MAT 1341 Test 2 solutions

2^{08}

1. If C is a $n \times 4$ matrix and $D = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and then the second column of the matrix CD is $= C \cdot$ (col 2 of D)

A. the same as the second column of C.
B. the sum of the first and second columns of C.
C. the sum of the second and fourth columns of C.
D. the same as the first column of D.
E. the same as the third row of D.
F. the sum of the first and the third columns of C.

$$CD = [C_1 C_2 C_3 C_4] \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} * & C_2 & * \end{bmatrix}$$

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2. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 2 & 1 & 2 \end{bmatrix}$. Then the second row of A^{-1} is:

A. $[-4 \ \ 1 \ \ 1]$
B. $[2 \ \ 1 \ \ -1]$
C. $[3 \ \ -1 \ \ 0]$
D. $[-1 \ \ 0 \ \ 2]$
E. $[2 \ \ 0 \ \ -1]$
F. $[0 \ \ -1 \ \ 1]$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 3 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 1 & -2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & -4 & 1 & 1 \end{array}\right]$$

(we could stop here since the second row won't change)

2^{11} Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 2 & 2 \end{bmatrix}$. Then the second row of A^{-1} is:

A. $[0 \ \ -1 \ \ 1]$
B. $[0 \ \ -1 \ \ 0]$
C. $[2 \ \ -1 \ \ 0]$
D. $[-1 \ \ 1 \ \ 1]$
E. $[2 \ \ 0 \ \ -1]$
F. $[0 \ \ -1 \ \ 1]$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

(again, we could stop here as subsequent row ops won't change the second row ..)

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & & * & \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

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3. Compute $\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{2012}$ = A

$A^2 = \begin{bmatrix} I & K \\ 0 & I \end{bmatrix}\begin{bmatrix} 1 & K \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 2K \\ 0 & I \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & K \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2K \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3K \\ 0 & 1 \end{bmatrix}$; so

$A^n = \begin{bmatrix} I & nK \\ 0 & I \end{bmatrix}$.

A. $\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 0 & 0 & -6036 \\ 0 & 1 & 6036 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 & -6036 & 0 \\ 0 & 1 & 0 & 6036 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 0 & 6036 \\ 0 & 1 & -6036 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

E. $\begin{bmatrix} 1 & 0 & 0 & -3^{2012} \\ 0 & 1 & 3^{2012} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

F. $\begin{bmatrix} 1 & 0 & 0 & 3^{2012} \\ 0 & 1 & -3^{2012} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

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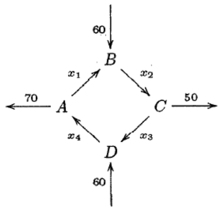
4. Consider the network of streets with intersections A, B, C and D below. The arrows indicate the direction of traffic flow along the **one way streets**, and the numbers refer to the number of cars observed to enter A or leave B,C and D **during one minute**. Each x_i denotes the unknown **number** of cars which passed along the indicated streets during the same period.



$\boxed{2}$ a) Write down a system of linear equations which describes the the traffic flow, **together with all the constraints** on the variables x_i, $i = 1, \ldots, 4$. (*Do not simply copy out the equations implicit in (b). You will not get any marks if you do this. Do not perform any operations on your equations: this is done for you in (b)!*)

		Flow in	=	Flow out
Intersection	A	x_4	=	$70 + x_1$
	B	$x_1 + 60$	=	x_2
	C	x_2	=	$x_3 + 50$
	D	$x_3 + 60$	=	x_4.

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$\frac{1}{2}$ Need $x_i \geq 0$ (one way streets)

$\frac{1}{2}$ $x_i \in \mathbb{Z}$ (whole numbers of cars)

$i = 1, \ldots, 4$

(Question 4 continued)

1 (b) Suppose The reduced row-echelon form of the augmented matrix from part (a) is

$$\begin{array}{c} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -70 \\ 0 & 1 & 0 & -1 & 10 \\ 0 & 0 & 1 & -1 & -40 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

(Actually, the correct RRE form, and correct answers are below in brackets. You will not lose marks if you assumed the RRE form as given on the left!)

$$\left(\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -70 \\ 0 & 1 & 0 & -1 & -10 \\ 0 & 0 & 1 & -1 & -60 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]\right)$$

Give the general solution. (Ignore the constraints at this point.)

$$x_1 = -70 + s$$
$$x_2 = 10 + s$$
$$x_3 = -40 + s \quad ; s \in \mathbb{R}$$
$$x_4 = s$$

$4 @ \frac{1}{4} = 1$

$$\left(\begin{array}{l} x_1 = -70 + s \\ x_2 = -10 + s \\ x_3 = -60 + s \\ x_4 = s \end{array} ; s \in \mathbb{R}\right)$$

3 (c) Using (b) and the constraints from (a), find the minimum traffic flows along

(i) $\overline{BC}$, and

(ii) $\overline{CD}$.

$x_1 \geq 0 \Leftrightarrow s \geq 70$; $x_3 \geq 0 \Leftrightarrow s \geq 40$ $\therefore s \geq 70$

$x_2 \geq 0 \Leftrightarrow s \geq -10$; $x_4 \geq 0 \Leftrightarrow s \geq 0$

(i) The flow along $\overline{BC}$ is $x_2 = 10 + s \geq \underline{80}$

(ii) the flow along $\overline{CD}$ is $x_3 = -40 + s \geq \underline{30}$

(i) Correct answer ½ + Justification 1

(ii) ditto.

$$\left(\begin{array}{l} x_1 \geq 0 \Leftrightarrow s \geq 70 \\ x_2 \geq 0 \Leftrightarrow s \geq 10 \\ x_3 \geq 0 \Leftrightarrow s \geq 60 \\ x_4 \geq 0 \Leftrightarrow s \geq 0 \end{array} \right\} \therefore s \geq 70$$

Hence $x_2 = -10 + s \geq 60$

$x_3 = -60 + s \geq 10$

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5. Let $A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

[2] a) Find the reduced row echelon form of A. $\tilde{A}$

[2] b) Find a basis for ker A.

[2] c) Is A invertible?

[2] d) [Bonus: 2pts] Extend your basis of ker A to a basis of $\mathbf{R}^4$, if necessary.

[2] a) $\tilde{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(1) correct

(1) – some justification

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[2] b) $[\tilde{A} \mid 0] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $\quad x_1 = -s, \; x_2 = -s, \; x_3 = 0, \; x_4 = s$; $s \in \mathbb{R}$

(1) correct or consistent with (a)

(1) justification (work shown)

$\therefore$ ker $A = \{ s(-1,-1,0,1) \mid s \in \mathbb{R} \}$.
$= \text{span}\{(-1,-1,0,1)\}$. Moreover, $(-1,-1,0,1)$ is non-zero $\therefore$ $\{(-1,-1,0,1)\}$ is a basis for ker A.

(Students can simply present the "basic solution" $(-1,-1,0,1)$ and stop there, since a theorem from class asserts these give a basis of ker A.)

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[2] c) (No) (1), since rank $A = 3 < 4$ (A is 4×4!) Justification (1)

d) Note that rank $\begin{bmatrix} -1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 4$, so

(½) – any correct extension

(1½) – Justification

$\{(-1,-1,0,1), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$ is a basis of $\mathbb{R}^4$. such

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6. Suppose A is an $n \times n$ matrix and that,

there is a <u>*non-zero*</u> *vector* $x \in \mathbf{R}^n$, *for which* $Ax = 0$.

State whether each of the following is (always) true, or is (possibly) false, in the box after the statement.

- If you say the statement may be false, you must give an explicit example (with numbers!).
- If you say the statment is true, you must give a clear explanation, for example by quoting a theorem presented in class.

a) The rank of A is n.

eg $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\boxed{\text{False}}$

but rank $A = 1 < 2$ (A is 2×2 here)

b) The columns of A are linearly dependent.

True, since $Ax = 0$ has a unique $\boxed{\text{TRUE}}$

(zero) solution $\Leftrightarrow$ col. of A are l.i. (thm from class)

c) There is $b \in \mathbf{R}^n$ such that $Ax = b$ is inconsistent.

We know from class that $Ax = b$ is consistent $\boxed{\text{TRUE}}$

$\forall b \in \mathbb{R}^n \Leftrightarrow A$ is invertible

$\Leftrightarrow Ax = 0 \implies x = 0$

Hence $\exists b$ s.t. $Ax = b$ is inconsistent

Each part ① – correct answer + some justification.

① Correct Justification (Explanation or proper counterexample)

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7. Let $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and define a subspace $\mathbf{C} \subseteq \mathbf{M}_{2\,2}$ by

$$\mathbf{C} = \{A \in \mathbf{M}_{2\,2} \mid AJ = JA\}.$$

1 (a) Show that

$$\mathbf{C} = \left\{ \begin{bmatrix} a & -c \\ c & a \end{bmatrix} \mid a, c \in \mathbf{R} \right\}.$$

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(Hint: Write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, compute AJ and JA, and find the general solution to system of the equations in a, b, c and d that guarantees $AJ = JA$.)

2 (b) Find a basis for $\mathbf{C}$, and hence find $\dim \mathbf{C}$.

1 a) $AJ = \begin{bmatrix} b & -a \\ d & -c \end{bmatrix} = JA = \begin{bmatrix} -c & -d \\ a & b \end{bmatrix} \iff$

$b = -c$, $-a = -d$, $d = a$ & $b = -c$

$\therefore$ $b = -c$, $a = d$. $\therefore$ $A = \begin{bmatrix} a & -c \\ c & a \end{bmatrix}$, for $a, c \in \mathbb{R}$

So $C = \left\{ \begin{bmatrix} a & -c \\ c & a \end{bmatrix} \mid a, c \in \mathbb{R} \right\}$ 1

b) $C = \text{span}\left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I}, \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{J} \right\}$, and

$aI + cJ = 0 \iff \begin{bmatrix} a & -c \\ c & a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\iff a = c = 0$ $\therefore \{I, J\}$ is

l.i. and spans C, so is a basis of C.

Hence $\dim C = 2$

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1 any correct basis $\frac{1}{2}$ Justification $\frac{1}{2}$ Dimension (consistent w basis)

(Same scheme as other question 7)

7. Suppose $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and we define a subspace $\mathbf{T} \subseteq \mathbf{M}_{22}$ by

$$\mathbf{T} = \{A \in \mathbf{M}_{22} \mid AK = KA\}.$$

a) Show that $\mathbf{T} = \left\{ \begin{bmatrix} a & c \\ c & a \end{bmatrix} \mid a, c \in \mathbf{R} \right\}$.

(Hint: Write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$*, compute* AK *and* KA*, and find the general solution to system of the equations in* a, b, c *and* d *that guarantees* $AK = KA$*.)*

a) $AK = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix} = KA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

$\Longleftrightarrow b = c, \; a = d, \; (d = a, \; c = b).$

$\therefore A = \begin{bmatrix} a & c \\ c & a \end{bmatrix}, \; a, c \in \mathbb{R}$, so

$T = \left\{ \begin{bmatrix} a & c \\ c & a \end{bmatrix} \middle| \, a, c \in \mathbb{R} \right.$.

b) Find a basis for $\mathbf{T}$, and hence find $\dim \mathbf{T}$. From (a), $T = \text{Span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$

$= \text{Span}\{I, K\}$. Moreover, $aI + cK = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} a & c \\ c & a \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Leftrightarrow a = c = 0$. Hence $\{I, K\}$ is l.i. Thus

$\{I, K\}$ spans T and is l.i., and so $\{I, K\}$ is a basis of T.

Thus $\dim T = 2$.