Solution to Q1-Q4 of Practice Problems-2

Q1

(a) Recall the definition of $a \to b$ as the largest x s.t. $a \wedge x \leq b$

(i)

$a \wedge 1 = a \leq a$

$\Rightarrow a \wedge 1 \leq a$

$\Rightarrow a \to a \geq 1$ [By defn, of '$\to$']

$\Rightarrow a \to a = 1$

(ii)

$a \wedge (a \to b) \leq b$ [By defn, of '$\to$']

$a \wedge (a \to b) \leq a$ [By defn, of '$\wedge$']

$\Rightarrow a \wedge (a \to b) \leq a \wedge b \ \cdots (1)$

$a \wedge b \leq b$

$\Rightarrow b \leq a \to b$ [By defn, of '$\to$']

$\Rightarrow a \wedge b \leq a \wedge (a \to b) \ \cdots (2)$ [By monotonicity of $\wedge$]

By (1) and (2),

$a \wedge (a \to b) = a \wedge b$

(iii) Clearly,

$b \wedge (a \to b) \leq b \cdots (1)$

In **(ii)** we also saw

$b \leq (a \to b)$

$\Rightarrow b \leq b \wedge (a \to b) \cdots (2)$

By (1) and (2),

$b \wedge (a \to b) = b$

(iv) We first show $a \to (b \wedge c) \leq (a \to b)$

$a \wedge (a \to (b \wedge c)) = a \wedge b \wedge c \leq b$ [First equality is by **(i)**]

$\Rightarrow a \to (b \wedge c) \leq a \to b \cdots (1)$ [By defn, of '$\to$']

Similarly,

$a \to (b \wedge c) \leq a \to c \cdots (2)$

$a \to (b \wedge c) \leq (a \to b) \wedge (a \to c) \ \cdots (3)$ [By (1) and (2)]

For the other direction,

$a \wedge (a \to b) \wedge (a \to c)$

$= a \wedge b \wedge (a \to c)$ [By **(i)**]

$= b \wedge a \wedge (a \to c)$ [By commutativity of $\wedge$]

$\leq b \wedge c$ [By **(i)**]

$\Rightarrow a \wedge (a \to b) \wedge (a \to c) \leq b \wedge c$

$\Rightarrow (a \to b) \wedge (a \to c) \leq (a \to b \wedge c) \cdots (4)$ [By defn, of '$\to$']

By (3) and (4),

$a \to (b \wedge c) = (a \to b) \wedge (a \to c)$

(b) We need to show that $a \to b$ as the largest x s.t. $a \wedge x \leq b$.

This has two parts,

(I) $a \wedge (a \to b) = a \wedge b \leq b$

(II) Here we need to show that for any c,

$a \wedge c \leq b \Rightarrow c \leq a \to b$

We first show that $\to$ is monotone in the second argument.
Let $b \leq c$
$\Rightarrow b = b \wedge c$
$\Rightarrow a \to b = a \to (b \wedge c)$

$\Rightarrow a \to b = (a \to b) \wedge (a \to c)$ (by iv)

$\Rightarrow a \to b \leq a \to c$

Let c be s.t. $a \wedge c \leq b$

$\Rightarrow a \to a \wedge c \leq a \to b$

$\Rightarrow (a \to a) \wedge (a \to c) \leq a \to b$ (by iv)

$\Rightarrow (a \to c) \leq a \to b \cdots (A)$ (using i)

$c = c \wedge (a \to c) \leq (a \to c) \leq a \to b$

[First equality is by (iii), the last inequality is by (A)]

Q2

(a) Consider $\{a\} \to \{b\}$, for $a, b \in X$.

As, $\{a\} \cap X = \{a\} \not\subseteq \{b\}$, $\{a\} \to \{b\} \neq X$

$\Rightarrow \{a\} \to \{b\}$ is a finite subset of X.

Let this be Z.

By definition of '$\to$' in HA, Z should be the largest finite set not containing $\{a\}$.

Clearly, there is no *largest* such finite set in Y.

(b) Consider now Y under reverse ordering, that is $Y = (Y, \supseteq, \cap, \cup, X, \emptyset)$. Show that it is a Heyting algebra.

$(Y, \supseteq, \cap, \cup, X, \emptyset)$ is also a distributive lattice, with $\cup$ being the meet and $\cap$ being the join now.

For $A, B \in Y$, HA '$\to$' is now the smallest (in $\subseteq$ order but the largest in $\supseteq$ order) set $C \in Y$ s.t. $A \cup C \supseteq B$.

Clearly there is a smallest such set, namely $B - A$.

Q3

(a) $\Rightarrow$ direction: Immediate by definition of logical equivalence.

$\Leftarrow$ direction: We use contraposition.

Let $(B, [\![\cdot]\!]_B)$ be s.t. $[\![P]\!]_B \neq [\![Q]\!]_B$.

$\Rightarrow [\![P]\!]_B \not\leq [\![Q]\!]_B$ or $[\![Q]\!]_B \not\leq [\![P]\!]_B$.

$\Rightarrow [\![P]\!]_B \to [\![Q]\!]_B \neq 1$ or $[\![Q]\!]_B \to [\![P]\!]_B \neq 1$ [by a result in the first exam]

W.l.o.g. let former be the case.

$\Rightarrow P \to Q$ is not true in BA B.

$\Rightarrow P \to Q$ is not true in BA $\mathbf{2}$. [by strong completeness theorem]

$\Rightarrow$ There is a valuation $[\![\cdot]\!]_{\mathbf{2}}$, in BA $\mathbf{2}$, s.t. $[\![P \to Q]\!]_{\mathbf{2}} \neq 1$

$\Rightarrow$ There is a valuation $[\![\cdot]\!]_{\mathbf{2}}$, in BA $\mathbf{2}$, s.t. $[\![P]\!]_{\mathbf{2}} = 1$ and $[\![Q]\!] = 0$.

$\Rightarrow$ There is a valuation $[\![\cdot]\!]_{\mathbf{2}}$, in BA $\mathbf{2}$, s.t. $[\![P]\!]_{\mathbf{2}} \neq [\![Q]\!] = 0$.

(b) Truth table for P, Q enumerates $[\![P]\!]_{\mathbf{2}}, [\![Q]\!]_{\mathbf{2}}$ for *all* valuations $[\![\cdot]\!]_{\mathbf{2}}$, in BA $\mathbf{2}$.

Q4

(a) Yes.

Logical equivalence implies equi-satisfiability.

The converse does not hold in general. Literals p, $\neg p$ are both satisfiable but not logically equivalent.

(b) Consider valuation, $x = 0, y = 0, z = 1$.

$(x \vee y) \wedge (\neg x \vee z)$ is false under this valuation but $y \vee z$ is true.

So, both expressions are **not** logically equivalent.

Another valuation $x = 1, y = 0, z = 1$ makes $(x \vee y) \wedge (\neg x \vee z)$ true.

Therefore both propositions are satisfiale and hence **equi-satisfiable**.

(c) We only need to take care of the case in the induction step when $\phi = \phi_1 \vee \phi_2$ and ϕ_1, ϕ_2 are in CNF.

Let $\phi_1 \equiv C_1 \wedge \ldots \wedge C_m$ and let $\phi_2 \equiv E_1 \wedge \ldots \wedge E_n$, where all C_i and E_j are clauses.

Construct a CNF formula

$\psi \equiv (C_1 \cup \{x\}) \wedge \ldots \wedge (C_m \cup \{x\}) \wedge (E_1 \cup \{\neg x\}) \wedge \ldots \wedge (E_n \cup \{\neg x\})$, where x is a new variable not occurring in ϕ.

We leave the easy fact that ϕ and ψ are equi-satisfiable for the reader to verify.