# 5. Span

All vector spaces (with real scalars) are infinite (with one exception: the trivial vector space $V = \{\mathbf{0}\}$ is finite).

Now, we explore a way to describe an infinite vector space using a finite number of vectors, by considering the concept of "span".

## DESCRIBING VECTOR SPACES

**Example 5.1.** Consider the following two subspaces of $\mathbb{R}^3$: **EXERCISE!** Verify that $X$ and $Y$ are subspaces of $\mathbb{R}^3$.

$$U = \left\{ \begin{bmatrix} t-3s \\ t \\ s \end{bmatrix} : s,t \in \mathbb{R} \right\} \qquad W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x - y + 3z = 0 \right\}$$

U's description is of the form

{elements defined with parameters   parameters can be any real numbers}

this way makes constructing elements of U easy just plug in some parameter values and output is an element of U

ex set $s=1, t=-5$

$$\Rightarrow \begin{bmatrix} -5-3(1) \\ -5 \\ 1 \end{bmatrix} \in U$$

W's description is of the form

{elements from some set   required conditions}

this way makes checking whether a given element belongs to W easy just check whether the condition is satisfied

ex Is $\begin{bmatrix} -8 \\ -5 \\ 1 \end{bmatrix} \in W$? Test condition

$(-8)-(-5)+3(1)=0$ ✓

Yes!

In fact, $U = W$ Let's show they're equal

$$\underbrace{\left\{ \begin{bmatrix} t-3s \\ t \\ s \end{bmatrix} : s,t \in \mathbb{R} \right\}}_{U} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : \begin{matrix} x=t-3s \\ y=t \\ z=s \end{matrix} \right\} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x=y-3z \right\} = \underbrace{\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x-y+3z=0 \right\}}_{W}$$

We can go further

$$U = \left\{ \begin{bmatrix} t-3s \\ t \\ s \end{bmatrix} : s,t \in \mathbb{R} \right\} = \left\{ \underbrace{t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}} : s,t \in \mathbb{R} \right\}$$

this description shows that each vector in U is a linear combination of two vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

†These notes are solely for the personal use of students registered in MAT1341.

**Definition 5.2.** Let $V$ be a vector space and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be vectors in $V$.

- A vector $\mathbf{w}$ is called a **LINEAR COMBINATION OF $\mathbf{v}_1, \ldots, \mathbf{v}_m$** if

  there exist scalars $a_1, a_2, , a_m \in \mathbb{R}$ such that $\vec{w} = a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_m\vec{v}_m$

- The **SPAN OF $\mathbf{v}_1, \ldots, \mathbf{v}_m$**, denoted $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$, is

  the set of all linear combinations of $\vec{v}_1, \ldots, \vec{v}_m$

  That is, $\operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_m\} = \{a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_m\vec{v}_m : a_1, \ldots, a_m \in \mathbb{R}\}$

- Suppose $S = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$.

  Then the set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is called a **SPANNING SET** for $S$.

  We also say that "$S$ is **SPANNED BY** the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$".

  Or we sometimes say "the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ **SPAN** $S$".

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Return to the previous example:

$$U = \left\{ \begin{bmatrix} t-3s \\ t \\ s \end{bmatrix} : s,t \in \mathbb{R} \right\} = \left\{ t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} : s,t \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a spanning set for $U$

- $U$ is spanned by $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

- The vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ span $U$

**Example 5.3.** **Symmetric $2 \times 2$ Real Matrices** Let $S = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} : a, b, d \in \mathbb{R} \right\}$. Find a spanning set for $S$.

$$S = \left\{ a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} : a,b,d \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$\therefore \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a spanning set for $S$

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**Example 5.4.** Let $\mathbb{P}_2$ denote the set of **Polynomials of Degree At Most 2** . That is, let

$$\mathbb{P}_2 = \{ax^2 + bx + c : a, b, c \in \mathbb{R}\}$$

**Exercise:** Show that $\mathbb{P}_2$ is a vector space (hint: use the Subspace Test!)

Find a spanning set for $\mathbb{P}_2$.

$$\mathbb{P}^2 = \{ax^2+bx+c \;\; a,b,c \in \mathbb{R}\} = \{a\,x^2 + b\,x + c\,1 \;\; a,b,c \in \mathbb{R}\} = \text{span}\{\underset{\nwarrow}{x^2}, \underset{\uparrow}{x}, \underset{\nearrow}{1}\}$$

each of these is a polynomial

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**Example 5.5.** Now, let $U = \{p(x) \in \mathbb{P}_2 : p(1) = 0\}$. Find a spanning set for $U$.

$$U = \{\underbrace{p(x) \in \mathbb{P}_2}_{\text{elements from some set}} : \underbrace{p(1) = 0}_{\text{required condition}}\}$$

$$= \{ax^2+bx+c : a,b,c \in \mathbb{R} \text{ and } p(1)=0\}$$

$$= \{ax^2+bx+c : a,b,c \in \mathbb{R} \text{ and } a(1^2)+b(1)+c = 0\}$$

$$= \{ax^2+bx+c \;\; a,b,c \in \mathbb{R} \text{ and } a+b+c=0\}$$

$$= \{ax^2+bx+c : a,b,c \in \mathbb{R} \text{ and } a=-b-c\}$$

put the condition into the vector parameters

$$= \{(-b-c)x^2+bx+c : b,c \in \mathbb{R}\}$$

$$= \{b(x-x^2) + c(1-x^2) : b,c \in \mathbb{R}\}$$

$$= \text{span}\{x-x^2, 1-x^2\}$$

$\boxed{\text{Every polynomial } p(x) \text{ of degree} \leq 2 \text{ such that } p(1)=0 \text{ is a linear combination of polynomials } x-x^2 \text{ and } 1-x^2}$

$U$ is spanned by $\{x-x^2, 1-x^2\}$

Ex $q(x) = 4(x-x^2) - 5(1-x^2) = x^2 + 4x - 5 \in U$ Check $q(1) = 1^2 + 4(1) - 5 = 0$ ✓

so $q$ satisfies condition for $U$

Ex $f(x) = 1-x \in U$ since $f(1) = 0$

$\therefore\ 1-x \in \operatorname{span}\{x-x^2, 1-x^2\}$

Check find $a, b \in \mathbb{R}$ such that $f(x) = a(x-x^2) + b(1-x^2) \iff 1-x = a(x-x^2) + b(1-x^2)$

$$\iff 1-x = (-a-b)x^2 + ax + b$$

Compare coefficients LS $= 0x^2 - x + 1$ RS $= (-a-b)x^2 + ax + b$

$$\iff \begin{Bmatrix} 0 = -a-b \\ -1 = a \\ 1 = b \end{Bmatrix} \iff a = -1 \text{ and } b = 1$$

Check $(-1)(x-x^2) + (1)(1-x^2) = -x + 1 = 1-x = f(x)$ ✓

So $f$ is in this span

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**Example 5.6.** Find the "condition" description for $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$.

$$\underbrace{\operatorname{span}}_{\text{vectors in } \mathbb{R}^4 \text{ that are a linear combination of } \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}}\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\} = \left\{ s\begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} s \\ -2s+t \\ t \\ 3s-t \end{bmatrix} : s, t \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4 : \begin{matrix} x = s \\ y = -2s+t \\ z = t \\ w = 3s-t \end{matrix} \right\}$$

$$= \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4 \quad \begin{matrix} y = -2x + z \\ w = 3x - z \end{matrix} \right\}$$

$$= \underbrace{\left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4 : \begin{matrix} -2x - y + z = 0 \\ 3x - z - w = 0 \end{matrix} \right\}}_{\text{vectors in } \mathbb{R}^4 \text{ that are orthogonal to both } \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ 0 \\ -1 \\ -1 \end{bmatrix}}$$

Now we get to "the BIG THEOREM" about spans:

**Theorem 5.7.** **(THE BIG THEOREM ABOUT SPANS)** Let $V$ be a vector space.
Let $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ be set of vectors in some vector space $V$. Then

$$\text{span}\{\vec{v}_1, \dots, \vec{v}_m\} \text{ is a subspace of } V$$

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**Proof:**

Let $U = \text{span}\{\vec{v}_1, \dots, \vec{v}_m\}$ where $\vec{v}_1, \quad, \vec{v}_m$ are vectors in a vector space $V$

By def of span, each vector in $U$ is a linear combination $a_1\vec{v}_1 + \dots + a_m\vec{v}_m$

Since $V$ is closed under scalar multiplication, $a_i\vec{v}_i \in V$ for all $a_i \in \mathbb{R}$, for all $i$

Since $V$ is closed under addition, $\underbrace{a_1\vec{v}_1}_{\in V} + \dots + \underbrace{a_m\vec{v}_m}_{\in V} \in V$

$\therefore U \subseteq V$

Equip $U$ with the operations of $V$ and apply the Subspace Test

Let $\vec{u}, \vec{v} \in U$ and $k \in \mathbb{R}$

Then $\vec{u} = a_1\vec{v}_1 + \dots + a_m\vec{v}_m$ and $\vec{v} = b_1\vec{v}_1 + \dots + b_m\vec{v}_m$ for some $a_i, b_i \in \mathbb{R}$, $1 \le i \le m$

① $\vec{u} + \vec{v} = (a_1\vec{v}_1 + \quad + a_m\vec{v}_m) + (b_1\vec{v}_1 + \quad + b_m\vec{v}_m)$

$= \underbrace{(a_1 + b_1)}_{\in \mathbb{R}}\vec{v}_1 + \quad + \underbrace{(a_m + b_m)}_{\in \mathbb{R}}\vec{v}_m$

$\therefore$ $\vec{u} + \vec{v}$ is a linear combination of $\vec{v}_1, \quad, \vec{v}_m$, hence $\vec{u} + \vec{v} \in U = \text{span}\{\vec{v}_1, \quad, \vec{v}_m\}$

② $k\vec{u} = k(a_1\vec{v}_1 + \quad + a_m\vec{v}_m) = \underbrace{(ka_1)}_{\in \mathbb{R}}\vec{v}_1 + \quad + \underbrace{(ka_m)}_{\in \mathbb{R}}\vec{v}_m$

$\therefore$ $k\vec{u}$ is a linear combination of $\vec{v}_1, \quad, \vec{v}_m$, hence $k\vec{u} \in U = \text{span}\{\vec{v}_1, \quad, \vec{v}_m\}$

③ $\vec{0} = \underbrace{0}_{\in \mathbb{R}}\vec{v}_1 + \dots + \underbrace{0}_{\in \mathbb{R}}\vec{v}_m$

$\therefore$ $\vec{0}$ is a linear combination of $\vec{v}_1, \quad, \vec{v}_m$, hence $\vec{0} \in U = \text{span}\{\vec{v}_1, \quad, \vec{v}_m\}$

Conclusion $U = \text{span}\{\vec{v}_1, \quad, \vec{v}_m\}$ is a subspace of $V$ !

**Corollary 5.8.** Let $W$ be a subspace of a vector space $V$. Let $\mathbf{w}_1, \ldots, \mathbf{w}_m \in W$.

Then $\operatorname{span}\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ is a subspace of $W$.

- The above result is just repeating what the BIG THEOREM told us but where the vectors all live inside a subspace of $V$.
- Stated this way, however, we notice that it means spans don't cross subspace boundaries.
- In other words, if the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_m$ live inside a subspace $W$ of $V$, then the span of those vectors is also entirely contained in the subspace $W$.
- Another way to put this is that $\operatorname{span}\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ is the SMALLEST subspace of $V$ that contains the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_m$.

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**Example 5.9.** Is $D = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$ a subspace of $M_{3,3}(\mathbb{R})$ ?

Instead of using the Subspace Test, we notice $D$ is spanned by three matrices in $M_{3,3}(\mathbb{R})$

$$D = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

BIG THEOREM applies!

$\therefore$ Yes, $D$ is a subspace of $M_{3,3}(\mathbb{R})$ !

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**Example 5.10.** Show that $S = \{A \in M_{2,2}(\mathbb{R}) : A^\top = -A\}$ is a subspace of $M_{2,2}(\mathbb{R})$.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^T = -A \iff \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} \iff \left\{ \begin{array}{l} a = -a \\ b = -c \\ c = -b \\ d = -d \end{array} \right\} \iff \left\{ \begin{array}{l} a = 0 \\ b = -c \\ d = 0 \end{array} \right\}$$

$$\therefore S = \left\{ \begin{bmatrix} 0 & -c \\ c & 0 \end{bmatrix} : c \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$$

$\therefore S$ is a subspace (by BIG THEOREM!)

Let $U$ be an arbitrary subspace of $\mathbb{R}^n$. What are the possibilities for $U$ ?

It might be that $U=\{\vec{0}\}$ (the trivial subspace)

If $U \neq \{\vec{0}\}$, then $U$ must contain at least one non-zero vector, say $\vec{x}_1 \in U$, $\vec{x}_1 \neq \vec{0}$

Then $\text{span}\{\vec{x}_1\}$ is a subspace of $U$ (by BIG THEOREM)

Maybe $U=\text{span}\{\vec{x}_1\}$ ← $\text{span}\{\vec{x}_1\}$ is vector parametric form for a line passing through the origin

If not, $U$ must contain another vector $\vec{x}_2$ that doesn't belong to $\text{span}\{\vec{x}_1\}$

Then $\text{span}\{\vec{x}_1,\vec{x}_2\}$ is a subspace of $U$ (by BIG THEOREM)

Maybe $U=\text{span}\{\vec{x}_1,\vec{x}_2\}$ ← $\text{span}\{\vec{x}_1,\vec{x}_2\}$ st $\vec{x}_2 \notin \text{span}\{\vec{x}_1\}$ is vector parametric form for a plane passing through the origin

If not, $U$ must contain another vector $\vec{x}_3$ that doesn't belong to $\text{span}\{\vec{x}_1,\vec{x}_2\}$

Then $\text{span}\{\vec{x}_1,\vec{x}_2,\vec{x}_3\}$ is a subspace of $U$ (by BIG THEOREM)

Maybe $U=\text{span}\{\vec{x}_1,\vec{x}_2,\vec{x}_3\}$

If not, ...

$\vdots$

We run out of names but a subspace of $\mathbb{R}^n$ is

a point, line, plane, ... always passing through the origin

**Example 5.11.** Show that $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}3\\6\end{bmatrix}\right\}$.

$$\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right\} = \left\{a\begin{bmatrix}1\\2\end{bmatrix} + b\begin{bmatrix}0\\0\end{bmatrix} : a, b \in \mathbb{R}\right\} = \left\{a\begin{bmatrix}1\\2\end{bmatrix} : a \in \mathbb{R}\right\} = \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$$

$\begin{bmatrix}1\\2\end{bmatrix} \in \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$, $\begin{bmatrix}2\\4\end{bmatrix} \in \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$ and $\begin{bmatrix}3\\6\end{bmatrix} \in \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$

By BIG THEOREM, $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}3\\6\end{bmatrix}\right\}$ is a subspace, hence subset, of $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$

$$\Longrightarrow \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}3\\6\end{bmatrix}\right\} \subseteq \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$$

Likewise, $\begin{bmatrix}1\\2\end{bmatrix} \in \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}3\\6\end{bmatrix}\right\} \therefore \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\} \subseteq \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}3\\6\end{bmatrix}\right\}$

$$\therefore \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}\right\}$$

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**EXERCISE:** Show that $\operatorname{span}\left\{\begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}\right\}$.

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What do we notice from these examples?

- Having more vectors in a spanning set DOES NOT imply that the subspace they span is bigger.
- It's not easy to tell if two subspaces are equal just based on the spanning sets you're given.
- The same subspace can have many different spanning sets.