Lecture-4 Main Points

- A warm-up Lemma.
  - There exist two irrational real numbers a, b s.t. a^b is rational.
  - There is a (classical logic) proof of this lemma involving only a few lines.
  - The proof does not yield values of a, b (only proves their existence).
- Constructivism
  - Constructivist school of thought in mathematics rejects above proof.
  - Constructivism requires that existence of a mathematical object with property P, be proved by **concretely exhibiting an object** and showing that it has property P.
  - A look at the proof reveals that its non-constructive nature is due to the use of excluded middle principle.
- What is a constructive proof for propositional language?
  - BHK interpretation
    - Associates a **construction** to a proposition.
      $c : A$ (read as "c is a construction for A")
    - If A is atomic
      - $p : A$ is a primitive/given notion.
    - $A \equiv A_1 \vee A_2$
      - $inl(t) : A$, where $t : A_1$ or
      - $inr(t) : A$, where $t : A_2$
    - $A \equiv A_1 \wedge A_2$
      - $\langle t_1, t_2 \rangle : A$, where $t_1 : A_1$ and $t_2 : A_2$

    * $A \equiv A_1 \to A_2$
        · $t : A$, where t is a function s.t.
          for any input $c : A_1$, $t(c) : A_2$.
    * There is no construction for $\perp$.

- Examples of BHK interpretation for some simple propositions.
  - $A \to A \vee B$, $A \to \neg\neg A$.
- Exercise: Why is there no BHK interpretation for $A \vee \neg A$ in general?