Lecture-7 Main Points

- Boolean algebra examples continued
  - BA of bit vectors of fixed length n.
    - Isomorphic to $\underbrace{2 \times 2 \times \ldots \times 2}_{n \text{ times}}$.
    - Direct Product of BAs.
  - More generally, 2^X, for any set X.
    - $2^X = \{f : X \to 2\}$ (as sets)
    - In particular, $\underbrace{2 \times 2 \times \ldots \times 2}_{n \text{ times}} = 2^{\{0,1,\ldots,n-1\}}$.
    - 2^X is a BA with operations defined component-wise.
      E.g. for $f, g \in 2^X$
      - $(f \vee g)(y) = f(y) \vee g(y)$, for all $y \in X$
      - $(\neg f)(y) = \neg f(y)$, for all $y \in X$
      - $(1)(y) = 1$, for all $y \in X$
      - etc.
      - In the above equations, symbols $\vee, \neg, 1$ on *lhs* are operations in BA 2^X, the same symbols on *rhs* are operations in BA 2.
  - 2^X is isomorphic to $P(X)$ as BA.
    - $f : X \to 2$ defines a unique subset of X.
    - Conversely, a subset S of X defines a unique $f : X \to 2$
      - the characteristic function of S
    - This identification of elements of the two BAs also preserves BA operations.

- A useful Lemma for BA
  - For all $a, b \in B$, $[(a \wedge b = 0 \text{ and } a \vee b = 1) \Rightarrow (b = \neg a)]$
  - $\neg\neg a = a$
    - Immediate from definition of $\neg$ and part (i).
  - DeMorgan Laws
    - Using part (i)
- Interpreting Propositions in a BA
  - Let B be a BA and $[\![\cdot]\!]$ a valuation into B
    - $[\![\cdot]\!] : \{\text{atomic propositions}\} \to B$
  - The valuation can be extended to *all* propositions as follows.
    - $[\![A \wedge B]\!] = [\![A]\!] \wedge [\![B]\!]$
    - $[\![A \vee B]\!] = [\![A]\!] \vee [\![B]\!]$
    - $[\![A \to B]\!] = \neg[\![A]\!] \vee [\![B]\!]$
    - $[\![\bot]\!] = 0$
    - In the above equations, symbols $\vee, \neg, \to, \bot$ on *lhs* are part of propositions syntax, the same symbols on *rhs* are operations in BA B.
- A proposition A is true in $(B, [\![\cdot]\!])$ iff $[\![A]\!] = 1$.