DGD 4

Q1. Let $U = \left\{ \begin{bmatrix} 0 & a \\ b & 2a \end{bmatrix} \in M_{2,2}(\mathbb{R}) : a, b \in \mathbb{R} \right\}$.

(a) Verify that U is closed under addition.
(b) Show that U is a subspace of $M_{2,2}(\mathbb{R})$.
(c) Find a matrix $A \in M_{2,2}(\mathbb{R})$ such that $A \notin U$.

Q2. Determine if the following sets are linearly independent. Justify your answer.

$$A = \left\{ \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\} \qquad B = \left\{ \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix} \right\} \qquad C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

Q3. For each of the following statements, either explain why it is always true or give a counterexample that demonstrates it can be false:

(a) If $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ and X is a subspace of $\mathbb{R}^2$ with $\mathbf{v} + 2\mathbf{w} \in X$, then $\mathbf{v}, \mathbf{w} \in X$.

(b) If $\mathbf{v}$ and $\mathbf{w}$ are vectors in a vector space V, then $\operatorname{span}\{\mathbf{v}, \mathbf{w}\} = \operatorname{span}\{\mathbf{v} + \mathbf{w}, \mathbf{v}\}$.

(c) If $\mathbf{v}$ and $\mathbf{w}$ are LI vectors in a vector space V and $\{\mathbf{v}, \mathbf{w}\}$ is linearly independent, then $\{\mathbf{v} + \mathbf{w}, \mathbf{v}\}$ is also linearly independent.

Q4. Which of the following sets are linearly independent in $M_{2,2}(\mathbb{R})$? Justify your answer.

$$A = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\} \qquad B = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix} \right\}$$

Q5. Which of the following sets are linearly independent in $\mathbb{P}_2$?

$$A = \{1,\ 1 - x,\ 1 - 2x\} \qquad B = \{1,\ 1 + x,\ x^2\}$$

Q6. Justify your answers to each of the following:

(a) Suppose V is a vector space and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subseteq V$ is known to be linearly independent. Show that $\{\mathbf{v}_1, \mathbf{v}_2\}$ must be linearly independent.

(b) Give an example of a linearly independent subset $\{\mathbf{v}_1, \mathbf{v}_2\}$ in $\mathbb{R}^3$ and a vector $\mathbf{w} \in \mathbb{R}^3$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}\}$ is linearly dependent.