13. Integration by Parts and Trig Integrals

Lec 12 mini review.

Substitution: $\int F'(g(x))g'(x)dx = \int F'(u)du = F(u) + C = F(g(x)) + C$

Integrals with Even Symmetry: If $f(-x) = f(x)$, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

Integrals with Odd Symmetry: If $f(-x) = -f(x)$, then $\int_{-a}^{a} f(x)dx = 0$

"UNDOING" THE PRODUCT RULE

Example 13.1. $\int xe^x dx$

Try substitution?

$u = x$?

$\frac{du}{dx} = 1$

$du = dx$

$\Rightarrow \int ue^u du$

works, but unhelpful

$u = e^x$?

$\frac{du}{dx} = e^x \Rightarrow \frac{du}{e^x} = dx$

$\Rightarrow \int xu \frac{du}{e^x} = \int xu \frac{du}{u} = \int x du$

can't cancel x?

wait: $x = \ln u$, so

$\int x du = \int \ln u \, du = ?$

$u = xe^x$?

$\frac{du}{dx} = e^x + xe^x$

$dx = \frac{du}{e^x + xe^x}$

$\Rightarrow \int u \frac{du}{e^x + xe^x} = \int \frac{u du}{e^x + u}$?

can't cancel x?

Recall the Chain Rule: $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$.

Some clues that might indicate we should try integration by substitution:

- we might see a composition in the integrand
- we might notice a function that could be the "inner" function $g(x)$, and its derivative will also be a factor of the integrand.
- me might see what looks like the aftermath of a Chain Rule derivative (like power chain rule, exponential chain rule, log chain rule, etc.)

In contrast, the Product Rule is $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$.

"Undoing" the product rule is less obvious than "undoing" the chain rule:

- If we want to integrate $\int \big(f'(x)g(x) + f(x)g'(x)\big)dx$, we might be lucky enough to recognize f and g
- if the integrand is the result of a product rule, then the factors need not be related to each other whatsoever

* These notes are solely for the personal use of students registered in MAT1320.

$$(uv)' = u'v + uv'$$
$$\Rightarrow \int (uv)' = \int u'v + uv'$$
$$\Rightarrow \quad uv = \int u'v + \int uv'$$
$$\Rightarrow \int uv' = uv - \int u'v$$

Integration by Parts

$$\boxed{\int uv' = uv - \int u'v}$$

we get a new integral to consider

OR

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Guidelines for I.B.P.

$$\int f(x)\,dx$$

- the integrand must be rethought as a product $f(x) = u(x)v'(x)$ (although factors of the product could be 1)
  - you choose the "parts" u and v'
  - when you choose u, you need to be able to calculate its derivative u'
  - when you choose v', you need to be able to calculate its antiderivative v
  - **Goal**: the "new" integral $\int u'v$ should be no worse than the original integral $\int uv'$

Example 13.2. $\int xe^x\,dx$

Choice of Parts:

$u = x \quad v' = e^x$
$u' = 1 \quad v = e^x$

$$\int xe^x dx = xe^x - \underbrace{\int (1)e^x dx}_{\text{looks easier}}$$

$u = e^x \quad v' = x$
$u' = e^x \quad v = \frac{x^2}{2}$

$$\int xe^x dx = \frac{x^2}{2}e^x - \underbrace{\int \frac{x^2}{2} e^x dx}_{\text{looks worse}}$$

$$\begin{aligned}\int xe^x dx &= xe^x - \int (1)e^x dx \\ &= xe^x - \int e^x dx \\ &= xe^x - e^x + C\end{aligned}$$

Check!

$$\begin{aligned}\frac{d}{dx}\left[xe^x - e^x\right] &= (1)e^x + xe^x - e^x \\ &= xe^x \checkmark\end{aligned}$$

Example 13.3. $\int x^3 e^{2x}\,dx$

$$\int x^3 e^{2x}dx = x^3\left(\tfrac{1}{2}e^{2x}\right) - \int (3x^2)\left(\tfrac{1}{2}e^{2x}\right)dx$$
$$= \tfrac{1}{2}x^3e^{2x} - \int \tfrac{3}{2}x^2e^{2x}dx$$
$$= \tfrac{1}{2}x^3e^{2x} - \left[\tfrac{3}{2}x^2\left(\tfrac{1}{2}e^{2x}\right) - \int 3x\left(\tfrac{1}{2}e^{2x}\right)dx\right]$$
$$= \tfrac{1}{2}x^3e^{2x} - \tfrac{3}{4}x^2e^{2x} + \int \tfrac{3}{2}xe^{2x}dx$$
$$= \tfrac{1}{2}x^3e^{2x} - \tfrac{3}{4}x^2e^{2x} + \left[\left(\tfrac{3}{2}x\right)\left(\tfrac{1}{2}e^{2x}\right) - \int\left(\tfrac{3}{2}\right)\left(\tfrac{1}{2}e^{2x}\right)dx\right]$$
$$= \tfrac{1}{2}x^3e^{2x} - \tfrac{3}{4}x^2e^{2x} + \tfrac{3}{4}xe^{2x} - \int \tfrac{3}{4}e^{2x}dx$$
$$= \tfrac{1}{2}x^3e^{2x} - \tfrac{3}{4}x^2e^{2x} + \tfrac{3}{4}xe^{2x} - \tfrac{3}{8}e^{2x} + C$$

Parts I

$u = x^3 \quad v' = e^{2x}$

$u' = 3x^2 \quad v = \tfrac{1}{2}e^{2x}$

Parts II

$u = \tfrac{3}{2}x^2 \quad v' = e^{2x}$

$u' = 3x \quad v = \tfrac{1}{2}e^{2x}$

Parts I

$u = \tfrac{3}{2}x \quad v' = e^{2x}$

$u' = \tfrac{3}{2} \quad v = \tfrac{1}{2}e^{2x}$

Example 13.4. $\int \ln(x)\,dx$ ← this doesn't look like a product but...

$$\int \ln x\,dx = (\ln x) \quad - \int\left(\tfrac{1}{x}\right)(x)\,dx$$
$$= x\ln x \quad - \int dx$$
$$= x\ln x - x + C$$

$u = \ln x \quad v' = 1$

$u' = \tfrac{1}{x} \quad v = x$

Check! $\frac{d}{dx}\left[x\ln x - x\right] = (1)\ln x + x\left(\tfrac{1}{x}\right) - 1$
$= \ln x$ ✓

Example 13.5. $\int (x^4 + 2x - 9)\ln(x)\,dx$ Try similar parts:

$$\int (x^4+2x-9)\ln(x)\,dx$$
$$= (\ln x)\left(\tfrac{x^5}{5} + x^2 - 9x\right) - \int\left(\tfrac{x^5}{5} + x^2 - 9x\right)\left(\tfrac{1}{x}\right)dx$$
$$= (\ln x)\left(\tfrac{x^5}{5} + x^2 - 9x\right) - \int\left(\tfrac{x^4}{5} + x - 9\right)dx$$
$$= (\ln x)\left(\tfrac{x^5}{5} + x^2 - 9x\right) - \tfrac{x^5}{25} - \tfrac{x^2}{2} + 9x + C$$

$u = \ln x \quad dv = (x^4 + 2x - 9)dx$

$\frac{du}{dx} = \frac{1}{x} \quad \frac{dv}{dx} = x^4 + 2x - 9$

$du = \tfrac{1}{x}dx \quad v = \tfrac{x^5}{5} + x^2 - 9x$

Common "Parts"

$\int x^n e^{kx}\,dx$

$$\boxed{\begin{array}{ll} u = x^n & v' = e^{kx} \\ u' = nx^{n-1} & v = \frac{1}{k}e^{kx} \end{array}}$$

Each iteration of IBP

$\leftarrow$ reduces power of x by one

$\int x^n (\ln x)^m\,dx$

$$\boxed{\begin{array}{ll} u = (\ln x)^m & v' = x^n \\ u' = m(\ln x)^{m-1}\cdot\left(\frac{1}{x}\right) & v = \frac{1}{n+1}x^{n+1} \end{array}}$$

$\leftarrow$ reduces power of $\ln x$ by one

Example 13.6. $\int x^3(\ln x)^2 dx$

$$\boxed{\begin{array}{ll} u = (\ln x)^2 & v' = x^3 \\ u' = 2(\ln x)\left(\frac{1}{x}\right) & v = \frac{1}{4}x^4 \end{array}}$$

$$\boxed{\begin{array}{ll} u = \ln x & v' = x^3 \\ u' = \frac{1}{x} & v = \frac{1}{4}x^4 \end{array}}$$

$$\begin{aligned}
&\int x^3(\ln x)^2 dx \\
&= (\ln x)^2\left(\tfrac{1}{4}x^4\right) - \int 2(\ln x)\left(\tfrac{1}{x}\right)\left(\tfrac{1}{4}x^4\right)dx \\
&= \tfrac{1}{4}x^4(\ln x)^2 - \tfrac{1}{2}\int x^3 \ln x\,dx \\
&= \tfrac{1}{4}x^4(\ln x)^2 - \tfrac{1}{2}\left[(\ln x)\left(\tfrac{1}{4}x^4\right) - \int\left(\tfrac{1}{x}\right)\left(\tfrac{1}{4}x^4\right)dx\right] \\
&= \tfrac{1}{4}x^4(\ln x)^2 - \tfrac{1}{8}x^4\ln x + \tfrac{1}{8}\int x^3 dx \\
&= \tfrac{1}{4}x^4(\ln x)^2 - \tfrac{1}{8}x^4\ln x + \tfrac{1}{8}\left(\tfrac{1}{4}x^4\right) + C \\
&= \tfrac{1}{4}x^4(\ln x)^2 - \tfrac{1}{8}x^4\ln x + \tfrac{1}{32}x^4 + C \\
&= \tfrac{1}{4}x^4\left((\ln x)^2 - \tfrac{1}{2}\ln x + \tfrac{1}{8}\right) + C
\end{aligned}$$

check: $\frac{d}{dx}\left[\frac{1}{4}x^4\left((\ln x)^2 - \frac{1}{2}\ln x + \frac{1}{8}\right)\right] = x^3\left((\ln x)^2 - \frac{1}{2}\ln x + \frac{1}{8}\right) + \frac{1}{4}x^4\left(2(\ln x)\left(\frac{1}{x}\right) - \frac{1}{2}\left(\frac{1}{x}\right)\right)$

$= x^3(\ln x)^2 - \frac{1}{2}x^3\ln x + \frac{1}{8}x^3 + \frac{1}{2}x^3\ln x - \frac{1}{8}x^3$

$= x^3(\ln x)^2$ ✓

Sometimes, when choosing "parts" there seem to be ties...

? vs. ?

Example 13.7. $\int e^x \sin(x)dx$

$u=\sin x$ $v'=e^x$ $u'=\cos x$ $v=e^x$	$u=e^x$ $v'=\sin x$ $u'=e^x$ $v=-\cos x$

Let's try these parts

$$\int e^x \sin x\,dx = (\sin x)e^x - \int(\cos x)e^x dx$$

$$= e^x\sin x - \int \underbrace{e^x\cos x}\,dx$$

this new integral is no better/no worse than the original

$u=\cos x$ $v'=e^x$
$u'=-\sin x$ $v=e^x$

$$= e^x\sin x - \left[(\cos x)e^x - \int(-\sin x)e^x dx\right]$$

$$= e^x\sin x - e^x\cos x - \int \underbrace{e^x\sin x}\,dx$$

original integral is back!?!

$$\therefore \underbrace{\int e^x\sin x\,dx}_{I} = e^x\sin x - e^x\cos x - \underbrace{\int e^x \sin x\,dx}_{I}$$

← now, we have an equation with one unknown: $I = \int e^x \sin x\,dx$

$$\Rightarrow 2I = 2\int e^x\sin x\,dx = e^x\sin x - e^x\cos x$$

$$\Rightarrow I = \int e^x\sin x\,dx = \frac{1}{2}\left(e^x\sin x - e^x\cos x\right) + C$$

TRIG INTEGRALS

◇ For certain integrals involving trig functions, the trick is to make use of trig identities before integrating.

✓ • $\cos^2 x + \sin^2 x = 1$

$(\div \cos^2 x)$ ✓ • $1 + \tan^2 x = \sec^2 x$

double-angle formulas:

✓ • $\sin 2x = 2\sin x\cos x$

✓ • $\cos 2x = \cos^2 x - \sin^2 x$

use $\cos 2x = \cos^2 x - \sin^2 x$ and $\cos^2 x + \sin^2 x = 1$ to get half-angle formulas:

$$\cos 2x = 1 - \sin^2 x - \sin^2 x$$

$$\Rightarrow \boxed{\sin^2 x = \frac{1}{2}(1-\cos 2x)}$$

$$\cos 2x = \cos^2 x - (1-\cos^2 x)$$

$$\Rightarrow \boxed{\cos^2 x = \frac{1}{2}(1+\cos 2x)}$$

Example 13.8. $\int \sin^{17}(x)\,dx$

use trig identity:
$\sin^2 x + \cos^2 x = 1$

$$\int \sin^{17}(x)\,dx = \int \sin(x)\cdot(\sin^2 x)^8\,dx$$
$$= \int \sin(x)(1-\cos^2 x)^8\,dx$$
$$= \int \cancel{\sin(x)}(1-u^2)^8 \frac{du}{-\cancel{\sin x}}$$
$$= \int -\underbrace{(1-u^2)^8}\,du$$

if expanded, this is easy to integrate (just a polynomial)

use u-substitution:
$u = \cos x$
$\frac{du}{dx} = -\sin x$
$dx = \frac{du}{-\sin x}$

Okay, same idea but a baby version:

$$\int \sin^5 x\,dx = \int \sin x(1-\cos^2 x)^2\,dx = \int -(1-u^2)^2\,du = -\int(1-2u^2+u^4)\,du$$
$$= -\left(u - \tfrac{2}{3}u^3 + \tfrac{1}{5}u^5\right) + C = -\cos x + \tfrac{2}{3}\cos^3 x - \tfrac{1}{5}\cos^5 x + C$$

Example 13.9. $\int \sin^4(x)\,dx$

$$= \int (\sin^2 x)^2\,dx$$
$$= \int \left(\tfrac{1}{2}(1-\cos 2x)\right)^2 dx$$

half-angle formula! $\sin^2 x = \frac{1}{2}(1-\cos(2x))$

$$= \tfrac{1}{4}\int (1-\cos 2x)^2\,dx$$
$$= \tfrac{1}{4}\int \left(1 - 2\cos 2x + \cos^2(2x)\right)dx$$
$$= \tfrac{1}{4}\int \left(1 - 2\cos(2x) + \tfrac{1}{2}\left(1+\cos(4x)\right)\right)dx$$

half-angle formula! $\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$
with $\theta = 2x$

$$= \tfrac{1}{4}\int \left(1 - 2\cos(2x) + \tfrac{1}{2} + \tfrac{1}{2}\cos(4x)\right)dx$$
$$= \tfrac{1}{4}\left(x - \sin(2x) + \tfrac{1}{2}x + \tfrac{1}{8}\sin(4x)\right) + C$$

Study Guide

- ◇ **strategy for integration by substitution**
- ◇ **integration by parts**: $\int uv' = uv - \int u'v$
- ◇ **making use of trig identities before integrating**