CS202M Second Exam

Feb 12, 2024
Duration: 70 minutes
Max Marks: 40

Closed Book and Notes

Instructions

1. The exam is closed book and closed notes.
2. You are not allowed to have electronic devices such as mobile phones, laptops or tablet PCs with you during the exam.
3. Answers should be clear, and to the point. Also write your answers neatly and strike out any rough work.
4. In your derivation trees, for each inference rule applied please write its name alongside.
5. The notation $\Gamma \vdash_{\mathbf{Nt}} A$, where $\mathbf{t} \in \{\mathbf{m}, \mathbf{i}, \mathbf{c}\}$, states that the judgment $\Gamma \vdash A$ is derivable in logical system $\mathbf{Nt}$.
6. You may assume the following judgments as derivable in $\mathbf{Ni}$, for any propositions A, B. That is, you may use them in your derivations without any proof.
   (a) $A \vdash \neg\neg A \qquad \neg\neg\neg A \vdash \neg A$
   (b) $\neg\neg(A \wedge B) \vdash \neg\neg A \wedge \neg\neg B \qquad \neg\neg A \wedge \neg\neg B \vdash \neg\neg(A \wedge B)$
   (c) $\neg\neg(A \to B) \vdash \neg\neg A \to \neg\neg B \qquad \neg\neg A \to \neg\neg B \vdash \neg\neg(A \to B)$
   (d) $A \to B \vdash \neg B \to \neg A$
7. Backside of this page is empty. Questions are given on page 3 and 4.

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Q1(marks-4+6+2) Let *Prop* be the set of all propositions. Consider the translation $(\cdot)^\circ : Prop \to Prop$ defined as follows.

$(\bot)^\circ = \bot$

$(P)^\circ = \neg\neg P$, if P is atomic and $P \neq \bot$

$(A \wedge B)^\circ = A^\circ \wedge B^\circ$

$(A \to B)^\circ = A^\circ \to B^\circ$

$(A \vee B)^\circ = \neg(\neg A^\circ \wedge \neg B^\circ)$

It can be proved by induction on A that, for any A, the judgment $\neg\neg A^\circ \vdash A^\circ$ is **Ni** derivable. Towards this answer the following.

(i) Prove the base case.

(ii) Prove induction step, when A is $A_1 \to A_2$.

(iii) Prove the induction step, when A is $A_1 \vee A_2$.

Q2(marks-5+10) The translation $(\cdot)^\circ$ of **Q1** is extended to a sequence $\Gamma = \{A_1, \ldots, A_m\}$ in a straightforward way as $\Gamma^\circ = \{A_1^\circ, \ldots, A_m^\circ\}$.

We can show $\Gamma \vdash_{\mathbf{Nc}} A \;\; \Rightarrow \;\; \Gamma^\circ \vdash_{\mathbf{Ni}} A^\circ$ by induction on the derivation of $\Gamma \vdash_{\mathbf{Nc}} A$. You are asked to prove induction step when the last rule applied in the derivation of $\Gamma \vdash_{\mathbf{Nc}} A$ is :

(i) $\vee$ introduction

(ii) $\vee$ elimination

Note: You are allowed use the result stated in **Q1**, even if you have not answered **Q1**.

Q3(marks-4+7+2) Let $\mathbf{B} = (B, \leq, \vee, \wedge, \neg, 0, 1)$ be a Boolean algebra. For any $a \in B$, $a \neq 0$, define set $B \restriction a = \{x \in B \mid x \leq a\}$.

(i) Show that $B \restriction a$ is closed under $\vee$ and $\wedge$.

(ii) Show that structure $(B \restriction a, \leq, \vee, \wedge)$ can be turned into a Boolean algebra by adding remaining operations and constants to it.

(iii) Is your Boolean algebra of part **(ii)**, a sub-algebra of $\mathbf{B}$? Justify your answer.

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