Lecture-9 Main Points

- Completing the completeness proof

  Define canonical valuation $[\![\cdot]\!]_{\mathcal{A}}$ in Lindenbaum BA $\mathcal{A}$ as,

  $[\![P]\!]_{\mathcal{A}} = [P]$, for all atomic propositions P.

  - **Lemma (canonical valuation):** For all propositions P, $[\![P]\!]_{\mathcal{A}} = [P]$.

    Proof: By a simple induction on P. □

  Let $\Gamma \vdash A$ be true in all $(B, [\![\cdot]\!]_B)$.

  $\Rightarrow \Gamma \vdash A$ is true in $(\mathcal{A}, [\![\cdot]\!]_{\mathcal{A}})$.

  $\Rightarrow [\![\Gamma]\!]_{\mathcal{A}} \leq_{\mathcal{A}} [\![A]\!]_{\mathcal{A}}$

  [by def. of truth of $\Gamma \vdash A$ in $(\mathcal{A}, [\![\cdot]\!]_{\mathcal{A}})$]

  $\Rightarrow [\![A_1 \wedge \ldots \wedge A_m]\!]_{\mathcal{A}} \leq_{\mathcal{A}} [\![A]\!]_{\mathcal{A}}$,

  [letting $\Gamma \equiv A_1, \ldots, A_m$]

  $\Rightarrow [A_1 \wedge \ldots \wedge A_m] \leq_{\mathcal{A}} [A]$

  (By canonical valuation Lemma)

  $\Rightarrow A_1 \wedge \ldots \wedge A_m \vdash_{\mathbf{Nc}} A$

  (By definition of $\leq_{\mathcal{A}}$)

  $\Rightarrow A_1, \ldots, A_m \vdash_{\mathbf{Nc}} A$

  (easy to see)

  $\Rightarrow \Gamma \vdash_{\mathbf{Nc}} A$

  This shows completeness. □

- A stronger completeness
  - **Theorem 1:** Following are equivalent.
    1. $\vdash_{\mathbf{Nc}} A$
    2. A is true in BA $\mathbf{2}$.
    3. A is true in all BA.
  - We show this using a result known as Stone's theorem, stated below.
- Stone's representation theorem for BA.
  - **Theorem (Stone) :** Any BA is isomorphic to a sub-algebra of a power-set Boolean algebra.
  - We won't prove it here. A proof may be found in chapter 4 [1].
- Proof of Theorem 1.
  - We only need to show (2) $\Rightarrow$ (3) in the statement of Theorem 1.

    We show this by proving the contrapositive.

  For some BA B, and valuation $[\![\cdot]\!]_B$, let $[\![A]\!]_B \neq 1$.

  By stone's theorem, we may assume B to be a sub-algebra of $P(X)$, for some X.

  As BA $P(X)$ and BA 2^X are isomorphic, we may assume B to be a sub-algebra of 2^X.

  $\Rightarrow [\![A]\!]_B(x) \neq 1$, for some $x \in X$.

  $\Rightarrow [\![A]\!]_2 \neq 1$, where valuation $[\![.]\!]_2$ is given as $[\![P]\!]_2 = [\![P]\!]_B(x)$.

  [**Exercise:** Show by induction on A, that $[\![A]\!]_B(x) = [\![A]\!]_2$, where $[\![\cdot]\!]_B$ and $[\![\cdot]\!]_2$ are as above.]

  $\Rightarrow A$ is not true in BA $\mathbf{2}$. $\square$

References

[1] J. L. Bell and M. Machover: A Course in Mathematical Logic. Published by North-Holland Publishing Company, 1977.