Lecture-8 Main Points

- Interpretation of a proposition P in a Boolean Algebra
  - P is true in $(B, [\![\cdot]\!]_B)$ iff $[\![P]\!]_B = 1$.
  - P is true in a BA B iff for all valuations $[\![\cdot]\!]_B$ into B, $[\![P]\!]_B = 1$.
  - P is valid (or a tautology) iff P is true in all BA B.
    (equivalently, for all $(B, [\![\cdot]\!]_B)$, $[\![P]\!]_B = 1$)
- Extending interpretation to a judgment $\Gamma \vdash A$
  - Define $[\![\Gamma]\!]_B = \begin{cases} 1 & \text{if } \Gamma \text{ is empty} \\ [\![A_1]\!]_B \wedge \ldots \wedge [\![A_m]\!]_B & \text{if } \Gamma \equiv A_1, \ldots, A_m \end{cases}$
  - $\Gamma \vdash A$ is true in $(B, [\![\cdot]\!]_B)$ iff $[\![\Gamma]\!]_B \leq [\![A]\!]_B$.
  - $\Gamma \vdash A$ is true in B iff for all B valuations $[\![\cdot]\!]_B$, $[\![\Gamma]\!]_B \leq [\![A]\!]_B$.
  - $\Gamma \vdash A$ is valid iff $\Gamma \vdash A$ is true for all BA B.
- Syntax vs. Semantics
  - Soundness
  - Completeness
- **Theorem** (Soundness): If $\Gamma \vdash_{\mathbf{Nc}} A$ then $\Gamma \vdash A$ is valid.
  - Proof by induction on depth of derivation tree of $\Gamma \vdash_{\mathbf{Nc}} A$.
- Completeness
  - Given a valid P, how to relate it to derivability in $\vdash_{\mathbf{Nc}}$?
  - We use a standard technique of constructing a Lindenbaum algebra of propositional logic.

* Define an equivalence relation '$\sim$' on the set of all propositions as follows.
  - $A \sim B$ iff $A \vdash_{\mathbf{Nc}} B$ and $B \vdash_{\mathbf{Nc}} A$.
    (In the following we use just '$\vdash$' for '$\vdash_{\mathbf{Nc}}$').
  - We let $[A]$ stand for equivalence class of A w.r.t. relation '$\sim$'.
* Domain of Lindenbaum algebra $\mathcal{A}$ is $\{[A] \mid A \text{ is a proposition}\}$
* Show that relation $\sim$ is a congruence for operations $\vee, \wedge, \rightarrow$.
* Define ordering $\leq$ on equivalence classes of $\sim$ as follows.
  - $[A] \leq [B]$ iff $A \vdash B$.
  - Show that relation $\leq$ is well defined.
* Show that $\leq$ is a partial order and $\vee, \wedge$ are lub and glb operators w.r.t. this ordering.
* Define operations on $\mathcal{A}$, as
  - $[A] \vee [B] = [A \vee B]$, $[A] \rightarrow [B] = [A \rightarrow B]$,$\neg[A] = [\neg A]$ etc.
* Show that $\mathcal{A} = (\mathcal{A}, \leq, \wedge, \vee, \neg, [\perp], [\perp \rightarrow \perp])$ is a BA.