7. Linear Independence and Spanning Sets

Recall: Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be vectors in a vector space V.

The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is...

LI $\iff$ the *only* solution to the dependency equation

$$a_1\mathbf{v}_1 + \cdots + a_m\mathbf{v}_m = \mathbf{0}$$

is the *trivial solution* in which $a_i = 0$ for all i.

LD $\iff$ there exists a *non-trivial solution* to the dependency equation

$$a_1\mathbf{v}_1 + \cdots + a_m\mathbf{v}_m = \mathbf{0}$$

meaning this equation can be satisfied with *at least one nonzero scalar* $a_i \neq 0$.

a nontrivial solution to dependency equation is called a "dependence relation"

Aside from investigating several examples, we also gathered a list of important facts about LI and LD vectors in a vector space V:

Fact 1 $\{\mathbf{v}\}$ is LI $\iff$ $\mathbf{v} \neq \mathbf{0}$.

Fact 2 $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is LD $\implies$ any set containing $\mathbf{v}_1, \ldots, \mathbf{v}_m$ is also LD.

Fact 3 $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is LI $\implies$ any subset of $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is also LI.

Fact 4 $\{\mathbf{0}\}$ is LD.

Fact 5 $\mathbf{0} \in S$ $\implies$ S is LD.

Fact 6 $\{\mathbf{u}, \mathbf{v}\}$ is LD $\iff$ $\mathbf{u} = k\mathbf{v}$ or $\mathbf{v} = k\mathbf{u}$.

Fact 7 A set with three or more vectors can be LD even though no two vectors are multiples of one another.

Fact 8 $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is LD $\iff$ there is at least one vector $\mathbf{v}_i \in S$ such that $\mathbf{v}_i \in \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_m\}$.

A SET IS LD $\iff$ IT CONTAINS AT LEAST ONE VECTOR SPANNED BY THE OTHERS

Suppose you have $U = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$, where $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are vectors in some vector space V.

If the spanning set $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ for U is LD, then, in some sense, it contains some "useless" vectors that don't contribute anything new to the span.

Fact 8 gives us a mechanism to identify the "useless" vectors.

Let's revisit the idea behind the proof of Fact 8 using a concrete example.

† These notes are solely for the personal use of students registered in MAT1341.

Example 7.1. Let $S = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} -4 \\ -10 \end{bmatrix} \right\}$.

Fact 8 tells us S is LD $\iff$ S contains at least one vector spanned by the others.

($\Longrightarrow$)

- Show that S is LD.
- Then use the idea in the proof of the forward implication of Fact 8 to find a vector in S that belongs to the span of the others.

S is LD since $5\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 0\begin{bmatrix} -4 \\ -10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a dependence relation

(at least one $a_i \neq 0$)

$$\underset{a_1}{5}\vec{v}_1 + \underset{a_2}{1}\vec{v}_2 + \underset{a_3}{(-1)}\vec{v}_3 + \underset{a_4}{0}\vec{v}_4 = \vec{0}$$

Since $a_1 \neq 0$, we can isolate $\underbrace{\vec{v}_1 = -\frac{1}{5}\vec{v}_2 + \frac{1}{5}\vec{v}_3 + \frac{0}{5}\vec{v}_4}_{\vec{v}_1 \text{ is a linear combination of } \vec{v}_2, \vec{v}_3, \vec{v}_4}$ to conclude $\vec{v}_1 \in \text{span}\{\vec{v}_2, \vec{v}_3, \vec{v}_4\}$

($\Longleftarrow$)

- Conversely, show that one of the vectors in S belongs to the span of the other three vectors.
- Then use the idea in the proof of the converse of Fact 8 to show that S must be LD.

Conversely, $\underset{\vec{v}_3}{\begin{bmatrix} 2 \\ 5 \end{bmatrix}} = 5\underset{\vec{v}_1}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + 1\underset{\vec{v}_2}{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + 0\underset{\vec{v}_4}{\begin{bmatrix} -4 \\ -10 \end{bmatrix}}$ $\therefore$ $\vec{v}_3 \in \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$

Rewrite this to obtain a dependence relation $\vec{0} = 5\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \underset{a_2 \neq 0}{(-1)}\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 0 \end{bmatrix} + 0\begin{bmatrix} -4 \\ -10 \end{bmatrix}$

$\therefore$ S is LD

Exercise! Go back and reread the proof of Fact 8 to see if the proof in its full generality starts to make more sense after seeing a concrete example.

The next big theorem tells us that we can throw away each of the "useless" vectors in a spanning set (the ones that belong to the span of the other vectors) and the leftover vectors will still span the same subspace.

Theorem 7.2. **REDUCING SPANNING SETS** Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be vectors in a vector space V.

$$\text{If } \vec{v}_1 \in \operatorname{span}\{\vec{v}_2, \ldots, \vec{v}_m\}, \text{ then } \operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\} = \operatorname{span}\{\vec{v}_2, \ldots, \vec{v}_m\}$$

$\vec{v}_1$ is in the span of the other vectors, so we can throw away $\vec{v}_1$ and still span the same subspace

Proof: Assume $\vec{v}_1 \in \operatorname{span}\{\vec{v}_2, \ldots, \vec{v}_m\}$ (goal prove $\operatorname{span}\{\vec{v}_1, \ , \vec{v}_m\} = \operatorname{span}\{\vec{v}_2, \ , \vec{v}_m\}$)

We also have

$$\vec{v}_2 = 1\vec{v}_2 + 0\vec{v}_3 + \quad + 0\vec{v}_m \quad \text{so } \vec{v}_2 \in \operatorname{span}\{\vec{v}_2, \ , \vec{v}_m\}$$
$$\vdots$$
$$\vec{v}_m = 0\vec{v}_2 + 0\vec{v}_3 + \quad + 1\vec{v}_m \quad \text{so } \vec{v}_m \in \operatorname{span}\{\vec{v}_2, \ , \vec{v}_m\}$$

Thus, $\vec{v}_1, \vec{v}_2, \ , \vec{v}_m \in \operatorname{span}\{\vec{v}_2, \ , \vec{v}_m\}$

By BIG THEOREM, $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \ , \vec{v}_m\}$ is a subspace, hence subset, of $\operatorname{span}\{\vec{v}_2, \ , \vec{v}_m\}$

$$\therefore \operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\} \subseteq \operatorname{span}\{\vec{v}_2, \ldots, \vec{v}_m\} \quad ①$$

Similarly, $\vec{v}_2, \ldots, \vec{v}_m \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ $\therefore \operatorname{span}\{\vec{v}_2, \ldots, \vec{v}_m\} \subseteq \operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ ②

The set inclusions ① and ② prove that $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\} = \operatorname{span}\{\vec{v}_2, \ldots, \vec{v}_m\}$

CONCLUSION: We can decrease the size of any LD spanning set and the smaller spanning set still spans just as much as the original spanning set

Example 7.3. Once again, consider $S = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} -4 \\ -10 \end{bmatrix} \right\}$. Find the smallest subset of S that spans the same subspace as $\operatorname{span} S$.

Since $\vec{v}_1 = -\frac{1}{5}\vec{v}_2 + \frac{1}{5}\vec{v}_3 + \frac{0}{5}\vec{v}_4$ (see Ex 7.1) we have $v_1 \in \operatorname{span}\{\vec{v}_2, \vec{v}_3, \vec{v}_4\}$

By Theorem 7.2, $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\} = \operatorname{span}\{\vec{v}_2, \vec{v}_3, \vec{v}_4\}$

(we can throw away $\vec{v}_1$ and still span same subspace)

Now, notice that $\underset{\vec{v}_4}{\begin{bmatrix} -4 \\ -10 \end{bmatrix}} = 0\underset{\vec{v}_2}{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + (-2)\underset{\vec{v}_3}{\begin{bmatrix} 2 \\ 5 \end{bmatrix}}$ $\therefore \vec{v}_4 \in \operatorname{span}\{\vec{v}_2, \vec{v}_3\}$

By Theorem 7.2, $\operatorname{span}\{\vec{v}_2, \vec{v}_3, \vec{v}_4\} = \operatorname{span}\{\vec{v}_2, \vec{v}_3\}$

(we can throw away $\vec{v}_4$ and still span same subspace)

Now $\left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$ is LI by Fact 6 (two vectors that are <u>not</u> scalar multiples of each other)

now Theorem 7.2 no longer applies

We've reached a "minimal spanning set" that spans the same subspace as $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$

$$\operatorname{span}\{\vec{v}_2, \vec{v}_3\} = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$$

Enlarging Linearly Independent Sets

If we reduce a linearly DEPENDENT spanning set down to a linearly INDEPENDENT one, we will have reached a point where Theorem 7.2 no longer applies.

Now let's start with a linearly INDEPENDENT set and add as many vectors as we can to it, while maintaining linear INDEPENDENCE.

Our goal is to keep adding vectors to a LI set, until we get a maximal independent set that spans the whole vector space.

First, let's give a mathematically equivalent restatement of Fact 8:

Theorem 7.4.

Fact 8 $S = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is LI $\iff$ for all $\mathbf{v}_i \in S$ we have $\mathbf{v}_i \notin \operatorname{span}\{\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_m\}$.

Now we show that you can enlarge a linearly independent set, while preserving linear independence (until your set spans the entire vector space in which the vectors live).

Theorem 7.5. ENLARGING LINEARLY INDEPENDENT SETS

Let $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ be a linearly INDEPENDENT set of vectors in a vector space V.

Let $\mathbf{v}_{m+1} \in V$. Then

$$\{\vec{v}_1, \dots, \vec{v}_m, \vec{v}_{m+1}\} \text{ is LI} \iff \vec{v}_{m+1} \notin \operatorname{span}\{\vec{v}_1, \dots, \vec{v}_m\}$$

we can add a new vector while preserving linear independence $\iff$ new vector is not in the span of the old vectors

Proof: Assume $\{\vec{v}_1, \dots, \vec{v}_m\}$ is LI

($\Longrightarrow$) Assume $\{\vec{v}_1, \dots, \vec{v}_m, \vec{v}_{m+1}\}$ is LI (goal prove $\vec{v}_{m+1} \notin \operatorname{span}\{\vec{v}_1, \ \vec{v}_m\}$)

Then, by Fact 8 (restated), $\vec{v}_{m+1} \notin \operatorname{span}\{\vec{v}_1, \dots, \vec{v}_m\}$

Why?

If $\vec{v}_{m+1} \in \operatorname{span}\{\vec{v}_1, \ \vec{v}_m\}$ then $\{\vec{v}_1, \ \vec{v}_m, \vec{v}_{m+1}\}$ would be LD by Fact 8

Thus, $\vec{v}_{m+1} \notin \operatorname{span}\{\vec{v}_1, \ \vec{v}_{m+1}\}$, as claimed

($\Longleftarrow$) Assume $\vec{v}_{m+1} \notin \operatorname{span}\{\vec{v}_1, \ \vec{v}_m\}$ (goal show $\{\vec{v}_1, \ \vec{v}_m, \vec{v}_{m+1}\}$ is LI)

Suppose $a_1\vec{v}_1 + \ + a_m\vec{v}_m + a_{m+1}\vec{v}_{m+1} = \vec{0}$ ①

We claim that a_{m+1} must be 0

Why?

If $a_{m+1} \neq 0$, then $\vec{v}_{m+1} = -\frac{a_1}{a_{m+1}}\vec{v}_1 - \ -\frac{a_m}{a_{m+1}}\vec{v}_m \in \operatorname{span}\{\vec{v}_1, \ \vec{v}_m\}$

But this contradicts our assumption that $\vec{v}_{m+1} \notin \operatorname{span}\{\vec{v}_1, \ \vec{v}_m\}$

Since $a_{m+1} \neq 0$ leads to a contradiction, we conclude that $a_{m+1} = 0$

Now, ① becomes

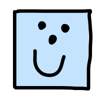
$$\vec{0} = a_1\vec{v}_1 + \quad + a_m\vec{v}_m + a_{m+1}\vec{v}_{m+1} = a_1\vec{v}_1 + \quad + a_m\vec{v}_m + 0\vec{v}_{m+1} = a_1\vec{v}_1 + \quad + a_m\vec{v}_m$$

Since $\{\vec{v}_1, \quad, \vec{v}_m\}$ is LI, only solution to dependency equation is trivial

so $a_1 = \quad = a_m = 0$

Since $a_1\vec{v}_1 + \quad + a_m\vec{v}_m + a_{m+1}\vec{v}_{m+1} = \vec{0} \Rightarrow a_1 = \quad = a_m = a_{m+1} = 0$,

we conclude that $\{\vec{v}_1, \quad, \vec{v}_m, \vec{v}_{m+1}\}$ is LI



CONCLUSION:

We can increase the size of any LI set until it spans the entire vector space

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Example 7.6. The set $\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ is LI. Enlarge it to a LI set with 3 elements.

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ are not scalar multiples of each other $\therefore$ they are LI (by Fact 6)

$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \notin \text{span}\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

since elements in the span will always have zeros in second row

$\therefore \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ is LI (by Theorem 75)