Solutions to Practice Problems Sheet-1

Q1

(a) Assume that there is a set $A = \{x \mid x \text{ is a set and } x \notin x\}$.

If $A \in A$ then by definition of A, $A \notin A$.

If $A \notin A$ then again by definition of A, $A \in A$.

Either $A \in A$ or $A \notin A$ (membership relation is classical) and we have contradiction in both cases.

Therefore our assumption that there is a set A as above, is untenable.

(Note that this argument holds both in naive set theory or in axiomatic set theory like ZF.)

(b) Assume that ϕ is provable in T then by definition of ϕ, ϕ is *False*. As T is sound this is not possible.

Therefore ϕ is not provable in T, again by definition of ϕ, it is *True*.

So there is a *True* sentence of T which is not provable in T.

Therefore T **is incomplete**.

Q2

(a) In minimal logic.

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\neg(A \vee B) \vdash \neg(A \vee B)$}
\AxiomC{}
\UnaryInfC{$A \vdash A$}
\UnaryInfC{$A \vdash A \vee B$}
\BinaryInfC{$\neg(A \vee B), A \vdash \bot$}
\UnaryInfC{$\neg(A \vee B) \vdash \neg A$}
\AxiomC{}
\UnaryInfC{$\neg(A \vee B) \vdash \neg(A \vee B)$}
\AxiomC{}
\UnaryInfC{$B \vdash B$}
\UnaryInfC{$B \vdash A \vee B$}
\BinaryInfC{$\neg(A \vee B), B \vdash \bot$}
\UnaryInfC{$\neg(A \vee B) \vdash \neg B$}
\BinaryInfC{$\neg(A \vee B), \neg(A \vee B) \vdash \neg A \wedge \neg B$}
\UnaryInfC{$\neg(A \vee B) \vdash \neg A \wedge \neg B$}
\UnaryInfC{$\vdash \neg(A \vee B) \to \neg A \wedge \neg B$}
\end{prooftree}
$$

(b) In minimal logic.

$$
\dfrac{\dfrac{\dfrac{}{A \vee B \vdash A \vee B} \qquad \dfrac{\dfrac{\dfrac{}{\neg A \wedge \neg B \vdash \neg A \wedge \neg B}}{\neg A \wedge \neg B \vdash \neg A} \quad \dfrac{}{A \vdash A}}{\neg A \wedge \neg B, A \vdash \bot} \qquad \dfrac{\dfrac{\dfrac{}{\neg A \wedge \neg B \vdash \neg A \wedge \neg B}}{\neg A \wedge \neg B \vdash \neg B} \quad \dfrac{}{B \vdash B}}{\neg A \wedge \neg B, B \vdash \bot}}{\neg A \wedge \neg B, A \vee B \vdash \bot}}{\vdash \neg A \wedge \neg B \to \neg(A \vee B)}
$$

(c) In classical logic.

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(A \wedge B) \vdash \neg(A \wedge B)} \qquad \dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\neg(\neg A \vee \neg B) \vdash A} \quad \dfrac{\mathcal{D}_2}{\neg(\neg A \vee \neg B) \vdash B}}{\neg(\neg A \vee \neg B), \neg(\neg A \vee \neg B) \vdash A \wedge B}}{\neg(\neg A \vee \neg B) \vdash A \wedge B}}{\neg(A \wedge B), \neg(\neg A \vee \neg B) \vdash \bot}}{\neg(A \wedge B) \vdash \neg A \vee \neg B}\,\textbf{PBC}}{\vdash \neg(A \wedge B) \to \neg A \vee \neg B}
$$

- where $\mathcal{D}_1$ is

$$
\dfrac{\dfrac{\dfrac{}{\neg(\neg A \vee \neg B) \vdash \neg(\neg A \vee \neg B)} \qquad \dfrac{\dfrac{}{\neg A \vdash \neg A}}{\neg A \vdash \neg A \vee \neg B}}{\neg(\neg A \vee \neg B), \neg A \vdash \bot}}{\neg(\neg A \vee \neg B) \vdash A}\ \text{PBC}
$$

(PBC stands for proof by contradiction.)

- and $\mathcal{D}_2$ is

$$
\dfrac{\dfrac{\dfrac{}{\neg(\neg A \vee \neg B) \vdash \neg(\neg A \vee \neg B)} \qquad \dfrac{\dfrac{}{\neg B \vdash \neg B}}{\neg B \vdash \neg A \vee \neg B}}{\neg(\neg A \vee \neg B), \neg B \vdash \bot}}{\neg(\neg A \vee \neg B) \vdash B}\ \text{PBC}
$$

(d) In minimal logic.

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\neg A \vee \neg B \vdash \neg A \vee \neg B} \qquad \dfrac{\dfrac{}{\neg A \vdash \neg A} \quad \dfrac{\dfrac{}{A \wedge B \vdash A \wedge B}}{A \wedge B \vdash A}}{A \wedge B, \neg A \vdash \bot} \qquad \dfrac{\dfrac{}{\neg B \vdash \neg B} \quad \dfrac{\dfrac{}{A \wedge B \vdash A \wedge B}}{A \wedge B \vdash B}}{A \wedge B, \neg B \vdash \bot}}{\neg A \vee \neg B, A \wedge B \vdash \bot}}{\neg A \vee \neg B \vdash \neg(A \wedge B)}}{\vdash \neg A \vee \neg B \to \neg(A \wedge B)}
$$

(e) In minimal logic.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\neg\neg A \vdash \neg\neg\neg A} \qquad \dfrac{\dfrac{\dfrac{}{\neg A \vdash \neg A} \qquad \dfrac{}{A \vdash A}}{A, \neg A \vdash \bot}}{A \vdash \neg\neg A}}{\neg\neg\neg A, A \vdash \bot}}{\neg\neg\neg A \vdash \neg A}}{\vdash \neg\neg\neg A \to \neg A}$$

Q3

(a)

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{A \to B \vdash (\neg B \to \neg A)} \qquad \dfrac{\dfrac{\mathcal{D}_2}{\neg B \to \neg A \vdash (\neg\neg A \to \neg\neg B)}}{\vdash (\neg B \to \neg A) \to (\neg\neg A \to \neg\neg B)}}{(A \to B) \vdash (\neg\neg A \to \neg\neg B)}}{\vdash (A \to B) \to (\neg\neg A \to \neg\neg B)}$$

- where $\mathcal{D}_1$ is

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \to B \vdash A \to B} \qquad \dfrac{}{A \vdash A}}{A \to B, A \vdash B} \qquad \dfrac{}{\neg B \vdash \neg B}}{A \to B, A, \neg B \vdash \bot}}{A \to B, \neg B \vdash \neg A}}{A \to B \vdash (\neg B \to \neg A)}$$

- $\mathcal{D}_2$ is similar, with A, B replaced by $\neg B, \neg A$ respectively.

(b) Forward implication

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\neg\neg(A \wedge B) \vdash \neg\neg A} \qquad \dfrac{\mathcal{D}_2}{\neg\neg(A \wedge B) \vdash \neg\neg B}}{\neg\neg(A \wedge B), \neg\neg(A \wedge B) \vdash \neg\neg A \wedge \neg\neg B}}{\neg\neg(A \wedge B) \vdash \neg\neg A \wedge \neg\neg B}}{\vdash \neg\neg(A \wedge B) \to \neg\neg A \wedge \neg\neg B}$$

- where $\mathcal{D}_1$ is

$$\begin{array}{cc} \hline A \wedge B \vdash A \wedge B & \text{axiom} \\ \hline A \wedge B \vdash A & \wedge e2 \\ \hline \neg A \vdash \neg(A \wedge B) & \text{CPOS} \\ \hline \neg\neg(A \wedge B) \vdash \neg\neg A & \text{CPOS} \end{array}$$

In which we have used a derived inference rule

$$\frac{\Gamma, A \vdash B}{\Gamma, \neg B \vdash \neg A} \text{ CPOS [CPOS stands for contra position]}$$

whose derivation is as follows.

$$\dfrac{\dfrac{\Gamma, A \vdash B \qquad \dfrac{}{\neg B \vdash \neg B}}{\Gamma, \neg B, A \vdash \bot}}{\Gamma, \neg B \vdash \neg A}$$

- Derivation $\mathcal{D}_2$ is similar.

(b) Backward implication,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{A, B, \neg(A \wedge B) \vdash \bot}}{\neg(A \wedge B), B \vdash \neg A} \qquad \dfrac{\dfrac{}{\neg\neg A \wedge \neg\neg B \vdash \neg\neg A \wedge \neg\neg B}}{\neg\neg A \wedge \neg\neg B \vdash \neg\neg A}}{\dfrac{\neg(A \wedge B), B, \neg\neg A \wedge \neg\neg B \vdash \bot}{\neg(A \wedge B), \neg\neg A \wedge \neg\neg B \vdash \neg B}} \qquad \dfrac{\dfrac{}{\neg\neg A \wedge \neg\neg B \vdash \neg\neg A \wedge \neg\neg B}}{\neg\neg A \wedge \neg\neg B \vdash \neg\neg B}}{\neg\neg A \wedge \neg\neg B, \neg\neg A \wedge \neg\neg B, \neg(A \wedge B) \vdash \bot}}{\neg\neg A \wedge \neg\neg B, \neg(A \wedge B) \vdash \bot}}{\dfrac{\neg\neg A \wedge \neg\neg B \vdash \neg\neg(A \wedge B)}{\vdash \neg\neg A \wedge \neg\neg B \to \neg\neg(A \wedge B)}}$$

- where $\mathcal{D}_1$ is

$$\dfrac{\dfrac{\dfrac{}{A \vdash A} \qquad \dfrac{}{B \vdash B}}{A, B \vdash (A \wedge B)} \qquad \dfrac{}{\neg(A \wedge B) \vdash \neg(A \wedge B)}}{A, B, \neg(A \wedge B) \vdash \bot}$$

Q4

(a) Labeling is as follows.

$$\wedge i \ \frac{\Gamma_1 \vdash t_1 : A \quad \Gamma_2 \vdash t_2 : B}{\Gamma_1, \Gamma_2 \vdash \langle t_1, t_2 \rangle : A \wedge B}$$

$$\vee i1 \ \frac{\Gamma \vdash t : A}{\Gamma \vdash inl(t) : A \vee B}$$

$$\vee i2 \ \frac{\Gamma \vdash t : B}{\Gamma \vdash inr(t) : A \vee B}$$

$$\rightarrow i \ \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x : A.t : A \rightarrow B}$$

$$\rightarrow e \ \frac{\Gamma_1 \vdash t : A \rightarrow B \quad \Gamma_2 \vdash s : A}{\Gamma_1, \Gamma_2 \vdash ts : B}$$

(b) Precise meaning of these construction terms is given by the following equations:

1. $\pi_1 \langle t, s \rangle = t$
2. $\pi_2 \langle t, s \rangle = s$
3. case $inl(v)$ of $inl(a) \Rightarrow t_1 |\ inr(b) \Rightarrow t_2 = t_1[v/a]$,
   where $t_1[v/a]$ stands for substituting all (free) occurrences a in t_1 by v.
4. case $inr(v)$ of $inl(a) \Rightarrow t_1 |\ inr(b) \Rightarrow t_2 = t_2[v/b]$

(c) Labeling for structural rules is as follows.

1. $$\frac{\Gamma \vdash t : A}{\Gamma, x : B \vdash t : A} \text{ weakening}$$

2. $$\frac{\Gamma, x : B, y : B \vdash t : A}{\Gamma, z : B \vdash t[z/x, z/y] : A} \text{ contraction}$$