CS202M First Exam Solutions

Q1

$$\cfrac{A \vee \neg A \vdash A \vee \neg A \qquad \cfrac{\cfrac{\cfrac{\overline{A \vdash A}}{A, \neg\neg A \vdash A}\textbf{weakening}}{A \vdash \neg\neg A \to A} \to i \qquad \cfrac{\cfrac{\cfrac{\overline{\neg A, \vdash \neg A} \quad \overline{\neg\neg A \vdash \neg\neg A}}{\neg A, \neg\neg A \vdash \bot} \to e}{\neg A, \neg\neg A \vdash A}\bot_i}{\neg A \vdash \neg\neg A \to A} \to i}{A \vee \neg A \vdash \neg\neg A \to A} \vee e}{}$$

Q2 Let $\mathcal{D}_1$ be the derivation as below.

$$\cfrac{\overline{\neg(\neg A \vee B) \vdash \neg(\neg A \vee B)} \qquad \cfrac{\cfrac{\cfrac{\overline{A \to B \vdash A \to B} \quad \overline{A \vdash A}}{A \to B, A \vdash B} \to e}{A \to B, A \vdash \neg A \vee B} \vee i}{A \to B, \neg(\neg A \vee B), A \vdash \bot} \to e}{A \to B, \neg(\neg A \vee B) \vdash \neg A} \to i$$

Then derivation of the proposition asked as follows.

$$\cfrac{\cfrac{\overline{\neg(\neg A \vee B) \vdash \neg(\neg A \vee B)} \qquad \cfrac{\cfrac{\mathcal{D}_1}{A \to B, \neg(\neg A \vee B) \vdash \neg A}}{A \to B, \neg(\neg A \vee B) \vdash \neg A \vee B} \vee i}{\cfrac{\neg(\neg A \vee B), A \to B, \neg(\neg A \vee B) \vdash \bot}{A \to B, \neg(\neg A \vee B) \vdash \bot}\textbf{contraction}} \to e}{A \to B \vdash \neg A \vee B}\textbf{PBC}$$

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Q3(a)

$$\cfrac{\cfrac{}{\neg B \vdash \neg B} \qquad \cfrac{\cfrac{}{A \to B \vdash A \to B} \qquad \cfrac{}{A \vdash A}}{A, A \to B \vdash B} \to e}{\cfrac{A, \neg B, A \to B \vdash \bot}{A, \neg B \vdash \neg(A \to B)} \to i} \to e$$

Q3(b)

$$\cfrac{\cfrac{}{x : \neg B \vdash x : \neg B} \qquad \cfrac{\cfrac{}{y : A \to B \vdash y : A \to B} \qquad \cfrac{}{z : A \vdash z : A}}{z : A, y : A \to B \vdash yz : B} \to e}{\cfrac{z : A, x : \neg B, y : A \to B \vdash x(yz) : \bot}{z : A, x : \neg B \vdash \lambda y : A \to B.x(yz) : \neg(A \to B)} \to i} \to e$$

Q3(c)

$$\cfrac{\cfrac{\mathbf{axiom}}{\neg\neg A \vdash \neg\neg A} \qquad \cfrac{\cfrac{\cfrac{\mathbf{axiom}}{\neg\neg(A \to B) \vdash \neg\neg(A \to B)} \qquad \cfrac{\mathbf{part\ (a)}}{A, \neg B \vdash \neg(A \to B)}}{\cfrac{\neg\neg(A \to B), A, \neg B \vdash \bot}{\neg\neg(A \to B), \neg B \vdash \neg A} \to i}}{\cfrac{\neg\neg(A \to B), \neg\neg A, \neg B \vdash \bot}{\neg\neg(A \to B), \neg\neg A \vdash \neg\neg B} \to i}}{\cfrac{}{\neg\neg(A \to B) \vdash \neg\neg A \to \neg\neg B} \to i} \to e$$

Q4(i)

$\Rightarrow$ direction

Let $a \leq b$.

$\neg b \wedge \neg a$

$= \neg(b \vee a)$ (By DeMorgan's law.)

$= \neg b$ ($\because b \vee a = b$ as $a \leq b$)

As $\neg b \wedge \neg a = \neg b$ it follows that $\neg b \leq \neg a$.

$\Leftarrow$ direction

Assume $\neg b \leq \neg a$

$\Rightarrow \neg\neg a \leq \neg\neg b$ (by $\Rightarrow$ direction)

$\Rightarrow a \leq b$ (by involution of negation).

Q4(ii)

$\Rightarrow$ direction

$a \leq b$.

$\Rightarrow a \vee b = b$

$\Rightarrow \neg a \vee (a \vee b) = \neg a \vee b$

$\Rightarrow (\neg a \vee a) \vee b = a \rightarrow b$

$\Rightarrow 1 \vee b = a \rightarrow b$

$\Rightarrow 1 = a \rightarrow b$

$\Leftarrow$ direction

$a \rightarrow b = 1$.

$\Rightarrow \neg a \vee b = 1$

$\Rightarrow a \wedge (\neg a \vee b) = a \wedge 1$

$\Rightarrow (a \wedge \neg a) \vee (a \wedge b) = a$

$\Rightarrow 0 \vee (a \wedge b) = a$

$\Rightarrow a \wedge b = a$

$\Rightarrow a \leq b$

———————————End———————————