15

Free Variables in a FO formula

Def: (By induction on Formula Construction)

- $FV(R(t_1 \dots t_n))$ = Variables among $t_1 \dots t_n$
- $FV(\phi_1 \cap \phi_2) = FV(\phi_1 \cup \phi_2) = FV(\phi_1 \to \phi_2) = FV(\phi_1) \cup FV(\phi_2)$
- $FV(\exists_x \phi_1) = FV(\forall_x \phi) = FV(\phi) - \{x\}$.

<u>Examples</u>

$$\left.\begin{array}{lcl}
\exists x\ \phi \vee \psi & \Leftrightarrow & \exists x (\phi \vee \psi) \\
\exists x\ \phi \wedge \psi & \Leftrightarrow & \exists x (\phi \wedge \psi) \\
\forall x\ \phi \wedge \psi & \Leftrightarrow & \forall x (\phi \wedge \psi) \\
\forall x\ \phi \vee \psi & \Leftrightarrow & \forall x (\phi \vee \psi)
\end{array}\right\}\quad x \notin FV(\psi)$$

to prove: $\exists x\ \phi \wedge \psi \vdash \exists x (\phi \wedge \psi)$

i) $\rightarrow$

$$\dfrac{\dfrac{}{\exists x \phi \vdash \exists x \phi}\,Ax \qquad \dfrac{\dfrac{\dfrac{}{\phi[y/x] \vdash \phi[y/x]}\,Ax \quad \dfrac{}{\psi \vdash \psi}\,Ax}{\phi[y/x], \psi \vdash \phi[y/x] \wedge \psi}\,\wedge i}{\phi[y/x], \psi \vdash \exists x [\phi \wedge \psi]}\,\exists i}{\exists x \phi, \psi \vdash \exists x (\phi \wedge \psi)}\,\exists e$$

$\psi[y/x] = \psi$

$$\frac{\exists x\; \phi\wedge\psi \vdash \exists x\, \phi\wedge\psi}{\exists x\, \phi\wedge\psi \vdash \exists x\, \phi} \quad , \quad \frac{\exists x\, \phi\wedge\psi \vdash \exists x\, \phi\wedge\psi}{\exists x\, \phi\wedge\psi \vdash \exists x\, \phi}$$

Use $\wedge_e$ and cut rule twice to convert ';' on the left into the '$\wedge$' on the right.

ii) $\leftarrow$

$\exists x\, (\phi\wedge\psi) \vdash$

$$\dfrac{\dfrac{\exists x(\phi\wedge\psi)\vdash \exists x(\phi\wedge\psi) \qquad \dfrac{\dfrac{\dfrac{}{\phi\wedge\psi\vdash\phi\wedge\psi}\,Ax}{\phi\vee\psi\vdash\phi}\,\wedge e}{\phi\wedge\psi\vdash\exists x\,\phi}\,\exists i}{\exists x\,(\phi\wedge\psi)\vdash \exists x\,\phi} \qquad \dfrac{\exists x(\phi\wedge\psi)\vdash \exists x(\phi\wedge\psi) \qquad \dfrac{\dfrac{\dfrac{}{\phi\wedge\psi\vdash\phi\wedge\psi}\,Ax}{\phi\vee\psi\vdash\psi}\,\wedge e}{\phi\wedge\psi\vdash\psi}\,\exists i}{\exists x\,(\phi\wedge\psi)\vdash\psi}}{\exists x(\phi\wedge\psi)\vdash \exists x\,\phi\wedge\psi}\,\wedge e$$

Contr^2 is missing

Condition that x is a FV, is used at several places.

Formal Semantics of FOL.

Structure ($\mathcal{A}$) : Consists of.

> Domain : $|\mathcal{A}|$ ← non-empty set
> $c_1^{\mathcal{A}} \cdots c_m^{\mathcal{A}}$ (Constant Symbols)
> $R_1^{\mathcal{A}} \cdots R_m^{\mathcal{A}}$ (Relations) or Predicate Symbols

for each R, a subset of n tuples over $|\mathcal{A}|$

$$R_i^{\mathcal{A}} \subseteq A^{a_i} \leftarrow \text{arity } a_i$$

$$c_i^{\mathcal{A}} \subseteq \mathcal{A}$$

Structure $\mathcal{A} = (A, R_1^{\mathcal{A}} \cdots R_m^{\mathcal{A}}, c_1^{\mathcal{A}} \cdots c_n^{\mathcal{A}})$

Valuation (I)

for each $f \in I$, a concrete f_i : $f_i^{\mathcal{A}} : A^{a_i} \to A$

↑ set of n-tuples over A.

(A, I) ← MODEL.

eg: A model for language of arithmetic

> $|A|$, set of elements
> 0^A, as an int^n of 0
> $'^A$, a one place fu^n $|A| \to |A|$
> $+^A, *^A$, two place fu^n from $|A|^2 \to |A|$

and

> $<^A$ a two place rel^n

Definition 2.18 Given a model $\mathcal{M}$ for a pair $(\mathcal{F}, \mathcal{P})$ and given an environment l, we define the satisfaction relation $\mathcal{M} \vDash_l \phi$ for each logical formula ϕ over the pair $(\mathcal{F}, \mathcal{P})$ and look-up table l by structural induction on ϕ. If $\mathcal{M} \vDash_l \phi$ holds, we say that ϕ computes to $\mathtt{T}$ in the model $\mathcal{M}$ with respect to the environment l.

P: If ϕ is of the form $P(t_1, t_2, \ldots, t_n)$, then we interpret the terms $t_1, t_2, \ldots, t_n$ in our set A by replacing all variables with their values according to l. In this way we compute concrete values $a_1, a_2, \ldots, a_n$ of A for each of these terms, where we interpret any function symbol $f \in \mathcal{F}$ by $f^{\mathcal{M}}$. Now $\mathcal{M} \vDash_l P(t_1, t_2, \ldots, t_n)$ holds iff $(a_1, a_2, \ldots, a_n)$ is in the set $P^{\mathcal{M}}$.

$\forall x$: The relation $\mathcal{M} \vDash_l \forall x\, \psi$ holds iff $\mathcal{M} \vDash_{l[x \mapsto a]} \psi$ holds for all $a \in A$.

$\exists x$: Dually, $\mathcal{M} \vDash_l \exists x\, \psi$ holds iff $\mathcal{M} \vDash_{l[x \mapsto a]} \psi$ holds for some $a \in A$.

$\neg$: The relation $\mathcal{M} \vDash_l \neg\psi$ holds iff it is not the case that $\mathcal{M} \vDash_l \psi$ holds.

$\vee$: The relation $\mathcal{M} \vDash_l \psi_1 \vee \psi_2$ holds iff $\mathcal{M} \vDash_l \psi_1$ or $\mathcal{M} \vDash_l \psi_2$ holds.

$\wedge$: The relation $\mathcal{M} \vDash_l \psi_1 \wedge \psi_2$ holds iff $\mathcal{M} \vDash_l \psi_1$ and $\mathcal{M} \vDash_l \psi_2$ hold.

$\to$: The relation $\mathcal{M} \vDash_l \psi_1 \to \psi_2$ holds iff $\mathcal{M} \vDash_l \psi_2$ holds whenever $\mathcal{M} \vDash_l \psi_1$ holds.

Truth

$(A, I) \models \phi$ ϕ is true in (A, I)

$(A, I) \models R(t_1 \cdots t_n)$ ← What does this evaluate to?

iff $R_i^A(t_1 \ldots t_n)$

where,

$$t_i^A = \begin{cases} I(x), & \text{if } t_i \equiv x \\ c^A & \text{if } t_i \equiv c \end{cases}$$

Suppose, we have a formula

$$(A, I) = \phi_1 \wedge \phi_2$$

with def^n,

two element BA.

iff $(A, I) = \phi_1$ and $(A, I) = \phi_2$

Similarly, $\{\vee, \to\}$ can be handled similarly

for $\forall$

$(A, I) \models \forall x \, \phi$ iff for all $a \in A$,

$(A, I[a/x]) \models \phi$

is known as the Perturbation Valuation

$$I[a/x](y) = \begin{cases} I(y) & \text{if } y \neq x \\ a & \text{if } y \equiv x \end{cases}$$

for $\exists$,

$(A, I) \models \exists x \, \phi$ iff there is an $a \in A$, st

$(A, I[a/x]) \models \phi$

So, this is the classical def^n of quantifiers.

Constructive Def^n of Quantifiers

↳ done by extending BHK interpretation

$c : \forall x . \phi$ st for all $a \in A$.

$$ca : \phi[a/x]$$

$f : A \to B$

(range depends on the input argument)

$\prod_{a \in A} B(a)$ → (notation of a dependent fu^n space)

$\equiv A \to B$, if $B(a)$ is indep^n of a.