10. Antiderivatives & Areas

ANTIDERIVATIVES

A function $F(x)$ is called an **antiderivative** of $f(x)$ if $F'(x) = f(x)$

Example 10.1. Guess an antiderivative for each of the following functions. Then check your answer by differentiating.

function $f(x)$	guess an antiderivative $F(x)$ of f	verify that $F'(x) = f(x)$
(a) $f(x) = 2x$	$F(x) = x^2$	$F'(x) = 2x = f(x)$ ✓
(b) $f(x) = 3x^2$	$F(x) = x^3$	$F'(x) = 3x^2 = f(x)$ ✓
(c) $f(x) = e^x$	$F(x) = e^x$	$F'(x) = e^x = f(x)$ ✓
(d) $f(x) = \cos(x)$	$F(x) = \sin(x)$	$F'(x) = \cos(x) = f(x)$ ✓
(e) $f(x) = 0$	$F(x) = C$ $C \in \mathbb{R}$, could be any constant!	$F'(x) = \frac{d}{dx}[C] = 0 = f(x)$ ✓

In fact, since the derivative of any constant $C \in \mathbb{R}$ is zero, we can always include a "$+C$" (an arbitrary constant) to any antiderivative.

* These notes are solely for the personal use of students registered in MAT1320.

Fact. If $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$, then $F(x) - G(x) = C$, for some constant C.

- Thus, if F and G are two distinct antiderivatives of $f(x)$, then the graphs of F and G are identical, up to a vertical translation.

Theorem 10.2. If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x) + C$ where C is an arbitrary constant.

the constant of integration

Integral Notation:

integral sign

should both be used like opening + closing parentheses

dx

$$\int \underbrace{f(x)}_{\text{the "integrand" (what we are integrating)}} dx = \underbrace{F(x) + C}_{\text{the most general antiderivative of the integrand } f(x)}$$

"UNDOING" BASIC RULES OF DIFFERENTIATION

Constants. $\boxed{\int 0 \, dx = C}$ because $\frac{d}{dx}[C] = 0$ ✓

$\boxed{\int k \, dx = kx + C}$ because $\frac{d}{dx}[kx + C] = k$ ✓

Powers. $\frac{d}{dx}[x^n] = nx^{n-1}$

"old" exponent becomes a constant multiple

exponent decreases by 1

- For the antiderivative of x^n we needed the "old" exponent to be $n+1$ (to get n after differentiating)
- We also need to compensate for the "missing" constant multiple of the "old" exponent...

For $n \neq -1$, → $\boxed{\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C}$

BEWARE OF $n = -1$

$\int x^{-1} dx \cancel{=} \frac{1}{0} x^0 + C$ undefined!

$$\frac{d}{dx}\left[\frac{1}{n+1} x^{n+1} + C\right] = \frac{1}{n+1}(n+1)x^{n+1-1} + 0 = x^n \checkmark$$

The special power $n=-1$

$x^{-1}=\frac{1}{x}$ is the derivative of $\ln x$ (for $x>0$)

$\frac{d}{dx}[\ln x]=\frac{1}{x}$

$x^{-1}=\frac{1}{x}$ is the derivative of $\ln|x|$ (for $x\neq 0$)

$$\boxed{\int \frac{1}{x}\,dx = \ln|x| + C}$$

$\frac{d}{dx}[\ln|x|+C]=\frac{1}{x}+0=\frac{1}{x}$ ✓

Constant multiples $(k\in\mathbb{R})$

$$\boxed{\int k\cdot f(x)\,dx = k\int f(x)\,dx}$$

Sums and differences.

$$\boxed{\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx}$$

$$\boxed{\int (f(x)-g(x))\,dx = \int f(x)\,dx - \int g(x)\,dx}$$

Example 10.3. Find $f(x)$ given that $f'(x)=\dfrac{5}{x^6}+8x^3-\dfrac{2}{x}$ and $f(1)=2$.

$f'(x)=5x^{-6}+8x^3-2x^{-1}$ so $f(x)$ is an antiderivative of $f'(x)$.

$\Rightarrow f(x)=5\left(\frac{1}{(-6+1)}x^{-6+1}\right)+8\left(\frac{1}{(3+1)}x^{3+1}\right)-2\ln|x|+C$

$\Rightarrow f(x)=5\left(\frac{1}{-5}x^{-5}\right)+8\left(\frac{1}{4}x^4\right)-2\ln|x|+C$

$\Rightarrow f(x)=-\frac{1}{x^5}+2x^4-2\ln|x|+C$

- Now, using the initial value $f(1)=2$, we can solve for the constant C.

$\Rightarrow 2=f(1)=-\frac{1}{1^5}+2(1^4)-2\ln|1|+C$

$\Rightarrow 2=-1+2-2(0)+C$

$\Rightarrow C=1.$

$$\boxed{\therefore f(x)=-\frac{1}{x^5}+2x^4-2\ln|x|+1}$$

Trig functions.

$$\int \cos(x)\,dx = \sin(x) + C \qquad \frac{d}{dx}[\sin x + C] = \cos x + 0 = \cos x \checkmark$$

$$\int \sin(x)\,dx = -\cos(x) + C \qquad \frac{d}{dx}[-\cos x + C] = -(-\sin x) + 0 = \sin x \checkmark$$

$$\int \sec^2(x)\,dx = \tan(x) + C \qquad \frac{d}{dx}[\tan x + C] = \sec^2 x + 0 = \sec^2 x \checkmark$$

⋮

* for any recognizable derivative, you should know its antiderivative...

Example 10.4. Verify that $F(x) = -\ln|\cos x| + C$ is an antiderivative of $\tan x$.

We must check that $F'(x) = \tan x$.

$$\frac{d}{dx}[F(x)] = \frac{d}{dx}\left[-\ln|\cos x| + C\right] = \underbrace{-\frac{1}{\cos x}(-\sin x)}_{\text{using } \ln \text{ chain rule}} = \frac{\sin x}{\cos x} = \tan x \checkmark$$

Exponential functions.

$$\int e^x\,dx = e^x + C \qquad \frac{d}{dx}[e^x + C] = e^x \checkmark$$

$$\int b^x\,dx = \frac{1}{\ln(b)} b^x + C \qquad \frac{d}{dx}\left[\frac{1}{\ln b} b^x + C\right] = \frac{1}{\ln b}(\ln b) b^x = b^x \checkmark$$

Keep in mind
$\frac{d}{dx}[b^x] = (\ln b) b^x$

Inverse trig functions.

$$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C \qquad \frac{d}{dx}[\arctan(x) + C] = \frac{1}{1+x^2} \checkmark$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C \qquad \frac{d}{dx}[\arcsin(x) + C] = \frac{1}{\sqrt{1-x^2}} \checkmark$$

⋮

* again, if you're faced with a recognizable derivative (or constant multiple of) then you should know its antiderivative.

Example 10.5. Verify that $x \ln x - x$ is an antiderivative of $\ln(x)$.

We must check that $F'(x) = \ln(x)$.

$$\frac{d}{dx}[F(x)] = \frac{d}{dx}[x \ln x - x] = \underbrace{(1)\ln x + x\left(\frac{1}{x}\right)}_{\text{product rule}} - 1 = \ln x + 1 - 1 = \ln x \checkmark$$

Example 10.6. Try to determine (guess!) an antiderivative for each of the following functions. Verify (check!) your answer by differentiating.

function	antiderivative?	Check
(a) $\underbrace{2xe^{x^2}}$ looks like the aftermath of an exp. chain rule?	e^{x^2}	$\frac{d}{dx}[e^{x^2}] = (e^{x^2})(2x) \checkmark$
(b) $\underbrace{3x^2 \sin x + x^3 \cos x}$ looks like aftermath of a product rule?	$x^3 \sin x$	$\frac{d}{dx}[x^3 \sin x] = (3x^2)(\sin x) + (x^3)(\cos x) \checkmark$
(c) $\cot(x) = \underbrace{\frac{\cos x}{\sin x}}$ looks like aftermath of a ln chain rule?	$\ln\lvert\sin x\rvert$	$\frac{d}{dx}[\ln\lvert\sin x\rvert] = \frac{1}{\sin x}(\cos x) = \cot x \checkmark$
(d) $\underbrace{e^{ax+b}}$ where a and b are constant real numbers. looks like aftermath of exp. chain rule?	$\frac{1}{a}e^{ax+b}$	$\frac{d}{dx}[\frac{1}{a}e^{ax+b}] = \frac{1}{a}(e^{ax+b})(a+0) = e^{ax+b} \checkmark$
(e) $\underbrace{\frac{g'(x)}{g(x)}}$ looks like aftermath of a ln chain rule?	$\ln\lvert g(x)\rvert$	$\frac{d}{dx}[\ln\lvert g(x)\rvert] = \frac{1}{g(x)} \cdot (g'(x)) = \frac{g'(x)}{g(x)} \checkmark$
(f) $\underbrace{e^x [e^x + 8]^{10}}$ looks like aftermath of a power chain rule?	$\frac{1}{11}(e^x+8)^{11}$	$\frac{d}{dx}[\frac{1}{11}(e^x+8)^{11}] = (\frac{1}{11})(11)(e^x+8)^{10}(e^x+0) = e^x(e^x+8)^{10} \checkmark$

Exercise 10.7. Determine an equation for the height of a ball at time t if the ball is thrown with an initial upwards velocity of v_0 m/s, from an initial height of h_0 m above the ground, on Earth where acceleration due to gravity is -9.8 m/s^2.

acceleration $a(t) = -9.8$ m/s^2

↳ is the rate of change of the ball's velocity with respect to time

i.e. $a(t) = v'(t)$

$\Rightarrow v(t)$ is an antiderivative of $a(t)$. Moreover, $v(0) = v_0$ m/s.

$\Rightarrow v(t) = -9.8t + C$ Using $v(0) = v_0$ we solve for C

$v_0 = -9.8(0) + C \Rightarrow C = v_0$

$\therefore$ Velocity is given by $v(t) = -9.8t + v_0$

↳ is the rate of change of the ball's height with respect to time

i.e. $v(t) = h'(t)$

$\Rightarrow h(t)$ is an antiderivative of $v(t)$. Moreover, $h(0) = h_0$ m/s.

$\Rightarrow h(t) = -9.8\left(\frac{1}{2}t^2\right) + v_0 t + K$ using $h(0) = h_0$ we solve for K

$= -4.9t^2 + v_0 t + K$

$h_0 = -4.9(0^2) + v_0(0) + K$

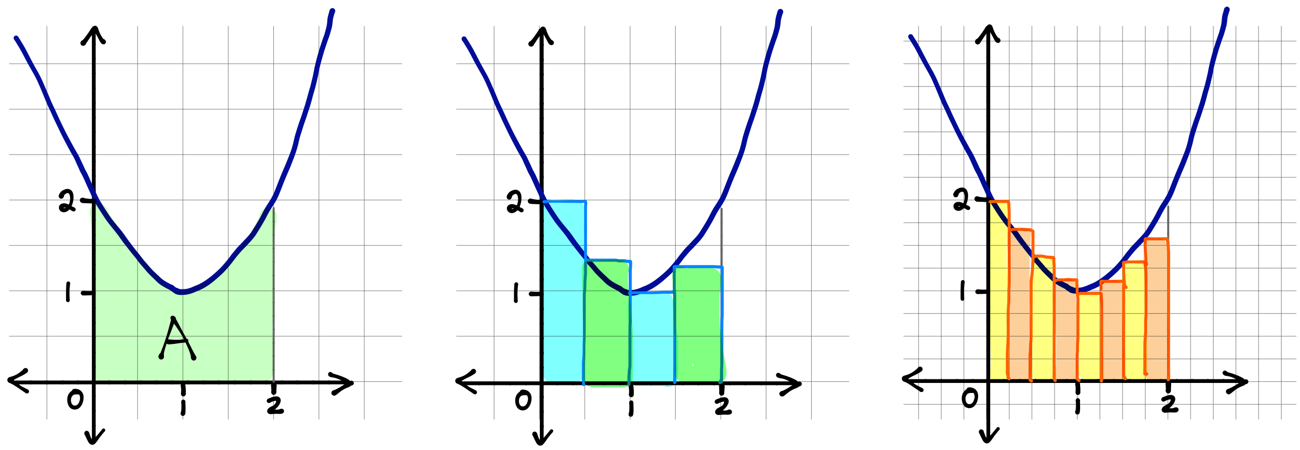
$\Rightarrow K = h_0$

$\therefore$ height is given by $h(t) = -4.9t^2 + v_0 t + h_0$

Example 10.8. Let $f(x) = (x-1)^2 + 1$.

What is the area of the region under the graph of f, above the x-axis between $x = 0$ and $x = 2$?

This is not a standard shape, so we do not have an area formula.



$f(x) = (x-1)^2 + 1$ | 4 rectangles using left endpoint | 8 rectangles using left endpoint

We can try to estimate the area using several rectangles:

- If we use $n = 4$ rectangles, then each rectangle has width $\Delta x = 0.5$ (this is because we are chopping the interval $[0, 2]$ into four equal pieces).
- If we use $n = 8$ rectangles instead, then each rectangle has width 0.25 (this is because we are chopping the interval $[0, 2]$ into eight equal pieces).
- For the height of each rectangle, we will use the y-coordinate of f at the appropriate value of x (for now, we'll use the x-coordinate at the bottom left corner of each rectangle).

$$L_4 = \begin{bmatrix} \text{Estimate of area using} \\ \text{4 rectangles with} \\ \text{left endpoint heights} \end{bmatrix}$$

$$\begin{aligned}
&\approx \begin{pmatrix} \text{area of 1st} \\ \text{rectangle} \end{pmatrix} + \begin{pmatrix} \text{area of 2nd} \\ \text{rectangle} \end{pmatrix} + \begin{pmatrix} \text{area of 3rd} \\ \text{rectangle} \end{pmatrix} + \begin{pmatrix} \text{area of 4th} \\ \text{rectangle} \end{pmatrix} \\
&= \left(\underbrace{0.5}_{\substack{\text{width} \\ \text{1st rect.}}} \times \underbrace{f(0)}_{\substack{\text{height} \\ \text{1st rect.}}} \right) + \left(\underbrace{0.5}_{\substack{\text{width} \\ \text{2nd rect.}}} \times \underbrace{f(0.5)}_{\substack{\text{height} \\ \text{2nd rect.}}} \right) + \left(\underbrace{0.5}_{\substack{\text{width} \\ \text{3rd rect.}}} \times \underbrace{f(1)}_{\substack{\text{height} \\ \text{3rd rect.}}} \right) + \left(\underbrace{0.5}_{\substack{\text{width} \\ \text{4th rect.}}} \times \underbrace{f(1.5)}_{\substack{\text{height} \\ \text{4th rect.}}} \right) \\
&= (0.5 \times 2) + (0.5 \times 1.25) + (0.5 \times 1) + (0.5 \times 1.25) \\
&= [2 + 1.25 + 1 + 1.25](0.5) = 2.75
\end{aligned}$$

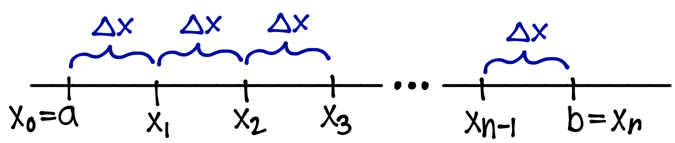
$$L_8 = \begin{bmatrix}\text{Estimate of area using} \\ \text{8 rectangles with} \\ \text{left endpoint heights}\end{bmatrix}$$

$$\begin{aligned}
&\approx \begin{pmatrix}\text{area of 1st} \\ \text{rectangle}\end{pmatrix} + \begin{pmatrix}\text{area of 2nd} \\ \text{rectangle}\end{pmatrix} + \begin{pmatrix}\text{area of 3rd} \\ \text{rectangle}\end{pmatrix} + \begin{pmatrix}\text{area of 4th} \\ \text{rectangle}\end{pmatrix} \\
&\quad + \begin{pmatrix}\text{area of 5th} \\ \text{rectangle}\end{pmatrix} + \begin{pmatrix}\text{area of 6th} \\ \text{rectangle}\end{pmatrix} + \begin{pmatrix}\text{area of 7th} \\ \text{rectangle}\end{pmatrix} + \begin{pmatrix}\text{area of 8th} \\ \text{rectangle}\end{pmatrix} \\
&= \big(0.25 \times f(0)\big) + \big(0.25 \times f(0.25)\big) + \big(0.25 \times f(0.5)\big) + \big(0.25 \times f(0.75)\big) \\
&\quad + \big(0.25 \times f(1)\big) + \big(0.25 \times f(1.25)\big) + \big(0.25 \times f(1.5)\big) + \big(0.25 \times f(1.75)\big) \\
&= \big[1 + 1.5625 + 1.25 + 1.0625 + 1 + 1.0625 + 1.25 + 1.5625\big]0.25 = 2.6875
\end{aligned}$$

General Observations:

- If we chopped the interval into even more (thinner) rectangles, we would get better and better estimates of the actual area.
- Using n rectangles, each of width Δx, where Δx is the interval length divided by n, we can estimate the net area between any continuous function and the x-axis from $x = a$ to $x = b$.

$$\Delta x = \frac{b-a}{n}$$



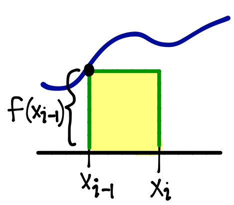
- for $i = 0, 1, 2, \ldots, n$, let $x_i = a + i\Delta x$

- The bigger n gets, the smaller Δx gets (more rectangles, but they are thinner).
- For the height of the ith rectangle, we could just as well have used the x-coordinate at the bottom right corner of each rectangle, or the x-coordinate at the midpoint of the rectangles two sides, or some other sample point x_i^*.

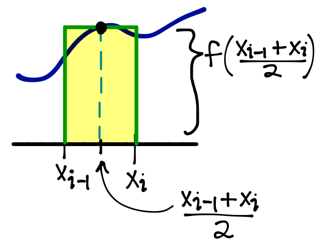
- base of ith rectangle is the ith subinterval $[x_{i-1}, x_i]$

- height of ith rectangle could be

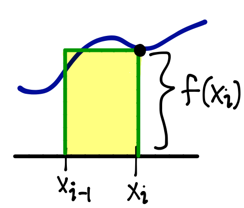
left endpoint



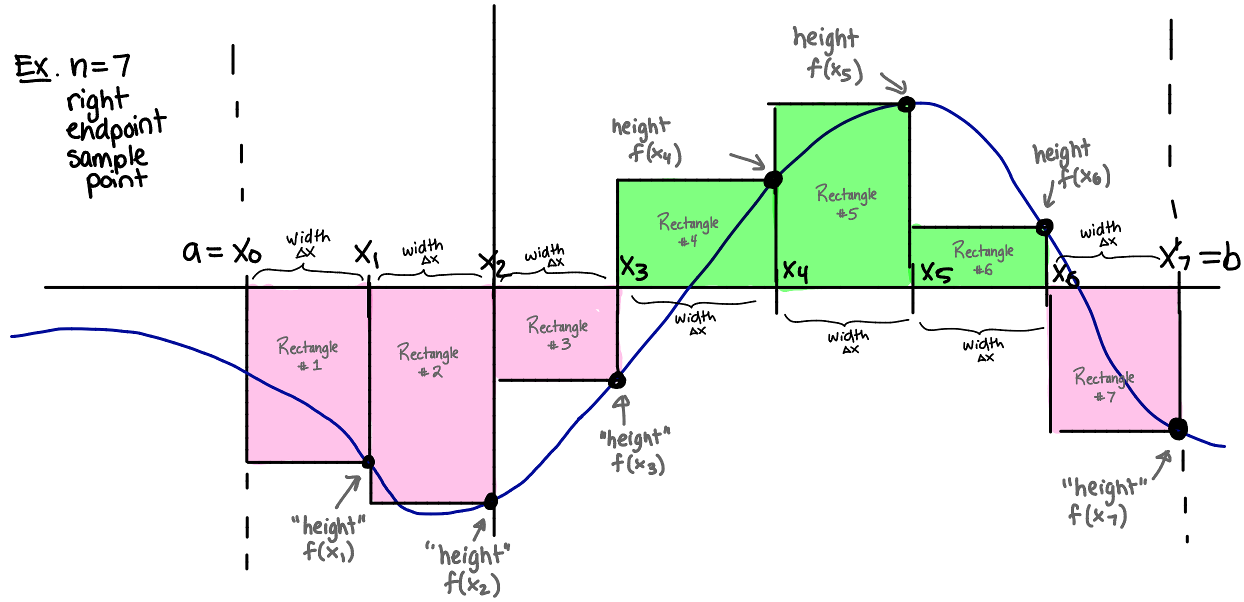
midpoint



right endpoint



- The widths of the rectangles did not all have to be equal (but having them all of equal width made the calculation more straightforward).



- for $i = 1, 2, \dots, n$, the ith subinterval is $\underbrace{[x_{i-1}, x_i]}$ ← the base of rectangle # i
- from each subinterval, choose a <u>sample point</u> $x_i^* \in [x_{i-1}, x_i]$
- for $i = 1, 2, \dots, n$, we will use $f(x_i^*)$ as the height of rectangle # i
- the sum $f(x_1^*)\Delta x + f(x_2^*)\Delta x + \dots + f(x_n^*)\Delta x$ is denoted $\sum_{i=1}^{n} f(x_i^*)\Delta x$
- $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called <u>a Riemann sum.</u>

  It gives an approximation of the net area between $f(x)$ and the x-axis on the interval $[a, b]$.

Ex $f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x + f(x_7)\Delta x \approx$ $\boxed{\text{Net Area between } f \text{ and x-axis on } [a, b]}$

−(area of rectangle #1) + −(area of rectangle #2) + −(area of rectangle #3) + (area of rectangle #4) + (area of rectangle #5) + (area of rectangle #6) −(area of rectangle #7)

Study Guide

- ◇ **an antiderivative** vs. **the most general antiderivative**
- ◇ **undoing basic rules of differentiation**
- ◇ **some ideas for undoing less basic rules of differentiation**
- ◇ **approximating area (or distance) using a Riemann sum**: $A \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$