6. Linear Dependence and Independence

We've now seen examples where we can describe a subspace as being the span of a finite list of vectors.

The span of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ in some vector space V is simply the set of all linear combinations that can be created from the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$.

The BIG THEOREM tells us that the span of a set of vectors in V is always a subspace of V.

However, spanning sets are not unique and we've seen an example where different spanning sets contain "useless" extra vectors, even though the span the same subspace.

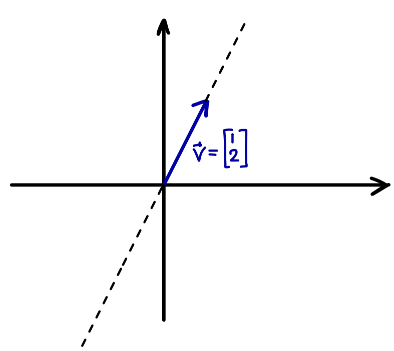
We now investigate this idea further via the concepts of linear dependence and linear independence.

DIFFERENT DIRECTIONS

We saw in Example 5.11 that $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}3\\6\end{bmatrix}\right\}$.

Since we're quite familiar with geometric vectors in $\mathbb{R}^2$, it is perhaps obvious why these spans are equal.

$\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\} = \left\{t\begin{bmatrix}1\\2\end{bmatrix} : t \in \mathbb{R}\right\}$ ← this corresponds to a line of slope 2 passing through the origin



Since $\begin{bmatrix}2\\4\end{bmatrix}$ and $\begin{bmatrix}3\\6\end{bmatrix}$ are parallel to $\begin{bmatrix}1\\2\end{bmatrix}$, they don't add any new directions to the span

†These notes are solely for the personal use of students registered in MAT1341.

Example 6.1. What if we add a new direction? What if our sets don't contain any obviously collinear vectors? Consider the following two spans:

$$U = \text{span}\left\{ \underbrace{\begin{bmatrix}1\\2\end{bmatrix}}_{\vec{u}_1}, \underbrace{\begin{bmatrix}1\\1\end{bmatrix}}_{\vec{u}_2} \right\} \qquad W = \text{span}\left\{ \underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{\vec{w}_1}, \underbrace{\begin{bmatrix}2\\1\end{bmatrix}}_{\vec{w}_2}, \underbrace{\begin{bmatrix}1\\3\end{bmatrix}}_{\vec{w}_3} \right\}$$

None of these vectors are scalar multiples of each other

Nonetheless, we'll show that $\vec{u}_1, \vec{u}_2 \in W$ and $\vec{w}_1, \vec{w}_2, \vec{w}_3 \in U$

① $\vec{u}_1 \in W \iff \begin{bmatrix}1\\2\end{bmatrix} = a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}2\\1\end{bmatrix} + c\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}a+2b+c\\b+3c\end{bmatrix}$ for some $a,b,c \in \mathbb{R}$

$\iff \begin{cases} a = 1-2b-c \\ b = 2-3c \end{cases}$ We are free to choose any value for c, say $c=0$

Then $\begin{matrix} a = 1-2b \\ b = 2 \end{matrix}$ so $\begin{matrix} a = 1-2(2) = -3 \\ b = 2 \end{matrix}$

So a solution exists! Check $\begin{bmatrix}1\\2\end{bmatrix} = -3\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}2\\1\end{bmatrix} + 0\begin{bmatrix}1\\3\end{bmatrix}$ ✓

so $\vec{u}_1 = (-3)\vec{w}_1 + 2\vec{w}_2 + 0\vec{w}_3$ $\therefore$ $\vec{u}_1 \in \text{span}\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$

Similarly, you can show that $\vec{u}_2 = (-1)\vec{w}_1 + (1)\vec{w}_2 + 0\vec{w}_3$ $\qquad \vec{w}_2 = (-1)\vec{u}_1 + 2\vec{u}_2$

$\vec{w}_1 = (-1)\vec{u}_1 + 2\vec{u}_2 \qquad \vec{w}_3 = 2\vec{u}_1 + (-1)\vec{u}_2$

BIG THEOREM tells us $U = \text{span}\{\vec{u}_1, \vec{u}_2\}$ is a subspace of W

and $W = \text{span}\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is a subspace of U

In particular, $U \subseteq W$ and $W \subseteq U$ $\qquad \therefore U = W$

- Both of the subspaces U and W are equal even though their spanning sets contain vectors that are not collinear.
- In fact, the spanning sets for U and W contain vectors that are coplanar (hence they actually span the same plane).
- The vectors $\mathbf{u}_1, \mathbf{u}_2$ share the same span as the vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$.
- We can describe all vectors in U as a linear combination of only 2 vectors.
- We can describe all vectors in W as a linear combination of 3 vectors.
- But since $U = W$, we prefer the description with fewer vectors (it's more efficient and describes the exact same subspace).

EXERCISE! Prove that $\mathbb{R}^2 = \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$. Next, show that $\begin{bmatrix}1\\0\end{bmatrix} \in U$ and $\begin{bmatrix}0\\1\end{bmatrix} \in U$.

Finally, conclude that $U = \mathbb{R}^2 = W$ (hint: use the BIG THEOREM and the fact that $U = W$).

What is going on in the previous example? Although the vectors in each of the spanning sets are not collinear, they are coplanar (meaning they lie on the same plane, actually, the *only* plane in $\mathbb{R}^2$, which is all of $\mathbb{R}^2$ itself).

So adding more vectors that still lie on the same plane did not make the space they spanned any bigger.

The only way we could have created a "bigger" subspace from a spanning set such as U or W would be to add another vector to the spanning set that was not coplanar with the vectors in the original spanning set (but this is not possible in $\mathbb{R}^2$ since all vectors in $\mathbb{R}^2$ are coplanar).

But in other vector spaces, this may be possible...

There is a general concept, called **LINEAR INDEPENDENCE**, that captures the essence of what it means for a collection of vectors to be genuinely in different "directions" (even if the vectors are non-geometric and the notion of "direction" has no visual interpretation).

Before we get to the definition of linear independence, let's explore the opposite idea: let's generalize the concept of vectors being collinear, coplanar, etc...

Definition 6.2. Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be vectors in some vector space V.

We say that the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are **LINEARLY DEPENDENT** if and only if

there exist scalars $a_1, \ , a_m \in \mathbb{R}$ that are <u>not</u> all zero such that

$$\underbrace{a_1\vec{v}_1 + \ldots + a_m\vec{v}_m = \vec{0}}_{\text{this is called a dependence equation}}$$

$\vec{v}_1, \ ,\vec{v}_m$ are LD $\iff$ there exists a non-trivial solution $a_1, \ , a_m$ to the dependence equation
(not all zero)

Example 6.3. Show that the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ are linearly dependent.

We must find $a_1, a_2 \in \mathbb{R}$ (not both zero) such that $a_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + a_2\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$a_1 = 3$ and $a_2 = -1$ work check $3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-1)\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

We can see $\vec{v}_1$ and $\vec{v}_2$ are collinear

The linear dependency equation confirms this algebraically

$3\vec{v}_1 + (-1)\vec{v}_2 = \vec{0} \implies \vec{v}_2 = 3\vec{v}_1$ So $\vec{v}_2$ is a scalar multiple of $\vec{v}_1$, hence collinear

Example 6.4. Are the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ linearly dependent?

$$a_1\vec{v}_1 + a_2\vec{v}_2 = \vec{0} \iff a_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + a_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\iff \begin{cases} a_1 + a_2 = 0 \\ 2a_1 + a_2 = 0 \end{cases}$$

$$\iff a_1 = 0 \text{ and } a_2 = 0$$

The <u>only</u> solution to the dependency equation is the trivial solution

$\therefore$ $\vec{v}_1$ and $\vec{v}_2$ are <u>not</u> LD

(but we already noticed that they are 2 vectors that are <u>not</u> collinear)

Example 6.5. Are the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ linearly dependent?

$$a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = \vec{0} \iff a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\iff \begin{cases} a + c = 0 \\ b + c = 0 \\ a + c = 0 \end{cases}$$

$$\iff \begin{cases} a = -c \\ b = -c \end{cases} \quad \text{(and } c \text{ is free)}$$

<u>Ex</u> take $c = 1$, $a = -1$, $b = -1$ and we get $(-1)\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (1)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore$ $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are LD

Now, when the answer is "the vectors are not linearly dependent", we will instead say "the vectors are linearly independent". But here is the precise definition:

Definition 6.6. Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be vectors in some vector space V.

We say that the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are **LINEARLY INDEPENDENT** if and only if

the only solution to the dependency equation

$$a_1\vec{v}_1 + \ldots + a_m\vec{v}_m = \vec{0}$$

is the trivial solution $a_1 = a_2 = \cdots = a_m = 0$

Example 6.7. Show that $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}} \in \mathbb{R}^3$ are linearly independent.

$$a\hat{i} + b\hat{j} + c\hat{k} = \vec{0} \iff a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix} + c\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \iff \begin{cases} a=0\\ b=0\\ c=0\end{cases}$$

Since the only solution is the trivial solution, $\hat{i}, \hat{j}, \hat{k}$ are indeed LI

Example 6.8. We have $0\begin{bmatrix}1\\0\\1\end{bmatrix} + 0\begin{bmatrix}0\\1\\0\end{bmatrix} + 0\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$.

So the dependency equation has a trivial solution. Does this mean $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$ are LI?

No! The trivial solution always satisfies the dependency equation

Vectors are LI $\iff$ the only solution is the trivial solution

In this example, non-trivial solutions also exist

Ex $(-1)\vec{v}_1 + (-1)\vec{v}_2 + (1)\vec{v}_3 = \vec{0}$ (see Ex 6.5)

Example 6.9. In $M_{2,2}(\mathbb{R})$, are $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ linearly independent?

$$a\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} + b\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} + c\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \iff \begin{array}{ll} a+b+2c=0 & (1) \\ a+2b+3c=0 & (2) \\ a+b=0 & (3) \\ 2a+2b+c=0 & (4) \end{array}$$

(1)-(2) $2c=0 \therefore c=0$ $\rightarrow$ sub into (2) $a+2b=0$ (5)

(5)-(3) $b=0$ $\rightarrow$ sub into (3) $a=0$

$\therefore$ only solution is trivial solution $a=b=c=0$

Conclusion these matrices are LI

Example 6.10. In the space $\mathscr{F}$ of real-valued functions with domain $\mathbb{R}$, are the functions $\sin(x), \cos(x)$ LI or LD ?

$$a\sin x + b\cos x = \underset{\text{the zero function}}{0} \iff a\sin(x) + b\cos(x) = 0 \text{ for all } x \in (-\infty, \infty)$$

In particular, for $x=0$, $a\sin(0)+b\cos(0)=0 \Rightarrow b=0$

for $x=\frac{\pi}{2}$ $a\sin(\frac{\pi}{2})+b\cos(\frac{\pi}{2})=0 \Rightarrow a=0$

} only solution is trivial solution

Conclusion $\sin(x)$ and $\cos(x)$ are LI

Example 6.11. In the space $\mathscr{F}$ of real-valued functions with domain $\mathbb{R}$, show that the functions $\sin^2(x),\ \cos^2(x),\ 1$ are LD.

1 is the constant function $f(x)=1$ for all $x \in \mathbb{R}$

Since $\cos^2 x + \sin^2 x = 1$ we have $1\cos^2 x + 1\sin^2 x + (-1)\,1 = 0$

non-trivial solution exists

$\therefore$ LD

Facts about Linear Independence and Linear Dependence

Here we collect some useful facts about linear independence and dependence.

Theorem 6.12. Let V be a vector space.

Fact 1: If $\mathbf{v} \in V$, then $\{\mathbf{v}\}$ is LI if and only if $\mathbf{v} \neq \mathbf{0}$.

Fact 2: If $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are LD, then any set containing $\mathbf{v}_1, \ldots, \mathbf{v}_m$ is also LD.

Fact 3: If $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are LI, then any subset of $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is also LI.

Fact 4: $\{\mathbf{0}\}$ is LD.

Fact 5: Any set of vectors that contains $\mathbf{0}$ is LD.

Fact 6: Vectors $\mathbf{u}, \mathbf{v} \in V$ are LD if and only if one of the vectors is a scalar multiple of the other.

Fact 7: A set with three or more vectors can be LD even though no two vectors are multiples of one another.

Fact 8: $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is LD if and only if there is at least one vector $\mathbf{v}_i \in S$ such that $\mathbf{v}_i \in \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_m\}$.

Proof of Fact 1 ($\Longrightarrow$) Assume $\vec{v} = \vec{0}$

Then $1\vec{v} = 1\vec{0} = \vec{0}$ so there is a nontrivial solution to $a\vec{v} = \vec{0}$ $\therefore \{\vec{v}\}$ is LD

($\Longleftarrow$) Assume $\vec{v} \neq \vec{0}$ Then $a\vec{v} = \vec{0} \Rightarrow a = 0$ or $\vec{v} = \vec{0}$ But $\vec{v} \neq \vec{0}$ $\therefore a = 0$

proof of Fact 2 Assume $\vec{v}_1, \ \ , \vec{v}_m$ is LD

Then there is a nontrivial solution $a_1\vec{v}_1 + \ldots + a_m\vec{v}_m = \vec{0}$ (at least one $a_1 \neq 0$)

For a bigger set of vectors $\{\vec{v}_1, \ \ , \vec{v}_m\} \cup \{\vec{u}_1, \ \ , \vec{u}_k\}$ we have

$a_1\vec{v}_1 + \ \ + a_2\vec{v}_2 + \ \ + a_m\vec{v}_m + 0\vec{u}_1 + \ \ + 0\vec{u}_k = \vec{0}$

with at least $a_1 \neq 0$ $\therefore \vec{v}_1, \ \ , \vec{v}_m, \vec{u}_1, \ \ , \vec{u}_k$ is also LD

Fact 3 is logically equivalent to fact 2

Proof of Fact 4 $\quad 1\vec{0} = \vec{0}, \; 1 \neq 0 \;\; \therefore \{\vec{0}\}$ is LD

Fact 5 follows from Facts 4 and 2

Proof of Fact 6 $\quad (\Longrightarrow)$ $\{\vec{u}, \vec{v}\}$ is LD $\Longrightarrow a\vec{u} + b\vec{v} = \vec{0}$ where $a \neq 0$ or $b \neq 0$

If $a \neq 0$, then $\vec{u} = -\frac{b}{a}\vec{v}$

If $b \neq 0$, then $\vec{v} = -\frac{a}{b}\vec{u}$

$(\Longleftarrow)$ Assume $\vec{u} = k\vec{v}$ Then $1\vec{u} + (-k)\vec{v} = \vec{0} \;\; \therefore \{\vec{u}, \vec{v}\}$ is LD

Assume $\vec{v} = c\vec{u}$ Then $(-c)\vec{u} + 1\vec{v} = \vec{0} \;\; \therefore \{\vec{u}, \vec{v}\}$ is LD

Justification of Fact 7 $\quad$ See Ex 65

Proof of Fact 8 $\quad (\Longrightarrow)$ Assume $\{\vec{v}_1, \; ,\vec{v}_m\}$ is LD

Then we have $a_1\vec{v}_1 + \ldots + a_\ell\vec{v}_\ell + \ldots + a_m\vec{v}_m = \vec{0}$ with at least one $a_\ell \neq 0$

Then $v_\ell = -\frac{a_1}{a_\ell}\vec{v}_1 - \quad - \frac{a_{\ell-1}}{a_\ell}\vec{v}_{\ell-1} - \frac{a_{\ell+1}}{a_\ell}\vec{v}_{\ell+1} + \ldots - \frac{a_m}{a_\ell}\vec{v}_m$

$\therefore \vec{v}_\ell \in \operatorname{span}\{\vec{v}_1, \; ,\vec{v}_{\ell-1}, \vec{v}_{\ell+1}, \ldots, \vec{v}_m\}$

$(\Longleftarrow)$ Assume $\vec{v}_\ell \in \operatorname{span}\{\vec{v}_1, \; ,\vec{v}_{\ell-1}, \vec{v}_{\ell+1}, \ldots, \vec{v}_m\}$

Then $\vec{v}_\ell = a_1\vec{v}_1 + . \; + a_{\ell-1}\vec{v}_{\ell-1} + a_{\ell+1}\vec{v}_{\ell+1} + \ldots + a_m\vec{v}_m$ for some $a_1, \; , a_{\ell-1}, a_{\ell+1}, \ldots, a_m \in \mathbb{R}$

Then $a_1\vec{v}_1 + \quad + a_{\ell-1}\vec{v}_{\ell-1} + (-1)\vec{v}_\ell + a_{\ell+1}\vec{v}_{\ell+1} + \quad + a_m\vec{v}_m = \vec{0}$

$-1 \neq 0$ so this is a nontrivial dependency equation

$\therefore \{\vec{v}_1, \; ,\vec{v}_m\}$ is LD
