11. Definite Integrals & The Fundamental Theorem of Calculus

Lec 10 mini review.

- ◇ **an antiderivative** vs. **the most general antiderivative**
- ◇ **undoing basic rules of differentiation**
- ◇ **some ideas for undoing less basic rules of differentiation**
- ◇ **setup for a Riemann sum with** n **rectangles on** $[a, b]$:

$$\Delta x = \tfrac{b-a}{n} \qquad x_i = a + i\Delta x \qquad \text{sample point} \quad x_i^* \in [x_{i-1}, x_i]$$

- ◇ **using a Riemann sum to approximate net area** A **between** f **and the** x**-axis on** $[a, b]$:

$$A \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

DEFINITE INTEGRALS

- Let f be a function defined for $a \le x \le b$.
- Divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b-a)/n$.
- Let $x_0 = a$ and, for $i = 0, \dots, n$, let $x_i = a + i\Delta x$ be the endpoints of these subintervals.
- Let x_i^* be *any* **sample point** from the ith subinterval $[x_{i-1}, x_i]$.

Then the **definite integral of** f **from** a **to** b is

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x}$$

provided that this limit exists and gives the same value for all possible choices of sample points.
If it does exist and is equal for all sample point choices, then we say that f is **INTEGRABLE** on $[a, b]$.

Theorem 11.1. If f is

continuous on $[a, b]$, or if f has only a finite number of jump discontinuities on $[a, b]$,

then the definite integral $\int_a^b f(x)\,dx$ (which is a limit!) exists, hence f is integrable on $[a, b]$.

Theorem 11.2. If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i)\,\Delta x\right)$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$ (that is, we can use right endpoints in our Riemann sum).

* These notes are solely for the personal use of students registered in MAT1320.

Riemann sum (with n rectangles, right endpoints) for $f(x)$ on $[a,b]$

Choose n.

Then $\Delta x = \dfrac{b-a}{n}$, $x_i = a + i\Delta x$, "height" of rectangle $\#i$ $= f(x_i)$

$$\sum_{i=1}^{n} f(x_i)\Delta x = \left(\begin{array}{c}\text{approx.}\\ \text{net area}\\ \text{between}\\ f \text{ and } x\text{-axis}\\ \text{on } [a,b]\end{array}\right)$$

$\downarrow$

Choose bigger n.

more rectangles $\Rightarrow$ better approximation

$\downarrow$

$$\sum_{i=1}^{n} f(x_i)\Delta x = \left(\begin{array}{c}\text{(better)}\\ \text{approx.}\\ \text{net area}\\ \text{between}\\ f \text{ and } x\text{-axis}\\ \text{on } [a,b]\end{array}\right)$$

$\downarrow$

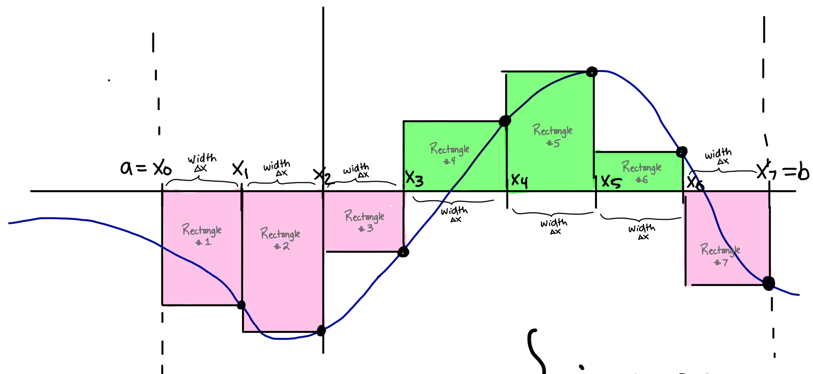
limit as $n \to \infty$

gives the definite integral

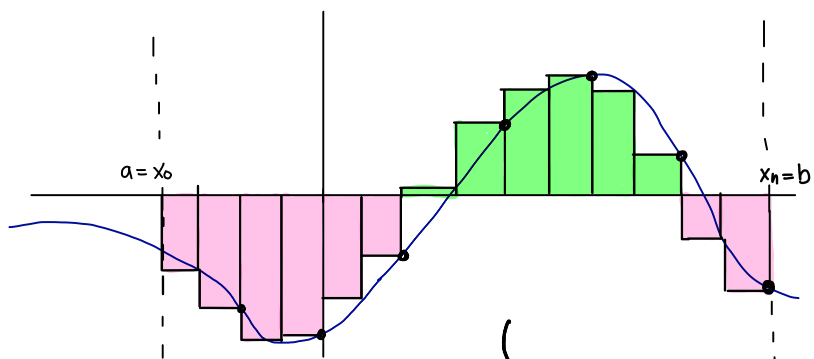
$\downarrow$

$$\underbrace{\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x}_{\int_a^b f(x)\,dx} = \left(\begin{array}{c}\text{exact}\\ \text{net area}\\ \text{between}\\ f \text{ and } x\text{-axis}\\ \text{on } [a,b]\end{array}\right)$$

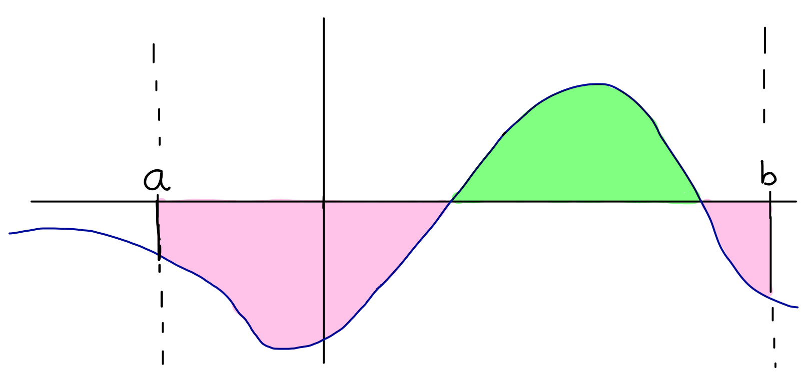
$$\int_a^b f(x)\,dx = \left(\begin{array}{c}\text{exact}\\ \text{net area}\\ \text{between}\\ f \text{ and } x\text{-axis}\\ \text{on } [a,b]\end{array}\right)$$



increase n



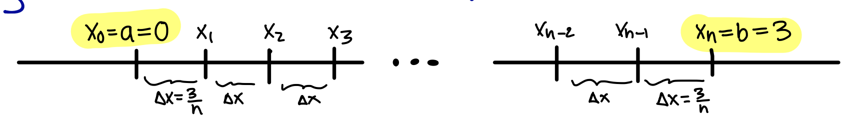
$\lim\limits_{n\to\infty}$



Example 11.3. Evaluate $\displaystyle\int_0^3 (x^3 - 6x)\,dx$ using the (limit) definition of a definite integral.

Setup integrand $f(x) = x^3 - 6x$ $\qquad \Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$

$a = 0,\ b = 3$

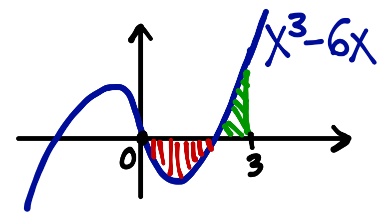


sample point (use right endpoint) $\quad x_i^* = x_i = a + i\Delta x = 0 + i\left(\frac{3}{n}\right) = \frac{3i}{n}$

height of ith rectangle $\quad f(x_i^*) = f\left(\frac{3i}{n}\right) = \left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right) = \frac{27i^3}{n^3} - \frac{18i}{n}$

So $\displaystyle\int_0^3 f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\left(\frac{27i^3}{n^3} - \frac{18i}{n}\right)}_{f(x_i^*)} \underbrace{\left(\frac{3}{n}\right)}_{\Delta x}$$



$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{81i^3}{n^4} - \frac{54i}{n^2}\right)$$

$$= \lim_{n\to\infty} \left(\frac{81}{n^4} \sum_{i=1}^{n} i^3 - \frac{54}{n^2} \sum_{i=1}^{n} i\right)$$ (using basic properties of sums)

$$= \lim_{n\to\infty} \left(\frac{81}{n^4}\left(\frac{n(n+1)}{2}\right)^2 - \frac{54}{n^2}\left(\frac{n(n+1)}{2}\right)\right)$$

Using these 2 useful formulas

$$\boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}}$$

$$\boxed{\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2}$$

$$= \frac{81}{4}\left(\lim_{n\to\infty} \frac{n^2(n+1)^2}{n^4}\right) - \frac{54}{2}\left(\lim_{n\to\infty} \frac{n(n+1)}{n^2}\right)$$

$$= \frac{81}{4}(1) - \frac{54}{2}(1)$$

$$= -\frac{27}{4}$$

$$= -6.75$$

← We will be happy when we learn the FTC (Fundamental Theorem of Calculus)

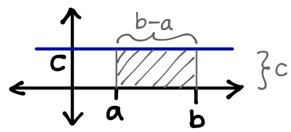
$$\boxed{\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx}$$

$\Delta x = \frac{b-a}{n}$ vs $\Delta x = \frac{a-b}{n} = -\frac{(b-a)}{n}$

$$\boxed{\int_a^a f(x)\,dx = 0}$$

$\Delta x = \frac{a-a}{n} = 0$

1. If $c \in \mathbb{R}$, then $\boxed{\int_a^b c\,dx = c(b-a)}$.



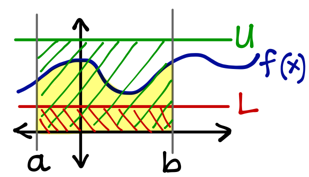
2./4. $\boxed{\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx}$

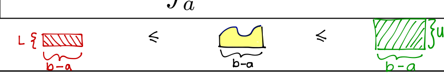
3. If $c \in \mathbb{R}$, then $\boxed{\int_a^b \big(cf(x)\big)\,dx = c\int_a^b f(x)\,dx}$

5. $\boxed{\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx}$

7. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\boxed{\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx}$

8. If $L \leq f(x) \leq U$ for $a \leq x \leq b$, then $\boxed{L(b-a) \leq \int_a^b f(x)\,dx \leq U(b-a)}$





Exercise 11.4. If $\int_1^4 f(x)\,dx = 5$ and $\int_1^4 [2f(x) + 3g(x)]\,dx = 7$, find $\int_4^1 g(x)\,dx$.

$$\int_1^4 [2f(x)+3g(x)]\,dx = 2\int_1^4 f(x)\,dx + 3\int_1^4 g(x)\,dx \quad \text{(by properties 2,3,4)}$$

$$\Rightarrow \quad 7 = 2(5) + 3\int_1^4 g(x)\,dx$$

$$\Rightarrow \quad -1 = \int_1^4 g(x)\,dx$$

$$\text{(by prop. 1)} \Rightarrow \quad -1 = -\int_4^1 g(x)\,dx \qquad \therefore \int_4^1 g(x)\,dx = 1.$$

The Fundamental Theorem of Calculus, Part 2

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, F is any function such that $F' = f$.

$\Longrightarrow$ We can forget about computing difficult limits of Riemann sums! FTC 2 gives us a quick way to evaluate definite integrals:

1. find an antiderivative of the integrand
2. subtract our antiderivative at the limits of integration

Notation: If $F(x)$ is an antiderivative of $f(x)$, then we have a 2-step notation:

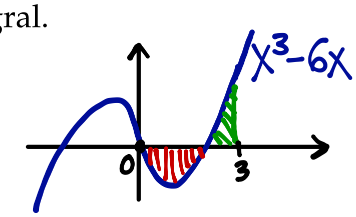
$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b$$

1 we find antiderivative

$$= F(b) - F(a)$$

$\leftarrow$ 2 subtract at limits of integration

Example 11.5. Evaluate the definite integral $\displaystyle\int_0^3 (x^3 - 6x)\,dx$ using FTC 2. Compare this procedure with the limit we used in Example 11.4 to compute the same definite integral.



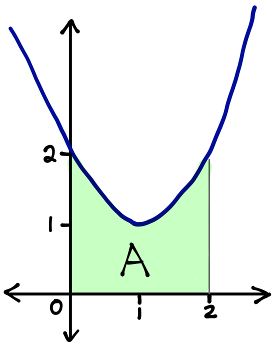
$$\int_0^3 (x^3-6x)\,dx = \left[\frac{x^4}{4} - 6\left(\frac{x^2}{2}\right)\right]_0^3 \quad (1.)$$

$$= \left[\frac{x^4}{4} - 3x^2\right]_0^3$$

$$= \left(\frac{3^4}{4} - 3(3^2)\right) - \left(\frac{0^4}{4} - 3(0^2)\right) \quad (2.)$$

$$= \frac{81}{4} - 27 - 0 \quad = -6.75$$

Example 11.6. Evaluate the definite integral $\int_0^2 \left((x-1)^2+1\right)dx$ using FTC 2. Compare this with the Riemann sum approximations we obtained in Example 10.8.



$$\int_0^2 \left((x-1)^2+1\right)dx = \int_0^2 (x^2-2x+2)\,dx$$

$$= \left[\frac{x^3}{3} - x^2 + 2x\right]_0^2$$

$$= \left(\frac{2^3}{3} - 2^2 + 2(2)\right) - \left(\frac{0^3}{3} - 0^2 + 2(0)\right)$$

$$= \frac{8}{3}$$

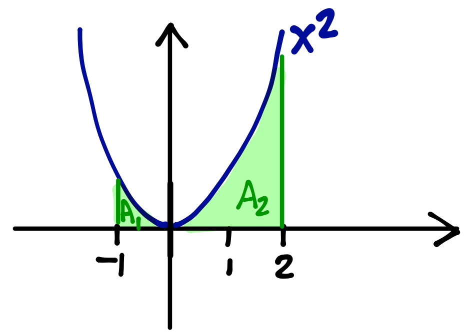
$$= 2.\overline{6}$$

Example 11.7. $\int_{-1}^{2} x^2\,dx = \left[\frac{x^3}{3}\right]_{-1}^{2}$

$$= \frac{2^3}{3} - \frac{(-1)^3}{3}$$

$$= \frac{8}{3} + \frac{1}{3}$$

$$= 3$$

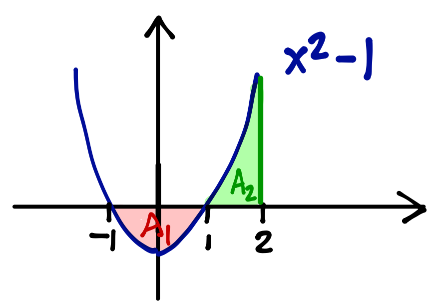


(net area $= A_1 + A_2 = 3$)

Example 11.8. $\int_{-1}^{2} (x^2-1)\,dx = \left[\frac{x^3}{3} - x\right]_{-1}^{2}$

$$= \left(\frac{2^3}{3} - 2\right) - \left(\frac{(-1)^3}{3} - (-1)\right)$$

$$= 0$$

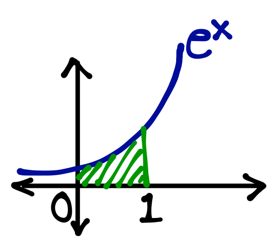


(net area $= -A_1 + A_2 = 0$)

Example 11.9. $\int_0^1 e^x\,dx = \left[e^x\right]_0^1$

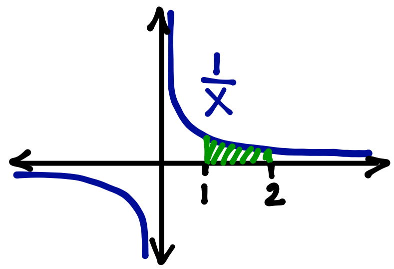
$$= e^1 - e^0$$

$$= e - 1$$



Example 11.10. $\int_1^2 \frac{dx}{x}$ $\leftarrow$ short for $\int_1^2 \frac{1}{x}\,dx$

$$\int_1^2 \frac{1}{x}\,dx = \Big[\ln|x|\Big]_1^2$$
$$= \ln|2| - \ln|1|$$
$$= \ln 2$$

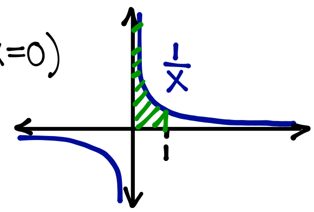


Example 11.11. $\int_0^1 \frac{dx}{x} \cancel{=} \Big[\ln|x|\Big]_0^1$ because $\frac{1}{x}$ is not continuous on $[0,1]$

($\frac{1}{x}$ has a vertical asymptote at $x=0$)

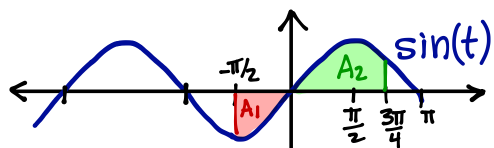
$\therefore$ FTC is Not Applicable ☠

This is called an improper integral
You will learn about improper integrals
if you take MAT1322



Example 11.12. $\int_{-\pi/2}^{3\pi/4} \sin(t)\,dt = \Big[-\cos(t)\Big]_{-\pi/2}^{3\pi/4}$

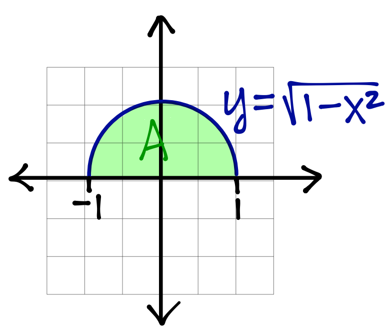
$$= -\cos\left(\tfrac{3\pi}{4}\right) - \left(-\cos\left(-\pi/2\right)\right)$$
$$= -\left(-\tfrac{1}{\sqrt{2}}\right) - \left(-(0)\right)$$
$$= \frac{1}{\sqrt{2}}$$



(net area is $-A_1 + A_2 = \frac{1}{\sqrt{2}}$)

Example 11.13. $\int_{-1}^{1} \sqrt{1-x^2}\,dx$ *hint: draw a picture of the net area represented by this definite integral.*

$$= \tfrac{1}{2}(\text{area of unit circle})$$
$$= \tfrac{1}{2}\pi(1^2)$$
$$= \frac{\pi}{2}$$



(net area $= A = \frac{\pi}{2}$)

From now on, we will use our integral notation in two ways:

- ◇ We write $\displaystyle\int f(x)\,dx$ to represent **THE MOST GENERAL ANTIDERIVATIVE OF $f(x)$.** That is,

$$\int f(x)dx = F(x) + C \quad \text{means} \quad F'(x) = f(x)$$

  The integral $\displaystyle\int f(x)\,dx$ is also called **AN INDEFINITE INTEGRAL**.

  In particular, an indefinite integral represents an infinite family of functions, each member of which has derivative equal to $f(x)$.

- ◇ If there are **LIMITS OF INTEGRATION**, $\int_a^b$, then $\displaystyle\int_a^b f(x)\,dx$ is called a **DEFINITE INTEGRAL**.

  By FTC (assuming $f(x)$ is continuous on $[a, b]$), the definite integral $\int_a^b f(x)dx$ equals the difference $F(b) - F(a)$, where F is any antiderivative of f. Thus, a definite integral is a **number**, *not* a family of functions. This number corresponds to the net area between $f(x)$ and the x-axis on the interval $[a, b]$.

Example 11.14. Evaluate each of the following integrals:

$$\int \frac{2}{x^2+1}\,dx = 2\arctan(x) + C$$

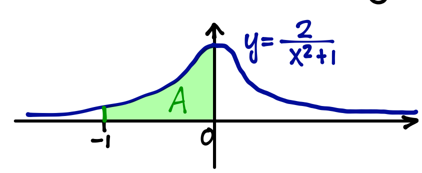
(indefinite integral)

$$\int_{-1}^{0} \frac{2}{x^2+1}\,dx = \Big[2\arctan(x)\Big]_{-1}^{0}$$

$$= 2\arctan(0) - 2\arctan(-1)$$

$$= 2(0) - 2\left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

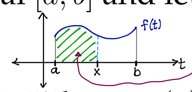
(definite integral)



(= net area = A)

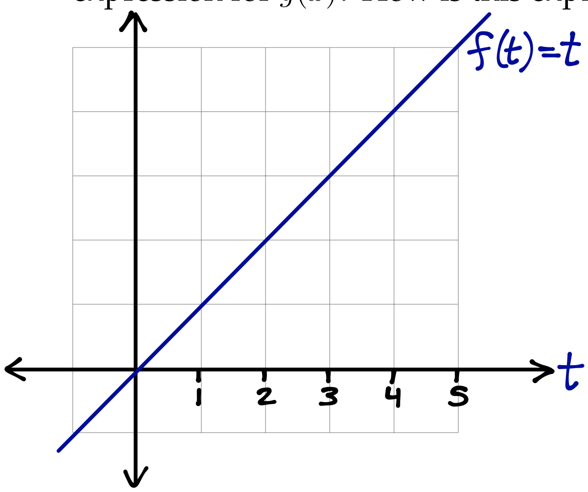
- The **Fundamental Theorem of Calculus (FTC)** is so called because it relates the two main branches of calculus: differential & integral
- The FTC has two parts. The first part tells us the derivative of a function defined by a definite integral. The second part tells us an easy way to evaluate definite integrals using antiderivatives.

Suppose $f(t)$ is a continuous function on the interval $[a,b]$ and let $g(x)$ be a function defined for all $x \in [a,b]$ as follows: $g(x) = \int_a^x f(t)\,dt \quad (a \le x \le b)$

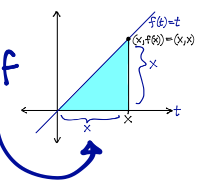


g(x) represents this net area
where $a \le x \le b$
(x is the upper limit of integration of g's integral)

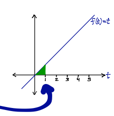
Example 11.15. Let $f(t) = t$. Define $g(x) = \int_0^x f(t)dt$. What is $g(1)$? What is $g(4)$? Can you find an expression for $g(x)$? How is this expression related to $f(t)$?

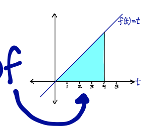


In fact, for any $x \ge 0$,

$g(x) = \int_0^x t\,dt =$ (area of )

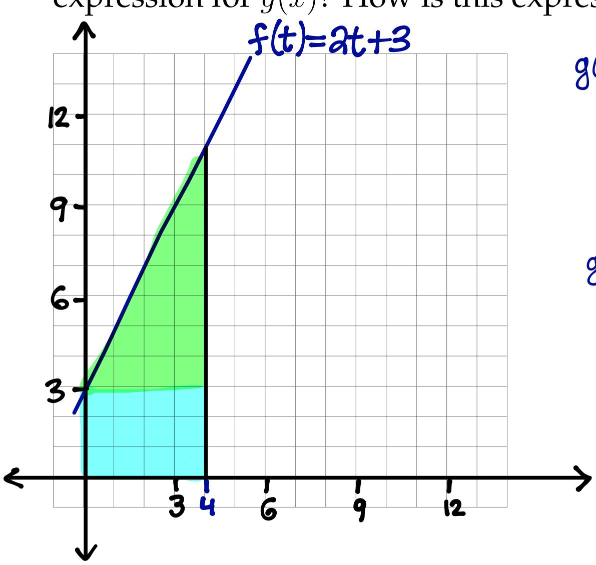
$= \frac{1}{2}(\text{base})(\text{height})$
$= \frac{1}{2}(x)(x)$
$= \frac{1}{2}x^2$

$g(1) = \int_0^1 t\,dt =$ (area of ) $= \frac{1}{2}(1)(1) = \frac{1}{2}$

$g(4) = \int_0^4 t\,dt =$ (area of ) $= \frac{1}{2}(4)(4) = 8$

Notice
$\frac{1}{2}x^2$ is an antiderivative of $f(x)$

Exercise 11.16. Let $f(t) = 2t+3$. Define $g(x) = \int_0^x f(t)dt$. What is $g(1)$? What is $g(4)$? Can you find an expression for $g(x)$? How is this expression related to $f(t)$?



$g(4) = \int_0^4 (2t+3)dt = \left(\text{area } \underbrace{\quad}_{4} \}3\right) + \left(\text{area } \underbrace{\quad}_{4} \}2(4)\right) = 4(3) + \frac{1}{2}(4)(8) = 28$

$g(x) = \int_0^x (2t+3)dt = \left(\text{area } \underbrace{\quad}_{x} \}3\right) + \left(\text{area } \underbrace{\quad}_{x} \}2x\right) = 3x + \frac{1}{2}(x)(2x) = x^2+3x$

Note x^2+3x is an antiderivative of $f(x)=2x+3$

The Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a,b]$, then the function g defined by

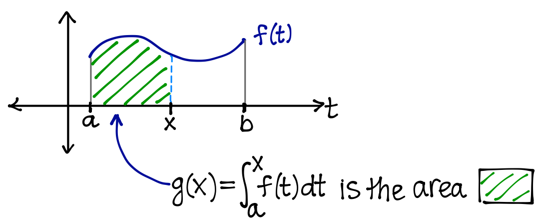
$$g(x) = \int_a^x f(t)\,dt \qquad a \le x \le b$$

is

- continuous on $[a,b]$,
- differentiable on (a,b), and
- $g'(x) = f(x)$. That is, $\dfrac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$

Idea behind the proof:

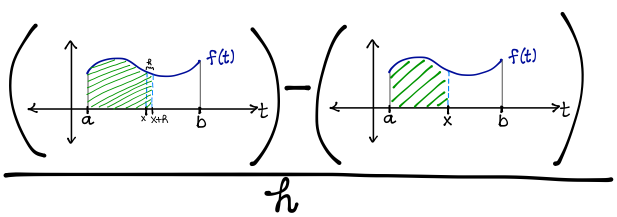
For simplicity, we'll consider the case when $f(t)$ lies above the t-axis on $[a,b]$



By definition,

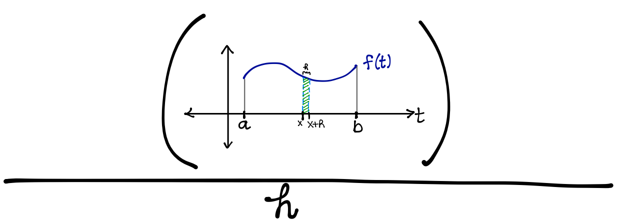
$$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$= \lim_{h\to 0}$

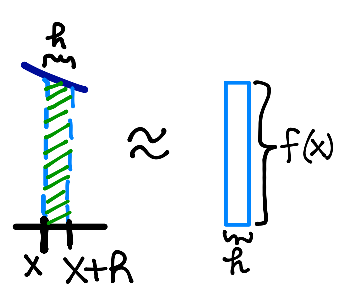


* this is very informal! (it's only to give you the idea)

$= \lim_{h\to 0}$



Since h is tiny ($h \to 0$) the area of the strip that remains is approximately equal to a rectangle of width h and height $f(x)$



$$\approx \lim_{h\to 0} \frac{\cancel{h}f(x)}{\cancel{h}}$$

$$= f(x)$$

*Note Many applications of integration in MAT 1322 use this idea of approximating the area of a thin strip under a function

Example 11.17. Find $\dfrac{d}{dx}\left[\int_2^x \sqrt{1+t^2}dt\right]$

integrand
$f(t)=\sqrt{1+t^2}$

By FTC1, $\dfrac{d}{dx}\left[\int_2^x f(t)dt\right] = f(x) = \sqrt{1+x^2}$

Example 11.18. Find the derivative of the function $g(x) = \displaystyle\int_1^{x^3} \sin(t)dt$.

integrand
$f(t)= \sin(t)$

Note $g(x)=g(u(x))$ where $u(x)=x^3$

Thus, $g(u)=\int_1^u \sin(t)dt$ and $u'(x)=3x^2$

By FTC1, $g'(u)=f(u)=\sin(u)$

By Chain Rule, $\dfrac{d}{dx}[g(u(x))]=g'(u(x))\,u'(x)$

$=\sin(u(x))(3x^2) = \sin(x^3)(3x^2)$

Net Change Theorem

By FTC2, we know that

$$\int_a^b f(x)dx = F(b) - F(a) \qquad \text{where } F' = f$$

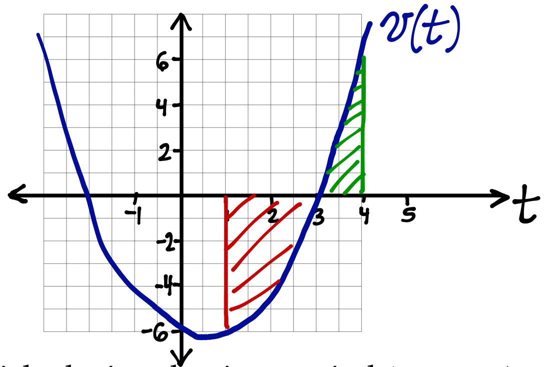
Thus, we can rewrite FTC2 as follows:

$$\int_a^b F'(x)dx = F(b) - F(a)$$

$\Longrightarrow$ the integral of a rate of change is the **net change**.

Example 11.19. A particle moves along a line so that its velocity at time t is given by $v(t) = t^2 - t - 6$ (measured in m/s).

(a) Sketch the graph of $v(t)$ over the time interval $[0, 5]$



(b) Find the displacement of the particle during the time period $1 \leq t \leq 4$

net change in position (on line) from $1 \leq t \leq 4$

Let $s(t)$ denote the particle's position on the line as a function of time t

$\therefore$ particle's displacement for $1 \leq t \leq 4$ is the net change $s(4) - s(1)$

<u>Note:</u> $v(t)$ = rate of change of position with respect to time <u>ie</u> $v(t) = s'(t)$

$\therefore$ by the Net Change Theorem,

$$\text{displacement} = s(4) - s(1) = \int_1^4 s'(t)\,dt$$

$$= \int_1^4 v(t)\,dt$$

$$= \int_1^4 (t^2 - t - 6)\,dt$$

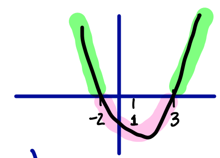
$$= \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4$$

$$= \left(\frac{4^3}{3} - \frac{4^2}{3} - 6(4)\right) - \left(\frac{1^3}{3} - \frac{1^2}{3} - 6(1)\right)$$

$$= -\frac{9}{2}\,\text{m}$$

(c) Find the distance travelled by the particle during the time period $1 \le t \le 4$

if we want to know the total distance travelled by particle, we would like to add the distance travelled by the particle in the negative direction instead of subtracting it.



$$|v(t)| = |t^2-t-6| = \begin{cases} t^2-t-6 & \text{if } t \geq 3 \\ -(t^2-t-6) & \text{if } -2<t<3 \\ t^2-t-6 & \text{if } t \leq -2 \end{cases}$$

Thus, we want

$$\int_1^4 |v(t)|dt = \overbrace{\int_1^3 -v(t)dt}^{\text{add the distance travelled by particle in negative direction}} + \overbrace{\int_3^4 v(t)dt}^{\text{add the distance travelled by particle in positive direction}}$$

$$= \int_1^3 (-t^2+t+6)dt + \int_3^4 (t^2-t-6)dt$$

$$= \left[-\frac{t^3}{3}+\frac{t^2}{2}+6t\right]_1^3 + \left[\frac{t^3}{3}-\frac{t^2}{2}-6t\right]_3^4$$

$$= \left(-\frac{3^3}{3}+\frac{3^2}{2}+6(3)\right)-\left(-\frac{1}{3}+\frac{1}{2}+6\right) + \left(\frac{4^3}{3}-\frac{4^2}{2}-24\right)-\left(\frac{3^3}{3}-\frac{3^2}{2}-18\right)$$

$$= \frac{22}{3} + 2.8\overline{3}$$

$$\approx 10.17 \text{ m}$$

STUDY GUIDE

- ◇ **definition of the definite integral**(if it exists): $\displaystyle\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$
- ◇ **net area interpretation of definite integral**
- ◇ **evaluating definite integrals using known sums and properties of integrals**
- ◇ **FTC1**: If $g(x) = \displaystyle\int_a^x f(t)dt$, then $g'(x) = f(x)$.
- ◇ **FTC2**: If f is continuous on $[a,b]$, then
$$\int_a^b f(x)dx = F(b) - F(a)$$
where F is any antiderivative of f.
- ◇ **indefinite integral** vs. **definite integral**
- ◇ **the Net Change Theorem**: $\displaystyle\int_a^b F'(x)dx = F(b)-F(a)$
(integral of a rate of change is the net change)