# 14

# First Order Logic.

(1) All men are mortal
(2) Socrates is a man
(3) $\Rightarrow$ Socrates is mortal.

In propositional logic, these sentences can be classified as,

$$\left.\begin{array}{l} (1)\ M \\ \underline{(2)\ S} \\ \quad H \end{array}\right\} \text{clearly invalid}$$

Predicate logic lets us translate and construct derivat^n for arguments whose validity depends on the components of simple statements.

$$\begin{array}{l} \text{man}(X) \leftarrow X \text{ is a man} \\ \text{mortal}(X) \leftarrow Y \text{ is a mortal} \end{array}$$

$$\Rightarrow \forall x\ (\text{man}(x) \rightarrow \text{mortal}(x))$$

Atomic proposition man(X) and mortal(X), are parametrized by variable (X).

Variables are still not sufficient
$\Rightarrow$ We need two more quantifiers, $\forall x$ and $\exists x$

Thus the sentence,
'Every student $x$, is younger than some instructor $y$'

$$\forall x\,(S(x) \rightarrow \exists y\ (I(y) \wedge Y(x,y)))$$

## Context to evaluate FOL

Variables range over a set of objects, (also called the universe).
↳ Collection of objects of interest

It is possible for predicates to have multiple variables

"Alice and Bob have the same grandmother".
a b

$$\exists w \exists u \exists v \begin{pmatrix} \text{mother}(u, \text{Alice}) \wedge \text{mother}(v, \text{Bob}) \wedge \\ \text{mother}(w, u) \wedge \text{mother}(w, v) \end{pmatrix}$$

## Language of FOL

Non logical symbols : Constants, Relations
$C_1 \ldots C_n$ $\quad$ $R_1 \ldots R_n$
↳ $R_i$ is $a_i$ place rel$^n$ ($a_i$: arity)

Different choice of non-logical symbols, leads to different FOL, but they are still defined uniformly.

Terms ← A constant or a variable

> A variable is a term
> If $c \in F$ is a nullary fu$^n$, then $c$ is a term
> If $t_1, t_2 \ldots t_m$ are terms and $f \in F$ has an arity $n > 0$, then $f(t_1, t_2 \ldots t_n)$ is a term.

## Atomic Formula

If $R_i$ is a relational symbol of arity $a_i$, and $t_1, \ldots t_{a_i}$ are terms then $R_i(t_1, \ldots t_{a_i})$ is atomic formula.

## Formula

> Every atomic formula is a formula.
> If $\phi_1, \phi_2$ are formula, then,

$$\phi_1 \vee \phi_2, \quad \phi_1 \wedge \phi_2, \quad \phi_1 \rightarrow \phi_2 \text{ are also formulas.}$$

> $\forall x\, \phi$, and $\exists x\, \phi$ are also formulas.

## FREE AND BOUND VARIABLES

$$\forall x \underbrace{[\text{man}(x) \longrightarrow \text{mortal}(x)]}_{\text{scope of } \forall}$$

$\Rightarrow$ bound occurance

$$\psi(x) = \forall x \underbrace{[\ldots]}_{\text{bound}} \wedge \text{philosopher}(x)$$

free

## CONVENTION:

$\forall, \exists, \neg$ bind the tightest.

$$\forall x\, \phi \wedge \psi \equiv (\forall x\, \phi) \wedge \psi$$

$$\not\equiv$$

$$\forall x\, (\phi \wedge \psi)$$

# Quantifier Rules

$$\frac{\Gamma \vdash \phi[t/x]}{\Gamma \vdash \exists x\, \phi} \quad \exists_i$$

$$\frac{\Gamma \vdash \exists x\, \phi \qquad \Gamma_1, \phi[y/x] \vdash \psi}{\Gamma, \Gamma_1 \vdash \psi} \quad \exists_e \qquad (y \notin FV(\Gamma_1, \psi, \exists x\, \phi))$$

$$\frac{\Gamma \vdash \phi[y/x]}{\Gamma \vdash \forall x\, \phi} \quad \forall_i \qquad (x \notin FV(\Gamma, \forall x\, \phi))$$

$$\frac{\Gamma \vdash \forall x\, \phi}{\Gamma \vdash \phi[t/x]} \quad \forall_e$$

Q. $\vdash \exists x\, (P(x) \wedge Q(x)) \to (\exists x P(x) \wedge \exists x Q(x))$

Sol:

$$\dfrac{\dfrac{Ax}{\exists x (P(x) \wedge Q(x)) \vdash \exists x (P(x) \wedge Q(x))} \qquad \dfrac{\dfrac{\dfrac{Ax}{P[y/x] \wedge Q[y/x] \vdash P[y/x] \wedge Q[y/x]}}{P[y/x] \wedge Q[y/x] \vdash P[y/x]}\, \wedge_e}{P[y/x] \wedge Q[y/x] \vdash \exists x P[x]}\, \exists_i}{\exists x\, (P(x) \wedge Q(x)) \vdash \exists x P(x)}\, \exists_e$$

$y \notin FV(\exists (P(x) \wedge Q(x)), \exists x P(x))$

$$\dfrac{\dfrac{D_1}{\exists x (P(x) \wedge Q(x)) \vdash \exists x P(x)} \qquad \dfrac{D_2}{\exists x (P(x) \wedge Q(x)) \vdash \exists x Q(x)}}{}$$

$\exists x\, (P \wedge Q) \vdash \exists x P \wedge \exists x Q$

$\exists x\, (P \wedge Q) \vdash \exists x$

Q. $\exists x\, P(x) \vdash \neg \forall x\, \neg P(x)$

$$\dfrac{\dfrac{Ax}{\exists x P(x) \vdash \exists x P(x)} \qquad \dfrac{\dfrac{\dfrac{\forall x \neg P(x) \vdash \forall x \neg P(x)}{\forall x \neg P(x) \vdash \neg P[y/x]}\, \forall_e \qquad \dfrac{Ax.}{P[y/x] \vdash P[y/x]}}{\forall x \neg P(x),\, P[y/x] \vdash \bot}\, \to_e}{\exists x P(x),\, \forall x \neg P(x) \vdash \bot}\, \exists_e}{\exists x P(x) \vdash \neg \forall x (\neg P)}$$