12. Integration by Substitution

Lec 11 mini review.

FTC Suppose f is continuous on $[a, b]$.

FTC1 If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.

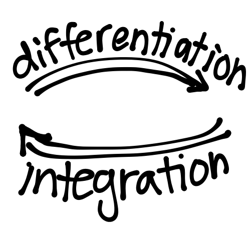
FTC2 $\int_a^b f(x)dx = F(b) - F(a)$ where F is any antiderivative of f (that is, $F' = f$).

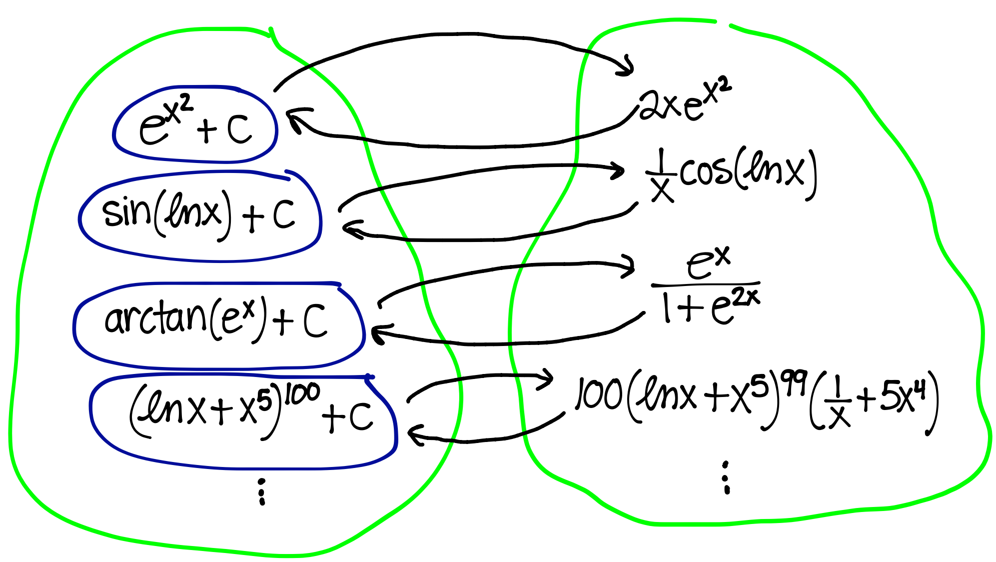
◇ **indefinite integral** vs. **definite integral**

◇ **the Net Change Theorem for the integral of a rate of change**: $\displaystyle\int_a^b F'(x)dx = F(b) - F(a)$

"UNDOING" THE CHAIN RULE

- If we think of differentiation as an operation on the set of differentiable functions, then integration (informally, "anti-differentiation") does the opposite.
- What does "undoing" the Chain Rule look like?
- There are many different instances of the Chain Rule...





Guidelines for Substitution

0. Inspect the integrand. Is it familiar because it's a function's derivative (give or take a constant multiple) ? If so, you should know what to do.

1. If the integrand is not an "obvious" familiar function's derivative, then consider the possibility that the integrand may be the aftermath of a Chain Rule:

Does integrand look roughly like the aftermath...

...of a power chain rule ? eg. $\int k(g(x))^n g'(x)\,dx$

...of an exp. chain rule? eg. $\int k g'(x) e^{g(x)}\,dx$

...of a ln chain rule ? eg. $\int k \frac{g'(x)}{g(x)}\,dx$

...of a trig chain rule ? eg. $\int k\cos(g(x))g'(x)\,dx$ etc...

2. If you see a potential "inner" function $g(x)$, then call it u (you might be wrong, but it doesn't hurt to try).

Now, compute your substitution ingredients:

Try $u=g(x)$ Then $\frac{du}{dx}=g'(x)$

$$\Longrightarrow dx = \frac{du}{g'(x)}$$

3. Rewrite your integral in terms of u and du. $\int f(g(x))g'(x)\,dx = \int f(u)\,du$

Note. The "old" variable (let's say it was x) should completely cancel. We should get a new (and hopefully easier) integral with respect to the "new" variable u.

4. Evaluate the new integral. Then don't forget to rewrite your answer in terms of the "old" variable.

$\int f(u)\,du = F(u)+C$ where $F'=f$

$= F(g(x))+C$ since $u=g(x)$

Example 12.1. $\int 2xe^{x^2}\,dx.$

u-substitution
$u=x^2$
$\frac{du}{dx}=2x$
$dx=\frac{du}{2x}$

$$=\int \cancel{2x}e^u\cdot\frac{du}{\cancel{2x}}$$
$$=\int e^u\,du$$
$$=e^u+C$$
$$=e^{x^2}+C$$

Check
$\frac{d}{dx}[e^{x^2}+C]=2xe^{x^2}$ ✓

Example 12.2. $\int x^2e^{x^3-9}\,dx.$

x^2 is "roughly" the derivative of x^3-9

Try $u=x^3-9$
$\Rightarrow \frac{du}{dx}=3x^2$
$\Rightarrow dx=\frac{du}{3x^2}$

$$=\int x^2e^u\left(\frac{du}{3x^2}\right)$$
$$=\int \frac{1}{3}e^u\,du$$
$$=\frac{1}{3}e^u+C$$
$$=\frac{1}{3}e^{x^3-9}+C$$

Check:
$\frac{d}{dx}\left[\frac{1}{3}e^{x^3-9}+C\right]=\left(\frac{1}{3}e^{x^3-9}\right)(3x^2-0)$
$=x^2e^{x^3-9}$ ✓

Example 12.3. $\int \sin(2t)\,dt.$

We see an "inner" function $2t$ in sin( )

Try $u=2t$
$\Rightarrow \frac{du}{dt}=2$
$\Rightarrow dt=\frac{1}{2}du$

$$=\int \sin(u)\left(\tfrac{1}{2}du\right)$$
$$=\frac{1}{2}\int \sin(u)\,du$$
$$=\frac{1}{2}(-\cos(u))+C$$
$$=-\frac{1}{2}\cos(2t)+C$$

Check!

Example 12.4. $\int \tan(\theta)\,d\theta.$

$$=\int \frac{\sin\theta}{\cos\theta}\,d\theta$$

← numerator is "roughly" the derivative of the denominator... this could be the aftermath of a ln( ) chain rule...

try $u=\cos\theta$
$\Rightarrow \frac{du}{d\theta}=-\sin\theta$
$\Rightarrow d\theta=\frac{du}{-\sin\theta}$

$$=\int \frac{\cancel{\sin\theta}}{(u)}\cdot\frac{du}{(-\cancel{\sin\theta})}$$
$$=\int -\frac{1}{u}\,du$$
$$=-\ln|u|+C$$

Note
$-\ln|\cos\theta|=\ln|(\cos\theta)^{-1}|=\ln|\sec\theta|$

$\therefore \int \tan\theta\,d\theta=\ln|\sec\theta|+C$

Check!

Example 12.5. $\int \overset{?}{\sec(x)}dx.$ $= \int \frac{1}{\overset{?}{\cos x}}dx$ (seemingly unhelpful...)

a clever trick: $\int \sec x\,dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right)dx$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}dx$$

$\boxed{\begin{array}{l}\text{Now try u-sub:}\\ u = \sec x + \tan x\\ \frac{du}{dx} = \sec x \tan x + \sec^2 x\end{array}}$

$$= \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$= \ln|\sec x + \tan x| + C$$

Example 12.6. $\int_{-\pi/2}^{2\pi} \cos(x)\sin^3(x)dx.$

$\int \cos x (\sin x)^3 dx$ (indefinite for now...)

$\boxed{\begin{array}{l}u = \sin x\\ \frac{du}{dx} = \cos x\\ \Rightarrow dx = \frac{du}{\cos x}\end{array}}$

$$= \int \cancel{\cos x}\cdot u^3 \cdot \frac{du}{\cancel{\cos x}}$$

$$= \int u^3 du$$

$$= \frac{u^4}{4}$$

$$= \frac{\sin^4(x)}{4}$$ ← now we know antiderivative

go back to definite integral

$$\therefore \int_{-\frac{\pi}{2}}^{2\pi} \cos x \sin^3 x\,dx = \left[\frac{\sin^4 x}{4}\right]_{-\pi/2}^{2\pi}$$

$$= \frac{1}{4}(\sin 2\pi)^4 - \frac{1}{4}\left(\sin\left(-\frac{\pi}{2}\right)\right)^4$$

$$= \frac{1}{4}(0^2) - \frac{1}{4}(-1)^4$$

$$= -\frac{1}{4}$$

Substitution Methods for Definite Integrals.

1. Solve the indefinite integral first, and completely (in terms of "old" variable), then subtract at the limits of integration.

$$\int_a^b F'(g(x))g'(x)dx = \left[\int F'(u)du\right]_{x=a}^{x=b} = \left[F(g(x))\right]_{x=a}^{x=b} = F(g(b)) - F(g(a))$$

2. Substitute the limits of integration at the same time as you perform the substitution.

$$\int_a^b F'(g(x))g'(x)dx = \int_{u=g(a)}^{u=g(b)} F'(u)du = \left[F(u)\right]_{u=g(a)}^{u=g(b)} = F(g(b)) - F(g(a))$$

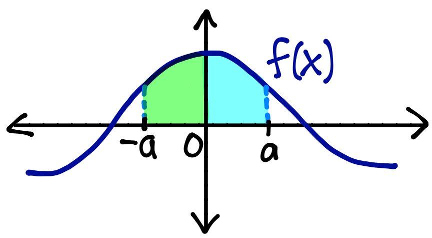
Example 12.7. $\displaystyle\int_1^e \frac{\ln x}{x}\,dx. = \int_{u=0}^{u=1} \frac{u}{x}(x\,du) = \int_{u=0}^{u=1} u\,du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2}$

$$\boxed{\begin{array}{ll} u = \ln x & \\ \frac{du}{dx} = \frac{1}{x} & x = e \Rightarrow u = \ln e = 1 \\ dx = x\,du & x = 1 \Rightarrow u = \ln 1 = 0 \end{array}}$$

Integrals of Functions with Even/Odd Symmetry

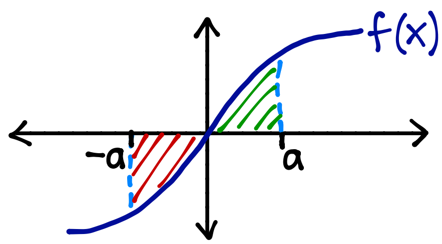
Let $a \in \mathbb{R}$. Suppose f is continuous on $[-a, a]$.

◊ If f has even symmetry, that is $f(-x) = f(x)$ for all $x \in [-a, a]$, then



$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

♦ If f has odd symmetry, that is $f(-x) = -f(x)$ for all $x \in [-a, a]$, then



$$\int_{-a}^{a} f(x)\,dx = 0$$

Example 12.8. $\displaystyle\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^6}\,dx = 0$

* limits of integration are symmetric about the y-axis

* the integrand has odd symmetry since $\dfrac{\tan(-x)}{1+(-x)^2+(-x)^6} = -\dfrac{\tan x}{1+x^2+x^6}$

- $\tan(-x) = \dfrac{\sin(-x)}{\cos(-x)} = \dfrac{-\sin(x)}{\cos(x)} = -\tan(x)$
- $1+(-x)^2+(-x)^6 = 1+x^2+x^6$

Study Guide

- ◇ **strategy for integration by substitution**
- ◇ **two ways to evaluate definite integrals via substitution**
- ◇ **using even/odd symmetry to evaluate certain integrals**