1. Vectors in $\mathbb{R}^n$

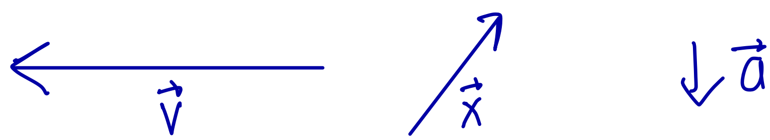
We begin MAT1341 with a friendly review of vectors, starting with concrete geometric examples in 2- and 3-space. Once comfortable in those spaces, we will be ready to explore the abstract realm of "vector spaces". By reframing vectors in terms of their abstract definitions and properties, we will be able to capture the *true essence* of what vectors really are, and we will discover that there are many surprising and interesting examples, beyond $\mathbb{R}^n$.

VECTORS (AS PRESENTED IN HIGH SCHOOL)

We start with the high school definition:

Definition 1.1. A **VECTOR** is an object that has:
1. a size ("magnitude")
2. a direction

We picture vectors as arrows, where the length of the arrow corresponds to its magnitude and the direction the arrow points is the direction of the vector.



We can represent a vector by its picture, or give it a name, such as $\mathbf{v}$.

Note: Usually, the name of a vector is typed using **boldface**: ex: $\mathbf{v}$ $\mathbf{x}$ $\mathbf{a}$

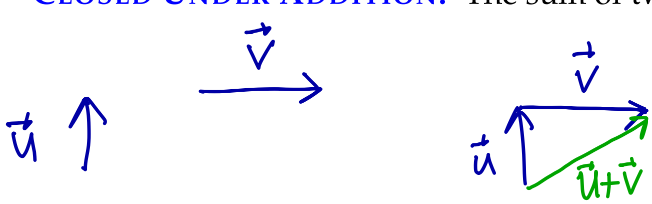
But, if we are writing a vector's name by hand, we tend to add an arrow on top (because it's difficult to convey boldface by hand).

Two vectors are **EQUAL** if they have the same magnitude and direction.

We have two operations: **VECTOR ADDITION** and **SCALAR MULTIPLICATION**.

- We add vectors tail-to-tip.

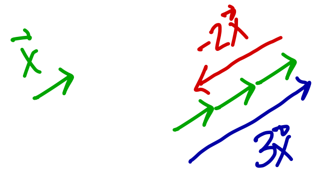
  CLOSED UNDER ADDITION: The sum of two vectors is again a vector.



†These notes are intended for students registered in MAT1341.

- We multiply a vector **v** by a **SCALAR** k (which is a real number) by creating a parallel vector with the correspondingly scaled length. If the scalar is positive, the new vector has the same direction as **v**. If the scalar is negative, the new vector has the opposite direction as **v**.

  CLOSED UNDER SCALAR MULTIPLICATION:
  A vector multiplied by a scalar is again a vector.



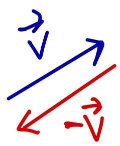
$\vec{0}$

Vectors have a "zero concept" called **THE ZERO VECTOR**, denoted **0**. The **ZERO VECTOR** has no magnitude (i.e. zero magnitude) and it does not have a well-defined direction.

What if we add a vector **v** to the zero vector **0** ?

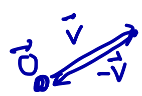
$$\vec{0} + \vec{v} = \vec{v}$$

Each vector **v** has a **NEGATIVE**, denoted $-\mathbf{v}$. The **NEGATIVE** of **v** has the same magnitude as **v** but the opposite direction.



What if we add a vector **v** to its negative $-\mathbf{v}$?

$$\vec{v} + (-\vec{v}) = \vec{0}$$

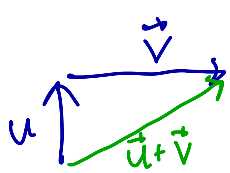


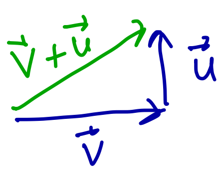
Recall: Vector addition and scalar multiplication also satisfy the following properties.

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors and let $c, d \in \mathbb{R}$ be scalars.

VECTOR ADDITION IS COMMUTATIVE

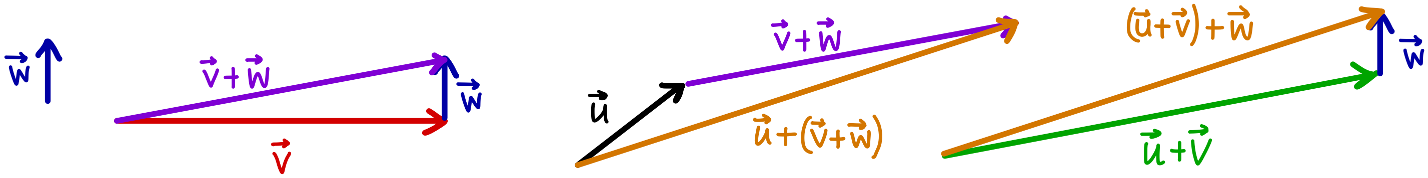
$$\vec{u} + \vec{v} = \vec{v} + \vec{u}$$



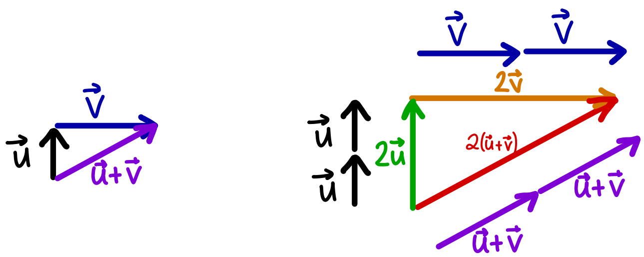


Vector Addition is Associative

$$\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$$



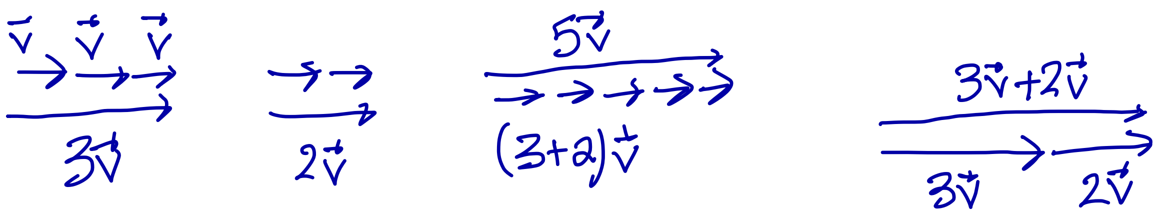
Scalar Multiplication Distributes over Vector Addition



$$c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$$

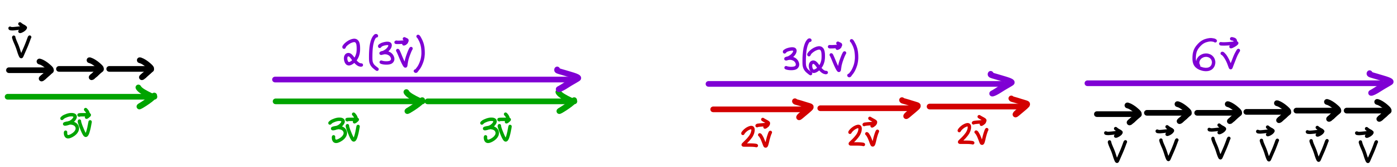
Addition of Scalars Distributes over Scalar Multiplication

$$(c+d)\vec{v} = c\vec{v} + d\vec{v}$$



Multiplication of Scalars is Compatible with Scalar Multiplication

$$c(d\vec{v}) = (cd)\vec{v}$$



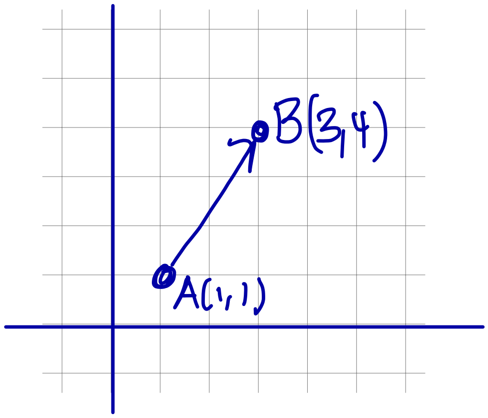
Unity Law

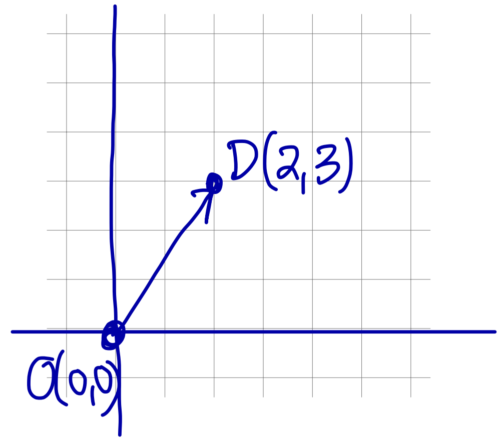
$$1 \cdot \vec{v} = \vec{v}$$

Note: What we're really saying is that the arrows we draw, equipped with our geometric rules for vector addition and scalar multiplication, satisfy the **Axioms of a Vector Space (over $\mathbb{R}$)** !

For now, we will focus on how these properties work in $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$,. . . , $\mathbb{R}^n$, . . . , both algebraically and geometrically. Later, we will reconsider these properties in terms of abstract vector spaces.

We also use coordinate systems to help us represent vectors.





- We can refer to a vector using its **INITIAL POINT** (the **TAIL** of the arrow) and its **TERMINAL POINT** (the **TIP** or **HEAD** of the arrow). The coordinates of these two points indicate the vector's position relative to the axes.

  Ex. Points: $A(1,1)$ and $B(3,4)$ Vector $\overrightarrow{AB}$

  Ex. Points: $O(0,0)$ and $D(2,3)$ Vector $\overrightarrow{OD}$

  Note: $\overrightarrow{AB} = \overrightarrow{OD}$ since they have the same magnitude and direction.

- We can refer to a vector using its **COORDINATES**, which indicate the position of its tip relative to its tail.

  Ex. $\mathbf{v} = (2,3)$

- We often write vectors as a **COLUMN VECTOR** which is a matrix with one column:

  Ex. $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

  Using a column vector is nicer and this notation is compatible with matrix multiplication (we'll see more about this later in the course).

- But, it saves space on paper to write vectors as comma-separated rows, or by using the matrix-transpose notation (the T stands for **TRANSPOSE**, which changes the rows of a matrix into columns and vice versa):

$$\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = (2,3) = \begin{bmatrix} 2 & 3 \end{bmatrix}^T$$

We can see by the Pythagorean theorem that the **LENGTH** of the vector $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is

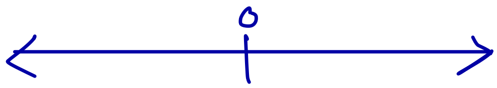
$$\|\vec{v}\| = \sqrt{2^2+3^2} = \sqrt{13}$$

For $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, we have: $\|\vec{x}\| = \sqrt{(x_1)^2 + (x_2)^2}$

$\mathbb{R}^n$

While the arrows we draw work best on **The Plane** (i.e. $\mathbb{R}^2$), or in **3-space** (i.e. $\mathbb{R}^3$), the coordinate system representation of vectors generalizes quite naturally to higher (or lower) dimensions!

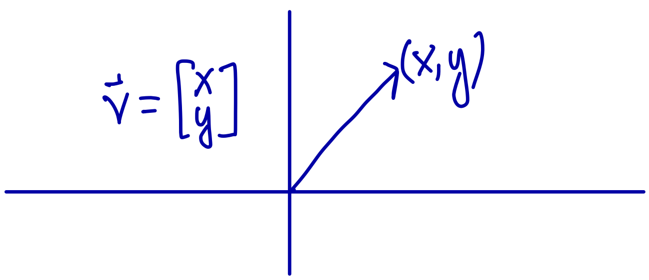
$n = 1$ $\quad$ $\mathbb{R}$ $\quad$ **The Real Line**



$\mathbb{R} = \{x : x \in \mathbb{R}\}$ scalars

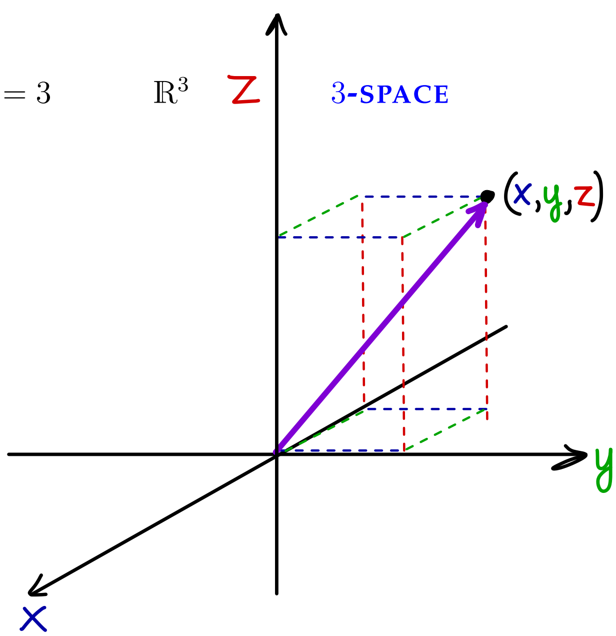
$n = 2$ $\quad$ $\mathbb{R}^2$ $\quad$ **The Plane**

$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$



$$\mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

$n = 3$ $\quad$ $\mathbb{R}^3$ $\quad$ **3-space**



$$\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \mathbb{R}^3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$$

$n = 4$ $\quad$ $\mathbb{R}^4$ $\quad$ **4-space**

$$\mathbb{R}^4 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : x_1, x_2, x_3, x_4 \in \mathbb{R} \right\}$$

$x_i \in \mathbb{R}$ for $1 \le i \le 4$

$\vdots$

$n \in \mathbb{Z}$, $n > 0$ $\quad$ $\mathbb{R}^n$ $\quad$ **n-space**

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_i \in \mathbb{R} \text{ for } 1 \le i \le n \right\}$$

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ be vectors in $\mathbb{R}^n$. Let $k \in \mathbb{R}$ be a scalar.

Note: The **ENTRIES** or **COMPONENTS** of these vectors are scalars, i.e. $u_1, \ldots, u_n, v_1, \ldots, v_n \in \mathbb{R}$. Again, we have two operations: **VECTOR ADDITION** and **SCALAR MULTIPLICATION**.

- We add vectors in $\mathbb{R}^n$ **COMPONENT-WISE**.

$$\underset{\vec{u}}{\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}} + \underset{\vec{v}}{\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}} = \underset{\vec{u}+\vec{v}}{\begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix}}$$

CLOSED UNDER ADDITION: The sum of two vectors in $\mathbb{R}^n$ is again a vector in $\mathbb{R}^n$.

- We scalar-multiply a vector in $\mathbb{R}^n$ **COMPONENT-WISE**.

$$k\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \underset{k\vec{u}}{\begin{bmatrix} ku_1 \\ \vdots \\ ku_n \end{bmatrix}}$$

CLOSED UNDER SCALAR MULTIPLICATION:
A vector in $\mathbb{R}^n$ multiplied by a scalar in $\mathbb{R}$ is again a vector in $\mathbb{R}^n$.

What is the **ZERO VECTOR** in $\mathbb{R}^n$? It should satisfy $\mathbf{0} + \mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$.

$$\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

For a vector $\mathbf{v} \in \mathbb{R}^n$, what is its **NEGATIVE** $-\mathbf{v}$? The negative of $\mathbf{v}$ should satisfy $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} -u_1 \\ \vdots \\ -u_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\vec{u} + (-\vec{u}) = \vec{0}$$

Proposition 1.2. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and let $c, d \in \mathbb{R}$. Then the following properties hold:

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
- $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
- $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
- $c(d\mathbf{u}) = (cd)\mathbf{u}$
- $1\mathbf{u} = \mathbf{u}$

Proof that scalar multiplication distributes over vector addition:

Let $c \in \mathbb{R}$ and let $\vec{u} = (u_1, \ldots, u_n)$ and $\vec{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$

Then

$$
\begin{aligned}
c(\vec{u} + \vec{v}) &= c \cdot \left(\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \right) \\
&= c \cdot \left(\begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix} \right) && \text{def of vector + in } \mathbb{R}^n \\
&= \begin{bmatrix} c(u_1 + v_1) \\ \vdots \\ c(u_n + v_n) \end{bmatrix} && \text{def. of scal. mult. by } c \\
&= \begin{bmatrix} cu_1 + cv_1 \\ \vdots \\ cu_n + cv_n \end{bmatrix} && \text{mult. of real \#s distributes over +} \\
&= \begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix} + \begin{bmatrix} cv_1 \\ \vdots \\ cv_n \end{bmatrix} && \text{def of vector +}
\end{aligned}
$$

EXERCISE: Prove that the other properties hold.

$$= c\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + c\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$ def of scal. mult.

$$= c \cdot \vec{u} + c\vec{v}$$

$\square$

With vector addition and scalar multiplication, what can we accomplish?

More precisely, if we are given m vectors in $\mathbb{R}^n$, say $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m \in \mathbb{R}^n$, what other vectors can we create by scaling and/or adding these vectors together? That is, what vectors $\mathbf{v}$ can be expressed as a **LINEAR COMBINATION** of the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_m$?

Definition 1.3. Let $\mathbf{u}_1, \ldots, \mathbf{u}_m$ be vectors in $\mathbb{R}^n$ and let $k_1, \ldots, k_m$ be scalars in $\mathbb{R}$.

A **LINEAR COMBINATION** of $\mathbf{u}_1, \ldots, \mathbf{u}_m$ is a vector $\mathbf{v}$ of the form

$$k_1\vec{u}_1 + k_2\vec{u}_2 + \ldots + k_m\vec{u}_m$$

Example 1.4. In $\mathbb{R}^2$, is $\mathbf{v} = \underline{\begin{bmatrix} -3 \\ 12.6 \end{bmatrix}}$ a linear combination of the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$?

Can we find scalars k_1, k_2 such that $\vec{v} = k_1\vec{u}_1 + k_2\vec{u}_2$?

yes $\iff \begin{bmatrix} -3 \\ 12.6 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\iff \begin{bmatrix} -3 \\ 12.6 \end{bmatrix} = \begin{bmatrix} k_1 \cdot 1 + k_2 \cdot 2 \\ k_1 \cdot 0 + k_2 \cdot 3 \end{bmatrix} = \begin{bmatrix} k_1 + 2k_2 \\ 3k_2 \end{bmatrix}$

$\iff 3k_2 = 12.6$ and $k_1 + 2k_2 = -3$

$\iff k_2 = 4.2$ and $k_1 = -3 - 2k_2$

$= -3 - 2(4.2)$

$= -3 - 8.4$

$= -11.4$

Check! $-11.4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4.2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 12.6 \end{bmatrix}$

yes $\begin{bmatrix} -3 \\ 12.6 \end{bmatrix}$ is a linear combination of $\vec{u}_1$ and $\vec{u}_2$.

EXERCISE! Show that every vector in $\mathbb{R}^3$ is a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Example 1.5. Is $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ a linear combination of $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$?

$$\vec{w} = c\vec{v}_1 + d\vec{v}_2 \iff \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = c\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$\iff \begin{cases} 1 = 2c \\ 1 = d \\ 1 = c + 2d \end{cases} \iff \begin{cases} c = \frac{1}{2} \\ d = 1 \\ \boxed{1 = \frac{1}{2} + 2(1)} \end{cases} \leftarrow \text{inconsistent}$$

there are no sol= for c, d that express $\vec{w}$ as a linear comb. of $\vec{v}_1, \vec{v}_2$; $\therefore$ NO

DOT PRODUCT, NORM, AND DISTANCE

Recall: for two vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ in $\mathbb{R}^3$, their **DOT PRODUCT** is defined as:

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + x_3y_3$$

We can then view the **LENGTH** or **NORM** of x as follows:

$$\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{{x_1}^2 + {x_2}^2 + {x_3}^2}$$

We then generalize this to $\mathbb{R}^n$.

Definition 1.6. Let $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n)$ be vectors in $\mathbb{R}^n$. Their **DOT PRODUCT** is defined as:

$$\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2 + \ldots + x_ny_n$$

Note $\vec{x} \cdot \vec{y}$ is a scalar
ie $\vec{x} \cdot \vec{y} \in \mathbb{R}$

The **NORM** of x is defined as:

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{(x_1)^2 + \ldots + (x_n)^2}$$

The dot product and norm, as defined above, have some nice properties:

Proposition 1.7. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and let $k \in \mathbb{R}$. Then

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- $\vec{\mathbf{u}} \cdot \vec{\mathbf{0}} = 0$ (a dot product with the zero vector yields $0 \leftarrow$ the number zero!)
- $(k\mathbf{u}) \cdot \mathbf{v} = k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$
- $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
- $\|\mathbf{v}\| \geq 0$
- $\|\vec{\mathbf{v}}\| = 0 \iff \vec{\mathbf{v}} = \vec{\mathbf{0}}$
- $\|k\mathbf{v}\| = |k| \, \|\mathbf{v}\|$

EXERCISE! Verify each of the statements in Prop. 1.7.

Example 1.8. Let $\mathbf{u} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

90° angle

Find $\mathbf{u} \cdot \mathbf{v}$. $= 2 \cdot 2 + 4 \cdot (-1) = 0 \quad (\therefore \vec{u} \perp \vec{v})$

Find $\|\mathbf{v}\|$. $= \sqrt{2^2 + (-1)^2} = \sqrt{5}$

has norm = 1

Find a **UNIT VECTOR** parallel to **v**.

$$\frac{1}{\|\vec{v}\|} \vec{v} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$$

In general, for $\mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{v} \neq \mathbf{0}$, we can create a **UNIT VECTOR** $\hat{\mathbf{v}}$ in the same direction as **v** as follows:

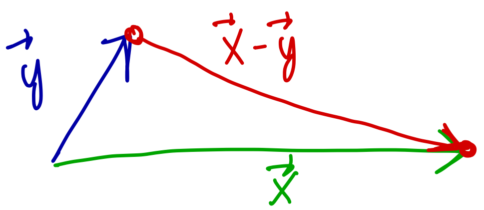
$$\hat{v} = \frac{1}{\|\vec{v}\|} \cdot \vec{v} \qquad (\text{So } \|\hat{v}\| = 1)$$

Although vectors are "portable", using the norm, we define the **DISTANCE** between two vectors as follows:

Definition 1.9. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. The **DISTANCE** between **x** and **y** is defined

$$\|\vec{x} - \vec{y}\| = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}$$

In two dimensions, we see that $\|\mathbf{x} - \mathbf{y}\|$ represents the distance between the tips of the vectors, when they are placed tail-to-tail:



$\|\vec{x} - \vec{y}\|$

EXERCISE! Find the distance between $\underset{\vec{x}}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$ and $\underset{\vec{y}}{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$. Find $\|\vec{x} - \vec{y}\| = \|(2, 2, 2)\|$

$= \sqrt{2^2 + 2^2 + 2^2}$

$= \sqrt{12}$

Not only does the dot product give us a way to compute norms of vectors and distances between vectors, but it also allows us to find **ANGLES** between vectors.

We'll start with one of the nicest angles to consider: $\pi/2 \quad (90°)$

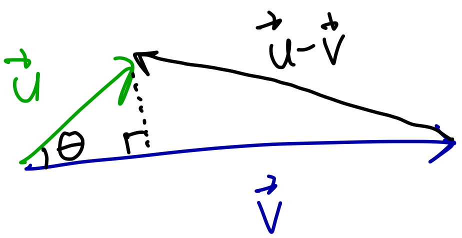
Definition 1.10. Let $\mathbf{u}, \mathbf{v}$ be two vectors in $\mathbb{R}^n$.

We say that $\mathbf{u}$ and $\mathbf{v}$ are **ORTHOGONAL** (or **PERPENDICULAR**) if $\vec{u}\cdot\vec{v} = 0$

Note: if $\mathbf{u} = \mathbf{0}$ or if $\mathbf{v} = \mathbf{0}$, then $\mathbf{u}\cdot\mathbf{v} = 0$.

Thus, the zero vector $\mathbf{0}$ is considered to be orthogonal to all other vectors (even though in the picture, we cannot convey a right angle between a vector and the zero vector).

What about other angles? We can imagine placing two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ tail-to-tail and measuring the angle formed between them as follows:



Cosine Law: $\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\cdot\|\vec{v}\|\cos\theta$

Also: $\|\vec{u}-\vec{v}\|^2 = \left(\sqrt{(u_1-v_1)^2 + \dots + (u_n-v_n)^2}\right)^2$

so $\|\vec{u}-\vec{v}\|^2 = (u_1-v_1)^2 + \dots + (u_n-v_n)^2$

$$\Rightarrow \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\cdot\|\vec{v}\|\cos\theta = (u_1-v_1)^2 + \dots + (u_n-v_n)^2$$

$$\Rightarrow (\cancel{u_1^2} + \dots + \cancel{u_n^2}) + (\cancel{v_1^2} + \dots + \cancel{v_n^2}) - 2\|\vec{u}\|\|\vec{v}\|\cos\theta = (\cancel{u_1^2} - 2u_1v_1 + \cancel{v_1^2}) + \dots + (\cancel{u_n^2} - 2u_nv_n + \cancel{v_n^2})$$

$$\Rightarrow -\cancel{2}\|\vec{u}\|\cdot\|\vec{v}\|\cos\theta = -\cancel{2}u_1v_1 - \cancel{2}u_2v_2 - \dots - \cancel{2}u_nv_n \quad \therefore \|\vec{u}\|\cdot\|\vec{v}\|\cos\theta = \vec{u}\cdot\vec{v}$$

$$= -\cancel{2}(u_1v_1 + u_2v_2 + \dots + u_nv_n)$$

We use these ideas to generalize what we mean by "angle" between two vectors in spaces beyond $\mathbb{R}^2$ and $\mathbb{R}^3$.

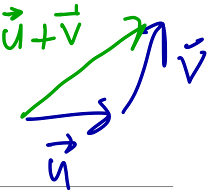
Definition 1.11. Let $\mathbf{u}, \mathbf{v}$ be two nonzero vectors in $\mathbb{R}^n$. The **ANGLE BETWEEN u AND v** is defined to be the number θ such that:

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|} \quad \text{and} \quad 0 \leq \theta \leq \pi$$

In order for the above definition to make sense, we need $\|\mathbf{u}\| \neq 0$, $\|\mathbf{v}\| \neq 0$ and $-1 \leq \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \leq 1$. The norms will both be nonzero since $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$. The bounds on $\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$ follow from the theorem below.

Theorem 1.12. (**CAUCHY–SCHWARZ INEQUALITY**) If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, then $\quad |\mathbf{u}\cdot\mathbf{v}| \leq \|\mathbf{u}\|\cdot\|\mathbf{v}\|$ proof in 2nd year MAT214)

Corollary 1.13. (TRIANGLE INEQUALITY) If If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, then $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$



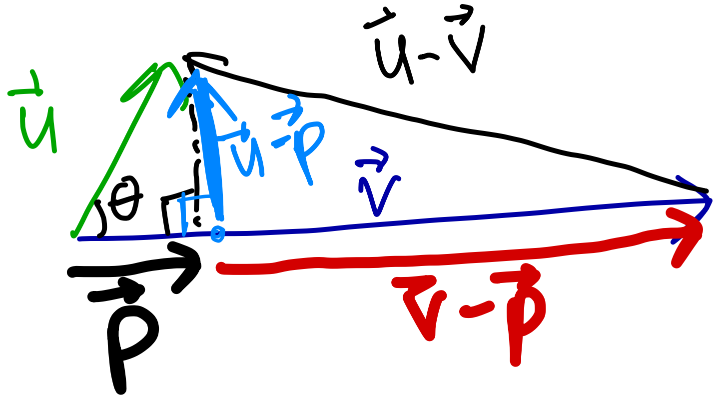
Proof:

Example 1.14. Find the angle between $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$.

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|} = \frac{(1,-1,2)\cdot(0,3,1)}{\sqrt{1^2+1^2+2^2}\,\sqrt{0^2+3^2+1^2}} = \frac{0-3+2}{\sqrt{6}\,\sqrt{10}} = -\frac{1}{\sqrt{60}}$$

$$\Rightarrow \theta = \arccos\left(\frac{-1}{\sqrt{60}}\right) \quad \left(\approx 97^\circ \text{ or } \approx 0.54\pi \text{ rad}\right)$$

The way we defined the angle between two vectors generalizes what we see in triangles in $\mathbb{R}^2$:



$$\cos\theta = \frac{\|\vec{p}\|}{\|\vec{u}\|}$$

In the above illustration, we decomposed the "**v**-edge of the triangle" into the vector **p** and **v** − **p** exactly so that **p** and **u** − **p** were orthogonal (in the picture, we dropped a perpendicular to show the height of the triangle, making a $90°$ angle with the base).

Now, with our definition for angles between vectors in $\mathbb{R}^n$, we want to generalize this idea, as follows:

Question: Given nonzero vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$, how do we find $\vec{p}$ so that

- $\vec{p}$ is parallel to $\vec{v}$
- $\vec{p}$ is orthogonal to $\vec{u}-\vec{p}$

Answer: the Projection of $\vec{u}$ onto $\vec{v}$

Definition 1.15. Let **u** and **v** be nonzero vectors in $\mathbb{R}^n$. Then the **PROJECTION OF u ONTO v**, denoted $\mathrm{proj}_{\mathbf{v}}(\mathbf{u})$, is the unique vector which satisfies:

- $\mathrm{proj}_{\mathbf{v}}(\mathbf{u})$ is parallel to **v** (i.e. $\mathrm{proj}_{\mathbf{v}}(\mathbf{u})$ is a scalar multiple of **v**), and
- $\mathbf{u} - \mathrm{proj}_{\mathbf{v}}(\mathbf{u})$ is orthogonal to **v** (i.e. $\mathbf{v} \cdot (\mathbf{u} - \mathrm{proj}_{\mathbf{v}}(\mathbf{u})) = 0$)

Proposition 1.16. Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in $\mathbb{R}^n$. Then

$$\text{proj}_{\mathbf{v}}(\mathbf{u}) = k \cdot \mathbf{v} \quad \text{where} \quad k = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}$$

That is $\boxed{\text{proj}_{\mathbf{v}}(\mathbf{u}) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}}$

Proof:

Example 1.17. Let $\mathbf{u} = (1, -3)$ and $\mathbf{v} = (2, 1)$. Find $\text{proj}_{\mathbf{v}}(\mathbf{u})$ and find $\text{proj}_{\mathbf{u}}(\mathbf{v})$.

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \frac{(1,-3)\cdot(2,1)}{(2,1)\cdot(2,1)}\begin{bmatrix}2\\1\end{bmatrix} = \frac{-1}{5}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}-2/5\\-1/5\end{bmatrix}$$