

CS201

MATHEMATICS FOR COMPUTER SCIENCE I

LECTURE 5

# INFINITE SETS OF CARDINALITY $\aleph_0$

- Suppose  $A$  is infinite with  $|A| = \aleph_0$ .
- Then there is a bijection between  $A$  and  $\mathbb{N}$ .
- Let this be given by mapping  $f$ ,  $f : A \mapsto \mathbb{N}$ .
- Define relation  $R$  on  $A$  as:

$$aRb \text{ iff } f(a) \leq f(b).$$

- $R$  is a total order on  $A$ .

# INFINITE SETS OF CARDINALITY $\aleph_1$

- If  $|A| = \aleph_1$ , there is a bijection between  $A$  and  $\mathbb{R}$ .
- Use this bijection in the same way as above to define a total order on  $A$ .
- But this cannot be done for all infinite cardinalities.
- **Axiom of Choice** ensures that it is possible.

# AXIOM OF CHOICE

## AXIOM OF CHOICE

Let  $A$  be a set whose elements are non-empty subsets of set  $U$ . Then there exists a mapping  $f$ ,  $f : A \mapsto U$  such that  $f(X) \in X$  for all  $X \in A$ .

- The axiom says that it is possible to ‘pick’ one element from every set in  $A$ , no matter what  $A$  is.
- This axiom has two alternative forms which are also very useful.

# WELL-ORDERING

## DEFINITION

Relation  $R$  on  $A$  is a **well-ordering** if for every  $B \subseteq A$ ,  $(B, R)$  has a minimal element.

- $\leq$  is a well-ordering on  $\mathbb{N}$  but not on  $\mathbb{Z}$ .
- A well-ordering is a total order also:
  - ▶ Let  $R$  be a well-ordering on  $A$ , and consider  $a, b \in A$ .
  - ▶ There is a minimal element of  $(\{a, b\}, R)$  since  $R$  is a well-ordering.
  - ▶ Hence, either  $aRb$  or  $bRa$ .

# WELL-ORDERING PRINCIPLE

## WELL-ORDERING PRINCIPLE

There exists a well-ordering on every set.

- Well-Ordering Principle immediately implies that there exists a total order on every set.

# ZORN'S LEMMA

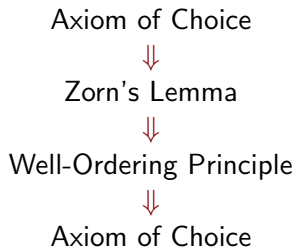
## DEFINITION (UPPER BOUND)

Let  $R$  be a partial order on  $A$  and  $C$  be a chain of  $(A, R)$ . An **upper bound** of  $C$  is an element  $u \in A$  such that for every  $a \in C$ ,  $aRu$ .

## ZORN'S LEMMA

Let  $R$  be a partial order on  $A$  such that every chain  $C$  of  $(A, R)$  has an upper bound. Then  $(A, R)$  has a maximal element.

# THREE EQUIVALENCES





## AXIOM OF CHOICE $\Rightarrow$ ZORN'S LEMMA

- Let  $R$  be a partial order on  $A$  such that every chain of  $(A, R)$  has an upper bound.
- For the sake of contradiction, assume that there is no maximal element of  $(A, R)$ .
- For any well-ordered subset  $C$  of  $A$ , let  $u_C$  be an upper bound of  $C$ , and define the set

$$X_C = \{v \in A \setminus C \mid u_C R v\}.$$

- By our assumption,  $X_C$  is non-empty.
- Define

$$X = \{X_C \mid C \text{ is a well-ordered subset of } A\}.$$

- By Axiom of Choice, there exists a function  $f, f : X \mapsto A$ , such that  $f(X_C) \in X_C$ .

# AXIOM OF CHOICE $\Rightarrow$ ZORN'S LEMMA

- Let  $g(C) = f(X_C)$ .
- By definition of  $X_C$  and  $f$  it follows that:

$$g(C) \notin C \text{ and for every } a \in C: aRg(C).$$

- Define a  $g$ -set to be a subset  $G$  of  $A$  such that  $G$  is well-ordered and for every  $a \in G$ :

$$g(\{c \mid cRa \text{ and } c \in G \text{ and } c \neq a\}) = a.$$

- Such sets exists. For example,  $\{g(\emptyset)\}$  is a  $g$ -set.

# AXIOM OF CHOICE $\Rightarrow$ ZORN'S LEMMA

## DEFINITION (INITIAL SEGMENT)

Let  $X$  and  $Y$  be two well-ordered subsets of  $A$ .  $X$  is an **initial segment** of  $Y$  if  $X \subseteq Y$  and if  $a \in X$  then for every  $b \in Y$  such that  $bRa$ ,  $b \in X$ .

- Let  $G$  and  $H$  be two  $g$ -sets.
- We show that either  $G$  is an initial segment of  $H$ , or vice versa.
- Consider sets  $X$  that are initial segments of both  $G$  and  $H$  and let  $W$  be the union of all such  $X$ 's.
- Clearly,  $W \subseteq G \cap H$ .

# AXIOM OF CHOICE $\Rightarrow$ ZORN'S LEMMA

- $W$  is also an initial segment of both  $G$  and  $H$ :
  - ▶ Consider  $a \in W$  and  $b \in G$  such that  $bRa$ .
  - ▶ Element  $a$  belongs to a set  $X$  that is an initial segment of both  $G$  and  $H$ , therefore  $b \in X$ .
  - ▶ Since  $W$  is union of all such initial segments,  $b \in W$ .
  - ▶ Same argument for  $H$ .
- Clearly,  $W$  is the largest initial segment of  $G$  and  $H$ .