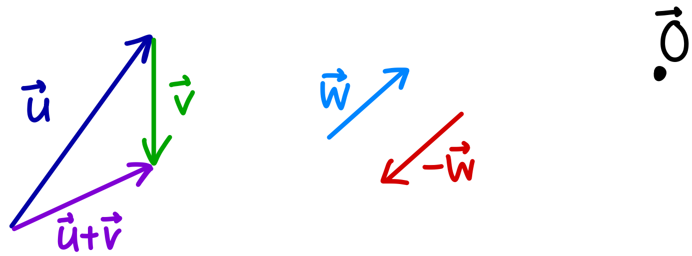
3. Vector Spaces

- We first reacquainted ourselves with vectors by depicting them as arrows with geometric rules for addition and scalar multiplication. We observed several "nice" properties that these operations satisfy.



- Next, we converted our arrow pictures into algebraic structures by using coordinate systems to represent those arrows. In terms of coordinates, the vector addition operation and scalar multiplication operation involved adding or multiplying coordinates and we didn't need pictures to "see" the result of those operations (although the pictures still helped us to develop intuition).

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad 8\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

- The algebraic way of adding and scalar-multiplying vectors also satisfied the same "nice" properties that the geometric way did. On top of that, the algebraic way of representing vectors also made it very natural to generalize from 2 and 3 dimensions into n dimensions for $n \geq 4$ (whereas the depiction of vectors as arrows doesn't).

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_i \in \mathbb{R} \text{ for } 1 \leq i \leq n \right\}$$

Now, we will see that generalizing beyond $\mathbb{R}^n$ is also possible! What we need is to study the essential ingredients that make a collection of things behave (in some abstract way) the same as $\mathbb{R}^n$.

†These notes are solely for the personal use of students registered in MAT1341.

Definition 3.1. Let V be a set, whose elements we will call "vectors", equipped with

- a rule for adding two vectors together
- a rule for scalar-multiplying a vector by a real number.

If V, with its two operations, satisfies the following 10 axioms, then V is called a **VECTOR SPACE** :

CLOSURE

(1) $\vec{u},\vec{v}\in V \Rightarrow \vec{u}+\vec{v}\in V$ [V is closed under addition]

the sum of any two vectors in V is again a vector in V

(2) $\vec{u}\in V$ and $c\in\mathbb{R} \Rightarrow c\vec{u}\in V$ [V is closed under scalar multiplication]

a scalar multiple of any vector in V is again a vector in V

EXISTENCE

(3) There exists $\vec{0}\in V$ such that $\vec{0}+\vec{v}=\vec{v}$ for every vector $\vec{v}\in V$ [Existence of zero vector]

V has a zero vector

(4) For each vector $\vec{v}\in V$, there exists $-\vec{v}\in V$ such that $\vec{v}+(-\vec{v})=\vec{0}$ [Existence of vector negatives]

the negative of $\vec{v}$ exists and is also a vector in $\vec{v}$

ARITHMETIC PROPERTIES

For all $\vec{u},\vec{v},\vec{w}\in V$ and for all $c,d\in\mathbb{R}$

(5) $\vec{u}+\vec{v}=\vec{v}+\vec{u}$ [Vector addition is commutative]

(6) $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$ [Vector addition is associative]

(7) $c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$ [Scalar multiplication distributes over vector addition]

(8) $(c+d)\vec{u}=c\vec{u}+d\vec{u}$ [Addition of scalars distributes over scalar multiplication]

(9) $c(d\vec{u})=(cd)\vec{u}$ [Multiplication of scalars is compatible with scalar multiplication]

(10) $1\vec{u}=\vec{u}$ [Unity Law]

Let's explore various examples.

Example 3.2. SPACES OF EQUATIONS
Consider the set of all linear equations in three variables, x, y, z.

$$\mathcal{E} = \{ ax+by+cz=d \quad a,b,c,d \in \mathbb{R} \}$$

What are some elements in the set $\mathcal{E}$?

$E_1 \quad 2x+3y-5z=3$

$E_2 \quad x-z=6$

$E_3 \quad 0=1$

We are simply considering the equations, not solving them, so E_3 is a valid equation even if it has no solutions

Note: we don't need all three variables to appear.

$x-z=6$ means $x+0y-z=6$

We can create new equations from old ones:

We can add equations

$E_1+E_2: \ 3x+3y-6z=9$

$E_2+E_3: \ x-z=7$

Note $E_1+E_2 \in \mathcal{E}$, $E_2+E_3 \in \mathcal{E}$

We can scalar-multiply equations

$\pi E_2 \quad \pi x - \pi z = 6\pi$

$(-1)E_1 \quad -2x-3y+5z=-3$

$\pi E_2 \in \mathcal{E}$, $(-1)E_1 \in \mathcal{E}$

Rule for adding elements of $\mathcal{E}$

$$\underbrace{(a_1x+b_1y+c_1z=d_1)}_{E_1} + \underbrace{(a_2x+b_2y+c_2z=d_2)}_{E_2} = \underbrace{(a_1+a_2)x+(b_1+b_2)y+(c_1+c_2)z=(d_1+d_2)}_{E_1+E_2}$$

Note $E_1, E_2 \in \mathcal{E} \implies E_1+E_2 \in \mathcal{E}$

1. $\mathcal{E}$ is closed under "vector" (ie equation) addition ✓

Rule for scalar-multiplying elements of $\mathcal{E}$

$$k\underbrace{(ax+by+cz=d)}_{E} = \underbrace{(ka)x+(kb)y+(kc)z=(kd)}_{kE}$$

Note $k \in \mathbb{R}$ and $E \in \mathcal{E} \implies kE \in \mathcal{E}$

2. $\mathcal{E}$ is closed under scalar multiplication ✓

(3) What is the zero "vector" (equation) for $\mathcal{E}$?

Let Z be the equation $Z \quad 0=0$

Let $E \in \mathcal{E}$. Then
$$\begin{aligned} Z+E &= (0=0)+(ax+by+cz=d) \\ &= ((0+a)x+(0+b)y+(0+c)z=(0+d)) \\ &= ax+by+cz=d \\ &= E \end{aligned}$$

$\vec{0}$ for $\mathcal{E}$ is Z!

(4) For $E \in \mathcal{E}$ what is its negative?

$E+(-E)=Z$ so $-E$ must be the equation $-ax-by-cz=-d$

(5) Let $E_1, E_2 \in \mathcal{E}$. Then
$$\begin{aligned} E_1+E_2 &= (a_1x+b_1x+c_1x=d_1)+(a_2x+b_2y+c_2z=d_2) \\ &= ((a_1+a_2)x+(b_1+b_2)y+(c_1+c_2)z=(d_1+d_2)) && \text{def of eq}^{\text{n}}\text{ addition} \\ &= ((a_2+a_1)x+(b_2+b_1)y+(c_2+c_1)z=(d_2+d_1)) && \text{commutativity of scalar addition} \\ &= (a_2x+b_2y+c_2z=d_2)+(a_1x+b_1x+c_1x=d_1) && \text{def of eq}^{\text{n}}\text{ addition} \\ &= E_2+E_1 \end{aligned}$$

(6) Let $E_1, E_2, E_3 \in \mathcal{E}$, $E_i=(a_ix+b_iy+c_iz=d_i)$ for $1 \le i \le 3$. Then
$$\begin{aligned} E_1+(E_2+E_3) &= (a_1x+b_1y+c_1z=d_1)+\left[(a_2x+b_2y+c_2z=d_2)+(a_3x+b_3y+c_3z=d_3)\right] \\ &= (a_1x+b_1y+c_1z=d_1)+\left[(a_2+a_3)x+(b_2+b_3)y+(c_2+c_3)z=(d_2+d_3)\right] \\ &= ((a_1+(a_2+a_3))x+(b_1+(b_2+b_3))y+(c_1+(c_2+c_3))z=(d_1+(d_2+d_3))z) \\ &= (((a_1+a_2)+a_3)x+((b_1+b_2)+b_3)y+((c_1+c_2)+c_3)z=((d_1+d_2)+d_3)) \\ &= \left[(a_1+a_2)x+(b_1+b_2)y+(c_1+c_2)z=(d_1+d_2)\right]+(a_3x+b_3y+c_3z=d_3) \\ &= \left[(a_1x+b_1x+c_1z=d_1)+(a_2x+b_2y+c_2z=d_2)\right]+(a_3x+b_3y+c_3z=d_3) \\ &= (E_1+E_2)+E_3 \end{aligned}$$

Try the rest yourself! (7)–(10)

Example 3.3. FUNCTION SPACES
Let $\mathscr{F}$ denote the set of all real-valued functions with domain $(-\infty, \infty)$.

$$\mathscr{F} = \{ f \mid f\, \mathbb{R} \to \mathbb{R} \}$$

For two functions $f, g \in \mathscr{F}$, we have $f = g$ if and only if $f(x) = f(g)$ for all $x \in (-\infty, \infty)$.

Ex $f(x) = x^2 \in \mathscr{F}$ $\quad g(x) = \sin(x) \in \mathscr{F}$ $\quad h(x) = \underline{\tan(x)} \notin \mathscr{F}$

domain is not all $x \in (-\infty, \infty)$

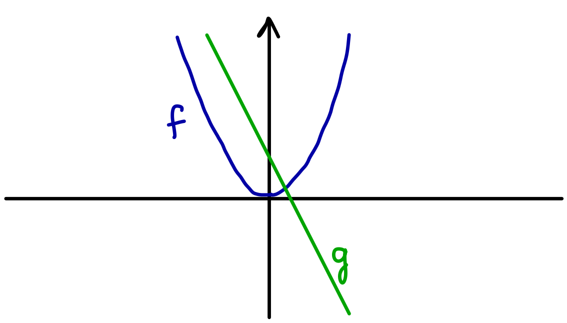
We can add two functions together:

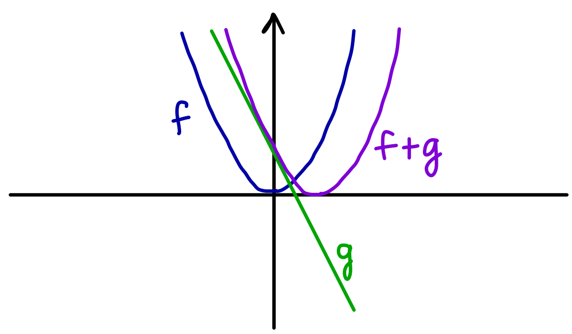
Let $f, g \in \mathscr{F}$ Then $f+g$ is the function defined by the rule

$$(f+g)(x) = f(x) + g(x) \quad \text{for all } x \in (-\infty, \infty)$$

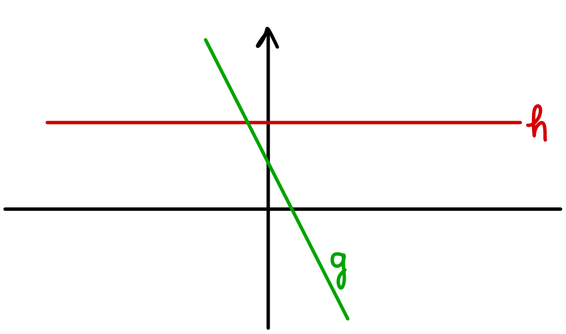
graphically, the function $f+g$ has y-values equal to the sum of the y-values of f and g at each point $x \in (-\infty, \infty)$

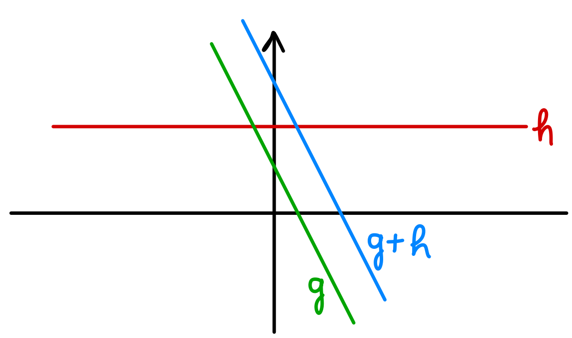
Ex $f(x) = x^2$, $g(x) = -2x+1$, $h(x) = 2$ $\qquad (f+g)(x) = x^2 - 2x + 1 = (x-1)^2$





$(g+h)(x) = -2x + 1 + 2 = -2x + 3$





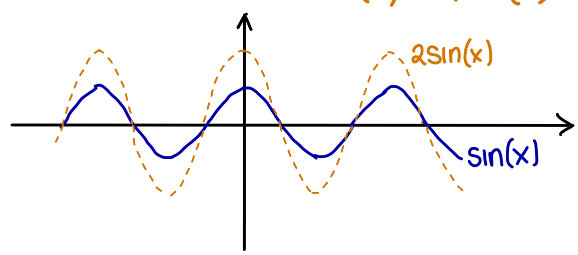
We can scale a function by a real scalar $k \in \mathbb{R}$:

Let $f \in \mathcal{F}$ and let $k \in \mathbb{R}$. Then kf is the function defined by the rule

$$(kf)(x) = kf(x) \text{ for all } x \in (-\infty, \infty)$$

Graphically, kf is obtained from f by vertically stretching by a factor of k.

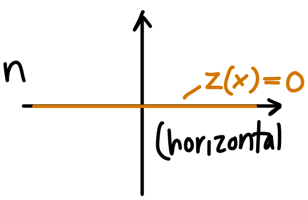
<u>Ex</u> $f(x) = \sin(x)$ $k = 2$ $2f(x) = 2\sin(x)$



<u>Ex</u> $z(x) = 0$ for all $x \in (-\infty, \infty)$ is in $\mathcal{F}$

↑ the zero function

Graph of zero function



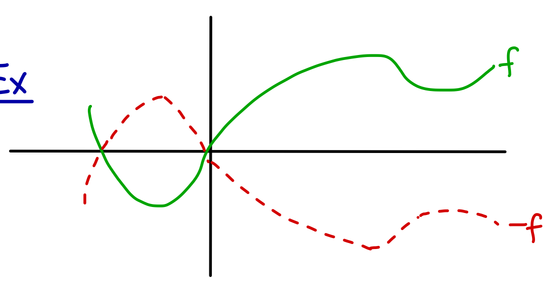
Then $\mathcal{F}$ is a vector space!

Let $f \in \mathcal{F}$. Then for all $x \in (-\infty, \infty)$,

$$\begin{aligned}(z+f)(x) &= z(x) + f(x) \\ &= 0 + f(x) \\ &= f(x) \quad \therefore z+f = f \qquad \text{Axiom } 2 \checkmark\end{aligned}$$

The graph of the negative of f is a reflection of f over the x-axis. Algebraically, $-f$ is the function defined by the rule $(-f)(x) = -f(x)$ for all $x \in (-\infty, \infty)$

<u>Ex</u>



$$\begin{aligned}(f+(-f))(x) &= f(x) + (-f(x)) \\ &= 0\end{aligned}$$

$\therefore f + (-f) = z$ Axiom 4 ✓

<u>Ex</u> Let $c, d \in \mathbb{R}$, and let $f \in \mathcal{F}$. Then, for all $x \in (-\infty, \infty)$,

$$\begin{aligned}((c+d)f)(x) &= (c+d)f(x) \\ &= cf(x) + df(x)\end{aligned}$$

$\therefore (c+d)f = cf + df$ Axiom 8 ✓

Try the rest yourself!

Example 3.4. THE TRIVIAL VECTOR SPACE Let V be a set with only one element, which we denote by $\mathbf{0}$. That is, let $V = \{\mathbf{0}\}$.

We can use the following "rule" for addition (of the one and only vector in V):

$$\vec{0} + \vec{0} = \vec{0}$$

We also have the following "rule" for multiplication of a (the!) vector in V by a scalar $k \in \mathbb{R}$:

$$k\vec{0} = \vec{0}$$

Then $V = \{\mathbf{0}\}$ is a vector space!

(even though $V = \{\mathbf{0}\}$ and its operations seem a bit degenerate, V nonetheless satisfies the axioms!)

Example 3.5. Consider the set $W = \{\} = \emptyset$ (the empty set). Is W a vector space?

No! The empty set doesn't contain a zero vector Axiom 3 ✗

Example 3.6. Consider the set $L = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : y = 2x \right\}$.

Using ordinary vector addition and scalar multiplication from $\mathbb{R}^2$, is L a vector space?

Let $\underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{u}}, \underbrace{\begin{bmatrix} c \\ d \end{bmatrix}}_{\vec{v}}, \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{w}} \in L$ and let $k, \ell \in \mathbb{R}$

$\therefore\ b = 2a,\ d = 2c,\ y = 2x$

1 $\vec{u} + \vec{v} = \begin{bmatrix} a + c \\ b + d \end{bmatrix} = \begin{bmatrix} a + c \\ 2a + 2c \end{bmatrix} = \underbrace{\begin{bmatrix} a + c \\ 2(a+c) \end{bmatrix}}_{\text{this is an element of the set } L}$

$\vec{u}, \vec{v} \in L \implies \vec{u} + \vec{v} \in L$ $\therefore$ L is closed under addition

2 $k\vec{u} = \begin{bmatrix} ka \\ kb \end{bmatrix} = \begin{bmatrix} ka \\ k(2a) \end{bmatrix} = \underbrace{\begin{bmatrix} ka \\ 2(ka) \end{bmatrix}}_{\text{this is an element of the set } L}$

$\vec{u} \in L, k \in \mathbb{R} \implies k\vec{u} \in L$ $\therefore$ L is closed under scalar multiplication

3 $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 2(0) \end{bmatrix}}_{\text{this is an element of the set } L}$ Just like in $\mathbb{R}^2$, $\vec{0} + \vec{u} = \vec{u}$ for all $\vec{u} \in L$

4 $-\vec{u} = \begin{bmatrix} -a \\ -b \end{bmatrix} = \begin{bmatrix} -a \\ -(2a) \end{bmatrix} = \underbrace{\begin{bmatrix} -a \\ 2(-a) \end{bmatrix}}_{\text{this is an element of the set } L}$ Just like in $\mathbb{R}^2$, $\vec{u} + (-\vec{u}) = \vec{0}$ for all $\vec{u} \in L$

5 Let $\vec{u},\vec{v}\in L$ Then $\underline{\vec{u}+\vec{v}=\vec{v}+\vec{u}}$

Why? Because this is already true for <u>all</u> vectors in $\mathbb{R}^2$
$\therefore$ It's true for all vectors in $\underbrace{L\subseteq\mathbb{R}^2}_{L \text{ is a subset of } \mathbb{R}^2}$

6 Let $\vec{u},\vec{v},\vec{w}\in L$ Then $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$

7 Let $\vec{u},\vec{v}\in L$, $k\in\mathbb{R}$ Then $k(\vec{u}+\vec{v})=k\vec{u}+k\vec{v}$

8 Let $\vec{u}\in L$, $k,\ell\in\mathbb{R}$ Then $(k+\ell)\vec{u}=k\vec{u}+\ell\vec{u}$

9 Let $\vec{u}\in L$, $k,\ell\in\mathbb{R}$ Then $k(\ell\vec{u})=(k\ell)\vec{u}$

10 Let $\vec{u}\in L$ Then $1\vec{u}=\vec{u}$

Why? Because this is already true for <u>all</u> vectors in $\mathbb{R}^2$
$\therefore$ It's true for all vectors in $\underbrace{L\subseteq\mathbb{R}^2}_{L \text{ is a subset of } \mathbb{R}^2}$

$\therefore$ L is a vector space!

Example 3.7. Let $W=\left\{\begin{bmatrix}x\\x+2\end{bmatrix}:x\in\mathbb{R}\right\}$. Using ordinary vector addition and scalar multiplication from $\mathbb{R}^2$, is W a vector space?

Let $\vec{u},\vec{v}\in W$, $k\in\mathbb{R}$

Then $\vec{u}=\begin{bmatrix}a\\a+2\end{bmatrix}$, $\vec{v}=\begin{bmatrix}b\\b+2\end{bmatrix}$ for some $a,b\in\mathbb{R}$ (def of W)

$\vec{u}+\vec{v}=\begin{bmatrix}a+b\\(a+2)+(b+2)\end{bmatrix}=\begin{bmatrix}a+b\\(a+b)+4\end{bmatrix}\notin W$ $\therefore$ W is <u>not</u> closed under addition ~~1~~

$\therefore$ W is not a vector space

Likewise, W is not closed under scalar multiplication ~~2~~

$\vec{0}=\begin{bmatrix}0\\0\end{bmatrix}\notin W$ since $\begin{bmatrix}a\\a+2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}\Longleftrightarrow\begin{cases}a=0\\a=-2\end{cases}$ ↯ no solution ~~3~~

For $\vec{u}\in W$, $-\vec{u}\notin W$ $\quad -\vec{u}=\begin{bmatrix}-a\\-(a+2)\end{bmatrix}=\begin{bmatrix}-a\\-a-2\end{bmatrix}$ ~~4~~

But, W does satisfy Axioms 5 – 10 since it inherits the arithmetic properties of $\mathbb{R}^2$