CS201

Mathematics for Computer Science I

Lecture 12

Counting Regions

- Suppose n lines are drawn on a place such that:
  - No two lines are parallel, and
  - No three lines intersect at a single point
- What is the number of regions in the plane?
- Solved through recurrence relations.

Counting Regions

- Let $R(n)$ be the number of regions formed by n lines.
- With $n-1$ lines, the number of regions are $R(n-1)$.
- The nth line intersects each line at a distinct point.
- Just before any intersection, it is passing through a region that gets split into two.
- The region after last intersection also gets split into two.
- Hence:
$$R(n) = R(n-1) + n,$$
and $R(0) = 1$.
- The recurrence relation can be solved directly to obtain:
$$R(n) = \frac{1}{2}(n^2 + n + 2).$$

Counting Regions

- Alternately, let $G(x) = \sum_{n \geq 0} R(n) x^n$.
- Then:

$$
\begin{aligned}
G(x) &= 1 + \sum_{n \geq 1} R(n-1) x^n + \sum_{n \geq 1} n x^n \\
&= 1 + xG(x) + x \frac{d}{dx} \sum_{n \geq 0} x^n \\
&= 1 + xG(x) + \frac{x}{(1-x)^2}
\end{aligned}
$$

Counting Regions

- Therefore,

$$
\begin{aligned}
G(x) &= \frac{1-x+x^2}{(1-x)^3} \\
&= (1-x+x^2)\sum_{n\geq 0}(-1)^n\binom{-3}{n}x^n \\
&= (1-x+x^2)\sum_{n\geq 0}\frac{1}{2}(n+1)(n+2)x^n \\
&= \sum_{n\geq 0}\frac{1}{2}((n+1)(n+2)-n(n+1)+(n-1)n)x^n \\
&= \sum_{n\geq 0}\frac{1}{2}(n^2+n+2)x^n
\end{aligned}
$$

General Linear Recurrence

- Suppose we have recurrence relation:

$$A(n) = \alpha_1 A(n-1) + \alpha_2 A(n-2) + \cdots + \alpha_k A(n-k)$$

for constants $\alpha_1, \ldots, \alpha_k$, with values of $A(n)$ also given for $0 \leq n < k$.

- Let $G(x) = \sum_{n \geq 0} A(n) x^n$.
- Then:

$$\begin{aligned}
G(x) &= \sum_{0 \leq r < k} A(r) x^r + \sum_{n \geq k} (\sum_{i=1}^{k} \alpha_i A(n-i)) x^n \\
&= \sum_{0 \leq r < k} A(r) x^r + \sum_{i=1}^{k} \alpha_i x^i \sum_{n \geq k} A(n-i) x^{n-i} \\
&= \sum_{0 \leq r < k} A(r) x^r + \sum_{i=1}^{k} \alpha_i x^i (G(x) - \sum_{j=0}^{k-i-1} A(j) x^j) \\
&= \sum_{0 \leq r < k} A(r) x^r + (\sum_{i=1}^{k} \alpha_i x^i) G(x) - \sum_{r=1}^{k-1} (\sum_{i=1}^{r} \alpha_i A(r-i)) x^r
\end{aligned}$$

General Linear Recurrence

- This give:

$$(1 - \sum_{i=1}^{k} \alpha_i x^i) G(x) = A(0) + \sum_{r=1}^{k-1} (A(r) - \sum_{i=1}^{r} \alpha_i A(r-i)) x^r.$$

- Therefore,

$$G(x) = \frac{A(0) + \sum_{r=1}^{k-1} (A(r) - \sum_{i=1}^{r} \alpha_i A(r-i)) x^r}{1 - \sum_{i=1}^{k} \alpha_i x^i}.$$

- To get the value of $A(n)$ from this, find all the k roots of the polynomial $1 - \sum_{i=1}^{k} \alpha_i x^i$.
- Then using the partial fraction expansion, write $A(n)$ as linear combination of nth powers of these roots.

Linear Recurrences with Non-constant Coefficients

- Consider the recurrence:

$$A(n) = f(n)A(n-1) + g(n),$$

where f and g are functions of n, $f(1) \neq 0$, and $A(0) = a$.

- Define $F(n) = \prod_{m=1}^{n} f(m)$, $F(0) = 1$, and divide the equation by $F(n)$:

$$\frac{1}{F(n)}A(n) = \frac{1}{F(n-1)}A(n-1) + \frac{g(n)}{F(n)}.$$

Linear Recurrences with Non-constant Coefficients

- Let $B(n) = \frac{A(n)}{F(n)}$. Then:

$$\begin{aligned} B(n) &= B(n-1) + \frac{g(n)}{F(n)} \\ &= a + \sum_{m=1}^{n} \frac{g(m)}{F(m)}. \end{aligned}$$

- Hence:

$$A(n) = aF(n) + F(n) \sum_{m=1}^{n} \frac{g(m)}{F(m)}.$$

Example: Time Complexity of Mergesort

```
Mergesort(A, n) { // A is an array of n numbers
      if (n <= 1) return;
      m = n/2;
      Mergesort(A, m); // sort first m numbers
      Mergesort(A+m, n-m); // sort last n-m numbers
      Merge(A, m, A+m, n-m); // merge sorted arrays
}
```

Example: Time Complexity of Mergesort

- Let $T(n)$ be time taken by `Mergesort` algorithm to sort an array of n numbers.
- We get recurrence:

$$T(n) \leq 2T(n/2) + cn$$

  for some constant c.
- Let $n = 2^r$, and define $S(r)$ such that:

$$S(r) = 2S(r-1) + c2^r.$$

- It can be proved using induction that $T(n) \leq S(\lceil \log n \rceil)$.

Example: Time Complexity of `Mergesort`

- We have:

$$\begin{aligned} S(r) &= S(0)2^r + 2^r \sum_{m=1}^{r} \frac{c2^m}{2^m} \\ &= S(0)2^r + cr2^r \\ &\leq \tilde{c}r2^r. \end{aligned}$$

- This gives:

$$T(n) \leq S(\lceil \log n \rceil) \leq \tilde{c} \lceil \log n \rceil 2^{\lceil \log n \rceil} \leq \hat{c} n \log n.$$