Lecture-10 Main Points

- Heyting Algebra $H = (H, \leq, \vee, \wedge, \rightarrow, 0, 1)$
  - $(H, \leq, \vee, \wedge)$ is a distributive lattice.
  - $a \rightarrow b$ is the largest x s.t. $a \wedge x \leq b$.
    (such a x must exist for each a, b in H for it to be a HA)
  - $0, 1$ are the smallest and the largest elements respectively, of H.
- Examples
  1. Every BA is a HA.
     $a \rightarrow b$ defined as $\neg a \vee b$ in BA satisfies the property of '$\rightarrow$' in HA.
  2. A five element linear order is a HA.
     $A \vee \neg A$ is not true in this HA, for any $A \neq 0, 1$.
- Soundness of $\vdash_i$ w.r.t. HA.
  - Given $(H, [\![\cdot]\!]_H)$, for any proposition A, $[\![A]\!]_H$ is defined as for a Boolean algebra except for $[\![A_1 \rightarrow A_2]\!]_H = [\![A_1]\!]_H \rightarrow [\![A_2]\!]_H$ (instead of $\neg[\![A_1]\!]_H \vee [\![A_2]\!]_H$).
  - Truth and validity extend as in the case of BA.
  - **Soundness Theorem:** If $\Gamma \vdash_{Ni} A$ then for any $(H, [\![\cdot]\!]_H)$, $[\![\Gamma]\!]_H \leq [\![A]\!]_H$.
    * Proof is the same as in BA case, except for treating $\rightarrow i$ and $\rightarrow e$ rules. These rules are easily handled as per definition of $a \rightarrow b$ given above for a HA.

- Completeness of $\vdash_i$ w.r.t. HA
  - **Completeness Theorem:** If $\Gamma \vdash A$ is true in all HA then $\Gamma \vdash_{Ni} A$.
  - Proof steps are completely analogous to the BA case.
    * Define Lindenbaum algebra for $\vdash_i$.
    * Show it to be a Heyting algebra.
    * Define canonical valuation for Lindenbaum algebra and prove its property.
    * Complete the proof as in BA case.