Completeness of $\vdash_{NC}$ w.r.t. BA 2

We need to prove that if P is true in 2, then it is true in any BA B.

Proof: We show that if P is not true in any BA B, then it is not true in 2.

By Stone's theorem B is a subalgebra of $P(X)$, some X.
As $P(X) \cong 2^X$, we take B as a subalgebra of 2^X.

$f, g \in 2^X$, $f : X \to 2 = \{0, 1\}$

$(\neg f)(x) = \neg(f(x))$
$(f \vee g)(x) = f(x) \vee g(x)$
$(f \to g)(x) = f(x) \to g(x)$

operations in 2

operations of 2^X

0 in 2^X is f s.t. $f(x) = 0 \ \forall x \in X$.
1 " " " g " $g(x) = 1 \ \forall x \in X$.

p is false in B $\Rightarrow$ there is a valuation of $[\![\cdot]\!]_B$ s.t. $[\![p]\!]_B \neq 1$ ($X \to 2$)
$\Rightarrow \exists x \in X, [\![p]\!]_B(x) \neq 1$

let $A_1, \ldots, A_n$ be atomic propositions in P.
$[\![p]\!]_B(x) = [\![p]\!]_2$ where $[\![\cdot]\!]_2 = [\![\cdot]\!]_B(x)$
$\Rightarrow$ there is a valuation $[\![\cdot]\!]$ s.t. $[\![p]\!]_2 \neq 1$
$\Rightarrow$ P is false in 2.

$A \vee (B \wedge C)$

$$f \vee (g \wedge h)(x) = f(x) \wedge (g(x) \wedge h(x))$$
$$= A \vee (B \wedge C) \text{ in } 2$$

with valuation $[\![A]\!]_2 = f(x)$, $[\![B]\!]_2 = g(x)$ and $[\![C]\!]_2 = h(x)$.

Algebraic Semantics of Intuitionistic Propositional Logic

Use Heyting algebra

def: A Heytingalgebra $H = (H, \leq, \vee, \wedge, \rightarrow, 0, 1)$ such that $(H, \leq, \vee, \wedge)$ is a distributive lattice. 0 and 1 are the smallest and largest elements of H

for $a, b \in H$, $a \rightarrow b$ is the largest x s.t. $a \wedge x \leq b$.

examples - (i) every BA is HA

Given $B = (B, \leq, \vee, \wedge, \neg, 0, 1)$

define $a \rightarrow b$ as $\neg a \vee b$

$$a \wedge (\neg a \vee b) = (a \wedge \neg a) \vee (a \wedge b)$$
$$= (a \wedge b) \leq b.$$

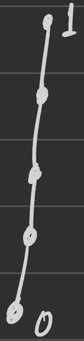
Now we need to show that $(\neg a \vee b)$ is the largest x.

consider any c s.t. $a \wedge c \leq b$

$$\neg a \vee (a \wedge c) \leq \neg a \vee b$$
$$\Rightarrow (\neg a \vee a) \wedge (\neg a \vee c) \leq \neg a \vee b$$
$$\Rightarrow \neg a \vee c \leq \neg a \vee b.$$
$$\Rightarrow c \leq \neg a \vee c \leq \neg a \vee b$$
$$\Rightarrow c \leq \neg a \vee b.$$

$\because$ this definition of $\rightarrow$ satisfies the conditions of Heyting Algebra.

(ii)



$a \vee b = \max\{a, b\}$

$a \wedge b = \min\{a, b\}$

exercise: verify that this is a distributive lattice

$$a \rightarrow b = \begin{cases} 1 & a \leq b \\ b & b \leq a \end{cases}$$

Now this is a Heyting algebra.

all boolean algebras are isomorphic to 2^X. $\because$ there is no BA with 5 elements. However, we just saw that there is a Heyting algebra with 5 elements.

$\neg a = a \rightarrow 0$ $\rightarrow$ definition of $\neg a$ in every Heyting algebra

$\neg a = 0$ $\rightarrow$ evaluation of $\neg a$ in this Heyting algebra

when $a \neq a$, $a \vee \neg a = a$ $\rightarrow$ An example of a HA that is not a BA.

Interpretation of PL in HA.

$(H, [\![\cdot]\!]_H)$ $[\![A \wedge B]\!]_H = [\![A]\!]_H \wedge [\![B]\!]_H$

$[\![A \vee B]\!]_H = [\![A]\!]_H \vee [\![B]\!]_H$

$[\![A \rightarrow B]\!]_H = [\![A]\!]_H \rightarrow [\![B]\!]_H$

$[\![\bot]\!]_H = 0$

definition: A is true in $(H, [\![\cdot]\!]_H)$ iff $[\![A]\!]_H = 1$

soundness:

If $\Gamma \vdash_i A$ then $\Gamma \vdash A$ is true in all HA.

($\Gamma \vdash A$ is true iff $[\![\Gamma]\!]_H \leq [\![A]\!]_H$ → definition

Proof: By induction on the derivation $\Gamma \vdash_i A$.

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \qquad [\![\Gamma]\!] \wedge [\![A]\!] \leq [\![B]\!]$$

$$\Rightarrow [\![\Gamma]\!] \leq [\![A]\!] \to [\![B]\!]$$ → difinition of HA

exercise: Prove the other cases.