Solution to Q5 Practice Problems Sheet-1

Q5(a)

Part-2(a)

- Consider labeled derivation $\mathcal{D}_1$ below.

$$\dfrac{\dfrac{\dfrac{}{y:\neg(A\vee B)\vdash y:\neg(A\vee B)} \qquad \dfrac{\dfrac{}{x:A\vdash x:A}}{x:A\vdash inl(x):A\vee B}}{y:\neg(A\vee B),x:A\vdash y(inl(x)):\bot}}{y:\neg(A\vee B)\vdash \lambda x:A.y(inl(x)):\neg A}$$

- Similarly, we have

  $z:\neg(A\vee B)\vdash \lambda x:B.z(inr(x)):\neg B$

- Following is the labeled main derivation.

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{y:\neg(A\vee B)\vdash \lambda x:A.y(inl(x)):\neg A} \qquad \dfrac{\mathcal{D}_2}{z:\neg(A\vee B)\vdash \lambda x:B.z(inr(x)):\neg B}}{y:\neg(A\vee B),z:\neg(A\vee B)\vdash \langle \lambda x:A.y(inl(x)),\lambda x:B.z(inr(x))\rangle:\neg A\wedge\neg B}}{y:\neg(A\vee B)\vdash \langle \lambda x:A.y(inl(x)),\lambda x:B.y(inr(x))\rangle:\neg A\wedge\neg B}}{\vdash \lambda y:\neg(A\vee B).\langle \lambda x:A.y(inl(x)),\lambda x:B.y(inr(x))\rangle:\neg(A\vee B)\to\neg A\wedge\neg B}$$

Part-2(b)

- Consider labeled derivation $\mathcal{D}_1$ below.

$$\dfrac{\dfrac{\dfrac{}{x:\neg A\wedge\neg B\vdash x:\neg A\wedge\neg B}}{x:\neg A\wedge\neg B\vdash \pi_1 x:\neg A} \qquad \dfrac{}{y:A\vdash y:A}}{x:\neg A\wedge\neg B,y:A\vdash(\pi_1 x)y:\bot}$$

- Similarly, we have

  $x:\neg A\wedge\neg B,y:B\vdash(\pi_2 x)y:\bot$

- Following is the labeled main derivation.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{u : A \vee B \vdash u : A \vee B} \quad \dfrac{\mathcal{D}_1}{x : \neg A \wedge \neg B, y : A \vdash (\pi_1 x)y : \bot} \quad \dfrac{\mathcal{D}_2}{x : \neg A \wedge \neg B, y : B \vdash (\pi_2 x)y : \bot}}{x : \neg A \wedge \neg B, u : A \vee B \vdash \textbf{case } u \textbf{ of } inl(y) \Rightarrow (\pi_1 x)y | inr(y) \Rightarrow (\pi_2 x)y : \bot}}{x : \neg A \wedge \neg B \vdash \lambda u : A \vee B.\textbf{case } u \textbf{ of } inl(y) \Rightarrow (\pi_1 x)y | inr(y) \Rightarrow (\pi_2 x)y : \neg(A \vee B)}}{\vdash \lambda x : \neg A \wedge \neg B.\lambda u : A \vee B.\textbf{case } u \textbf{ of } inl(y) \Rightarrow (\pi_1 x)y | inr(y) \Rightarrow (\pi_2 x)y : \neg A \wedge \neg B \rightarrow \neg(A \vee B)}$$

Part-2(d)

- Consider labeled derivation $\mathcal{D}_1$ below.

$$\dfrac{\dfrac{}{x : \neg A \vdash x : \neg A} \quad \dfrac{\dfrac{}{y : A \wedge B \vdash y : A \wedge B}}{y : A \wedge B \vdash \pi_1 y : A}}{y : A \wedge B, x : \neg A \vdash x(\pi_1 y) : \bot}$$

- Similarly, we have

  $y : A \wedge B, x : \neg B \vdash x(\pi_2 y) : \bot$

- Following is the labeled main derivation.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{u : \neg A \vee \neg B \vdash u : \neg A \vee \neg B} \quad \dfrac{\mathcal{D}_1}{y : A \wedge B, x : \neg A \vdash x(\pi_1 y) : \bot} \quad \dfrac{\mathcal{D}_2}{y : A \wedge B, x : \neg B \vdash x(\pi_2 y) : \bot}}{u : \neg A \vee \neg B, y : A \wedge B \vdash \textbf{case } u \textbf{ of } inl(x) \Rightarrow x(\pi_1 y) | inr(x) \Rightarrow x(\pi_2 y) : \bot}}{u : \neg A \vee \neg B \vdash \lambda y : A \wedge B.\textbf{case } u \textbf{ of } inl(x) \Rightarrow x(\pi_1 y) | inr(x) \Rightarrow x(\pi_2 y) : \neg(A \wedge B)}}{\vdash \lambda u : \neg A \vee \neg B.\lambda y : A \wedge B.\textbf{case } u \textbf{ of } inl(x) \Rightarrow x(\pi_1 y) | inr(x) \Rightarrow x(\pi_2 y) : \neg A \vee \neg B \rightarrow \neg(A \wedge B)}$$

Part-2(e)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{z : \neg\neg\neg A \vdash z : \neg\neg\neg A} \quad \dfrac{\dfrac{\dfrac{}{x : \neg A \vdash x : \neg A} \quad \dfrac{}{y : A \vdash y : A}}{y : A, x : \neg A \vdash xy : \bot}}{y : A \vdash \lambda x : \neg A.xy : \neg\neg A}}{z : \neg\neg\neg A, y : A \vdash z(\lambda x : \neg A.xy) : \bot}}{z : \neg\neg\neg A \vdash \lambda y : A.z(\lambda x : \neg A.xy) : \neg A}}{\vdash \lambda z : \neg\neg\neg A.\lambda y : A.z(\lambda x : \neg A.xy) : \neg\neg\neg A \rightarrow \neg A}$$

Q5(b)

Part-3(a)

- Following is labeled deduction $\mathcal{D}_1$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{x : A \to B \vdash x : A \to B} \quad \overline{y : A \vdash y : A}}{x : A \to B, y : A \vdash xy : B} \quad \overline{z : \neg B \vdash z : \neg B}}{x : A \to B, y : A, z : \neg B \vdash z(xy) : \bot}}{x : A \to B, z : \neg B \vdash \lambda y : A.z(xy) : \neg A}}{x : A \to B \vdash \lambda z : \neg B.\lambda y : A.z(xy) : (\neg B \to \neg A)}$$

- Similar labeling of $\mathcal{D}_2$ gives

  $u : \neg B \to \neg A \vdash \lambda w : \neg\neg A.\lambda v : \neg B.w(uv) : (\neg\neg A \to \neg\neg B)$

- Following is the labeled main derivation, where we let

  $t_1 \equiv \lambda z : \neg B.\lambda y : A.z(xy)$ and $t_2 \equiv \lambda w : \neg\neg A.\lambda v : \neg B.w(uv)$.

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{x : A \to B \vdash t_1 : \neg B \to \neg A} \quad \dfrac{\dfrac{\mathcal{D}_2}{u : \neg B \to \neg A \vdash t_2 : \neg\neg A \to \neg\neg B}}{\vdash \lambda u : \neg B \to \neg A.t_2 : (\neg B \to \neg A) \to (\neg\neg A \to \neg\neg B)}}{x : A \to B \vdash (\lambda u : \neg B \to \neg A.t_2)t_1 : \neg\neg A \to \neg\neg B}}{\vdash \lambda x : A \to B.(\lambda u : \neg B \to \neg A.t_2)t_1 : (A \to B) \to (\neg\neg A \to \neg\neg B)}$$

- Putting for t_1 and t_2 back, we see that the construction term associated with conclusion is

  $\lambda x : A \to B.(\lambda u : \neg B \to \neg A.\lambda w : \neg\neg A.\lambda v : \neg B.w(uv))(\lambda z : \neg B.\lambda y : A.z(xy))$

Part-3(b) Forward implication

- Labeled derivation of derived rule CPOS is

$$\dfrac{\dfrac{\Gamma, x : A \vdash t : B \qquad \dfrac{}{y : \neg B \vdash y : \neg B}}{\Gamma, y : \neg B, x : A \vdash yt : \bot}}{\Gamma, y : \neg B \vdash \lambda x : A.yt : \neg A}$$

- Labeled CPOS can therefore be written as

$$\dfrac{\Gamma, x : A \vdash t : B}{\Gamma, y : \neg B \vdash \lambda x : A.yt : \neg A}\ \text{CPOS}$$

- Labeled derivation $\mathcal{D}_1$ is as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{u : A \wedge B \vdash u : A \wedge B}\ \text{axiom}}{u : A \wedge B \vdash \pi_1 u : A}\ \wedge e2}{v : \neg A \vdash \lambda u : A \wedge B.v(\pi_1 u) : \neg(A \wedge B)}\ \text{CPOS}}{z : \neg\neg(A \wedge B) \vdash \lambda v : \neg A.z(\lambda u : A \wedge B.v(\pi_1 u)) : \neg\neg A}\ \text{CPOS}$$

- Similarly we get

  $x : \neg\neg(A \wedge B) \vdash \lambda v : \neg B.x(\lambda u : A \wedge B.v(\pi_2 u)) : \neg\neg B$

- Following is the labeled main derivation, where we let

  $t_1(z) \equiv \lambda v : \neg A.z(\lambda u : A \wedge B.v(\pi_1 u))$ and

  $t_2(x) \equiv \lambda v : \neg B.x(\lambda u : A \wedge B.v(\pi_2 u))$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{z : \neg\neg(A \wedge B \vdash t_1(z) : \neg\neg A} \qquad \dfrac{\mathcal{D}_2}{x : \neg\neg(A \wedge B) \vdash t_2(x) : \neg\neg B}}{z : \neg\neg(A \wedge B), x : \neg\neg(A \wedge B) \vdash \langle t_1(z), t_2(x)\rangle : \neg\neg A \wedge \neg\neg B}}{x : \neg\neg(A \wedge B) \vdash \langle t_1(x/z), t_2(x)\rangle : \neg\neg A \wedge \neg\neg B}}{\vdash \lambda x : \neg\neg(A \wedge B).\langle t_1(x/z), t_2(x)\rangle : \neg\neg(A \wedge B) \to \neg\neg A \wedge \neg\neg B}$$

- Putting for t_1 and t_2 back, we see that the construction term associated with conclusion is

  $\lambda x : \neg\neg(A \wedge B).\langle \lambda v : \neg A.x(\lambda u : A \wedge B.v(\pi_1 u)), \lambda v : \neg B.x(\lambda u : A \wedge B.v(\pi_2 u))\rangle$

Part-3(b) Backward implication,

- Labeled derivation $\mathcal{D}_1$ is as below.

$$\dfrac{\dfrac{\dfrac{}{x:A\vdash x:A}\qquad\dfrac{}{y:B\vdash y:B}}{x:A,y:B\vdash\langle x,y\rangle:A\wedge B}\qquad\dfrac{}{z:\neg(A\wedge B)\vdash z:\neg(A\wedge B)}}{x:A,y:B,z:\neg(A\wedge B)\vdash z\langle x,y\rangle:\bot}$$

- Labeled derivation $\mathcal{D}_2$ is as below.

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{x:A,y:B,z:\neg(A\wedge B)\vdash z\langle x,y\rangle:\bot}}{z:\neg(A\wedge B),y:B\vdash \lambda x:A.z\langle x,y\rangle:\neg A}\qquad \dfrac{\dfrac{}{u:\neg\neg A\wedge\neg\neg B\vdash u:\neg\neg A\wedge\neg\neg B}}{u:\neg\neg A\wedge\neg\neg B\vdash \pi_1 u:\neg\neg A}}{z:\neg(A\wedge B),y:B,u:\neg\neg A\wedge\neg\neg B\vdash(\pi_1u)(\lambda x:A.z\langle x,y\rangle):\bot}}{z:\neg(A\wedge B),u:\neg\neg A\wedge\neg\neg B\vdash\lambda y:B.(\pi_1u)(\lambda x:A.z\langle x,y\rangle):\neg B}$$

- Following is the labeled main derivation, where we let

  $t=\lambda y:B.(\pi_1u)(\lambda x:A.z\langle x,y\rangle)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{v:\neg\neg A\wedge\neg\neg B\vdash v:\neg\neg A\wedge\neg\neg B}{v:\neg\neg A\wedge\neg\neg B\vdash\pi_2v:\neg\neg B}\qquad\dfrac{\mathcal{D}_2}{z:\neg(A\wedge B),u:\neg\neg A\wedge\neg\neg B\vdash t:\neg B}}{u:\neg\neg A\wedge\neg\neg B,v:\neg\neg A\wedge\neg\neg B,z:\neg(A\wedge B)\vdash(\pi_2v)t:\bot}}{u:\neg\neg A\wedge\neg\neg B,z:\neg(A\wedge B)\vdash(\pi_2u)t:\bot}}{u:\neg\neg A\wedge\neg\neg B\vdash\lambda z:\neg(A\wedge B).(\pi_2u)t:\neg\neg(A\wedge B)}}{\vdash\lambda u:\neg\neg A\wedge\neg\neg B.\lambda z:\neg(A\wedge B).(\pi_2u)t:\neg\neg A\wedge\neg\neg B\to\neg\neg(A\wedge B)}$$

- Putting back for t, we see that the construction term associated with the conclusion is $\lambda u:\neg\neg A\wedge\neg\neg B.\lambda z:\neg(A\wedge B).(\pi_2u)(\lambda y:B.(\pi_1u)(\lambda x:A.z\langle x,y\rangle))$.