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Algebraic Semantics of Classical Logic

Usually semantics for classical propositional formulas is defined in terms of two truth values.

Let $\mathbb{B} = \{0,1\}$.

A valuation in $\mathbb{B}$, is a map $v : PV \to \mathbb{B}$, such a map is also called a 0,1 valuation

Given a 0-1 valuation, define the map, $[\![\bullet]\!]_v : \Phi \to \mathbb{B}$,

a) $[\![p]\!]_v = v(p)$ for $p \in PV$
b) $[\![\bot]\!]_v = 0$
c) $[\![\phi \vee \psi]\!]_v = \max\{[\![\phi]\!]_v, [\![\psi]\!]_v\}$
d) $[\![\phi \wedge \psi]\!]_v = \min\{[\![\phi]\!]_v, [\![\psi]\!]_v\}$
e) $[\![\phi \to \psi]\!]_v = \max\{1 - [\![\phi]\!]_v, [\![\psi]\!]_v\}$

* $\varphi \in \Phi$, is considered as a tautology, if, $v(\varphi) = 1$, for all valuations in $\mathbb{B}$!

Field of sets

A field of sets (over X), is a non empty family $\mathcal{R}$ of subsets, closed over union, intersection and compliments (to X).

2.3.3. DEFINITION. Let $\mathcal{R}$ be a field of sets over X.

(i) A *valuation in* $\mathcal{R}$ is a map $v : PV \to \mathcal{R}$.

(ii) Given a valuation v in $\mathcal{R}$, define the map $[\![\bullet]\!]_v : \Phi \to X$ by:

$$\begin{array}{lcll} [\![p]\!]_v & = & v(p) & \text{for } p \in PV \\ [\![\bot]\!]_v & = & \{\} & \\ [\![\varphi \vee \psi]\!]_v & = & [\![\varphi]\!]_v \cup [\![\psi]\!]_v & \\ [\![\varphi \wedge \psi]\!]_v & = & [\![\varphi]\!]_v \cap [\![\psi]\!]_v & \\ [\![\varphi \to \psi]\!]_v & = & (X - [\![\varphi]\!]_v) \cup [\![\psi]\!]_v & \end{array}$$

We also write $v(\varphi)$ for $[\![\varphi]\!]_v$.

Proposition

These two semantics are equivalent, ie following are equivalent for each field of subsets R over X.

> ϕ is a tautology
> $v(\phi) = X$, for all valuations v in R.

proof:

(1) $\Rightarrow$ (2)

suppose that $v(\phi) \neq X$, then there is an element $a \in X$, such that $a \notin v(\phi)$. Define a 0-1 valuations w, so that $w(p) = 1$, iff $a \in v(p)$
By induction,

$$w(\psi) = 1, \text{ iff } a \in v(\psi)$$

Then $w(\phi) \neq 1$.

(2) $\Rightarrow$ (1)

A 0-1 valuation can be seen as a valuation in R that assigns only X and $\{\}$ to propositional variables

We can generalize this set of semantics to arbitrary Boolean Algebras, by replacing valuation in a field of sets by valuations in a Boolean algebra, in the obvious way.

In fact, every Boolean algebra is isomorphic to a field of sets, so this generalization does not change our semantic.

Given any B.A, $(B, \leq, \vee, \wedge, 0, 1, \neg)$, we would like to associate with each prop^n, an element of BA

Let B, be a BA, and $[\![\cdot]\!]$ a valuation into B.

* $[\![\cdot]\!] : \{\text{atomic prop}^n\} \to B$.

$$[\![A \wedge B]\!] = [\![A]\!] \wedge [\![B]\!]$$

boolean algebra operations

$$[\![A \vee B]\!] = [\![A]\!] \vee [\![B]\!]$$

$$[\![A \to B]\!] = \neg[\![A]\!] \vee [\![B]\!]$$

$$[\![\bot]\!] = 0 \quad \text{(part of definition)}$$

$$[\![\neg A]\!] = [\![A \to \bot]\!] = \neg[\![A]\!] \vee [\![\bot]\!] = \neg[\![A]\!] \vee 0$$
$$= \neg[\![A]\!].$$

operations on prop^n

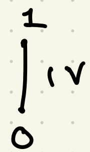
* A is true in this representation if $[\![A]\!] = 1$.

Boolean Algebra $(B, \leq, \vee, \wedge, 0, 1, \neg)$

- $(B, \leq, \vee, \wedge)$ is a distributive lattice.
- $1, 0$ are the largest and smallest element respectively.
- '$\neg$' is a unary operation

$$\forall a \in B, \quad \neg a \wedge a = 0 \quad \neg a \vee a = 1$$

Examples

$2 \equiv$ 

Not a number 2, just name of this algebra

- $P(x) = (X, \subseteq, \cup, \cap, \phi, X, \bar{\ })$
- Boolean vectors of length n (for some fixed $n \in \mathbb{N}$)

operations are defined bitwise

$(0101) \wedge (1100) = 0100$ $\neg(0101) = 1010$

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$?

$2 \times 2 \times 2 \times 2$

$\{0,1\}^n$

i^{th} bit on lhs

$a_i \wedge (b_i \vee c_i)$

$= (a_i \wedge b_i) \vee (a_i \wedge c_i)$

$= i^{th}$ bit on the rhs

$B_1 \times B_2 = \{ (b_1, b_2) \mid b_1 \in B_1, b_2 \in B_2 \}$

Define operations componentwise

$(b_1, b_2) \wedge (c_1, c_2) = (b_1 \wedge c_1, b_2 \wedge c_2)$

Ex: Verify that $B_1 \times B_2$ is a BA

2^X for any set X.

Domain is the set of fns : $X \to 2$

Operations are defined componentwise (that is independently at each index)

Ex: show that 2^X is a BA

Ex: 2^X is isomorphic to $P(X)$.

$X = \{1, 2, \dots, 5\}$

$f_A(y) = 0$ if $y \notin A$
$= 1$ if $y \in A$

$A = \{2, 4\}$

$f : X \to \{0, 1\}$

$f(1) = 0,\ f(2) = 1,\ f(3) = 0,\ f(4) = 1,\ f(5) = 0$

$f_1, f_2 \in 2^X$

$(f_1 \wedge f_2)(y) = f_1(y) \wedge f_2(y)$ for $y \in X$

Lemma :

1) For any $a, b \in B$
   if $a \wedge b = 0$ and $a \vee b = 1$
   then $b = \neg a$

2) $\neg\neg a = a$

3) $\neg(a \vee b) = \neg a \wedge \neg b$
   $\neg(a \wedge b) = \neg a \vee \neg b$

Proof : 1) $a \wedge b = 0$

$\Rightarrow \neg a \vee (a \wedge b) = \neg a \wedge 0$

$\Rightarrow (\neg a \vee a) \wedge (\neg a \vee b) = \neg a$

$\Rightarrow 1 \wedge (\neg a \vee b) = \neg a$

$\Rightarrow \neg a \vee b = \neg a$

$\Rightarrow b \leq \neg a$

$a \vee b = 1$

$\neg a \wedge (a \vee b) =$
$\neg a \wedge 1$

Similar simplification leads to
$\neg a \leq b$

2) By application of 1

$$\left.\begin{aligned} \neg a \wedge a &= 0 \\ \neg a \vee a &= 1 \end{aligned}\right\} \Rightarrow \quad \neg\neg a = a$$

by part 1.

3) Strategy is to show

$$\left.\begin{aligned} (a \vee b) \wedge (\neg a \wedge \neg b) &= 0 \\ (a \vee b) \vee (\neg a \wedge \neg b) &= 1 \end{aligned}\right\} \underline{\underline{Ex}}.$$

we can conclude by part 1 that

$$\neg(a \vee b) = \neg a \wedge \neg b$$

The second De Morgan law can be shown similarly.