8

Soundness of $\vdash_{N_C}$ (wrt BAs)

P is true in $(B, [\![\cdot]\!])$ iff $[\![P]\!] = 1$.

P is true in B, iff for all valuations $[\![\cdot]\!]$ into B, $[\![P]\!]_B$ is true in B.

P is valid, or a tautology iff for all BAs, B, P is true

Extending Truth / Validity to a Judgement

$$\Gamma \vdash A$$

$$[\![\Gamma]\!] = \begin{cases} 1 & \text{if } \Gamma \text{ is empty} \\ [\![A_1]\!] \wedge [\![A_2]\!] \cdots [\![A_m]\!] & \text{if } \Gamma = \{A_1 \ldots A_m\} \end{cases}$$

Γ is valid in $(B, [\![\cdot]\!])$, iff $[\![\Gamma]\!] \leq [\![A]\!]$

Soundness

Only valid statements are derivable

If $\Gamma \vdash_{N_C} A$, then $\Gamma \vdash A$ is valid.

Completeness

All valid statements are derivable

PROOF OF SOUNDNESS.

Proof $\leftarrow$ proof by induction on depth of derivation tree

Base Case: $A \vdash_{N_C} A$, $[\![A]\!] \leq [\![A]\!]$

Induction: Cases based on the last rule applied in the derivation $\Gamma \vdash A$.

i)
$$\frac{T \vdash^{D} A \wedge C}{T \vdash A} \rightarrow e$$

$$[\![T]\!] \leq [\![A \wedge C]\!] = [\![A]\!] \wedge [\![C]\!] \leq [\![A]\!]$$

ii)
$$\frac{T_1 \vdash A_1 , \quad T_2 \vdash A_2}{T_1, T_2 \vdash A_1 \wedge A_2}$$

$$T = T_1 , T_2$$
$$A = A_1 , A_2$$

$$\left.\begin{array}{l} [\![T_1]\!] = [\![A_1]\!] \\ [\![T_2]\!] = [\![A_2]\!] \end{array}\right\} \text{By Induction Hypothesis}$$

To Show,
$$[\![T_1, T_2]\!] \leq [\![A_1 \wedge A_2]\!]$$

$$[\![T_1, T_2]\!] = [\![T_1]\!] \wedge [\![T_2]\!] \leq [\![A_1]\!] \wedge [\![A_2]\!] = [\![A_1 \wedge A_2]\!].$$

iii)
$$\frac{T_1 \vdash^{D_1} A_1 \rightarrow A \qquad T_2 \vdash^{D_2} A_1}{T_1, T_2 \vdash A_1} \rightarrow e.$$

$$\left.\begin{array}{l} [\![T_1]\!] \leq [\![A_1 \rightarrow A_2]\!] = \neg [\![A_1]\!] \vee [\![A]\!] \\ [\![T_2]\!] \leq [\![A_1]\!] \end{array}\right\} \text{I.H.}$$

$$\begin{aligned} [\![T_1]\!] \wedge [\![T_2]\!] &\leq [\![A_1]\!] \wedge (\neg [\![A_1]\!] \vee [\![A]\!]) \\ &= [\![A_1]\!] \wedge [\![A]\!] \leq [\![A]\!] \end{aligned}$$

Semantic Entailment

If for all valuations, in which all $\phi_1, \dots \phi_n$ eval. to True, and ψ evaluates to T as well,

$$\phi_1, \dots \phi_n \models \psi \text{ holds}$$

Sem^{n} entail^{n}

* Given a proposition, true in all $(B, [\![\cdot]\!])$. How to relate it to derivability in N_c.

Construct B·A, from the syntax of PL and $\vdash_{N_c}$.

Define '$\sim$' on all prop^n, as follows.
→ an eq^n relation

$$A \sim B, \text{ iff } A \vdash_{N_c} B \text{ and } B \vdash_{N_c} A.$$

Let $[A]$ stand for equivalence class of A, wrt relation $\sim$.

* '$\sim$' is a congruence for operations $\wedge, \vee, \rightarrow$

Define, ordering $\leq$, on eq^n classes as,

$$[A_1] \leq [A_2] \text{ iff } A_1 \vdash_{N_c} A_2.$$

$$\boxed{\left.\begin{matrix} A_1 \vdash B_1 \;, & A_2 \vdash B_2 \\ B_1 \vdash A_1 \;, & B_2 \vdash A_2 \end{matrix}\right\} \Rightarrow A_1 \vdash A_2, \text{ iff } B_1 \vdash B_2.}$$

$$\dfrac{\dfrac{B_1 \vdash A_1 \;,\quad A_1 \vdash A_2}{B_1 \vdash A_2}\,\text{cut} \qquad A_2 \vdash B_2}{B_1 \vdash B_2}\,\text{cut}$$

$\mathcal{A} = \{ [A] \mid A \text{ is a proposition} \}$

$\mathcal{A} = \{ A, \leq, \wedge, \vee, \neg, [\perp], [\perp \to \perp] \}$

oper^n on A are defined as,

$[A] \vee [B] = [A \vee B]$

$[A] \wedge [B] = [A \wedge B]$

$[A] \to [B] = [A \to B]$

$\neg [A] = [\neg A]$

$[A_1] \vee [A_2] = [A_1 \vee A_2]$ is lub (w.r.t $\leq$) of $[A_1]$ and $[A_2]$

$[A_1] \leq [A_1 \vee A_2]$ $\quad A_1 \vdash A_1 \vee A_2$

$[A_1 \vee A_2]$ is an ub of $[A_1]$ and $[A_2]$

to show that $[A_1 \vee A_2]$

if $[A_1] \leq [C]$, $[A_2] \leq [C]$

then $[A_1 \vee A_2] \leq [C]$

Lindenbaum Algebra

$A_1 \vee A_2 \vdash C$ $\left\{ \begin{array}{l} \text{given } A_1 \vdash C \\ \text{and } A_2 \vdash C \end{array} \right\}$ or elim

We define $\phi_1, \phi_2, \ldots \phi_n \models \psi$ to mean that any valuation that makes $\phi_1, \phi_2, \ldots \phi_n$ true also makes ψ true. $\models$ is also called the **semantic entailment** relation.

Our goal is to prove that this is equivalent to our previous notion of valid sequents.

The property that if $\phi_1, \phi_2, \ldots \phi_n \vdash \psi$, then $\phi_1, \phi_2, \ldots \phi_n \models \psi$ is called **soundness**.

The property that if $\phi_1, \phi_2, \ldots \phi_n \models \psi$, then $\phi_1, \phi_2, \ldots \phi_n \vdash \psi$ is called **completeness**.

Soundness states that any valid sequent says something about the relationship between the formulas involved, under all valuations.

Completeness states that given a relationship between the valuations of formulas, there must exist a proof of validity of the corresponding sequent.

It helps to think about the case where n (the number of formulas on the left-hand side) is zero. Soundness states that any formula that is a theorem is true under all valuations. Completeness says that any formula that is true under all valuations is a theorem.

We are going to prove these two properties for our system of natural deduction and our system of valuations.

Soundness

Recall our definition of soundness: if $\phi_1, \phi_2, \ldots \phi_n \vdash \psi$, then $\phi_1, \phi_2, \ldots \phi_n \models \psi$.

Soundness asserts that a valid sequent says something about valuations. A sequent is shown valid by means of a natural deduction proof. Our proof of soundness will proceed by induction on the length of this proof (the number of lines).

Note that our proofs are just flattened trees. We could have given a recursive definition of what a proof was (an application of a rule to one or more subproofs) in which case our proof by induction becomes structural induction. It might help to keep this in mind.

Thm: If $\phi_1, \phi_2, \ldots \phi_n \vdash \psi$, then $\phi_1, \phi_2, \ldots \phi_n \models \psi$.

Pf: If a sequent has a proof, then it has a proof that uses only the core deduction rules (i.e., no MT, LEM, PBC, or $\neg\neg$i). We shall therefore assume that that the given proof of ϕ uses only core deduction rules and proceed by strong induction on the length k of a proof of the sequent $\phi_1, \phi_2, \ldots \phi_n \vdash \psi$.

If the proof has one line, it must be of the form

1	ϕ	premise

So the sequent is $\phi \vdash \phi$, and clearly $\phi \models \phi$.

For the inductive step, we assume that soundness holds for sequents with proofs of length less than k, and prove that it holds for a sequent $\phi_1, \phi_2, \ldots \phi_n \vdash \psi$ with a proof of length k (again, our attention is restricted to deductions involving only core rules).

The kth line of the proof must look like

k	ψ	justification

We have a number of cases depending on which rule is applied in this line. We'll do enough cases to get the general idea, since we have a lot of rules in natural deduction. (This proof is easier for other systems with fewer rules, but then using such systems is more difficult.)

If the kth line looks like

k	ψ	$\wedge$i k_1, k_2

then ψ must be of the form $\psi_1 \wedge \psi_2$, where ψ_1 appears on line k_1, and ψ_2 appears on line k_2.

The sequent $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi_1$ has a proof of length at most k_1 (the first k_1 lines of the proof we started with), so $\phi_1, \phi_2, \ldots, \phi_n \models \psi_1$. Similarly, $\phi_1, \phi_2, \ldots, \phi_n \models \psi_2$.

What this means is that for every interpretation Φ such that $\Phi(\phi_1) = T, \ldots, \Phi(\phi_n) = T$, we also have $\Phi(\psi_1) = T$, and whenever $\Phi(\phi_1) = T, \ldots, \Phi(\phi_n) = T$, we have $\Phi(\psi_2) = T$.

So let $\Phi(\phi_1) = \cdots = \Phi(\phi_n) = T$.

Then $\Phi(\psi_1) = \Phi(\psi_2) = T$. Therefore,

$$\Phi(\psi_1 \wedge \psi_2) = \text{meaning}(\wedge)(\Phi(\psi_1), \Phi(\psi_2)) = \text{meaning}(\wedge)(T, T) = T$$

Thus, $\phi_1, \phi_2, \ldots, \phi_n \models \psi$, as required.

This sort of reasoning works well for many other rules, such as $\wedge$e$_1$ and $\rightarrow$e. But it gets a little complicated if the last line in our proof closes off a box, as with $\rightarrow$i. Suppose we have such a proof of $\phi_1, \phi_2, \ldots, \phi_n \vdash \alpha \rightarrow \beta$.

1		
$\vdots$		
i	α	assumption
$\vdots$		
$k-1$	β	justification
k	$\alpha \rightarrow \beta$	$\rightarrow$i i–$(k-1)$

If we delete the last line, this is not a complete proof.

But we can turn it into one by making the assumption of the open box into a premise.

1		
$\vdots$		
i	α	premise
$\vdots$		
$k-1$	β	justification

This is a proof of the sequent $\phi_1, \phi_2, \ldots, \phi_n, \alpha \vdash \beta$.

Since our proof of $\phi_1, \phi_2, \ldots, \phi_n, \alpha \vdash \beta$ has $k-1$ lines, we can apply the inductive hypothesis to conclude that $\phi_1, \phi_2, \ldots, \phi_n, \alpha \models \beta$.

We need to conclude that $\phi_1, \phi_2, \ldots, \phi_n \models \alpha \to \beta$. So consider an interpretation Φ that makes $\phi_1, \phi_2, \ldots, \phi_n$ true. The only way we could have $\Phi(\alpha \to \beta) = F$ would be if $\Phi(\alpha) = T$ and $\Phi(\beta) = F$ (this is by examination of the truth table that defines $\text{meaning}(\to)$). But we have

$$\phi_1, \phi_2, \ldots, \phi_n, \alpha \models \beta\,,$$

so if $\Phi(\phi_1) = \cdots = \Phi(\phi_n) = \Phi(\alpha) = T$, then $\Phi(\beta) = T$. Hence $\Phi(\alpha \to \beta) = T$, and we have $\phi_1, \phi_2, \ldots, \phi_n \models \alpha \to \beta$.

This sort of reasoning works for the cases where the last line closes off a proof box.

The full proof involves a complete examination of each of the dozen or so rules of natural deduction, but we have already seen the two main ideas.

Soundness gives us a method of showing that a sequent $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ does **not** have a proof in natural deduction; we simply have to find a valuation that makes $\phi_1, \phi_2, \ldots, \phi_n$ true but ψ false.

2. *Both* $\varphi \vdash \varphi \lor \psi$ *and* $\psi \vdash \varphi \lor \psi$.

Proof. 1. Consider the following derivation:

$$ {}_1\,\dfrac{\varphi \lor \psi \qquad \dfrac{\neg\varphi \qquad [\varphi]^1}{\bot}\,\neg\text{Elim} \qquad \dfrac{\neg\psi \qquad [\psi]^1}{\bot}\,\neg\text{Elim}}{\bot}\,\lor\text{Elim} $$

This is a derivation of $\bot$ from undischarged assumptions $\varphi \lor \psi$, $\neg\varphi$, and $\neg\psi$.

2. We can derive both

$$ \dfrac{\varphi}{\varphi \lor \psi}\,\lor\text{Intro} \qquad\qquad \dfrac{\psi}{\varphi \lor \psi}\,\lor\text{Intro} $$

□

Proposition 10.21. *1.* $\varphi, \varphi \to \psi \vdash \psi$.

2. *Both* $\neg\varphi \vdash \varphi \to \psi$ *and* $\psi \vdash \varphi \to \psi$.

Proof. 1. We can derive:

$$ \dfrac{\varphi \to \psi \qquad \varphi}{\psi}\,\to\text{Elim} $$

2. This is shown by the following two derivations:

$$ {}_1\,\dfrac{\dfrac{\dfrac{\neg\varphi \qquad [\varphi]^1}{\bot}\,\neg\text{Elim}}{\psi}\,\bot_I}{\varphi \to \psi}\,\to\text{Intro} \qquad\qquad \dfrac{\psi}{\varphi \to \psi}\,\to\text{Intro} $$

Note that $\to$Intro may, but does not have to, discharge the assumption φ.
□

10.8 Soundness

A derivation system, such as natural deduction, is *sound* if it cannot derive things that do not actually follow. Soundness is thus a kind of guaranteed safety property for derivation systems. Depending on which proof theoretic property is in question, we would like to know for instance, that

1. every derivable sentence is a tautology;

2. if a sentence is derivable from some others, it is also a consequence of them;

3. if a set of sentences is inconsistent, it is unsatisfiable.

These are important properties of a derivation system. If any of them do not hold, the derivation system is deficient—it would derive too much. Consequently, establishing the soundness of a derivation system is of the utmost importance.

Theorem 10.22 (Soundness). *If φ is derivable from the undischarged assumptions Γ, then $\Gamma \vDash \varphi$.*

Proof. Let δ be a derivation of φ. We proceed by induction on the number of inferences in δ.

For the induction basis we show the claim if the number of inferences is 0. In this case, δ consists only of a single sentence φ, i.e., an assumption. That assumption is undischarged, since assumptions can only be discharged by inferences, and there are no inferences. So, any valuation $\mathfrak{v}$ that satisfies all of the undischarged assumptions of the proof also satisfies φ.

Now for the inductive step. Suppose that δ contains n inferences. The premise(s) of the lowermost inference are derived using sub-derivations, each of which contains fewer than n inferences. We assume the induction hypothesis: The premises of the lowermost inference follow from the undischarged assumptions of the sub-derivations ending in those premises. We have to show that the conclusion φ follows from the undischarged assumptions of the entire proof.

We distinguish cases according to the type of the lowermost inference. First, we consider the possible inferences with only one premise.

1. Suppose that the last inference is $\lnot$Intro: The derivation has the form

$$\dfrac{\begin{array}{c}\Gamma, [\varphi]^n \\ \vdots\ \delta_1 \\ \bot\end{array}}{\lnot\varphi}\ {\lnot}\mathrm{Intro}\quad (n)$$

   By inductive hypothesis, $\bot$ follows from the undischarged assumptions $\Gamma \cup \{\varphi\}$ of δ_1. Consider a valuation $\mathfrak{v}$. We need to show that, if $\mathfrak{v} \vDash \Gamma$, then $\mathfrak{v} \vDash \lnot\varphi$. Suppose for reductio that $\mathfrak{v} \vDash \Gamma$, but $\mathfrak{v} \nvDash \lnot\varphi$, i.e., $\mathfrak{v} \vDash \varphi$. This would mean that $\mathfrak{v} \vDash \Gamma \cup \{\varphi\}$. This is contrary to our inductive hypothesis. So, $\mathfrak{v} \vDash \lnot\varphi$.

2. The last inference is $\wedge$Elim: There are two variants: φ or ψ may be inferred from the premise $\varphi \wedge \psi$. Consider the first case. The derivation δ looks like this:

$$\dfrac{\begin{array}{c}\Gamma \\ \vdots\, \delta_1 \\ \varphi \wedge \psi\end{array}}{\varphi}\;\wedge\text{Elim}$$

By inductive hypothesis, $\varphi \wedge \psi$ follows from the undischarged assumptions Γ of δ_1. Consider a structure $\mathfrak{v}$. We need to show that, if $\mathfrak{v} \vDash \Gamma$, then $\mathfrak{v} \vDash \varphi$. Suppose $\mathfrak{v} \vDash \Gamma$. By our inductive hypothesis ($\Gamma \vDash \varphi \wedge \psi$), we know that $\mathfrak{v} \vDash \varphi \wedge \psi$. By definition, $\mathfrak{v} \vDash \varphi \wedge \psi$ iff $\mathfrak{v} \vDash \varphi$ and $\mathfrak{v} \vDash \psi$. (The case where ψ is inferred from $\varphi \wedge \psi$ is handled similarly.)

3. The last inference is $\vee$Intro: There are two variants: $\varphi \vee \psi$ may be inferred from the premise φ or the premise ψ. Consider the first case. The derivation has the form

$$\dfrac{\begin{array}{c}\Gamma \\ \vdots\, \delta_1 \\ \varphi\end{array}}{\varphi \vee \psi}\;\vee\text{Intro}$$

By inductive hypothesis, φ follows from the undischarged assumptions Γ of δ_1. Consider a valuation $\mathfrak{v}$. We need to show that, if $\mathfrak{v} \vDash \Gamma$, then $\mathfrak{v} \vDash \varphi \vee \psi$. Suppose $\mathfrak{v} \vDash \Gamma$; then $\mathfrak{v} \vDash \varphi$ since $\Gamma \vDash \varphi$ (the inductive hypothesis). So it must also be the case that $\mathfrak{v} \vDash \varphi \vee \psi$. (The case where $\varphi \vee \psi$ is inferred from ψ is handled similarly.)

4. The last inference is $\rightarrow$Intro: $\varphi \rightarrow \psi$ is inferred from a subproof with assumption φ and conclusion ψ, i.e.,

$$n\;\dfrac{\begin{array}{c}\Gamma, [\varphi]^n \\ \vdots\, \delta_1 \\ \psi\end{array}}{\varphi \rightarrow \psi}\;\rightarrow\text{Intro}$$

By inductive hypothesis, ψ follows from the undischarged assumptions of δ_1, i.e., $\Gamma \cup \{\varphi\} \vDash \psi$. Consider a valuation $\mathfrak{v}$. The undischarged assumptions of δ are just Γ, since φ is discharged at the last inference. So we need to show that $\Gamma \vDash \varphi \rightarrow \psi$. For reductio, suppose that for some valuation $\mathfrak{v}$, $\mathfrak{v} \vDash \Gamma$ but $\mathfrak{v} \nvDash \varphi \rightarrow \psi$. So, $\mathfrak{v} \vDash \varphi$ and $\mathfrak{v} \nvDash \psi$. But by hypothesis, ψ is a consequence of $\Gamma \cup \{\varphi\}$, i.e., $\mathfrak{v} \vDash \psi$, which is a contradiction. So, $\Gamma \vDash \varphi \rightarrow \psi$.

5. The last inference is $\bot_I$: Here, δ ends in

$$\cfrac{\begin{matrix}\Gamma \\ \vdots\,\delta_1 \\ \bot\end{matrix}}{\varphi}\;\bot_I$$

   By induction hypothesis, $\Gamma \vDash \bot$. We have to show that $\Gamma \vDash \varphi$. Suppose not; then for some $\mathfrak{v}$ we have $\mathfrak{v} \vDash \Gamma$ and $\mathfrak{v} \nvDash \varphi$. But we always have $\mathfrak{v} \nvDash \bot$, so this would mean that $\Gamma \nvDash \bot$, contrary to the induction hypothesis.

6. The last inference is $\bot_C$: Exercise.

Now let's consider the possible inferences with several premises: $\lor$Elim, $\land$Intro, and $\rightarrow$Elim.

1. The last inference is $\land$Intro. $\varphi \land \psi$ is inferred from the premises φ and ψ and δ has the form

$$\cfrac{\begin{matrix}\Gamma_1 \\ \vdots\,\delta_1 \\ \varphi\end{matrix} \qquad \begin{matrix}\Gamma_2 \\ \vdots\,\delta_2 \\ \psi\end{matrix}}{\varphi \land \psi}\;\land\text{Intro}$$

   By induction hypothesis, φ follows from the undischarged assumptions Γ_1 of δ_1 and ψ follows from the undischarged assumptions Γ_2 of δ_2. The undischarged assumptions of δ are $\Gamma_1 \cup \Gamma_2$, so we have to show that $\Gamma_1 \cup \Gamma_2 \vDash \varphi \land \psi$. Consider a valuation $\mathfrak{v}$ with $\mathfrak{v} \vDash \Gamma_1 \cup \Gamma_2$. Since $\mathfrak{v} \vDash \Gamma_1$, it must be the case that $\mathfrak{v} \vDash \varphi$ as $\Gamma_1 \vDash \varphi$, and since $\mathfrak{v} \vDash \Gamma_2$, $\mathfrak{v} \vDash \psi$ since $\Gamma_2 \vDash \psi$. Together, $\mathfrak{v} \vDash \varphi \land \psi$.

2. The last inference is $\lor$Elim: Exercise.

3. The last inference is $\rightarrow$Elim. ψ is inferred from the premises $\varphi \rightarrow \psi$ and φ. The derivation δ looks like this:

$$\cfrac{\begin{matrix}\Gamma_1 \\ \vdots\,\delta_1 \\ \varphi \rightarrow \psi\end{matrix} \qquad \begin{matrix}\Gamma_2 \\ \vdots\,\delta_2 \\ \varphi\end{matrix}}{\psi}\;\rightarrow\text{Elim}$$

□

By induction hypothesis, $\varphi \to \psi$ follows from the undischarged assumptions Γ_1 of δ_1 and φ follows from the undischarged assumptions Γ_2 of δ_2. Consider a valuation $\mathfrak{v}$. We need to show that, if $\mathfrak{v} \vDash \Gamma_1 \cup \Gamma_2$, then $\mathfrak{v} \vDash \psi$. Suppose $\mathfrak{v} \vDash \Gamma_1 \cup \Gamma_2$. Since $\Gamma_1 \vDash \varphi \to \psi$, $\mathfrak{v} \vDash \varphi \to \psi$. Since $\Gamma_2 \vDash \varphi$, we have $\mathfrak{v} \vDash \varphi$. This means that $\mathfrak{v} \vDash \psi$ (For if $\mathfrak{v} \nvDash \psi$, since $\mathfrak{v} \vDash \varphi$, we'd have $\mathfrak{v} \nvDash \varphi \to \psi$, contradicting $\mathfrak{v} \vDash \varphi \to \psi$).

4. The last inference is $\lnot$Elim: Exercise.

Corollary 10.23. *If $\vdash \varphi$, then φ is a tautology.*

Corollary 10.24. *If Γ is satisfiable, then it is consistent.*

Proof. We prove the contrapositive. Suppose that Γ is not consistent. Then $\Gamma \vdash \bot$, i.e., there is a derivation of $\bot$ from undischarged assumptions in Γ. By Theorem 10.22, any valuation $\mathfrak{v}$ that satisfies Γ must satisfy $\bot$. Since $\mathfrak{v} \nvDash \bot$ for every valuation $\mathfrak{v}$, no $\mathfrak{v}$ can satisfy Γ, i.e., Γ is not satisfiable. □

Problems

Problem 10.1. Give derivations that show the following:

1. $\varphi \land (\psi \land \chi) \vdash (\varphi \land \psi) \land \chi$.
2. $\varphi \lor (\psi \lor \chi) \vdash (\varphi \lor \psi) \lor \chi$.
3. $\varphi \to (\psi \to \chi) \vdash \psi \to (\varphi \to \chi)$.
4. $\varphi \vdash \lnot\lnot\varphi$.

Problem 10.2. Give derivations that show the following:

1. $(\varphi \lor \psi) \to \chi \vdash \varphi \to \chi$.
2. $(\varphi \to \chi) \land (\psi \to \chi) \vdash (\varphi \lor \psi) \to \chi$.
3. $\vdash \lnot(\varphi \land \lnot\varphi)$.
4. $\psi \to \varphi \vdash \lnot\varphi \to \lnot\psi$.
5. $\vdash (\varphi \to \lnot\varphi) \to \lnot\varphi$.
6. $\vdash \lnot(\varphi \to \psi) \to \lnot\psi$.
7. $\varphi \to \chi \vdash \lnot(\varphi \land \lnot\chi)$.
8. $\varphi \land \lnot\chi \vdash \lnot(\varphi \to \chi)$.
9. $\varphi \lor \psi, \lnot\psi \vdash \varphi$.

Chapter 13

The Completeness Theorem

13.1 Introduction

The completeness theorem is one of the most fundamental results about logic. It comes in two formulations, the equivalence of which we'll prove. In its first formulation it says something fundamental about the relationship between semantic consequence and our derivation system: if a sentence φ follows from some sentences Γ, then there is also a derivation that establishes $\Gamma \vdash \varphi$. Thus, the derivation system is as strong as it can possibly be without proving things that don't actually follow.

In its second formulation, it can be stated as a model existence result: every consistent set of sentences is satisfiable. Consistency is a proof-theoretic notion: it says that our derivation system is unable to produce certain derivations. But who's to say that just because there are no derivations of a certain sort from Γ, it's guaranteed that there is valuation $\mathfrak{v}$ with $\mathfrak{v} \vDash \Gamma$? Before the completeness theorem was first proved—in fact before we had the derivation systems we now do—the great German mathematician David Hilbert held the view that consistency of mathematical theories guarantees the existence of the objects they are about. He put it as follows in a letter to Gottlob Frege:

> If the arbitrarily given axioms do not contradict one another with all their consequences, then they are true and the things defined by the axioms exist. This is for me the criterion of truth and existence.

Frege vehemently disagreed. The second formulation of the completeness theorem shows that Hilbert was right in at least the sense that if the axioms are consistent, then *some* valuation exists that makes them all true.

These aren't the only reasons the completeness theorem—or rather, its proof—is important. It has a number of important consequences, some of which we'll discuss separately. For instance, since any derivation that shows $\Gamma \vdash \varphi$ is finite and so can only use finitely many of the sentences in Γ, it follows by the completeness theorem that if φ is a consequence of Γ, it is already a consequence of a finite subset of Γ. This is called *compactness*. Equivalently, if every finite subset of Γ is consistent, then Γ itself must be consistent.

Although the compactness theorem follows from the completeness theorem via the detour through derivations, it is also possible to use the *the proof of* the completeness theorem to establish it directly. For what the proof does is take a set of sentences with a certain property—consistency—and constructs a structure out of this set that has certain properties (in this case, that it satisfies the set). Almost the very same construction can be used to directly establish compactness, by starting from "finitely satisfiable" sets of sentences instead of consistent ones.

13.2 Outline of the Proof

The proof of the completeness theorem is a bit complex, and upon first reading it, it is easy to get lost. So let us outline the proof. The first step is a shift of perspective, that allows us to see a route to a proof. When completeness is thought of as "whenever $\Gamma \vDash \varphi$ then $\Gamma \vdash \varphi$," it may be hard to even come up with an idea: for to show that $\Gamma \vdash \varphi$ we have to find a derivation, and it does not look like the hypothesis that $\Gamma \vDash \varphi$ helps us for this in any way. For some proof systems it is possible to directly construct a derivation, but we will take a slightly different approach. The shift in perspective required is this: completeness can also be formulated as: "if Γ is consistent, it is satisfiable." Perhaps we can use the information in Γ together with the hypothesis that it is consistent to construct a valuation that satisfies every formula in Γ. After all, we know what kind of valuation we are looking for: one that is as Γ describes it!

If Γ contains only propositional variables, it is easy to construct a model for it. All we have to do is come up with a valuation $\mathfrak{v}$ such that $\mathfrak{v} \vDash p$ for all $p \in \Gamma$. Well, let $\mathfrak{v}(p) = \mathbb{T}$ iff $p \in \Gamma$.

Now suppose Γ contains some formula $\neg\psi$, with ψ atomic. We might worry that the construction of $\mathfrak{v}$ interferes with the possibility of making $\neg\psi$ true. But here's where the consistency of Γ comes in: if $\neg\psi \in \Gamma$, then $\psi \notin \Gamma$, or else Γ would be inconsistent. And if $\psi \notin \Gamma$, then according to our construction of $\mathfrak{v}$, $\mathfrak{v} \nvDash \psi$, so $\mathfrak{v} \vDash \neg\psi$. So far so good.

What if Γ contains complex, non-atomic formulas? Say it contains $\varphi \wedge \psi$. To make that true, we should proceed as if both φ and ψ were in Γ. And if $\varphi \vee \psi \in \Gamma$, then we will have to make at least one of them true, i.e., proceed as if one of them was in Γ.

This suggests the following idea: we add additional formulas to Γ so as to (a) keep the resulting set consistent and (b) make sure that for every possible atomic sentence φ, either φ is in the resulting set, or $\neg\varphi$ is, and (c) such that, whenever $\varphi \wedge \psi$ is in the set, so are both φ and ψ, if $\varphi \vee \psi$ is in the set, at least one of φ or ψ is also, etc. We keep doing this (potentially forever). Call the set of all formulas so added Γ^*. Then our construction above would provide us with a valuation $\mathfrak{v}$ for which we could prove, by induction, that it satisfies all sentences in Γ^*, and hence also all sentence in Γ since $\Gamma \subseteq \Gamma^*$. It turns out that guaranteeing (a) and (b) is enough. A set of sentences for which (b) holds is called *complete*. So our task will be to extend the consistent set Γ to a consistent and complete set Γ^*.

So here's what we'll do. First we investigate the properties of complete consistent sets, in particular we prove that a complete consistent set contains $\varphi \wedge \psi$ iff it contains both φ and ψ, $\varphi \vee \psi$ iff it contains at least one of them, etc. (Proposition 13.2). We'll then take the consistent set Γ and show that it can be extended to a consistent and complete set Γ^* (Lemma 13.3). This set Γ^* is what we'll use to define our valuation $\mathfrak{v}(\Gamma^*)$. The valuation is determined by the propositional variables in Γ^* (Definition 13.4). We'll use the properties of complete consistent sets to show that indeed $\mathfrak{v}(\Gamma^*) \vDash \varphi$ iff $\varphi \in \Gamma^*$ (Lemma 13.5), and thus in particular, $\mathfrak{v}(\Gamma^*) \vDash \Gamma$.

13.3 Complete Consistent Sets of Sentences

Definition 13.1 (Complete set). A set Γ of sentences is *complete* iff for any sentence φ, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

Complete sets of sentences leave no questions unanswered. For any sentence φ, Γ "says" if φ is true or false. The importance of complete sets extends beyond the proof of the completeness theorem. A theory which is complete and axiomatizable, for instance, is always decidable.

Complete consistent sets are important in the completeness proof since we can guarantee that every consistent set of sentences Γ is contained in a complete consistent set Γ^*. A complete consistent set contains, for each sentence φ, either φ or its negation $\neg\varphi$, but not both. This is true in particular for propositional variables, so from a complete consistent set, we can construct a valuation where the truth value assigned to propositional variables is defined according to which propositional variables are in Γ^*. This valuation can then be shown to make all sentences in Γ^* (and hence also all those in Γ) true. The proof of this latter fact requires that $\neg\varphi \in \Gamma^*$ iff $\varphi \notin \Gamma^*$, $(\varphi \vee \psi) \in \Gamma^*$ iff $\varphi \in \Gamma^*$ or $\psi \in \Gamma^*$, etc.

In what follows, we will often tacitly use the properties of reflexivity, monotonicity, and transitivity of $\vdash$ (see sections 9.6, 10.5, 11.5 and 12.4).

Proposition 13.2. *Suppose Γ is complete and consistent. Then:*

1. *If $\Gamma \vdash \varphi$, then $\varphi \in \Gamma$.*

2. *$\varphi \wedge \psi \in \Gamma$ iff both $\varphi \in \Gamma$ and $\psi \in \Gamma$.*

3. *$\varphi \vee \psi \in \Gamma$ iff either $\varphi \in \Gamma$ or $\psi \in \Gamma$.*

4. *$\varphi \to \psi \in \Gamma$ iff either $\varphi \notin \Gamma$ or $\psi \in \Gamma$.*

Proof. Let us suppose for all of the following that Γ is complete and consistent.

1. If $\Gamma \vdash \varphi$, then $\varphi \in \Gamma$.

   Suppose that $\Gamma \vdash \varphi$. Suppose to the contrary that $\varphi \notin \Gamma$. Since Γ is complete, $\neg\varphi \in \Gamma$. By Propositions 9.19, 10.17, 11.17 and 12.24, Γ is inconsistent. This contradicts the assumption that Γ is consistent. Hence, it cannot be the case that $\varphi \notin \Gamma$, so $\varphi \in \Gamma$.

2. Exercise.

3. First we show that if $\varphi \lor \psi \in \Gamma$, then either $\varphi \in \Gamma$ or $\psi \in \Gamma$. Suppose $\varphi \lor \psi \in \Gamma$ but $\varphi \notin \Gamma$ and $\psi \notin \Gamma$. Since Γ is complete, $\neg\varphi \in \Gamma$ and $\neg\psi \in \Gamma$. By Propositions 9.22, 10.20, 11.20 and 12.27, item (1), Γ is inconsistent, a contradiction. Hence, either $\varphi \in \Gamma$ or $\psi \in \Gamma$.

   For the reverse direction, suppose that $\varphi \in \Gamma$ or $\psi \in \Gamma$. By Propositions 9.22, 10.20, 11.20 and 12.27, item (2), $\Gamma \vdash \varphi \lor \psi$. By (1), $\varphi \lor \psi \in \Gamma$, as required.

4. Exercise. □

13.4 Lindenbaum's Lemma

We now prove a lemma that shows that any consistent set of sentences is contained in some set of sentences which is not just consistent, but also complete. The proof works by adding one sentence at a time, guaranteeing at each step that the set remains consistent. We do this so that for every φ, either φ or $\neg\varphi$ gets added at some stage. The union of all stages in that construction then contains either φ or its negation $\neg\varphi$ and is thus complete. It is also consistent, since we make sure at each stage not to introduce an inconsistency.

Lemma 13.3 (Lindenbaum's Lemma). *Every consistent set Γ in a language $\mathcal{L}$ can be extended to a complete and consistent set Γ^*.*

Proof. Let Γ be consistent. Let $\varphi_0, \varphi_1, \dots$ be an enumeration of all the sentences of $\mathcal{L}$. Define $\Gamma_0 = \Gamma$, and

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\varphi_n\} & \text{if } \Gamma_n \cup \{\varphi_n\} \text{ is consistent;} \\ \Gamma_n \cup \{\neg\varphi_n\} & \text{otherwise.} \end{cases}$$

Let $\Gamma^* = \bigcup_{n \geq 0} \Gamma_n$.

Each Γ_n is consistent: Γ_0 is consistent by definition. If $\Gamma_{n+1} = \Gamma_n \cup \{\varphi_n\}$, this is because the latter is consistent. If it isn't, $\Gamma_{n+1} = \Gamma_n \cup \{\neg\varphi_n\}$. We have to verify that $\Gamma_n \cup \{\neg\varphi_n\}$ is consistent. Suppose it's not. Then *both* $\Gamma_n \cup \{\varphi_n\}$ and $\Gamma_n \cup \{\neg\varphi_n\}$ are inconsistent. This means that Γ_n would be inconsistent by Propositions 9.20, 10.18, 11.18 and 12.25, contrary to the induction hypothesis.

For every n and every $i < n$, $\Gamma_i \subseteq \Gamma_n$. This follows by a simple induction on n. For $n = 0$, there are no $i < 0$, so the claim holds automatically. For the inductive step, suppose it is true for n. We have $\Gamma_{n+1} = \Gamma_n \cup \{\varphi_n\}$ or $= \Gamma_n \cup \{\neg\varphi_n\}$ by construction. So $\Gamma_n \subseteq \Gamma_{n+1}$. If $i < n$, then $\Gamma_i \subseteq \Gamma_n$ by inductive hypothesis, and so $\subseteq \Gamma_{n+1}$ by transitivity of $\subseteq$.

From this it follows that every finite subset of Γ^* is a subset of Γ_n for some n, since each $\psi \in \Gamma^*$ not already in Γ_0 is added at some stage i. If n is the last one of these, then all ψ in the finite subset are in Γ_n. So, every finite subset of Γ^* is consistent. By Propositions 9.16, 10.14, 11.14 and 12.18, Γ^* is consistent.

Every sentence of $\mathrm{Frm}(\mathcal{L})$ appears on the list used to define Γ^*. If $\varphi_n \notin \Gamma^*$, then that is because $\Gamma_n \cup \{\varphi_n\}$ was inconsistent. But then $\neg\varphi_n \in \Gamma^*$, so Γ^* is complete. □

13.5 Construction of a Model

We are now ready to define a valuation that makes all $\varphi \in \Gamma$ true. To do this, we first apply Lindenbaum's Lemma: we get a complete consistent $\Gamma^* \supseteq \Gamma$. We let the propositional variables in Γ^* determine $\mathfrak{v}(\Gamma^*)$.

Definition 13.4. Suppose Γ^* is a complete consistent set of formulas. Then we let

$$\mathfrak{v}(\Gamma^*)(p) = \begin{cases} \mathbb{T} & \text{if } p \in \Gamma^* \\ \mathbb{F} & \text{if } p \notin \Gamma^* \end{cases}$$

Lemma 13.5 (Truth Lemma). $\mathfrak{v}(\Gamma^*) \vDash \varphi$ *iff* $\varphi \in \Gamma^*$.

Proof. We prove both directions simultaneously, and by induction on φ.

1. $\varphi \equiv \bot$: $\mathfrak{v}(\Gamma^*) \nvDash \bot$ by definition of satisfaction. On the other hand, $\bot \notin \Gamma^*$ since Γ^* is consistent.

2. $\varphi \equiv p$: $\mathfrak{v}(\Gamma^*) \vDash p$ iff $\mathfrak{v}(\Gamma^*)(p) = \mathbb{T}$ (by the definition of satisfaction) iff $p \in \Gamma^*$ (by the construction of $\mathfrak{v}(\Gamma^*)$).

3. $\varphi \equiv \neg\psi$: $\mathfrak{v}(\Gamma^*) \vDash \varphi$ iff $\mathfrak{v}(\Gamma^*) \nvDash \psi$ (by definition of satisfaction). By induction hypothesis, $\mathfrak{v}(\Gamma^*) \nvDash \psi$ iff $\psi \notin \Gamma^*$. Since Γ^* is consistent and complete, $\psi \notin \Gamma^*$ iff $\neg\psi \in \Gamma^*$.

4. $\varphi \equiv \psi \wedge \chi$: exercise.

5. $\varphi \equiv \psi \vee \chi$: $\mathfrak{v}(\Gamma^*) \vDash \varphi$ iff $\mathfrak{v}(\Gamma^*) \vDash \psi$ or $\mathfrak{v}(\Gamma^*) \vDash \chi$ (by definition of satisfaction) iff $\psi \in \Gamma^*$ or $\chi \in \Gamma^*$ (by induction hypothesis). This is the case iff $(\psi \vee \chi) \in \Gamma^*$ (by Proposition 13.2(3)).

6. $\varphi \equiv \psi \rightarrow \chi$: exercise.

13.6 The Completeness Theorem

Let's combine our results: we arrive at the completeness theorem.

Theorem 13.6 (Completeness Theorem). *Let Γ be a set of sentences. If Γ is consistent, it is satisfiable.*

Proof. Suppose Γ is consistent. By Lemma 13.3, there is a $\Gamma^* \supseteq \Gamma$ which is consistent and complete. By Lemma 13.5, $\mathfrak{v}(\Gamma^*) \vDash \varphi$ iff $\varphi \in \Gamma^*$. From this it follows in particular that for all $\varphi \in \Gamma$, $\mathfrak{v}(\Gamma^*) \vDash \varphi$, so Γ is satisfiable. □

Corollary 13.7 (Completeness Theorem, Second Version). *For all Γ and sentences φ: if $\Gamma \vDash \varphi$ then $\Gamma \vdash \varphi$.*

Proof. Note that the Γ's in Corollary 13.7 and Theorem 13.6 are universally quantified. To make sure we do not confuse ourselves, let us restate Theorem 13.6 using a different variable: for any set of sentences Δ, if Δ is consistent, it is satisfiable. By contraposition, if Δ is not satisfiable, then Δ is inconsistent. We will use this to prove the corollary.

Suppose that $\Gamma \vDash \varphi$. Then $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable by Proposition 7.21. Taking $\Gamma \cup \{\neg\varphi\}$ as our Δ, the previous version of Theorem 13.6 gives us that $\Gamma \cup \{\neg\varphi\}$ is inconsistent. By Propositions 9.18, 10.16, 11.16 and 12.23, $\Gamma \vdash \varphi$. □

13.7 The Compactness Theorem

One important consequence of the completeness theorem is the compactness theorem. The compactness theorem states that if each *finite* subset of a set of sentences is satisfiable, the entire set is satisfiable—even if the set itself is infinite. This is far from obvious. There is nothing that seems to rule out, at first glance at least, the possibility of there being infinite sets of sentences which are contradictory, but the contradiction only arises, so to speak, from the infinite number. The compactness theorem says that such a scenario can be ruled out: there are no unsatisfiable infinite sets of sentences each finite subset of which is satisfiable. Like the completeness theorem, it has a version related to entailment: if an infinite set of sentences entails something, already a finite subset does.

Definition 13.8. A set Γ of formulas is *finitely satisfiable* iff every finite $\Gamma_0 \subseteq \Gamma$ is satisfiable.

Theorem 13.9 (Compactness Theorem). *The following hold for any sentences Γ and φ:*