CS202M Second Exam Solutions

Q1

(i) The base case has following two subcases.

(a) $A \equiv \bot$

We need to show that $\neg\neg\bot \vdash \bot$. This is shown below.

$$\dfrac{\dfrac{axiom}{\neg\neg\bot \vdash \neg\bot \to \bot} \qquad \dfrac{\dfrac{axiom}{\bot \vdash \bot}}{\vdash \neg\bot} \to i}{\neg\neg\bot \vdash \bot} \to e$$

(b) $A \equiv P$, where P an atomic proposition ($\neq \bot$).

$\Rightarrow A^\circ = \neg\neg P$

We need to show that $\neg\neg\neg\neg P \vdash \neg\neg P$

This follows by taking $A \equiv \neg P$ in the judgment (a.2) of the instructions.

(ii) By I.H. $\neg\neg A_1^\circ \vdash A_1^\circ$ and $\neg\neg A_2^\circ \vdash A_2^\circ$

We need to show

$\neg\neg(A_1 \to A_2)^\circ \vdash (A_1 \to A_2)^\circ$

or $\neg\neg(A_1^\circ \to A_2^\circ) \vdash A_1^\circ \to A_2^\circ$

Following is a derivaion of this.

$$\dfrac{\dfrac{\dfrac{\dfrac{c.1}{\neg\neg(A_1^\circ \to A_2^\circ) \vdash \neg\neg A_1^\circ \to \neg\neg A_2^\circ} \qquad \dfrac{a.1}{A_1^\circ \vdash \neg\neg A_1^\circ}}{\neg\neg(A_1^\circ \to A_2^\circ), A_1^\circ \vdash \neg\neg A_2^\circ} \to e \qquad \dfrac{IH}{\neg\neg A_2^\circ \vdash A_2^\circ}}{\neg\neg(A_1^\circ \to A_2^\circ), A_1^\circ \vdash A_2^\circ} \textbf{cut}}{\neg\neg(A_1^\circ \to A_2^\circ) \vdash A_1^\circ \to A_2^\circ} \to i$$

(iii) We need to show

$\neg\neg(A_1 \vee A_2)^\circ \vdash (A_1 \vee A_2)^\circ$

or $\neg\neg\neg(\neg A_1^\circ \wedge \neg A_2^\circ) \vdash \neg(\neg A_1^\circ \wedge \neg A_2^\circ)$

This follows by taking $A \equiv \neg A_1^\circ \wedge \neg A_2^\circ$ in the judgment (a.2) of the instructions.

Q2

(i) Let the derivation $\Gamma \vdash_{\mathbf{Nc}} A$ be

$$\dfrac{\dfrac{\mathcal{D}_1}{\Gamma \vdash_{\mathbf{Nc}} A_1}}{\Gamma \vdash_{\mathbf{Nc}} A_1 \vee A_2}.$$

By I.H. $\Gamma^\circ \vdash_{\mathbf{Ni}} A_1^\circ$,

we need to show that $\Gamma^\circ \vdash_{\mathbf{Ni}} \neg(\neg A_1^\circ \wedge \neg A_2^\circ)$.

Following derivation shows this.

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{axiom}}{\neg A_1^\circ \wedge \neg A_2^\circ \vdash \neg A_1^\circ \wedge \neg A_2^\circ}}{\neg A_1^\circ \wedge \neg A_2^\circ \vdash \neg A_1^\circ} \vee e \qquad \dfrac{\text{I.H.}}{\Gamma^\circ \vdash_{\mathbf{Ni}} A_1^\circ}}{\Gamma^\circ, \neg A_1^\circ \wedge \neg A_2^\circ \vdash_{\mathbf{Ni}} \bot} \to e}{\Gamma^\circ \vdash_{\mathbf{Ni}} \neg(\neg A_1^\circ \wedge \neg A_2^\circ)} \to i$$

(ii) Let the derivation $\Gamma \vdash_{\mathbf{Nc}} A$ be

$$\dfrac{\dfrac{\mathcal{D}_0}{\Gamma_1 \vdash_{\mathbf{Nc}} A_1 \vee A_2} \qquad \dfrac{\mathcal{D}_1}{A_1, \Gamma_2 \vdash_{\mathbf{Nc}} A} \qquad \dfrac{\mathcal{D}_2}{A_2, \Gamma_2 \vdash_{\mathbf{Nc}} A}}{\Gamma_1, \Gamma_2 \vdash_{\mathbf{Nc}} A}$$

By I.H. $\Gamma_1^\circ \vdash_{\mathbf{Ni}} \neg(\neg A_1^\circ \wedge \neg A_2^\circ)$,

$A_1^\circ,\ \Gamma_2^\circ \vdash_{\mathbf{Ni}} A^\circ$

and $A_2^\circ, \Gamma_2^\circ \vdash_{\mathbf{Ni}} A^\circ$

we need to show that $\Gamma_1^\circ, \Gamma_2^\circ \vdash_{\mathbf{Ni}} A^\circ$.

Let $\mathcal{E}_1$ be the derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{I.H.}}{A_1^\circ, \Gamma_2^\circ \vdash_{\mathbf{Ni}} A^\circ}}{\Gamma_2^\circ \vdash_{\mathbf{Ni}} A_1^\circ \to A^\circ}\to i \qquad \dfrac{2.d}{A_1^\circ \to A^\circ \vdash \neg A^\circ \to \neg A_1^\circ}}{\Gamma_2^\circ \vdash_{\mathbf{Ni}} \neg A^\circ \to \neg A_1^\circ}\text{cut} \qquad \dfrac{\text{axiom}}{\neg A^\circ \vdash \neg A^\circ}}{\Gamma_2^\circ, \neg A^\circ \vdash_{\mathbf{Ni}} \neg A_1^\circ}\to i$$

Similarly, let $\mathcal{E}_2$ be the derivation of $\Gamma_2^\circ, \neg A^\circ \vdash_{\mathbf{Ni}} \neg A_2^\circ$

Further, let $\mathcal{F}$ be the derivation

$$\dfrac{\dfrac{\dfrac{\mathcal{E}_1}{\Gamma_2^\circ, \neg A^\circ \vdash_{\mathbf{Ni}} \neg A_1^\circ} \qquad \dfrac{\mathcal{E}_2}{\Gamma_2^\circ, \neg A^\circ \vdash_{\mathbf{Ni}} \neg A_2^\circ}}{\Gamma_2^\circ, \neg A^\circ, \Gamma_2^\circ, \neg A^\circ \vdash_{\mathbf{Ni}} \neg A_1^\circ \wedge \neg A_2^\circ}\wedge i}{\Gamma_2^\circ, \neg A^\circ \vdash_{\mathbf{Ni}} \neg A_1^\circ \wedge \neg A_2^\circ}\text{contraction}$$

The required derivation is

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{I.H.}}{\Gamma_1^\circ \vdash_{\mathbf{Ni}} \neg(\neg A_1^\circ \wedge \neg A_2^\circ)} \qquad \dfrac{\mathcal{F}}{\Gamma_2^\circ, \neg A^\circ \vdash_{\mathbf{Ni}} \neg A_1^\circ \wedge \neg A_2^\circ}}{\Gamma_1^\circ, \Gamma_2^\circ, \neg A^\circ \vdash_{\mathbf{Ni}} \bot}\to e}{\Gamma_1^\circ, \Gamma_2^\circ \vdash_{\mathbf{Ni}} \neg\neg A^\circ}\to i \qquad \dfrac{\text{Q1}}{\neg\neg A^\circ \vdash A^\circ}}{\Gamma_1^\circ, \Gamma_2^\circ \vdash_{\mathbf{Ni}} A^\circ}\text{cut}$$

Q3

(i) Let $c, d \in B \restriction a$.

$\Rightarrow c, d \leq a$

$\Rightarrow c \vee d \leq a$ ($\because$ a is an ub for c, d and $c \vee d$ is the least such ub)

$\Rightarrow c \vee d \in B \restriction a$.

Also,

$c \wedge d \leq c \leq a$

$\Rightarrow c \wedge d \in B \restriction a$

(ii) We turn this into a BA as $(B \restriction a, \leq, \vee, \wedge, \sim, 0, a)$, where

- symbol '$\sim$' stands for negation in $B \restriction a$ which is defined as $\sim x = \neg x \wedge a$.
- element a being the largest in $B \restriction a$, stands for 1 of the BA.

We need to verify the axioms for negation '$\sim$'.

For any $x \in B \restriction a$,

- $x \vee \sim x = x \vee (\neg x \wedge a) = (x \vee \neg x) \wedge (x \vee a) = a$
- $x \wedge \sim x = x \wedge (\neg x \wedge a) = (x \wedge \neg x) \wedge a = 0 \wedge a = 0$

(iii) No.

This is because constant 1 of BA $B \restriction a$ is different from constant 1 of parent BA B.

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