8. Implicit Differentiation

Lec 7 mini review.

Two Special Limits: $\boxed{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1}$ $\boxed{\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}=0}$

Trig Rules:

$\boxed{\frac{d}{dx}[\sin x]=\cos x}$ $\boxed{\frac{d}{dx}[\cos x]=-\sin x}$ $\boxed{\frac{d}{dx}[\tan x]=\sec^2 x}$

$\boxed{\frac{d}{dx}[\csc x]=-\csc x\cot x}$ $\boxed{\frac{d}{dx}[\sec x]=\sec x\tan x}$ $\boxed{\frac{d}{dx}[\cot x]=-\csc^2 x}$

The Chain Rule: $\boxed{\frac{d}{dx}\Big[f\big(g(x)\big)\Big]=f'\big(g(x)\big)g'(x)}$

Power Chain Rule: $\boxed{\frac{d}{dx}\Big[\big(g(x)\big)^n\Big]=n\big(g(x)\big)^{n-1}g'(x)}$

Exponential Chain Rule: $\boxed{\frac{d}{dx}\Big[e^{g(x)}\Big]=e^{g(x)}g'(x)}$

WARM-UP FOR IMPLICIT DIFFERENTIATION

Differentiate each of the following expressions:

$f(x)=x^3g(x)+[h(x)]^5$

$$f'(x)=3x^2g(x)+x^3\cdot g'(x)+5[h(x)]^4\cdot h'(x)$$

$V(t)=\pi[R(t)]^2H(t)$

$$V'(t)=\pi(2)[R(t)]^1\cdot R'(t)\cdot H(t)+\pi[R(t)]^2\cdot H'(t)$$

$y=\left(\sqrt[3]{v(x)}\right)\Big[u(x)\Big]^{10}$

$$y'=\frac{1}{3}[v(x)]^{-2/3}\cdot v'(x)[u(x)]^{10}+\left(\sqrt[3]{v(x)}\right)\cdot(10)[u(x)]^9\cdot u'(x)$$

OR $$\frac{dy}{dx}=\frac{1}{3}v^{-2/3}\cdot\frac{dv}{dx}\cdot u^{10}+\left(\sqrt[3]{v}\right)\left(10u^9\cdot\frac{du}{dx}\right)$$

$$p(x) = e^{x^3} + e^{g(x)/h(x)} + \frac{h(x)}{\cos(x)} + \sin\left(f(x) + g(x)\right)$$

$$p'(x) = \left(e^{x^3}\right)(3x^2) + \left(e^{g(x)/h(x)}\right)\left[\frac{g'(x)h(x) - g(x)h'(x)}{[h(x)]^2}\right] + \cos\left(f(x)+g(x)\right)\left[f'(x) + g'(x)\right]$$

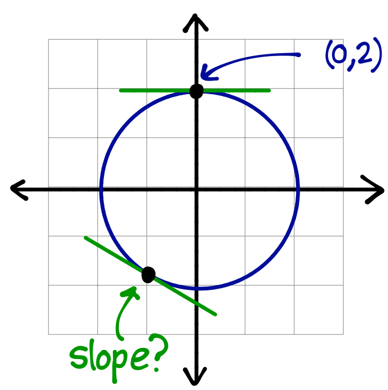
<u>Observation</u> we don't need explicit formulas for f(x), g(x), etc, in order to find expressions for derivatives.

GRAPHS

- ◇ Any equation in two variables (let's use x and y) has a graph.
- ◇ The graph consists of all pairs of the form (x, y) that satisfy the equation.
- ◇ It might not be possible to isolate y and write an *explicit* formula $y = f(x)$.
- ◇ Nonetheless, we still think of y as an **implicit** function of x (where "function" is being used loosely; the graph of the equation could fail the vertical line test, hence not technically be a function of x)

Example 8.1. Consider the equation $x^2 + y^2 = 4$.

circle of radius 2 centred at the origin



(0,2) is a point on the graph because $0^2+2^2=4$ (satisfies the equation)

y is not a function of x

$y = \pm\sqrt{4-x^2}$ (fails V.L.T.)

- even though it's not the graph of a function, we can still wonder what the slope of the tangent is at certain points.

<u>ex</u>. slope of tangent at (0,2) is zero because it's horizontal.

<u>ex</u> slope of tangent at $x=-1$? $\to$ $(-1,\sqrt{3})$ or $(-1,-\sqrt{3})$ (we need to specify both coordinates)

less obvious than slope at (0,2)...

IMPLICIT DIFFERENTIATION

1. Start with some equation with x's and y's.

2. Implicitly differentiate both sides of the equation:

- Treat y as a "mystery" function of x (an implicitly defined function)
- When you need to write the derivative of y, just write $\frac{dy}{dx}$
- Differentiate x's as usual.
- When you are done differentiating both sides, you will have a new equation that may contain some x's, some y's, and some $\frac{dy}{dx}$'s. Because you performed the same operation (differentiation) to both sides of the original equation, the new equation is still a valid equation.

3. Isolate $\frac{dy}{dx}$ from your new equation:

- Put all terms that have a $\frac{dy}{dx}$ on one side of the equation, and put all other terms on the other side of the equation.
- Factor out $\frac{dy}{dx}$ from all terms on the $\frac{dy}{dx}$-side of the equation, then divide to isolate $\frac{dy}{dx}$.

Example 8.2. **a.** Find $\frac{dy}{dx}$ for the equation $x^2 + y^2 = 4$.

(original eq$^{\underline{n}}$) $\quad x^2 + y^2 = 4$

(imp. diff.) $\quad \Rightarrow 2x + 2y \cdot \frac{dy}{dx} = 0$

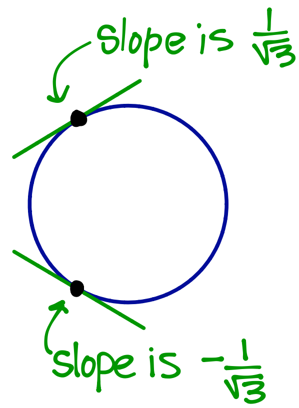
$$\Rightarrow 2y \cdot \frac{dy}{dx} = -2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

b. What is the slope of the tangent line to the graph of $x^2 + y^2 = 4$ at the point $(-1, \sqrt{3})$? What is it at $(-1, -\sqrt{3})$?

At $(-1, \sqrt{3})$, $\frac{dy}{dx} = \frac{-(-1)}{\sqrt{3}} = \frac{1}{\sqrt{3}}$

At $(-1, -\sqrt{3})$, $\frac{dy}{dx} = \frac{-(-1)}{-\sqrt{3}} = -\frac{1}{\sqrt{3}}$



Example 8.3. For the following equation, find $\frac{dy}{dx}$ at the point $(1, 0)$: $\quad e^{2y+x} + x^2y^3 = e^x$

$$e^{2y+x} + x^2y^3 = e^x$$

$$\Rightarrow \left(e^{2y+x}\right)\left(2\frac{dy}{dx} + 1\right) + (2x)y^3 + x^2\left(3y^2\frac{dy}{dx}\right) = e^x$$

We could isolate $\frac{dy}{dx}$ then plug in $(x,y)=(1,0)$. $\rightarrow$

$$\boxed{\begin{aligned} &\Rightarrow 2e^{2y+x}\left(\frac{dy}{dx}\right) + e^{2y+x} + 2xy^3 + 3x^2y^2\left(\frac{dy}{dx}\right) = e^x \\ &\Rightarrow 2e^{2y+x}\left(\frac{dy}{dx}\right) + 3x^2y^2\left(\frac{dy}{dx}\right) = e^x - e^{2y+x} - 2xy^3 \\ &\Rightarrow \left[2e^{2y+x} + 3x^2y^2\right]\left(\frac{dy}{dx}\right) = e^x - e^{2y+x} - 2xy^3 \\ &\Rightarrow \frac{dy}{dx} = \frac{e^x - e^{2y+x} - 2xy^3}{2e^{2y+x} + 3x^2y^2} \end{aligned}}$$

Since we're only interested in $\frac{dy}{dx}$ at the point $(1,0)$, a more efficient solution is to plug in $(1,0)$ first, then isolate dy/dx (for that point)

$$\left(e^{2(0)+1}\right)\left(2\frac{dy}{dx} + 1\right) + 2(1)(0^3) + (1^2)(3)(0^2)\frac{dy}{dx} = e^1$$

$$\Rightarrow 2e\frac{dy}{dx} + e + 0 + 0 = e \qquad \Rightarrow \text{at } (1,0)\ \frac{dy}{dx} = 0$$

Example 8.4. Find $\frac{dy}{dx}$ if $\sin(x + y) = y^2 \cos(x)$.

$$\sin(x+y) = y^2\cos(x)$$

$$\Rightarrow \cos(x+y)\left[1 + \frac{dy}{dx}\right] = 2y\frac{dy}{dx}\cos(x) + y^2(-\sin(x))$$

$$\Rightarrow \cos(x+y) + \cos(x+y)\frac{dy}{dx} = 2y\cos(x)\frac{dy}{dx} - y^2\sin(x)$$

$$\Rightarrow \cos(x+y)\frac{dy}{dx} - 2y\cos(x)\frac{dy}{dx} = -y^2\sin(x) - \cos(x+y)$$

$$\Rightarrow \left[\cos(x+y) - 2y\cos(x)\right]\frac{dy}{dx} = -y^2\sin(x) - \cos(x+y)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-y^2\sin(x) - \cos(x+y)}{\cos(x+y) - 2y\cos(x)}$$

Inverse Trig Derivatives

Derivative of Arcsine $y=\arcsin(x) \iff \sin(y)=x$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

we want dy/dx so we use implicit differentiation on the equation

$$\sin(y)=x$$

$$\Rightarrow \cos(y)\cdot\frac{dy}{dx}=1$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\cos(y)}$$

in terms of x?

$$\therefore \frac{d}{dx}[\arcsin(x)]=\frac{1}{\sqrt{1-x^2}}$$

Note. $\cos^2 y+\sin^2 y=1$

$$\Rightarrow \cos(y)=\pm\sqrt{1-\sin^2(y)}$$

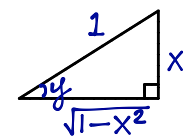
but we assumed $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$

on this interval, $\cos(y)\geq 0$

so, on this interval,

$$\cos(y)=\sqrt{1-\sin^2(y)}$$

Finally, Keep in mind that $\sin(y)=x$



Derivative of Arctangent $y=\arctan(x) \iff \tan(y)=x$ and $-\frac{\pi}{2}<y<\frac{\pi}{2}$

Same strategy:

$$\tan(y)=x$$

$$\Rightarrow \sec^2(y)\frac{dy}{dx}=1$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sec^2(y)}$$

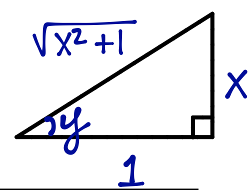
$$\frac{d}{dx}[\arctan(x)]=\frac{1}{1+x^2}$$

Note. $1+\tan^2(y)=\sec^2(y)$

$$\Rightarrow \frac{1}{\sec^2(y)}=\frac{1}{1+\tan^2(y)}$$

and Keep in mind that

$$x=\tan(y)$$



Other **Inverse Trig Rules**:

$\frac{d}{dx}[\cos^{-1}(x)]=-\frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}[\sec^{-1}(x)]=\frac{1}{x\sqrt{x^2-1}}$
$\frac{d}{dx}[\cot^{-1}(x)]=-\frac{1}{1+x^2}$	$\frac{d}{dx}[\csc^{-1}(x)]=-\frac{1}{x\sqrt{x^2-1}}$

Recall $\boxed{y = \log_b(x) \iff b^y = x \quad \text{and } x > 0}$

To find $\frac{d}{dx}[\log_b(x)]$, we'll use the same strategy. We will find dy/dx with implicit differentiation of the equation $b^y = x$

$$\Rightarrow (\ln b)(b^y)\left(\frac{dy}{dx}\right) = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(\ln b)(b^y)} \leftarrow \text{Keep in mind } b^y = x$$

$$\Rightarrow \boxed{\frac{d}{dx}[\log_b(x)] = \left(\frac{1}{\ln b}\right)\left(\frac{1}{x}\right)}$$

and, since $\ln e = 1$, for base $b = e$, the rule is $\rightarrow$ $\boxed{\frac{d}{dx}[\ln(x)] = \frac{1}{x}}$

Log Chain Rules

base b $\boxed{\frac{d}{dx}[\log_b(g(x))] = \left(\frac{1}{\ln b}\right)\left(\frac{1}{g(x)}\right)g'(x) = \frac{g'(x)}{(\ln b)g(x)}}$

base e $\boxed{\frac{d}{dx}[\ln(g(x))] = \left(\frac{1}{g(x)}\right)g'(x) = \frac{g'(x)}{g(x)}}$

Example 8.5. Find $f'(x)$ if $f(x) = \ln|x|$.

Thus, $f(x) = \begin{cases} \ln(x) & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$

$$\Rightarrow f'(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ \underbrace{\frac{1}{(-x)}(-1)}_{\ln \text{ chain rule with } g(x) = -x} = \frac{1}{x} & \text{if } x < 0 \end{cases}$$

$\therefore \boxed{\frac{d}{dx}[\ln|x|] = \frac{1}{x}}$

Example 8.6. Differentiate each of the following:

$f(x) = \arctan(3e^x - 2x^5)$ (we need an arctangent instance of the chain rule)

$$f'(x) = \left(\frac{1}{1+(3e^x-2x^5)^2}\right)(3e^x - 10x^4)$$

$$\frac{d}{dx}[\arctan(g(x))] = \frac{g'(x)}{1+[g(x)]^2}$$

$$= \frac{3e^x - 10x^4}{1+(3e^x-2x^5)^2}$$

$y = \ln(x)\sin^{-1}(x)$

$$y' = \left(\frac{1}{x}\right)\sin^{-1}(x) + \ln(x)\left(\frac{1}{\sqrt{1-x^2}}\right)$$

LOGARITHMIC DIFFERENTIATION

Example 8.7. Find $f'(x)$ where $f(x) = x^x$.

exponent is variable $\Rightarrow$ not a power function

base is variable $\Rightarrow$ not an exponential function

we have no rule for this!

<u>Strategy for logarithmic differentiation:</u>

- "$\ln$" the absolute value of both sides (or "$\ln$" both sides <u>assuming</u> they're >0)
- use nice $\ln$ properties
- implicitly differentiate to find dy/dx
- use the fact that we know $y=f(x)$ explicitly.

<u>eq$^{\underline{n}}$</u> $y = x^x$ (assume $x>0$)

$$\Rightarrow \ln y = \ln(x^x)$$

$$\Rightarrow \ln y = x \ln x$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = (1)\ln x + x\left(\frac{1}{x}\right)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \ln x + 1$$

$$\Rightarrow \frac{dy}{dx} = (\ln x + 1)y$$ but we know $y = x^x$ $\therefore \frac{dy}{dx} = (\ln x + 1)x^x$

Exercise 8.8. Use logarithmic differentiation to prove that the Power Rule (which we've been using for all sorts of powers $n \in \mathbb{R}$) is in fact valid.

That is, prove $\frac{d}{dx}[x^n] = nx^{n-1}$ for all $n \in \mathbb{R}$.

We already "know" $\frac{dy}{dx} = nx^{n-1}$, but this Exercise is a proof that the power rule (that we've been happily using for quite a while) actually holds true for any power $n \in \mathbb{R}$.

(eq$^{\underline{n}}$) Let $y = x^n$ (assume $x \neq 0$)

('ln' both sides) Then $\ln|y| = \ln|x^n|$

(simplify) $\Rightarrow \ln|y| = \ln(|x|^n)$ ← think about this repositioning of absolute value and n...

(ln property) $\Rightarrow \ln|y| = n\ln|x|$

(imp. diff.) $\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = n\left(\frac{1}{x}\right)$

(isolate $\frac{dy}{dx}$) $\Rightarrow \frac{dy}{dx} = n\left(\frac{1}{x}\right)y$ but we know $y = x^n$ explicitly...

(use $y = x^n$) $\Rightarrow \frac{dy}{dx} = n\left(\frac{1}{x}\right)x^n$

(simplify) $\Rightarrow \boxed{\frac{dy}{dx} = nx^{n-1}}$ ✓ (the power rule really is true for all $n \in \mathbb{R}$)

STUDY GUIDE

- ◇ **implicit differentiation strategy**
- ◇ **derivative rules for inverse trig functions:** $\boxed{\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}}$ $\boxed{\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2}}$ (and others!)
- ◇ **derivative rules for logs:** $\boxed{\frac{d}{dx}[\ln(x)] = \frac{1}{x}}$ $\boxed{\frac{d}{dx}[\log_b(x)] = \left(\frac{1}{\ln b}\right)\frac{1}{x}}$
- ◇ **log chain rule:** $\boxed{\frac{d}{dx}[\ln(g(x))] = \frac{g'(x)}{g(x)}}$
- ◇ **logarithmic differentiation strategy**

9. Related Rates & Linear Approximations

Lec 8 mini review.

implicit differentiation strategy

derivative rules for inverse trig functions:

$$\boxed{\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}}$$

$$\boxed{\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2}}$$

(and others!)

logarithmic differentiation strategy

derivative rules for logs:

$$\boxed{\frac{d}{dx}[\ln(x)] = \frac{1}{x}}$$

$$\boxed{\frac{d}{dx}[\log_b(x)] = \left(\frac{1}{\ln b}\right)\frac{1}{x}}$$

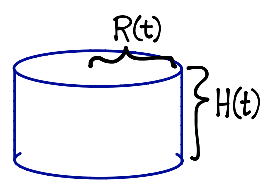
$$\boxed{\frac{d}{dx}[\ln\left(g(x)\right)] = \frac{g'(x)}{g(x)}}$$

WARM-UP TO RELATED RATES

Example 9.1. For each of the following equations, implicitly differentiate both sides with respect to the time variable t.

(volume of a cylinder whose dimensions might be changing as a function of time)

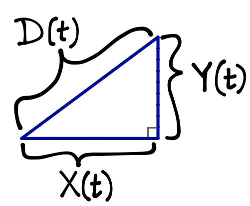
$$V(t) = \pi\big[R(t)\big]^2 H(t)$$



$$V'(t) = \pi(2)[R(t)]^1 \cdot R'(t)H(t) + \pi[R(t)]^2 \cdot H'(t)$$

(sides of a right-angled triangle, which are changing as a function of time)

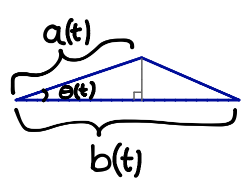
$$D^2 = X^2 + Y^2$$



$$2D \cdot \frac{dD}{dt} = 2X\frac{dX}{dt} + 2Y\frac{dY}{dt}$$

(the area of a triangle whose sides and angle are changing as time goes on)

$$A = \frac{1}{2}ab\sin\theta$$

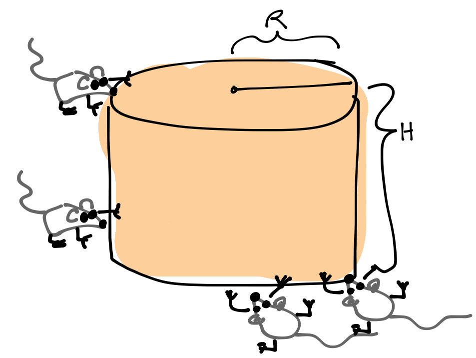


$$\frac{dA}{dt} = \frac{1}{2}\frac{da}{dt}b\sin\theta + \frac{1}{2}a\frac{db}{dt}\sin\theta + \frac{1}{2}ab\cos\theta \cdot \frac{d\theta}{dt}$$

RELATED RATES STRATEGY

- Identify the variables in the problem and draw a diagram if you can.
- Determine what rates of change (derivatives) are given, and what the question is asking for (usually a rate of change at a given point in time).
- Find an equation that relates the variables to each other at all times.
- Implicitly differentiate the equation, with respect to time.
- Use the equation and the implicitly differentiated equation to solve for the desired quantity or rate.

Example 9.2. Mice are systematically eating a huge cylindrical wheel of cheese. The cheese is shrinking! The radius shrinks at a rate of 2cm/min and the height of the cheese cylinder shrinks 5cm / min. At what rate is the volume of the cheese changing when, at some point in time, its radius is 10 cm and its height is 20 cm?



Variables
R = radius of cheese
H = height of cheese
V = volume of cheese

Given
$$\frac{dR}{dt} = -2 \text{ cm/min}$$
$$\frac{dH}{dt} = -5 \text{cm/min}$$

want $\frac{dV}{dt}$ when $R = 10$cm and $H = 20$ cm

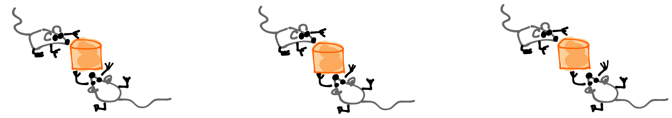
eq$^{\underline{n}}$ $V = \pi R^2 H$ ← (true at all times, assuming cheese is always a cylinder, even though V, R, and H are changing as time passes)

imp. diff. $$\frac{dV}{dt} = 2\pi R \frac{dR}{dt} H + \pi R^2 \frac{dH}{dt}$$

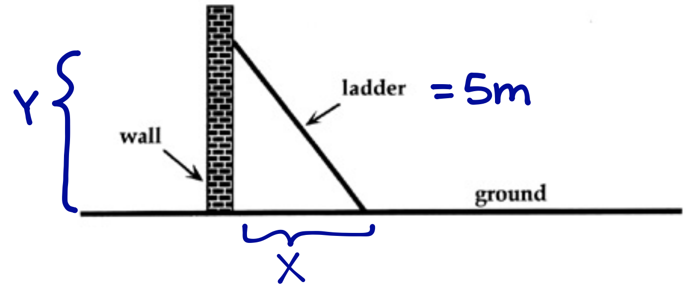
Plug in what we know:
(at time when $R=10$ and $H=20$)
$$\frac{dV}{dt} = 2\pi(10)(-2)(20) + \pi(10^2)(-5)$$

$$\Rightarrow \frac{dV}{dt} = -1300\pi \text{ cm}^3/\text{min}$$

∴ at that moment in time, the cheese's volume is decreasing at a rate of 1300π cm^3/min.



Example 9.3. A ladder is leaning against a wall. The ladder is 5 m long. The top of the ladder is sliding down the wall at a rate of 2 m/s. At the same time, the bottom of the ladder is sliding away from the wall. When the bottom of the ladder is 4 m away from the wall, how fast is it sliding away from the wall?



Variables X = distance between wall and bottom of ladder

Y = distance between top of ladder and the ground below

Given ladder = 5 m $\quad \frac{dY}{dt} = -2$ m/s (Y is decreasing as time goes on)

Want $\frac{dX}{dt}$ when $X = 4$m

eqⁿ $\quad X^2 + Y^2 = 5^2$

imp. diff. $\quad 2X\frac{dX}{dt} + 2Y\frac{dY}{dt} = 0$

when $X=4$: $\quad 2(4)\frac{dX}{dt} + 2Y(-2) = 0$

when $X=4$, solve for Y using eqⁿ $\Rightarrow Y = \sqrt{25-4^2} = 3$

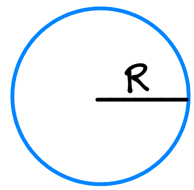
$$\Rightarrow 2(4)\frac{dX}{dt} + 2(3)(-2) = 0$$

$$\Rightarrow \frac{dX}{dt} = \frac{12}{8} = 1.5 \text{ m/s}$$

$\therefore$ when $X = 4$m, the bottom of the ladder is sliding away from the wall at a rate of 1.5 m/s

Example 9.4. A stone is tossed into a pond, creating a circular ripple that grows outward from the centre. If the radius of the circle is growing at a rate of 10 cm per second, how fast is the area of the circle growing when the radius is $\frac{100}{\pi}$ cm ? How fast is the area growing when the radius is 1 cm ?

Variables A = area of ripple's circle
R = radius of ripple's circle



Given $\frac{dR}{dt} = 10\text{cm/s}$

Want $\frac{dA}{dt}$ when $R = \frac{100}{\pi}$ (≈ 31.8) cm

eqn $A = \pi R^2$

imp. diff. $\frac{dA}{dt} = 2\pi R \cdot \frac{dR}{dt}$

when $R = 100/\pi$ cm $\quad \frac{dA}{dt} = 2\pi\left(\frac{100}{\pi}\right)(10) = 2000 \text{ cm}^2/\text{s}$

$\therefore$ when $R = 100/\pi$ cm, the area of the ripple's circle is increasing at a rate of $2000 \text{ cm}^2/\text{s}$.

when $R = 1$ cm $\quad \frac{dA}{dt} = 2\pi(1)(10) = 20\pi$ (≈ 62.8) cm^2/s

$\therefore$ when $R = 1$ cm, the area of the ripple's circle is increasing at a rate of $20\pi \text{ cm}^2/\text{s}$.

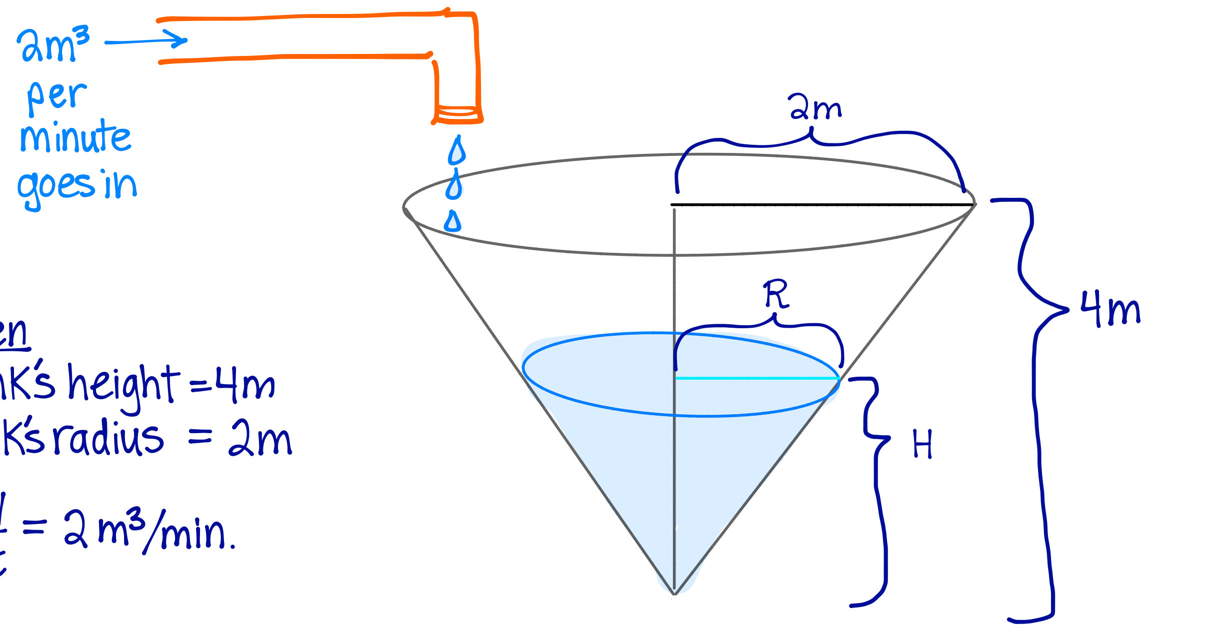
Example 9.5. A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.

<u>Variables</u>

H = height of water level in tank (the height of the tank is constant)

R = radius of water's upper surface (the radius of the tank is constant)

V = volume of water in tank (the volume of the tank itself is constant)



<u>Given</u>

tank's height = 4m

tank's radius = 2m

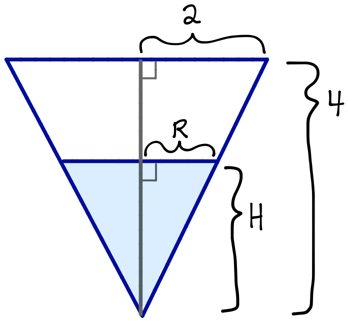
$\frac{dV}{dt} = 2\ m^3/min.$

<u>want</u> $\frac{dH}{dt}$ when $H = 3m$

<u>eqn</u> $V = \frac{1}{3}\pi R^2 H$

<u>imp. diff.</u> $\frac{dV}{dt} = \frac{2}{3}\pi R \cdot \frac{dR}{dt} H + \frac{1}{3}\pi R^2 \frac{dH}{dt}$

when $H = 3$, $2 = \frac{2}{3}\pi R \frac{dR}{dt}(3) + \frac{1}{3}\pi R^2 \frac{dH}{dt}$ but what is R? What is dR/dt?



There is a relationship between R and H based on similar triangles:

$$\frac{R}{H} = \frac{2}{4}$$

$$\therefore R = \frac{1}{2}H$$

by imp. diff. this eq^{n}

when $H = 3\,m$, $R = \frac{1}{2}H = \frac{1}{2}(3) = 1.5\,m$ and $\frac{dR}{dt} = \frac{1}{2}\frac{dH}{dt}$

$\therefore$ when $H = 3$, we have $2 = \frac{2}{3}\pi(1.5)\left(\frac{1}{2}\frac{dH}{dt}\right)(3) + \frac{1}{3}\pi(1.5)^2\frac{dH}{dt}$

Solving for $\frac{dH}{dt}$ we get $\frac{dH}{dt} = \frac{8}{9\pi}$ m/min.

$\therefore$ at the moment in time when the water level is 3m, the water level is rising at a rate of $\frac{8}{9\pi}$ (≈ 0.283) m/min.

Alternatively, we could notice relationship between R and H before imp. diff.

Since $R = \frac{1}{2}H$, for this cone, the volume equation can be rewritten as $V = \frac{1}{3}\pi\left(\frac{1}{2}H\right)^2 H = \frac{1}{12}\pi H^3$

<u>New eq^{n}</u> $V = \frac{1}{12}\pi H^3$

<u>New imp. diff.</u> $\frac{dV}{dt} = \frac{3\pi}{12}H^2\frac{dH}{dt}$

now plug in $H = 3$, $\frac{dV}{dt} = 2$ to find $\frac{dH}{dt}$

observation: Zoom in toward a point on the graph of a differentiable function. The curve looks almost the same as a line (near the point of tangency)

idea: use the tangent line to the graph of f at x=a
to approximate f(x) at x-values <u>near</u> a.

Linear Approximation of $f(x)$ near $x = a$:

Near x=a, $\boxed{f(x) \approx f(a)+f'(a)(x-a)}$

the equation of the tangent line to f at a.

This is called the <u>Linear Approximation of f at a</u> (also called the <u>Tangent Line Approximation of f at a</u>).

The line $\boxed{y=L(x)=f(a)+f'(a)(x-a)}$ is called the <u>Linearization of f at a.</u>

Note. In order for the linearization $L(x)$ of f at a to be of practical use, we need $f(a)$ and $f'(a)$ to be easy to compute; otherwise, the linearization would be just as difficult to obtain as finding exact values of $f(x)$ near a.

Example 9.6. Use a linear approximation to estimate $\sin(0.1)$.

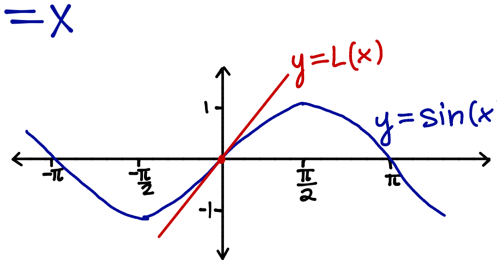
$f(x)=\sin(x)$ use $a=0$ $f(a)=\sin(0)=0$
$f'(x)=\cos(x)$ $f'(a)=\cos(0)=1$

both easy to compute

$\Rightarrow L(x)=f(a)+f'(a)(x-a) = 0+(1)(x-0) = x$

Near $a=0$, $\sin(x)\approx L(x)=x$

so $\sin(0.1)\approx L(0.1) = 0.1$



Example 9.7. Find the linearization of $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$.

$$f(1) = \sqrt{1+3} = \sqrt{4} = 2 \qquad f'(x) = \frac{1}{2\sqrt{x+3}}$$

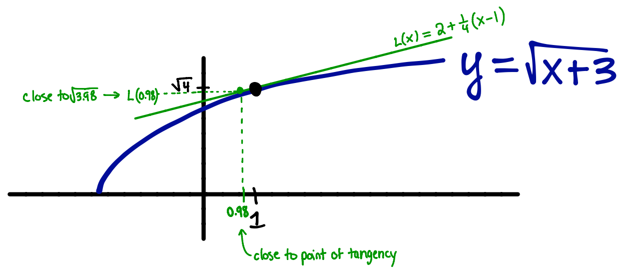
$$f'(1) = \frac{1}{2\sqrt{1+4}} = \frac{1}{4}$$

The linear approximation of f at 1 is

$$L(x) = f(1) + f'(1)(x-1) = 2 + \frac{1}{4}(x-1) = \frac{1}{4}x + \frac{7}{4}$$

$$\sqrt{3.98} = \sqrt{0.98+3} = f(0.98) \approx L(0.98) = \frac{1}{4}(0.98) + \frac{7}{4} = 1.995$$

$$\sqrt{4.05} = \sqrt{1.05+3} = f(1.05) \approx L(1.05) = \frac{1}{4}(1.05) + \frac{7}{4} = 2.0125$$



STUDY GUIDE

◇ **related rates strategy**:

1. read problem carefully
2. identify the variables and draw a diagram if possible
3. determine what rates of change are given, and what is being asked
4. find an equation that relates the variables to each other at all times
5. implicitly differentiate that equation with respect to time
6. use the equation and the implicitly differentiated equation to solve for the desired quantity or rate

◇ **Linear Approximation of f at a**: $\boxed{f(x) \approx L(x) = f(a) + f'(a)(x-a)}$

10. Antiderivatives & Areas

ANTIDERIVATIVES

A function $F(x)$ is called an **antiderivative** of $f(x)$ if $F'(x) = f(x)$

Example 10.1. Guess an antiderivative for each of the following functions. Then check your answer by differentiating.

function $f(x)$	guess an antiderivative $F(x)$ of f	verify that $F'(x) = f(x)$
(a) $f(x) = 2x$	$F(x) = x^2$	$F'(x) = 2x = f(x)$ ✓
(b) $f(x) = 3x^2$	$F(x) = x^3$	$F'(x) = 3x^2 = f(x)$ ✓
(c) $f(x) = e^x$	$F(x) = e^x$	$F'(x) = e^x = f(x)$ ✓
(d) $f(x) = \cos(x)$	$F(x) = \sin(x)$	$F'(x) = \cos(x) = f(x)$ ✓
(e) $f(x) = 0$	$F(x) = C$ $C \in \mathbb{R}$, could be any constant!	$F'(x) = \frac{d}{dx}[C] = 0 = f(x)$ ✓

In fact, since the derivative of any constant $C \in \mathbb{R}$ is zero, we can always include a "$+C$" (an arbitrary constant) to any antiderivative.

* These notes are solely for the personal use of students registered in MAT1320.

Fact. If $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$, then $F(x) - G(x) = C$, for some constant C.

- Thus, if F and G are two distinct antiderivatives of $f(x)$, then the graphs of F and G are identical, up to a vertical translation.

Theorem 10.2. If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x) + C$ where C is an arbitrary constant.

← the constant of integration

Integral Notation:

integral sign → ← should both be used like opening + closing parentheses → dx

$$\int f(x)\, dx = \underbrace{F(x) + C}$$

the "integrand" (what we are integrating)

the most general antiderivative of the integrand $f(x)$

"UNDOING" BASIC RULES OF DIFFERENTIATION

Constants. $\boxed{\int 0\, dx = C}$ because $\frac{d}{dx}[C] = 0$ ✓

$\boxed{\int k\, dx = kx + C}$ because $\frac{d}{dx}[kx + C] = k$ ✓

Powers. $\frac{d}{dx}[x^n] = nx^{n-1}$

"old" exponent becomes a constant multiple

exponent decreases by 1

- For the antiderivative of x^n we needed the "old" exponent to be $n+1$ (to get n after differentiating)
- We also need to compensate for the "missing" constant multiple of the "old" exponent...

For $n \neq -1$, → $\boxed{\int x^n\, dx = \frac{1}{n+1} x^{n+1} + C}$

BEWARE OF $n=-1$

$\int x^{-1} dx \cancel{=} \frac{1}{0} x^0 + C$ undefined!

$$\frac{d}{dx}\left[\frac{1}{n+1}x^{n+1} + C\right] = \frac{1}{n+1}(n+1)x^{n+1-1} + 0 = x^n \checkmark$$

The special power $n=-1$ $\quad x^{-1}=\frac{1}{x}$ is the derivative of $\ln x$ (for $x>0$)

$\frac{d}{dx}[\ln x]=\frac{1}{x}$ $\quad x^{-1}=\frac{1}{x}$ is the derivative of $\ln|x|$ (for $x\neq 0$)

$$\boxed{\int \frac{1}{x}\,dx = \ln|x| + C}$$

$\frac{d}{dx}[\ln|x|+C]=\frac{1}{x}+0=\frac{1}{x}$ ✓

Constant multiples $(k\in\mathbb{R})$

$$\boxed{\int k\cdot f(x)\,dx = k\int f(x)\,dx}$$

Sums and differences.

$$\boxed{\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx}$$

$$\boxed{\int (f(x)-g(x))\,dx = \int f(x)\,dx - \int g(x)\,dx}$$

Example 10.3. Find $f(x)$ given that $f'(x)=\frac{5}{x^6}+8x^3-\frac{2}{x}$ and $f(1)=2$.

$f'(x)=5x^{-6}+8x^3-2x^{-1}$ so $f(x)$ is an antiderivative of $f'(x)$.

$\Rightarrow f(x)=5\left(\frac{1}{(-6+1)}x^{-6+1}\right)+8\left(\frac{1}{(3+1)}x^{3+1}\right)-2\ln|x|+C$

$\Rightarrow f(x)=5\left(\frac{1}{-5}x^{-5}\right)+8\left(\frac{1}{4}x^{4}\right)-2\ln|x|+C$

$\Rightarrow f(x)=-\frac{1}{x^5}+2x^4-2\ln|x|+C$

- Now, using the initial value $f(1)=2$, we can solve for the constant C.

$\Rightarrow 2=f(1)=-\frac{1}{1^5}+2(1^4)-2\ln|1|+C$

$\Rightarrow 2=-1+2-2(0)+C$

$\Rightarrow C=1.$

$$\boxed{\therefore f(x)=-\frac{1}{x^5}+2x^4-2\ln|x|+1}$$

Trig functions.

$$\int \cos(x)\,dx = \sin(x) + C \qquad \frac{d}{dx}[\sin x + C] = \cos x + 0 = \cos x \checkmark$$

$$\int \sin(x)\,dx = -\cos(x) + C \qquad \frac{d}{dx}[-\cos x + C] = -(-\sin x) + 0 = \sin x \checkmark$$

$$\int \sec^2(x)\,dx = \tan(x) + C \qquad \frac{d}{dx}[\tan x + C] = \sec^2 x + 0 = \sec^2 x \checkmark$$

$\vdots$

* for any recognizable derivative, you should know its antiderivative...

Example 10.4. Verify that $F(x) = -\ln|\cos x| + C$ is an antiderivative of $\tan x$.

We must check that $F'(x) = \tan x$.

$$\frac{d}{dx}[F(x)] = \frac{d}{dx}\left[-\ln|\cos x| + C\right] = \underbrace{-\frac{1}{\cos x}(-\sin x)}_{\text{using } \ln \text{ chain rule}} = \frac{\sin x}{\cos x} = \tan x \checkmark$$

Exponential functions.

$$\int e^x\,dx = e^x + C \qquad \frac{d}{dx}[e^x + C] = e^x \checkmark$$

$$\int b^x\,dx = \frac{1}{\ln(b)} b^x + C \qquad \frac{d}{dx}\left[\frac{1}{\ln b} b^x + C\right] = \frac{1}{\ln b}(\ln b) b^x = b^x \checkmark$$

Keep in mind
$\frac{d}{dx}[b^x] = (\ln b) b^x$

Inverse trig functions.

$$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C \qquad \frac{d}{dx}[\arctan(x) + C] = \frac{1}{1+x^2} \checkmark$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C \qquad \frac{d}{dx}[\arcsin(x) + C] = \frac{1}{\sqrt{1-x^2}} \checkmark$$

$\vdots$

* again, if you're faced with a recognizable derivative (or constant multiple of) then you should know its antiderivative.

Example 10.5. Verify that $x \ln x - x$ is an antiderivative of $\ln(x)$.

We must check that $F'(x) = \ln(x)$.

$$\frac{d}{dx}[F(x)] = \frac{d}{dx}[x\ln x - x] = \underbrace{(1)\ln x + x\left(\frac{1}{x}\right)}_{\text{product rule}} - 1 = \ln x + 1 - 1 = \ln x \checkmark$$

Example 10.6. Try to determine (guess!) an antiderivative for each of the following functions. Verify (check!) your answer by differentiating.

function	antiderivative?	Check
(a) $\underbrace{2xe^{x^2}}$ looks like the aftermath of an exp. chain rule?	e^{x^2}	$\frac{d}{dx}[e^{x^2}] = (e^{x^2})(2x) \checkmark$
(b) $\underbrace{3x^2 \sin x + x^3 \cos x}$ looks like aftermath of a product rule?	$x^3 \sin x$	$\frac{d}{dx}[x^3 \sin x] = (3x^2)(\sin x) + (x^3)(\cos x) \checkmark$
(c) $\cot(x) = \underbrace{\frac{\cos x}{\sin x}}$ looks like aftermath of a ln chain rule?	$\ln\lvert\sin x\rvert$	$\frac{d}{dx}[\ln\lvert\sin x\rvert] = \frac{1}{\sin x}(\cos x) = \cot x \checkmark$
(d) $\underbrace{e^{ax+b}}$ where a and b are constant real numbers. looks like aftermath of exp. chain rule?	$\frac{1}{a}e^{ax+b}$	$\frac{d}{dx}[\frac{1}{a}e^{ax+b}] = \frac{1}{a}(e^{ax+b})(a+0) = e^{ax+b} \checkmark$
(e) $\underbrace{\frac{g'(x)}{g(x)}}$ looks like aftermath of a ln chain rule?	$\ln\lvert g(x)\rvert$	$\frac{d}{dx}[\ln\lvert g(x)\rvert] = \frac{1}{g(x)} \cdot (g'(x)) = \frac{g'(x)}{g(x)} \checkmark$
(f) $\underbrace{e^x[e^x+8]^{10}}$ looks like aftermath of a power chain rule?	$\frac{1}{11}(e^x+8)^{11}$	$\frac{d}{dx}[\frac{1}{11}(e^x+8)^{11}] = (\frac{1}{11})(11)(e^x+8)^{10}(e^x+0) = e^x(e^x+8)^{10} \checkmark$

Exercise 10.7. Determine an equation for the height of a ball at time t if the ball is thrown with an initial upwards velocity of v_0 m/s, from an initial height of h_0 m above the ground, on Earth where acceleration due to gravity is -9.8 m/s^2.

acceleration $a(t) = -9.8$ m/s^2

is the rate of change of the ball's velocity with respect to time

i.e. $a(t) = v'(t)$

$\Rightarrow v(t)$ is an antiderivative of $a(t)$. Moreover, $v(0) = v_0$ m/s.

$\Rightarrow v(t) = -9.8t + C$ Using $v(0) = v_0$ we solve for C

$v_0 = -9.8(0) + C \Rightarrow C = v_0$

$\therefore$ Velocity is given by $v(t) = -9.8t + v_0$

is the rate of change of the ball's height with respect to time

i.e. $v(t) = h'(t)$

$\Rightarrow h(t)$ is an antiderivative of $v(t)$. Moreover, $h(0) = h_0$ m/s.

$\Rightarrow h(t) = -9.8\left(\frac{1}{2}t^2\right) + v_0 t + K$ using $h(0) = h_0$ we solve for K

$= -4.9t^2 + v_0 t + K$

$h_0 = -4.9(0^2) + v_0(0) + K$

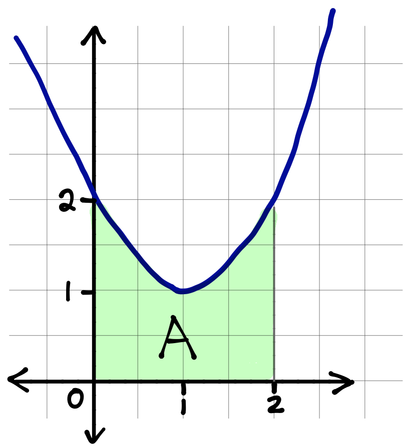
$\Rightarrow K = h_0$

$\therefore$ height is given by $h(t) = -4.9t^2 + v_0 t + h_0$

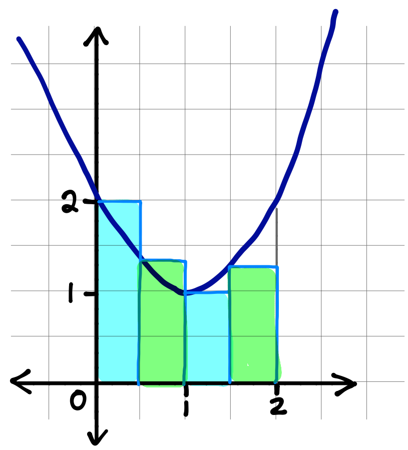
Example 10.8. Let $f(x) = (x-1)^2 + 1$.

What is the area of the region under the graph of f, above the x-axis between $x = 0$ and $x = 2$?

This is not a standard shape, so we do not have an area formula.

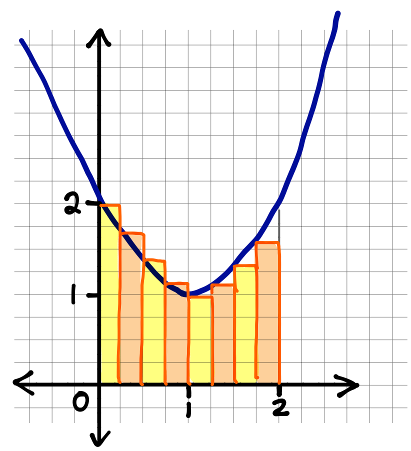


$f(x) = (x-1)^2 + 1$



4 rectangles

using left endpoint



8 rectangles

using left endpoint

We can try to estimate the area using several rectangles:

- If we use $n = 4$ rectangles, then each rectangle has width $\Delta x = 0.5$ (this is because we are chopping the interval $[0, 2]$ into four equal pieces).
- If we use $n = 8$ rectangles instead, then each rectangle has width 0.25 (this is because we are chopping the interval $[0, 2]$ into eight equal pieces).
- For the height of each rectangle, we will use the y-coordinate of f at the appropriate value of x (for now, we'll use the x-coordinate at the bottom left corner of each rectangle).

$$L_4 = \begin{bmatrix} \text{Estimate of area using} \\ \text{4 rectangles with} \\ \text{left endpoint heights} \end{bmatrix}$$

$$\begin{aligned}
&\approx \begin{pmatrix} \text{area of 1st} \\ \text{rectangle} \end{pmatrix} + \begin{pmatrix} \text{area of 2nd} \\ \text{rectangle} \end{pmatrix} + \begin{pmatrix} \text{area of 3rd} \\ \text{rectangle} \end{pmatrix} + \begin{pmatrix} \text{area of 4th} \\ \text{rectangle} \end{pmatrix} \\
&= \left(\underbrace{0.5}_{\substack{\text{width} \\ \text{1st rect.}}} \times \underbrace{f(0)}_{\substack{\text{height} \\ \text{1st rect.}}} \right) + \left(\underbrace{0.5}_{\substack{\text{width} \\ \text{2nd rect.}}} \times \underbrace{f(0.5)}_{\substack{\text{height} \\ \text{2nd rect.}}} \right) + \left(\underbrace{0.5}_{\substack{\text{width} \\ \text{3rd rect.}}} \times \underbrace{f(1)}_{\substack{\text{height} \\ \text{3rd rect.}}} \right) + \left(\underbrace{0.5}_{\substack{\text{width} \\ \text{4th rect.}}} \times \underbrace{f(1.5)}_{\substack{\text{height} \\ \text{4th rect.}}} \right) \\
&= (0.5 \times 2) + (0.5 \times 1.25) + (0.5 \times 1) + (0.5 \times 1.25) \\
&= [2 + 1.25 + 1 + 1.25](0.5) = 2.75
\end{aligned}$$

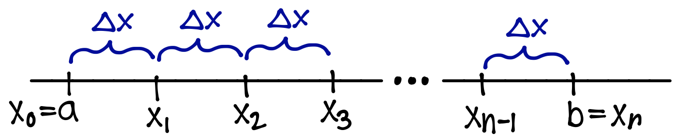
$$L_8 = \begin{bmatrix}\text{Estimate of area using} \\ \text{8 rectangles with} \\ \text{left endpoint heights}\end{bmatrix}$$

$$\begin{aligned}
&\approx \begin{pmatrix}\text{area of 1st} \\ \text{rectangle}\end{pmatrix} + \begin{pmatrix}\text{area of 2nd} \\ \text{rectangle}\end{pmatrix} + \begin{pmatrix}\text{area of 3rd} \\ \text{rectangle}\end{pmatrix} + \begin{pmatrix}\text{area of 4th} \\ \text{rectangle}\end{pmatrix} \\
&\quad + \begin{pmatrix}\text{area of 5th} \\ \text{rectangle}\end{pmatrix} + \begin{pmatrix}\text{area of 6th} \\ \text{rectangle}\end{pmatrix} + \begin{pmatrix}\text{area of 7th} \\ \text{rectangle}\end{pmatrix} + \begin{pmatrix}\text{area of 8th} \\ \text{rectangle}\end{pmatrix} \\
&= \big(0.25 \times f(0)\big) + \big(0.25 \times f(0.25)\big) + \big(0.25 \times f(0.5)\big) + \big(0.25 \times f(0.75)\big) \\
&\quad + \big(0.25 \times f(1)\big) + \big(0.25 \times f(1.25)\big) + \big(0.25 \times f(1.5)\big) + \big(0.25 \times f(1.75)\big) \\
&= \big[1 + 1.5625 + 1.25 + 1.0625 + 1 + 1.0625 + 1.25 + 1.5625\big]0.25 = 2.6875
\end{aligned}$$

General Observations:

- If we chopped the interval into even more (thinner) rectangles, we would get better and better estimates of the actual area.
- Using n rectangles, each of width Δx, where Δx is the interval length divided by n, we can estimate the net area between any continuous function and the x-axis from $x = a$ to $x = b$.

$$\Delta x = \frac{b-a}{n}$$



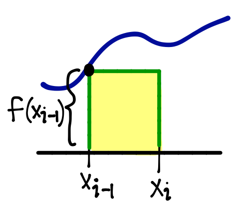
- for $i = 0, 1, 2, \ldots, n$, let $x_i = a + i\Delta x$

- The bigger n gets, the smaller Δx gets (more rectangles, but they are thinner).
- For the height of the ith rectangle, we could just as well have used the x-coordinate at the bottom right corner of each rectangle, or the x-coordinate at the midpoint of the rectangles two sides, or some other sample point x_i^*.

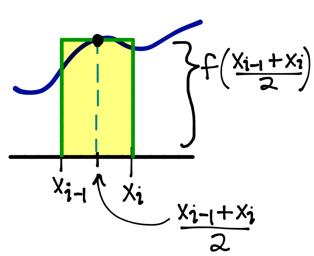
- base of ith rectangle is the ith subinterval $[x_{i-1}, x_i]$

- height of ith rectangle could be

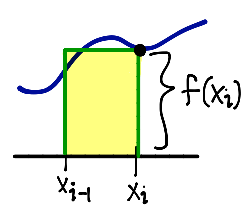
left endpoint



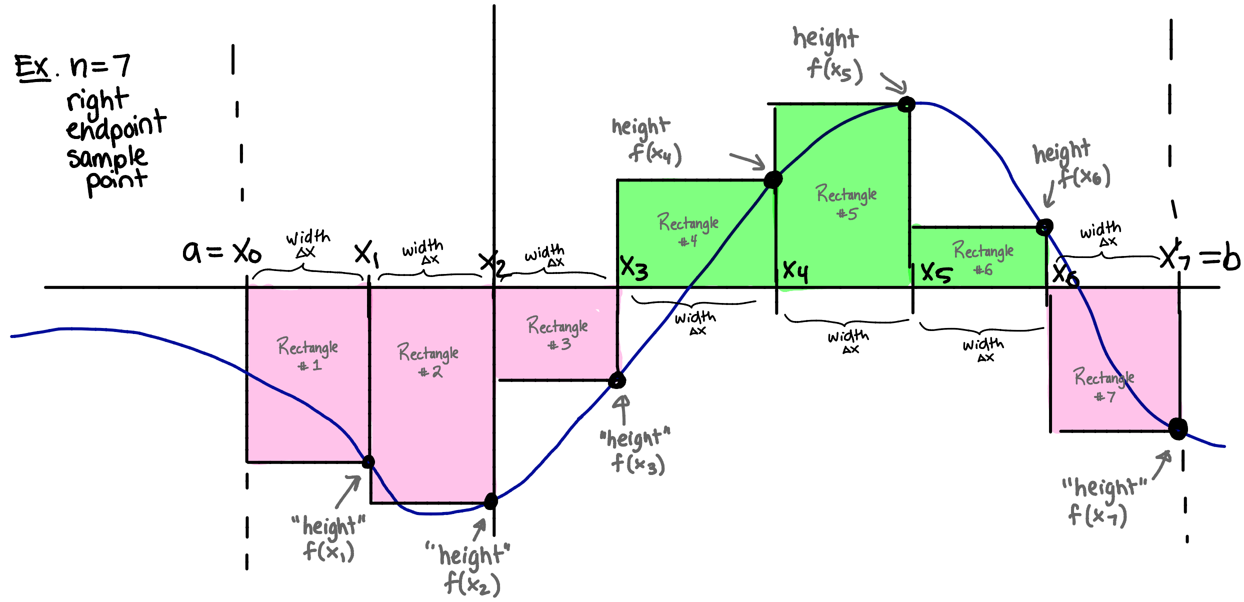
midpoint



right endpoint



- The widths of the rectangles did not all have to be equal (but having them all of equal width made the calculation more straightforward).



- for $i = 1, 2, \ldots, n$, the ith subinterval is $\underbrace{[x_{i-1}, x_i]}$ ← the base of rectangle # i
- from each subinterval, choose a <u>sample point</u> $x_i^* \in [x_{i-1}, x_i]$
- for $i = 1, 2, \ldots, n$, we will use $f(x_i^*)$ as the height of rectangle # i
- the sum $f(x_1^*)\Delta x + f(x_2^*)\Delta x + \ldots + f(x_n^*)\Delta x$ is denoted $\sum_{i=1}^{n} f(x_i^*)\Delta x$
- $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called <u>a Riemann sum.</u>

  It gives an approximation of the net area between $f(x)$ and the x-axis on the interval $[a, b]$.

Ex $\underbrace{f(x_1)\Delta x}_{-(\text{area of rectangle \#1})} + \underbrace{f(x_2)\Delta x}_{-(\text{area of rectangle \#2})} + \underbrace{f(x_3)\Delta x}_{-(\text{area of rectangle \#3})} + \underbrace{f(x_4)\Delta x}_{(\text{area of rectangle \#4})} + \underbrace{f(x_5)\Delta x}_{(\text{area of rectangle \#5})} + \underbrace{f(x_6)\Delta x}_{(\text{area of rectangle \#6})} + \underbrace{f(x_7)\Delta x}_{-(\text{area of rectangle \#7})} \approx$ [Net Area between f and x-axis on [a, b]]

Study Guide

- ◇ **an antiderivative** vs. **the most general antiderivative**
- ◇ **undoing basic rules of differentiation**
- ◇ **some ideas for undoing less basic rules of differentiation**
- ◇ **approximating area (or distance) using a Riemann sum**: $A \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$

11. Definite Integrals & The Fundamental Theorem of Calculus

Lec 10 mini review.

- ◇ **an antiderivative** vs. **the most general antiderivative**
- ◇ **undoing basic rules of differentiation**
- ◇ **some ideas for undoing less basic rules of differentiation**
- ◇ **setup for a Riemann sum with n rectangles on** $[a, b]$:

$$\Delta x = \tfrac{b-a}{n} \qquad x_i = a + i\Delta x \qquad \text{sample point} \quad x_i^* \in [x_{i-1}, x_i]$$

- ◇ **using a Riemann sum to approximate net area A between f and the x-axis on** $[a, b]$:

$$A \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

DEFINITE INTEGRALS

- Let f be a function defined for $a \leq x \leq b$.
- Divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b-a)/n$.
- Let $x_0 = a$ and, for $i = 0, \ldots, n$, let $x_i = a + i\Delta x$ be the endpoints of these subintervals.
- Let x_i^* be *any* **sample point** from the ith subinterval $[x_{i-1}, x_i]$.

Then the **definite integral of f from a to b** is

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x}$$

provided that this limit exists and gives the same value for all possible choices of sample points.
If it does exist and is equal for all sample point choices, then we say that f is **INTEGRABLE** on $[a, b]$.

Theorem 11.1. If f is

continuous on $[a, b]$, or if f has only a finite number of jump discontinuities on $[a, b]$,

then the definite integral $\int_a^b f(x)\,dx$ (which is a limit!) exists, hence f is integrable on $[a, b]$.

Theorem 11.2. If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \left(\sum_{i=1}^{n} f(x_i)\,\Delta x \right)$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$ (that is, we can use right endpoints in our Riemann sum).

---

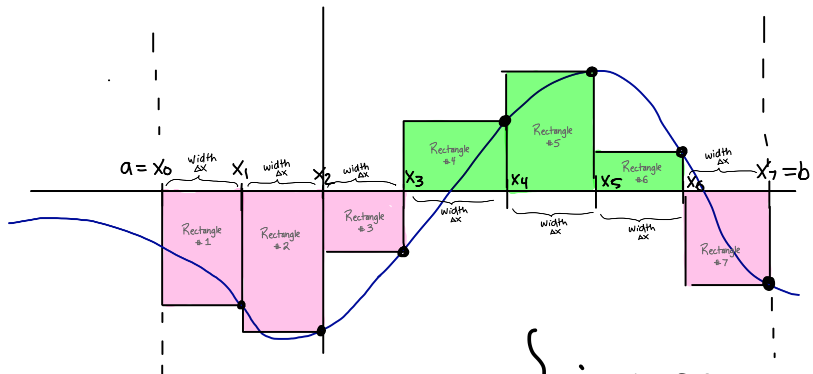
* These notes are solely for the personal use of students registered in MAT1320.

Riemann sum (with n rectangles, right endpoints) for $f(x)$ on $[a,b]$

Choose n.

Then $\Delta x = \dfrac{b-a}{n}$, $x_i = a + i\Delta x$, "height" of rectangle #i $= f(x_i)$

$$\sum_{i=1}^{n} f(x_i)\Delta x = \left(\begin{array}{c}\text{approx.}\\ \text{net area}\\ \text{between}\\ f \text{ and } x\text{-axis}\\ \text{on } [a,b]\end{array}\right)$$

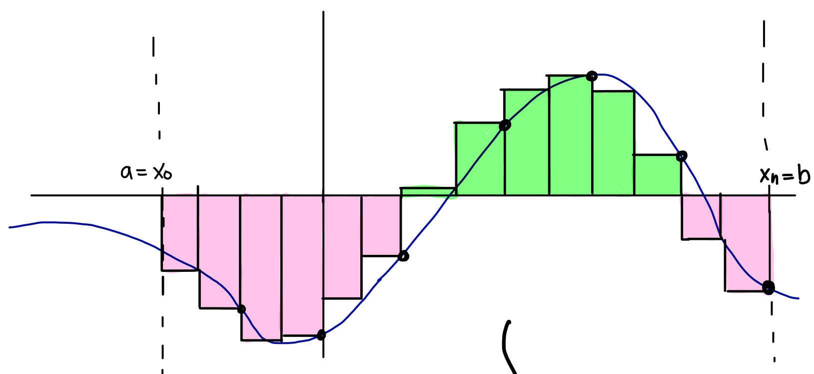


increase n

Choose bigger n.

more rectangles $\Rightarrow$ better approximation

$$\sum_{i=1}^{n} f(x_i)\Delta x = \left(\begin{array}{c}\text{(better)}\\ \text{approx.}\\ \text{net area}\\ \text{between}\\ f \text{ and } x\text{-axis}\\ \text{on } [a,b]\end{array}\right)$$



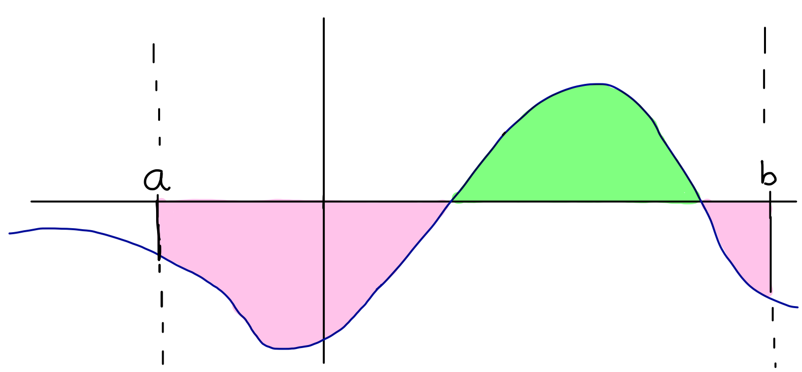
$\lim_{n\to\infty}$

limit as $n \to \infty$

gives the definite integral

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \left(\begin{array}{c}\text{exact}\\ \text{net area}\\ \text{between}\\ f \text{ and } x\text{-axis}\\ \text{on } [a,b]\end{array}\right)$$

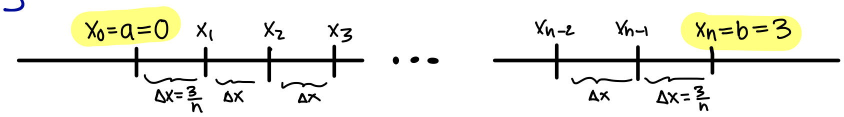
$$\int_a^b f(x)\,dx = \left(\begin{array}{c}\text{exact}\\ \text{net area}\\ \text{between}\\ f \text{ and } x\text{-axis}\\ \text{on } [a,b]\end{array}\right)$$



Example 11.3. Evaluate $\displaystyle\int_0^3 (x^3 - 6x)\,dx$ using the (limit) definition of a definite integral.

Setup integrand $f(x) = x^3 - 6x$ $\qquad \Delta x = \dfrac{b-a}{n} = \dfrac{3-0}{n} = \dfrac{3}{n}$

$a = 0,\ b = 3$

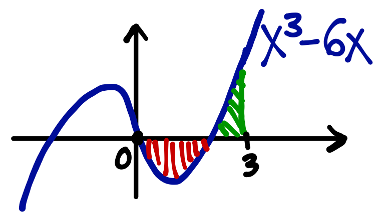


sample point (use right endpoint) $\quad x_i^* = x_i = a + i\Delta x = 0 + i\left(\frac{3}{n}\right) = \frac{3i}{n}$

height of ith rectangle $\quad f(x_i^*) = f\left(\frac{3i}{n}\right) = \left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right) = \frac{27i^3}{n^3} - \frac{18i}{n}$

So $\displaystyle\int_0^3 f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\left(\frac{27i^3}{n^3} - \frac{18i}{n}\right)}_{f(x_i^*)} \underbrace{\left(\frac{3}{n}\right)}_{\Delta x}$$



$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{81i^3}{n^4} - \frac{54i}{n^2}\right)$$

$$= \lim_{n\to\infty} \left(\frac{81}{n^4} \sum_{i=1}^{n} i^3 - \frac{54}{n^2} \sum_{i=1}^{n} i\right) \quad \text{(using basic properties of sums)}$$

Using these 2 useful formulas

$$\boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}}$$

$$\boxed{\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2}$$

$$= \lim_{n\to\infty} \left(\frac{81}{n^4}\left(\frac{n(n+1)}{2}\right)^2 - \frac{54}{n^2}\left(\frac{n(n+1)}{2}\right)\right)$$

$$= \frac{81}{4}\left(\lim_{n\to\infty} \frac{n^2(n+1)^2}{n^4}\right) - \frac{54}{2}\left(\lim_{n\to\infty} \frac{n(n+1)}{n^2}\right)$$

$$= \frac{81}{4}(1) - \frac{54}{2}(1)$$

$$= -\frac{27}{4}$$

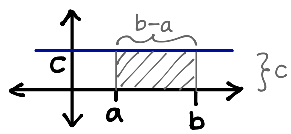
$$= -6.75$$

← We will be happy when we learn the FTC (Fundamental Theorem of Calculus)

$$\boxed{\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx}$$ $\Delta x = \frac{b-a}{n}$ vs $\Delta x = \frac{a-b}{n} = -\frac{(b-a)}{n}$

$$\boxed{\int_a^a f(x)\,dx = 0}$$ $\Delta x = \frac{a-a}{n} = 0$

1. If $c \in \mathbb{R}$, then $\boxed{\int_a^b c\,dx = c(b-a)}$.



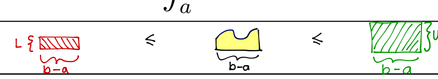
2./4. $\boxed{\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx}$

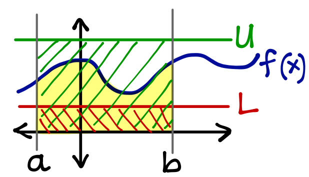
3. If $c \in \mathbb{R}$, then $\boxed{\int_a^b \big(cf(x)\big)\,dx = c\int_a^b f(x)\,dx}$

5. $\boxed{\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx}$

7. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\boxed{\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx}$

8. If $L \leq f(x) \leq U$ for $a \leq x \leq b$, then $\boxed{L(b-a) \leq \int_a^b f(x)\,dx \leq U(b-a)}$





Exercise 11.4. If $\int_1^4 f(x)\,dx = 5$ and $\int_1^4 [2f(x) + 3g(x)]\,dx = 7$, find $\int_4^1 g(x)\,dx$.

$\int_1^4 [2f(x)+3g(x)]\,dx = 2\int_1^4 f(x)\,dx + 3\int_1^4 g(x)\,dx$ (by properties 2,3,4)

$\Rightarrow \quad 7 = 2(5) + 3\int_1^4 g(x)\,dx$

$\Rightarrow \quad -1 = \int_1^4 g(x)\,dx$

(by prop. 1) $\Rightarrow \quad -1 = -\int_4^1 g(x)\,dx$, $\quad \therefore \int_4^1 g(x)\,dx = 1.$

FTC 2

The Fundamental Theorem of Calculus, Part 2

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, F is any function such that $F' = f$.

$\Longrightarrow$ We can forget about computing difficult limits of Riemann sums! FTC 2 gives us a quick way to evaluate definite integrals:

1. find an antiderivative of the integrand
2. subtract our antiderivative at the limits of integration

Notation: If $F(x)$ is an antiderivative of $f(x)$, then we have a 2-step notation:

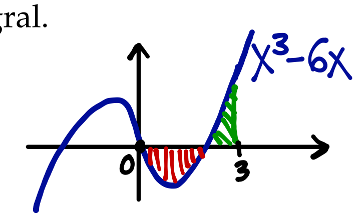
$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b$$

1 we find antiderivative

$$= F(b) - F(a)$$

$\leftarrow$ 2 subtract at limits of integration

Example 11.5. Evaluate the definite integral $\displaystyle\int_0^3 (x^3 - 6x)\,dx$ using FTC 2. Compare this procedure with the limit we used in Example 11.4 to compute the same definite integral.



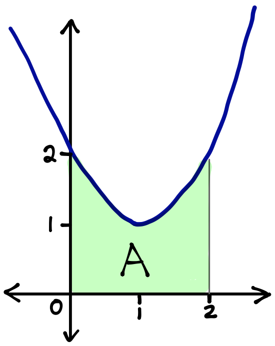
$$\int_0^3 (x^3 - 6x)\,dx = \left[\frac{x^4}{4} - 6\left(\frac{x^2}{2}\right)\right]_0^3$$ (1.)

$$= \left[\frac{x^4}{4} - 3x^2\right]_0^3$$

$$= \left(\frac{3^4}{4} - 3(3^2)\right) - \left(\frac{0^4}{4} - 3(0^2)\right)$$ (2.)

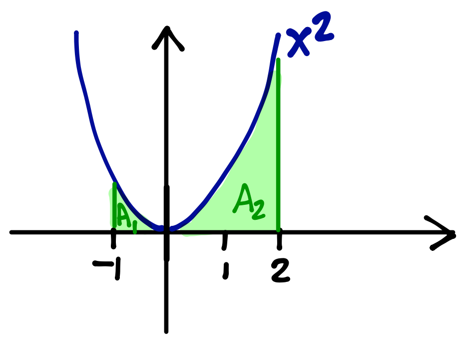
$$= \frac{81}{4} - 27 - 0 \qquad = -6\,75$$

Example 11.6. Evaluate the definite integral $\displaystyle\int_0^2 \left((x-1)^2+1\right)dx$ using FTC 2. Compare this with the Riemann sum approximations we obtained in Example 10.8.



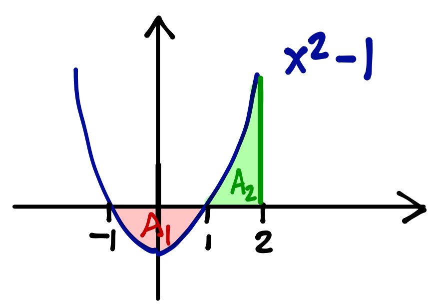
$$\int_0^2 \left((x-1)^2+1\right)dx = \int_0^2 (x^2-2x+2)\,dx$$
$$= \left[\frac{x^3}{3} - x^2 + 2x\right]_0^2$$
$$= \left(\frac{2^3}{3} - 2^2 + 2(2)\right) - \left(\frac{0^3}{3} - 0^2 + 2(0)\right)$$
$$= \frac{8}{3}$$
$$= 2.\overline{6}$$

Example 11.7. $\displaystyle\int_{-1}^2 x^2\,dx = \left[\frac{x^3}{3}\right]_{-1}^2$
$$= \frac{2^3}{3} - \frac{(-1)^3}{3}$$
$$= \frac{8}{3} + \frac{1}{3}$$
$$= 3$$



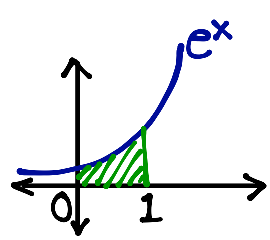
(net area $= A_1 + A_2 = 3$)

Example 11.8. $\displaystyle\int_{-1}^2 (x^2-1)\,dx = \left[\frac{x^3}{3} - x\right]_{-1}^2$
$$= \left(\frac{2^3}{3} - 2\right) - \left(\frac{(-1)^3}{3} - (-1)\right)$$
$$= 0$$



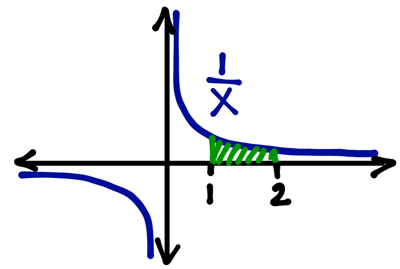
(net area $= -A_1 + A_2 = 0$)

Example 11.9. $\displaystyle\int_0^1 e^x\,dx = \left[e^x\right]_0^1$
$$= e^1 - e^0$$
$$= e - 1$$



Example 11.10. $\displaystyle\int_1^2 \frac{dx}{x}$ $\leftarrow$ short for $\displaystyle\int_1^2 \frac{1}{x}\,dx$

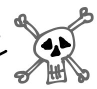
$$\int_1^2 \frac{1}{x}\,dx = \Big[\ln|x|\Big]_1^2$$
$$= \ln|2| - \ln|1|$$
$$= \ln 2$$



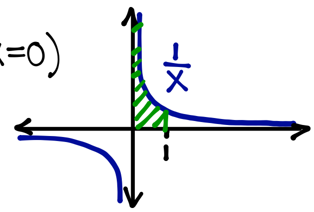
Example 11.11. $\displaystyle\int_0^1 \frac{dx}{x} \cancel{=} \Big[\ln|x|\Big]_0^1$ because $\frac{1}{x}$ is not continuous on $[0,1]$

($\frac{1}{x}$ has a vertical asymptote at $x=0$)

$\therefore$ FTC is Not Applicable

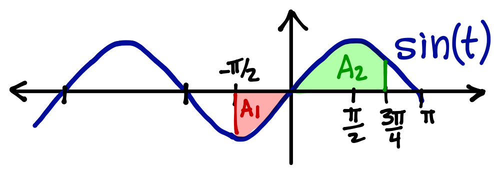


This is called an improper integral
You will learn about improper integrals
if you take MAT1322



Example 11.12. $\displaystyle\int_{-\pi/2}^{3\pi/4} \sin(t)\,dt = \Big[-\cos(t)\Big]_{-\pi/2}^{3\pi/4}$

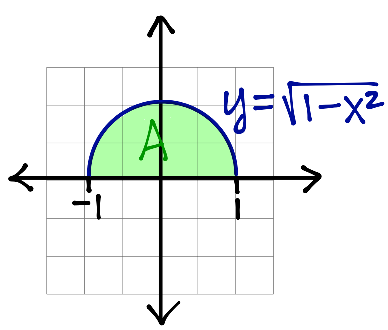
$$= -\cos\left(\tfrac{3\pi}{4}\right) - \left(-\cos(-\pi/2)\right)$$
$$= -\left(-\tfrac{1}{\sqrt{2}}\right) - \left(-(0)\right)$$
$$= \frac{1}{\sqrt{2}}$$



(net area is $-A_1 + A_2 = \frac{1}{\sqrt{2}}$)

Example 11.13. $\displaystyle\int_{-1}^1 \sqrt{1-x^2}\,dx$ *hint: draw a picture of the net area represented by this definite integral.*

$$= \tfrac{1}{2}(\text{area of unit circle})$$
$$= \tfrac{1}{2}\pi(1^2)$$
$$= \frac{\pi}{2}$$



(net area $= A = \frac{\pi}{2}$)

From now on, we will use our integral notation in two ways:

- We write $\displaystyle\int f(x)\,dx$ to represent **THE MOST GENERAL ANTIDERIVATIVE OF $f(x)$.** That is,

$$\int f(x)dx = F(x) + C \quad \text{means} \quad F'(x) = f(x)$$

The integral $\displaystyle\int f(x)\,dx$ is also called **AN INDEFINITE INTEGRAL** .

In particular, an indefinite integral represents an infinite family of functions, each member of which has derivative equal to $f(x)$.

- If there are **LIMITS OF INTEGRATION** , $\int_a^b$, then $\displaystyle\int_a^b f(x)\,dx$ is called a **DEFINITE INTEGRAL** .

By FTC (assuming $f(x)$ is continuous on $[a, b]$), the definite integral $\int_a^b f(x)dx$ equals the difference $F(b) - F(a)$, where F is any antiderivative of f. Thus, a definite integral is a **number**, *not* a family of functions. This number corresponds to the net area between $f(x)$ and the x-axis on the interval $[a, b]$.

Example 11.14. Evaluate each of the following integrals:

$$\int \frac{2}{x^2+1}\,dx = 2\arctan(x) + C$$

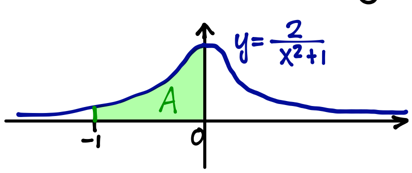
(indefinite integral)

$$\int_{-1}^{0} \frac{2}{x^2+1}\,dx = \Big[2\arctan(x)\Big]_{-1}^{0}$$

$$= 2\arctan(0) - 2\arctan(-1)$$

$$= 2(0) - 2\left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

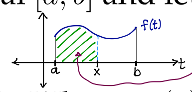
(definite integral)



(= net area = A)

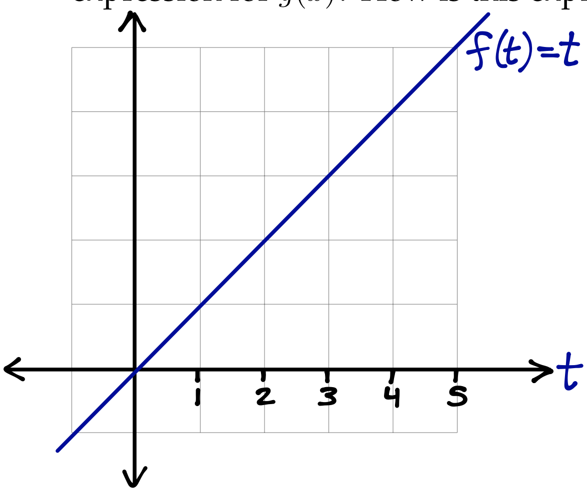
- The **Fundamental Theorem of Calculus (FTC)** is so called because it relates the two main branches of calculus: differential & integral
- The FTC has two parts. The first part tells us the derivative of a function defined by a definite integral. The second part tells us an easy way to evaluate definite integrals using antiderivatives.

Suppose $f(t)$ is a continuous function on the interval $[a, b]$ and let $g(x)$ be a function defined for all $x \in [a, b]$ as follows: $g(x) = \int_a^x f(t)\,dt \quad (a \leq x \leq b)$



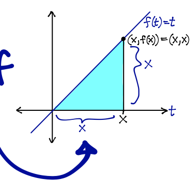
g(x) represents this net area
where $a \leq x \leq b$
(x is the upper limit of integration of g's integral)

Example 11.15. Let $f(t) = t$. Define $g(x) = \int_0^x f(t)dt$. What is $g(1)$? What is $g(4)$? Can you find an expression for $g(x)$? How is this expression related to $f(t)$?



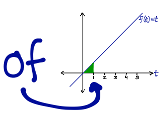
In fact, for any $x \geq 0$,

$$g(x) = \int_0^x t\,dt = \left(\text{area of}\right)$$

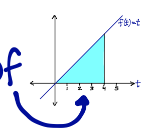


$$= \tfrac{1}{2}(\text{base})(\text{height})$$
$$= \tfrac{1}{2}(x)(x)$$
$$= \tfrac{1}{2}x^2$$

$$g(1) = \int_0^1 t\,dt = \left(\text{area of}\right) = \tfrac{1}{2}(1)(1) = \tfrac{1}{2}$$

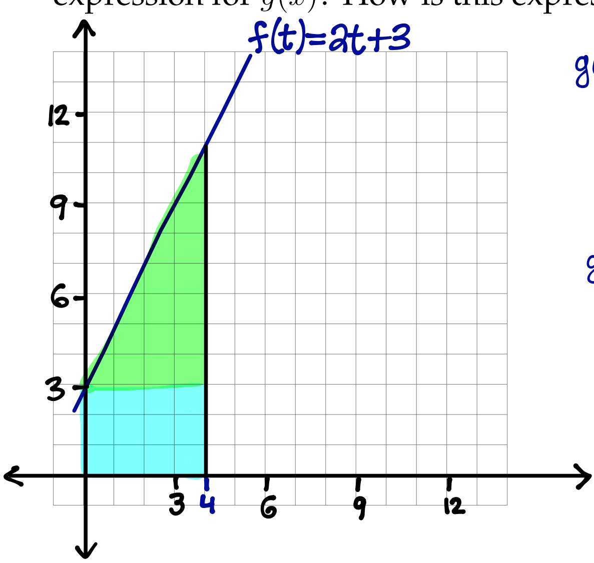


$$g(4) = \int_0^4 t\,dt = \left(\text{area of}\right) = \tfrac{1}{2}(4)(4) = 8$$



Notice
$\frac{1}{2}x^2$ is an antiderivative of $f(x)$

Exercise 11.16. Let $f(t) = 2t + 3$. Define $g(x) = \int_0^x f(t)dt$. What is $g(1)$? What is $g(4)$? Can you find an expression for $g(x)$? How is this expression related to $f(t)$?



$$g(4) = \int_0^4 (2t+3)dt = \left(\text{area } \underbrace{}_{4} \}3\right) + \left(\text{area } \underbrace{}_{4} \}2(4)\right) = 4(3) + \tfrac{1}{2}(4)(8) = 28$$

$$g(x) = \int_0^x (2t+3)dt = \left(\text{area } \underbrace{}_{x} \}3\right) + \left(\text{area } \underbrace{}_{x} \}2x\right) = 3x + \tfrac{1}{2}(x)(2x) = x^2 + 3x$$

Note x^2+3x is an antiderivative of $f(x) = 2x+3$

The Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a,b]$, then the function g defined by

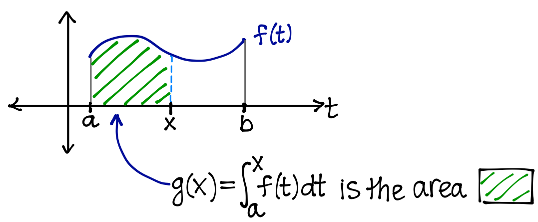
$$g(x) = \int_a^x f(t)dt \qquad\qquad a \le x \le b$$

is

- continuous on $[a,b]$,
- differentiable on (a,b), and
- $g'(x) = f(x)$. That is, $\dfrac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$

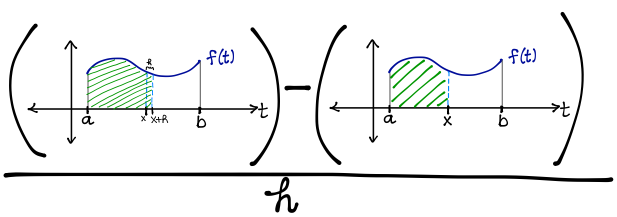
Idea behind the proof:

For simplicity, we'll consider the case when $f(t)$ lies above the t-axis on $[a,b]$

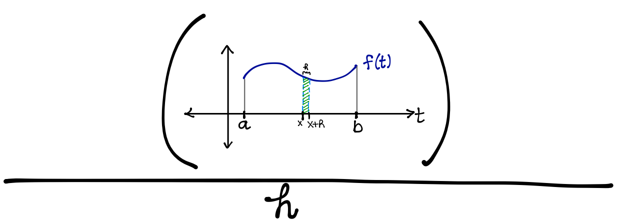


By definition,

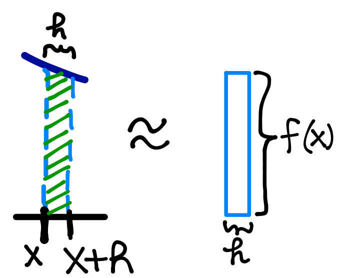
$$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$= \lim_{h\to 0}$ 

* this is very informal!
(it's only to give you the idea)

$= \lim_{h\to 0}$ 

Since h is tiny ($h \to 0$) the area of the strip that remains is approximately equal to a rectangle of width h and height $f(x)$



$$\approx \lim_{h\to 0} \frac{\cancel{h} f(x)}{\cancel{h}}$$

$$= f(x)$$

*Note Many applications of integration in MAT 1322 use this idea of approximating the area of a thin strip under a function

Example 11.17. Find $\frac{d}{dx}\left[\int_2^x \sqrt{1+t^2}dt\right]$

integrand
$f(t)=\sqrt{1+t^2}$

By FTC1, $\frac{d}{dx}\left[\int_2^x f(t)dt\right] = f(x) = \sqrt{1+x^2}$

Example 11.18. Find the derivative of the function $g(x) = \int_1^{x^3} \sin(t)dt$.

integrand
$f(t)= \sin(t)$

Note $g(x) = g(u(x))$ where $u(x) = x^3$

Thus, $g(u) = \int_1^u \sin(t)dt$ and $u'(x) = 3x^2$

By FTC1, $g'(u) = f(u) = \sin(u)$

By Chain Rule, $\frac{d}{dx}[g(u(x))] = g'(u(x))\, u'(x)$

$= \sin(u(x))(3x^2) = \sin(x^3)(3x^2)$

Net Change Theorem

By FTC2, we know that

$$\int_a^b f(x)dx = F(b) - F(a) \qquad \text{where } F' = f$$

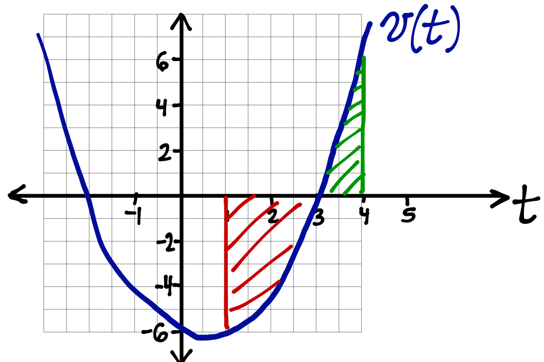
Thus, we can rewrite FTC2 as follows:

$$\int_a^b F'(x)dx = F(b) - F(a)$$

$\Longrightarrow$ the integral of a rate of change is the **net change**.

Example 11.19. A particle moves along a line so that its velocity at time t is given by $v(t) = t^2 - t - 6$ (measured in m/s).

(a) Sketch the graph of $v(t)$ over the time interval $[0, 5]$



(b) Find the displacement of the particle during the time period $1 \le t \le 4$

net change in position (on line) from $1 \le t \le 4$

Let $s(t)$ denote the particle's position on the line as a function of time t

$\therefore$ particle's displacement for $1 \le t \le 4$ is the net change $s(4) - s(1)$

Note: $v(t)$ = rate of change of position with respect to time ie $v(t) = s'(t)$

$\therefore$ by the Net Change Theorem,

$$\text{displacement} = s(4) - s(1) = \int_1^4 s'(t)\,dt$$

$$= \int_1^4 v(t)\,dt$$

$$= \int_1^4 (t^2 - t - 6)\,dt$$

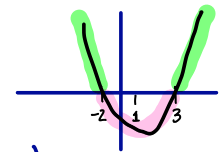
$$= \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4$$

$$= \left(\frac{4^3}{3} - \frac{4^2}{3} - 6(4)\right) - \left(\frac{1^3}{3} - \frac{1^2}{3} - 6(1)\right)$$

$$= -\frac{9}{2}\text{ m}$$

(c) Find the distance travelled by the particle during the time period $1 \le t \le 4$

if we want to know the total distance travelled by particle, we would like to add the distance travelled by the particle in the negative direction instead of subtracting it.



Thus, we want

$$\int_1^4 |v(t)|dt = \overbrace{\int_1^3 -v(t)dt}^{\text{add the distance travelled by particle in negative direction}} + \overbrace{\int_3^4 v(t)dt}^{\text{add the distance travelled by particle in positive direction}}$$

$$|v(t)| = |t^2 - t - 6| = \begin{cases} t^2 - t - 6 & \text{if } t \geq 3 \\ -(t^2 - t - 6) & \text{if } -2 < t < 3 \\ t^2 - t - 6 & \text{if } t \leq -2 \end{cases}$$

$$= \int_1^3 (-t^2 + t + 6)dt + \int_3^4 (t^2 - t - 6)dt$$

$$= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 6t\right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4$$

$$= \left(-\frac{3^3}{3} + \frac{3^2}{2} + 6(3)\right) - \left(-\frac{1}{3} + \frac{1}{2} + 6\right) + \left(\frac{4^3}{3} - \frac{4^2}{2} - 24\right) - \left(\frac{3^3}{3} - \frac{3^2}{2} - 18\right)$$

$$= \frac{22}{3} + 2.8\overline{3}$$

$$\approx 10.17 \text{ m}$$

STUDY GUIDE

- ◇ **definition of the definite integral**(if it exists): $\displaystyle\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$
- ◇ **net area interpretation of definite integral**
- ◇ **evaluating definite integrals using known sums and properties of integrals**
- ◇ **FTC1**: If $g(x) = \displaystyle\int_a^x f(t)dt$, then $g'(x) = f(x)$.
- ◇ **FTC2**: If f is continuous on $[a, b]$, then
$$\int_a^b f(x)dx = F(b) - F(a)$$
where F is any antiderivative of f.
- ◇ **indefinite integral** vs. **definite integral**
- ◇ **the Net Change Theorem**: $\displaystyle\int_a^b F'(x)dx = F(b) - F(a)$
(integral of a rate of change is the net change)

12. Integration by Substitution

Lec 11 mini review.

FTC Suppose f is continuous on $[a, b]$.

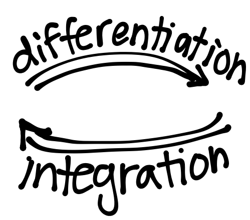
FTC1 If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.

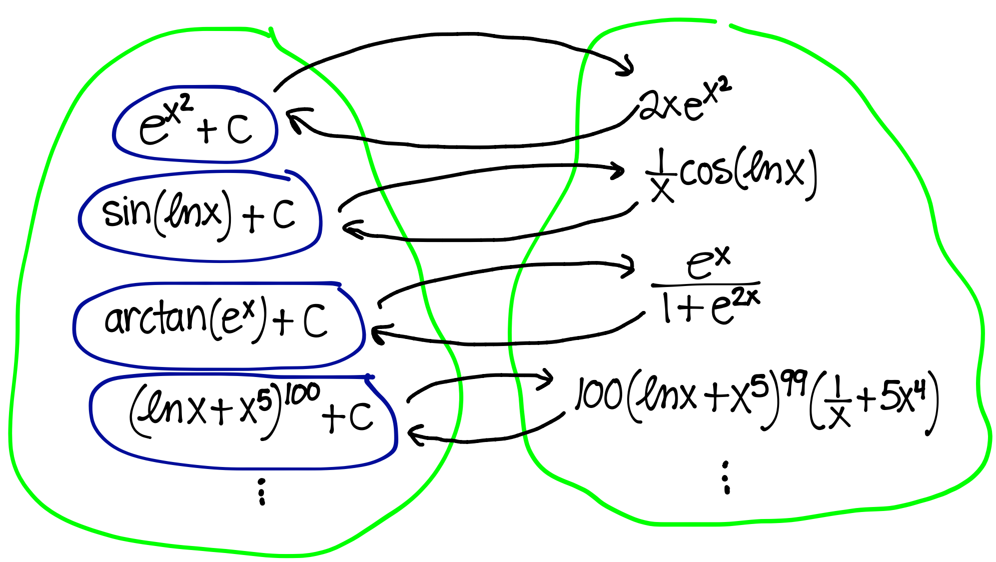
FTC2 $\int_a^b f(x)dx = F(b) - F(a)$ where F is any antiderivative of f (that is, $F' = f$).

- ◇ **indefinite integral** vs. **definite integral**
- ◇ **the Net Change Theorem for the integral of a rate of change**: $\displaystyle\int_a^b F'(x)dx = F(b) - F(a)$

"UNDOING" THE CHAIN RULE

- If we think of differentiation as an operation on the set of differentiable functions, then integration (informally, "anti-differentiation") does the opposite.
- What does "undoing" the Chain Rule look like?
- There are many different instances of the Chain Rule...





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GUIDELINES FOR SUBSTITUTION

0. Inspect the integrand. Is it familiar because it's a function's derivative (give or take a constant multiple) ? If so, you should know what to do.

1. If the integrand is not an "obvious" familiar function's derivative, then consider the possibility that the integrand may be the aftermath of a Chain Rule:

Does integrand look roughly like the aftermath...

...of a power chain rule ? e.g. $\int k(g(x))^n g'(x)\,dx$

...of an exp. chain rule? e.g. $\int k g'(x) e^{g(x)}\,dx$

...of a ln chain rule ? e.g. $\int k \frac{g'(x)}{g(x)}\,dx$

...of a trig chain rule ? e.g. $\int k\cos(g(x))g'(x)\,dx$ etc...

2. If you see a potential "inner" function $g(x)$, then call it u (you might be wrong, but it doesn't hurt to try).

Now, compute your substitution ingredients:

Try $u = g(x)$ Then $\frac{du}{dx} = g'(x)$

$$\Longrightarrow dx = \frac{du}{g'(x)}$$

3. Rewrite your integral in terms of u and du. $\int f(g(x))g'(x)\,dx = \int f(u)\,du$

Note. The "old" variable (let's say it was x) should completely cancel. We should get a new (and hopefully easier) integral with respect to the "new" variable u.

4. Evaluate the new integral. Then don't forget to rewrite your answer in terms of the "old" variable.

$\int f(u)\,du = F(u) + C$ where $F' = f$

$= F(g(x)) + C$ since $u = g(x)$

Example 12.1. $\int 2xe^{x^2}\,dx.$

$$\boxed{\begin{array}{l}\text{u-substitution}\\ u = x^2\\ \frac{du}{dx} = 2x\\ dx = \frac{du}{2x}\end{array}}$$

$$= \int \cancel{2x} e^u \cdot \frac{du}{\cancel{2x}}$$

$$= \int e^u\,du$$

$$= e^u + C$$

$$= e^{x^2} + C$$

Check

$\frac{d}{dx}[e^{x^2} + C] = 2xe^{x^2}$ ✓

Example 12.2. $\int x^2 e^{x^3-9}\,dx.$

x^2 is "roughly" the derivative of x^3-9

$$\boxed{\begin{array}{l}\text{Try } u = x^3-9\\ \Rightarrow \frac{du}{dx} = 3x^2\\ \Rightarrow dx = \frac{du}{3x^2}\end{array}}$$

$$= \int x^2 e^u \left(\frac{du}{3x^2}\right)$$

$$= \int \frac{1}{3} e^u\,du$$

$$= \frac{1}{3}e^u + C$$

$$= \frac{1}{3}e^{x^3-9} + C$$

Check:

$\frac{d}{dx}\left[\frac{1}{3}e^{x^3-9} + C\right] = \left(\frac{1}{3}e^{x^3-9}\right)(3x^2-0)$

$= x^2e^{x^3-9}$ ✓

Example 12.3. $\int \sin(2t)\,dt.$

We see an "inner" function $2t$ in $\sin(\ )$

$$\boxed{\begin{array}{l}\text{Try } u = 2t\\ \Rightarrow \frac{du}{dt} = 2\\ \Rightarrow dt = \frac{1}{2}du\end{array}}$$

$$= \int \sin(u)\left(\frac{1}{2}du\right)$$

$$= \frac{1}{2}\int \sin(u)\,du$$

$$= \frac{1}{2}(-\cos(u)) + C$$

$$= -\frac{1}{2}\cos(2t) + C$$

Check!

Example 12.4. $\int \tan(\theta)\,d\theta.$

$$= \int \frac{\sin\theta}{\cos\theta}\,d\theta$$

← numerator is "roughly" the derivative of the denominator... this could be the aftermath of a $\ln(\ )$ chain rule...

$$\boxed{\begin{array}{l}\text{try } u = \cos\theta\\ \Rightarrow \frac{du}{d\theta} = -\sin\theta\\ \Rightarrow d\theta = \frac{du}{-\sin\theta}\end{array}}$$

$$= \int \frac{\cancel{\sin\theta}}{(u)} \cdot \frac{du}{(-\cancel{\sin\theta})}$$

$$= \int -\frac{1}{u}\,du$$

$$= -\ln|u| + C$$

Note

$-\ln|\cos\theta| = \ln|(\cos\theta)^{-1}| = \ln|\sec\theta|$

$$\therefore \int \tan\theta\,d\theta = \ln|\sec\theta| + C$$

Check!

Example 12.5. $\int \overset{?}{\sec}(x)dx.$ $= \int \frac{\overset{?}{1}}{\cos x}dx$ (seemingly unhelpful...)

a clever trick: $\int \sec x\,dx = \int \sec x\left(\frac{\sec x + \tan x}{\sec x + \tan x}\right)dx$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}dx$$

$$\boxed{\begin{array}{l}\text{Now try u-sub:}\\ u = \sec x + \tan x\\ \frac{du}{dx} = \sec x \tan x + \sec^2 x\end{array}}$$

$$= \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$= \ln|\sec x + \tan x| + C$$

Example 12.6. $\int_{-\pi/2}^{2\pi} \cos(x)\sin^3(x)dx.$

$\int \cos x(\sin x)^3 dx$ (indefinite for now...)

$$\boxed{\begin{array}{l}u = \sin x\\ \frac{du}{dx} = \cos x\\ \Rightarrow dx = \frac{du}{\cos x}\end{array}}$$

$$= \int \cancel{\cos x}\cdot u^3 \cdot \frac{du}{\cancel{\cos x}}$$

$$= \int u^3 du$$

$$= \frac{u^4}{4}$$

$$= \frac{\sin^4(x)}{4}$$ ← now we know antiderivative

go back to definite integral

$$\therefore \int_{-\frac{\pi}{2}}^{2\pi} \cos x \sin^3 x\,dx = \left[\frac{\sin^4 x}{4}\right]_{-\pi/2}^{2\pi}$$

$$= \frac{1}{4}(\sin 2\pi)^4 - \frac{1}{4}\left(\sin\left(-\frac{\pi}{2}\right)\right)^4$$

$$= \frac{1}{4}(0^2) - \frac{1}{4}(-1)^4$$

$$= -\frac{1}{4}$$

Substitution Methods for Definite Integrals.

1. Solve the indefinite integral first, and completely (in terms of "old" variable), then subtract at the limits of integration.

$$\int_a^b F'(g(x))g'(x)dx = \left[\int F'(u)du\right]_{x=a}^{x=b} = \left[F(g(x))\right]_{x=a}^{x=b} = F(g(b)) - F(g(a))$$

2. Substitute the limits of integration at the same time as you perform the substitution.

$$\int_a^b F'(g(x))g'(x)dx = \int_{u=g(a)}^{u=g(b)} F'(u)du = \left[F(u)\right]_{u=g(a)}^{u=g(b)} = F(g(b)) - F(g(a))$$

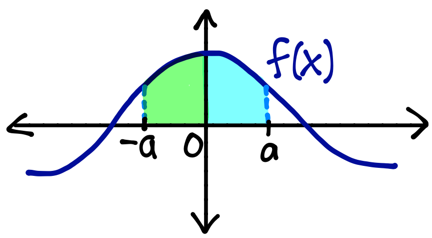
Example 12.7. $\displaystyle\int_1^e \frac{\ln x}{x}\,dx. = \int_{u=0}^{u=1} \frac{u}{x}(x\,du) = \int_{u=0}^{u=1} u\,du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2}$

$$\boxed{\begin{array}{ll} u = \ln x & \\ \dfrac{du}{dx} = \dfrac{1}{x} & x = e \Rightarrow u = \ln e = 1 \\ dx = x\,du & x = 1 \Rightarrow u = \ln 1 = 0 \end{array}}$$

Integrals of Functions with Even/Odd Symmetry

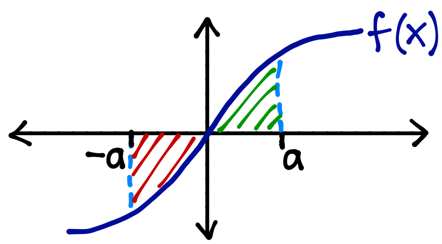
Let $a \in \mathbb{R}$. Suppose f is continuous on $[-a, a]$.

- ◇ If f has even symmetry, that is $f(-x) = f(x)$ for all $x \in [-a, a]$, then



$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

- ◆ If f has odd symmetry, that is $f(-x) = -f(x)$ for all $x \in [-a, a]$, then



$$\int_{-a}^{a} f(x)\,dx = 0$$

Example 12.8. $\displaystyle\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^6}\,dx = 0$

* limits of integration are symmetric about the y-axis

* the integrand has odd symmetry since $\dfrac{\tan(-x)}{1+(-x)^2+(-x)^6} = -\dfrac{\tan x}{1+x^2+x^6}$

- $\tan(-x) = \dfrac{\sin(-x)}{\cos(-x)} = \dfrac{-\sin(x)}{\cos(x)} = -\tan(x)$
- $1+(-x)^2+(-x)^6 = 1+x^2+x^6$

Study Guide

- ◇ **strategy for integration by substitution**
- ◇ **two ways to evaluate definite integrals via substitution**
- ◇ **using even/odd symmetry to evaluate certain integrals**

13. Integration by Parts and Trig Integrals

Lec 12 mini review.

Substitution: $\int F'(g(x))g'(x)dx = \int F'(u)du = F(u) + C = F(g(x)) + C$

Integrals with Even Symmetry: If $f(-x) = f(x)$, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

Integrals with Odd Symmetry: If $f(-x) = -f(x)$, then $\int_{-a}^{a} f(x)dx = 0$

"UNDOING" THE PRODUCT RULE

Example 13.1. $\int xe^x dx$

Try substitution?

$u = x$?

$\frac{du}{dx} = 1$

$du = dx$

$\Rightarrow \int ue^u du$

works, but unhelpful

$u = e^x$?

$\frac{du}{dx} = e^x \Rightarrow \frac{du}{e^x} = dx$

$\Rightarrow \int xu\frac{du}{e^x} = \int xu\frac{du}{u} = \int x du$

can't cancel x?

wait: $x = \ln u$, so

$\int x du = \int \ln u \, du = ?$

$u = xe^x$?

$\frac{du}{dx} = e^x + xe^x$

$dx = \frac{du}{e^x + xe^x}$

$\Rightarrow \int u\frac{du}{e^x + xe^x} = \int \frac{u du}{e^x + u}$?

can't cancel x?

Recall the Chain Rule: $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$.

Some clues that might indicate we should try integration by substitution:

- we might see a composition in the integrand
- we might notice a function that could be the "inner" function $g(x)$, and its derivative will also be a factor of the integrand.
- me might see what looks like the aftermath of a Chain Rule derivative (like power chain rule, exponential chain rule, log chain rule, etc.)

In contrast, the Product Rule is $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$.

"Undoing" the product rule is less obvious than "undoing" the chain rule:

- If we want to integrate $\int (f'(x)g(x) + f(x)g'(x))dx$, we might be lucky enough to recognize f and g
- if the integrand is the result of a product rule, then the factors need not be related to each other whatsoever

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$$(uv)' = u'v + uv'$$
$$\Rightarrow \int (uv)' = \int u'v + uv'$$
$$\Rightarrow \quad uv = \int u'v + \int uv'$$
$$\Rightarrow \int uv' = uv - \int u'v$$

Integration by Parts

$$\boxed{\int uv' = uv - \int u'v}$$

we get a new integral to consider

OR

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Guidelines for I.B.P.

$$\int f(x)\,dx$$

- the integrand must be rethought as a product $f(x) = u(x)v'(x)$ (although factors of the product could be 1)
  - you choose the "parts" u and v'
  - when you choose u, you need to be able to calculate its derivative u'
  - when you choose v', you need to be able to calculate its antiderivative v
  - **Goal**: the "new" integral $\int u'v$ should be no worse than the original integral $\int uv'$

Example 13.2. $\int xe^x\,dx$

Choice of Parts:

$u = x \quad v' = e^x$ $u' = 1 \quad v = e^x$ $\int xe^x dx = xe^x - \underbrace{\int (1)e^x dx}_{\text{looks easier}}$	$u = e^x \quad v' = x$ $u' = e^x \quad v = \frac{x^2}{2}$ $\int xe^x dx = \frac{x^2}{2}e^x - \underbrace{\int \frac{x^2}{2} e^x dx}_{\text{looks worse}}$

$$\int xe^x dx = xe^x - \int (1)e^x dx$$
$$= xe^x - \int e^x dx$$
$$= xe^x - e^x + C$$

Check!

$$\frac{d}{dx}\left[xe^x - e^x\right] = (1)e^x + xe^x - e^x$$
$$= xe^x \checkmark$$

Example 13.3. $\int x^3 e^{2x}\, dx$

$$\int x^3e^{2x}dx = x^3\left(\tfrac{1}{2}e^{2x}\right) - \int (3x^2)\left(\tfrac{1}{2}e^{2x}\right)dx$$
$$= \tfrac{1}{2}x^3e^{2x} - \int \tfrac{3}{2}x^2e^{2x}dx$$
$$= \tfrac{1}{2}x^3e^{2x} - \left[\tfrac{3}{2}x^2\left(\tfrac{1}{2}e^{2x}\right) - \int 3x\left(\tfrac{1}{2}e^{2x}\right)dx\right]$$
$$= \tfrac{1}{2}x^3e^{2x} - \tfrac{3}{4}x^2e^{2x} + \int \tfrac{3}{2}xe^{2x}dx$$
$$= \tfrac{1}{2}x^3e^{2x} - \tfrac{3}{4}x^2e^{2x} + \left[\left(\tfrac{3}{2}x\right)\left(\tfrac{1}{2}e^{2x}\right) - \int\left(\tfrac{3}{2}\right)\left(\tfrac{1}{2}e^{2x}\right)dx\right]$$
$$= \tfrac{1}{2}x^3e^{2x} - \tfrac{3}{4}x^2e^{2x} + \tfrac{3}{4}xe^{2x} - \int \tfrac{3}{4}e^{2x}dx$$
$$= \tfrac{1}{2}x^3e^{2x} - \tfrac{3}{4}x^2e^{2x} + \tfrac{3}{4}xe^{2x} - \tfrac{3}{8}e^{2x} + C$$

Parts I
$u = x^3$ $v' = e^{2x}$
$u' = 3x^2$ $v = \frac{1}{2}e^{2x}$

Parts II
$u = \frac{3}{2}x^2$ $v' = e^{2x}$
$u' = 3x$ $v = \frac{1}{2}e^{2x}$

Parts I
$u = \frac{3}{2}x$ $v' = e^{2x}$
$u' = \frac{3}{2}$ $v = \frac{1}{2}e^{2x}$

Example 13.4. $\int \ln(x)\, dx$ ← this doesn't look like a product but...

$$\int \ln x\, dx = (\ln x) \quad - \int \left(\tfrac{1}{x}\right)(x)\, dx$$
$$= x\ln x - \int dx$$
$$= x\ln x - x + C$$

$u = \ln x$ $v' = 1$
$u' = \frac{1}{x}$ $v = x$

Check! $\frac{d}{dx}[x\ln x - x] = (1)\ln x + x\left(\frac{1}{x}\right) - 1$
$= \ln x$ ✓

Example 13.5. $\int (x^4 + 2x - 9)\ln(x)\, dx$

Try similar parts:

$$\int (x^4+2x-9)\ln(x)\,dx$$
$$= (\ln x)\left(\tfrac{x^5}{5} + x^2 - 9x\right) - \int\left(\tfrac{x^5}{5} + x^2 - 9x\right)\left(\tfrac{1}{x}\right)dx$$
$$= (\ln x)\left(\tfrac{x^5}{5} + x^2 - 9x\right) - \int\left(\tfrac{x^4}{5} + x - 9\right)dx$$
$$= (\ln x)\left(\tfrac{x^5}{5} + x^2 - 9x\right) - \tfrac{x^5}{25} - \tfrac{x^2}{2} + 9x + C$$

$u = \ln x$ $dv = (x^4+2x-9)dx$
$\frac{du}{dx} = \frac{1}{x}$ $\frac{dv}{dx} = x^4+2x-9$
$du = \frac{1}{x}dx$ $v = \frac{x^5}{5} + x^2 - 9x$

Common "Parts"

Each iteration of IBP

$\int x^n e^{kx}\,dx$

$$\boxed{\begin{array}{ll} u = x^n & v' = e^{kx} \\ u' = nx^{n-1} & v = \frac{1}{k}e^{kx} \end{array}}$$

$\leftarrow$ reduces power of x by one

$\int x^n (\ln x)^m\,dx$

$$\boxed{\begin{array}{ll} u = (\ln x)^m & v' = x^n \\ u' = m(\ln x)^{m-1}\cdot\left(\frac{1}{x}\right) & v = \frac{1}{n+1}x^{n+1} \end{array}}$$

$\leftarrow$ reduces power of $\ln x$ by one

Example 13.6. $\int x^3(\ln x)^2 dx$

$$\boxed{\begin{array}{ll} u = (\ln x)^2 & v' = x^3 \\ u' = 2(\ln x)\left(\frac{1}{x}\right) & v = \frac{1}{4}x^4 \end{array}}$$

$$\boxed{\begin{array}{ll} u = \ln x & v' = x^3 \\ u' = \frac{1}{x} & v = \frac{1}{4}x^4 \end{array}}$$

$$\begin{aligned}
&\int x^3(\ln x)^2 dx \\
&= (\ln x)^2\left(\tfrac{1}{4}x^4\right) - \int 2(\ln x)\left(\tfrac{1}{x}\right)\left(\tfrac{1}{4}x^4\right)dx \\
&= \tfrac{1}{4}x^4(\ln x)^2 - \tfrac{1}{2}\int x^3 \ln x\, dx \\
&= \tfrac{1}{4}x^4(\ln x)^2 - \tfrac{1}{2}\left[(\ln x)\left(\tfrac{1}{4}x^4\right) - \int\left(\tfrac{1}{x}\right)\left(\tfrac{1}{4}x^4\right)dx\right] \\
&= \tfrac{1}{4}x^4(\ln x)^2 - \tfrac{1}{8}x^4\ln x + \tfrac{1}{8}\int x^3 dx \\
&= \tfrac{1}{4}x^4(\ln x)^2 - \tfrac{1}{8}x^4\ln x + \tfrac{1}{8}\left(\tfrac{1}{4}x^4\right) + C \\
&= \tfrac{1}{4}x^4(\ln x)^2 - \tfrac{1}{8}x^4\ln x + \tfrac{1}{32}x^4 + C \\
&= \tfrac{1}{4}x^4\left((\ln x)^2 - \tfrac{1}{2}\ln x + \tfrac{1}{8}\right) + C
\end{aligned}$$

check: $\frac{d}{dx}\left[\frac{1}{4}x^4\left((\ln x)^2 - \frac{1}{2}\ln x + \frac{1}{8}\right)\right] = x^3\left((\ln x)^2 - \frac{1}{2}\ln x + \frac{1}{8}\right) + \frac{1}{4}x^4\left(2(\ln x)\left(\frac{1}{x}\right) - \frac{1}{2}\left(\frac{1}{x}\right)\right)$

$= x^3(\ln x)^2 - \frac{1}{2}x^3\ln x + \frac{1}{8}x^3 + \frac{1}{2}x^3\ln x - \frac{1}{8}x^3$

$= x^3(\ln x)^2$ ✓

Sometimes, when choosing "parts" there seem to be ties... ? vs. ?

Example 13.7. $\int e^x \sin(x)dx$

$\boxed{\begin{array}{ll} u=\sin x & v'=e^x \\ u'=\cos x & v=e^x \end{array}}$ $\boxed{\begin{array}{ll} u=e^x & v'=\sin x \\ u'=e^x & v=-\cos x \end{array}}$

Let's try these parts

$$\int e^x \sin x\,dx = (\sin x)e^x - \int(\cos x)e^x dx$$

$$= e^x \sin x - \int \underline{e^x \cos x}\, dx$$

this new integral is no better/no worse than the original

$\boxed{\begin{array}{ll} u=\cos x & v'=e^x \\ u'=-\sin x & v=e^x \end{array}}$

$$= e^x \sin x - \left[(\cos x)e^x - \int(-\sin x)e^x dx\right]$$

$$= e^x \sin x - e^x \cos x - \int \underline{e^x \sin x}\, dx$$

original integral is back!?!

$$\therefore \underbrace{\int e^x \sin x\, dx}_{I} = e^x \sin x - e^x \cos x - \underbrace{\int e^x \sin x\, dx}_{I}$$

← now, we have an equation with one unknown: $I = \int e^x \sin x\, dx$

$$\Rightarrow 2I = 2\int e^x \sin x\, dx = e^x \sin x - e^x \cos x$$

$$\Rightarrow I = \int e^x \sin x\, dx = \frac{1}{2}\left(e^x \sin x - e^x \cos x\right) + C$$

Trig Integrals

◇ For certain integrals involving trig functions, the trick is to make use of trig identities before integrating.

✓ • $\cos^2 x + \sin^2 x = 1$

$(\div \cos^2 x)$ ✓ • $1 + \tan^2 x = \sec^2 x$

double-angle formulas:

✓ • $\sin 2x = 2\sin x \cos x$

✓ • $\cos 2x = \cos^2 x - \sin^2 x$

use $\cos 2x = \cos^2 x - \sin^2 x$ and $\cos^2 x + \sin^2 x = 1$ to get half-angle formulas:

$$\cos 2x = 1 - \sin^2 x - \sin^2 x$$

$$\Rightarrow \boxed{\sin^2 x = \frac{1}{2}(1 - \cos 2x)}$$

$$\cos 2x = \cos^2 x - (1 - \cos^2 x)$$

$$\Rightarrow \boxed{\cos^2 x = \frac{1}{2}(1 + \cos 2x)}$$

Example 13.8. $\int \sin^{17}(x)\,dx$

use trig identity:

$\sin^2 x + \cos^2 x = 1$

$$\int \sin^{17}(x)\,dx = \int \sin(x)\cdot(\sin^2 x)^8\,dx$$
$$= \int \sin(x)(1-\cos^2 x)^8\,dx$$
$$= \int \cancel{\sin(x)}(1-u^2)^8 \frac{du}{-\cancel{\sin x}}$$
$$= \int -\underbrace{(1-u^2)^8}\,du$$

use u-substitution:

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$dx = \frac{du}{-\sin x}$

if expanded, this is easy to integrate (just a polynomial)

Okay, same idea but a baby version:

$$\int \sin^5 x\,dx = \int \sin x(1-\cos^2 x)^2\,dx = \int -(1-u^2)^2\,du = -\int(1-2u^2+u^4)\,du$$
$$= -\left(u - \tfrac{2}{3}u^3 + \tfrac{1}{5}u^5\right) + C = -\cos x + \tfrac{2}{3}\cos^3 x - \tfrac{1}{5}\cos^5 x + C$$

Example 13.9. $\int \sin^4(x)\,dx$ $= \int (\sin^2 x)^2\,dx$

$$= \int \left(\tfrac{1}{2}(1-\cos 2x)\right)^2 dx$$

half-angle formula! $\sin^2 x = \frac{1}{2}(1-\cos(2x))$

$$= \tfrac{1}{4}\int (1-\cos 2x)^2\,dx$$
$$= \tfrac{1}{4}\int \left(1 - 2\cos 2x + \cos^2(2x)\right)dx$$
$$= \tfrac{1}{4}\int \left(1 - 2\cos(2x) + \tfrac{1}{2}(1+\cos(4x))\right)dx$$

half-angle formula! $\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$

with $\theta = 2x$

$$= \tfrac{1}{4}\int \left(1 - 2\cos(2x) + \tfrac{1}{2} + \tfrac{1}{2}\cos(4x)\right)dx$$
$$= \tfrac{1}{4}\left(x - \sin(2x) + \tfrac{1}{2}x + \tfrac{1}{8}\sin(4x)\right) + C$$

Study Guide

- ◇ **strategy for integration by substitution**
- ◇ **integration by parts**: $\int uv' = uv - \int u'v$
- ◇ **making use of trig identities before integrating**