4. Subspaces

We now know the 10 axioms for a vector space and we showed that there is a wide variety of vector spaces other than $\mathbb{R}^n$. On the one hand, all vector spaces are instances of the same thing (because they all have operations that satisfy the same axioms). On the other hand, each vector space can contain fundamentally different types of "vectors" (e.g. functions in the vector space of functions can't be visualized as arrows the way that vectors in $\mathbb{R}^3$).

Now, given a vector space, we will investigate its most closely-related vector spaces: that is, we will look at the vector spaces that "live inside" other bigger vector spaces.

SUBSPACES

Definition 4.1. Let V be a vector space, equipped with its rules for vector addition $(+)$ and scalar multiplication $(\cdot)$.

A set U is called a **SUBSPACE OF V** if

- $\underbrace{U \subseteq V}_{U \text{ is a subset of } V}$ (ie for all x, $x \in U \Rightarrow x \in V$)

and

- equipped with the same vector addition and scalar multiplication operations as V, U is a vector space in its own right

In other words, U is a subspace of V if all of the vectors in U already belong to the vector space V, and the operations for U are inherited from V, and with V's operations, the set U satisfies all 10 vector space axioms.

Let's revisit an example we previously looked at:

Example 4.2. $L = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : y = 2x \right\}$, equipped with the operations of $\mathbb{R}^2$.

We showed L satisfies closure axioms and existence axioms

We noticed that L satisfies the Arithmetic Properties because those properties hold for <u>all</u> vectors in $\mathbb{R}^2$, including the vectors in L

So L itself is a vector space that "lives inside" $\mathbb{R}^2$

<u>ie</u> L is a subspace of $\mathbb{R}^2$!

†These notes are solely for the personal use of students registered in MAT1341.

Suppose U is a subset of a vector space V, equipped with the same operations as V. If we want to check whether U is a subspace of V, we can verify all 10 axioms.

However, U "inherits" the arithmetic properties from V.

What do we mean by this? Well, we don't really have to check those 6 axioms for U if we already know they hold for *all* vectors in V. Why? Because every vector in U is a vector in V. We already know V is a vector space. So V and its operations definitely satisfy those axioms.

What about the first 4 axioms?

Knowing that V satisfies those 4 axioms does not guarantee that U will satisfy them.

Recall the following example:

Example 4.3. $W = \left\{ \begin{bmatrix} x \\ x+2 \end{bmatrix} : x \in \mathbb{R} \right\}$ equipped with the operations of $\mathbb{R}^2$.

W is a subset of $\mathbb{R}^2$ ✓

W is equipped with the operations of $\mathbb{R}^2$ ✓

But W is not a vector space itself

Closure Existence Arithmetic

Axioms for W ~~1~~ ~~2~~ ~~3~~ ~~4~~ 5 6 7 8 9 10 ✓

$\therefore$ W is not a subspace of $\mathbb{R}^2$

Observations:

- Adding two vectors in W does not always result in a vector in W.
- Adding two vectors in W does always result in a vector in $\mathbb{R}^2$ because $\mathbb{R}^2$ *is* closed under vector addition.
- Multiplying a vector in W by a scalar $k \in \mathbb{R}$ does not always result in a vector in W.
- Multiplying a vector in W by a scalar $k \in \mathbb{R}$ does always result in a vector in $\mathbb{R}^2$ because $\mathbb{R}^2$ *is* closed under scalar multiplication.
- The zero vector $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is not in W.
- For all $\mathbf{w} \in W$, we still have $\mathbf{0} + \mathbf{w} = \mathbf{w}$ because this holds for all vectors in $\mathbb{R}^2$ which includes all vectors $\mathbf{w} \in W$.
- It is not the case that for every $\mathbf{w} \in W$, there exists $-\mathbf{w} \in W$ such that $\mathbf{w} + (-\mathbf{w}) = \mathbf{0}$.
- But, for every $\mathbf{w} \in W$, there exists $-\mathbf{w} \in \mathbb{R}^2$ such that $\mathbf{w} + (-\mathbf{w}) = \mathbf{0}$.
- All vectors in $\mathbb{R}^2$ satisfy the Arithmetic Axioms. Since all vectors of W are vectors in $\mathbb{R}^2$, the vectors of W satisfy the Arithmetic Axioms too.

Suppose V is a vector space, and U is a subset of V equipped with the operations of V. If we want to find out whether U is a subspace of V, we need only worry about checking Axioms 1–4.

In fact, the next results tell us (among other things) that for $U \subseteq V$, we don't need to worry about Axiom 4 either, so long as the first three axioms are satisfied.

Proposition 4.4. Let V be a vector space. Let $a \in \mathbb{R}$ and let $\mathbf{u} \in V$. Then

(a) $a\vec{0} = \vec{0}$

(b) If $a\vec{u} = \vec{0}$, then $a = 0$ or $\vec{u} = \vec{0}$

(c) $0\vec{u} = \vec{0}$

(d) $(-1)\vec{u} = -\vec{u}$ (proof in DGD 2)

Proposition 4.5. Let U be a subset of a vector space V, equipped with the operations of V.

If U is closed under scalar multiplication, then U satisfies Axiom 4 (Existence of Vector Negatives).

Proof:

- Assume U is a subset of a vector space V.
- Assume U is equipped with the same operations as V.
- Assume U is closed under scalar multiplication.

Let $\vec{w} \in W$ Then $(-1)\vec{w} \in W$ (since W is closed under scalar multiplication)

By Prop 49(d), $-\vec{w} = (-1)\vec{w}$ $\therefore -\vec{w} \in W$

The Subspace Test

Now, given a subset U of a vector space V, where U is equipped with the operations of V, if we want to know whether U is a subspace of V, we have an efficient test, summarized by the following theorem:

Theorem 4.6. The Subspace Test Let U be a subset of a vector space V, equipped with the same operations as V.

Then W is a subspace of V if and only if the following 3 conditions hold:

1. W is closed under vector addition, i.e., $\vec{u}, \vec{v} \in W \Rightarrow \vec{u} + \vec{v} \in W$
2. W is closed under scalar multiplication, i.e., $k \in \mathbb{R}$ and $\vec{u} \in W \Rightarrow k\vec{u} \in W$
3. $\vec{0} \in W$

Example 4.7. Let $T = \{\mathbf{u} \in \mathbb{R}^3 : \mathbf{u} \cdot (1,2,3) = 0\}$, with the usual operations of $\mathbb{R}^3$. Is T a subspace of $\mathbb{R}^3$?

First, T is a subset of $\mathbb{R}^3$ equipped with the operations of $\mathbb{R}^3$

$\Longrightarrow$ we can attempt the subspace test

More specifically,

$$T = \{(x,y,z) \in \mathbb{R}^3 : (x,y,z)\cdot(1,2,3) = 0\} = \{(x,y,z) \in \mathbb{R}^3 : \underbrace{x+2y+3z=0}_{T \text{ is a plane passing through the origin}}\}$$

Onto the Subspace Test !

① Let $\vec{u}, \vec{v} \in T$ (goal show $\vec{u}+\vec{v} \in T$)

Then $\vec{u}\cdot(1,2,3) \overset{①}{=} 0$ and $\vec{v}\cdot(1,2,3) \overset{②}{=} 0$ (by def of T)

Now, $(\vec{u}+\vec{v})\cdot(1,2,3) = \vec{u}\cdot(1,2,3) + \vec{v}\cdot(1,2,3)$ (property of dot product)

$= 0 + 0$ (by ① and ②)

$= 0$

$\Longrightarrow \underbrace{\vec{u}+\vec{v} \in T}_{\text{(goal!)}}$ (by def of T) $\therefore$ T is closed under vector addition ✓

② Let $k \in \mathbb{R}$ and $\vec{u} \in T$ (goal show $k\vec{u} \in T$)

Now $(k\vec{u})\cdot(1,2,3) = k(u\cdot(1,2,3))$ (property of dot product)

$= k\,0$ (since $\vec{u}\,(1,2,3) = 0$ by def of T)

$= 0$

$\Longrightarrow \underbrace{k\vec{u} \in T}_{\text{(goal!)}}$ $\therefore$ T is closed under scalar multiplication ✓

③ (goal show $\vec{0} \in T$)

Since $\vec{0}\cdot(1,2,3) = (0,0,0)\cdot(1,2,3) = 0$, we conclude that $\underbrace{\vec{0} \in T}_{\text{(goal!)}}$ (by def of T)

Conclusion T is a subspace of $\mathbb{R}^3$!

Remark 4.8. Let $(a,b,c) \in \mathbb{R}^3$ be any nonzero vector. Consider the plane

$$P = \{(x,y,z) \in \mathbb{R}^3 : (x,y,z)\cdot(a,b,c) = 0\} = \{(x,y,z) \in \mathbb{R}^3 : ax+by+cz = 0\}.$$

Just as in the previous example, the subspace test shows that P is a subspace of $\mathbb{R}^3$.

Conclusion: Any plane passing through the origin in $\mathbb{R}^3$ is a subspace!

Example 4.9. **POLYNOMIALS** Let $\mathbb{P}$ denote the set of all polynomial functions (in variable x). Each element in $\mathbb{P}$ is of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

for some $n \in \mathbb{Z}$, $n \geq 0$ and some $a_0, a_1, \ldots, a_n \in \mathbb{R}$ (the "coefficients").

Is $\mathbb{P}$ a vector space?

First, observe that $\mathbb{P} \subseteq \mathcal{F}$, the space of real-valued functions with domain $(-\infty, \infty)$

We add and scalar-multiply polynomials just like any functions

For $p(x), q(x) \in \mathbb{P}$ and $k \in \mathbb{R}$, $p+q$ and kp are defined by the rules

$(p+q)(x) = p(x) + q(x)$ and $(kp)(x) = kp(x)$ for all $x \in (-\infty, \infty)$

(i.e. $\mathbb{P}$ is equipped with the same operations as $\mathcal{F}$)

So we can use the Subspace Test on $\mathbb{P}$

1. If we add two polynomials together, the result is still a polynomial ✓
2. If we scale a polynomial by $k \in \mathbb{R}$, the result is still a polynomial ✓
3. The zero function $z(x) = 0$ is a polynomial, so $z \in \mathbb{P}$ ✓

$\therefore$ $\mathbb{P}$ is a subspace of $\mathcal{F}$, hence $\mathbb{P}$ is itself a vector space!

Example 4.10. Is the set $S = \{p \in \mathbb{P} : p(5) = 1\}$ a subspace of $\mathbb{P}$ (when equipped with the usual operations for adding and scaling functions)?

We can apply the Subspace Test!

1. Assume $p, q \in S$ Then $p(5) = 1$ and $q(5) = 1$ (def of S)

Now $p+q$ satisfies $(p+q)(5) = p(5) + q(5) = 1 + 1 = 2 \neq 1$

$\Rightarrow p+q \notin S$ $\therefore$ S is not closed under addition

$\therefore$ S is not a subspace of $\mathbb{P}$

Example 4.11. Is the set $T = \{p \in \mathbb{P} : p(3) = 0\}$ a subspace of $\mathbb{P}$ (when equipped with the usual operations for adding and scaling functions)?

(1) Assume $p, q \in T$ Then $p(3) = 0$ and $q(3) = 0$ (def of T)

Now, $(p+q)(3) = p(3) + q(3) = 0 + 0 = 0 \quad \therefore p+q \in T$ closed under + ✓

(2) Assume $p \in T$ and $k \in \mathbb{R}$ Then $p(3) = 0$ (def of T)

Now $(kp)(3) = k\,p(3) = k\,0 = 0 \quad \therefore kp \in T$ closed under scalar mult ✓

(3) The zero polynomial satisfies $z(x) = 0$ for all $x \in \mathbb{R}$

$\therefore z(3) = 0 \qquad \therefore z \in T$ contains zero vector ✓

$\therefore T$ is a subspace of $\mathbb{P}$!

Example 4.12. **MATRICES** Let $m, n \in \mathbb{Z}$ such that $m, n \geq 1$. Let $M_{m,n}(\mathbb{R})$ denote the set of $m \times n$ matrices with real entries.

We can add two matrices of the same size (by adding their entries together):

Note a_{ij} denotes the entry of A in the ith row and jth column

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}}_{A+B}$$

We can multiply a matrix by a scalar $k \in \mathbb{R}$ (by scaling every entry by k):

$$k\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & & & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix}}_{kA}$$

EXERCISE! Convince yourself that $M_{m,n}(\mathbb{R})$ is a vector space (consider all 10 axioms).

Definition 4.13. Let $A \in M_{m,n}(\mathbb{R})$ be a matrix. The **TRANSPOSE OF** A, denoted A^T, is the $n \times m$ matrix whose rows are the columns of A.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{mn} \end{bmatrix}_{n \times m}$$

ie $(A^T)_{i,j} = (A)_{j,i}$

Ex $A = \begin{bmatrix} 4 & 3 & -1 \\ 0 & 1 & \pi \end{bmatrix}_{2 \times 3}$ $\quad A^T = \begin{bmatrix} 4 & 0 \\ 3 & 1 \\ -1 & \pi \end{bmatrix}_{3 \times 2}$

Example 4.14. Let $S = \{A \in M_{2,2}(\mathbb{R}) : A = A^T\}$. Is S a subspace of $M_{2,2}(\mathbb{R})$?

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2,2}(\mathbb{R})$ Then $A = A^T \Leftrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \Leftrightarrow b = c$

Thus, $S = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} : a, b, d \in \mathbb{R} \right\}$ Ex $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \in S$

1. Let $A, B \in S$ Then $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, $B = \begin{bmatrix} d & e \\ e & f \end{bmatrix}$ for some a,b,c,d,e,f

   Now $A+B = \begin{bmatrix} a+d & b+e \\ b+e & c+f \end{bmatrix}$ satisfies $(A+B) = (A+B)^T$

   $\therefore A+B \in S$ closed under addition ✓

2. Let $A \in S$ and $k \in \mathbb{R}$ Then $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ for some $a,b,c \in \mathbb{R}$

   Now $kA = \begin{bmatrix} ka & kb \\ kb & kc \end{bmatrix}$ satisfies $(kA) = (kA)^T$

   $\therefore kA \in S$ closed under scalar mult ✓

3. The zero matrix of $M_{2,2}(\mathbb{R})$ is $Z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ← check that vector space Axiom 3 for $M_{2,2}(\mathbb{R})$ is satisfied with $Z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

   Z satisfies $Z = Z^T$ $\therefore Z \in S$ contains zero vector ✓

S is a subspace of $M_{2,2}(\mathbb{R})$!