1. Vectors in $\mathbb{R}^n$

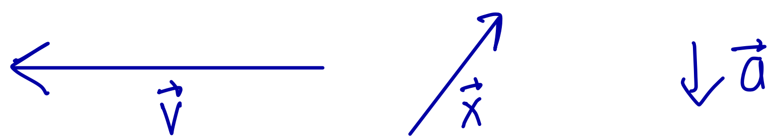
We begin MAT1341 with a friendly review of vectors, starting with concrete geometric examples in 2- and 3-space. Once comfortable in those spaces, we will be ready to explore the abstract realm of “vector spaces”. By reframing vectors in terms of their abstract definitions and properties, we will be able to capture the *true essence* of what vectors really are, and we will discover that there are many surprising and interesting examples, beyond $\mathbb{R}^n$.

VECTORS (AS PRESENTED IN HIGH SCHOOL)

We start with the high school definition:

Definition 1.1. A **VECTOR** is an object that has: ① a size ("magnitude") ② a direction

We picture vectors as arrows, where the length of the arrow corresponds to its magnitude and the direction the arrow points is the direction of the vector.

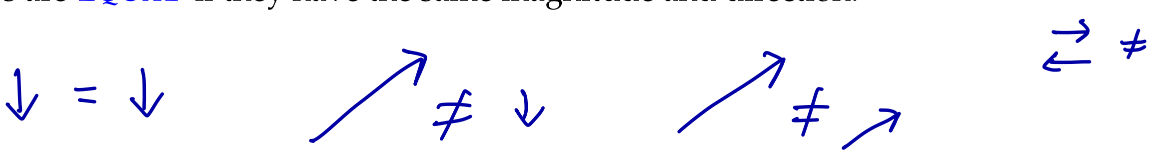


We can represent a vector by its picture, or give it a name, such as **v**.

Note: Usually, the name of a vector is typed using **boldface**: ex: **v** **x** **a**

But, if we are writing a vector’s name by hand, we tend to add an arrow on top (because it’s difficult to convey boldface by hand).

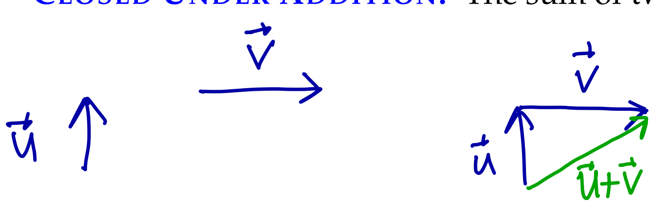
Two vectors are **EQUAL** if they have the same magnitude and direction.



We have two operations: **VECTOR ADDITION** and **SCALAR MULTIPLICATION** .

- We add vectors tail-to-tip.

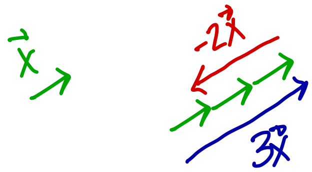
  CLOSED UNDER ADDITION: The sum of two vectors is again a vector.



[†]These notes are intended for students registered in MAT1341.

- We multiply a vector $\mathbf{v}$ by a **Scalar** k (which is a real number) by creating a parallel vector with the correspondingly scaled length. If the scalar is positive, the new vector has the same direction as $\mathbf{v}$. If the scalar is negative, the new vector has the opposite direction as $\mathbf{v}$.

  Closed Under Scalar Multiplication:
  A vector multiplied by a scalar is again a vector.

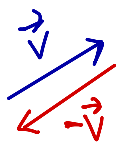


Vectors have a "zero concept" called **the Zero Vector**, denoted $\mathbf{0}$. The **Zero Vector** has no magnitude (i.e. zero magnitude) and it does not have a well-defined direction.

What if we add a vector $\mathbf{v}$ to the zero vector $\mathbf{0}$?

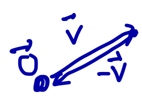
$$\vec{0}+\vec{v}=\vec{v}$$

Each vector $\mathbf{v}$ has a **Negative**, denoted $-\mathbf{v}$. The **Negative** of $\mathbf{v}$ has the same magnitude as $\mathbf{v}$ but the opposite direction.



What if we add a vector $\mathbf{v}$ to its negative $-\mathbf{v}$?

$$\vec{v}+(-\vec{v})=\vec{0}$$

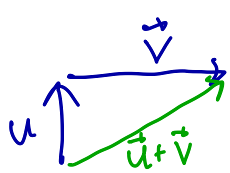


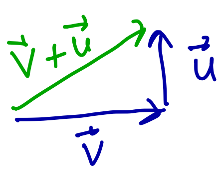
Recall: Vector addition and scalar multiplication also satisfy the following properties.

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors and let $c, d \in \mathbb{R}$ be scalars.

Vector Addition is Commutative

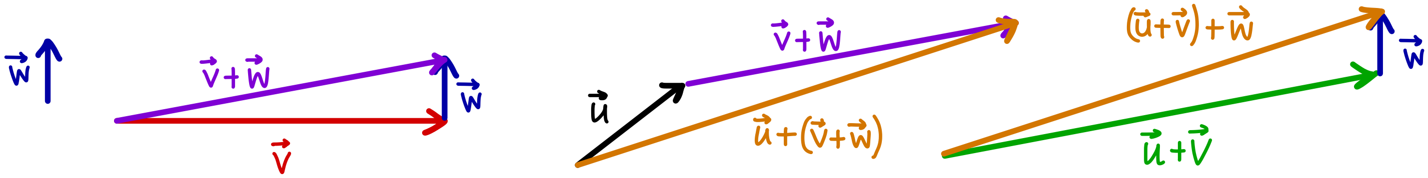
$$\vec{u}+\vec{v}=\vec{v}+\vec{u}$$





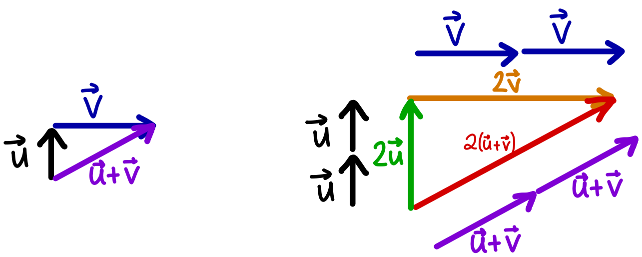
Vector Addition is Associative

$$\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$$



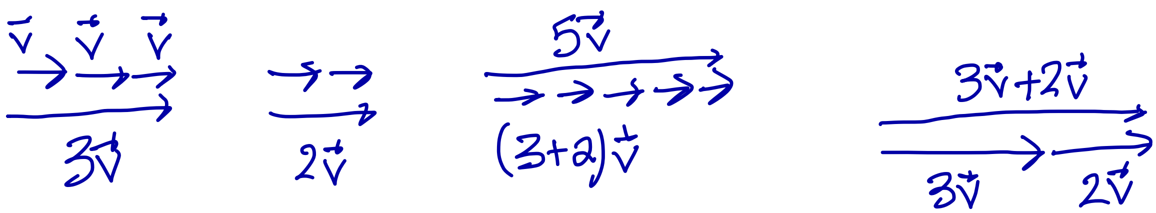
Scalar Multiplication Distributes over Vector Addition

$$c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$$



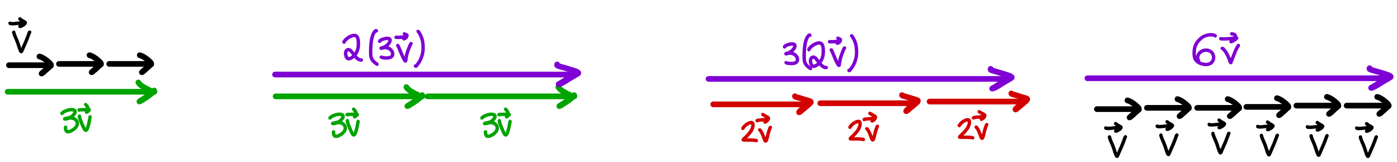
Addition of Scalars Distributes over Scalar Multiplication

$$(c + d)\vec{v} = c\vec{v} + d\vec{v}$$



Multiplication of Scalars is Compatible with Scalar Multiplication

$$c(d\vec{v}) = (cd)\vec{v}$$



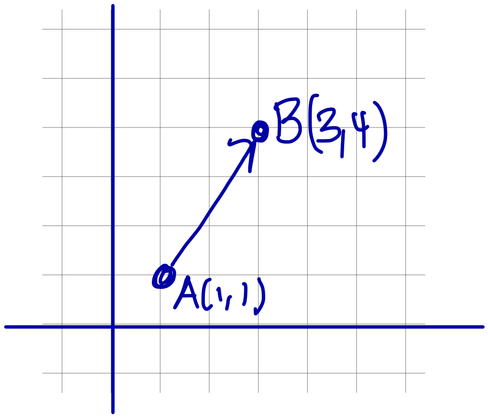
Unity Law

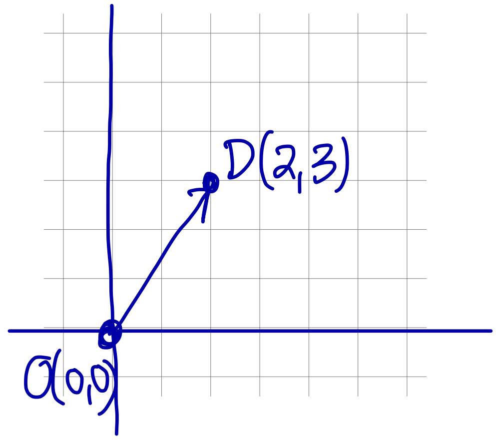
$$1 \cdot \vec{v} = \vec{v}$$

Note: What we're really saying is that the arrows we draw, equipped with our geometric rules for vector addition and scalar multiplication, satisfy the **Axioms of a Vector Space (over $\mathbb{R}$)**!

For now, we will focus on how these properties work in $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$,…, $\mathbb{R}^n$, …, both algebraically and geometrically. Later, we will reconsider these properties in terms of abstract vector spaces.

We also use coordinate systems to help us represent vectors.





- We can refer to a vector using its **INITIAL POINT** (the **TAIL** of the arrow) and its **TERMINAL POINT** (the **TIP** or **HEAD** of the arrow). The coordinates of these two points indicate the vector's position relative to the axes.

  Ex. Points: $A(1,1)$ and $B(3,4)$ Vector $\overrightarrow{AB}$

  Ex. Points: $O(0,0)$ and $D(2,3)$ Vector $\overrightarrow{OD}$

  Note: $\overrightarrow{AB} = \overrightarrow{OD}$ since they have the same magnitude and direction.

- We can refer to a vector using its **COORDINATES**, which indicate the position of its tip relative to its tail.

  Ex. $\mathbf{v} = (2,3)$

- We often write vectors as a **COLUMN VECTOR** which is a matrix with one column:

  Ex. $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

  Using a column vector is nicer and this notation is compatible with matrix multiplication (we'll see more about this later in the course).

- But, it saves space on paper to write vectors as comma-separated rows, or by using the matrix-transpose notation (the T stands for **TRANSPOSE**, which changes the rows of a matrix into columns and vice versa):

$$\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = (2,3) = \begin{bmatrix} 2 & 3 \end{bmatrix}^T$$

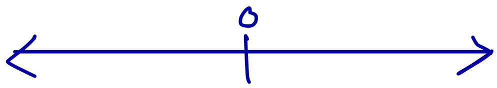
  We can see by the Pythagorean theorem that the **LENGTH** of the vector $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is

$$\|\vec{v}\| = \sqrt{2^2+3^2}$$
$$= \sqrt{13}$$

  For $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, we have: $\|\vec{x}\| = \sqrt{(x_1)^2 + (x_2)^2}$

While the arrows we draw work best on **THE PLANE** (i.e. $\mathbb{R}^2$), or in **3-SPACE** (i.e. $\mathbb{R}^3$), the coordinate system representation of vectors generalizes quite naturally to higher (or lower) dimensions!

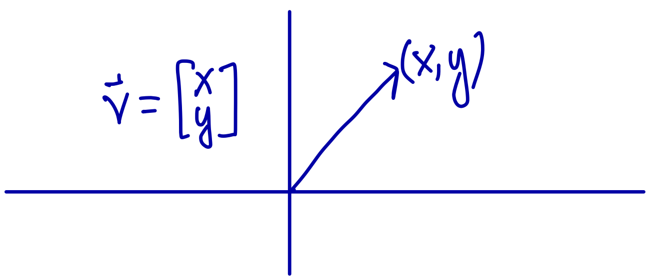
$n = 1$ $\quad$ $\mathbb{R}$ $\quad$ **THE REAL LINE**



$\mathbb{R} = \{x : x \in \mathbb{R}\}$ scalars

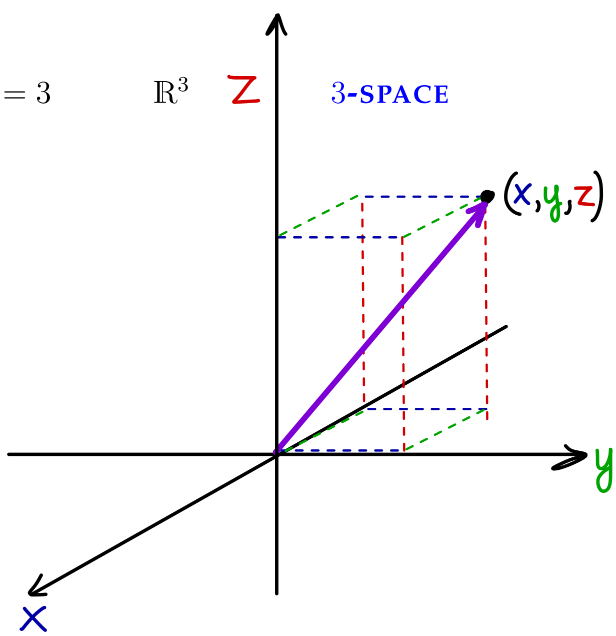
$n = 2$ $\quad$ $\mathbb{R}^2$ $\quad$ **THE PLANE**

$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$



$\mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x, y \in \mathbb{R} \right\}$

$n = 3$ $\quad$ $\mathbb{R}^3$ $\quad$ **3-SPACE**



$\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\qquad$ $\mathbb{R}^3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$

$n = 4$ $\quad$ $\mathbb{R}^4$ $\quad$ **4-SPACE**

$\mathbb{R}^4 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : x_1, x_2, x_3, x_4 \in \mathbb{R} \right\}$

$x_i \in \mathbb{R}$ for $1 \le i \le 4$

$\vdots$

$n \in \mathbb{Z}$, $n > 0$ $\quad$ $\mathbb{R}^n$ $\quad$ **n-SPACE**

$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_i \in \mathbb{R} \text{ for } 1 \le i \le n \right\}$

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ be vectors in $\mathbb{R}^n$. Let $k \in \mathbb{R}$ be a scalar.

Note: The **ENTRIES** or **COMPONENTS** of these vectors are scalars, i.e. $u_1, \ldots, u_n, v_1, \ldots, v_n \in \mathbb{R}$.
Again, we have two operations: **VECTOR ADDITION** and **SCALAR MULTIPLICATION** .

- We add vectors in $\mathbb{R}^n$ **COMPONENT-WISE** .

$$\underbrace{\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}}_{\vec{u}} + \underbrace{\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}}_{\vec{v}} = \underbrace{\begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix}}_{\vec{u}+\vec{v}}$$

CLOSED UNDER ADDITION: The sum of two vectors in $\mathbb{R}^n$ is again a vector in $\mathbb{R}^n$.

- We scalar-multiply a vector in $\mathbb{R}^n$ **COMPONENT-WISE** .

$$k\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \underbrace{\begin{bmatrix} ku_1 \\ \vdots \\ ku_n \end{bmatrix}}_{k\vec{u}}$$

CLOSED UNDER SCALAR MULTIPLICATION:
A vector in $\mathbb{R}^n$ multiplied by a scalar in $\mathbb{R}$ is again a vector in $\mathbb{R}^n$.

What is the **ZERO VECTOR** in $\mathbb{R}^n$? It should satisfy $\mathbf{0} + \mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$.

$$\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

For a vector $\mathbf{v} \in \mathbb{R}^n$, what is its **NEGATIVE** $-\mathbf{v}$? The negative of $\mathbf{v}$ should satisfy $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} -u_1 \\ \vdots \\ -u_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\vec{u} + (-\vec{u}) = \vec{0}$$

Proposition 1.2. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and let $c, d \in \mathbb{R}$. Then the following properties hold:

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
- $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
- $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
- $c(d\mathbf{u}) = (cd)\mathbf{u}$
- $1\mathbf{u} = \mathbf{u}$

Proof that scalar multiplication distributes over vector addition:

Let $c \in \mathbb{R}$ and let $\vec{u} = (u_1, \dots, u_n)$ and $\vec{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$

Then

$$c(\vec{u} + \vec{v}) = c \cdot \left(\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \right)$$

$$= c \cdot \left(\begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix} \right) \quad \text{def of vector + in } \mathbb{R}^n$$

$$= \begin{bmatrix} c(u_1 + v_1) \\ \vdots \\ c(u_n + v_n) \end{bmatrix} \quad \text{def. of scal. mult. by } c$$

$$= \begin{bmatrix} cu_1 + cv_1 \\ \vdots \\ cu_n + cv_n \end{bmatrix} \quad \text{mult. of real \#s distributes over +}$$

$$= \begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix} + \begin{bmatrix} cv_1 \\ \vdots \\ cv_n \end{bmatrix} \quad \text{def of vector +}$$

EXERCISE: Prove that the other properties hold.

$$= c\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + c\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$ def of scal. mult.

$$= c \cdot \vec{u} + c\vec{v}$$

$\square$

With vector addition and scalar multiplication, what can we accomplish?

More precisely, if we are given m vectors in $\mathbb{R}^n$, say $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m \in \mathbb{R}^n$, what other vectors can we create by scaling and/or adding these vectors together? That is, what vectors $\mathbf{v}$ can be expressed as a **LINEAR COMBINATION** of the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_m$?

Definition 1.3. Let $\mathbf{u}_1, \ldots, \mathbf{u}_m$ be vectors in $\mathbb{R}^n$ and let $k_1, \ldots, k_m$ be scalars in $\mathbb{R}$.

A **LINEAR COMBINATION** of $\mathbf{u}_1, \ldots, \mathbf{u}_m$ is a vector $\mathbf{v}$ of the form

$$k_1\vec{u}_1 + k_2\vec{u}_2 + \cdots + k_m\vec{u}_m$$

Example 1.4. In $\mathbb{R}^2$, is $\mathbf{v} = \underline{\begin{bmatrix} -3 \\ 12.6 \end{bmatrix}}$ a linear combination of the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$?

Can we find scalars k_1, k_2 such that $\vec{v} = k_1\vec{u}_1 + k_2\vec{u}_2$?

yes $\iff \begin{bmatrix} -3 \\ 12.6 \end{bmatrix} = k_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + k_2\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\iff \begin{bmatrix} -3 \\ 12.6 \end{bmatrix} = \begin{bmatrix} k_1 \cdot 1 + k_2 \cdot 2 \\ k_1 \cdot 0 + k_2 \cdot 3 \end{bmatrix} = \begin{bmatrix} k_1 + 2k_2 \\ 3k_2 \end{bmatrix}$

$\iff 3k_2 = 12.6$ and $k_1 + 2k_2 = -3$

$\iff k_2 = 4.2$ and $k_1 = -3 - 2k_2$

$= -3 - 2(4.2)$

$= -3 - 8.4$

$= -11.4$

Check! $-11.4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4.2\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 12.6 \end{bmatrix}$

yes $\begin{bmatrix} -3 \\ 12.6 \end{bmatrix}$ is a linear combination of $\vec{u}_1$ and $\vec{u}_2$.

EXERCISE! Show that every vector in $\mathbb{R}^3$ is a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Example 1.5. Is $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ a linear combination of $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$?

$$\vec{w} = c\vec{v}_1 + d\vec{v}_2 \iff \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = c\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$\iff \begin{cases} 1 = 2c \\ 1 = d \\ 1 = c + 2d \end{cases} \iff \begin{cases} c = \frac{1}{2} \\ d = 1 \\ \boxed{1 = \frac{1}{2} + 2(1)} \end{cases} \leftarrow \text{inconsistent}$$

there are no sol$^{\underline{n}}$ for c, d that express $\vec{w}$ as a linear comb. of $\vec{v}_1, \vec{v}_2$: $\therefore$ NO

Dot Product, Norm, and Distance

Recall: for two vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ in $\mathbb{R}^3$, their **Dot Product** is defined as:

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + x_3 y_3$$

We can then view the **Length** or **Norm** of x as follows:

$$\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{{x_1}^2 + {x_2}^2 + {x_3}^2}$$

We then generalize this to $\mathbb{R}^n$.

Definition 1.6. Let $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n)$ be vectors in $\mathbb{R}^n$. Their **Dot Product** is defined as:

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$$

Note $\vec{x} \cdot \vec{y}$ is a scalar ie $\vec{x} \cdot \vec{y} \in \mathbb{R}$

The **Norm** of x is defined as:

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{(x_1)^2 + \ldots + (x_n)^2}$$

The dot product and norm, as defined above, have some nice properties:

Proposition 1.7. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and let $k \in \mathbb{R}$. Then

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- $\vec{\mathbf{u}} \cdot \vec{\mathbf{0}} = 0$ (a dot product with the zero vector yields $0 \leftarrow$ the number zero!)
- $(k\mathbf{u}) \cdot \mathbf{v} = k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$
- $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
- $\|\mathbf{v}\| \geq 0$
- $\|\vec{\mathbf{v}}\| = 0 \iff \vec{\mathbf{v}} = \vec{\mathbf{0}}$
- $\|k\mathbf{v}\| = |k| \, \|\mathbf{v}\|$

Exercise! Verify each of the statements in Prop. 1.7.

Example 1.8. Let $\mathbf{u} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

90° angle

Find $\mathbf{u} \cdot \mathbf{v}$. $= 2 \cdot 2 + 4 \cdot (-1) = 0 \quad (\therefore \vec{u} \perp \vec{v})$

Find $\|\mathbf{v}\|$. $= \sqrt{2^2 + (-1)^2} = \sqrt{5}$

has norm = 1

Find a **UNIT VECTOR** parallel to $\mathbf{v}$.

$$\frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$$

In general, for $\mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{v} \neq \mathbf{0}$, we can create a **UNIT VECTOR** $\hat{\mathbf{v}}$ in the same direction as $\mathbf{v}$ as follows:

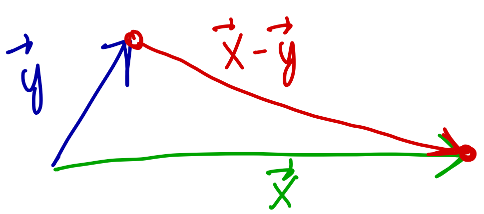
$$\hat{v} = \frac{1}{\|\vec{v}\|} \cdot \vec{v} \qquad (\text{So } \|\hat{v}\| = 1)$$

Although vectors are "portable", using the norm, we define the **DISTANCE** between two vectors as follows:

Definition 1.9. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. The **DISTANCE** between $\mathbf{x}$ and $\mathbf{y}$ is defined

$$\|\vec{x} - \vec{y}\| = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}$$

In two dimensions, we see that $\|\mathbf{x} - \mathbf{y}\|$ represents the distance between the tips of the vectors, when they are placed tail-to-tail:



$\|\vec{x} - \vec{y}\|$

EXERCISE! Find the distance between $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

$\vec{x}$ $\quad$ $\vec{y}$

Find $\|\vec{x} - \vec{y}\| = \|(2, 2, 2)\|$

$= \sqrt{2^2 + 2^2 + 2^2}$

$= \sqrt{12}$

Not only does the dot product give us a way to compute norms of vectors and distances between vectors, but it also allows us to find **ANGLES** between vectors.

We'll start with one of the nicest angles to consider: $\pi/2 \quad (90^\circ)$

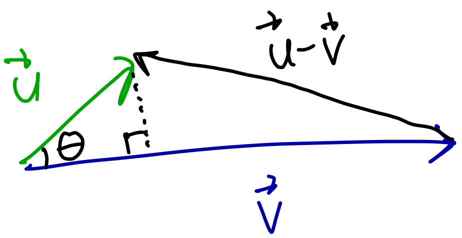
Definition 1.10. Let $\mathbf{u}, \mathbf{v}$ be two vectors in $\mathbb{R}^n$.

We say that $\mathbf{u}$ and $\mathbf{v}$ are **ORTHOGONAL** (or **PERPENDICULAR**) if $\vec{u}\cdot\vec{v} = 0$

Note: if $\mathbf{u} = \mathbf{0}$ or if $\mathbf{v} = \mathbf{0}$, then $\mathbf{u}\cdot\mathbf{v} = 0$.

Thus, the zero vector $\mathbf{0}$ is considered to be orthogonal to all other vectors (even though in the picture, we cannot convey a right angle between a vector and the zero vector).

What about other angles? We can imagine placing two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ tail-to-tail and measuring the angle formed between them as follows:



Cosine Law: $\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\cdot\|\vec{v}\|\cos\theta$

Also: $\|\vec{u}-\vec{v}\|^2 = \left(\sqrt{(u_1-v_1)^2 + \dots + (u_n-v_n)^2}\right)^2$

so $\|\vec{u}-\vec{v}\|^2 = (u_1-v_1)^2 + \dots + (u_n-v_n)^2$

$$\Rightarrow \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\cdot\|\vec{v}\|\cos\theta = (u_1-v_1)^2 + \dots + (u_n-v_n)^2$$

$$\Rightarrow (\cancel{u_1^2} + \dots + \cancel{u_n^2}) + (\cancel{v_1^2} + \dots + \cancel{v_n^2}) - 2\|\vec{u}\|\|\vec{v}\|\cos\theta = (\cancel{u_1^2} - 2u_1v_1 + \cancel{v_1^2}) + \dots + (\cancel{u_n^2} - 2u_nv_n + \cancel{v_n^2})$$

$$\Rightarrow -\cancel{2}\|\vec{u}\|\cdot\|\vec{v}\|\cos\theta = -\cancel{2}u_1v_1 - \cancel{2}u_2v_2 - \dots - \cancel{2}u_nv_n \quad \therefore \|\vec{u}\|\cdot\|\vec{v}\|\cos\theta = \vec{u}\cdot\vec{v}$$

$$= -\cancel{2}(u_1v_1 + u_2v_2 + \dots + u_nv_n)$$

We use these ideas to generalize what we mean by "angle" between two vectors in spaces beyond $\mathbb{R}^2$ and $\mathbb{R}^3$.

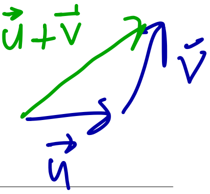
Definition 1.11. Let $\mathbf{u}, \mathbf{v}$ be two nonzero vectors in $\mathbb{R}^n$. The **ANGLE BETWEEN u AND v** is defined to be the number θ such that:

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|} \quad \text{and} \quad 0 \le \theta \le \pi$$

In order for the above definition to make sense, we need $\|\mathbf{u}\| \neq 0$, $\|\mathbf{v}\| \neq 0$ and $-1 \le \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \le 1$. The norms will both be nonzero since $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$. The bounds on $\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$ follow from the theorem below.

Theorem 1.12. (**CAUCHY–SCHWARZ INEQUALITY**) If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, then $|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\cdot\|\mathbf{v}\|$ proof in 2nd year MAT214)

Corollary 1.13. (**Triangle Inequality**) If If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, then $\quad \|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$



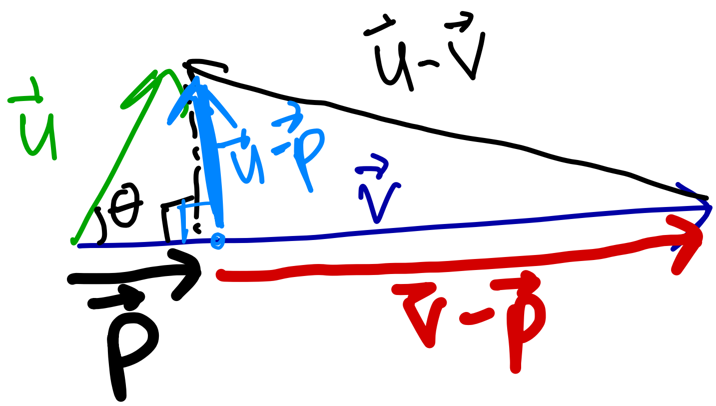
Proof:

Example 1.14. Find the angle between $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$.

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|} = \frac{(1,-1,2)\cdot(0,3,1)}{\sqrt{1^2+1^2+2^2}\,\sqrt{0^2+3^2+1^2}} = \frac{0-3+2}{\sqrt{6}\,\sqrt{10}} = -\frac{1}{\sqrt{60}}$$

$$\Rightarrow \theta = \arccos\left(\frac{-1}{\sqrt{60}}\right) \quad (\approx 97^\circ \text{ or } \approx 0.54\pi \text{ rad})$$

The way we defined the angle between two vectors generalizes what we see in triangles in $\mathbb{R}^2$:



$$\cos\theta = \frac{\|\vec{p}\|}{\|\vec{u}\|}$$

In the above illustration, we decomposed the "v-edge of the triangle" into the vector $\mathbf{p}$ and $\mathbf{v} - \mathbf{p}$ exactly so that $\mathbf{p}$ and $\mathbf{u} - \mathbf{p}$ were orthogonal (in the picture, we dropped a perpendicular to show the height of the triangle, making a $90°$ angle with the base).

Now, with our definition for angles between vectors in $\mathbb{R}^n$, we want to generalize this idea, as follows:

Question: Given nonzero vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$, how do we find $\vec{p}$ so that

- $\vec{p}$ is parallel to $\vec{v}$
- $\vec{p}$ is orthogonal to $\vec{u}-\vec{p}$

Answer: the Projection of $\vec{u}$ onto $\vec{v}$

Definition 1.15. Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in $\mathbb{R}^n$. Then the **PROJECTION OF u ONTO v**, denoted $\mathrm{proj}_{\mathbf{v}}(\mathbf{u})$, is the unique vector which satisfies:

- $\mathrm{proj}_{\mathbf{v}}(\mathbf{u})$ is parallel to $\mathbf{v}$ (i.e. $\mathrm{proj}_{\mathbf{v}}(\mathbf{u})$ is a scalar multiple of $\mathbf{v}$), and
- $\mathbf{u} - \mathrm{proj}_{\mathbf{v}}(\mathbf{u})$ is orthogonal to $\mathbf{v}$ (i.e. $\mathbf{v} \cdot (\mathbf{u} - \mathrm{proj}_{\mathbf{v}}(\mathbf{u})) = 0$)

Proposition 1.16. Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in $\mathbb{R}^n$. Then

$$\text{proj}_{\mathbf{v}}(\mathbf{u}) = k \cdot \mathbf{v} \quad \text{where} \quad k = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}$$

That is $\boxed{\text{proj}_{\mathbf{v}}(\mathbf{u}) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}}$

Proof:

Example 1.17. Let $\mathbf{u} = (1, -3)$ and $\mathbf{v} = (2, 1)$. Find $\text{proj}_{\mathbf{v}}(\mathbf{u})$ and find $\text{proj}_{\mathbf{u}}(\mathbf{v})$.

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\vec{v} = \frac{(1,-3)\cdot(2,1)}{(2,1)\cdot(2,1)}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{-1}{5}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/5 \\ -1/5 \end{bmatrix}$$

2. Lines and Planes

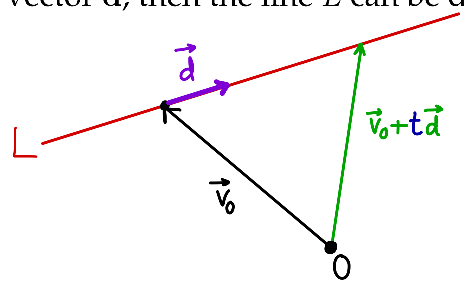
Here, we review lines and planes in $\mathbb{R}^2$ and $\mathbb{R}^3$, as well as the cross product (in $\mathbb{R}^3$).

Describing Lines

A line L is completely determined by its direction $\mathbf{d}$ and a point P on the line.

We think of the point P as being the tip of a vector with its tail at the origin O. Say, $\vec{OP} = \mathbf{v}_0$. If the direction of the line is given by the direction vector $\mathbf{d}$, then the line L can be described as the set:

$$L = \{ \vec{v}_0 + t\vec{d} : t \in \mathbb{R} \}$$

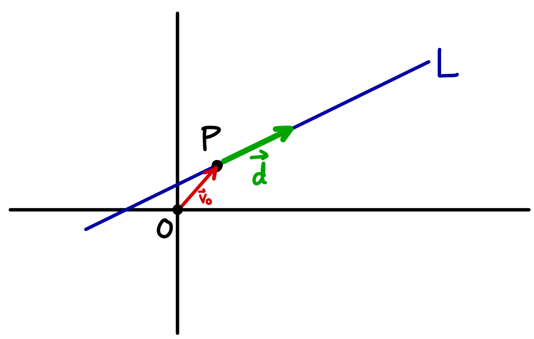


Any point on the line L can be expressed as $\mathbf{v}_0 + t\mathbf{d}$, where $t \in \mathbb{R}$ is a **Parameter**. This way of representing points on the line is called **Vector Parametric Form** or a **Parametric Equation** for the line.

Example 2.1. Find a parametric equation for the line in $\mathbb{R}^2$ passing through the points $(1,1)$ and $(3,2)$.

direction $\vec{d} = \vec{QP} = \begin{bmatrix} 3-1 \\ 2-1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

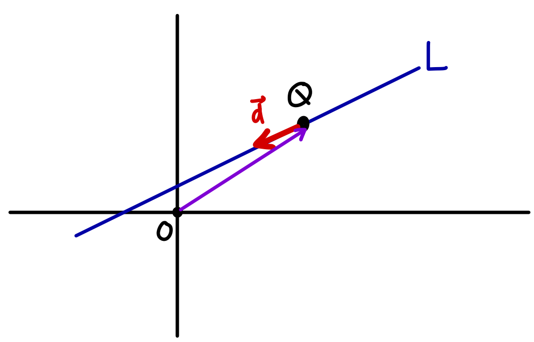
$$L = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$



Alternatives $L = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$

$$L = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix} + t\begin{bmatrix} -1 \\ -0.5 \end{bmatrix} : t \in \mathbb{R} \right\}$$

$$\vdots$$



†These notes are solely for the personal use of students registered in MAT1341.

Example 2.2. Find a parametric equation for the line $y = 3x + 2$.

Point on line

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 3x+2 \end{bmatrix} = \begin{bmatrix} 0+x \\ 2+3x \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + x\begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad L = \left\{ \begin{bmatrix} 0 \\ 2 \end{bmatrix} + t\begin{bmatrix} 1 \\ 3 \end{bmatrix} \ t \in \mathbb{R} \right\}$$

x is the parameter

Alternative $y = 3x+2 \Rightarrow x = \frac{1}{3}y - \frac{2}{3}$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3}y - \frac{2}{3} \\ y \end{bmatrix} = \begin{bmatrix} -2/3 \\ 0 \end{bmatrix} + y\begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$$

y is the parameter

$$L = \left\{ \begin{bmatrix} -2/3 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1/3 \\ 1 \end{bmatrix} \ s \in \mathbb{R} \right\}$$

Example 2.3. Find the intersection of lines $L_1 = \left\{ t\begin{bmatrix} 1 \\ 2 \end{bmatrix} : t \in \mathbb{R} \right\}$ and $L_2 = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 3 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$.

Note It's wrong to set $t\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and solve for t !

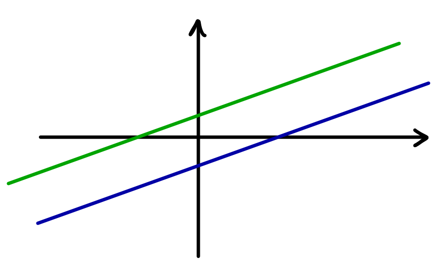
The parameter-value for L_1 need not be equal to the parameter-value for L_2 at POI

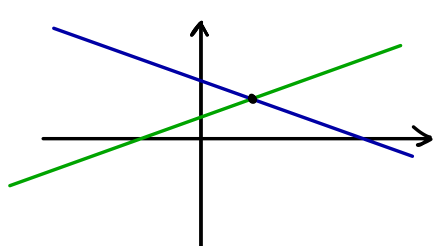
$$t\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} 3 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} t \\ 2t \end{bmatrix} = \begin{bmatrix} 3s \\ 1+s \end{bmatrix} \Rightarrow 2(3s) = 1+s \Rightarrow s = \frac{1}{5} \Rightarrow t = \frac{3}{5}$$

Use L_1 and t to find POI: $\frac{3}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 6/5 \end{bmatrix}$

or L_2 and s $\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 6/5 \end{bmatrix}$

Remark 2.4. In $\mathbb{R}^2$, two distinct lines are either parallel or they intersect at a point.





Remark 2.5. In $\mathbb{R}^3$, two distinct lines are either parallel or they intersect at a point, or they are skew. If the lines are parallel or if they intersect, then there exists a plane that contains both lines. If the lines are skew, then they are not both contained in the same plane, but you can find two parallel planes such that each contains one of the lines.

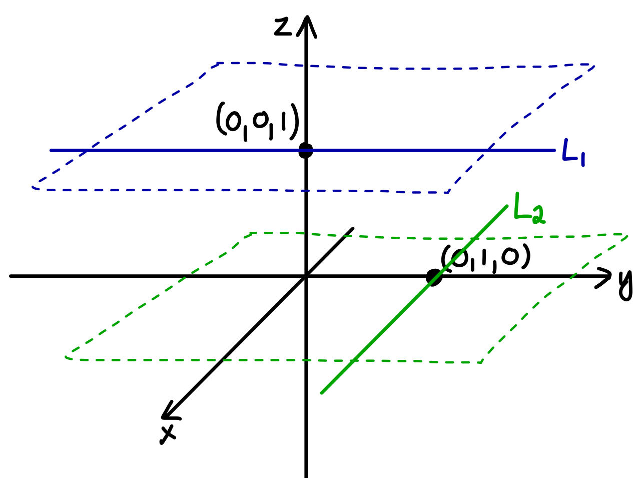
Example 2.6. Consider the lines $L_1 = \{(0, t, 1) : t \in \mathbb{R}\}$ and $L_2 = \{(t, 1, 0) : t \in \mathbb{R}\}$. Are they parallel, skew, or do they intersect?

$$\begin{bmatrix} 0 \\ t \\ 1 \end{bmatrix} = \begin{bmatrix} s \\ 1 \\ 0 \end{bmatrix} \leftarrow \text{no solution} \quad \text{no intersection}$$

$$\vec{d}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{d}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{d}_1 \text{ is not a scalar multiple of } \vec{d}_2 \quad \therefore \text{ not parallel}$$

$\therefore$ L_1 and L_2 are skew

For L_1, $x=0, y=t, z=1$



For L_2, $x=s, y=1, z=0$

A plane in $\mathbb{R}^3$ can be expressed as an equation in **POINT-NORMAL FORM** or **CARTESIAN FORM**, which looks as follows:

$$ax + by + cz = d$$

$a, b, c, d \in \mathbb{R}$

$\vec{n} = (a, b, c)$ is a NORMAL VECTOR to the plane

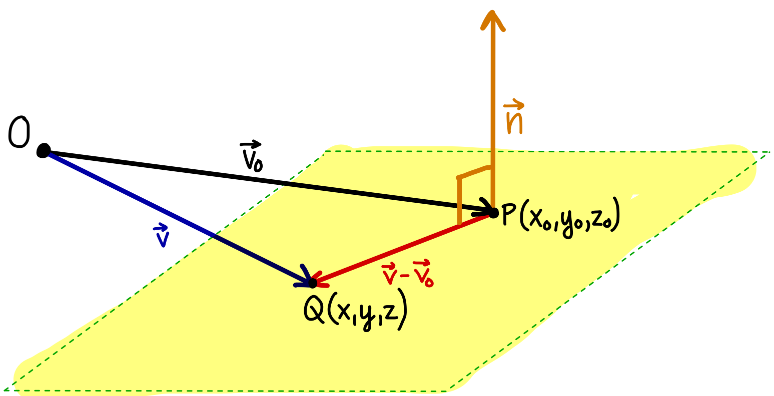
Unlike a line, a plane has infinitely many directions. But in $\mathbb{R}^3$, a plane has a single perpendicular direction, which is the plane's **NORMAL VECTOR**.

How does the normal vector end up in the coefficients of the equation?

Let $P(x_0, y_0, z_0)$ be a particular point on the plane and let $\overrightarrow{OP} = \vec{v}_0$

Let $Q(x, y, z)$ denote an arbitrary point on the plane and let $\overrightarrow{OQ} = \vec{v}$

Then $\vec{v} - \vec{v}_0$ is on the plane so $(\vec{v} - \vec{v}_0) \cdot \vec{n} = 0$



$$\Rightarrow \overbrace{(x - x_0, y - y_0, z - z_0)}^{\vec{v}-\vec{v}_0} \cdot \overbrace{(a, b, c)}^{\vec{n}} = 0$$

$$\Rightarrow ax - ax_0 + by - by_0 + cz - cz_0 = 0$$

$$\Rightarrow ax + by + cz = \underbrace{ax_0 + by_0 + cz_0}_{\text{call this } d} = d$$

In fact, d is simply $\vec{v}_0 \cdot \vec{n}$

$$\vec{v} - \vec{v}_0 \text{ lies on plane} \iff (\vec{v} - \vec{v}_0)\,\vec{n} = 0$$
$$\iff \vec{v}\,\vec{n} - \vec{v}_0\,\vec{n} = 0$$
$$\iff \vec{v}\,\vec{n} = \vec{v}_0\,\vec{n}$$

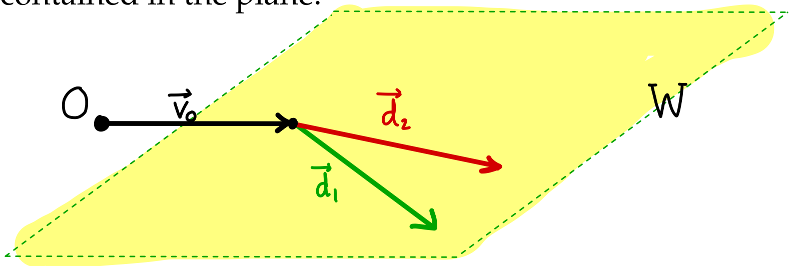
Example 2.7. Find the Cartesian equation for the plane that contains the point $(1, 2, 3)$ with normal vector $\mathbf{n} = (-1, 5, 2)$.

$P(1,2,3)$ $\quad \vec{v}_0 = \overrightarrow{OP} = (1,2,3)$ $\qquad \vec{n} = (-1,5,2)$

$$\vec{n} \cdot \vec{v}_0 = (-1,5,2) \cdot (1,2,3) = -1 + 10 + 6 = 15$$

$\therefore$ plane's equation is $-x + 5y + 2z = 15$

A plane in $\mathbb{R}^3$ can also be expressed in **PARAMETRIC FORM**. We need two vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ that are parallel to the plane but not parallel to each other, i.e. we need two distinct "direction vectors" contained in the plane.



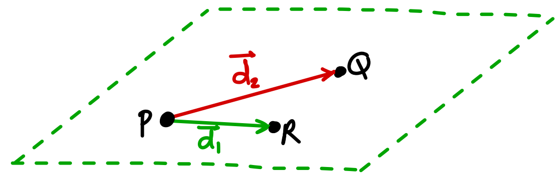
$$W = \{ \vec{v}_0 + s\vec{d}_1 + t\vec{d}_2 \ : s,t \in \mathbb{R} \}$$

2 parameters

Example 2.8. Find a parametric equation for the plane given by $3x - y + 2z = -12$.

Strategy:

① Isolate a variable
② The other two become the free parameters
③ Extract the point and direction vectors



① $y = 3\underset{s}{x} + 2\underset{t}{z} + 12$

② $x = s$, $z = t$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s \\ 3s+2t+12 \\ t \end{bmatrix} \overset{③}{=} \underbrace{\begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}}_{\text{parametric form}}$$

EXERCISE! Find a parametric equation for the same plane as the previous example using the following alternative method:

1. Find three points on the plane.
2. Create two direction vectors from the points.

Example 2.9. Find a parametric equation for the plane given by $y = 2x + 3$.

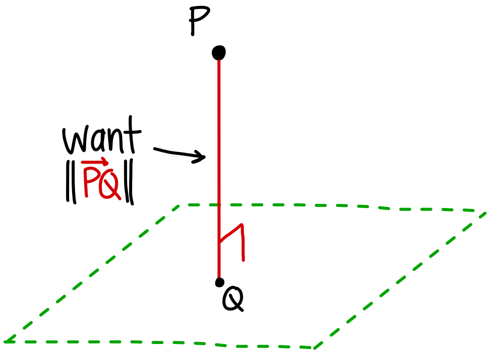
Note In $\mathbb{R}^3$, the equation $y = 2x+3$ does NOT define a line !

In $\mathbb{R}^3$, lines don't have Cartesian equations

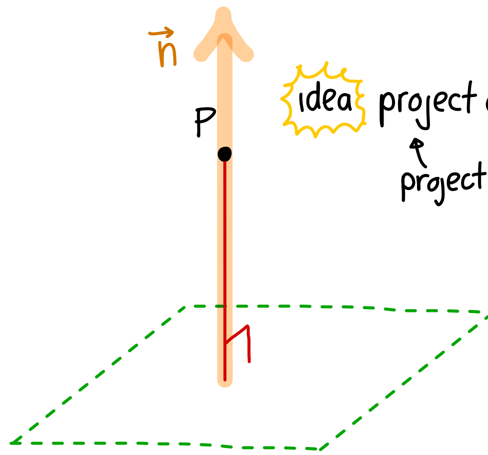
The equation $y=2x+3$ doesn't care what z equals, so z is a free variable !

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 2x+3 \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} x \\ 2x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} + x\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{parametric form}}$$

Question: How would we find the **DISTANCE FROM A POINT TO A PLANE** ?



where Q is the unknown point on plane closest to P



idea project onto $\vec{n}$

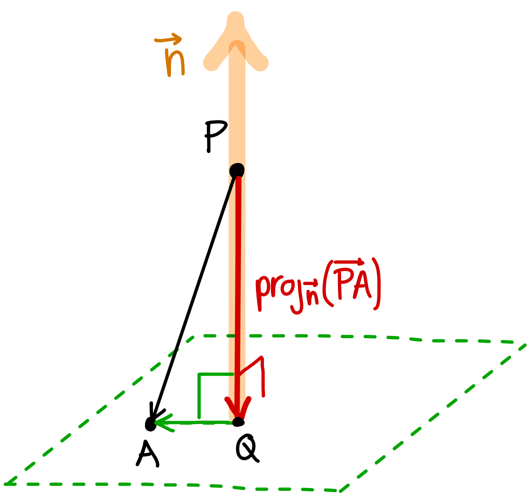
project what?

Pick any point A on the plane

Create vector $\overrightarrow{PA}$

Project $\overrightarrow{PA}$ onto normal vector $\vec{n}$

Distance from P to plane is then $\|\text{proj}_{\vec{n}}(\overrightarrow{PA})\|$



Why is Q (tip of $\text{proj}_{\vec{n}}(\overrightarrow{PA})$) the closest point to P on the plane ?

By defintion, $\text{proj}_{\vec{n}}(\overrightarrow{PA})$ satisfies $\overrightarrow{PA} = \underbrace{\overbrace{\text{proj}_{\vec{n}}(\overrightarrow{PA})}^{\overrightarrow{PQ}}}_{\text{parallel to } \vec{n}} + \underbrace{\overbrace{(\overrightarrow{PA} - \text{proj}_{\vec{n}}(\overrightarrow{PA}))}^{\overrightarrow{QA}}}_{\text{perpendicular to } \vec{n}}$

$\Rightarrow \triangle PQA$ is a right triangle so Pythagorean Theorem applies

$\Rightarrow \underbrace{\|\overrightarrow{PA}\|^2}_{(\text{dist from P to A})^2} = \underbrace{\|\overrightarrow{PQ}\|^2}_{(\text{dist from P to Q})^2} + \underbrace{\|\overrightarrow{QA}\|^2}_{\geq 0}$

$\Rightarrow$ (distance from P to A) $\geq$ (distance from P to Q)

$\Rightarrow$ Q is as close to P as any point A on plane

Example 2.10. Find the distance from the point $P(1, 2, 3)$ to the plane W with Cartesian equation $3x - 4z = -1$.

$\vec{n} = (3, 0, -4)$

Pick any point on plane eg set $x=0$ Then $-4z = -1 \Rightarrow z = \frac{1}{4}$

y is free so set $y = 0$ (easy option)

$A(0, 0, \frac{1}{4})$ is on plane

$\Rightarrow \overrightarrow{PA} = (0-1, 0-2, \frac{1}{4} - 3) = (-1, -2, \frac{-11}{4})$

$$\text{proj}_{\vec{n}}(\overrightarrow{PA}) = \frac{(\overrightarrow{PA}) \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \vec{n} = \frac{(-1, -2, -\frac{11}{4}) \cdot (3, 0, -4)}{(3, 0, -4) \cdot (3, 0, -4)}(3, 0, -4) = \frac{-3 + 0 + \frac{44}{4}}{9 + 0 + 16}(3, 0, -4) = \frac{8}{25}(3, 0, -4)$$

$$\Rightarrow \text{dist from P to W} = \|\text{proj}_{\vec{n}}(\overrightarrow{PA})\| = \left\|\frac{8}{25}(3, 0, -4)\right\| = \left|\frac{8}{25}\right| \sqrt{3^2 + 0^2 + (-4)^2} = \frac{8}{25}\sqrt{25} = \frac{8}{5}$$

Remark 2.11. In $\mathbb{R}^3$:

- a line is either entirely contained on a plane, or it intersects the plane in exactly one point, or there is no intersection and the line is parallel to the plane
- two distinct planes can be parallel, or else they intersect in a line

---

Example 2.12. Find the intersection of the line L and each of the planes given below:

$$L = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} : t \in \mathbb{R} \right\} \quad \Pi_1 : 2x - y + z = 3 \quad \Pi_2 : 2x - y + 2z = -2 \quad \Pi_3 : 2x - y + z = -5$$

points on line satisfy $x = 1 - t$ $\quad y = 2 + 2t$ $\quad z = 3 + 4t$

sub into Π_1 $\quad 2(1-t) - (2+2t) + (3+4t) = 3$

$3 = 3$ $\quad$ so L satisfies Π_1 eq no matter what t is

L lies on Π_1

sub into Π_2 $\quad 2(1-t) - (2+2t) + 2(3+4t) = -2$

$6 + 4t = -2$

$\Rightarrow t = -2$ $\quad \Rightarrow$ POI when $t = -2$ $\quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (-2) \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix}$

sub into Π_3 $\quad 2(1-t) - (2+2t) + (3+4t) = -5$

$3 = -5$ $\lightning$ no solution $\therefore$ no intersection

---

Example 2.13. Find the **Intersection of the Planes** $2x + 4y + 4z = 7$ and $6x - 3y + 2z = 1$.

① $2x + 4y + 4z = 7$

② $6x - 3y + 2z = 1$

planes aren't parallel (non-parallel normal vectors)

eliminate x

$$\begin{array}{r} 6x + 12y + 12z = 21 \\ 6x - 3y + 2z = 1 \\ \hline 15y + 10z = 20 \end{array}$$

← 3x eq①

eliminate z

$$\begin{array}{r} 2x + 4y + 4z = 7 \\ -\ \ 12x - 6y + 4z = 2 \\ \hline -10x + 10y \quad = 5 \end{array}$$

← 2x eq②

express x and z in terms of y

$10z = 20 - 15y$

$z = 2 - \frac{3}{2}y$

$-10x = 5 - 10y$

$x = -\frac{1}{2} + y$

Intersection $\quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} + y \\ y \\ 2 - \frac{3}{2}y \end{bmatrix} = \begin{bmatrix} -1/2 \\ 0 \\ 2 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \\ -3/2 \end{bmatrix}$

Example 2.14. Find the Cartesian equation for the plane $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + s \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} : s, t \in \mathbb{R} \right\}$.

We have two direction vectors for plane but we need a normal vector

---

CROSS PRODUCT

Recall: in $\mathbb{R}^3$, we have an operation that takes two vectors and creates a vector orthogonal to both: the **CROSS PRODUCT** .

Definition 2.15. Given $\mathbf{u} = (a, b, c)$ and $\mathbf{v} = (x, y, z)$ in $\mathbb{R}^3$, the **CROSS PRODUCT OF u AND v** , denoted $\mathbf{u} \times \mathbf{v}$, is defined as follows:

$$\boxed{\vec{u} \times \vec{v} = (a,b,c) \times (d,e,f) = (bf-ce, -(af-cd), ae-bd)}$$

$$(a,b,c) \times (d,e,f) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a & b & c \\ d & e & f \end{vmatrix} = \hat{\imath} \begin{vmatrix} b & c \\ e & f \end{vmatrix} - \hat{\jmath} \begin{vmatrix} a & c \\ d & f \end{vmatrix} + \hat{k} \begin{vmatrix} a & b \\ d & e \end{vmatrix} \qquad \hat{\imath} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \hat{\jmath} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

memory aid
determinant
notation

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a & b & c \\ d & e & f \end{vmatrix} \quad \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a & b & c \\ d & e & f \end{vmatrix} \quad \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a & b & c \\ d & e & f \end{vmatrix}$$

Back to Example 2.14!

normal vector $\vec{n} \perp \vec{d}_1$ and $\vec{n} \perp \vec{d}_2$ Find $\vec{n} = \vec{d}_1 \times \vec{d}_2$

$$\vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 8 & 1 \\ 1 & 2 & 2 \end{vmatrix} = \hat{\imath} \begin{vmatrix} 8 & 1 \\ 2 & 2 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} + \hat{k} \begin{vmatrix} 4 & 8 \\ 1 & 2 \end{vmatrix}$$

$$= (8(2)-(1)(2), -(4(2)-(1)(1)), 4(2)-8(1))$$

$$= (14, -7, 0)$$

point on plane $(1,-1,2)$ $\Rightarrow$ $d = \vec{n} \cdot \vec{v}_0 = (14,-7,0) \cdot (1,-1,2) = 21$

$$\boxed{\text{Cartesian eq} \quad 14x - 7y + 0z = 21}$$

Example 2.16. Compute $\hat{\mathbf{i}} \times \hat{\mathbf{j}}$ and $\hat{\mathbf{j}} \times \hat{\mathbf{i}}$.

$$\hat{\imath} \times \hat{\jmath} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = (0-0, -(0-0), 1-0) = (0,0,1) = \hat{k}$$

$$\hat{\jmath} \times \hat{\imath} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = (0-0, -(0-0), 1-0) = (0,0,-1) = -\hat{k}$$

$\hat{\imath} \times \hat{\jmath} \neq \hat{\jmath} \times \hat{\imath}$

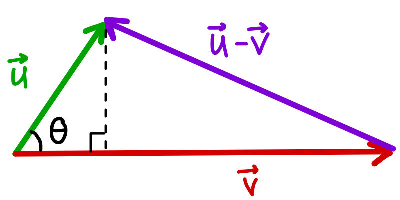
Cross product is NOT commutative!

But, cross product does have other nice properties

Proposition 2.17. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Then

- $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
- $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} =$
- $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} =$
- $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$
- $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\| \sin\theta$ where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ and $0 \leq \theta \leq \pi$.

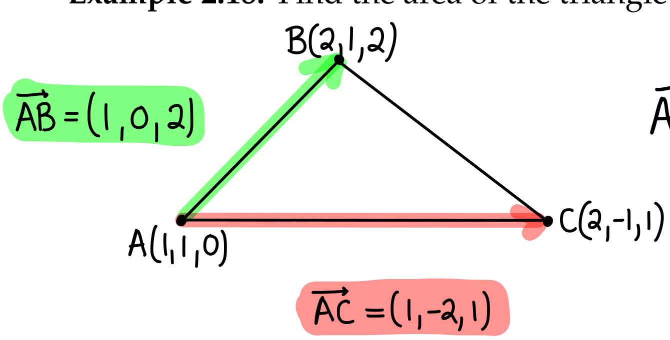
AREA OF TRIANGLE The last property above allows us to use cross products to compute the area of the triangle formed between two vectors (or the area of the parallelogram created by the vectors).



$$\sin\theta = \frac{(\text{height})}{\|\vec{u}\|} \Rightarrow (\text{height}) = \|\vec{u}\| \sin\theta$$

$$\begin{aligned} \text{Area } \triangle &= \tfrac{1}{2}(\text{base}) \times (\text{height}) \\ &= \tfrac{1}{2}\|\vec{v}\| \, \|\vec{u}\| \sin\theta \\ &= \tfrac{1}{2}\|\vec{u} \times \vec{v}\| \quad (\text{by Prop } 2.17) \end{aligned}$$

Example 2.18. Find the area of the triangle formed by the points $A(1,1,0), B(2,1,2), C(2,-1,1)$.



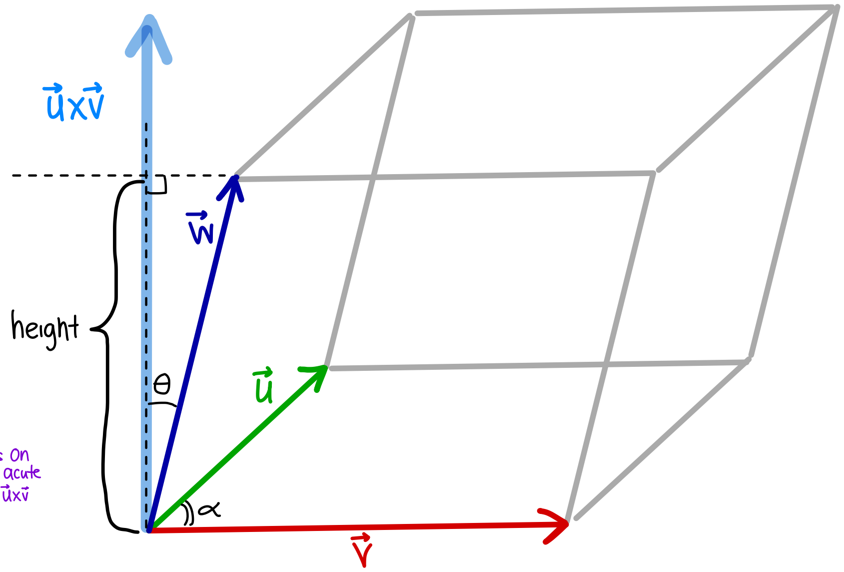
$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & 2 \\ 1 & -2 & 1 \end{vmatrix} = (4, -(-1), -2) = (4,1,-2)$$

$$\|\vec{AB} \times \vec{AC}\| = \sqrt{(4)^2 + 1^2 + (-2)^2} = \sqrt{21}$$

$$\text{Area } \triangle ABC = \tfrac{1}{2}\|\vec{AB} \times \vec{AC}\| = \tfrac{1}{2}\sqrt{21}$$

Volume of a Parallelepiped Using both cross and dot product, we also have the following way to compute the volume of a parallelepiped!

$\theta =$ angle between $\vec{u}\times\vec{v}$ and $\vec{w}$



$$|\cos\theta| = \frac{\text{height}}{\|\vec{w}\|}$$

Sign of $\cos\theta$ depends on whether $\vec{w}$ makes an acute or obtuse angle with $\vec{u}\times\vec{v}$

$$\Rightarrow \text{height} = \|\vec{w}\|\,|\cos\theta|$$

$$\text{Area of } \text{parallelogram}(\vec{u},\vec{v}) = \|\vec{u}\times\vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sin\alpha$$

$$\Rightarrow \text{Vol Parallelepiped} = \left(\text{Area of parallelogram}(\vec{u},\vec{v})\right)\left(\text{height}\right)$$

$$= \|\vec{u}\times\vec{v}\|\,\|\vec{w}\|\,|\cos\theta|$$

$$= |(\vec{u}\times\vec{v})\cdot\vec{w}|$$

Recall $\vec{x}\cdot\vec{y} = \|\vec{x}\|\|\vec{y}\|\cos\theta$

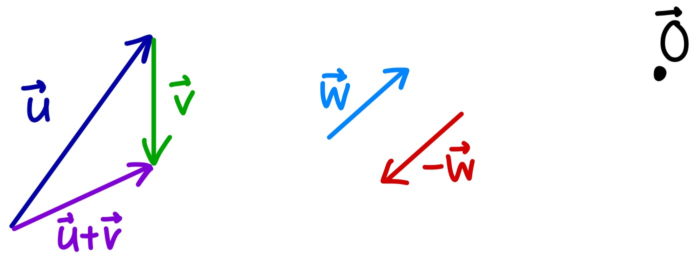
Example 2.19. Find the volume of the parallelepiped formed by $\mathbf{u} = (1, 2, 3)$, $\mathbf{v} = (1, 3, 2)$, $\mathbf{w} = (1, 2, 2)$.

$$\text{Volume} = |(\vec{u}\times\vec{v})\cdot\vec{w}| = |(-5,1,1)\cdot(1,2,2)| = |-5+2+2| = |-1| = 1$$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ 1 & 3 & 2 \end{vmatrix} = (-5, 1, 1)$$

3. Vector Spaces

- We first reacquainted ourselves with vectors by depicting them as arrows with geometric rules for addition and scalar multiplication. We observed several "nice" properties that these operations satisfy.



- Next, we converted our arrow pictures into algebraic structures by using coordinate systems to represent those arrows. In terms of coordinates, the vector addition operation and scalar multiplication operation involved adding or multiplying coordinates and we didn't need pictures to "see" the result of those operations (although the pictures still helped us to develop intuition).

$$\begin{bmatrix}u_1\\u_2\end{bmatrix}+\begin{bmatrix}v_1\\v_2\end{bmatrix}=\begin{bmatrix}u_1+v_1\\u_2+v_2\end{bmatrix} \qquad \begin{bmatrix}0\\0\end{bmatrix} \qquad 8\begin{bmatrix}w_1\\w_2\end{bmatrix}$$

- The algebraic way of adding and scalar-multiplying vectors also satisfied the same "nice" properties that the geometric way did. On top of that, the algebraic way of representing vectors also made it very natural to generalize from 2 and 3 dimensions into n dimensions for $n \geq 4$ (whereas the depiction of vectors as arrows doesn't).

$$\mathbb{R}^n = \left\{ \begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix} : x_i \in \mathbb{R} \text{ for } 1 \leq i \leq n \right\}$$

Now, we will see that generalizing beyond $\mathbb{R}^n$ is also possible! What we need is to study the essential ingredients that make a collection of things behave (in some abstract way) the same as $\mathbb{R}^n$.

† These notes are solely for the personal use of students registered in MAT1341.

Definition 3.1. Let V be a set, whose elements we will call "vectors", equipped with

- a rule for adding two vectors together
- a rule for scalar-multiplying a vector by a real number.

If V, with its two operations, satisfies the following 10 axioms, then V is called a **VECTOR SPACE**:

CLOSURE

(1) $\vec{u}, \vec{v} \in V \Rightarrow \vec{u}+\vec{v} \in V$ [V is closed under addition]

the sum of any two vectors in V is again a vector in V

(2) $\vec{u} \in V$ and $c \in \mathbb{R} \Rightarrow c\vec{u} \in V$ [V is closed under scalar multiplication]

a scalar multiple of any vector in V is again a vector in V

EXISTENCE

(3) There exists $\vec{0} \in V$ such that $\vec{0}+\vec{v}=\vec{v}$ for every vector $\vec{v} \in V$ [Existence of zero vector]

V has a zero vector

(4) For each vector $\vec{v} \in V$, there exists $-\vec{v} \in V$ such that $\vec{v}+(-\vec{v})=\vec{0}$ [Existence of vector negatives]

the negative of $\vec{v}$ exists and is also a vector in $\vec{v}$

ARITHMETIC PROPERTIES

For all $\vec{u}, \vec{v}, \vec{w} \in V$ and for all $c, d \in \mathbb{R}$

(5) $\vec{u}+\vec{v}=\vec{v}+\vec{u}$ [Vector addition is commutative]

(6) $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$ [Vector addition is associative]

(7) $c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$ [Scalar multiplication distributes over vector addition]

(8) $(c+d)\vec{u}=c\vec{u}+d\vec{u}$ [Addition of scalars distributes over scalar multiplication]

(9) $c(d\vec{u})=(cd)\vec{u}$ [Multiplication of scalars is compatible with scalar multiplication]

(10) $1\vec{u}=\vec{u}$ [Unity Law]

Let's explore various examples.

Example 3.2. SPACES OF EQUATIONS
Consider the set of all linear equations in three variables, x, y, z.

$$\mathcal{E} = \{ ax+by+cz=d \quad a,b,c,d \in \mathbb{R} \}$$

What are some elements in the set $\mathcal{E}$?

$E_1 \quad 2x+3y-5z=3$

$E_2 \quad x-z=6$

$E_3 \quad 0=1$

We are simply considering the equations, not solving them, so E_3 is a valid equation even if it has no solutions

Note: we don't need all three variables to appear.

$x-z=6$ means $x+0y-z=6$

We can create new equations from old ones:

We can add equations

$E_1+E_2: \; 3x+3y-6z=9$

$E_2+E_3: \; x-z=7$

Note $E_1+E_2 \in \mathcal{E}$, $E_2+E_3 \in \mathcal{E}$

We can scalar-multiply equations

$\pi E_2 \quad \pi x - \pi z = 6\pi$

$(-1)E_1 \quad -2x-3y+5z=-3$

$\pi E_2 \in \mathcal{E}$, $(-1)E_1 \in \mathcal{E}$

Rule for adding elements of $\mathcal{E}$

$$\underbrace{(a_1x+b_1y+c_1z=d_1)}_{E_1} + \underbrace{(a_2x+b_2y+c_2z=d_2)}_{E_2} = \underbrace{(a_1+a_2)x+(b_1+b_2)y+(c_1+c_2)z=(d_1+d_2)}_{E_1+E_2}$$

Note $E_1, E_2 \in \mathcal{E} \Rightarrow E_1+E_2 \in \mathcal{E}$

1. $\mathcal{E}$ is closed under "vector" (ie equation) addition ✓

Rule for scalar-multiplying elements of $\mathcal{E}$

$$k\underbrace{(ax+by+cz=d)}_{E} = \underbrace{(ka)x+(kb)y+(kc)z=(kd)}_{kE}$$

Note $k \in \mathbb{R}$ and $E \in \mathcal{E} \Rightarrow kE \in \mathcal{E}$

2. $\mathcal{E}$ is closed under scalar multiplication ✓

(3) What is the zero "vector" (equation) for $\mathcal{E}$?

Let Z be the equation $Z \quad 0=0$

Let $E \in \mathcal{E}$. Then
$$\begin{aligned} Z+E &= (0=0)+(ax+by+cz=d) \\ &= ((0+a)x+(0+b)y+(0+c)z=(0+d)) \\ &= ax+by+cz=d \\ &= E \end{aligned}$$

$\vec{0}$ for $\mathcal{E}$ is Z!

(4) For $E \in \mathcal{E}$ what is its negative?

$E+(-E)=Z$ so $-E$ must be the equation $-ax-by-cz=-d$

(5) Let $E_1, E_2 \in \mathcal{E}$. Then

$$\begin{aligned} E_1+E_2 &= (a_1x+b_1x+c_1x=d_1)+(a_2x+b_2y+c_2z=d_2) \\ &= ((a_1+a_2)x+(b_1+b_2)y+(c_1+c_2)z=(d_1+d_2)) && \text{def of eq}^{\underline{n}} \text{ addition} \\ &= ((a_2+a_1)x+(b_2+b_1)y+(c_2+c_1)z=(d_2+d_1)) && \text{commutativity of scalar addition} \\ &= (a_2x+b_2y+c_2z=d_2)+(a_1x+b_1x+c_1x=d_1) && \text{def of eq}^{\underline{n}} \text{ addition} \\ &= E_2+E_1 \end{aligned}$$

(6) Let $E_1, E_2, E_3 \in \mathcal{E}$, $E_i=(a_ix+b_iy+c_iz=d_i)$ for $1 \leq i \leq 3$. Then

$$\begin{aligned} E_1+(E_2+E_3) &= (a_1x+b_1y+c_1z=d_1)+[(a_2x+b_2y+c_2z=d_2)+(a_3x+b_3y+c_3z=d_3)] \\ &= (a_1x+b_1y+c_1z=d_1)+[(a_2+a_3)x+(b_2+b_3)y+(c_2+c_3)z=(d_2+d_3)] \\ &= ((a_1+(a_2+a_3))x+(b_1+(b_2+b_3))y+(c_1+(c_2+c_3))z=(d_1+(d_2+d_3))z) \\ &= (((a_1+a_2)+a_3)x+((b_1+b_2)+b_3)y+((c_1+c_2)+c_3)z=((d_1+d_2)+d_3)) \\ &= [(a_1+a_2)x+(b_1+b_2)y+(c_1+c_2)z=(d_1+d_2)]+(a_3x+b_3y+c_3z=d_3) \\ &= [(a_1x+b_1x+c_1z=d_1)+(a_2x+b_2y+c_2z=d_2)]+(a_3x+b_3y+c_3z=d_3) \\ &= (E_1+E_2)+E_3 \end{aligned}$$

Try the rest yourself! (7)–(10)

Example 3.3. FUNCTION SPACES
Let $\mathscr{F}$ denote the set of all real-valued functions with domain $(-\infty, \infty)$.

$$\mathscr{F} = \{ f \mid f\ \mathbb{R} \to \mathbb{R} \}$$

For two functions $f, g \in \mathscr{F}$, we have $f = g$ if and only if $f(x) = f(g)$ for all $x \in (-\infty, \infty)$.

<u>Ex</u> $f(x) = x^2 \in \mathscr{F}$ $\quad g(x) = \sin(x) \in \mathscr{F}$ $\quad h(x) = \underline{\tan(x)} \notin \mathscr{F}$

domain is <u>not</u> all $x \in (-\infty, \infty)$

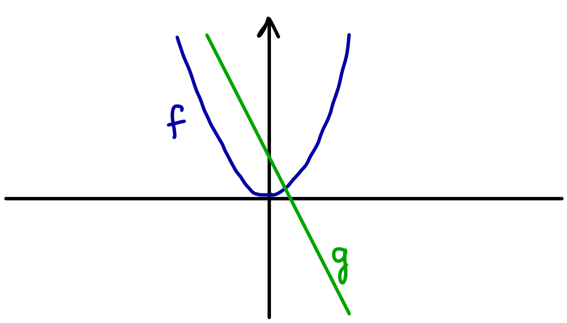
We can add two functions together:

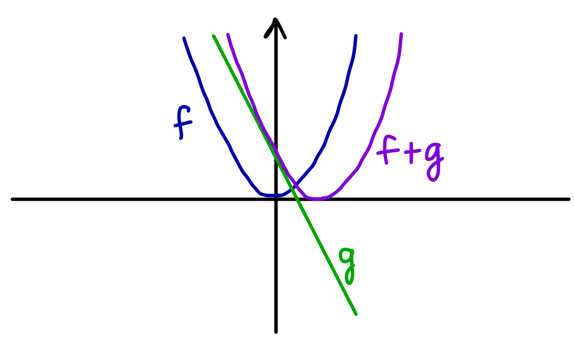
Let $f, g \in \mathscr{F}$ Then $f+g$ is the function defined by the rule

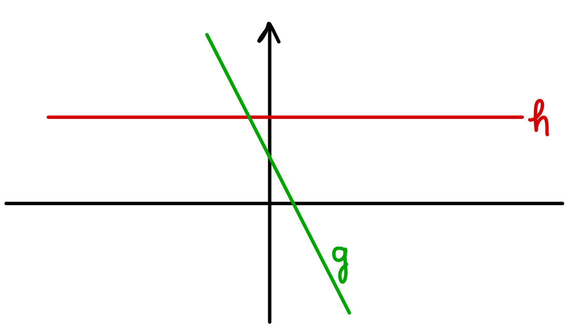
$$(f+g)(x) = f(x) + g(x) \quad \text{for all } x \in (-\infty, \infty)$$

graphically, the function $f+g$ has y-values equal to the sum of the y-values of f and g at each point $x \in (-\infty, \infty)$

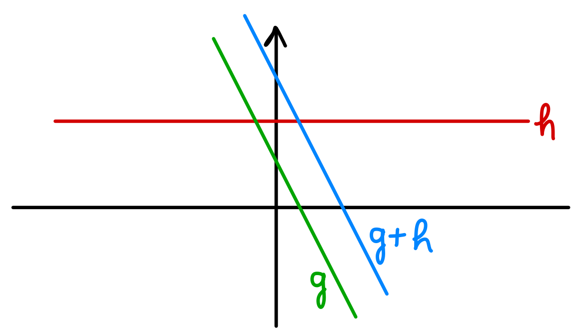
<u>Ex</u> $f(x) = x^2$, $g(x) = -2x + 1$, $h(x) = 2$ $\qquad (f+g)(x) = x^2 - 2x + 1 = (x-1)^2$







$(g+h)(x) = -2x + 1 + 2 = -2x + 3$



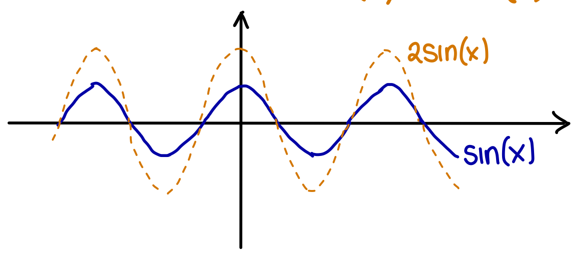
We can scale a function by a real scalar $k \in \mathbb{R}$:

Let $f \in \mathscr{F}$ and let $k \in \mathbb{R}$. Then kf is the function defined by the rule

$$(kf)(x) = k f(x) \quad \text{for all } x \in (-\infty, \infty)$$

Graphically, kf is obtained from f by vertically stretching by a factor of k

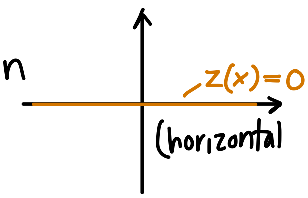
Ex $f(x) = \sin(x)$ $\quad k = 2$ $\qquad 2f(x) = 2\sin(x)$



Ex $z(x) = 0$ for all $x \in (-\infty, \infty)$ is in $\mathscr{F}$

the zero function

Graph of zero function



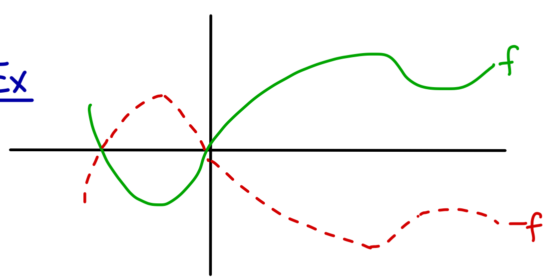
Then $\mathscr{F}$ is a vector space!

Let $f \in \mathscr{F}$. Then for all $x \in (-\infty, \infty)$,

$$\begin{aligned}(z+f)(x) &= z(x) + f(x)\\ &= 0 + f(x)\\ &= f(x) \quad \therefore\ z + f = f \qquad \text{Axiom } 2 \ \checkmark\end{aligned}$$

The graph of the negative of f is a reflection of f over the x-axis. Algebraically, $-f$ is the function defined by the rule $(-f)(x) = -f(x)$ for all $x \in (-\infty, \infty)$

Ex



$$\begin{aligned}(f + (-f))(x) &= f(x) + (-f(x))\\ &= 0\end{aligned}$$

$\therefore\ f + (-f) = z \qquad$ Axiom 4 $\checkmark$

Ex Let $c, d \in \mathbb{R}$, and let $f \in \mathscr{F}$. Then, for all $x \in (-\infty, \infty)$,

$$\begin{aligned}((c+d)f)(x) &= (c+d)f(x)\\ &= cf(x) + df(x)\end{aligned}$$

$\therefore\ (c+d)f = cf + df \qquad$ Axiom 8 $\checkmark$

Try the rest yourself!

Example 3.4. **The Trivial Vector Space** Let V be a set with only one element, which we denote by $\mathbf{0}$. That is, let $V = \{\mathbf{0}\}$.

We can use the following "rule" for addition (of the one and only vector in V):

$$\vec{0} + \vec{0} = \vec{0}$$

We also have the following "rule" for multiplication of a (the!) vector in V by a scalar $k \in \mathbb{R}$:

$$k\vec{0} = \vec{0}$$

Then $V = \{\mathbf{0}\}$ is a vector space!

(even though $V = \{\mathbf{0}\}$ and its operations seem a bit degenerate, V nonetheless satisfies the axioms!)

Example 3.5. Consider the set $W = \{\} = \emptyset$ (the empty set). Is W a vector space?

No! The empty set doesn't contain a zero vector Axiom 3 $\times$

Example 3.6. Consider the set $L = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : y = 2x \right\}$.

Using ordinary vector addition and scalar multiplication from $\mathbb{R}^2$, is L a vector space?

Let $\underset{\vec{u}}{\begin{bmatrix} a \\ b \end{bmatrix}}, \underset{\vec{v}}{\begin{bmatrix} c \\ d \end{bmatrix}}, \underset{\vec{w}}{\begin{bmatrix} x \\ y \end{bmatrix}} \in L$ and let $k, \ell \in \mathbb{R}$

$\therefore\ b = 2a,\ d = 2c,\ y = 2x$

1 $\vec{u} + \vec{v} = \begin{bmatrix} a + c \\ b + d \end{bmatrix} = \begin{bmatrix} a + c \\ 2a + 2c \end{bmatrix} = \underbrace{\begin{bmatrix} a + c \\ 2(a+c) \end{bmatrix}}_{\text{this is an element of the set } L}$

$\vec{u}, \vec{v} \in L \Longrightarrow \vec{u} + \vec{v} \in L$ $\therefore$ L is closed under addition

2 $k\vec{u} = \begin{bmatrix} ka \\ kb \end{bmatrix} = \begin{bmatrix} ka \\ k(2a) \end{bmatrix} = \underbrace{\begin{bmatrix} ka \\ 2(ka) \end{bmatrix}}_{\text{this is an element of the set } L}$

$\vec{u} \in L, k \in \mathbb{R} \Longrightarrow k\vec{u} \in L$ $\therefore$ L is closed under scalar multiplication

3 $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 2(0) \end{bmatrix}}_{\text{this is an element of the set } L}$ Just like in $\mathbb{R}^2$, $\vec{0} + \vec{u} = \vec{u}$ for all $\vec{u} \in L$

4 $-\vec{u} = \begin{bmatrix} -a \\ -b \end{bmatrix} = \begin{bmatrix} -a \\ -(2a) \end{bmatrix} = \underbrace{\begin{bmatrix} -a \\ 2(-a) \end{bmatrix}}_{\text{this is an element of the set } L}$ Just like in $\mathbb{R}^2$, $\vec{u} + (-\vec{u}) = \vec{0}$ for all $\vec{u} \in L$

(5) Let $\vec{u}, \vec{v} \in L$ Then $\vec{u}+\vec{v} = \vec{v}+\vec{u}$

Why? Because this is already true for all vectors in $\mathbb{R}^2$
$\therefore$ It's true for all vectors in $\underbrace{L \subseteq \mathbb{R}^2}_{L \text{ is a subset of } \mathbb{R}^2}$

(6) Let $\vec{u}, \vec{v}, \vec{w} \in L$ Then $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$

(7) Let $\vec{u}, \vec{v} \in L$, $k \in \mathbb{R}$ Then $k(\vec{u}+\vec{v}) = k\vec{u}+k\vec{v}$

(8) Let $\vec{u} \in L$, $k, \ell \in \mathbb{R}$ Then $(k+\ell)\vec{u} = k\vec{u}+\ell\vec{u}$

(9) Let $\vec{u} \in L$, $k, \ell \in \mathbb{R}$ Then $k(\ell\vec{u}) = (k\ell)\vec{u}$

(10) Let $\vec{u} \in L$ Then $1\vec{u} = \vec{u}$

Why? Because this is already true for all vectors in $\mathbb{R}^2$
$\therefore$ It's true for all vectors in $\underbrace{L \subseteq \mathbb{R}^2}_{L \text{ is a subset of } \mathbb{R}^2}$

$\therefore$ L is a vector space!

Example 3.7. Let $W = \left\{ \begin{bmatrix} x \\ x+2 \end{bmatrix} : x \in \mathbb{R} \right\}$. Using ordinary vector addition and scalar multiplication from $\mathbb{R}^2$, is W a vector space?

Let $\vec{u}, \vec{v} \in W$, $k \in \mathbb{R}$

Then $\vec{u} = \begin{bmatrix} a \\ a+2 \end{bmatrix}, \vec{v} = \begin{bmatrix} b \\ b+2 \end{bmatrix}$ for some $a, b \in \mathbb{R}$ (def of W)

$\vec{u}+\vec{v} = \begin{bmatrix} a+b \\ (a+2)+(b+2) \end{bmatrix} = \begin{bmatrix} a+b \\ (a+b)+4 \end{bmatrix} \notin W$ $\therefore$ W is not closed under addition

$\cancel{1}$

$\therefore$ W is not a vector space

Likewise, W is not closed under scalar multiplication $\cancel{2}$

$\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin W$ since $\begin{bmatrix} a \\ a+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} a=0 \\ a=-2 \end{cases}$ ↯ no solution $\cancel{3}$

For $\vec{u} \in W$, $-\vec{u} \notin W$ $-\vec{u} = \begin{bmatrix} -a \\ -(a+2) \end{bmatrix} = \begin{bmatrix} -a \\ -a-2 \end{bmatrix}$ $\cancel{4}$

But, W does satisfy Axioms (5) − (10) since it inherits the arithmetic properties of $\mathbb{R}^2$

4. Subspaces

We now know the 10 axioms for a vector space and we showed that there is a wide variety of vector spaces other than $\mathbb{R}^n$. On the one hand, all vector spaces are instances of the same thing (because they all have operations that satisfy the same axioms). On the other hand, each vector space can contain fundamentally different types of "vectors" (e.g. functions in the vector space of functions can't be visualized as arrows the way that vectors in $\mathbb{R}^3$).

Now, given a vector space, we will investigate its most closely-related vector spaces: that is, we will look at the vector spaces that "live inside" other bigger vector spaces.

Subspaces

Definition 4.1. Let V be a vector space, equipped with its rules for vector addition (+) and scalar multiplication (·).

A set U is called a **Subspace of V** if

- $\underbrace{U \subseteq V}_{U \text{ is a subset of } V}$ (ie for all x, $x \in U \Rightarrow x \in V$)

and

- equipped with the same vector addition and scalar multiplication operations as V, U is a vector space in its own right

In other words, U is a subspace of V if all of the vectors in U already belong to the vector space V, and the operations for U are inherited from V, and with V's operations, the set U satisfies all 10 vector space axioms.

Let's revisit an example we previously looked at:

Example 4.2. $L = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : y = 2x \right\}$, equipped with the operations of $\mathbb{R}^2$.

We showed L satisfies closure axioms and existence axioms

We noticed that L satisfies the Arithmetic Properties because those properties hold for <u>all</u> vectors in $\mathbb{R}^2$, including the vectors in L

So L itself is a vector space that "lives inside" $\mathbb{R}^2$

<u>ie</u> L is a subspace of $\mathbb{R}^2$!

†These notes are solely for the personal use of students registered in MAT1341.

Suppose U is a subset of a vector space V, equipped with the same operations as V. If we want to check whether U is a subspace of V, we can verify all 10 axioms.

However, U "inherits" the arithmetic properties from V.

What do we mean by this? Well, we don't really have to check those 6 axioms for U if we already know they hold for *all* vectors in V. Why? Because every vector in U is a vector in V. We already know V is a vector space. So V and its operations definitely satisfy those axioms.

What about the first 4 axioms?

Knowing that V satisfies those 4 axioms does not guarantee that U will satisfy them.

Recall the following example:

Example 4.3. $W = \left\{ \begin{bmatrix} x \\ x+2 \end{bmatrix} : x \in \mathbb{R} \right\}$ equipped with the operations of $\mathbb{R}^2$.

W is a subset of $\mathbb{R}^2$ ✓

W is equipped with the operations of $\mathbb{R}^2$ ✓

But W is not a vector space itself

Closure Existence Arithmetic

Axioms for W ~~1~~ ~~2~~ ~~3~~ ~~4~~ 5 6 7 8 9 10 ✓

$\therefore$ W is not a subspace of $\mathbb{R}^2$

Observations:

- Adding two vectors in W does not always result in a vector in W.
- Adding two vectors in W does always result in a vector in $\mathbb{R}^2$ because $\mathbb{R}^2$ *is* closed under vector addition.
- Multiplying a vector in W by a scalar $k \in \mathbb{R}$ does not always result in a vector in W.
- Multiplying a vector in W by a scalar $k \in \mathbb{R}$ does always result in a vector in $\mathbb{R}^2$ because $\mathbb{R}^2$ *is* closed under scalar multiplication.
- The zero vector $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is not in W.
- For all $\mathbf{w} \in W$, we still have $\mathbf{0} + \mathbf{w} = \mathbf{w}$ because this holds for all vectors in $\mathbb{R}^2$ which includes all vectors $\mathbf{w} \in W$.
- It is not the case that for every $\mathbf{w} \in W$, there exists $-\mathbf{w} \in W$ such that $\mathbf{w} + (-\mathbf{w}) = \mathbf{0}$.
- But, for every $\mathbf{w} \in W$, there exists $-\mathbf{w} \in \mathbb{R}^2$ such that $\mathbf{w} + (-\mathbf{w}) = \mathbf{0}$.
- All vectors in $\mathbb{R}^2$ satisfy the Arithmetic Axioms. Since all vectors of W are vectors in $\mathbb{R}^2$, the vectors of W satisfy the Arithmetic Axioms too.

Suppose V is a vector space, and U is a subset of V equipped with the operations of V. If we want to find out whether U is a subspace of V, we need only worry about checking Axioms 1–4.

In fact, the next results tell us (among other things) that for $U \subseteq V$, we don't need to worry about Axiom 4 either, so long as the first three axioms are satisfied.

Proposition 4.4. Let V be a vector space. Let $a \in \mathbb{R}$ and let $\mathbf{u} \in V$. Then

(a) $a\vec{0} = \vec{0}$

(b) If $a\vec{u} = \vec{0}$, then $a = 0$ or $\vec{u} = \vec{0}$

(c) $0\vec{u} = \vec{0}$

(d) $(-1)\vec{u} = -\vec{u}$ (proof in DGD 2)

Proposition 4.5. Let U be a subset of a vector space V, equipped with the operations of V.

If U is closed under scalar multiplication, then U satisfies Axiom 4 (Existence of Vector Negatives).

Proof:

- Assume U is a subset of a vector space V.
- Assume U is equipped with the same operations as V.
- Assume U is closed under scalar multiplication.

Let $\vec{w} \in W$ Then $(-1)\vec{w} \in W$ (since W is closed under scalar multiplication)

By Prop 49(d), $-\vec{w} = (-1)\vec{w}$ $\therefore -\vec{w} \in W$

The Subspace Test

Now, given a subset U of a vector space V, where U is equipped with the operations of V, if we want to know whether U is a subspace of V, we have an efficient test, summarized by the following theorem:

Theorem 4.6. **The Subspace Test** Let U be a subset of a vector space V, equipped with the same operations as V.

Then W is a subspace of V if and only if the following 3 conditions hold:

1 W is closed under vector addition, i.e., $\vec{u}, \vec{v} \in W \Rightarrow \vec{u} + \vec{v} \in W$

2 W is closed under scalar multiplication, i.e., $k \in \mathbb{R}$ and $\vec{u} \in W \Rightarrow k\vec{u} \in W$

3 $\vec{0} \in W$

Example 4.7. Let $T = \{\mathbf{u} \in \mathbb{R}^3 : \mathbf{u} \cdot (1,2,3) = 0\}$, with the usual operations of $\mathbb{R}^3$. Is T a subspace of $\mathbb{R}^3$?

First, T is a subset of $\mathbb{R}^3$ equipped with the operations of $\mathbb{R}^3$

$\Longrightarrow$ we can attempt the subspace test

More specifically,

$$T = \{(x,y,z) \in \mathbb{R}^3 : (x,y,z)\cdot(1,2,3) = 0\} = \{(x,y,z) \in \mathbb{R}^3 : \underbrace{x+2y+3z=0}_{T \text{ is a plane passing through the origin}}\}$$

Onto the Subspace Test!

① Let $\vec{u}, \vec{v} \in T$ (goal show $\vec{u}+\vec{v} \in T$)

Then $\vec{u}\cdot(1,2,3) \overset{①}{=} 0$ and $\vec{v}\cdot(1,2,3) \overset{②}{=} 0$ (by def of T)

$$\begin{aligned} \text{Now, } (\vec{u}+\vec{v})\cdot(1,2,3) &= \vec{u}\cdot(1,2,3) + \vec{v}\cdot(1,2,3) && \text{(property of dot product)} \\ &= 0 + 0 && \text{(by ① and ②)} \\ &= 0 \end{aligned}$$

$\Longrightarrow \vec{u}+\vec{v} \in T$ (goal!) (by def of T) $\therefore$ T is closed under vector addition ✓

② Let $k \in \mathbb{R}$ and $\vec{u} \in T$ (goal show $k\vec{u} \in T$)

$$\begin{aligned} \text{Now } (k\vec{u})\cdot(1,2,3) &= k(u\cdot(1,2,3)) && \text{(property of dot product)} \\ &= k\,0 && (\text{since } \vec{u}\,(1,2,3) = 0 \text{ by def of } T) \\ &= 0 \end{aligned}$$

$\Longrightarrow k\vec{u} \in T$ (goal!) $\therefore$ T is closed under scalar multiplication ✓

③ (goal show $\vec{0} \in T$)

Since $\vec{0}\cdot(1,2,3) = (0,0,0)\cdot(1,2,3) = 0$, we conclude that $\vec{0} \in T$ (goal!) (by def of T)

Conclusion T is a subspace of $\mathbb{R}^3$!

Remark 4.8. Let $(a,b,c) \in \mathbb{R}^3$ be any nonzero vector. Consider the plane

$$P = \{(x,y,z) \in \mathbb{R}^3 : (x,y,z)\cdot(a,b,c) = 0\} = \{(x,y,z) \in \mathbb{R}^3 : ax+by+cz = 0\}.$$

Just as in the previous example, the subspace test shows that P is a subspace of $\mathbb{R}^3$.

Conclusion: Any plane passing through the origin in $\mathbb{R}^3$ is a subspace!

Example 4.9. POLYNOMIALS Let $\mathbb{P}$ denote the set of all polynomial functions (in variable x). Each element in $\mathbb{P}$ is of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

for some $n \in \mathbb{Z}$, $n \geq 0$ and some $a_0, a_1, \ldots, a_n \in \mathbb{R}$ (the "coefficients").

Is $\mathbb{P}$ a vector space?

First, observe that $\mathbb{P} \subseteq \mathcal{F}$, the space of real-valued functions with domain $(-\infty, \infty)$

We add and scalar-multiply polynomials just like any functions

For $p(x), q(x) \in \mathbb{P}$ and $k \in \mathbb{R}$, $p+q$ and kp are defined by the rules

$(p+q)(x) = p(x) + q(x)$ and $(kp)(x) = kp(x)$ for all $x \in (-\infty, \infty)$

(i.e. $\mathbb{P}$ is equipped with the same operations as $\mathcal{F}$)

So we can use the Subspace Test on $\mathbb{P}$

1. If we add two polynomials together, the result is still a polynomial ✓
2. If we scale a polynomial by $k \in \mathbb{R}$, the result is still a polynomial ✓
3. The zero function $z(x) = 0$ is a polynomial, so $z \in \mathbb{P}$ ✓

$\therefore$ $\mathbb{P}$ is a subspace of $\mathcal{F}$, hence $\mathbb{P}$ is itself a vector space!

Example 4.10. Is the set $S = \{p \in \mathbb{P} : p(5) = 1\}$ a subspace of $\mathbb{P}$ (when equipped with the usual operations for adding and scaling functions)?

We can apply the Subspace Test!

1. Assume $p, q \in S$ Then $p(5) = 1$ and $q(5) = 1$ (def of S)

Now $p+q$ satisfies $(p+q)(5) = p(5) + q(5) = 1 + 1 = 2 \neq 1$

$\Rightarrow p+q \notin S$ $\therefore$ S is not closed under addition

$\therefore$ S is not a subspace of $\mathbb{P}$

Example 4.11. Is the set $T = \{p \in \mathbb{P} : p(3) = 0\}$ a subspace of $\mathbb{P}$ (when equipped with the usual operations for adding and scaling functions)?

1) Assume $p, q \in T$ Then $p(3)=0$ and $q(3)=0$ (def of T)

Now, $(p+q)(3) = p(3)+q(3) = 0+0 = 0$ $\therefore p+q \in T$ closed under $+$ ✓

2) Assume $p \in T$ and $k \in \mathbb{R}$ Then $p(3)=0$ (def of T)

Now $(kp)(3) = k\,p(3) = k\,0 = 0$ $\therefore kp \in T$ closed under scalar mult ✓

3) The zero polynomial satisfies $z(x)=0$ for $\underline{\underline{\text{all}}}$ $x \in \mathbb{R}$

$\therefore z(3)=0$ $\therefore z \in T$ contains zero vector ✓

$\therefore T$ is a subspace of $\mathbb{P}$!

Example 4.12. **Matrices** Let $m, n \in \mathbb{Z}$ such that $m, n \geq 1$. Let $M_{m,n}(\mathbb{R})$ denote the set of $m \times n$ matrices with real entries.

We can add two matrices of the same size (by adding their entries together):

Note a_{ij} denotes the entry of A in the ith row and jth column

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}}_{A+B}$$

We can multiply a matrix by a scalar $k \in \mathbb{R}$ (by scaling every entry by k):

$$k\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & & & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix}}_{kA}$$

Exercise! Convince yourself that $M_{m,n}(\mathbb{R})$ is a vector space (consider all 10 axioms).

Definition 4.13. Let $A \in M_{m,n}(\mathbb{R})$ be a matrix. The **Transpose of A**, denoted A^T, is the $n \times m$ matrix whose rows are the columns of A.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{mn} \end{bmatrix}_{n\times m}$$

ie $(A^T)_{i,j} = (A)_{j,i}$

Ex $A = \begin{bmatrix} 4 & 3 & -1 \\ 0 & 1 & \pi \end{bmatrix}_{2\times 3}$ $\quad A^T = \begin{bmatrix} 4 & 0 \\ 3 & 1 \\ -1 & \pi \end{bmatrix}_{3\times 2}$

Example 4.14. Let $S = \{A \in M_{2,2}(\mathbb{R}) : A = A^T\}$. Is S a subspace of $M_{2,2}(\mathbb{R})$?

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2,2}(\mathbb{R})$ Then $A = A^T \iff \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \iff b = c$

Thus, $S = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} : a, b, d \in \mathbb{R} \right\}$ Ex $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \in S$

(1) Let $A, B \in S$ Then $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, $B = \begin{bmatrix} d & e \\ e & f \end{bmatrix}$ for some a,b,c,d,e,f

Now $A+B = \begin{bmatrix} a+d & b+e \\ b+e & c+f \end{bmatrix}$ satisfies $(A+B) = (A+B)^T$

$\therefore A+B \in S$ $\quad$ closed under addition ✓

(2) Let $A \in S$ and $k \in \mathbb{R}$ Then $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ for some $a,b,c \in \mathbb{R}$

Now $kA = \begin{bmatrix} ka & kb \\ kb & kc \end{bmatrix}$ satisfies $(kA) = (kA)^T$

$\therefore kA \in S$ $\quad$ closed under scalar mult ✓

(3) The zero matrix of $M_{2,2}(\mathbb{R})$ is $Z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\leftarrow$ check that vector space Axiom 3 for $M_{2,2}(\mathbb{R})$ is satisfied with $Z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Z satisfies $Z = Z^T$ $\therefore Z \in S$ $\quad$ contains zero vector ✓

S is a subspace of $M_{2,2}(\mathbb{R})$!

5. Span

All vector spaces (with real scalars) are infinite (with one exception: the trivial vector space $V = \{\mathbf{0}\}$ is finite).

Now, we explore a way to describe an infinite vector space using a finite number of vectors, by considering the concept of "span".

DESCRIBING VECTOR SPACES

Example 5.1. Consider the following two subspaces of $\mathbb{R}^3$: **EXERCISE!** Verify that X and Y are subspaces of $\mathbb{R}^3$.

$$U = \left\{ \begin{bmatrix} t-3s \\ t \\ s \end{bmatrix} : s,t \in \mathbb{R} \right\} \qquad W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x - y + 3z = 0 \right\}$$

U's description is of the form

{elements defined with parameters   parameters can be any real numbers}

this way makes constructing elements of U easy just plug in some parameter values and output is an element of U

ex set s=1, t=-5

$$\Rightarrow \begin{bmatrix} -5-3(1) \\ -5 \\ 1 \end{bmatrix} \in U$$

W's description is of the form

{elements from some set   required conditions}

this way makes checking whether a given element belongs to W easy just check whether the condition is satisfied

ex Is $\begin{bmatrix} -8 \\ -5 \\ 1 \end{bmatrix} \in W$? Test condition

$(-8)-(-5)+3(1)=0$ ✓

Yes!

In fact, U = W Let's show they're equal

$$\underbrace{\left\{ \begin{bmatrix} t-3s \\ t \\ s \end{bmatrix} : s,t \in \mathbb{R} \right\}}_{U} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : \begin{matrix} x=t-3s \\ y=t \\ z=s \end{matrix} \right\} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x=y-3z \right\} = \underbrace{\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x-y+3z=0 \right\}}_{W}$$

We can go further

$$U = \left\{ \begin{bmatrix} t-3s \\ t \\ s \end{bmatrix} : s,t \in \mathbb{R} \right\} = \left\{ \underbrace{t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}} : s,t \in \mathbb{R} \right\}$$

this description shows that each vector in U is a linear combination of two vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

†These notes are solely for the personal use of students registered in MAT1341.

Definition 5.2. Let V be a vector space and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be vectors in V.

- A vector $\mathbf{w}$ is called a **LINEAR COMBINATION OF** $\mathbf{v}_1, \ldots, \mathbf{v}_m$ if

  there exist scalars $a_1, a_2, \ldots, a_m \in \mathbb{R}$ such that $\vec{w} = a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_m\vec{v}_m$

- The **SPAN OF** $\mathbf{v}_1, \ldots, \mathbf{v}_m$, denoted $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$, is

  the set of <u>all</u> linear combinations of $\vec{v}_1, \ldots, \vec{v}_m$

  That is, $\operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_m\} = \{a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_m\vec{v}_m : a_1, \ldots, a_m \in \mathbb{R}\}$

- Suppose $S = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$.

  Then the set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is called a **SPANNING SET** for S.

  We also say that "S is **SPANNED BY** the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$".

  Or we sometimes say "the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ **SPAN** S".

Return to the previous example:

$$U = \left\{ \begin{bmatrix} t-3s \\ t \\ s \end{bmatrix} : s,t \in \mathbb{R} \right\} = \left\{ t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} : s,t \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a spanning set for U

- U is spanned by $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

- The vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ span U

Example 5.3. **SYMMETRIC 2×2 REAL MATRICES** Let $S = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} : a, b, d \in \mathbb{R} \right\}$. Find a spanning set for S.

$$S = \left\{ a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} : a,b,d \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$\therefore \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a spanning set for S

Example 5.4. Let $\mathbb{P}_2$ denote the set of **POLYNOMIALS OF DEGREE AT MOST 2** . That is, let

$$\mathbb{P}_2 = \{ax^2 + bx + c : a, b, c \in \mathbb{R}\}$$

EXERCISE: Show that $\mathbb{P}_2$ is a vector space (hint: use the Subspace Test!)

Find a spanning set for $\mathbb{P}_2$.

$$\mathbb{P}^2 = \{ax^2+bx+c \;\; a,b,c \in \mathbb{R}\} = \{a\,x^2 + b\,x + c\,1 \;\; a,b,c \in \mathbb{R}\} = \operatorname{span}\{x^2, x, 1\}$$

each of these is a polynomial

Example 5.5. Now, let $U = \{p(x) \in \mathbb{P}_2 : p(1) = 0\}$. Find a spanning set for U.

elements from some set

required condition

$$U = \{p(x) \in \mathbb{P}_2 : p(1) = 0\}$$

$$= \{ax^2+bx+c : a,b,c \in \mathbb{R} \text{ and } p(1) = 0\}$$

$$= \{ax^2+bx+c : a,b,c \in \mathbb{R} \text{ and } a(1^2)+b(1)+c = 0\}$$

$$= \{ax^2+bx+c \;\; a,b,c \in \mathbb{R} \text{ and } a+b+c = 0\}$$

$$= \{ax^2+bx+c : a,b,c \in \mathbb{R} \text{ and } a = -b-c\}$$

put the condition into the vector parameters

$$= \{(-b-c)x^2+bx+c : b,c \in \mathbb{R}\}$$

$$= \{b(x-x^2) + c(1-x^2) : b,c \in \mathbb{R}\}$$

$$= \operatorname{span}\{x-x^2, 1-x^2\}$$

Every polynomial $p(x)$ of degree ≤ 2 such that $p(1)=0$ is a linear combination of polynomials $x-x^2$ and $1-x^2$

U is spanned by $\{x-x^2, 1-x^2\}$

<u>Ex</u> $q(x) = 4(x-x^2) - 5(1-x^2) = x^2 + 4x - 5 \in U$ Check $q(1) = 1^2 + 4(1) - 5 = 0$ ✓

so q satisfies condition for U

<u>Ex</u> $f(x) = 1 - x \in U$ since $f(1) = 0$

$\therefore 1-x \in \operatorname{span}\{x-x^2, 1-x^2\}$

Check find $a, b \in \mathbb{R}$ such that $f(x) = a(x-x^2) + b(1-x^2) \iff 1-x = a(x-x^2) + b(1-x^2)$

$$\iff 1-x = (-a-b)x^2 + ax + b$$

Compare coefficients $LS = 0x^2 - x + 1$ $RS = (-a-b)x^2 + ax + b$

$$\iff \left\{ \begin{array}{l} 0 = -a-b \\ -1 = a \\ 1 = b \end{array} \right\} \iff a = -1 \text{ and } b = 1$$

Check $(-1)(x-x^2) + (1)(1-x^2) = -x + 1 = 1 - x = f(x)$ ✓

So f is in this span

Example 5.6. Find the "condition" description for $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$.

vectors in $\mathbb{R}^4$ that are a linear combination of $\begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}$

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\} = \left\{ s\begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} s \\ -2s+t \\ t \\ 3s-t \end{bmatrix} : s, t \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4 : \begin{array}{l} x = s \\ y = -2s+t \\ z = t \\ w = 3s-t \end{array} \right\}$$

$$= \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4 \quad \begin{array}{l} y = -2x + z \\ w = 3x - z \end{array} \right\}$$

$$= \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4 : \begin{array}{l} -2x - y + z = 0 \\ 3x - z - w = 0 \end{array} \right\}$$

vectors in $\mathbb{R}^4$ that are orthogonal to both $\begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ -1 \\ -1 \end{bmatrix}$

Now we get to "the BIG THEOREM" about spans:

Theorem 5.7. **(THE BIG THEOREM ABOUT SPANS)** Let V be a vector space.
Let $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ be set of vectors in some vector space V. Then

$$\text{span}\{\vec{v}_1, \dots, \vec{v}_m\} \text{ is a subspace of } V$$

Proof:

Let $U = \text{span}\{\vec{v}_1, \dots, \vec{v}_m\}$ where $\vec{v}_1, \ \ , \vec{v}_m$ are vectors in a vector space V

By def of span, each vector in U is a linear combination $a_1\vec{v}_1 + .. + a_m\vec{v}_m$

Since V is closed under scalar multiplication, $a_i\vec{v}_i \in V$ for all $a_i \in \mathbb{R}$, for all i

Since V is closed under addition, $\underbrace{a_1\vec{v}_1}_{\in V} + .. + \underbrace{a_m\vec{v}_m}_{\in V} \in V$

$\therefore U \subseteq V$

Equip U with the operations of V and apply the Subspace Test

Let $\vec{u}, \vec{v} \in U$ and $k \in \mathbb{R}$

Then $\vec{u} = a_1\vec{v}_1 + \dots + a_m\vec{v}_m$ and $\vec{v} = b_1\vec{v}_1 + \dots + b_m\vec{v}_m$ for some $a_i, b_i \in \mathbb{R}$, $1 \le i \le m$

① $$\vec{u} + \vec{v} = (a_1\vec{v}_1 + \ \ + a_m\vec{v}_m) + (b_1\vec{v}_1 + \ \ + b_m\vec{v}_m)$$
$$= \underbrace{(a_1 + b_1)}_{\in \mathbb{R}}\vec{v}_1 + \ \ + \underbrace{(a_m + b_m)}_{\in \mathbb{R}}\vec{v}_m$$

$\therefore$ $\vec{u} + \vec{v}$ is a linear combination of $\vec{v}_1, \ \ , \vec{v}_m$, hence $\vec{u} + \vec{v} \in U = \text{span}\{\vec{v}_1, \ \ , \vec{v}_m\}$

② $$k\vec{u} = k(a_1\vec{v}_1 + \ \ + a_m\vec{v}_m) = \underbrace{(ka_1)}_{\in \mathbb{R}}\vec{v}_1 + \ \ + \underbrace{(ka_m)}_{\in \mathbb{R}}\vec{v}_m$$

$\therefore$ $k\vec{u}$ is a linear combination of $\vec{v}_1, \ \ , \vec{v}_m$, hence $k\vec{u} \in U = \text{span}\{\vec{v}_1, \ \ , \vec{v}_m\}$

③ $$\vec{0} = \underbrace{0}_{\in \mathbb{R}}\vec{v}_1 + \dots + \underbrace{0}_{\in \mathbb{R}}\vec{v}_m$$

$\therefore$ $\vec{0}$ is a linear combination of $\vec{v}_1, \ \ , \vec{v}_m$, hence $\vec{0} \in U = \text{span}\{\vec{v}_1, \ \ , \vec{v}_m\}$

Conclusion $U = \text{span}\{\vec{v}_1, \ \ , \vec{v}_m\}$ is a subspace of V !

Corollary 5.8. Let W be a subspace of a vector space V. Let $\mathbf{w}_1, \ldots, \mathbf{w}_m \in W$.

Then $\operatorname{span}\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ is a subspace of W.

- The above result is just repeating what the BIG THEOREM told us but where the vectors all live inside a subspace of V.
- Stated this way, however, we notice that it means spans don't cross subspace boundaries.
- In other words, if the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_m$ live inside a subspace W of V, then the span of those vectors is also entirely contained in the subspace W.
- Another way to put this is that $\operatorname{span}\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ is the SMALLEST subspace of V that contains the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_m$.

Example 5.9. Is $D = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$ a subspace of $M_{3,3}(\mathbb{R})$?

Instead of using the Subspace Test, we notice D is spanned by three matrices in $M_{3,3}(\mathbb{R})$

$$D = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

BIG THEOREM applies!

$\therefore$ Yes, D is a subspace of $M_{3,3}(\mathbb{R})$!

Example 5.10. Show that $S = \{A \in M_{2,2}(\mathbb{R}) : A^\top = -A\}$ is a subspace of $M_{2,2}(\mathbb{R})$.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^T = -A \iff \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} \iff \left\{ \begin{array}{l} a = -a \\ b = -c \\ c = -b \\ d = -d \end{array} \right\} \iff \left\{ \begin{array}{l} a = 0 \\ b = -c \\ d = 0 \end{array} \right\}$$

$$\therefore S = \left\{ \begin{bmatrix} 0 & -c \\ c & 0 \end{bmatrix} : c \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$$

$\therefore$ S is a subspace (by BIG THEOREM!)

Let U be an arbitrary subspace of $\mathbb{R}^n$. What are the possibilities for U ?

It might be that $U = \{\vec{0}\}$ (the trivial subspace)

If $U \neq \{\vec{0}\}$, then U must contain at least one non-zero vector, say $\vec{x}_1 \in U$, $\vec{x}_1 \neq \vec{0}$

Then $\text{span}\{\vec{x}_1\}$ is a subspace of U (by BIG THEOREM)

Maybe $U = \text{span}\{\vec{x}_1\}$ ← $\text{span}\{\vec{x}_1\}$ is vector parametric form for a line passing through the origin

If not, U must contain another vector $\vec{x}_2$ that doesn't belong to $\text{span}\{\vec{x}_1\}$

Then $\text{span}\{\vec{x}_1, \vec{x}_2\}$ is a subspace of U (by BIG THEOREM)

Maybe $U = \text{span}\{\vec{x}_1, \vec{x}_2\}$ ← $\text{span}\{\vec{x}_1, \vec{x}_2\}$ st $\vec{x}_2 \notin \text{span}\{\vec{x}_1\}$ is vector parametric form for a plane passing through the origin

If not, U must contain another vector $\vec{x}_3$ that doesn't belong to $\text{span}\{\vec{x}_1, \vec{x}_2\}$

Then $\text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ is a subspace of U (by BIG THEOREM)

Maybe $U = \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$

If not, ...

⋮

We run out of names but a subspace of $\mathbb{R}^n$ is

a point, line, plane, ... always passing through the origin

Example 5.11. Show that $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}3\\6\end{bmatrix}\right\}$.

$$\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right\} = \left\{a\begin{bmatrix}1\\2\end{bmatrix} + b\begin{bmatrix}0\\0\end{bmatrix} : a, b \in \mathbb{R}\right\} = \left\{a\begin{bmatrix}1\\2\end{bmatrix} : a \in \mathbb{R}\right\} = \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$$

$$\begin{bmatrix}1\\2\end{bmatrix} \in \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}, \quad \begin{bmatrix}2\\4\end{bmatrix} \in \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\} \text{ and } \begin{bmatrix}3\\6\end{bmatrix} \in \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$$

By BIG THEOREM, $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}3\\6\end{bmatrix}\right\}$ is a subspace, hence subset, of $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$

$$\Longrightarrow \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}3\\6\end{bmatrix}\right\} \subseteq \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$$

Likewise, $\begin{bmatrix}1\\2\end{bmatrix} \in \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}3\\6\end{bmatrix}\right\} \therefore \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\} \subseteq \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}3\\6\end{bmatrix}\right\}$

$$\therefore \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}\right\}$$

EXERCISE: Show that $\operatorname{span}\left\{\begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}\right\}$.

What do we notice from these examples?

- Having more vectors in a spanning set DOES NOT imply that the subspace they span is bigger.
- It's not easy to tell if two subspaces are equal just based on the spanning sets you're given.
- The same subspace can have many different spanning sets.

6. Linear Dependence and Independence

We've now seen examples where we can describe a subspace as being the span of a finite list of vectors.

The span of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ in some vector space V is simply the set of all linear combinations that can be created from the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$.

The BIG THEOREM tells us that the span of a set of vectors in V is always a subspace of V.

However, spanning sets are not unique and we've seen an example where different spanning sets contain "useless" extra vectors, even though the span the same subspace.

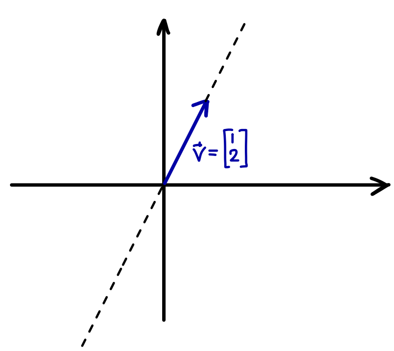
We now investigate this idea further via the concepts of linear dependence and linear independence.

DIFFERENT DIRECTIONS

We saw in Example 5.11 that $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}3\\6\end{bmatrix}\right\}$.

Since we're quite familiar with geometric vectors in $\mathbb{R}^2$, it is perhaps obvious why these spans are equal.

$\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\} = \left\{t\begin{bmatrix}1\\2\end{bmatrix} : t \in \mathbb{R}\right\} \leftarrow$ this corresponds to a line of slope 2 passing through the origin



Since $\begin{bmatrix}2\\4\end{bmatrix}$ and $\begin{bmatrix}3\\6\end{bmatrix}$ are parallel to $\begin{bmatrix}1\\2\end{bmatrix}$, they don't add any new directions to the span

†These notes are solely for the personal use of students registered in MAT1341.

Example 6.1. What if we add a new direction? What if our sets don't contain any obviously collinear vectors? Consider the following two spans:

$$U = \operatorname{span}\left\{ \underbrace{\begin{bmatrix}1\\2\end{bmatrix}}_{\vec{u}_1}, \underbrace{\begin{bmatrix}1\\1\end{bmatrix}}_{\vec{u}_2} \right\} \qquad W = \operatorname{span}\left\{ \underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{\vec{w}_1}, \underbrace{\begin{bmatrix}2\\1\end{bmatrix}}_{\vec{w}_2}, \underbrace{\begin{bmatrix}1\\3\end{bmatrix}}_{\vec{w}_3} \right\}$$

None of these vectors are scalar multiples of each other

Nonetheless, we'll show that $\vec{u}_1, \vec{u}_2 \in W$ and $\vec{w}_1, \vec{w}_2, \vec{w}_3 \in U$

① $\vec{u}_1 \in W \iff \begin{bmatrix}1\\2\end{bmatrix} = a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}2\\1\end{bmatrix} + c\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}a+2b+c\\b+3c\end{bmatrix}$ for some $a,b,c \in \mathbb{R}$

$\iff \begin{cases} a = 1-2b-c \\ b = 2-3c \end{cases}$ We are free to choose any value for c, say $c=0$

Then $\begin{matrix} a = 1-2b \\ b = 2 \end{matrix}$ so $\begin{matrix} a = 1-2(2) = -3 \\ b = 2 \end{matrix}$

So a solution exists! Check $\begin{bmatrix}1\\2\end{bmatrix} = -3\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}2\\1\end{bmatrix} + 0\begin{bmatrix}1\\3\end{bmatrix}$ ✓

so $\vec{u}_1 = (-3)\vec{w}_1 + 2\vec{w}_2 + 0\vec{w}_3$ $\therefore$ $\vec{u}_1 \in \operatorname{span}\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$

Similarly, you can show that $\vec{u}_2 = (-1)\vec{w}_1 + (1)\vec{w}_2 + 0\vec{w}_3$ $\quad \vec{w}_2 = (-1)\vec{u}_1 + 2\vec{u}_2$

$\vec{w}_1 = (-1)\vec{u}_1 + 2\vec{u}_2$ $\quad \vec{w}_3 = 2\vec{u}_1 + (-1)\vec{u}_2$

BIG THEOREM tells us $U = \operatorname{span}\{\vec{u}_1, \vec{u}_2\}$ is a subspace of W

and $W = \operatorname{span}\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is a subspace of U

In particular, $U \subseteq W$ and $W \subseteq U$ $\quad \therefore U = W$

- Both of the subspaces U and W are equal even though their spanning sets contain vectors that are not collinear.
- In fact, the spanning sets for U and W contain vectors that are coplanar (hence they actually span the same plane).
- The vectors $\mathbf{u}_1, \mathbf{u}_2$ share the same span as the vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$.
- We can describe all vectors in U as a linear combination of only 2 vectors.
- We can describe all vectors in W as a linear combination of 3 vectors.
- But since $U = W$, we prefer the description with fewer vectors (it's more efficient and describes the exact same subspace).

EXERCISE! Prove that $\mathbb{R}^2 = \operatorname{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$. Next, show that $\begin{bmatrix}1\\0\end{bmatrix} \in U$ and $\begin{bmatrix}0\\1\end{bmatrix} \in U$.

Finally, conclude that $U = \mathbb{R}^2 = W$ (hint: use the BIG THEOREM and the fact that $U = W$).

What is going on in the previous example? Although the vectors in each of the spanning sets are not collinear, they are coplanar (meaning they lie on the same plane, actually, the *only* plane in $\mathbb{R}^2$, which is all of $\mathbb{R}^2$ itself).

So adding more vectors that still lie on the same plane did not make the space they spanned any bigger.

The only way we could have created a "bigger" subspace from a spanning set such as U or W would be to add another vector to the spanning set that was not coplanar with the vectors in the original spanning set (but this is not possible in $\mathbb{R}^2$ since all vectors in $\mathbb{R}^2$ are coplanar).

But in other vector spaces, this may be possible...

There is a general concept, called **LINEAR INDEPENDENCE**, that captures the essence of what it means for a collection of vectors to be genuinely in different "directions" (even if the vectors are non-geometric and the notion of "direction" has no visual interpretation).

Before we get to the definition of linear independence, let's explore the opposite idea: let's generalize the concept of vectors being collinear, coplanar, etc...

Definition 6.2. Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be vectors in some vector space V.

We say that the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are **LINEARLY DEPENDENT** if and only if

there exist scalars $a_1, \,, a_m \in \mathbb{R}$ that are <u>not</u> all zero such that

$$\underbrace{a_1\vec{v}_1 + \ldots + a_m\vec{v}_m = \vec{0}}_{\text{this is called a dependence equation}}$$

$\vec{v}_1, \,, \vec{v}_m$ are LD $\Longleftrightarrow$ there exists a $\underbrace{\text{non-trivial}}_{\text{(not all zero)}}$ solution $a_1, \,, a_m$ to the dependence equation

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Example 6.3. Show that the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ are linearly dependent.

We must find $a_1, a_2 \in \mathbb{R}$ (not both zero) such that $a_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + a_2\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$a_1 = 3$ and $a_2 = -1$ work check $3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-1)\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

We can see $\vec{v}_1$ and $\vec{v}_2$ are collinear

The linear dependency equation confirms this algebraically

$3\vec{v}_1 + (-1)\vec{v}_2 = \vec{0} \implies \vec{v}_2 = 3\vec{v}_1$ So $\vec{v}_2$ is a scalar multiple of $\vec{v}_1$, hence collinear

Example 6.4. Are the vectors $\mathbf{v}_1 = \begin{bmatrix}1\\2\end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix}1\\1\end{bmatrix}$ linearly dependent?

$$a_1\vec{v}_1 + a_2\vec{v}_2 = \vec{0} \iff a_1\begin{bmatrix}1\\2\end{bmatrix} + a_2\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\iff \begin{cases} a_1 + a_2 = 0 \\ 2a_1 + a_2 = 0 \end{cases}$$

$$\iff a_1 = 0 \text{ and } a_2 = 0$$

The <u>only</u> solution to the dependency equation is the trivial solution

$\therefore$ $\vec{v}_1$ and $\vec{v}_2$ are <u>not</u> LD

(but we already noticed that they are 2 vectors that are <u>not</u> collinear)

Example 6.5. Are the vectors $\mathbf{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}$ linearly dependent?

$$a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = \vec{0} \iff a\begin{bmatrix}1\\0\\1\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix} + c\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\iff \begin{cases} a + c = 0 \\ b + c = 0 \\ a + c = 0 \end{cases}$$

$$\iff \begin{cases} a = -c \\ b = -c \end{cases} \quad \text{(and } c \text{ is free)}$$

<u>Ex</u> take $c = 1$, $a = -1$, $b = -1$ and we get $(-1)\begin{bmatrix}1\\0\\1\end{bmatrix} + (-1)\begin{bmatrix}0\\1\\0\end{bmatrix} + (1)\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$\therefore$ $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are LD

Now, when the answer is "the vectors are not linearly dependent", we will instead say "the vectors are linearly independent". But here is the precise definition:

Definition 6.6. Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be vectors in some vector space V.

We say that the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are **LINEARLY INDEPENDENT** if and only if

the <u>only</u> solution to the dependency equation

$$a_1\vec{v}_1 + \ldots + a_m\vec{v}_m = \vec{0}$$

is the trivial solution $a_1 = a_2 = \cdots = a_m = 0$

Example 6.7. Show that $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}} \in \mathbb{R}^3$ are linearly independent.

$$a\hat{i} + b\hat{j} + c\hat{k} = \vec{0} \iff a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix} + c\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \iff \begin{cases} a=0 \\ b=0 \\ c=0 \end{cases}$$

Since the <u>only</u> solution is the trivial solution, $\hat{i}, \hat{j}, \hat{k}$ are indeed LI

Example 6.8. We have $0\begin{bmatrix}1\\0\\1\end{bmatrix} + 0\begin{bmatrix}0\\1\\0\end{bmatrix} + 0\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$.

So the dependency equation has a trivial solution. Does this mean $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$ are LI?

No! The trivial solution <u>always</u> satisfies the dependency equation

Vectors are LI $\iff$ the <u>only</u> solution is the trivial solution

In this example, non-trivial solutions also exist

<u>Ex</u> $(-1)\vec{v}_1 + (-1)\vec{v}_2 + (1)\vec{v}_3 = \vec{0}$ (see Ex 6.5)

Example 6.9. In $M_{2,2}(\mathbb{R})$, are $\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}, \begin{bmatrix}1 & 2\\1 & 2\end{bmatrix}, \begin{bmatrix}2 & 3\\0 & 1\end{bmatrix}$ linearly independent?

$$a\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix} + b\begin{bmatrix}1 & 2\\1 & 2\end{bmatrix} + c\begin{bmatrix}2 & 3\\0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} \iff \begin{array}{ll} a+b+2c=0 & ① \\ a+2b+3c=0 & ② \\ a+b=0 & ③ \\ 2a+2b+c=0 & ④ \end{array}$$

①-② $2c=0 \therefore c=0$ → sub into ② $a+2b=0$ ⑤

⑤-③ $b=0$ → sub into ③ $a=0$

$\therefore$ only solution is trivial solution $a=b=c=0$

Conclusion these matrices are LI

Example 6.10. In the space $\mathscr{F}$ of real-valued functions with domain $\mathbb{R}$, are the functions $\sin(x), \cos(x)$ LI or LD ?

the zero function

$a\sin x + b\cos x = 0 \iff a\sin(x) + b\cos(x) = 0$ for all $x \in (-\infty, \infty)$

In particular, for $x=0$, $a\sin(0)+b\cos(0)=0 \Rightarrow b=0$

for $x=\frac{\pi}{2}$ $a\sin(\frac{\pi}{2})+b\cos(\frac{\pi}{2})=0 \Rightarrow a=0$

only solution is trivial solution

Conclusion $\sin(x)$ and $\cos(x)$ are LI

Example 6.11. In the space $\mathscr{F}$ of real-valued functions with domain $\mathbb{R}$, show that the functions $\sin^2(x),\ \cos^2(x),\ 1$ are LD.

1 is the constant function $f(x)=1$ for all $x\in\mathbb{R}$

Since $\cos^2 x + \sin^2 x = 1$ we have $1\cos^2 x + 1\sin^2 x + (-1)\,1 = 0$

non-trivial solution exists

$\therefore$ LD

FACTS ABOUT LINEAR INDEPENDENCE AND LINEAR DEPENDENCE

Here we collect some useful facts about linear independence and dependence.

Theorem 6.12. Let V be a vector space.

Fact 1: If $\mathbf{v}\in V$, then $\{\mathbf{v}\}$ is LI if and only if $\mathbf{v}\neq\mathbf{0}$.

Fact 2: If $\mathbf{v}_1,\ldots,\mathbf{v}_m$ are LD, then any set containing $\mathbf{v}_1,\ldots,\mathbf{v}_m$ is also LD.

Fact 3: If $\mathbf{v}_1,\ldots,\mathbf{v}_m$ are LI, then any subset of $\{\mathbf{v}_1,\ldots,\mathbf{v}_m\}$ is also LI.

Fact 4: $\{\mathbf{0}\}$ is LD.

Fact 5: Any set of vectors that contains $\mathbf{0}$ is LD.

Fact 6: Vectors $\mathbf{u},\mathbf{v}\in V$ are LD if and only if one of the vectors is a scalar multiple of the other.

Fact 7: A set with three or more vectors can be LD even though no two vectors are multiples of one another.

Fact 8: $\{\mathbf{v}_1,\ldots,\mathbf{v}_m\}$ is LD if and only if there is at least one vector $\mathbf{v}_i\in S$ such that $\mathbf{v}_i\in\operatorname{span}\{\mathbf{v}_1,\ldots,\mathbf{v}_{i-1},\mathbf{v}_{i+1},\ldots,\mathbf{v}_m\}$.

Proof of Fact 1 ($\Longrightarrow$) Assume $\vec{v}=\vec{0}$

Then $1\vec{v}=1\vec{0}=\vec{0}$ so there is a nontrivial solution to $a\vec{v}=\vec{0}$ $\therefore \{\vec{v}\}$ is LD

($\Longleftarrow$) Assume $\vec{v}\neq\vec{0}$ Then $a\vec{v}=\vec{0} \Rightarrow a=0$ or $\vec{v}=\vec{0}$ But $\vec{v}\neq\vec{0}$ $\therefore a=0$

proof of Fact 2 Assume $\vec{v}_1, \ , \vec{v}_m$ is LD

Then there is a nontrivial solution $a_1\vec{v}_1+..+a_m\vec{v}_m=\vec{0}$ (at least one $a_1\neq 0$)

For a bigger set of vectors $\{\vec{v}_1, \ ,\vec{v}_m\}\cup\{\vec{u}_1, \ ,\vec{u}_k\}$ we have

$a_1\vec{v}_1 + \ + a_2\vec{v}_2 + \ + a_m\vec{v}_m + 0\vec{u}_1 + \ + 0\vec{u}_k = \vec{0}$

with at least $a_1\neq 0$ $\therefore \vec{v}_1, \ ,\vec{v}_m,\vec{u}_1, \ ,\vec{u}_k$ is also LD

Fact 3 is logically equivalent to fact 2

Proof of Fact 4 $\quad 1\vec{0}=\vec{0}\,,\ 1\neq 0 \quad \therefore \{\vec{0}\}$ is LD

Fact 5 follows from Facts 4 and 2

Proof of Fact 6 $\quad (\Longrightarrow)\ \{\vec{u},\vec{v}\}$ is LD $\Longrightarrow a\vec{u}+b\vec{v}=\vec{0}$ where $a\neq 0$ or $b\neq 0$

If $a\neq 0$, then $\vec{u} = -\frac{b}{a}\vec{v}$

If $b\neq 0$, then $\vec{v} = -\frac{a}{b}\vec{u}$

$(\Longleftarrow)$ Assume $\vec{u}=k\vec{v}$ Then $1\vec{u}+(-k)\vec{v}=\vec{0} \quad \therefore \{\vec{u},\vec{v}\}$ is LD

Assume $\vec{v}=c\vec{u}$ Then $(-c)\vec{u}+1\vec{v}=\vec{0} \quad \therefore \{\vec{u},\vec{v}\}$ is LD

Justification of Fact 7 $\quad$ See Ex 65

Proof of Fact 8 $\quad (\Longrightarrow)$ Assume $\{\vec{v}_1, \;\; ,\vec{v}_m\}$ is LD

Then we have $a_1\vec{v}_1+\ldots+a_i\vec{v}_i+\ldots+a_m\vec{v}_m=\vec{0}$ with at least one $a_i\neq 0$

Then $v_i = -\frac{a_1}{a_i}\vec{v}_1 - \;\;\; - \frac{a_{i-1}}{a_i}\vec{v}_{i-1} - \frac{a_{i+1}}{a_i}\vec{v}_{i+1} + \ldots - \frac{a_m}{a_i}\vec{v}_m$

$\therefore \vec{v}_i \in \text{span}\{\vec{v}_1, \;\; ,\vec{v}_{i-1},\vec{v}_{i+1},\ldots,\vec{v}_m\}$

$(\Longleftarrow)$ Assume $\vec{v}_i \in \text{span}\{\vec{v}_1, \;\; ,\vec{v}_{i-1},\vec{v}_{i+1},\ldots,\vec{v}_m\}$

Then $\vec{v}_i = a_1\vec{v}_1 + \;\; + a_{i-1}\vec{v}_{i-1} + a_{i+1}\vec{v}_{i+1}+\ldots+a_m\vec{v}_m$ for some $a_1, \;\; ,a_{i-1},a_{i+1},\ldots,a_m\in\mathbb{R}$

Then $a_1\vec{v}_1 + \;\; + a_{i-1}\vec{v}_{i-1} + (-1)\vec{v}_i + a_{i+1}\vec{v}_{i+1} + \;\; + a_m\vec{v}_m = \vec{0}$

$-1\neq 0$ so this is a nontrivial dependency equation

$\therefore \{\vec{v}_1, \;\; ,\vec{v}_m\}$ is LD

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7. Linear Independence and Spanning Sets

Recall: Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be vectors in a vector space V.

The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is...

LI $\iff$ the *only* solution to the dependency equation

$$a_1\mathbf{v}_1 + \cdots + a_m\mathbf{v}_m = \mathbf{0}$$

is the *trivial solution* in which $a_i = 0$ for all i.

LD $\iff$ there exists a *non-trivial solution* to the dependency equation

$$a_1\mathbf{v}_1 + \cdots + a_m\mathbf{v}_m = \mathbf{0}$$

meaning this equation can be satisfied with *at least one nonzero scalar* $a_i \neq 0$.

a nontrivial solution to dependency equation is called a "dependence relation"

Aside from investigating several examples, we also gathered a list of important facts about LI and LD vectors in a vector space V:

Fact 1 $\{\mathbf{v}\}$ is LI $\iff$ $\mathbf{v} \neq \mathbf{0}$.

Fact 2 $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is LD $\implies$ any set containing $\mathbf{v}_1, \ldots, \mathbf{v}_m$ is also LD.

Fact 3 $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is LI $\implies$ any subset of $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is also LI.

Fact 4 $\{\mathbf{0}\}$ is LD.

Fact 5 $\mathbf{0} \in S$ $\implies$ S is LD.

Fact 6 $\{\mathbf{u}, \mathbf{v}\}$ is LD $\iff$ $\mathbf{u} = k\mathbf{v}$ or $\mathbf{v} = k\mathbf{u}$.

Fact 7 A set with three or more vectors can be LD even though no two vectors are multiples of one another.

Fact 8 $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is LD $\iff$ there is at least one vector $\mathbf{v}_i \in S$ such that $\mathbf{v}_i \in \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_m\}$.

A SET IS LD $\iff$ IT CONTAINS AT LEAST ONE VECTOR SPANNED BY THE OTHERS

Suppose you have $U = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$, where $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are vectors in some vector space V.

If the spanning set $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ for U is LD, then, in some sense, it contains some "useless" vectors that don't contribute anything new to the span.

Fact 8 gives us a mechanism to identify the "useless" vectors.

Let's revisit the idea behind the proof of Fact 8 using a concrete example.

†These notes are solely for the personal use of students registered in MAT1341.

Example 7.1. Let $S = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} -4 \\ -10 \end{bmatrix} \right\}$.

Fact 8 tells us S is LD $\iff$ S contains at least one vector spanned by the others.

($\Longrightarrow$)

- Show that S is LD.
- Then use the idea in the proof of the forward implication of Fact 8 to find a vector in S that belongs to the span of the others.

S is LD since $5\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 0\begin{bmatrix} -4 \\ -10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a dependence relation
(at least one $a_i \neq 0$)

$$\underset{a_1}{5}\vec{v}_1 + \underset{a_2}{1}\vec{v}_2 + \underset{a_3}{(-1)}\vec{v}_3 + \underset{a_4}{0}\vec{v}_4 = \vec{0}$$

Since $a_1 \neq 0$, we can isolate $\underbrace{\vec{v}_1 = -\frac{1}{5}\vec{v}_2 + \frac{1}{5}\vec{v}_3 + \frac{0}{5}\vec{v}_4}_{\vec{v}_1 \text{ is a linear combination of } \vec{v}_2, \vec{v}_3, \vec{v}_4}$ to conclude $\vec{v}_1 \in \text{span}\{\vec{v}_2, \vec{v}_3, \vec{v}_4\}$

($\Longleftarrow$)

- Conversely, show that one of the vectors in S belongs to the span of the other three vectors.
- Then use the idea in the proof of the converse of Fact 8 to show that S must be LD.

Conversely, $\underset{\vec{v}_3}{\begin{bmatrix} 2 \\ 5 \end{bmatrix}} = 5\underset{\vec{v}_1}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + 1\underset{\vec{v}_2}{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + 0\underset{\vec{v}_4}{\begin{bmatrix} -4 \\ -10 \end{bmatrix}}$ $\therefore$ $\vec{v}_3 \in \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$

Rewrite this to obtain a dependence relation $\vec{0} = 5\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \underset{a_2 \neq 0}{(-1)}\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 0 \end{bmatrix} + 0\begin{bmatrix} -4 \\ -10 \end{bmatrix}$

$\therefore$ S is LD

EXERCISE! Go back and reread the proof of Fact 8 to see if the proof in its full generality starts to make more sense after seeing a concrete example.

The next big theorem tells us that we can throw away each of the "useless" vectors in a spanning set (the ones that belong to the span of the other vectors) and the leftover vectors will still span the same subspace.

Theorem 7.2. **REDUCING SPANNING SETS** Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be vectors in a vector space V.

If $\vec{v}_1 \in \operatorname{span}\{\vec{v}_2, \ldots, \vec{v}_m\}$, then $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\} = \operatorname{span}\{\vec{v}_2, \ldots, \vec{v}_m\}$

$\vec{v}_1$ is in the span of the other vectors, so we can throw away $\vec{v}_1$ and still span the same subspace

Proof: Assume $\vec{v}_1 \in \operatorname{span}\{\vec{v}_2, \ldots, \vec{v}_m\}$ (goal prove $\operatorname{span}\{\vec{v}_1, \ \ ,\vec{v}_m\} = \operatorname{span}\{\vec{v}_2, \ \ ,\vec{v}_m\}$)

We also have

$\vec{v}_2 = 1\vec{v}_2 + 0\vec{v}_3 + \ \ + 0\vec{v}_m$ so $\vec{v}_2 \in \operatorname{span}\{\vec{v}_2, \ \ ,\vec{v}_m\}$

$\vdots$

$\vec{v}_m = 0\vec{v}_2 + 0\vec{v}_3 + \ \ + 1\vec{v}_m$ so $\vec{v}_m \in \operatorname{span}\{\vec{v}_2, \ \ ,\vec{v}_m\}$

Thus, $\vec{v}_1, \vec{v}_2, \ \ ,\vec{v}_m \in \operatorname{span}\{\vec{v}_2, \ \ ,\vec{v}_m\}$

By BIG THEOREM, $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \ \ ,\vec{v}_m\}$ is a subspace, hence subset, of $\operatorname{span}\{\vec{v}_2, \ \ ,\vec{v}_m\}$

$\therefore \operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\} \subseteq \operatorname{span}\{\vec{v}_2, \ldots, \vec{v}_m\}$ ①

Similarly, $\vec{v}_2, \ldots, \vec{v}_m \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ $\therefore \operatorname{span}\{\vec{v}_2, \ldots, \vec{v}_m\} \subseteq \operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ ②

The set inclusions ① and ② prove that $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\} = \operatorname{span}\{\vec{v}_2, \ldots, \vec{v}_m\}$

CONCLUSION: We can decrease the size of any LD spanning set and the smaller spanning set still spans just as much as the original spanning set

Example 7.3. Once again, consider $S = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} -4 \\ -10 \end{bmatrix} \right\}$. Find the smallest subset of S that spans the same subspace as $\operatorname{span} S$.

Since $\vec{v}_1 = -\frac{1}{5}\vec{v}_2 + \frac{1}{5}\vec{v}_3 + \frac{0}{5}\vec{v}_4$ (see Ex 7.1) we have $v_1 \in \operatorname{span}\{\vec{v}_2, \vec{v}_3, \vec{v}_4\}$

By Theorem 7.2, $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\} = \operatorname{span}\{\vec{v}_2, \vec{v}_3, \vec{v}_4\}$

(we can throw away $\vec{v}_1$ and still span same subspace)

Now, notice that $\underbrace{\begin{bmatrix} -4 \\ -10 \end{bmatrix}}_{\vec{v}_4} = 0\underbrace{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}_{\vec{v}_2} + (-2)\underbrace{\begin{bmatrix} 2 \\ 5 \end{bmatrix}}_{\vec{v}_3}$ $\therefore \vec{v}_4 \in \operatorname{span}\{\vec{v}_2, \vec{v}_3\}$

By Theorem 7.2, $\operatorname{span}\{\vec{v}_2, \vec{v}_3, \vec{v}_4\} = \operatorname{span}\{\vec{v}_2, \vec{v}_3\}$

(we can throw away $\vec{v}_4$ and still span same subspace)

Now $\left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$ is LI by Fact 6 (two vectors that are not scalar multiples of eachother)

now Theorem 7.2 no longer applies

We've reached a "minimal spanning set" that spans the same subspace as $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$

$$\operatorname{span}\{\vec{v}_2, \vec{v}_3\} = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$$

ENLARGING LINEARLY INDEPENDENT SETS

If we reduce a linearly DEPENDENT spanning set down to a linearly INDEPENDENT one, we will have reached a point where Theorem 7.2 no longer applies.

Now let's start with a linearly INDEPENDENT set and add as many vectors as we can to it, while maintaining linear INDEPENDENCE.

Our goal is to keep adding vectors to a LI set, until we get a maximal independent set that spans the whole vector space.

First, let's give a mathematically equivalent restatement of Fact 8:

Theorem 7.4.

Fact 8 $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is LI $\iff$ for all $\mathbf{v}_i \in S$ we have $\mathbf{v}_i \notin \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_m\}$.

Now we show that you can enlarge a linearly independent set, while preserving linear independence (until your set spans the entire vector space in which the vectors live).

Theorem 7.5. ENLARGING LINEARLY INDEPENDENT SETS

Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ be a linearly INDEPENDENT set of vectors in a vector space V.

Let $\mathbf{v}_{m+1} \in V$. Then

$\{\vec{v}_1, \ldots, \vec{v}_m, \vec{v}_{m+1}\}$ is LI $\iff$ $\vec{v}_{m+1} \notin \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_m\}$

we can add a new vector while preserving linear independence $\iff$ new vector is <u>not</u> in the span of the old vectors

Proof: Assume $\{\vec{v}_1, \ldots, \vec{v}_m\}$ is LI

($\Longrightarrow$) Assume $\{\vec{v}_1, \ldots, \vec{v}_m, \vec{v}_{m+1}\}$ is LI (goal prove $\vec{v}_{m+1} \notin \operatorname{span}\{\vec{v}_1, \ \vec{v}_m\}$)

Then, by Fact 8 (restated), $\vec{v}_{m+1} \notin \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_m\}$

Why?

If $\vec{v}_{m+1} \in \operatorname{span}\{\vec{v}_1, \ \vec{v}_m\}$ then $\{\vec{v}_1, \ \vec{v}_m, \vec{v}_{m+1}\}$ would be LD by Fact 8

Thus, $\vec{v}_{m+1} \notin \operatorname{span}\{\vec{v}_1, \ \vec{v}_{m+1}\}$, as claimed

($\Longleftarrow$) Assume $\vec{v}_{m+1} \notin \operatorname{span}\{\vec{v}_1, \ \vec{v}_m\}$ (goal show $\{\vec{v}_1, \ \vec{v}_m, \vec{v}_{m+1}\}$ is LI)

Suppose $a_1\vec{v}_1 + \ + a_m\vec{v}_m + a_{m+1}\vec{v}_{m+1} = \vec{0}$ ①

We claim that a_{m+1} must be 0

Why?

If $a_{m+1} \neq 0$, then $\vec{v}_{m+1} = -\frac{a_1}{a_{m+1}}\vec{v}_1 - \ - \frac{a_m}{a_{m+1}}\vec{v}_m \in \operatorname{span}\{\vec{v}_1, \ \vec{v}_m\}$

But this contradicts our assumption that $\vec{v}_{m+1} \notin \operatorname{span}\{\vec{v}_1, \ \vec{v}_m\}$

Since $a_{m+1} \neq 0$ leads to a contradiction, we conclude that $a_{m+1} = 0$

Now, ① becomes

$$\vec{0} = a_1\vec{v}_1 + \quad + a_m\vec{v}_m + a_{m+1}\vec{v}_{m+1} = a_1\vec{v}_1 + \quad + a_m\vec{v}_m + 0\vec{v}_{m+1} = a_1\vec{v}_1 + \quad + a_m\vec{v}_m$$

Since $\{\vec{v}_1, \quad ,\vec{v}_m\}$ is LI, only solution to dependency equation is trivial

so $a_1 = \quad = a_m = 0$

Since $a_1\vec{v}_1 + \quad + a_m\vec{v}_m + a_{m+1}\vec{v}_{m+1} = \vec{0} \Rightarrow a_1 = \quad = a_m = a_{m+1} = 0$,

we conclude that $\{\vec{v}_1, \quad ,\vec{v}_m, \vec{v}_{m+1}\}$ is LI

CONCLUSION:

We can increase the size of any LI set until it spans the entire vector space

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Example 7.6. The set $\left\{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right\}$ is LI. Enlarge it to a LI set with 3 elements.

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ are not scalar multiples of each other $\therefore$ they are LI (by Fact 6)

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \notin \text{span}\left\{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right\}$$

since elements in the span will always have zeros in second row

$$\therefore \left\{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\} \text{ is LI (by Theorem 75)}$$