Practice Problems Sheet-1

Q1

(a) Without using set theoretic axioms such as ZF, show that there is no set $A = \{x \mid x \text{ is a set and } x \notin x\}$.

(b) Consider some mathematical theory T which has a sentence ϕ that is logically equivalent to "ϕ is not provable in T". Assuming that T is sound what can you say about T?

Q2 For each of the following judgement, give its derivation in the weakest of the three systems Nm, Ni and Nc, in which it can be derived.

(a) $\vdash \neg(A \vee B) \to \neg A \wedge \neg B$

(b) $\vdash \neg A \wedge \neg B \to \neg(A \vee B)$

(c) $\vdash \neg(A \wedge B) \to \neg A \vee \neg B$

(d) $\vdash \neg A \vee \neg B \to \neg(A \wedge B)$

(e) $\vdash \neg\neg\neg A \to \neg A$

Q3 For propositions ϕ and ψ, define $\phi \leftrightarrow \psi$ as $(\phi \to \psi) \wedge (\psi \to \phi)$.

Show that the following are derivable in Ni.

(a) $\vdash (A \to B) \to (\neg\neg A \to \neg\neg B)$

(b) $\vdash \neg\neg(A \wedge B) \leftrightarrow (\neg\neg A \wedge \neg\neg B)$

Q4

(a) Label Nm inference rules $\wedge i, \vee i, \rightarrow i$ and $\rightarrow e$ with construction terms.

(b) We extend the language of construction terms to label Nm rules $\wedge e$ and $\vee e$ as follows.

$$\frac{\Gamma \vdash t : A \wedge B}{\Gamma \vdash \pi_1 t : A \wedge B} \wedge e1 \qquad \frac{\Gamma \vdash t : A \wedge B}{\Gamma \vdash \pi_2 t : A \wedge B} \wedge e2$$

$$\frac{\Gamma \vdash t : A \vee B \quad \Gamma', a : A \vdash t_1 : C \quad \Gamma', b : B \vdash t_2 : C}{\Gamma, \Gamma' \vdash \text{case } t \text{ of } inl(a) \Rightarrow t_1 |\ inr(b) \Rightarrow t_2 : C} \vee e$$

Here π_i , $i \in \{1, 2\}$, stands for i^{th} projection of a pair and "case" construct stands for reasoning by cases.

Give precise meaning of these constructs and explain why the above labeling of Nm inference rules is intuitively correct.

(c) Can you think of term assignment for structural rules of Ni?

Q5 You may use labeling given in **Q4** to answer the following.

- **(a)** Label your derivations of those parts of **Q2** which are in Nm. In particular, provide a construction term for the conclusion judgement.
- **(b)** Label your derivations of those parts of **Q3** which are in Nm. In particular, provide a construction term for the conclusion judgement.

Q6. Let $\vdash$ be either $\vdash_{Nm}$ or $\vdash_{Ni}$ or $\vdash_{Nc}$ in this question. Define a binary relation '$\sim$' on propositions (of our propositional language) as follows.

$\phi \sim \psi$ iff $\vdash (\phi \rightarrow \psi)$ and $\vdash (\psi \rightarrow \phi)$.

Show the following.

- **(i)** Relation $\sim$ is an equivalence relation.
- **(ii)** Relation $\sim$ is a congruence with respect to logical operations.
- **(iii)** Let $[\phi]$ be the equivalence class of ϕ w.r.t. relation $\sim$ above. Define a binary relation '$\leq$' on these equivalence classes as $[\phi] \leq [\psi]$ iff $\vdash (\phi \rightarrow \psi)$. Show that $\leq$ is a partial order.

Q7 Let $(A, \leq)$ and $(B, \leq)$ be partial orders. A function $f : A^n \to B$ is said be monotone in i^{th} argument if it preserves order of i^{th} argument. That is, for all $a_1, \ldots, a_i, \ldots, a_n$ and a'_i in A, $a_i \leq a'_i \Rightarrow f(a_1, \ldots, a_i, \ldots, a_n) \leq f(a_1, \ldots, a'_i, \ldots, a_n)$.

Show that in a lattice $(A, \leq, \vee, \wedge)$, both functions $\vee$ and $\wedge$ are monotone in both arguments.

Q8 Consider an algebraic structure $(A, \vee, \wedge)$ satisfying the following axioms.

$a \vee b = b \vee a$	$a \wedge b = b \wedge a$	**(Commutativity)**
$(a \vee b) \vee c = a \vee (b \vee c)$	$(a \wedge b) \wedge c = a \wedge (b \wedge c)$	**(Associativity)**
$a \vee a = a$	$a \wedge a = a$	**(Idempotence)**
$a \vee (a \wedge b) = a$	$a \wedge (a \vee b) = a$	**(Absorption)**

Define a binary relation R on A s.t. aRb iff $a \wedge b = a$.

(a) Show that R is a partial order on A.

(b) Show that $\vee$ and $\wedge$ are respectively *lub* and *glb* operations on (A, R).

(c) Conclude that the class of lattices can be defined using (above) equations only.

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