CS202M Final Exam solutions

Q1(marks-4+8)

(i) $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$ always holds.

Proof: Clearly, $b \wedge c \leq b$

$\Rightarrow a \vee (b \wedge c) \leq a \vee b$ [monotonicity of $\vee$]

Similarly, $a \vee (b \wedge c) \leq a \vee c$

$\Rightarrow a \vee (b \wedge c)$ is a lower bound for both $a \vee b$ and $a \vee c$

$\Rightarrow$ $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$ (by def. of $\wedge$) $\square$

(ii) Let $a \to b = \vee\{x \in L \mid a \wedge x \leq b\}$

(The $\vee$ shown exists as it is over a finite set. It can be obtained by finitely many uses of binary $\vee$)

We need to verify that $a \to b$ is the largest y s.t. $a \wedge y \leq b$.

This is done in the following two steps.

1. $a \wedge (a \to b) = a \wedge (\vee\{x \in L \mid a \wedge x \leq b\})$
   $= \vee\{a \wedge x \in L \mid a \wedge x \leq b\}$ [by distributing $\vee$ over $\wedge$]
   $\leq b$
2. Let y be s.t. $a \wedge y \leq b$
   $\Rightarrow y \in \{x \in L \mid a \wedge x \leq b\}$
   By def of $\vee$,
   $y \leq \vee\{x \in L \mid a \wedge x \leq b\} = a \to b$. $\square$

Q2(marks-3+4)

(i) Sub-algebra induced on subset $\{\emptyset, \{1,2,3\}, \{1\}, \{2,3\}\}$.

[Some other example subsets are $\{\emptyset, \{1,2,3\}, \{1,2\}, \{3\}\}$ and $\{\emptyset, \{1,2,3\}, \{1,3\}, \{2\}\}$].

(ii)

(a) Does **not** hold.

$p \to p = \neg p \vee p$

for valuation $p \mapsto n$, $\neg p \vee p = n$ which is not a designated value.

(b) **Holds**

Consider a valuation v, s.t. $v(\neg(p \wedge q)) \in \{b, 1\}$

$\Rightarrow v(p \wedge q) \in \{b, 0\}$

We Consider following two cases.

$v(p \wedge q) = b$

$\Rightarrow$ $v(p) = b$ or $v(q) = b$
$\Rightarrow$ $v(\neg p) = b$ or $v(\neg q) = b$
$\Rightarrow$ $v(\neg p \vee \neg q) \geq b$

$v(p \wedge q) = 0$

$\Rightarrow$ at least one of $v(p), v(q) = 0$ or at least one of $v(p), v(q) = b$
$\Rightarrow$ at least one of $v(\neg p), v(\neg q) = 1$ or at least one of $v(\neg p), v(\neg q) = b$
$\Rightarrow$ $v(\neg p \vee \neg q) \geq b$

Q3(marks-3+8+5)

(i) $\mathbf{B}_1 \times \mathbf{B}_2 = (B_1 \times B_2, \leq_{12}, \wedge_{12}, \vee_{12}, \neg_{12}, 0_{12}, 1_{12})$, where

$(a_1, a_2) \leq_{12} (b_1, b_2)$ iff $a_1 \leq_1 b_1$ and $a_2 \leq_2 b_2$

$(a_1, a_2) \wedge_{12} (b_1, b_2) = (a_1 \wedge_1 b_1, a_2 \wedge_2 b_2)$

$(a_1, a_2) \vee_{12} (b_1, b_2) = (a_1 \vee_1 b_1, a_2 \vee_2 b_2)$

$\neg_{12}(a_1, a_2) = (\neg_1 a_1, \neg_2 a_2)$

$0_{12} = (0_1, 0_2)$

$1_{12} = (1_1, 1_2)$

(ii)

I To show 1-1 property, we show that $f(x) = f(y) \Rightarrow x = y$.

$f(x) = f(y)$
$\Rightarrow (x \wedge a, x \wedge \neg a) = (y \wedge a, y \wedge \neg a)$
$\Rightarrow x \wedge a = y \wedge a$ and $x \wedge \neg a = y \wedge \neg a$
$\Rightarrow\ x = x \wedge (a \vee \neg a) = (x \wedge a) \vee (x \wedge \neg a)$
$= (y \wedge a) \vee (y \wedge \neg a) = y \wedge (a \vee \neg a) = y$

II To show that f is onto, let $(c, d) \in (B \restriction a) \times (B \restriction \neg a)$

$\Rightarrow\ c \leq a$ and $d \leq \neg a$
$\Rightarrow c \vee d \in B$
$\Rightarrow f(c \vee d) = ((c \vee d) \wedge a, (c \vee d) \wedge \neg a)$
$(c \vee d) \wedge a = (c \wedge a) \vee (d \wedge a)$
$c \wedge a = c\ \ (\because c \leq a)$
$d \leq \neg a\ \Rightarrow d \wedge a \leq \neg a \wedge a = 0$
$\Rightarrow\ (c \vee d) \wedge a = (c \wedge a) \vee (d \wedge a) = c \vee 0 = c.$

Similarly,

$(c \vee d) \wedge \neg a = (c \wedge \neg a) \vee (d \wedge \neg a)$
$d \wedge \neg a = d\ \ (\because d \leq \neg a)$
$c \leq a\ \Rightarrow c \wedge \neg a \leq a \wedge \neg a = 0$
$\Rightarrow\ (c \vee d) \wedge \neg a = (c \wedge \neg a) \vee (d \wedge \neg a) = 0 \vee d = d$

Therefore, $f(c \vee d) = (c, d)$.

(iii) We proof this by induction on the number of elements in a finite BA $\mathbf{B}$.

Base-Case: $\mathbf{B}$ has just two elements. Only algebra on two elements is BA $\mathbf{2}$.

Therefore $\mathbf{B} = \mathbf{2}$.

Induction-Step: If $\mathbf{B}$ has more that two elements then it has an a, s.t. $0 < a < 1$.

$0 < a < 1 \Rightarrow 0 < \neg a < 1$.

By part (ii), we know that $\mathbf{B}$ is isomorphic to $(\mathbf{B} \upharpoonright a) \times (\mathbf{B} \upharpoonright \neg a)$.

Further, as 1 of $\mathbf{B}$ does not belong to $\mathbf{B} \upharpoonright a$, the number of elements in $\mathbf{B} \upharpoonright a$ are strictly less than the number of elements in $\mathbf{B}$. Therefore, by induction hypothesis, $\mathbf{B} \upharpoonright a \simeq \underbrace{2 \times \ldots \times \mathbf{2}}_{r \text{ times}}$, for some r. Similarly, $\mathbf{B} \upharpoonright \neg a \simeq \underbrace{2 \times \ldots \times \mathbf{2}}_{s \text{ times}}$, for some s.

$\Rightarrow \mathbf{B} \simeq (\mathbf{B} \upharpoonright a) \times (\mathbf{B} \upharpoonright \neg a) \simeq (\underbrace{2 \times \ldots \times \mathbf{2}}_{r \textbf{ times}}) \times (\underbrace{2 \times \ldots \times \mathbf{2}}_{s \textbf{ times}}) \simeq \underbrace{2 \times \ldots \times \mathbf{2}}_{r+s \text{ times}}$.

Q4(marks-5)

Given a CNF in which each clause has at most one negative literal, we replace each variable by its negation and remove any two consecutive occurrences of $\neg$. The resulting CNF is equi-satisfiable with the original CNF (satisfying assignment of one is obtained from the satisfying assignment of the other by flipping 0's and 1's).

In the resulting CNF each clause has at most one positive literal. Consider any such clause $(\neg a_1, \ldots, \neg a_n, b)$.

$\neg a_1 \vee \ldots \vee \neg a_n \vee b$

$\Leftrightarrow \neg(a_1 \wedge \ldots \wedge a_n) \vee b$

$\Leftrightarrow (a_1 \wedge \ldots \wedge a_n) \rightarrow b$

$\Leftrightarrow a_1, \ldots, a_n \rightarrow b$

So each clause is a Horn clause.

We now apply Horn satisfiability algorithm done in class to the resulting CNF formula.

Q5(marks-20)

(i)

$$
\dfrac{\dfrac{}{\exists x \forall y.A(x,y) \vdash \exists x \forall y.A(x,y)} \qquad \dfrac{\dfrac{\dfrac{axiom}{\forall y.A(z,y) \vdash \forall y.A(z,y)}}{\forall y.A(z,y) \vdash A(z,y)}\forall e}{\forall y.A(z,y) \vdash \exists x.A(x,y)}\exists i}{\dfrac{\exists x \forall y.A(x,y) \vdash \exists x.A(x,y)}{\exists x \forall y.A(x,y) \vdash \forall y \exists x.A(x,y)}\forall i}\exists e
$$

(ii) $\exists x.(C(x) \to D(x)) \vdash \forall x.C(x) \to D(x)$

We define a falsifying interpretation as follows.

Structure $\mathcal{A}$, with domain($\mathcal{A}$)=$\{a,b\}$

$C^{\mathcal{A}} = \{a,b\}$, $D^{\mathcal{A}} = \{a\}$.

Interpretation I, with $I(x) = b$.

(iii) $\forall y \exists x.A(x,y) \vdash \exists x \forall y.A(x,y)$

We define a falsifying interpretation as follows.

Structure $\mathcal{A}$, with domain($\mathcal{A}$)=$\{a,b\}$

$A^{\mathcal{A}} = \{(a,a),(b,b)\}$

(v)

$$
\dfrac{\dfrac{\dfrac{axiom}{\forall x.A(x) \vdash \forall x.A(x)}}{\forall x.A(x) \vdash A(x)}\forall e}{\forall x.A(x) \vdash \exists x.A(x)}\exists i
$$

(iv) $\phi \to \exists x.\psi(x) \vdash \exists x.(\phi \to \psi(x))$, $x \notin FV(\phi)$

Let $\mathcal{D}_1$ be the following derivation

$$\dfrac{\dfrac{\text{prop. reasoning}}{\phi, \phi \to \exists x.\psi(x) \vdash \exists x.\psi(x)} \qquad \dfrac{\dfrac{\text{prop. reasoning}}{\psi[y/x] \vdash \phi \to \psi[y/x]}}{\psi[y/x] \vdash \exists x.(\phi \to \psi(x))}\exists i}{\phi, \phi \to \exists x.\psi(x) \vdash \exists x.(\phi \to \psi(x))}\exists e$$

Let $\mathcal{D}_2$ be the following derivation

$$\dfrac{\dfrac{\dfrac{\mathit{axiom}}{\neg\exists x.(\phi \to \psi(x)) \vdash \neg\exists x.(\phi \to \psi(x))} \qquad \dfrac{\mathcal{D}_1}{\phi, \phi \to \exists x.\psi(x) \vdash \exists x.(\phi \to \psi(x))}}{\phi, \phi \to \exists x.\psi(x), \neg\exists x.(\phi \to \psi(x)) \vdash \bot} \to e}{\phi \to \exists x.\psi(x), \neg\exists x.(\phi \to \psi(x)) \vdash \neg\phi} \to i$$

Let $\mathcal{D}_3$ be the following derivation

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_2}{\phi \to \exists x.\psi(x), \neg\exists x.(\phi \to \psi(x)) \vdash \neg\phi}}{\phi \to \exists x.\psi(x), \neg\exists x.(\phi \to \psi(x)) \vdash \phi \to \psi(x)} \text{ prop reasoning}}{\phi \to \exists x.\psi(x), \neg\exists x.(\phi \to \psi(x)) \vdash \exists x.(\phi \to \psi(x))}\exists i$$

Finally, the required derivation is

$$\dfrac{\dfrac{\dfrac{\mathit{axiom}}{\neg\exists x.(\phi \to \psi(x)) \vdash \neg\exists x.(\phi \to \psi(x))} \qquad \dfrac{\mathcal{D}_3}{\phi \to \exists x.\psi(x), \neg\exists x.(\phi \to \psi(x)) \vdash \exists x.(\phi \to \psi(x))}}{\phi \to \exists x.\psi(x), \neg\exists x.(\phi \to \psi(x)) \vdash \bot} \to e}{\phi \to \exists x.\psi(x) \vdash \exists x.(\phi \to \psi(x))}\bot_c$$

Q6(marks-5)

(i) $\models_1 (p \to q) \to \neg p \vee q$ **does not hold**.

Consider valuation v, with $v(p) = v(q) = 0.5$.

$v(p \to q) = 1,$

$v(\neg p \vee q) = max\{1 - 0.5, 0.5\} = 0.5$

$\Rightarrow v((p \to q) \to \neg p \vee q) = 1 - (1 - 0.5) = 0.5 \neq 1$

(ii) $\models_1 \neg p \vee q \to (p \to q)$ **holds**

Proof:

Let v be any valuation. We consider two cases.

(I) $v(p) \leq v(q)$

$\Rightarrow v(p \to q) = 1$

$\Rightarrow v(\neg p \vee q) \leq v(p \to q)$

(II) $v(p) > v(q)$

$v(\neg p \vee q) = max\{1 - v(p), v(q)\}$

$\leq 1 - v(p) + v(q)$

$= 1 - (v(p) - v(q))$

$= v(p \to q)$

$\Rightarrow v(\neg p \vee q) \leq v(p \to q)$

In both cases, $v(\neg p \vee q) \leq v(p \to q)$. Therefore, $v(\neg p \vee q \to (p \to q)) = 1$. $\square$

(iii) $p \to q \models_1 (p \wedge r \to q)$ **holds.**

Proof:

Let v be any valuation s.t. $v(p \to q) = 1$

$\Rightarrow v(p) \leq v(q)$

$\Rightarrow v(p \wedge r) = min(v(p), v(r)) \leq v(p) \leq v(q)$

$\Rightarrow v(p \wedge r \to q) = 1$. $\square$

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