound.
- For the sake of contradiction, assume that there is no maximal element of (A, R).
- For any well-ordered subset C of A, let u_C be an upper bound of C, and define the set

$$X_C = \{v \in A \setminus C \mid u_C R v\}.$$

- By our assumption, X_C is non-empty.
- Define

$$X = \{X_C \mid C \text{ is a well-ordered subset of } A\}.$$

- By Axiom of Choice, there exists a function $f, f : X \mapsto A$, such that $f(X_C) \in X_C$.

Axiom of Choice ⇒ Zorn's Lemma

- Let $g(C) = f(X_C)$.
- By definition of X_C and f it follows that:

$$g(C) \notin C \text{ and for every } a \in C\text{: } aRg(C).$$

- Define a g-set to be a subset G of A such that G is well-ordered and for every $a \in G$:

$$g(\{c \mid cRa \text{ and } c \in G \text{ and } c \neq a\}) = a.$$

- Such sets exists. For example, $\{g(\emptyset)\}$ is a g-set.

Axiom of Choice $\Rightarrow$ Zorn's Lemma

Definition (Initial Segment)

Let X and Y be two well-ordered subsets of A. X is an initial segment of Y if $X \subseteq Y$ and if $a \in X$ then for every $b \in Y$ such that bRa, $b \in X$.

- Let G and H be two g-sets.
- We show that either G is an initial segment of H, or vice versa.
- Consider sets X that are initial segments of both G and H and let W be the union of all such X's.
- Clearly, $W \subseteq G \cap H$.

Axiom of Choice ⇒ Zorn's Lemma

- W is also an initial segment of both G and H:
  - Consider $a \in W$ and $b \in G$ such that bRa.
  - Element a belongs to a set X that is an initial segment of both G and H, therefore $b \in X$.
  - Since W is union of all such initial segments, $b \in W$.
  - Same argument for H.
- Clearly, W is the largest initial segment of G and H.