CS201

Mathematics for Computer Science I

Lecture 14

Two Versions

Ramsey Theorem for Graphs

For any $c, n_1, n_2, \ldots, n_c \geq 1$, there exists a number $N(n_1, n_2, \ldots, n_c) > 0$ such that for any set X with $|X| \geq N(n_1, n_2, \ldots, n_c)$, and any mapping $f : X \times X \mapsto \{1, 2, \ldots, c\}$, there exists a $\tilde{c}$, $1 \leq \tilde{c} \leq c$ and subset $Y \subseteq X$, $|Y| = n_{\tilde{c}}$, with $f(Y \times Y) = \tilde{c}$.

Ramsey Theorem General Form

For any $c, n_1, n_2, \ldots, n_c, k \geq 1$, there exists a number $N(n_1, n_2, \ldots, n_c, k) > 0$ such that for any set X with $|X| \geq N_{n_1, n_2, \ldots, n_c, k}$, and any mapping $f : X^k \mapsto \{1, 2, \ldots, c\}$, there exists a $\tilde{c}$, $1 \leq \tilde{c} \leq c$ and a subset $Y \subseteq X$, $|Y| = n_{\tilde{c}}$, with $f(Y^k) = \tilde{c}$.

Proof of Ramsey Theorem for Graphs

- View f as assigning one of c colors to elements of set $X \times X$.
- Proof is by induction on c.
- For $c = 1$, $N(n_1) \leq n_1$.
- We need to prove the case $c = 2$ also as part of base step.
- This is because induction step requires to use $c = 2$ case.

Proof

- To prove $c = 2$ case, we do induction on $n_1 + n_2$:
  - Trivially true when $n_1 + n_2 = 2$.
  - Assume for $n_1 + n_2 - 1$ and consider for $n_1 + n_2$.
  - Consider X of size $N(n_1, n_2 - 1) + N(n_1 - 1, n_2)$.
  - Take $a \in X$ and let

$$Y_{c_1} = \{b \mid b \neq a, f(a,b) = c_1\}, Y_{c_2} = \{b \mid b \neq a, f(a,b) = c_2\}.$$

  - We have $|Y_{c_1}| \geq N(n_1 - 1, n_2)$ or $|Y_{c_2}| \geq N(n_1, n_2 - 1)$.
  - If former, then by induction, Y_{c_1} either has a subset Z_1 of size $n_1 - 1$ with $f(Z_1 \times Z_1) = c_1$ or a subset Z_2 of size n_2 with $f(Z_2 \times Z_2) = c_2$.
  - In the first case, $Z_1' = \{a\} \cup Z_1$ is a set of size n_1 with $f(Z_1' \times Z_1') = c_1$.
  - Same argument for the other case.

Proof

- Assume true for $c - 1$ colors.
- Consider any X of size $N(n_1, n_2, \ldots, n_{c-2}, N(n_{c-1}, n_c))$.
- Define mapping

$$g(a, b) = \begin{cases} f(a, b) & \text{if } f(a, b) \leq c - 2 \\ c - 1 & \text{if } f(a, b) = c - 1, c \end{cases}$$

Proof

- By induction hypothesis, there exists a $\hat{c}$, $1 \leq \hat{c} \leq c-1$ and subset $Y \subseteq X$:
$$|Y| = \begin{cases} n_{\hat{c}} & \text{if } \hat{c} \leq c-2 \\ N(n_{c-1}, n_c) & \text{if } \hat{c} = c-1 \end{cases}$$
with $g(Y \times Y) = \hat{c}$.
- If $\hat{c} \leq c-2$, we are done.
- If $\hat{c} = c-1$, we have $|Y| = N(n_{c-1}, n_c)$ and $g(Y \times Y) = c-1$.
- Therefore, $f(Y \times Y) \in \{c-1, c\}$ and $|Y| = N(n_{c-1}, n_c)$.
- By induction hypothesis for $c = 2$, we get that either there exists a set $Z \subseteq Y$, $|Z| = n_{c-1}$ and $f(Z \times Z) = c-1$ or there exists a set $Z \subseteq Y$, $|Z| = n_c$ and $f(Z \times Z) = c$.
- This completes the proof.

Counting Number of Primes

- Let $\pi(n)$ equal the number of prime numbers $\leq n$.
- How do we estimate $\pi(n)$?
- This question led to development of number theory and complex analysis!
- It is done through L-functions.

L-FUNCTIONS

- Given an infinite sequence of numbers a_0, a_1, $\ldots$, we defined its generating function as:

$$G(x) = \sum_{n \geq 0} a_n x^n.$$

- Define its L-function as:

$$L(z) = \sum_{n \geq 1} a_n n^{-z} = \sum_{n \geq 1} \frac{a_n}{n^z}.$$

- It is taken to be defined over complex numbers.

L-FUNCTIONS

- Even simple *L*-functions have very interesting properties.
- The simplest one is:

$$\zeta(z) = \sum_{n \geq 1} \frac{1}{n^z},$$

  called the zeta function.
- We have:

$$\begin{aligned}\zeta(z) &= \prod_{\text{prime } p} \left(1 + \frac{1}{p^z} + \frac{1}{p^{2z}} + \frac{1}{p^{3z}} + \cdots\right) \\ &= \prod_{\text{prime } p} \frac{1}{1 - \frac{1}{p^z}}\end{aligned}$$

ζ-FUNCTION

- Function $\zeta(z)$ converges for $|z| > 1$ for complex z.
- Using methods from complex analysis, its definition can be extended to entire complex plane except for $z = 1$ where it is undefined.
- It is done using the following functional equation for ζ:

$$\pi^{-z/2}\zeta(z)\Gamma(z/2) = \pi^{(1-z)/2}\zeta(1-z)\Gamma((1-z)/2),$$

  where $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$ is a generalization of factorial function.
- Zeroes of $\zeta(z)$ are at $z = -2m$ for positive integers m, and in the region $0 \leq \Re(z) \leq 1$.

Estimating $\pi(n)$

- Riemann showed that

$$\pi(n) = \frac{n}{\ln n} - \sum_{\substack{\rho \\ \zeta(\rho)=0 \\ 0 \leq \Re(\rho) \leq 1}} \frac{n^\rho}{\rho} + O(1).$$

- He conjectured that $\Re(\rho) = \frac{1}{2}$ for every ρ with $\zeta(\rho) = 0$ and $0 \leq \Re(\rho) \leq 1$. It is called Riemann Hypothesis.
- Hadamard and Vallee Poussin proved that $\Re(\rho) < 1$. This implied:

$$\lim_{n \to \infty} \pi(n) = \frac{n}{\ln n}.$$