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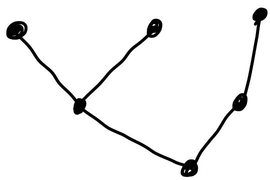
10th Jan'23

PROPOSITIONAL MINIMAL LOGIC

Inference rules to derive judgements of the form

$$\Gamma \vdash A$$

'turnstile'



$$(A \to B) \vdash (\neg B \to \neg A)$$

$\neg B$: $A \to \bot$

proof:

$$\dfrac{\dfrac{\overset{Ax}{A \to B \vdash A \to B}, \quad \overset{Ax}{A \vdash A}}{A, A \to B \vdash B}\,{\to}e \quad , \quad \overset{Ax}{B \to \bot \vdash B \to \bot}}{A, A \to B, B \to \bot \vdash \bot}\,{\to}e$$

$$\dfrac{A, A \to B, B \to \bot \vdash \bot}{A \to B, B \to \bot \vdash A \to \bot}\,{\to}i$$

$$\dfrac{A \to B, B \to \bot \vdash A \to \bot}{A \to B \vdash (B \to \bot) \to (A \to \bot)}\,{\to}i$$

$$\dfrac{A \to B \vdash (B \to \bot) \to (A \to \bot)}{A \to B \vdash (\neg B) \to (\neg A)}$$

$$\dfrac{\dfrac{(A \to B) \vdash (A \to B) \qquad A \vdash A}{A, A \to B \vdash B} \qquad B \to \bot \vdash B \to \bot}{A, A \to B, B \to \bot \vdash \bot}$$

$$\downarrow$$

$$A \to B, (B \to \bot) \vdash (A \to \bot)$$

$$\downarrow$$

$$A \to B \vdash (B \to \bot) \to (A \to \bot)$$

Whole set of inference rule and derivation are syntactic (formal) notations

It is useful to use the inference rule,

$$\dfrac{\Gamma \vdash A, \quad \Gamma', A \to B}{\Gamma, \Gamma' \to B} \quad \text{Cut}$$

$$\dfrac{\Gamma \vdash A, \qquad \dfrac{\Gamma', A \vdash B}{\Gamma' \vdash A \to B}}{\Gamma, \Gamma' \vdash B}$$

even though it is not part of axi= or rules in min^2 logic.

It is known as DERIVED RULES

Derive

$$A \wedge B \vdash \neg(\neg A \vee \neg B)$$

Proof:

$$\dfrac{\dfrac{\neg A \vee \neg B \vdash \neg A \vee \neg B \quad , \quad \dfrac{\neg A \vdash \neg A \quad \dfrac{A \to B \vdash A \to B}{A \wedge B \vdash A}}{\neg A, A \wedge B \vdash \bot} \quad , \quad \dfrac{\neg B \vdash \neg B \quad \dfrac{A \to B \vdash A \to B}{A \wedge B \vdash B}}{\neg B, A \wedge B \vdash \bot}}{\neg A \vee \neg B, A \wedge B \vdash \bot}}{A \wedge B \vdash \neg(\neg A \vee \neg B)}$$

In logic that is used in mathematics if there is a contradiction

(B, $\neg B$, can be derived from B).

then every proposition is derivable

$$\bot_i \left(\dfrac{\Gamma \vdash \bot}{\Gamma \vdash A} \quad \text{for any proposition } A. \right)$$ Principle of Explosion

Intuitionistic Logic $\equiv$ Minimal Logic $+$ $\bot_i$

$$\dfrac{\dfrac{\Gamma \vdash B, \quad \Gamma' \vdash B \to \bot}{\Gamma, \Gamma' \vdash \bot}}{\Gamma, \Gamma' \vdash A}$$

$\bot$ in I.L is a propositional constant

$$\left.\begin{array}{l} A \vdash_{N_m} \neg\neg A \\ A \vdash_{N_i} \neg\neg A \end{array}\right\} \rightarrow \text{is derivable in IL.}$$

But we can not derive, $\neg\neg A \vdash A$ in IL.

Also,

LEM

$\vdash A \vee \neg A$ is not derivable in IL.

proof:

$A \perp A \quad , \neg A \vdash (A \rightarrow \perp)$

$A, \neg A \vdash \perp.$

$A \vdash \neg A \rightarrow \perp$

$A \vdash (\neg\neg A)$.

Reasoning by contradiction

$$\frac{\Gamma, \neg A \vdash \bot}{\Gamma \vdash A}\ , \ \bot_c$$

Intutionistic logic $+ \ \bot_c \ \equiv$ Classical logic

derive

$$\neg\neg A \vdash_{N_c} A$$

proof:

$$\cfrac{\cfrac{\overset{Ax}{\neg\neg A \vdash \neg\neg A} \qquad \overset{Ax}{\neg A \vdash \neg A}}{\neg\neg A, \neg A \vdash \bot}\ \to e}{\neg\neg A \vdash A}\ \bot_e$$

derive

1. Show that in classical logic,

$$\vdash_{N_c} A \vee (\neg A)$$

Sol:

$$\cfrac{\cfrac{\cfrac{A \vdash A \ , \ \neg A \vdash \neg A}{A, \neg A \vdash \bot}}{A \wedge \neg A \vdash \bot}}{\vdash \neg (A \wedge \neg A)}$$