CS201

Mathematics for Computer Science I

Lecture 13

Pigeon Hole Principle

If $n+1$ objects are kept in n boxes then at least one box has more than one object.

- Obvious statement, but has non-obvious applications.

Example 1

- Given any $n + 1$ numbers from the set $[1, 2n] = \{1, 2, \ldots, 2n\}$, there exists one number that is a multiple of another.
  - Represent any number m from the set $[1, 2n]$ as $m = 2^k s$ where s is odd.
  - Clearly, $s \in \{1, 3, 5, \ldots, 2n - 1\}$.
  - There are exactly n possibilities for s.
  - By Pigeon Hole Principle, given $n + 1$ numbers from $[1, 2n]$, at least two will have the same s.
  - One of these numbers is a multiple of other.

Example 2

- Let A be a finite set with partial order $\leq$ defined on A and $|A| \geq n^2 + 1$. Then A has either a chain of size $n+1$ or an antichain of size $n+1$.
- An antichain is a subset of A such that no two elements of the subset are related.
  - Suppose the size of largest chain of A is k.
  - Let $B_1 \subseteq A$ be the set of all minimal elements of A.
  - Set B_1 is clearly an antichain and also a maximal one.

Example 2

- Proof continued...
  - Now consider the set $A \backslash B_1$ and let B_2 be the set of all minimal elements of $A \backslash B_1$.
  - Set B_2 is also a maximal antichain.
  - Continuing this way, suppose we get antichains B_1, B_2, $\ldots$, B_r and $A = \cup_{i=1}^{r} B_i$.
  - Consider a maximal chain C of A.
  - The smallest element of C will be in B_1, next smallest in B_2, and so on.
  - Hence, elements of C will be present in first $|C|$ antichains.

Example 2

- Proof continued...
  - Since the longest chain of A has size k by assumption, $r = k$.
  - Therefore, A is a union of k disjoint antichains.
  - By Pigeon Hole Principle, at least one antichain has size

$$\geq \frac{n^2+1}{k}.$$

  - Hence, either $k \geq n+1$ or $\frac{n^2+1}{k} \geq n+1$.

Two Versions

Ramsey Theorem for Graphs

For any $c, n_1, n_2, \ldots, n_c \geq 1$, there exists a number $N(n_1, n_2, \ldots, n_c) > 0$ such that for any set X with $|X| \geq N(n_1, n_2, \ldots, n_c)$, and any mapping $f : X \times X \mapsto \{1, 2, \ldots, c\}$, there exists a $\tilde{c}$, $1 \leq \tilde{c} \leq c$ and subset $Y \subseteq X$, $|Y| = n_{\tilde{c}}$, with $f(Y \times Y) = \tilde{c}$.

Ramsey Theorem General Form

For any $c, n_1, n_2, \ldots, n_c, k \geq 1$, there exists a number $N(n_1, n_2, \ldots, n_c, k) > 0$ such that for any set X with $|X| \geq N(n_1, n_2, \ldots, n_c, k)$, and any mapping $f : X^k \mapsto \{1, 2, \ldots, c\}$, there exists a $\tilde{c}$, $1 \leq \tilde{c} \leq c$ and a subset $Y \subseteq X$, $|Y| = n_{\tilde{c}}$, with $f(Y^k) = \tilde{c}$.

Applications

- Non-trivial generalization of Pigeon Hole Principle ($n_1 = 2 = n_2 = \cdots = n_c$, $k = 1$).
- Any group of six persons either has three mutual acquaintances, or three mutual strangers ($c = 2$, $n_1 = 3 = n_2$, $k = 2$).
- Schur's Theorem: for any $m > 1$, there exists a number N such that for any prime $p \geq N$, equation $x^m + y^m = z^m \mod p$ has a solution.
- Erdos-Szekeres Theorem: for any $m \geq 4$, there exists a number N such that given any N points on a place with no three points on a line, there exists a convex polygon of m points.

Applications

- Schur's Theorem: to be done later in the course.
- Erdos-Szekeres Theorem: Choose $c = 2$, $n_1 = m$, $n_2 = 5$, $k = 4$ and $N = N(m, 5, 4)$ in the General Form.
  - Consider a set X of N points on the plane with no three points on a line.
  - Define coloring of X^4 as: $f(p_1, p_2, p_3, p_4) = 1$ if the four points make a convex polygon, 2 otherwise.

Applications

- Proof of Erdos-Szekeres Theorem:
  - By the Ramsey Theorem, there exists a set $Y \subseteq X$, such that $|Y| = m$ and $f(Y^4) = 1$, or $|Y| = 5$ and $f(Y^4) = 2$.
  - Second is not possible since four out of any five points of X make a convex polygon.
  - So there are m points such that any four of them make a convex polygon.
  - Then all m points must make a convex polygon.