2. Lines and Planes

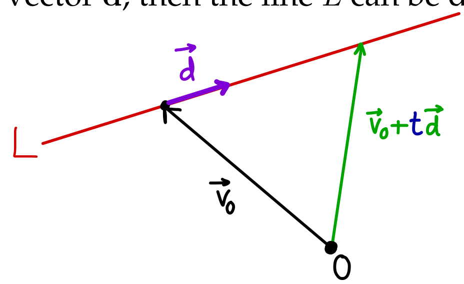
Here, we review lines and planes in $\mathbb{R}^2$ and $\mathbb{R}^3$, as well as the cross product (in $\mathbb{R}^3$).

DESCRIBING LINES

A line L is completely determined by its direction $\mathbf{d}$ and a point P on the line.

We think of the point P as being the tip of a vector with its tail at the origin O. Say, $\vec{OP} = \mathbf{v}_0$. If the direction of the line is given by the direction vector $\mathbf{d}$, then the line L can be described as the set:

$$L = \{\vec{v}_0 + t\vec{d} : t \in \mathbb{R}\}$$

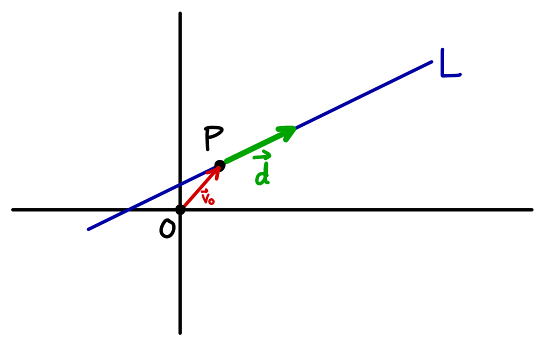


Any point on the line L can be expressed as $\mathbf{v}_0 + t\mathbf{d}$, where $t \in \mathbb{R}$ is a **PARAMETER** . This way of representing points on the line is called **VECTOR PARAMETRIC FORM** or a **PARAMETRIC EQUATION** for the line.

Example 2.1. Find a parametric equation for the line in $\mathbb{R}^2$ passing through the points $(1, 1)$ and $(3, 2)$.

direction $\vec{d} = \vec{QP} = \begin{bmatrix} 3-1 \\ 2-1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

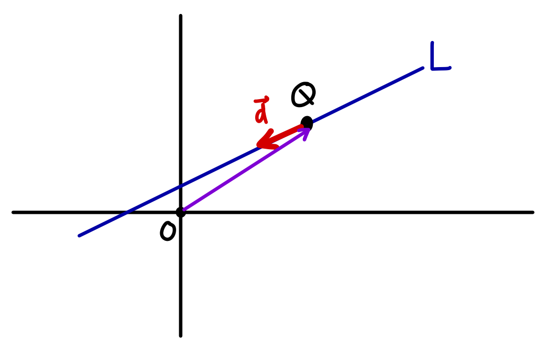
$$L = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$



Alternatives $L = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$

$$L = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix} + t\begin{bmatrix} -1 \\ -0.5 \end{bmatrix} : t \in \mathbb{R} \right\}$$

$$\vdots$$



†These notes are solely for the personal use of students registered in MAT1341.

Example 2.2. Find a parametric equation for the line $y = 3x + 2$.

Point on line

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 3x+2 \end{bmatrix} = \begin{bmatrix} 0+x \\ 2+3x \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + x\begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad L = \left\{ \begin{bmatrix} 0 \\ 2 \end{bmatrix} + t\begin{bmatrix} 1 \\ 3 \end{bmatrix} \; t \in \mathbb{R} \right\}$$

x is the parameter

Alternative $y = 3x+2 \Rightarrow x = \frac{1}{3}y - \frac{2}{3}$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3}y - \frac{2}{3} \\ y \end{bmatrix} = \begin{bmatrix} -2/3 \\ 0 \end{bmatrix} + y\begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$$

y is the parameter

$$L = \left\{ \begin{bmatrix} -2/3 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1/3 \\ 1 \end{bmatrix} \; s \in \mathbb{R} \right\}$$

Example 2.3. Find the intersection of lines $L_1 = \left\{ t\begin{bmatrix} 1 \\ 2 \end{bmatrix} : t \in \mathbb{R} \right\}$ and $L_2 = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 3 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$.

Note It's wrong to set $t\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and solve for t !

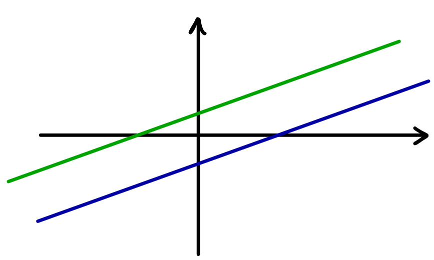
The parameter-value for L_1 need not be equal to the parameter-value for L_2 at POI

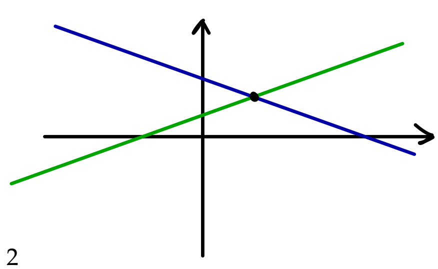
$$t\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} 3 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} t \\ 2t \end{bmatrix} = \begin{bmatrix} 3s \\ 1+s \end{bmatrix} \Rightarrow 2(3s) = 1+s \Rightarrow s = \frac{1}{5} \Rightarrow t = \frac{3}{5}$$

Use L_1 and t to find POI: $\frac{3}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 6/5 \end{bmatrix}$

or L_2 and s $\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 6/5 \end{bmatrix}$

Remark 2.4. In $\mathbb{R}^2$, two distinct lines are either parallel or they intersect at a point.





Remark 2.5. In $\mathbb{R}^3$, two distinct lines are either parallel or they intersect at a point, or they are skew. If the lines are parallel or if they intersect, then there exists a plane that contains both lines. If the lines are skew, then they are not both contained in the same plane, but you can find two parallel planes such that each contains one of the lines.

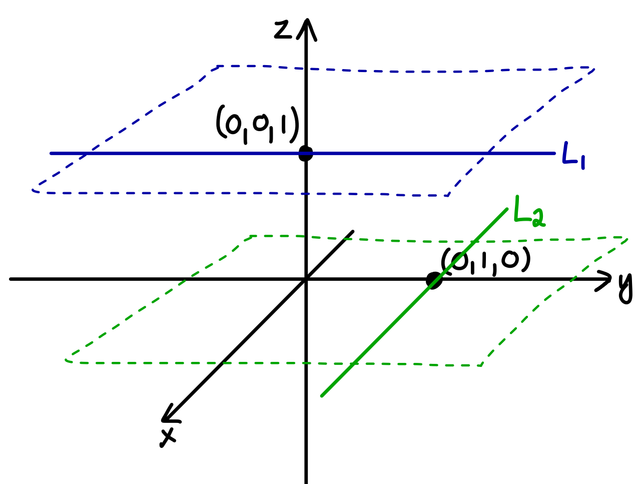
Example 2.6. Consider the lines $L_1 = \{(0, t, 1) : t \in \mathbb{R}\}$ and $L_2 = \{(t, 1, 0) : t \in \mathbb{R}\}$. Are they parallel, skew, or do they intersect?

$$\begin{bmatrix} 0 \\ t \\ 1 \end{bmatrix} = \begin{bmatrix} s \\ 1 \\ 0 \end{bmatrix} \leftarrow \text{no solution} \qquad \text{no intersection}$$

$$\vec{d}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{d}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{d}_1 \text{ is not a scalar multiple of } \vec{d}_2 \quad \therefore \text{ not parallel}$$

$\therefore$ L_1 and L_2 are skew

For L_1, $x=0, y=t, z=1$



For L_2, $x=s, y=1, z=0$

A plane in $\mathbb{R}^3$ can be expressed as an equation in **POINT-NORMAL FORM** or **CARTESIAN FORM**, which looks as follows:

$$ax + by + cz = d$$

$a, b, c, d \in \mathbb{R}$

$\vec{n} = (a, b, c)$ is a NORMAL VECTOR to the plane

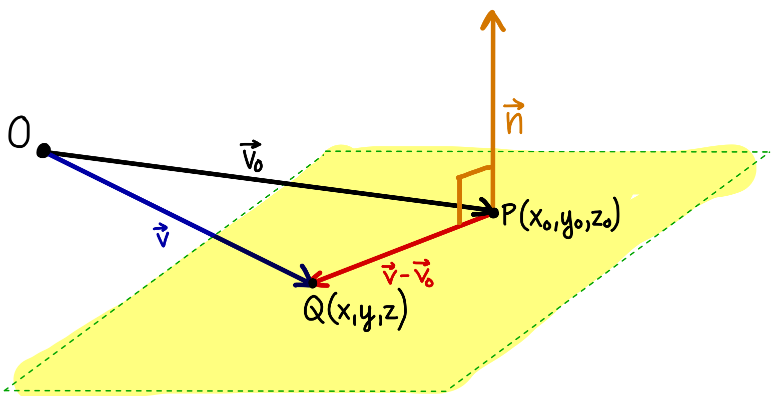
Unlike a line, a plane has infinitely many directions. But in $\mathbb{R}^3$, a plane has a single perpendicular direction, which is the plane's **NORMAL VECTOR**.

How does the normal vector end up in the coefficients of the equation?

Let $P(x_0, y_0, z_0)$ be a particular point on the plane and let $\vec{OP} = \vec{v}_0$

Let $Q(x, y, z)$ denote an arbitrary point on the plane and let $\vec{OQ} = \vec{v}$

Then $\vec{v} - \vec{v}_0$ is on the plane so $(\vec{v} - \vec{v}_0) \cdot \vec{n} = 0$



$$\Rightarrow \underbrace{(x - x_0, y - y_0, z - z_0)}_{\vec{v}-\vec{v}_0} \cdot \underbrace{(a, b, c)}_{\vec{n}} = 0$$

$$\Rightarrow ax - ax_0 + by - by_0 + cz - cz_0 = 0$$

$$\Rightarrow ax + by + cz = \underbrace{ax_0 + by_0 + cz_0}_{\text{call this } d} = d$$

In fact, d is simply $\vec{v}_0 \cdot \vec{n}$

$$\vec{v} - \vec{v}_0 \text{ lies on plane} \iff (\vec{v} - \vec{v}_0)\vec{n} = 0$$

$$\iff \vec{v}\vec{n} - \vec{v}_0\vec{n} = 0$$

$$\iff \vec{v}\vec{n} = \vec{v}_0\vec{n}$$

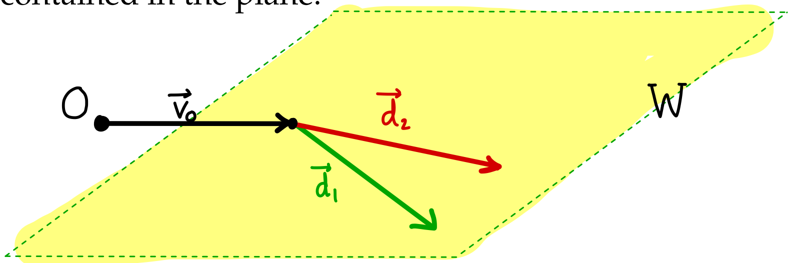
Example 2.7. Find the Cartesian equation for the plane that contains the point $(1, 2, 3)$ with normal vector $\mathbf{n} = (-1, 5, 2)$.

$P(1,2,3)$ $\vec{v}_0 = \vec{OP} = (1,2,3)$ $\vec{n} = (-1,5,2)$

$$\vec{n} \cdot \vec{v}_0 = (-1,5,2) \cdot (1,2,3) = -1 + 10 + 6 = 15$$

$\therefore$ plane's equation is $-x + 5y + 2z = 15$

A plane in $\mathbb{R}^3$ can also be expressed in **PARAMETRIC FORM** . We need two vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ that are parallel to the plane but not parallel to each other, i.e. we need two distinct "direction vectors" contained in the plane.



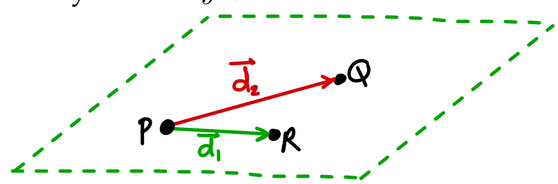
$$W = \{ \vec{v}_0 + s\vec{d}_1 + t\vec{d}_2 \ : s,t \in \mathbb{R} \}$$

2 parameters

Example 2.8. Find a parametric equation for the plane given by $3x - y + 2z = -12$.

Strategy:

1. Isolate a variable
2. The other two become the free parameters
3. Extract the point and direction vectors



① $y = 3\underset{s}{x} + 2\underset{t}{z} + 12$

② $x = s$, $z = t$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s \\ 3s+2t+12 \\ t \end{bmatrix} \overset{③}{=} \underbrace{\begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}}_{\text{parametric form}}$$

EXERCISE! Find a parametric equation for the same plane as the previous example using the following alternative method:

1. Find three points on the plane.
2. Create two direction vectors from the points.

Example 2.9. Find a parametric equation for the plane given by $y = 2x + 3$.

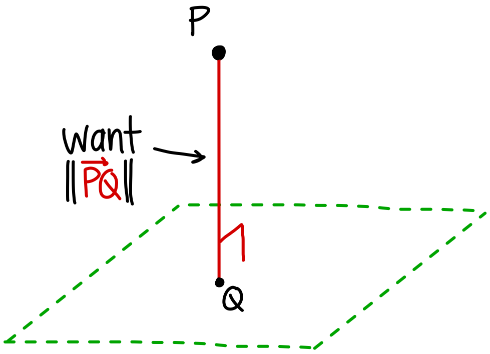
Note In $\mathbb{R}^3$, the equation $y=2x+3$ does NOT define a line !

In $\mathbb{R}^3$, lines don't have Cartesian equations

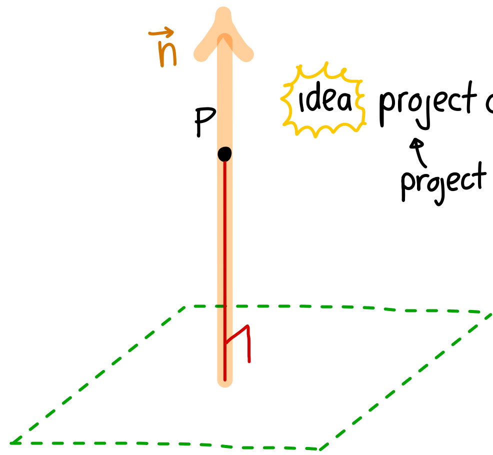
The equation $y=2x+3$ doesn't care what z equals, so z is a free variable !

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 2x+3 \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} x \\ 2x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} + x\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{parametric form}}$$

Question: How would we find the **DISTANCE FROM A POINT TO A PLANE** ?



where Q is the unknown point on plane closest to P



idea: project onto $\vec{n}$

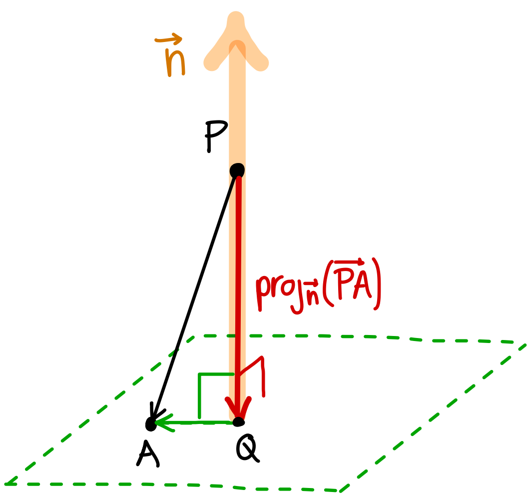
project what?

Pick any point A on the plane

Create vector $\overrightarrow{PA}$

Project $\overrightarrow{PA}$ onto normal vector $\vec{n}$

Distance from P to plane is then $\|\text{proj}_{\vec{n}}(\overrightarrow{PA})\|$



Why is Q (tip of $\text{proj}_{\vec{n}}(\overrightarrow{PA})$) the closest point to P on the plane?

By definition, $\text{proj}_{\vec{n}}(\overrightarrow{PA})$ satisfies $\overrightarrow{PA} = \underbrace{\overbrace{\text{proj}_{\vec{n}}(\overrightarrow{PA})}^{\overrightarrow{PQ}}}_{\text{parallel to } \vec{n}} + \underbrace{\overbrace{(\overrightarrow{PA} - \text{proj}_{\vec{n}}(\overrightarrow{PA}))}^{\overrightarrow{QA}}}_{\text{perpendicular to } \vec{n}}$

$\Rightarrow \triangle PQA$ is a right triangle so Pythagorean Theorem applies

$\Rightarrow \underbrace{\|\overrightarrow{PA}\|^2}_{(\text{dist from P to A})^2} = \underbrace{\|\overrightarrow{PQ}\|^2}_{(\text{dist from P to Q})^2} + \underbrace{\|\overrightarrow{QA}\|^2}_{\geq 0}$

$\Rightarrow$ (distance from P to A) $\geq$ (distance from P to Q)

$\Rightarrow$ Q is as close to P as any point A on plane

Example 2.10. Find the distance from the point $P(1, 2, 3)$ to the plane W with Cartesian equation $3x - 4z = -1$.

$\vec{n} = (3, 0, -4)$

Pick any point on plane eg set $x = 0$ Then $-4z = -1 \Rightarrow z = \frac{1}{4}$

y is free so set $y = 0$ (easy option)

$A(0, 0, \frac{1}{4})$ is on plane

$\Rightarrow \overrightarrow{PA} = (0-1, 0-2, \frac{1}{4} - 3) = (-1, -2, \frac{-11}{4})$

$$\text{proj}_{\vec{n}}(\overrightarrow{PA}) = \frac{(\overrightarrow{PA}) \cdot \vec{n}}{\vec{n}\ \vec{n}} \vec{n} = \frac{(-1, -2, -\frac{11}{4}) \cdot (3, 0, -4)}{(3, 0, -4) \cdot (3, 0, -4)} (3, 0, -4) = \frac{-3 + 0 + \frac{44}{4}}{9 + 0 + 16} (3, 0, -4) = \frac{8}{25} (3, 0, -4)$$

$$\Rightarrow \text{dist from P to W} = \|\text{proj}_{\vec{n}}(\overrightarrow{PA})\| = \left\| \frac{8}{25}(3, 0, -4) \right\| = \left|\frac{8}{25}\right| \sqrt{3^2 + 0^2 + (-4)^2} = \frac{8}{25}\sqrt{25} = \frac{8}{5}$$

Remark 2.11. In $\mathbb{R}^3$:

- a line is either entirely contained on a plane, or it intersects the plane in exactly one point, or there is no intersection and the line is parallel to the plane
- two distinct planes can be parallel, or else they intersect in a line

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Example 2.12. Find the intersection of the line L and each of the planes given below:

$$L = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} : t \in \mathbb{R} \right\} \quad \Pi_1 : 2x - y + z = 3 \quad \Pi_2 : 2x - y + 2z = -2 \quad \Pi_3 : 2x - y + z = -5$$

points on line satisfy $x = 1 - t$ $\quad y = 2 + 2t$ $\quad z = 3 + 4t$

sub into Π_1 $\quad 2(1-t) - (2+2t) + (3+4t) = 3$

$3 = 3$ $\quad$ so L satisfies Π_1 eq no matter what t is

L lies on Π_1

sub into Π_2 $\quad 2(1-t) - (2+2t) + 2(3+4t) = -2$

$6 + 4t = -2$

$\Rightarrow t = -2 \quad \Rightarrow$ POI when $t = -2$ $\quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (-2)\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix}$

sub into Π_3 $\quad 2(1-t) - (2+2t) + (3+4t) = -5$

$3 = -5$ ⚡ no solution $\therefore$ no intersection

Example 2.13. Find the **Intersection of the Planes** $2x + 4y + 4z = 7$ and $6x - 3y + 2z = 1$.

① $2x + 4y + 4z = 7$ $\quad$ planes aren't parallel (non-parallel normal vectors)
② $6x - 3y + 2z = 1$

eliminate x

$$\begin{array}{rl} 6x + 12y + 12z = 21 & \leftarrow 3\text{x eq①} \\ 6x - 3y + 2z = 1 & \\ \hline 15y + 10z = 20 & \end{array}$$

eliminate z

$$\begin{array}{rl} 2x + 4y + 4z = 7 & \\ -\ \ 12x - 6y + 4z = 2 & \leftarrow 2\text{x eq②} \\ \hline -10x + 10y = 5 & \end{array}$$

express x and z in terms of y

$10z = 20 - 15y$
$z = 2 - \frac{3}{2}y$

$-10x = 5 - 10y$
$x = -\frac{1}{2} + y$

Intersection $\quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} + y \\ y \\ 2 - \frac{3}{2}y \end{bmatrix} = \begin{bmatrix} -1/2 \\ 0 \\ 2 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \\ -3/2 \end{bmatrix}$

Example 2.14. Find the Cartesian equation for the plane $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + s \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} : s, t \in \mathbb{R} \right\}$.

We have two direction vectors for plane but we need a normal vector

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CROSS PRODUCT

Recall: in $\mathbb{R}^3$, we have an operation that takes two vectors and creates a vector orthogonal to both: the **CROSS PRODUCT** .

Definition 2.15. Given $\mathbf{u} = (a, b, c)$ and $\mathbf{v} = (x, y, z)$ in $\mathbb{R}^3$, the **CROSS PRODUCT OF u AND v** , denoted $\mathbf{u} \times \mathbf{v}$, is defined as follows:

$$\boxed{\vec{u} \times \vec{v} = (a,b,c) \times (d,e,f) = (bf-ce, -(af-cd), ae-bd)}$$

$$(a,b,c) \times (d,e,f) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a & b & c \\ d & e & f \end{vmatrix} = \hat{\imath} \begin{vmatrix} b & c \\ e & f \end{vmatrix} - \hat{\jmath} \begin{vmatrix} a & c \\ d & f \end{vmatrix} + \hat{k} \begin{vmatrix} a & b \\ d & e \end{vmatrix} \qquad \hat{\imath} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \hat{\jmath} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

memory aid
determinant
notation

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a & b & c \\ d & e & f \end{vmatrix} \qquad \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a & b & c \\ d & e & f \end{vmatrix} \qquad \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a & b & c \\ d & e & f \end{vmatrix}$$

Back to Example 2.14!

normal vector $\vec{n} \perp \vec{d}_1$ and $\vec{n} \perp \vec{d}_2$ Find $\vec{n} = \vec{d}_1 \times \vec{d}_2$

$$\vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 8 & 1 \\ 1 & 2 & 2 \end{vmatrix} = \hat{\imath} \begin{vmatrix} 8 & 1 \\ 2 & 2 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} + \hat{k} \begin{vmatrix} 4 & 8 \\ 1 & 2 \end{vmatrix}$$

$$= (8(2)-(1)(2), -(4(2)-(1)(1)), 4(2)-8(1))$$

$$= (14, -7, 0)$$

point on plane $(1,-1,2)$ $\Rightarrow$ $d = \vec{n} \cdot \vec{v}_0 = (14,-7,0) \cdot (1,-1,2) = 21$

$\boxed{\text{Cartesian eq} \quad 14x - 7y + 0z = 21}$

Example 2.16. Compute $\hat{\mathbf{i}} \times \hat{\mathbf{j}}$ and $\hat{\mathbf{j}} \times \hat{\mathbf{i}}$.

$$\hat{i} \times \hat{j} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = (0-0, -(0-0), 1-0) = (0,0,1) = \hat{k}$$

$$\hat{j} \times \hat{i} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = (0-0, -(0-0), 1-0) = (0,0,-1) = -\hat{k}$$

$\hat{i} \times \hat{j} \neq \hat{j} \times \hat{i}$

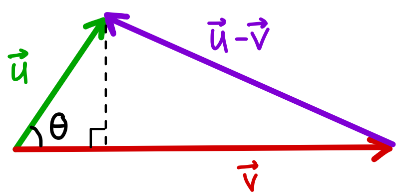
Cross product is <u>NOT</u> commutative !

But, cross product does have other nice properties

Proposition 2.17. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Then

- $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
- $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} =$
- $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} =$
- $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$
- $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin\theta$ where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ and $0 \leq \theta \leq \pi$.

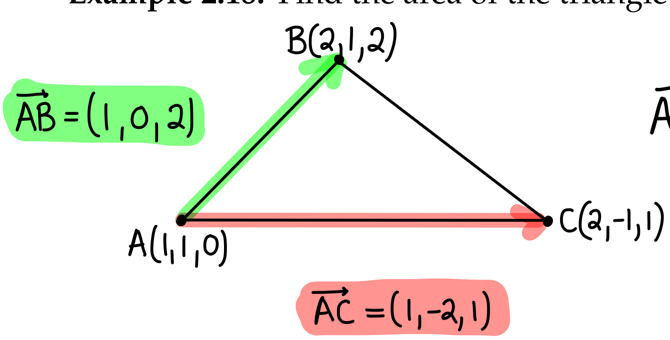
Area of Triangle The last property above allows us to use cross products to compute the area of the triangle formed between two vectors (or the area of the parallelogram created by the vectors).



$$\sin\theta = \frac{(\text{height})}{\|\vec{u}\|} \Rightarrow (\text{height}) = \|\vec{u}\|\sin\theta$$

$$\begin{aligned} \text{Area } \triangle &= \tfrac{1}{2}(\text{base}) \times (\text{height}) \\ &= \tfrac{1}{2}\|\vec{v}\|\,\|\vec{u}\|\sin\theta \\ &= \tfrac{1}{2}\|\vec{u} \times \vec{v}\| \quad (\text{by Prop } 2.17) \end{aligned}$$

Example 2.18. Find the area of the triangle formed by the points $A(1,1,0), B(2,1,2), C(2,-1,1)$.

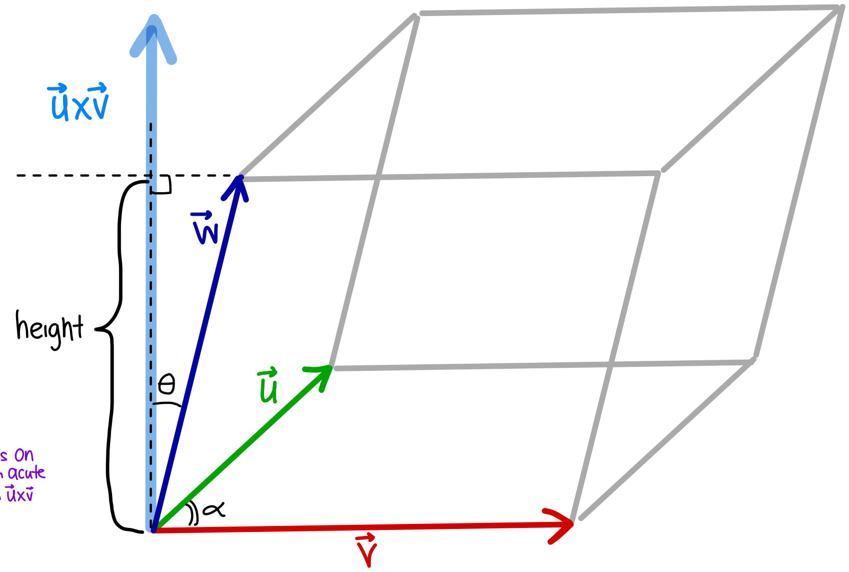


$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ 1 & -2 & 1 \end{vmatrix} = (4, -(-1), -2) = (4,1,-2)$$

$$\|\vec{AB} \times \vec{AC}\| = \sqrt{(4)^2 + 1^2 + (-2)^2} = \sqrt{21}$$

$$\text{Area } \triangle ABC = \tfrac{1}{2}\|\vec{AB} \times \vec{AC}\| = \tfrac{1}{2}\sqrt{21}$$

Volume of a Parallelepiped Using both cross and dot product, we also have the following way to compute the volume of a parallelepiped!

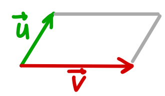


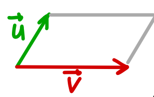
θ = angle between $\vec{u}\times\vec{v}$ and $\vec{w}$

$$|\cos\theta| = \frac{\text{height}}{\|\vec{w}\|}$$

Sign of $\cos\theta$ depends on whether $\vec{w}$ makes an acute or obtuse angle with $\vec{u}\times\vec{v}$

$$\Rightarrow \text{height} = \|\vec{w}\|\,|\cos\theta|$$

Area of  $= \|\vec{u}\times\vec{v}\| = \|\vec{u}\|\,\|\vec{v}\|\sin\alpha$

$\Rightarrow$ Vol Parallelepiped $=$ (Area of )(height)

$$= \|\vec{u}\times\vec{v}\|\,\|\vec{w}\|\,|\cos\theta|$$

$$= |(\vec{u}\times\vec{v})\cdot\vec{w}|$$

Recall $\vec{x}\cdot\vec{y} = \|\vec{x}\|\,\|\vec{y}\|\cos\theta$

Example 2.19. Find the volume of the parallelepiped formed by $\mathbf{u} = (1,2,3)$, $\mathbf{v} = (1,3,2)$, $\mathbf{w} = (1,2,2)$.

$$\text{Volume} = |(\vec{u}\times\vec{v})\cdot\vec{w}| = |(-5,1,1)\cdot(1,2,2)| = |-5+2+2| = |-1| = 1$$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ 1 & 3 & 2 \end{vmatrix} = (-5,1,1)$$