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Completness of $\vdash_{NC}$ w.r.t. Boolean Algebra

Recall $A = \{[A] \mid A \text{ is a prop}^n\}$

Lindenbaum algebra $\mathcal{A} = (A, \leq, \vee, \wedge, \rightarrow, \neg, [\bot], [\bot \rightarrow \bot])$

$A \sim B \iff A \vdash B$ and $B \vdash A$

$[A] \leq [B] \iff A \vdash B$

$\leq, \vee, \wedge, \neg \rightarrow$ are well defined

operations are defined as follows,

$[A_1 \vee A_2] = [A_1] \vee [A_2]$

$[A_1] \rightarrow [A_2] = [A_1 \rightarrow A_2]$

$\neg [A_1] = [\neg A_1]$

$\vee \rightarrow$ lub

$\wedge \rightarrow$ glb

Q. Prove $[A] \vee [\neg A] = [\bot \rightarrow \bot]$

Sol.

$A \vee \neg A \vdash \bot \rightarrow \bot$

$\bot \vdash \bot$ (Axioms)

$\Rightarrow \vdash \bot \rightarrow \bot$ $(\rightarrow i)$

$A \vee \neg A \vdash \bot \rightarrow \bot$ (weakning)

$\therefore [A \vee \neg A] = [\bot \rightarrow \bot]$

$[A] \vee \neg [A] = [\bot \rightarrow \bot]$

$\bot \rightarrow \bot \vdash A \vee \neg A$

$\vdash A \vee \neg A$ (LEM)

$\bot \rightarrow \bot \vdash A \vee \neg A$ (weaken)

Cannonical Valuation into $\mathcal{A}$ $\longrightarrow (\mathcal{A}, [\![\cdot]\!]_{\mathcal{A}})$

$$[\![\cdot]\!]_{\mathcal{A}}(A) = [A]$$

$$\Downarrow$$

$$[\![A]\!]_{\mathcal{A}}$$

Cannonical Valuation

For all propositions P, $[\![P]\!]_A = [P]$

<u>Proof</u>:
By induction on A

base: A is atomic, by definition, $A \equiv \bot$, $[\![\bot]\!]_A = 0 = [\bot]$

induction: $A \equiv A_1 \to A_2$

$$\underbrace{[\![A]\!]_A = \neg [\![A_1]\!]_A \vee [\![A_2]\!]_A}_{\text{def}^n \text{ of } [\![\ ]\!]_A} = \neg [A_1] \vee [A_2]$$
$$= [\neg A_1 \vee A_2]$$
$$= [A_1 \to A_2]$$

whenever prop^n are mapped to any BA, some rules need to be followed. We have shown that these rules are being followed with this mapping.

Since $\dot{H}$ is valid, $\forall$ BA, and valuations.
$\dot{H}$ needs to be valid for this particular BA & val^n.

Completness of N_c

Let $\Gamma \vdash A$, be true in all $(B, [\![\cdot]\!]_B)$

$\Rightarrow$ $\Gamma \vdash A$ is true in $(\mathcal{A}, [\![\cdot]\!]_{\mathcal{A}})$

$\Rightarrow$ $[\![\Gamma]\!]_{\mathcal{A}} \leqslant [\![A]\!]_{\mathcal{A}}$

$\Rightarrow$ $[\![A_1 \wedge A_2 \cdots A_m]\!]_{\mathcal{A}} \leqslant [\![A]\!]_{\mathcal{A}}$

$\Rightarrow$ $[A_1 \wedge A_2 \cdots A_m] \leqslant [A]$

$\Rightarrow$ $A_1 \wedge A_2 \cdots A_m \vdash_{N_c} A$ $\Rightarrow$ $\Gamma \vdash_{N_c} A$ $\therefore$ This shows completness

Stones Representation Theorem

Any BA is isomorphic to sub-algebras of a power set Boolean Algebra

Stronger Form of Completness

The following are equivalent:

- $\Gamma \vdash_{NC} A$
- $\Gamma \vdash A$ is true in 2.
- $\Gamma \vdash A$ is true in BAs B.

proof:

For some BA B, and val^{n} $[\![\cdot]\!]_B$, let $[\![A]\!]_B \neq 1$

By stones theorem, we assume B to be a sub-algebra of $P(X)$ for some X.

As $P(X)$ BA and BA 2^X are isomorphic. We may assume B to be sub-algebra of BA 2^X.

$[\![A]\!]_B(x) \neq 1$ for some $x \in X$
$[\![A]\!]_2 \neq 1$, where $[\![\cdot]\!]_2$ is given as $[\![P]\!]_2 = [\![P]\!]_B(x)$

$\Rightarrow$ A is not true in BA 2.

> Subalgebra of $(B, \leq, \vee, \wedge, \neg, 0, 1)$ is the BA induced on a subset $C \subseteq B$, st $0, 1 \in C$, and C is closed wrt $\wedge, \vee$ and $\neg$