11

Boolean Expressions

Constructed from variables and connectives of B.A.
($\vee, \wedge, \neg$ and $\rightarrow$)

eg: $x \vee y \rightarrow z$

Same as propositions, with atomic propositions being called as variables.

Can be interpreted in any B.A.

NORMAL FORMS.

Definition 1.40 Let ϕ and ψ be formulas of propositional logic. We say that ϕ and ψ are *semantically equivalent* iff $\phi \vDash \psi$ and $\psi \vDash \phi$ hold. In that case we write $\phi \equiv \psi$. Further, we call ϕ *valid* if $\vDash \phi$ holds.

eg: $p \rightarrow q \equiv \neg q \rightarrow \neg p$
$p \rightarrow q \equiv \neg p \vee q$
$p \wedge q \rightarrow q \equiv r \vee \neg r$ ← tautology.

Lemma 1.41 *Given formulas $\phi_1, \phi_2, \ldots, \phi_n$ and ψ of propositional logic, $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds iff $\vDash \phi_1 \rightarrow (\phi_2 \rightarrow (\phi_3 \rightarrow \cdots \rightarrow (\phi_n \rightarrow \psi)))$ holds.*

PROOF: First, suppose that $\vDash \phi_1 \rightarrow (\phi_2 \rightarrow (\phi_3 \rightarrow \cdots \rightarrow (\phi_n \rightarrow \psi)))$ holds. If $\phi_1, \phi_2, \ldots, \phi_n$ are all true under some valuation, then ψ has to be true as well for that same valuation. Otherwise, $\vDash \phi_1 \rightarrow (\phi_2 \rightarrow (\phi_3 \rightarrow \cdots \rightarrow (\phi_n \rightarrow \psi)))$ would not hold (compare this with Figure 1.11). Second, if $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds, we have already shown that $\vDash \phi_1 \rightarrow (\phi_2 \rightarrow (\phi_3 \rightarrow \cdots \rightarrow (\phi_n \rightarrow \psi)))$ follows in step 1 of our completeness proof. $\square$

Negation Normal Form

A boolean expression is in NNF, if every occourance of '$\neg$' is in front of a variable.

a) remove '$\rightarrow$' using $a \rightarrow b \equiv \neg a \vee b$
b) Use De Morgan's Laws.

eg: $\neg(p \rightarrow q \wedge r) \vee \neg\neg r$.

$$\neg(\neg p \vee (q \wedge r)) \vee \neg\neg r$$
$$\equiv (\neg\neg p \wedge (\neg q \vee \neg r)) \vee \neg\neg r$$
$$\equiv (p \wedge (\neg q \vee \neg r)) \vee r \quad \leftarrow \text{NNF}$$

Lemma

Every boolean expression is equal to an expression in NNF

<u>proof</u>: by induction on size of expression

Base Case: $e \equiv x$, $e \equiv 0$, $e \equiv 1$

Induction Step:
i) $e = e_1 \vee e_2$
ii) $e = e_1 \wedge e_2$
iii) $e = \neg f$
↳ $f \equiv \neg g \Rightarrow e \equiv g$
↳ $f \equiv f_1 \wedge f_2 \Rightarrow e \equiv \neg f_1 \vee \neg f_2$
↳ $f \equiv f_1 \vee f_2 \Rightarrow e \equiv \neg f_1 \wedge \neg f_2$

$\Rightarrow$ This therefore yields an algorithm to convert to NNF in $O(n)$ <u>time</u>.

CONJUNCTIVE NORMAL FORM.

<u>Literal</u> : A variable or its negation

<u>Clause</u> : A disjunction of literals

A boolean expⁿ is in CNF, if it is a conjunction of clauses

eg: $(p \wedge (\neg q \vee \neg r)) \vee r$

$$\equiv (p \vee r) \wedge ((\neg q \vee \neg r) \vee r)$$
$$\equiv (p \vee r) \wedge (\cancel{\neg q \vee r}) \vee 1)$$
$$\equiv (p \vee r)$$

Lemma

Every boolean expression is equal to an expression in CNF

<u>proof</u>:

for an input ϕ, convert to NNF ψ

→ if $\psi = \psi_1 \wedge \psi_2$, apply IH on ψ_1 & ψ_2
→ if $\psi = \psi_1 \vee \psi_2$

let $\psi_1 = c_1 \wedge c_2 \ldots c_n$
$\psi_2 = d_1 \wedge d_2 \ldots d_n$.

then, $\psi = \bigwedge_{\substack{1 \le i \le n \\ 1 \le j \le n}} (c_i \vee d_j)$

DISJUNCTIVE NORMAL FORM

A boolean expression is in DNF, if it can be written as,

$D_1 \vee D_2 \cdots D_n$, where each D_i is a conj^n of literals.

* Every boolean expression can be converted to DNF.

* All conversions so far, hold in all BA, because the rules used are valid in all BAs.

Lemma 1.43 *A disjunction of literals $L_1 \vee L_2 \vee \cdots \vee L_m$ is valid iff there are $1 \le i, j \le m$ such that L_i is $\neg L_j$.*

PROOF: If L_i equals $\neg L_j$, then $L_1 \vee L_2 \vee \cdots \vee L_m$ evaluates to T for all valuations. For example, the disjunct $p \vee q \vee r \vee \neg q$ can never be made false.

To see that the converse holds as well, assume that no literal L_k has a matching negation in $L_1 \vee L_2 \vee \cdots \vee L_m$. Then, for each k with $1 \le k \le n$, we assign F to L_k, if L_k is an atom; or T, if L_k is the negation of an atom. For example, the disjunct $\neg q \vee p \vee r$ can be made false by assigning F to p and r and T to q. □

Definition 1.44 Given a formula ϕ in propositional logic, we say that ϕ is *satisfiable* if it has a valuation in which is evaluates to T.

For example, the formula $p \vee q \to p$ is satisfiable since it computes T if we assign T to p. Clearly, $p \vee q \to p$ is not valid. Thus, satisfiability is a weaker concept since every valid formula is by definition also satisfiable but not vice versa. However, these two notions are just mirror images of each other, the mirror being negation.

Proposition 1.45 *Let ϕ be a formula of propositional logic. Then ϕ is satisfiable iff $\neg\phi$ is not valid.*

PROOF: First, assume that ϕ is satisfiable. By definition, there exists a valuation of ϕ in which ϕ evaluates to T; but that means that $\neg\phi$ evaluates to F for that same valuation. Thus, $\neg\phi$ cannot be valid.

Second, assume that $\neg\phi$ is not valid. Then there must be a valuation of $\neg\phi$ in which $\neg\phi$ evaluates to F. Thus, ϕ evaluates to T and is therefore satisfiable. (Note that the valuations of ϕ are exactly the valuations of $\neg\phi$.) □

2.4 Normal Form Reduction

Exercise 2.32.

Reduce to Negative Normal Form (NNF) the formula

$$\neg(\neg p \vee q) \vee (r \to \neg s)$$

Solution.

1. $\neg(\neg p \vee q) \vee (\neg r \vee \neg s)$
2. $(\neg\neg p \wedge \neg q) \vee (\neg r \vee \neg s)$
3. $(p \wedge \neg q) \vee (\neg r \vee \neg s)$

*

Exercise 2.33.

Reduce to NNF the formula

$$(\neg p \to q) \to (q \to \neg r)$$

Solution.

1. $\neg(\neg p \to q) \vee (q \to \neg r)$
2. $\neg(p \vee q) \vee (\neg q \vee \neg r)$
3. $(\neg p \wedge \neg q) \vee (\neg q \vee \neg r)$

*

Exercise 2.34.

Reduce to Conjunctive Normal Form (CNF) the formula

$$\neg(\neg p \vee q) \vee (r \rightarrow \neg s)$$

Solution.

1. $\neg(\neg p \vee q) \vee (\neg r \vee \neg s)$
2. $(\neg\neg p \wedge \neg q) \vee (\neg r \vee \neg s)$
3. $(p \wedge \neg q) \vee (\neg r \vee \neg s)$ NNF
4. $(p \vee \neg r \vee \neg s) \wedge (\neg q \vee \neg r \vee \neg s)$

*

Exercise 2.35.

Reduce to CNF the formula

$$(\neg p \rightarrow q) \rightarrow (q \rightarrow \neg r)$$

Solution.

1. $\neg(\neg p \rightarrow q) \vee (q \rightarrow \neg r)$
2. $\neg(p \vee q) \vee (\neg q \vee \neg r)$
3. $(\neg p \wedge \neg q) \vee (\neg q \vee \neg r)$ NNF
4. $(\neg p \vee \neg q \vee \neg r) \wedge (\neg q \vee \neg r)$

*

Exercise 2.36.

Reduce to CNF the following formulas: