Practice Problems Sheet-2

Q1

(a) Show that in a Heyting algebra $H = (H, \leq, \vee, \wedge, \rightarrow, 0, 1)$ following properties hold.

(i) $a \rightarrow a = 1$

(ii) $a \wedge (a \rightarrow b) = a \wedge b$

(iii) $b \wedge (a \rightarrow b) = b$

(iv) $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$

(b) Show that if in a distributive lattice $L = (L, \leq, \vee, \wedge, 0, 1)$ there is a binary operation '$\rightarrow$' satisfying properties **(i)-(iv)** of part **(a)** then $L = (L, \leq, \vee, \wedge, \rightarrow, 0, 1)$ is a Heyting algebra.

Q2 Let X be an infinite set. Let $Y = \{A \subseteq X \mid A \text{ is finite}\} \cup \{X\}$.

(a) Clearly $Y = (Y, \subseteq, \cup, \cap, \emptyset, X)$ is a distributive lattice. Show that it can't be extended to a Heyting algebra.

(b) Consider now Y under reverse ordering, that is $Y = (Y, \supseteq, \cap, \cup, X, \emptyset)$. Show that it is a Heyting algebra.

Q3 Definition: Two propositions P, Q are *logically equivalent* iff for any Boolean algebra B and any valuation $[\![\cdot]\!]_B$, $[\![P]\!]_B = [\![Q]\!]_B$.

(a) Show that P, Q are logically equivalent iff for any valuation $[\![\cdot]\!]_{\mathbf{2}}$, in BA $\mathbf{2}$, $[\![P]\!]_{\mathbf{2}} = [\![Q]\!]_{\mathbf{2}}$.

(b) Note that equivalence of P and Q in BA $\mathbf{2}$ is equivalent to saying that P, Q have the same truth table.

Q4 In this question all valuations are in BA **2.**

Definition: Two propositions P, Q are *equi-satisfiable* iff

P is satisfiable $\Leftrightarrow$ Q is satisfiable.

(a) Are logical equivalence and equi-satisfiable different relations on propositions?

(b) For any atomic propositions x, y and z are propositions $(x \vee y) \wedge (\neg x \vee z)$ and $y \vee z$ logically equivalent? Are they equi-satisfiable?

(c) In class, we saw that a proposition may explode exponentially in size when converting it to a logically equivalent CNF formula. Using idea in example of part **(b)** show that any proposition P can be efficiently converted to a equi-satisfiable proposition which is in CNF.

Q5 Choose an appropriate first order language and give a first order sentence for each of the statement below.

(a) Some people can be fooled all the time.

(b) Everyone can be fooled sometime.

(c) Not all people can be fooled all the time.

Q6 For each of the following judgment, give its derivation in the weakest of the three systems Nm, Ni and Nc, in which it can be derived.

(a) $\vdash \forall x(A \vee B) \leftrightarrow \forall x A \vee B$, $x \notin FV(B)$

(b) $\vdash \forall x(A \wedge B) \leftrightarrow \forall x A \wedge B$, $x \notin FV(B)$

(c) $\vdash \forall x A \leftrightarrow \neg \exists x \neg A$

(d) $\vdash \exists x A \leftrightarrow \neg \forall x \neg A$

Q7 In **Q6(a)** and **Q6(d),** label those deductions which are in minimal logic, with constructions as per BHK interpretation.

Q8 Show that each quantifier inference rule given in class is sound w.r.t. classical semantics of quantifiers.

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