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Many valued Logics

Generalization of two valued classical logic.
(BA 2)

Form

$$(V, f_{c_1}, \ldots, f_{c_n}, D)$$

Set of Values

connective define

set of designated values

BA 2

$$(\{0,1\}, f_\wedge, f_\vee, f_\neg, \{1\})$$

Kleene's 3 valued logic

$\{0, 1, i\}$

i: undecided

$\wedge, \vee, \neg, \to$

$\wedge$	0	i	1
0	0	0	0
i	0	i	i
1	0	i	1

$\neg$	
0	1
1	0
i	i

$\vee$	0	1	i
0	0	1	i
1	1	1	1
i	i	1	i

b

$a \to b$	0	1	i
0	1	1	1
1	0	1	i
i	i	1	i

a

Consequence Relation

$\Sigma \models A$

Set of proposition — Σ; prop — A

For all valuation v if for every proposition $p \in \Sigma$ $v(p) =$ true then $v(A)$ is true.

A prop p is True if it has one of the designated values (w.r.t. a valuation)

In K_3 $D = \{1\}$

There are no tautolgies in K_3.

$A \to A$ $\quad v(A) = i$

$i \to i$ then $v(A \to A) = i$

$A \to B \models_{K_3} \neg B \to \neg A$

$v(A \to B) = 1$

iff $v(A) = 0 \vee \Leftrightarrow v(\neg B) = 0$

$v(B) = 1 \Leftrightarrow v(\neg A) = 1$

$A, \neg A \models B$

There is no valuation, s.t. $v(A), v(\neg A) \in D = \{1\}$

$A, \quad A \to B \models B$ ✓ (Exercise : verify this)

$A \not\models B \vee \neg B \qquad v(A) = 1, \; v(B) = i$

A variation LK_3

is same as K_3 but $D = \{1, i\}$

<u>Rationale</u> : i is thought of as both true and false

$A \vee \neg A$ is a tautology

$A \to A$ "

$A, \neg A \not\models_{LK_3} B \qquad v(A) = i, v(B) = 0$

para-consistency

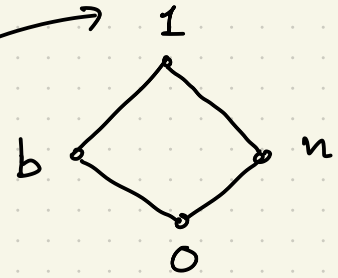
$A \models B \vee \neg B$

$A, A \to B \not\models_{LK_3} B \qquad v(B) = 0$

$v(A) = i$

A four valued logic
(due to Belnap)

Designated value



both true & false

$\neg b = b$

$\neg n = n$

$\to$ neither true or false

$A \to B = \neg A \vee B$

Exercise: To explore tautologies and $\models$ in this logic

Sorites Paradox

Ann is a 2 years old girl, hence she is a child.

$A_i \equiv$ Ann is a child after i seconds

$$\frac{A_0 \qquad A_i \to A_{i+1}}{\text{for all } i,\ A_i}$$

tall, young, small, drunk

Vague Predicates

Truth value changes gradually

$$L = ([0,1], f_\wedge, f_\vee, f_\neg, f_\rightarrow, D)$$

$$a \wedge b = \min\{a, b\}$$

$$a \vee b = \max\{a, b\}$$

$$\neg a = 1 - a$$

$$a \rightarrow b = \begin{cases} 1 & a \leq b \\ 1-(a-b) & \text{if } a > b \end{cases}$$

$a \leq b$ $\quad a \rightarrow b$ is true

$1 - (a-b)$

Fuzzy Logic

$$D_\varepsilon = \{x \mid x \geq \varepsilon\}$$

$\models_\varepsilon$ $\qquad \Sigma \models A$ iff for every ε $\Sigma \models_\varepsilon A$.

Exercise 3.1.

Non Logical symbols:

constants a, b; *functions* f^1, g^2; *predicates* p^1, r^2, q^3.

Say whether the following strings of symbols are well formed FOL formulas or terms:

1. $q(a)$
2. $p(y)$
3. $p(g(b))$
4. $\neg r(x, a)$
5. $q(x, p(a), b)$
6. $p(g(f(a), g(x, f(x))))$
7. $q(f(a), f(f(x)), f(g(f(z), g(a, b))))$
8. $r(a, r(a, a))$

WFF.

Exercise 3.4.

Find free variables in the following formulas:

1. $p(x) \wedge \neg r(y, a)$ x, y
2. $\exists x.r(x, y)$ y
3. $\forall x.p(x) \to \exists y.\neg q(f(x), y, f(y))$ x
4. $\forall x \exists y.r(x, f(y))$ –
5. $\forall x \exists y.r(x, f(y)) \to r(x, y)$ x, y

Exercise 3.5.

Find free variables in the following formulas:

1. $\forall x.(p(x) \to \exists y.\neg q(f(x), y, f(y)))$ –
2. $\forall x(\exists y.r(x, f(y)) \to r(x, y))$ y
3. $\forall z.(p(z) \to \exists y.(\exists x.q(x, y, z) \vee q(z, y, x)))$ x
4. $\forall z \exists u \exists y.(q(z, u, g(u, y)) \vee r(u, g(z, u)))$ –
5. $\forall z \exists x \exists y(q(z, u, g(u, y)) \vee r(u, g(z, u)))$ u

1. $bought(Frank, dvd)$ Frank bought a dvd

2. $\exists x.bought(Frank, x)$ Frank bought somthing

3. $\forall x.(bought(Frank, x) \rightarrow bought(Susan, x))$ Susan bought everything that frank bought

4. $\forall x.bought(Frank, x) \rightarrow \forall x.bought(Susan, x)$ If frank bought smthg susan bought it

5. $\forall x \exists y.bought(x, y)$ Everyone bought smth

6. $\exists x \forall y.bought(x, y)$ someone bought everything

Define an appropriate language and formalize the following sentences using FOL formulas.

1. *All Students are smart.* $\forall x \ (Stud(x) \rightarrow Smart(x))$
2. *There exists a student.* $\exists x \ Stud(x)$
3. *There exists a smart student.* $\exists x \ (Stu(x) \wedge Smar(x))$
4. *Every student loves some student.* $\forall x \ (Stud(x) \rightarrow \exists y \ (Stud(y) \wedge \neg(x=y) \wedge Loves(x,y)))$
5. *Every student loves some other student.*
6. *There is a student who is loved by every other student.* $\exists x(Stud\ x \wedge \forall y \ (Stud(y) \wedge \neg(x=y) \rightarrow Loves(y,x)))$
7. *Bill is a student.* $Stud(Bill)$
8. *Bill takes either Analysis or Geometry (but not both).*
9. *Bill takes Analysis and Geometry.*
10. *Bill doesn't take Analysis.*
11. *No students love Bill.*

1. *"A is above C, D is above F and on E."*

   $\phi_1 : Above(A, C) \wedge Above(E, F) \wedge On(D, E)$

2. *"A is green while C is not."*

   $\phi_2 : Green(A) \wedge \neg Green(C)$

3. *"Everything is on something."*

   $\phi_3 : \forall x \exists y.On(x, y)$

4. *"Everything that is free has nothing on it."*

   $\phi_4 : \forall x.(Free(x) \rightarrow \neg \exists y.On(y, x))$

5. *"Everything that is green is free."*

   $\phi_5 : \forall x.(Green(x) \rightarrow Free(x))$

6. *"There is something that is red and is not free."*

   $\phi_6 : \exists x.(Red(x) \wedge \neg Free(x))$

7. *"Everything that is not green and is above B, is red."*

   $\phi_7 : \forall x.(\neg Green(x) \wedge Above(x, B) \rightarrow Red(x))$

Exercise 3.14.

Consider the following sentences:

1. *All actors and journalists invited to the party are late.*
2. *There is at least a person who is on time.*
3. *There is at least an invited person who is neither a journalist nor an actor.*

Formalize the sentences and prove that 3. is not a logical consequence of 1. and 2.

Solution.

1. $\forall x.((a(x) \vee j(x)) \wedge i(x) \rightarrow l(x))$
2. $\exists x.\neg l(x)$
3. $\exists x.(i(x) \wedge \neg a(x) \wedge \neg j(x))$

It's sufficient to find an interpretation $\mathcal{I}$ for which the logical consequence does not hold:

	l(x)	a(x)	j(x)	i(x)
Bob	F	T	F	F
Tom	T	T	F	T
Mary	T	F	T	T

Q. x doesn't occour free in A,

$$\vdash \forall x(A \to B) \to (A \to \forall x.B)$$

Sol:

$$\dfrac{\dfrac{\forall x\,(A \to B) \vdash \forall x\,(A \to B)}{\forall x(A \to B) \vdash (A \to B)[y/x]} \qquad A \vdash A}{\dfrac{A, \forall x\,(A \to B) \vdash B}{A, \forall x\,(A \to B) \vdash \forall x.B}}$$

Q. $\exists x\; P(x) \vdash \neg \forall x \neg P(x)$

$$\dfrac{\dfrac{Ax}{\exists x\, P(x) \vdash \exists x\, P(x)} \qquad \dfrac{\dfrac{\forall x \neg P(x) \vdash \forall x \neg P(x)}{\forall x \neg P(x) \vdash \neg P[y/x]}\,\forall_e \qquad \dfrac{Ax.}{P[y/x] \vdash P[y/x]}}{\forall x \neg P(x), P[y/x] \vdash \bot}\,\to_e}{\dfrac{\exists x\, P(x), \forall x \neg P(x) \vdash \bot}{\exists x\, P(x) \vdash \neg \forall x(\neg P)}}\,\exists_e$$

Q. $\neg \exists x \neg A \to \forall x. A$

$$\dfrac{\dfrac{Ax}{\neg \exists x \neg A(x) \vdash \neg \exists x \neg A(x)} \qquad \dfrac{\neg A[y/x] \vdash \neg A[y/x]}{\neg A[y/x] \vdash \exists x \neg A(x)}\,\exists_i}{\dfrac{\dfrac{\neg \exists x \neg A(x), \neg A[y/x] \vdash \bot}{\neg \exists x \neg A(x) \vdash A[y/x]}}{\neg \exists x \neg A(x) \vdash \forall x. A(x)}\,\forall_i}\,\to E$$

Q. $\exists x\,(A(x) \wedge B(x)) \vdash \exists x\, A(x)$

$$
\cfrac{
  \cfrac{
    \cfrac{A[y/x] \wedge B[y/x] \vdash A[y/x] - \cdot}{A[y/x] \wedge B[y/x] \vdash A[y/x]}\;\wedge_e
  }{A[y/x] \wedge B[y/x] \vdash \exists x\, A(x)}\;\exists_i
  \qquad
  \exists x\,(A(x) - -) \vdash \exists x\,(A(x) - -)
}{\exists x\,(A(x) \wedge B(x)) \vdash \exists x\,(A(x))}\;\exists_e
$$

Q. $\exists x\, A(x),\ \forall x\,(A(x) \to A(f(x))),\ \forall x\,(A(x) \to A(g(x))) \vDash \exists x\, A(f(g(x)))$

$$
\cfrac{
  \exists x.A(x)
  \qquad
  \cfrac{
    \cfrac{
      \cfrac{
        \cfrac{
          A(x)^1
          \quad
          \cfrac{\forall x.(A(x) \to A(g(x)))}{A(x) \to A(g(x))}\;\forall E
        }{A(g(x))}\;\to E
        \quad
        \cfrac{\forall x.(A(x) \to A(f(x)))}{A(g(x)) \to A(f(g(x)))}\;\forall E
      }{A(f(g(x)))}\;\to E
    }{\exists x.A(f(g(x)))}\;\exists I
  }{}
}{\exists x.A(f(g(x)))}\;\exists E^1
$$

Q. $\neg\neg\, \forall x \cdot A(x) \vdash \forall x\, \neg\neg \cdot A(x)$

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{
        \neg\neg\forall x.A(x)^3
        \quad
        \cfrac{
          \cfrac{
            \neg A(x)^2
            \quad
            \cfrac{\forall x.A(x)^1}{A(x)}\;\forall E
          }{\bot}\;\to E
        }{\neg\forall x.A(x)}\;\bot_c^1
      }{\bot}\;\to E
    }{\neg\neg A(x)}\;\bot_c^2
  }{\forall x.\neg\neg A(x)}\;\forall I
}{\neg\neg\forall x.A(x) \to \forall x.\neg\neg A(x)}\;\to I^3
$$

Q. $\forall z((\neg A(z) \vee B(z)) \to A(z) \to B(z))$

$$\dfrac{\dfrac{\neg A[y/z]\vee B[y/z] \vdash \cdots \qquad \dfrac{\dfrac{\neg A[y/z] \vdash \neg A[y/z] \quad , \quad A[y/z]\vdash A[y/z]}{A, \neg A \vdash \bot}}{A, \neg A \vdash B[y/z]} \qquad B[y/z]\vdash B[y/z]}{\neg A[y/z] \vee B[y/z], \cancel{A[y/z]} \vdash B[y/z]}\vee_e}{\vdash \neg(A(y/z) \text{ --- } \text{ --- })}$$

$$\dfrac{}{}\ \forall_i$$

Q. $\forall x \forall y\, A(x,y) \vdash \neg \forall x.\neg A(x,x)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x\forall y.A(x,y)^2}{\forall y.A(x,y)}\forall E}{A(x,x)}\forall E \qquad \dfrac{\forall x.\neg A(x,x)^1}{\neg A(x,x)}\forall E}{\bot}\to E}{\neg\forall x.\neg A(x,x)}\bot_c^1}{\forall x\forall y.A(x,y)\to\neg\forall x.\neg A(x,x)}\to I^2$$

Q. $\exists y(B(y) \to C(y)), \forall x.B(x) \vDash \exists x.C(x)$

Sol^n:

$$\dfrac{\exists y(B(y)\to C(y))\vdash \cdots \qquad \dfrac{\dfrac{B(x)\to C(x)\vdash B(x)\to C(x) \qquad \dfrac{\forall x\, B(x)\vdash\forall x\, B(x)}{\forall x\, B(x)\vdash B(x)}\forall_e}{\forall x\, B(x), B(x)\to C(x)\vdash C(x)}\to E}{\forall x\, B(x), B(x)\to C(x)\vdash \exists x\, C(x)}\exists_i}{\exists y(B(y)\to C(y)), \forall x\, B(x) \vdash \exists x\, C(x)}\exists_e$$

Q. $\exists x . A(x) \to \forall x\, B(x) \vDash \forall x\, (A(x) \to B(x))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\exists x.A(x) \to \forall x.B(x) \qquad \dfrac{A(x)^1}{\exists x.A(x)}\,\exists I}{\forall x.B(x)}\,\to E}{B(x)}\,\forall E}{A(x) \to B(x)}\,\to I^1}{\forall x.(A(x) \to B(x))}\,\forall I$$

Q. $\forall x\, (P(x) \to \exists y\, P(y))$

$$\dfrac{\dfrac{P(x) \vdash P(x)}{P(x) \vdash \exists y\, P(y)}\,\exists i}{\vdash P(x) \to \exists y\, P(y) \; \text{---} \; \forall i}$$

Q. $\exists x\, (P(x) \to \forall y\, P(y))$

$$\dfrac{\dfrac{\dfrac{\dfrac{P(x), \neg P(x) \vdash \bot}{P(x), \neg P(x) \vdash \forall y\, P(y)}}{\neg P(x) \vdash P(x) \to \forall y\, P(y)}}{\neg P(x) \vdash \exists x\, (P(x) \to \forall y (P(y))) \qquad \neg(\exists x\, (P(x) \to \forall y (P(y)))) \vdash \ldots}\,(\to e)$$

$$\dfrac{\dfrac{\neg P(x), \neg(\exists x (P(x) \to \forall y (P(y))) \vdash \bot}{\neg(\exists x (P(x) \text{ --- }) \vdash P(x)}}{}\,\forall i$$

$$\dfrac{\dfrac{\dfrac{P(x) \to \forall y P(y)}{\exists x(P(x) \to \forall y P(y))}\,\exists I \qquad \neg(\exists x(P(x) \to \forall y P(y)))^3}{\bot}\,\to E}{\exists x(P(x) \to \forall y P(y))}\,\bot_{c(3)}$$

Q. $\neg\neg\forall x\ P(x) \vdash \forall x.\neg\neg P(x)$

$$
\begin{prooftree}
\AxiomC{$\neg\neg\forall x.P(x)^3$}
\AxiomC{$\neg P(x)^2$}
\AxiomC{$\forall x.P(x)^1$}
\RightLabel{$\forall E$}
\UnaryInfC{$P(x)$}
\RightLabel{$\rightarrow E$}
\BinaryInfC{$\perp$}
\RightLabel{$\perp_c^1$}
\UnaryInfC{$\neg\forall x.P(x)$}
\RightLabel{$\rightarrow E$}
\BinaryInfC{$\perp$}
\RightLabel{$\perp_c^2$}
\UnaryInfC{$\neg\neg P(x)$}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall x.\neg\neg P(x)$}
\RightLabel{$\rightarrow I^3$}
\UnaryInfC{$\neg\neg\forall x.P(x) \rightarrow \forall x.\neg\neg P(x)$}
\end{prooftree}
$$