8th Jan '24

Propositional Language

We start with a stock of variables, like $p_0, p_1 \ldots p_n$ that we take to range over propositions.

they can mean anything
"the sky is blue"
"$2+2=5$"

More intuitively, propositions are things that are either true or false

Inductive Definition

> Each propositional variable is a formula } Atomic Propositions
> $\top$ or $\bot$ are formulas

> If A is a formula, $\neg A$ is a formula
> If A and B are formulas,

$A \wedge B$
$A \vee B$
$A \to B$
$A \iff B$
} are formulas

Since these formulas are generated inductively, we can define complexity, as the number of connective in a formula,

complexity (Atomic Prop^s) = 0
complexity ($\neg A$) = complexity (A) + 1

complexity ($A \wedge B$ / $A \vee B$ / $A \to B$ / $A \iff B$) = complexity (A) + complexity (B) + 1

eg: 1.) $\vee, \wedge$ associate from the left

$$A \vee B \vee C \Rightarrow (A \vee B) \vee C$$

2.) $\rightarrow$ associates from the right.

$$A \rightarrow B \rightarrow C \Rightarrow A \rightarrow (B \rightarrow C)$$

$\Gamma \vdash A$ ← judgement; A ← proposition; Γ ← set of propositions

$\Gamma \vdash A \;,\; \Gamma' \vdash B \Rightarrow \Gamma, \Gamma' \vdash A \wedge B$ (And introduction) $\wedge i$

$\Gamma \vdash (A \wedge B) \Rightarrow \Gamma \vdash A$ (And elimination) $\wedge e_1$

Inference Rules

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash (A \rightarrow B)}$$

$$\frac{\Gamma \vdash (A \rightarrow B) \;,\; \Gamma' \vdash A}{\Gamma, \Gamma' \vdash B}$$

premises ← $\Gamma \vdash (A \rightarrow B) \;,\; \Gamma' \vdash A$

inference ← $\Gamma, \Gamma' \vdash B$

inference rules.

1.) $A \vdash A$ AXIOM

2.) $\dfrac{\Gamma \vdash B}{\Gamma, A \vdash B}$ WEAKNING

3.) $\dfrac{\Gamma, A, A \vdash B}{\Gamma \vdash B}$ CONTRADICTION

4.) $\dfrac{\Gamma \vdash A \quad , \quad \Gamma' \vdash B}{\Gamma, \Gamma \vdash (A \wedge B)}$ (And Introduction) $\wedge i$

5.) $\dfrac{\Gamma \vdash (A \wedge B)}{\Gamma \vdash A}$ (And Elimination) $\wedge e_1$

6.) $\dfrac{\Gamma \vdash (A \wedge B)}{\Gamma \vdash B}$ (And Elimination) $\wedge e_2$

Logical Rules come in pairs for each pair of conn^2 (implication & elimination)

7.) $\dfrac{\Gamma \vdash A}{\Gamma \vdash (A \vee B)}$, $\dfrac{\Gamma \vdash B}{\Gamma \vdash (A \vee B)}$ (Or Introduction)

8.) $\dfrac{\Gamma \vdash (A \vee B) \quad , \quad \Gamma', A \vdash C \quad , \quad \Gamma', B \vdash C}{\Gamma, \Gamma' \vdash C}$ (Or Eliminations)

9) $\dfrac{\Upsilon, A \vdash B}{\Upsilon \vdash (A \to B)}$ $(\to i)$

10) $\dfrac{\Upsilon \vdash (A \to B) \quad \Upsilon' \vdash A}{\Upsilon, \Upsilon' \vdash B}$ $(\to e)$

11) $\dfrac{\Gamma \vdash A, \quad \Gamma', A \vdash B}{\Gamma, \Gamma' \vdash B}$ (Cut)

12.)

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A}$$ (Principle of Explosion) IL

$$\frac{\Gamma, \neg A \vdash \bot}{\Gamma \vdash A}$$ (Contraductions) CL

$$\Downarrow$$

$$\vdash (A \vee \neg A)$$ LEM

$$\frac{\Gamma, A \vdash B}{\Gamma, \neg B \vdash \neg A}$$ [Contraposition]

der:

$$\dfrac{\dfrac{\Gamma, A \vdash B \quad , \quad \dfrac{}{\neg B \vdash \neg B}}{\Gamma, A, \neg B \vdash \bot}}{\Gamma, \neg B \vdash \neg A}$$

Q.
Prove that,

$$A \to (B \to C) \vdash (A \wedge B) \to C$$

Sol^n

$$\dfrac{\dfrac{\dfrac{A \wedge B \vdash A \wedge B}{A \wedge B \vdash B}, \; A \wedge B, \; A \to (B \to C) \vdash (B \to C)}{A \wedge B, \; A \wedge B, \; A \to (B \to C) \vdash C}}{A \to (B \to C) \vdash (A \wedge B) \to C}$$

Q. Prove,

i) $A \wedge B \to C \vdash (A \to B) \to C$

Sol^n.

$$\dfrac{\dfrac{A \wedge B \to C \vdash A \wedge B \to C, \; \dfrac{A \vdash A, \; B \vdash B}{A, B \vdash A \wedge B}\,\wedge_i}{A \wedge B \to C, A, B \vdash C}\,{\to}e}{A \wedge B \to C \vdash (A \to B \to C)}\,(\to i)_{x_2}$$

ii) $A \wedge (B \wedge C) \vdash (A \wedge B) \wedge C$

$\neg$, is negation

↳ this is a derived connective

$\neg A \equiv A \to \bot$ ($\bot$ is some fixed atomic propositions)

Q. $A \vdash \neg\neg A$

$$\dfrac{\dfrac{A \vdash A \quad , \quad A \to \bot \vdash A \to \bot}{A, A \to \bot \vdash \bot}}{A}$$

$$\dfrac{\dfrac{\dfrac{A \vdash A \quad , \quad A \to \bot \vdash A \to \bot}{A, A \to \bot \vdash \bot}}{A \vdash (A \to \bot) \to \bot}}{A \vdash (\neg\neg A)}$$

Q 1.13 $(p \wedge q) \to r \vdash p \to (q \to r)$

Sol?

$$\dfrac{\dfrac{(p \wedge q) \to r \vdash (p \wedge q) \to r \quad , \quad \dfrac{p \vdash p \quad q \vdash q}{p, q \vdash p \wedge q}}{(p \wedge r) \to r, p, q \vdash r}}{(p \wedge r) \to r \vdash p \to (q \to r)}$$

Q 1.14 $p \to (q \to r) \vdash (p \wedge q) \to r$

Sol?

$$\dfrac{\dfrac{p \to (q \to r) \vdash p \to (q \to r) \qquad p \wedge q \vdash p}{p \to (q \to r), p \vdash (q \to r) \qquad , p \wedge q \vdash q}}{p \to (q \to r), p \wedge q \vdash r}$$

$$p \to (q \to r) \vdash (p \wedge q) \to r$$

Q 1.15 $p \to q \vdash (p \wedge r) \to (q \wedge r)$

Sol?

$$\dfrac{\dfrac{p \to q \vdash p \to q \quad , \quad \dfrac{p \wedge r \vdash p \wedge r}{p \wedge r \vdash p} \qquad \dfrac{p \wedge r \vdash p \wedge r}{p \wedge r \vdash r}}{p \to q, p \wedge r \vdash q}}{p \to q, p \wedge r \vdash (q \wedge r)}$$

$$p \to q \vdash (p \wedge r) \to (q \wedge r)$$

Q 1.16 $q \to r \vdash (p \vee q) \to (p \vee r)$

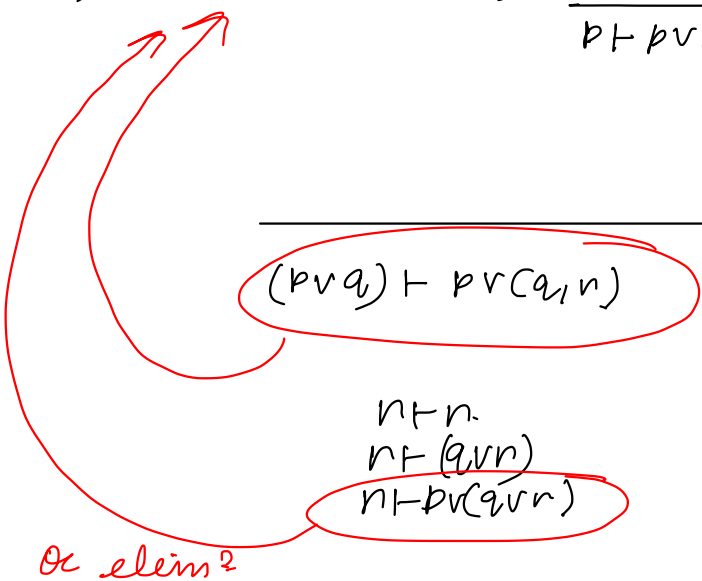
$$\dfrac{\dfrac{\dfrac{q \to r \vdash q \to r \quad , \quad q \vdash q}{q \to r, q \vdash r}}{q \to r, q \vdash p \vee r} \quad , \quad \dfrac{p \vdash p}{p \vdash p \vee r} \qquad (p \vee q) \vdash (p \vee q)}{q \to r, (p \vee q) \vdash (p \vee r)}$$

$$q \to r \vdash (p \vee q) \to (p \vee r)$$

Q.1.17 $(p \vee q) \vee r \vdash p \vee (q \vee r)$

Sol³.

$(p \vee q) \vee r \vdash (p \vee q) \vee r$ $\quad (p \vee q) \vdash (p \vee q)$ $\quad \dfrac{p \vdash p}{p \vdash p \vee (q \vee r)}$ $\quad \dfrac{\dfrac{q \vdash q}{q \vdash q \vee r}}{q \vdash p \vee (q \vee r)}$



Q.1.20 $\neg p \vee q \vdash p \rightarrow q$

Sol²

$$\dfrac{\neg p \vee q \vdash \neg p \vee q, \quad \dfrac{\dfrac{\neg p \vdash \neg p, q \vdash q}{p, \neg p \vdash \bot}}{\dfrac{p, \neg p \vdash q}{\neg p \vdash (p \rightarrow q)}}, \quad \dfrac{q \vdash q}{p, q \vdash q}}{\dfrac{}{q \vdash (p \rightarrow q)}}$$

$$\neg p \vee q \vdash (p \rightarrow q)$$

Q1.21 $p \to q,\ p \to \neg q \vdash \neg p$

Sol^n

$$\frac{p \to q \vdash p \to q}{p \to q,\ p \vdash q} \qquad \frac{p \to \neg q \vdash p \to \neg q \qquad p \vdash p}{p \to \neg q,\ p \vdash (q \to \bot)}$$

$$\frac{}{\dfrac{p \to q,\ p,\ p,\ p \to \neg q \vdash \bot}{p \to q,\ p \to \neg q \vdash (p \to \bot)}}$$

Q1.22 $p \to (q \to r),\ p,\ \neg r \vdash \neg q$

Sol^n

$$\frac{p \to (q \to r) \vdash p \to (q \to r),\quad p \vdash p,\quad q \vdash q}{p \to (q \to r),\ p,\ q \vdash r} \quad ,\ \neg r \vdash (r \to \bot)$$

$$\frac{}{p \to (q \to r,\ p,\ \neg r \vdash q}$$

Q1.23 $p \wedge \neg q \to r,\ \neg r,\ p \vdash q$

$$(p \wedge \neg q) \to r \vdash (p \wedge \neg q) \to r \qquad \frac{p \vdash p,\quad \neg q \vdash \neg q}{p, \neg q \vdash p \wedge \neg q}$$

$$\frac{}{(p \wedge \neg q) \to r,\ p, \neg q \vdash r \qquad \neg r \vdash (r \to \bot)}$$

$$\frac{}{\dfrac{(p \wedge \neg q) \to r,\ p, \neg q, \neg r \vdash \bot}{(p \wedge \neg q) \to r,\ p,\ \neg r \perp q}}$$

Q 1.24 $p \to q \vdash \neg p \vee q$, using LEM

$$\dfrac{\dfrac{p \vdash p \quad , \quad p \to q \vdash q \to p}{p, p \to q \vdash p \wedge q} \quad , \quad \vdash \neg p \vee p \quad , \quad \dfrac{q \vdash q}{\neg q \vdash \neg p \vee q}}{p \to q \vdash \neg p \vee q}$$

$$\dfrac{\dfrac{\dfrac{}{q \to p \vdash q \to p} \quad , \quad \dfrac{}{q \vdash q}}{\begin{array}{l} p, p \to q \vdash q \\ q, p \to q \vdash \neg p \vee q \end{array}} \quad , \quad \dfrac{}{\begin{array}{l} p \wedge q \vdash q \\ q \vdash q \end{array}} \quad , \quad \vdash (\neg p \vee p)}{p \to q \vdash \neg p \vee q}$$