8. Implicit Differentiation

Lec 7 mini review.

Two Special Limits: $\boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1}$ $\boxed{\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0}$

Trig Rules:

$\boxed{\frac{d}{dx}[\sin x] = \cos x}$ $\boxed{\frac{d}{dx}[\cos x] = -\sin x}$ $\boxed{\frac{d}{dx}[\tan x] = \sec^2 x}$

$\boxed{\frac{d}{dx}[\csc x] = -\csc x \cot x}$ $\boxed{\frac{d}{dx}[\sec x] = \sec x \tan x}$ $\boxed{\frac{d}{dx}[\cot x] = -\csc^2 x}$

The Chain Rule: $\boxed{\frac{d}{dx}\Big[f\big(g(x)\big)\Big] = f'\big(g(x)\big)g'(x)}$

Power Chain Rule: $\boxed{\frac{d}{dx}\Big[\big(g(x)\big)^n\Big] = n\big(g(x)\big)^{n-1}g'(x)}$

Exponential Chain Rule: $\boxed{\frac{d}{dx}\Big[e^{g(x)}\Big] = e^{g(x)}g'(x)}$

WARM-UP FOR IMPLICIT DIFFERENTIATION

Differentiate each of the following expressions:

$f(x) = x^3 g(x) + [h(x)]^5$

$$f'(x) = 3x^2 g(x) + x^3 \cdot g'(x) + 5[h(x)]^4 \cdot h'(x)$$

$V(t) = \pi[R(t)]^2 H(t)$

$$V'(t) = \pi(2)[R(t)]^1 \cdot R'(t) \cdot H(t) + \pi[R(t)]^2 \cdot H'(t)$$

$y = \left(\sqrt[3]{v(x)}\right)\Big[u(x)\Big]^{10}$

$$y' = \frac{1}{3}[v(x)]^{-2/3} \cdot v'(x)[u(x)]^{10} + \left(\sqrt[3]{v(x)}\right) \cdot (10)[u(x)]^9 \cdot u'(x)$$

OR $$\frac{dy}{dx} = \frac{1}{3}v^{-2/3} \cdot \frac{dv}{dx} \cdot u^{10} + \left(\sqrt[3]{v}\right)\left(10u^9 \cdot \frac{du}{dx}\right)$$

$$p(x) = e^{x^3} + e^{g(x)/h(x)} + \frac{h(x)}{\cos(x)} + \sin\left(f(x) + g(x)\right)$$

$$p'(x) = \left(e^{x^3}\right)(3x^2) + \left(e^{g(x)/h(x)}\right)\left[\frac{g'(x)h(x) - g(x)h'(x)}{[h(x)]^2}\right] + \cos\left(f(x)+g(x)\right)\left[f'(x) + g'(x)\right]$$

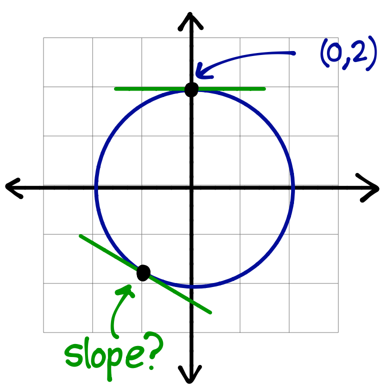
Observation we don't need explicit formulas for f(x), g(x), etc, in order to find expressions for derivatives.

GRAPHS

- ◇ Any equation in two variables (let's use x and y) has a graph.
- ◇ The graph consists of all pairs of the form (x, y) that satisfy the equation.
- ◇ It might not be possible to isolate y and write an *explicit* formula $y = f(x)$.
- ◇ Nonetheless, we still think of y as an **implicit** function of x (where "function" is being used loosely; the graph of the equation could fail the vertical line test, hence not technically be a function of x)

Example 8.1. Consider the equation $x^2 + y^2 = 4$.

circle of radius 2 centred at the origin



(0,2) is a point on the graph because $0^2+2^2=4$ (satisfies the equation)

y is not a function of x

$y = \pm\sqrt{4-x^2}$ (fails V.L.T.)

- even though it's not the graph of a function, we can still wonder what the slope of the tangent is at certain points.

ex. slope of tangent at (0,2) is zero because it's horizontal.

ex slope of tangent at $x=-1$? $\to$ $(-1,\sqrt{3})$ or $(-1,-\sqrt{3})$ (we need to specify both coordinates)

less obvious than slope at (0,2)...

Implicit Differentiation

1. Start with some equation with x's and y's.

2. Implicitly differentiate both sides of the equation:

- Treat y as a "mystery" function of x (an implicitly defined function)
- When you need to write the derivative of y, just write $\frac{dy}{dx}$
- Differentiate x's as usual.
- When you are done differentiating both sides, you will have a new equation that may contain some x's, some y's, and some $\frac{dy}{dx}$'s. Because you performed the same operation (differentiation) to both sides of the original equation, the new equation is still a valid equation.

3. Isolate $\frac{dy}{dx}$ from your new equation:

- Put all terms that have a $\frac{dy}{dx}$ on one side of the equation, and put all other terms on the other side of the equation.
- Factor out $\frac{dy}{dx}$ from all terms on the $\frac{dy}{dx}$-side of the equation, then divide to isolate $\frac{dy}{dx}$.

Example 8.2. **a.** Find $\frac{dy}{dx}$ for the equation $x^2 + y^2 = 4$.

(original eq$^{\underline{n}}$) $\quad x^2 + y^2 = 4$

(imp. diff.) $\quad \Rightarrow 2x + 2y \cdot \frac{dy}{dx} = 0$

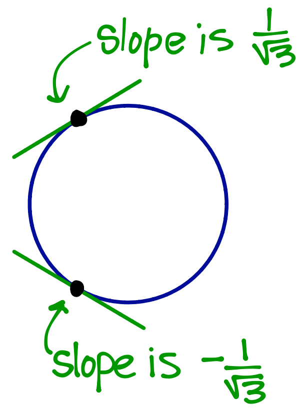
$$\Rightarrow 2y \cdot \frac{dy}{dx} = -2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

b. What is the slope of the tangent line to the graph of $x^2 + y^2 = 4$ at the point $(-1, \sqrt{3})$? What is it at $(-1, -\sqrt{3})$?

At $(-1, \sqrt{3})$, $\frac{dy}{dx} = \frac{-(-1)}{\sqrt{3}} = \frac{1}{\sqrt{3}}$

At $(-1, -\sqrt{3})$, $\frac{dy}{dx} = \frac{-(-1)}{-\sqrt{3}} = -\frac{1}{\sqrt{3}}$



Example 8.3. For the following equation, find $\frac{dy}{dx}$ at the point $(1, 0)$: $e^{2y+x} + x^2y^3 = e^x$

$$e^{2y+x} + x^2y^3 = e^x$$

$$\Rightarrow \left(e^{2y+x}\right)\left(2\frac{dy}{dx}+1\right) + (2x)y^3 + x^2\left(3y^2\frac{dy}{dx}\right) = e^x$$

We could isolate $\frac{dy}{dx}$ then plug in $(x,y)=(1,0)$. $\rightarrow$

$$\boxed{\begin{aligned}&\Rightarrow 2e^{2y+x}\left(\frac{dy}{dx}\right) + e^{2y+x} + 2xy^3 + 3x^2y^2\left(\frac{dy}{dx}\right) = e^x\\ &\Rightarrow 2e^{2y+x}\left(\frac{dy}{dx}\right) + 3x^2y^2\left(\frac{dy}{dx}\right) = e^x - e^{2y+x} - 2xy^3\\ &\Rightarrow \left[2e^{2y+x} + 3x^2y^2\right]\left(\frac{dy}{dx}\right) = e^x - e^{2y+x} - 2xy^3\\ &\Rightarrow \frac{dy}{dx} = \frac{e^x - e^{2y+x} - 2xy^3}{2e^{2y+x} + 3x^2y^2}\end{aligned}}$$

Since we're only interested in $\frac{dy}{dx}$ at the point $(1,0)$, a more efficient solution is to plug in $(1,0)$ first, then isolate dy/dx (for that point)

$$\left(e^{2(0)+1}\right)\left(2\frac{dy}{dx}+1\right) + 2(1)(0^3) + (1^2)(3)(0^2)\frac{dy}{dx} = e^1$$

$$\Rightarrow 2e\frac{dy}{dx} + e + 0 + 0 = e \qquad \Rightarrow \text{at } (1,0)\ \frac{dy}{dx} = 0$$

Example 8.4. Find $\frac{dy}{dx}$ if $\sin(x + y) = y^2\cos(x)$.

$$\sin(x+y) = y^2\cos(x)$$

$$\Rightarrow \cos(x+y)\left[1+\frac{dy}{dx}\right] = 2y\frac{dy}{dx}\cos(x) + y^2(-\sin(x))$$

$$\Rightarrow \cos(x+y) + \cos(x+y)\frac{dy}{dx} = 2y\cos(x)\frac{dy}{dx} - y^2\sin(x)$$

$$\Rightarrow \cos(x+y)\frac{dy}{dx} - 2y\cos(x)\frac{dy}{dx} = -y^2\sin(x) - \cos(x+y)$$

$$\Rightarrow \left[\cos(x+y) - 2y\cos(x)\right]\frac{dy}{dx} = -y^2\sin(x) - \cos(x+y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y^2\sin(x) - \cos(x+y)}{\cos(x+y) - 2y\cos(x)}$$

Derivative of Arcsine $\boxed{y=\arcsin(x) \iff \sin(y)=x \text{ and } -\frac{\pi}{2}\leq y\leq \frac{\pi}{2}}$

we want dy/dx so we use implicit differentiation on the equation

$$\sin(y)=x$$

$$\Rightarrow \cos(y)\cdot\frac{dy}{dx}=1$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\cos(y)}$$

in terms of x ?

$$\boxed{\therefore \frac{d}{dx}[\arcsin(x)]=\frac{1}{\sqrt{1-x^2}}}$$

Note. $\cos^2 y+\sin^2 y=1$

$$\Rightarrow \cos(y)=\pm\sqrt{1-\sin^2(y)}$$

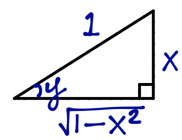
but we assumed $-\frac{\pi}{2}\leq y\leq\frac{\pi}{2}$

on this interval, $\cos(y)\geq 0$

so, on this interval,

$$\cos(y)=\sqrt{1-\sin^2(y)}$$

Finally, keep in mind that $\sin(y)=x$



Derivative of Arctangent $\boxed{y=\arctan(x)\iff\tan(y)=x \text{ and } -\frac{\pi}{2}<y<\frac{\pi}{2}}$

Same strategy:

$$\tan(y)=x$$

$$\Rightarrow \sec^2(y)\frac{dy}{dx}=1$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sec^2(y)}$$

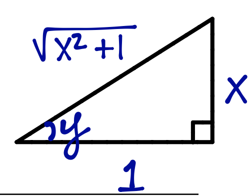
$$\boxed{\frac{d}{dx}[\arctan(x)]=\frac{1}{1+x^2}}$$

Note. $1+\tan^2(y)=\sec^2(y)$

$$\Rightarrow \frac{1}{\sec^2(y)}=\frac{1}{1+\tan^2(y)}$$

and keep in mind that

$$x=\tan(y)$$



Other **Inverse Trig Rules**:

$\frac{d}{dx}[\cos^{-1}(x)]=-\frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}[\sec^{-1}(x)]=\frac{1}{x\sqrt{x^2-1}}$
$\frac{d}{dx}[\cot^{-1}(x)]=-\frac{1}{1+x^2}$	$\frac{d}{dx}[\csc^{-1}(x)]=-\frac{1}{x\sqrt{x^2-1}}$

Recall $\boxed{y = \log_b(x) \iff b^y = x \quad \text{and } x > 0}$

To find $\frac{d}{dx}[\log_b(x)]$, we'll use the same strategy. We will find dy/dx with implicit differentiation of the equation $b^y = x$

$$\Rightarrow (\ln b)(b^y)\left(\frac{dy}{dx}\right) = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(\ln b)(b^y)} \leftarrow \text{Keep in mind } b^y = x$$

$$\Rightarrow \boxed{\frac{d}{dx}[\log_b(x)] = \left(\frac{1}{\ln b}\right)\left(\frac{1}{x}\right)}$$

and, since $\ln e = 1$, for base $b = e$, the rule is $\rightarrow \boxed{\frac{d}{dx}[\ln(x)] = \frac{1}{x}}$

Log Chain Rules

base b $\quad \boxed{\frac{d}{dx}[\log_b(g(x))] = \left(\frac{1}{\ln b}\right)\left(\frac{1}{g(x)}\right)g'(x) = \frac{g'(x)}{(\ln b)g(x)}}$

base e $\quad \boxed{\frac{d}{dx}[\ln(g(x))] = \left(\frac{1}{g(x)}\right)g'(x) = \frac{g'(x)}{g(x)}}$

Example 8.5. Find $f'(x)$ if $f(x) = \ln|x|$.

Thus, $f(x) = \begin{cases} \ln(x) & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$

$$\Rightarrow f'(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ \underbrace{\frac{1}{(-x)}(-1)}_{\ln \text{ chain rule with } g(x) = -x} = \frac{1}{x} & \text{if } x < 0 \end{cases}$$

$$\therefore \boxed{\frac{d}{dx}[\ln|x|] = \frac{1}{x}}$$

Example 8.6. Differentiate each of the following:

$f(x) = \arctan(3e^x - 2x^5)$ (we need an arctangent instance of the chain rule)

$$f'(x) = \left(\frac{1}{1+(3e^x-2x^5)^2}\right)(3e^x-10x^4)$$

$$\frac{d}{dx}[\arctan(g(x))] = \frac{g'(x)}{1+[g(x)]^2}$$

$$= \frac{3e^x-10x^4}{1+(3e^x-2x^5)^2}$$

$y = \ln(x)\sin^{-1}(x)$

$$y' = \left(\frac{1}{x}\right)\sin^{-1}(x) + \ln(x)\left(\frac{1}{\sqrt{1-x^2}}\right)$$

Logarithmic Differentiation

Example 8.7. Find $f'(x)$ where $f(x) = x^x$.

exponent is variable $\Rightarrow$ not a power function

base is variable $\Rightarrow$ not an exponential function

we have no rule for this!

Strategy for logarithmic differentiation:

- "$\ln$" the absolute value of both sides (or "$\ln$" both sides assuming they're >0)
- use nice $\ln$ properties
- implicitly differentiate to find dy/dx
- use the fact that we know $y=f(x)$ explicitly.

eq$^{\underline{n}}$ $y = x^x$ (assume $x>0$)

$$\Rightarrow \ln y = \ln(x^x)$$

$$\Rightarrow \ln y = x\ln x$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = (1)\ln x + x\left(\frac{1}{x}\right)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \ln x + 1$$

$$\Rightarrow \frac{dy}{dx} = (\ln x + 1)y$$ but we know $y = x^x$ $\therefore \frac{dy}{dx} = (\ln x + 1)x^x$

Exercise 8.8. Use logarithmic differentiation to prove that the Power Rule (which we've been using for all sorts of powers $n \in \mathbb{R}$) is in fact valid.

That is, prove $\frac{d}{dx}[x^n] = nx^{n-1}$ for all $n \in \mathbb{R}$.

We already "know" $\frac{dy}{dx} = nx^{n-1}$, but this Exercise is a proof that the power rule (that we've been happily using for quite a while) actually holds true for any power $n \in \mathbb{R}$.

(eq$^{\underline{n}}$) Let $y = x^n$ (assume $x \neq 0$)

(ln' both sides) Then $\ln|y| = \ln|x^n|$

(simplify) $\Rightarrow \ln|y| = \ln(|x|^n)$ ← think about this repositioning of absolute value and n...

(ln property) $\Rightarrow \ln|y| = n\ln|x|$

(imp. diff.) $\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = n\left(\frac{1}{x}\right)$

(isolate $\frac{dy}{dx}$) $\Rightarrow \frac{dy}{dx} = n\left(\frac{1}{x}\right)y$ but we know $y = x^n$ explicitly...

(use $y = x^n$) $\Rightarrow \frac{dy}{dx} = n\left(\frac{1}{x}\right)x^n$

(simplify) $\Rightarrow \boxed{\frac{dy}{dx} = nx^{n-1}}$ ✓ (the power rule really is true for all $n \in \mathbb{R}$)

Study Guide

◇ **implicit differentiation strategy**

◇ **derivative rules for inverse trig functions:**

$\boxed{\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}}$ $\boxed{\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2}}$ (and others!)

◇ **derivative rules for logs:** $\boxed{\frac{d}{dx}[\ln(x)] = \frac{1}{x}}$ $\boxed{\frac{d}{dx}[\log_b(x)] = \left(\frac{1}{\ln b}\right)\frac{1}{x}}$

◇ **log chain rule:** $\boxed{\frac{d}{dx}[\ln(g(x))] = \frac{g'(x)}{g(x)}}$

◇ **logarithmic differentiation strategy**