9. Related Rates & Linear Approximations

Lec 8 mini review.

implicit differentiation strategy

derivative rules for inverse trig functions:

$$\boxed{\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}}$$

$$\boxed{\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2}}$$

(and others!)

logarithmic differentiation strategy

derivative rules for logs:

$$\boxed{\frac{d}{dx}[\ln(x)] = \frac{1}{x}}$$

$$\boxed{\frac{d}{dx}[\log_b(x)] = \left(\frac{1}{\ln b}\right)\frac{1}{x}}$$

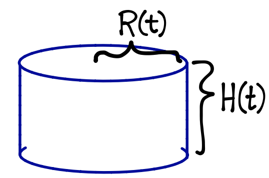
$$\boxed{\frac{d}{dx}[\ln\left(g(x)\right)] = \frac{g'(x)}{g(x)}}$$

WARM-UP TO RELATED RATES

Example 9.1. For each of the following equations, implicitly differentiate both sides with respect to the time variable t.

(volume of a cylinder whose dimensions might be changing as a function of time)

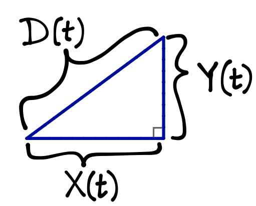
$$V(t) = \pi\big[R(t)\big]^2 H(t)$$



$$V'(t) = \pi(2)[R(t)]^1 \cdot R'(t)H(t) + \pi[R(t)]^2 \cdot H'(t)$$

(sides of a right-angled triangle, which are changing as a function of time)

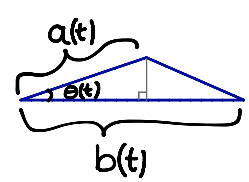
$$D^2 = X^2 + Y^2$$



$$2D \cdot \frac{dD}{dt} = 2X\frac{dX}{dt} + 2Y\frac{dY}{dt}$$

(the area of a triangle whose sides and angle are changing as time goes on)

$$A = \frac{1}{2}ab\sin\theta$$



$$\frac{dA}{dt} = \frac{1}{2}\frac{da}{dt}b\sin\theta + \frac{1}{2}a\frac{db}{dt}\sin\theta + \frac{1}{2}ab\cos\theta \cdot \frac{d\theta}{dt}$$

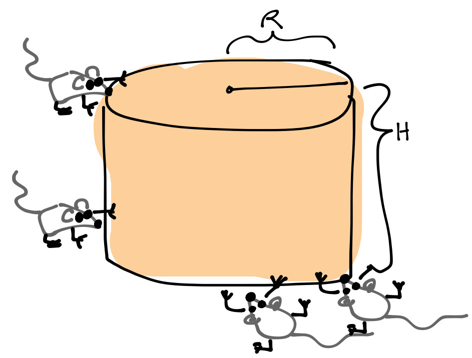
Related Rates Strategy

- Identify the variables in the problem and draw a diagram if you can.
- Determine what rates of change (derivatives) are given, and what the question is asking for (usually a rate of change at a given point in time).
- Find an equation that relates the variables to each other at all times.
- Implicitly differentiate the equation, with respect to time.
- Use the equation and the implicitly differentiated equation to solve for the desired quantity or rate.

Example 9.2. Mice are systematically eating a huge cylindrical wheel of cheese. The cheese is shrinking! The radius shrinks at a rate of 2cm/min and the height of the cheese cylinder shrinks 5cm / min. At what rate is the volume of the cheese changing when, at some point in time, its radius is 10 cm and its height is 20 cm?

Variables
R = radius of cheese
H = height of cheese
V = volume of cheese

Given
$\frac{dR}{dt} = -2$ cm/min
$\frac{dH}{dt} = -5$ cm/min



want $\frac{dV}{dt}$ when $R = 10$ cm and $H = 20$ cm

eqn $V = \pi R^2 H$ ← (true at all times, assuming cheese is always a cylinder, even though V, R, and H are changing as time passes)

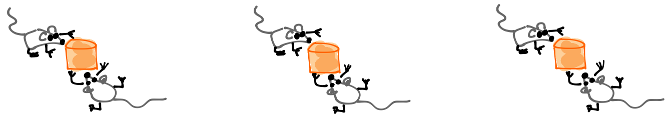
imp. diff. $\frac{dV}{dt} = 2\pi R \frac{dR}{dt} H + \pi R^2 \frac{dH}{dt}$

Plug in what we know:
(at time when $R = 10$ and $H = 20$)

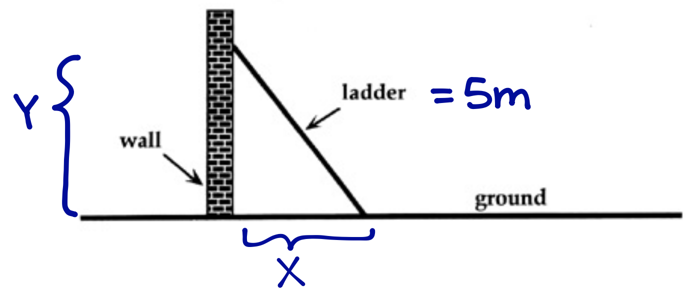
$$\frac{dV}{dt} = 2\pi(10)(-2)(20) + \pi(10^2)(-5)$$

$$\Rightarrow \frac{dV}{dt} = -1300\pi \text{ cm}^3/\text{min}$$

$\therefore$ at that moment in time, the cheese's volume is decreasing at a rate of 1300π cm^3/min.



Example 9.3. A ladder is leaning against a wall. The ladder is 5 m long. The top of the ladder is sliding down the wall at a rate of 2 m/s. At the same time, the bottom of the ladder is sliding away from the wall. When the bottom of the ladder is 4 m away from the wall, how fast is it sliding away from the wall?



Variables X = distance between wall and bottom of ladder
Y = distance between top of ladder and the ground below

Given ladder = 5 m $\quad \frac{dY}{dt} = -2$ m/s (Y is decreasing as time goes on)

want $\frac{dX}{dt}$ when $X = 4$m

eqn $$X^2 + Y^2 = 5^2$$

imp. diff. $$2X\frac{dX}{dt} + 2Y\frac{dY}{dt} = 0$$

when $X=4$: $$2(4)\frac{dX}{dt} + 2Y(-2) = 0$$

when $X=4$, solve for Y using eqn
$\Rightarrow Y = \sqrt{25-4^2} = 3$

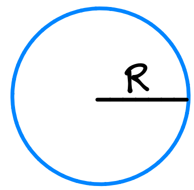
$$\Rightarrow 2(4)\frac{dX}{dt} + 2(3)(-2) = 0$$

$$\Rightarrow \frac{dX}{dt} = \frac{12}{8} = 1.5 \text{ m/s}$$

$\therefore$ when $X = 4$m, the bottom of the ladder is sliding away from the wall at a rate of 1.5 m/s

Example 9.4. A stone is tossed into a pond, creating a circular ripple that grows outward from the centre. If the radius of the circle is growing at a rate of 10 cm per second, how fast is the area of the circle growing when the radius is $\frac{100}{\pi}$ cm ? How fast is the area growing when the radius is 1 cm ?

Variables A = area of ripple's circle
R = radius of ripple's circle



Given $\frac{dR}{dt} = 10 \text{ cm/s}$

want $\frac{dA}{dt}$ when $R = \frac{100}{\pi}$ (≈ 31.8) cm

eqn $A = \pi R^2$

imp. diff. $\frac{dA}{dt} = 2\pi R \cdot \frac{dR}{dt}$

when $R = 100/\pi$ cm $\quad \frac{dA}{dt} = 2\pi\left(\frac{100}{\pi}\right)(10) = 2000 \text{ cm}^2/\text{s}$

$\therefore$ when $R = 100/\pi$ cm, the area of the ripple's circle is increasing at a rate of 2000 cm^2/s.

when $R = 1$ cm $\quad \frac{dA}{dt} = 2\pi(1)(10) = 20\pi$ (≈ 62.8) cm^2/s

$\therefore$ when $R = 1$ cm, the area of the ripple's circle is increasing at a rate of 20π cm^2/s.

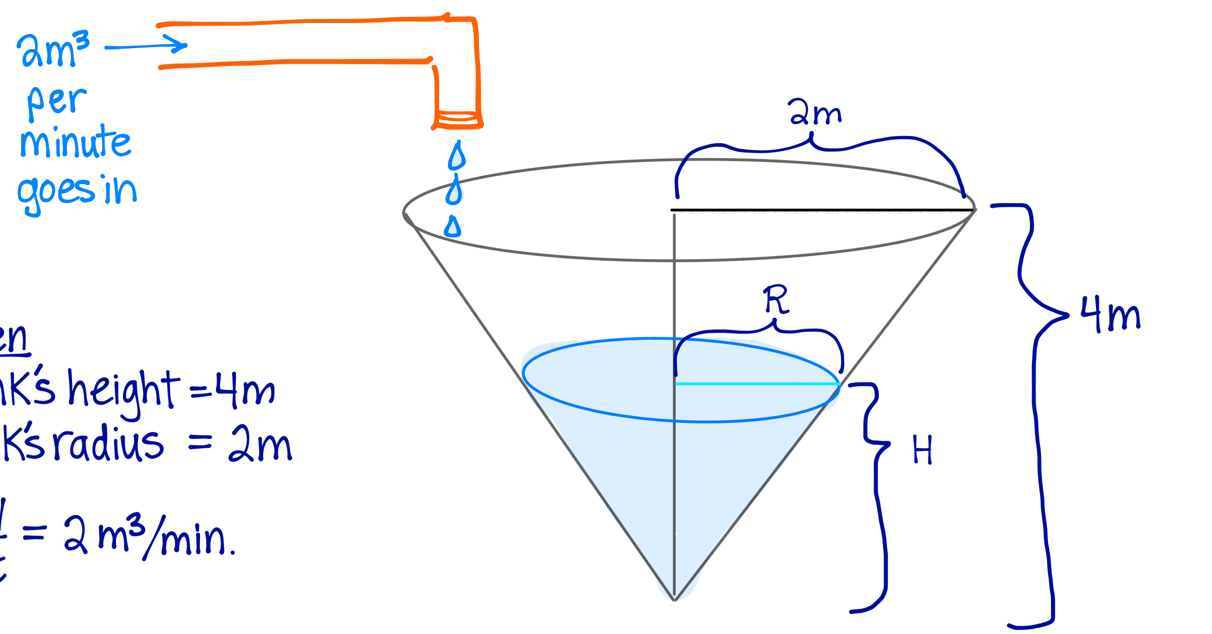
Example 9.5. A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.

Variables

H = height of water level in tank (the height of the tank is constant)

R = radius of water's upper surface (the radius of the tank is constant)

V = volume of water in tank (the volume of the tank itself is constant)



Given

tank's height = 4m

tank's radius = 2m

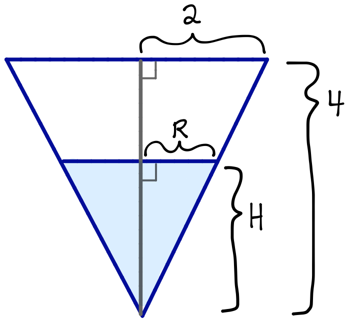
$\frac{dV}{dt} = 2\ m^3/min.$

want $\frac{dH}{dt}$ when $H = 3m$

eqn $V = \frac{1}{3}\pi R^2 H$

imp. diff. $\frac{dV}{dt} = \frac{2}{3}\pi R \cdot \frac{dR}{dt} H + \frac{1}{3}\pi R^2 \frac{dH}{dt}$

when $H=3$, $2 = \frac{2}{3}\pi R \frac{dR}{dt}(3) + \frac{1}{3}\pi R^2 \frac{dH}{dt}$ but what is R?
What is dR/dt ?



There is a relationship between R and H based on similar triangles:

$$\frac{R}{H} = \frac{2}{4}$$

$$\therefore R = \frac{1}{2}H$$

by imp. diff. this eqⁿ

When $H = 3\,m$, $R = \frac{1}{2}H = \frac{1}{2}(3) = 1.5\,m$ and $\frac{dR}{dt} = \frac{1}{2}\frac{dH}{dt}$

$\therefore$ when $H = 3$, we have $2 = \frac{2}{3}\pi(1.5)\left(\frac{1}{2}\frac{dH}{dt}\right)(3) + \frac{1}{3}\pi(1.5)^2\frac{dH}{dt}$

Solving for $\frac{dH}{dt}$ we get $\frac{dH}{dt} = \frac{8}{9\pi}$ m/min.

$\therefore$ at the moment in time when the water level is 3m, the water level is rising at a rate of $\frac{8}{9\pi}$ (≈ 0.283) m/min.

Alternatively, we could notice relationship between R and H before imp. diff.

Since $R = \frac{1}{2}H$, for this cone, the volume equation can be rewritten as $V = \frac{1}{3}\pi\left(\frac{1}{2}H\right)^2 H = \frac{1}{12}\pi H^3$

New eqⁿ $V = \frac{1}{12}\pi H^3$

New imp. diff. $\frac{dV}{dt} = \frac{3\pi}{12}H^2\frac{dH}{dt}$

now plug in $H = 3$, $\frac{dV}{dt} = 2$ to find $\frac{dH}{dt}$

observation: Zoom in toward a point on the graph of a differentiable function. The curve looks almost the same as a line (near the point of tangency)

idea: use the tangent line to the graph of f at x=a to approximate f(x) at x-values <u>near</u> a.

Linear Approximation of $f(x)$ near $x = a$:

Near x=a, $\boxed{f(x) \approx f(a)+f'(a)(x-a)}$

the equation of the tangent line to f at a.

This is called the <u>Linear Approximation of f at a</u> (also called the <u>Tangent Line Approximation of f at a</u>).

The line $\boxed{y = L(x) = f(a)+f'(a)(x-a)}$ is called the <u>Linearization of f at a.</u>

Note. In order for the linearization $L(x)$ of f at a to be of practical use, we need $f(a)$ and $f'(a)$ to be easy to compute; otherwise, the linearization would be just as difficult to obtain as finding exact values of $f(x)$ near a.

Example 9.6. Use a linear approximation to estimate $\sin(0.1)$.

$f(x) = \sin(x)$ use $a=0$ $f(a) = \sin(0) = 0$

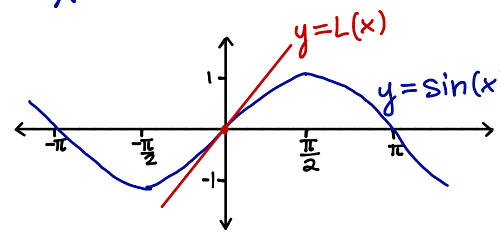
$f'(x) = \cos(x)$ $f'(a) = \cos(0) = 1$

both easy to compute

$$\Rightarrow L(x) = f(a)+f'(a)(x-a) = 0+(1)(x-0) = x$$

Near $a=0$, $\sin(x) \approx L(x) = x$

so $\sin(0.1) \approx L(0.1) = 0.1$



Example 9.7. Find the linearization of $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$.

$$f(1) = \sqrt{1+3} = \sqrt{4} = 2$$

$$f'(x) = \frac{1}{2\sqrt{x+3}}$$

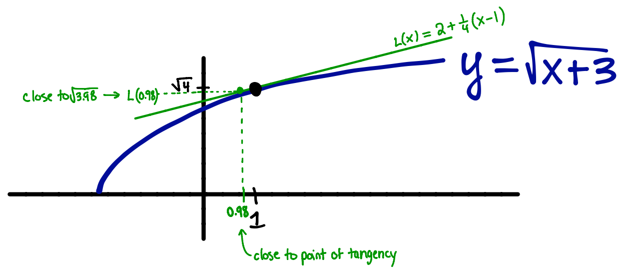
$$f'(1) = \frac{1}{2\sqrt{1+4}} = \frac{1}{4}$$

The linear approximation of f at 1 is

$$L(x) = f(1) + f'(1)(x-1) = 2 + \frac{1}{4}(x-1) = \frac{1}{4}x + \frac{7}{4}$$

$$\sqrt{3.98} = \sqrt{0.98+3} = f(0.98) \approx L(0.98) = \frac{1}{4}(0.98) + \frac{7}{4} = 1.995$$

$$\sqrt{4.05} = \sqrt{1.05+3} = f(1.05) \approx L(1.05) = \frac{1}{4}(1.05) + \frac{7}{4} = 2.0125$$



STUDY GUIDE

- **related rates strategy**:
  1. read problem carefully
  2. identify the variables and draw a diagram if possible
  3. determine what rates of change are given, and what is being asked
  4. find an equation that relates the variables to each other at all times
  5. implicitly differentiate that equation with respect to time
  6. use the equation and the implicitly differentiated equation to solve for the desired quantity or rate
- **Linear Approximation of f at a:** $\boxed{f(x) \approx L(x) = f(a) + f'(a)(x-a)}$