DGD 1

Q1. Let $\mathbf{u} \in \mathbb{R}^n$ and let $c, d \in \mathbb{R}$. Prove that $(c+d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.

Let $c, d \in \mathbb{R}$ and $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^n$

Then

$$(c+d)\vec{u} = (c+d)\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

$$= \begin{bmatrix} (c+d)u_1 \\ \vdots \\ (c+d)u_n \end{bmatrix} \quad \text{def. of scalar mult.}$$

$$= \begin{bmatrix} cu_1 + du_1 \\ \vdots \\ cu_n + du_n \end{bmatrix} \quad \text{distrib. of real numbers}$$

$$= \begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix} + \begin{bmatrix} du_1 \\ \vdots \\ du_n \end{bmatrix} \quad \text{def. of vector addition}$$

$$= c\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + d\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \quad \text{def. of scalar mult.}$$

$$= c \cdot \vec{u} + d\vec{u}$$

$\square$

Q2. Show that every vector in $\mathbb{R}^3$ can be expressed as a linear combination of the vectors $\hat{\mathbf{i}} = (1,0,0)$, $\hat{\mathbf{j}} = (0,1,0)$, $\hat{\mathbf{k}} = (0,0,1)$.

Let $\vec{u} = (a,b,c)$ be an arbitrary vector in $\mathbb{R}^3$.

Then $\vec{u} = a\hat{i} + b\hat{j} + c\hat{k}$ $\therefore \vec{u}$ is a linear combination of $\hat{i}, \hat{j}, \hat{k}$

Q3. Let $\mathbf{u} = (1, -1, 0)$ and $\mathbf{v} = (2, -1, -2)$ be two vectors in $\mathbb{R}^3$.

a) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} = \frac{(1,-1,0)\cdot(2,-1,-2)}{\sqrt{1^2+(-1)^2+0^2}\,\sqrt{2^2+(-1)^2+(-2)^2}}$$

$$= \frac{2+1+0}{\sqrt{2}\,\sqrt{9}} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

b) Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \frac{(1,-1,0)\cdot(2,-1,-2)}{2^2+(-1)^2+(-2)^2}\cdot(2,-1,-2)$$

$$= \frac{2+1+0}{9}(2,-1,-2)$$

$$= \frac{1}{3}(2,-1,-2)$$

$$= \left(\tfrac{2}{3}, -\tfrac{1}{3}, -\tfrac{2}{3}\right)$$

c) Find a vector in $\mathbb{R}^3$ that is orthogonal to $\mathbf{v}$.

There are many possible answers. We need a vector (x,y,z) such that

$(x,y,z)\cdot\vec{v} = 0$

$\Rightarrow (x,y,z)\cdot(2,-1,-2) = 0$

$\Rightarrow 2x - y - 2z = 0$

$\Rightarrow x = \frac{1}{2}y + z$ (and y,z are free)

$\Rightarrow (x,y,z)$ is any vector of the form $\left(\frac{1}{2}y+z, y, z\right)$

e.g. $\left(\frac{1}{2}(0)+0, 0, 0\right) = (0,0,0)$ Check: $(0,0,0)\cdot\vec{v} = (0,0,0)\cdot(2,-1,-2) = 0$ ✓

$\left(\frac{1}{2}(1)+1, 1, 1\right) = \left(\frac{3}{2}, 1, 1\right)$ $\left(\frac{3}{2},1,1\right)\cdot(2,-1,-2) = 3-1-2 = 0$ ✓

$\left(\frac{1}{2}(5)+3, 5, 3\right) = \left(\frac{11}{2}, 5, 3\right)$ etc. $\left(\frac{11}{2},5,3\right)\cdot(2,-1,-2) = 11-5-6 = 0$ ✓ etc...

Q4. Let $\mathbf{u}, \mathbf{v}$ be two orthogonal vectors in $\mathbb{R}^n$. Show that $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ (i.e. show that the Theorem of Pythagoras holds in $\mathbb{R}^n$).

Assume $\vec{u}, \vec{v} \in \mathbb{R}^n$ are orthogonal. Thus $\vec{u}\cdot\vec{v} = 0$.

Then

$$\begin{aligned}
LS = \|\vec{u}+\vec{v}\|^2 &= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) && \text{def of norm } \|\vec{x}\| = \sqrt{\vec{x}\cdot\vec{x}} \\
&= (\vec{u}+\vec{v})\cdot\vec{u} + (\vec{u}+\vec{v})\cdot\vec{v} && \left.\right\} \text{properties of dot product} \\
&= \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v} && \\
&= \vec{u}\cdot\vec{u} + 0 + 0 + \vec{v}\cdot\vec{v} && \text{since } \vec{u}\perp\vec{v} \text{ so } \vec{u}\cdot\vec{v} = 0 = \vec{v}\cdot\vec{u} \\
&= \|\vec{u}\|^2 + \|\vec{v}\|^2 && \text{def. of norm} \\
&= RS
\end{aligned}$$

$\therefore \|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$

$\square$

Q5. Find the intersection of the lines $L_1 = \{(4, 13, -7) + t(-1, -6, 4) : t \in \mathbb{R}\}$ and $L_2 = \{(2, 0, 2) + t(0, 1, -1) : t \in \mathbb{R}\}$.

$$\begin{bmatrix}4\\13\\-7\end{bmatrix} + t\begin{bmatrix}-1\\-6\\4\end{bmatrix} = \begin{bmatrix}2\\0\\2\end{bmatrix} + s\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

direction vectors of L_1 and L_2 are not parallel

$$\begin{bmatrix}4-t\\13-6t\\-7+4t\end{bmatrix} = \begin{bmatrix}2\\s\\2-s\end{bmatrix} \implies \begin{array}{l} t=2 \\ s = 13-6t = 13-6(2) = 1 \\ -7+4t = -7+4(2) = 1 \\ \text{and } 2-s = 2-1 = 1 \end{array}$$

P.O.I: when $t=2$ (using L_1) $\begin{bmatrix}4\\13\\-7\end{bmatrix} + 2\begin{bmatrix}-1\\-6\\4\end{bmatrix} = \begin{bmatrix}2\\1\\1\end{bmatrix} = (2,1,1)$ ✓

(or when $s=1$ (using L_2) $\begin{bmatrix}2\\0\\2\end{bmatrix} + (1)\begin{bmatrix}0\\1\\-1\end{bmatrix} = \begin{bmatrix}2\\1\\1\end{bmatrix}$)

Q6. Find a vector parametric equation for the line $2x + 3y - 13 = 0$ of $\mathbb{R}^2$.

$2x+3y-13=0 \Rightarrow x = \frac{13-3y}{2} = \frac{13}{2} - \frac{3}{2}y$

$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13/2 - 3/2\, y \\ y \end{bmatrix} = \begin{bmatrix} 13/2 \\ 0 \end{bmatrix} + y \begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$

Alternative: $y = \frac{13-2x}{3} = \frac{13}{3} - \frac{2}{3}x$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 13/3 - \frac{2}{3}x \end{bmatrix} = \begin{bmatrix} 0 \\ 13/3 \end{bmatrix} + x \begin{bmatrix} 1 \\ -2/3 \end{bmatrix}$

direction vectors are parallel

They can also be scaled if we prefer to avoid fractions

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 13/3 \end{bmatrix} + s \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ e.g.

Q7. Find a vector parametric equation and an Cartesian equation for the line passing through $(1, 3)$ and $(2, 7)$.

Vector parametric: $\vec{d} = (2-1, 7-3) = (1,4)$

$L = \{(1,3) + t(1,4) : t \in \mathbb{R}\}$

Cartesian: $\frac{\Delta y}{\Delta x} = \frac{7-3}{2-1} = 4$

$y - 3 = 4(x-1)$

$y = 4x - 4 + 3 = 4x - 1$

$y - 4x = -1$

Q8. Find all values of k for which $(k, k, 1)$ and $(k, -2, -3)$ are orthogonal.

orthogonal $\Leftrightarrow (k,k,1)\cdot(k,-2,-3) = 0$

$\Leftrightarrow k^2 - 2k - 3 = 0$

$\Leftrightarrow (k-3)(k+1) = 0$

$\Leftrightarrow k = 3$ or $k = -1$

Q9. Find the point of intersection of the plane $P = \{(x, y, z) : 2x + 2y - z = 5\}$ and the line with parametric equations $x = 4 - t$, $y = 13 - 6t$, $z = -7 + 4t$.

$2x + 2y - z = 5$

plug in line parametrics: $2(4-t) + 2(13-6t) - (-7+4t) = 5$

solve for t: $8 - 2t + 26 - 12t + 7 - 4t = 5$

$\Rightarrow \quad 36 = 18t$

$\Rightarrow \quad t = 2$

use t-value to find point: $x = 4 - 2 = 2$

$y = 13 - 6(2) = 1$

$z = -7 + 4(2) = 1$

$\therefore$ POI $(2, 1, 1)$

Q10. Find an equation for the plane that passes through the point $(1, 1, 1)$ and which is perpendicular to the line with parametric equations

$$x = -6 + 2t, y = 1 - 4t, z = -3 + 3t \quad t \in \mathbb{R}$$

plane perpendicular to line $\Rightarrow$ normal of plane is $\parallel$ to direction of line

$L = (-6, 1, -3) + t(2, -4, 3)$

$\vec{d} = (2, -4, 3)$

$\Rightarrow \vec{n} = (2, -4, 3)$

For Cartesian eq: $\vec{n} \cdot \vec{v} = \vec{n} \cdot \vec{v}_0$

$\vec{v}_0 = (1, 1, 1)$ ← corresponds to point on plane

$\vec{n} \cdot (x, y, z) = \vec{n} \cdot (1, 1, 1)$

$\vec{v} = (x, y, z)$ is an arbitrary point on plane

$(2, -4, 3) \cdot (x, y, z) = (2, -4, 3) \cdot (1, 1, 1)$

$2x - 4y + 3z = 2 - 4 + 3$

$\boxed{2x - 4y + 3z = 1}$

Q11. Find an equation for the line containing $(-5, 0, 1)$ and which is parallel to the two planes with Cartesian equations $2x - 4y + z = 0$ and $x - 3y - 2z = 1$.

Line that is parallel to plane has direction that is orthogonal to plane's normal vector.

$\Rightarrow$ direction of line must be orthogonal to both plane's normals

$$\Rightarrow \vec{d} = \vec{n}_1 \times \vec{n}_2 = (2,-4,1) \times (1,-3,-2)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 1 \\ 1 & -3 & -2 \end{vmatrix}$$

$$= (8+3, -(-4-1), -6-(-4))$$

$$= (11, 5, -2)$$

$$L = \{(-5,0,1) + t(11,5,-2) : t \in \mathbb{R}\}$$

Q12. Find a Cartesian equation for the plane $W = \{(0, 2, -2) + s(1, -1, 2) + t(2, -4, -1) : s, t \in \mathbb{R}\}$

normal is $\perp$ to each direction vector of plane

$$\vec{n} = \vec{d}_1 \times \vec{d}_2 = (1,-1,2) \times (2,-4,-1)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ 2 & -4 & -1 \end{vmatrix}$$

$$= (1-(-8), -(-1-4), -4-(-2))$$

$$= (9, 5, -2)$$

pt on plane

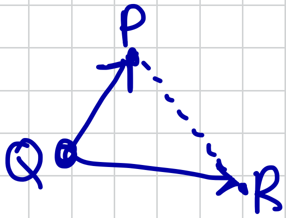
$$\vec{n}\vec{v} = \vec{n}\vec{v}_0 \Rightarrow (9,5,-2)(x,y,z) = (9,5,-2)\cdot(0,2,-2)$$

$$\Rightarrow 9x + 5y - 2z = 0 + 10 + 4 = 14$$

$$\Rightarrow 9x + 5y - 2z = 14$$

Q13. Find the area of the triangle whose vertices are the points $P(3,0,-2)$, $Q(5,2,-1)$ and $R(5,9,0)$.

$A=(3,0,-2) \qquad B(5,2,-1) \qquad C=(5,9,0)$



$\vec{QP} = (-2,-2,-1)$

$\vec{QR} = (0,7,1)$

$$\vec{QP} \times \vec{QR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -2 & -1 \\ 0 & 7 & 1 \end{vmatrix}$$

$$= (5, -(-2), -14)$$

$$= (5, 2, -14)$$

$$\text{Area } \Delta ABC = \frac{1}{2}\|\vec{QP} \times \vec{QR}\| = \frac{1}{2}\sqrt{25+4+196}$$

$$= \frac{1}{2}\sqrt{225}$$

$$= 15/2$$

Q14. Find the volume of the parallelepiped determined by $\mathbf{u} = (1,-2,3)$, $\mathbf{v} = (1,3,1)$ and $\mathrm{w} = (2,1,2)$.

$\vec{u} = (1,-2,3) \quad \vec{v} = (1,3,1) \quad \vec{w} = (2,1,2)$

$$\text{vol. parallelepiped} = |(\vec{u} \times \vec{v}) \cdot \vec{w}|$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 1 & 3 & 1 \end{vmatrix} = (-11, -(-2), 5) = (-11, 2, 5)$$

$$\text{Vol} = |(\vec{u} \times \vec{v}) \cdot \vec{w}| = |(-11,2,5) \cdot (2,1,2)| = |-22+2+10| = |-10| = 10$$