6

So far we have considered proof theoretic aspects of logic, now we consider (algebraic) semantics of our logic.

Partial Order

A partial order R, on a set X, is a binary relation R on X, which is

- reflexive, $\forall\ x \in X,\ xRx$
- anti-symmetric, $\forall\ x, y \in X : (xRy \wedge yRx) \Rightarrow x = y$
- transitive, $\forall\ x, y, z \in X : (xRy \wedge yRz) \Rightarrow xRz$

Strict Partial Order

A strict partial order, $<$, on X is a binary rel^n is,

- irreflexive $\forall\ x \in X : \neg(x < x)$
- transitive $\forall\ x, y, z \in X : (x<y \wedge y<z) \Rightarrow (x<z)$

Theorem. If $<$ is a strict partial order on X then $\leq$ defined by $x \leq y \iff x < y \vee x = y$ is a partial order on X ■

Proof. – $x \leq x \iff x < x \vee x = x = \text{ff} \vee \text{tt} = \text{tt}$

– $x \leq y \wedge y \leq x \iff (x < y \vee x = y) \wedge (y < x \vee x = y)$

1. if $x = y$ antisymmetry is proved
2. if $x \neq y$ we have $x < y \wedge y < x$ whence $x < x$ by transitivity, in contradiction with irreflexivity, so this case is impossible.

– If $x \leq y \wedge y \leq z$ then $(x < y \vee x = y) \wedge (y < z \vee y = z)$

1. if $x = y$ then $x < z \vee x = z$ so $x \leq z$
2. if $y = z$ then $x < z \vee x = z$ so $x \leq z$
3. Otherwise $x < y \wedge y < z$ so by transitivity $x < z$

Poset

A poset $\langle X, \leq \rangle$ is a set equipped with a partial order $\leq$ on X.

Hasse Diagram

Let $\langle X, \leq \rangle$ be a finite poset. Its Hasse diagram is a set of points

$$\{p(a) \mid a \in X\}$$

in the euclidean plane, R^2 and a set of lines

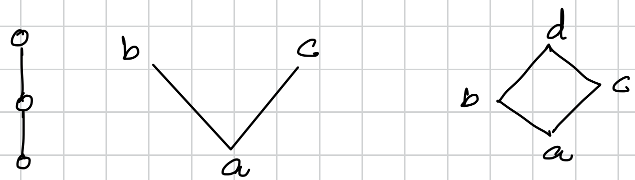
$$\{l(a,b) \mid a,b \in X \wedge a \prec b\}$$

joining $p(a)$ and $p(b)$, such that,

- if $a \prec b$ then $p(a)$ is lower than $p(b)$
- no point $p(c)$ belongs to line $l(a,b)$, $c \neq a,b$.

Theorem: A poset has no directed cycles other than self-loops

eg:



Upper Bound

if $a,b \in A$, is a P.O, $c \in A$, is an upper bound, of a,b, if $a \leq c$, and $b \leq c$.

An ub. c, is the least ub (lub) of a,b if for any ub d of (a,b) $c \leq d$

Lower Bound

↳ glb is least common ancestor

notations: $a \vee b = lub(a,b)$
$a \wedge b = glb(a,b)$

Lattice

A lattice is an algebraic system, (partial order) $(A, \leq)$, with two binary operations, $\wedge$ and $\vee$.
ie, $(A, \leq, \wedge, \vee)$

Any arbitary element a, b, c in A, satisfy four axioms,

1) Idempotent Law, $a \vee a = a$, $a \wedge a = a$
2) Commutative Law, $a \vee b = b \vee a$
   $a \wedge b = b \wedge a$
3) Associative Law, $a \vee (b \vee c) = (a \vee b) \vee c$
   $a \wedge (b \wedge c) = (a \wedge b$
4) Absorption Laws, $a \vee (a \wedge b) = a$
   $a \wedge (a \vee b) = a$

proof: $(a \vee b) \vee c = a \vee (b \vee c)$

It suffices to show that $a \vee b \leq a \vee (b \vee c)$ and
$c \leq a \vee (b \vee c)$
to prove, $(a \vee b) \vee c = a \vee (b \vee c)$

$a \leq a$ (reflexivity) $\Rightarrow$ $a \vee b \leq a \vee (b \vee c)$
$b \leq b \vee c$ (def^n)

$c \leq b \vee c$ $\Rightarrow$ $c \leq a \vee (b \vee c)$ (transitivity)
$b \vee c \leq a \vee (b \vee c)$

1. Let $A = \{0, 1\}$. Let $a \vee b = \max\{a, b\}$ and $a \cdot b = \min\{a, b\}$ be binary operations on A. Then, the algebraic system $\langle A, \vee, \cdot \rangle$ satisfies the axioms of the lattice.
2. Let Z be the set the integers. Let $a \vee b = \max\{a, b\}$ and $a \cdot b = \min\{a, b\}$ be binary operations on Z. Then, the algebraic system $\langle Z, \vee, \cdot \rangle$ satisfies the axioms of the lattice.
3. Let $S = \{a, b\}$. Let $\cup$ and $\cap$ be binary operations on $P(S) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$. Then, the algebraic system $\langle P(S), \cup, \cap \rangle$ satisfies the axioms of the lattice.
4. Let $A = \{1, 2, 3, 6\}$. Let $a \vee b =$(least common multiple of a and b), and $a \wedge b =$(greatest common divisor of a and b) be binary operations on A. Then, the algebraic system $\langle A, \vee, \wedge \rangle$ satisfies the axioms of the lattice. ∎

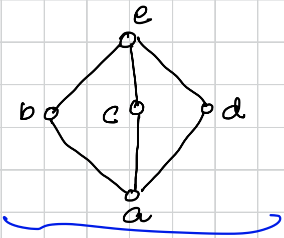
Destributive Lattice

A lattice satisfying, $(A, \vee, \wedge)$

5) Distributive Laws

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$
$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

eg:



non distributive

any lattic with ≤ 4 elements, is distributive.

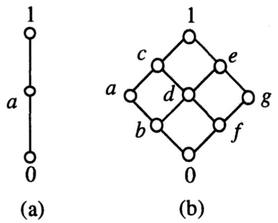


Figure 2.1 Examples of distributive lattice.

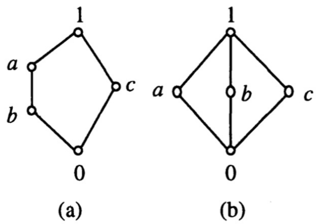


Figure 2.2 Examples of non-distributive lattice.

Let a lattice $(A, \vee, \wedge)$, have maximal element 1, and a minimal element 0.

For any element a in A, there exists an element xa such that, $a \vee xa = 1$ and $a \wedge xa = 0$
Then the lattice is a complemented lattice

xa is the complement of a

Boolean Algebra

A boolean algebra B, is a distributive lattice, with largest and smallest elements (1 and 0 respectively) and a unary operator, st $\forall\ a \in B$ $(a \vee \neg a = 1,\ a \wedge \neg a = 0)$.

(1) Idempotent laws: $a \vee a = a,\ a \cdot a = a;$
(2) Commutative laws: $a \vee b = b \vee a,\ a \cdot b = b \cdot a;$
(3) Associative laws: $a \vee (b \vee c) = (a \vee b) \vee c,$
$a \cdot (b \cdot c) = (a \cdot b) \cdot c;$
(4) Absorption laws: $a \vee (a \cdot b) = a,\ a \cdot (a \vee b) = a;$

(5) Distributive laws: $a \vee (b \cdot c) = (a \vee b) \cdot (a \vee c),$
$a \cdot (b \vee c) = (a \cdot b) \vee (a \cdot c);$
(6) Involution: $\bar{\bar{a}} = a;$
(7) Complements: $a \vee \bar{a} = 1,\ a \cdot \bar{a} = 0;$
(8) Identities: $a \vee 0 = a,\ a \cdot 1 = a;$
(9) $a \vee 1 = 1,\ a \cdot 0 = 0;$
(10) De Morgan's laws: $\overline{a \vee b} = \bar{a} \cdot \bar{b},\ \overline{a \cdot b} = \bar{a} \vee \bar{b}.$

$a \vee (a \wedge b) = a$

Ordered set
↓
lattice
↓
Distributive lattice
↓
Boolean Algebra

Huntington's Postulates

Boolean algebra is the algebra satisfying the ten axioms in Section 2.4.1. However, to verify whether the given algebra is Boolean algebra or not, we need only to check the following four axioms, the **Huntington's postulates**.

Identities: $a \vee 0 = a,\ a \cdot 1 = a;$
Commutative laws: $a \vee b = b \vee a,\ a \cdot b = b \cdot a;$
Distributive laws: $a \vee (b \cdot c) = (a \vee b) \cdot (a \vee c),\ a \cdot (b \vee c) = (a \cdot b) \vee (a \cdot c);$
Complements: $a \vee \bar{a} = 1,\ a \cdot \bar{a} = 0.$

From the above four axioms, we can derive the other axioms of the Boolean algebra.