7. Differentiation: Trig Rules and Chain Rule

Lec 6 mini review.

Constant Multiple Rule: for any $k \in \mathbb{R}$, $\boxed{\frac{d}{dx}[kf(x)] = kf'(x)}$

Sum/Difference Rule: $\boxed{\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)}$

Constant Rule: for any $c \in \mathbb{R}$, $\boxed{\frac{d}{dx}[c] = 0}$

Power Rule: for any $n \in \mathbb{R}$, $\boxed{\frac{d}{dx}[x^n] = nx^{n-1}}$

Derivative of e^x $\boxed{\frac{d}{dx}[e^x] = e^x}$

Product Rule: $\boxed{\frac{d}{dx}[fg] = f'g + fg'}$

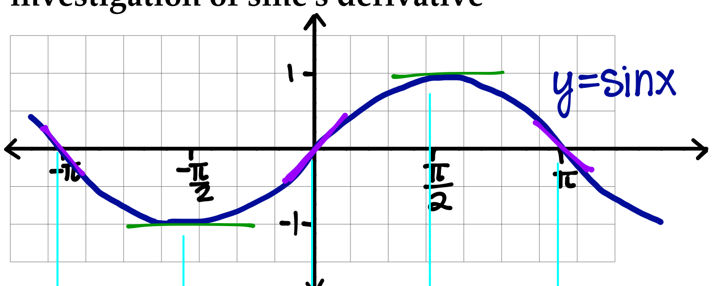
Quotient Rule: $\boxed{\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{f'g - fg'}{g^2}}$

TRIG RULES

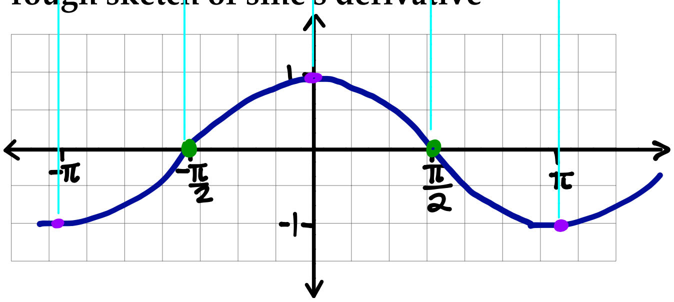
a few helpful trig identities to keep in mind:

- $\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$
- $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$
- $\cos^2(A) + \sin^2(A) = 1$
- $\tan(A) = \frac{\sin(A)}{\cos(A)}$ $\quad \cot(A) = \frac{\cos(A)}{\sin(A)}$ $\quad \csc(A) = \frac{1}{\sin(A)}$ $\quad \sec(A) = \frac{1}{\cos(A)}$

investigation of sine's derivative



rough sketch of sine's derivative



By measuring/eyeballing slopes at various points on the graph of y=sinx we get a rough idea of the shape of its derivative...

...which looks like cosx!

$$\frac{d}{dx}\left[\sin(x)\right] = \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

addition trig identity
$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$

$$= \lim_{h\to 0} \cos(x)\left(\frac{\sin(h)}{h}\right) + \sin(x)\left(\frac{\cos(h)-1}{h}\right)$$

$$= \cos(x)\left(\lim_{h\to 0}\left(\frac{\sin(h)}{h}\right)\right) + \sin(x)\left(\lim_{h\to 0}\left(\frac{\cos(h)-1}{h}\right)\right)$$

Two important limits lurking within the derivative of $\sin(x)$**:** $\boxed{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}}$ and $\boxed{\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta}}$

θ	$\frac{\sin\theta}{\theta}$	$\frac{\cos\theta-1}{\theta}$
−0.1	0.9983416...	0.049958...
−0.01	0.9999833...	0.00499...
−0.001	0.99999983...	0.00049...
0.001	0.99999983...	−0.00049...
0.01	0.9999833...	−0.00499...
0.1	0.9983416...	−0.049958...

0

numerical guess:

$\downarrow$? ? 1 ? ?

$\downarrow$? ? 0 ? ?

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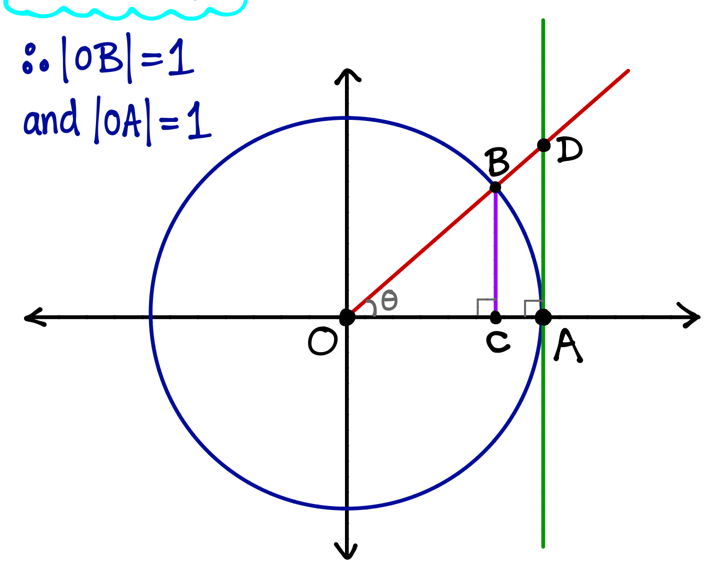
setup for a geometric proof that $\boxed{\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1}$

unit circle radius is 1

$\therefore |OB| = 1$

and $|OA| = 1$



2 trig ratios for angle θ:

$$\sin\theta = \frac{|BC|}{1} = |BC|$$

$$\cos\theta = \frac{|OC|}{1} = |OC|$$

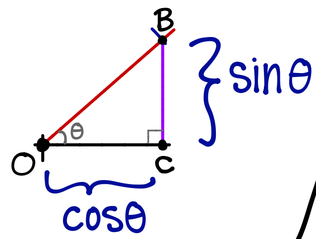
$\triangle BOC$ is similar to $\triangle DOA$

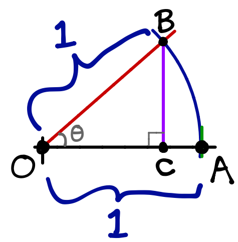
$$\therefore \frac{|BC|}{|OC|} = \frac{|DA|}{|OA|}$$

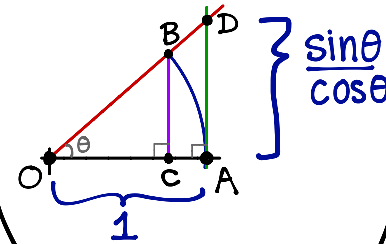
$$\Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{|DA|}{1} = |DA|$$

Geometric proof that $\boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1}$

$$\left(\text{area of } \triangle BOC\right) \leq \left(\text{area of sector } OAB\right) \leq \left(\text{area of } \triangle DOA\right)$$







area of triangle: $\frac{1}{2}$(base)(height) | area of sector of a circle: $\frac{(\text{angle of sector})}{2\pi}$(area of entire circle) | area of triangle: $\frac{1}{2}$(base)(height)

$$\Rightarrow \quad \frac{1}{2}\cos\theta\sin\theta \quad \leq \quad \frac{\theta}{2\pi}\left[\pi(1^2)\right] \quad \leq \quad \frac{1}{2}(1)\left(\frac{\sin\theta}{\cos\theta}\right)$$

$$\Rightarrow \quad \cos\theta\sin\theta \quad \leq \quad \theta \quad \leq \quad \frac{\sin\theta}{\cos\theta}$$

• For now, we will assume θ is a tiny positive angle, approaching zero $\therefore \sin\theta > 0$

(for tiny $\theta < 0$, the proof is similar but $\sin\theta$ will be negative)

$$\Rightarrow \quad \cos\theta \leq \frac{\theta}{\sin\theta} \leq \frac{1}{\cos\theta}$$

(all terms divided by $\sin\theta$ which is > 0 when $\theta \to 0^+$)

Squeeze Theorem!

$$\Rightarrow \lim_{\theta\to0^+} \cos\theta \leq \lim_{\theta\to0^+} \frac{\theta}{\sin\theta} \leq \lim_{\theta\to0^+} \frac{1}{\cos\theta}$$

$$\Rightarrow \quad 1 \quad \leq \lim_{\theta\to0^+} \frac{\theta}{\sin\theta} \leq \quad 1$$

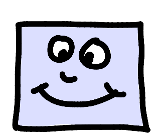
$$\boxed{\therefore \lim_{\theta\to0^+} \frac{\theta}{\sin\theta} = 1}$$

Finally,

$$\lim_{\theta\to0^+} \frac{\sin\theta}{\theta} = \lim_{\theta\to0^+} \frac{1}{\frac{\theta}{\sin\theta}} = \frac{\lim_{\theta\to0^+} 1}{\lim_{\theta\to0^+} \frac{\theta}{\sin\theta}} = \frac{1}{1} = 1 \; !$$

$$\boxed{\therefore \lim_{\theta\to0^+} \frac{\sin\theta}{\theta} = 1}$$

(and similarly, the left-side limit = 1)



Proof that $\boxed{\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}=0}$ **using** $\boxed{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1}$

$$\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}=\lim_{\theta\to 0}\left(\frac{\cos\theta-1}{\theta}\right)\left(\frac{\cos\theta+1}{\cos\theta+1}\right)$$

using difference of squares

$$=\lim_{\theta\to 0}\frac{\cos^2\theta-1}{\theta(\cos\theta+1)}$$

$$=\lim_{\theta\to 0}\frac{-\sin^2\theta}{\theta(\cos\theta+1)}$$

using $\cos^2\theta+\sin^2\theta=1$

$$=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\frac{(-\sin\theta)}{\cos\theta+1}$$

$$=\left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right)\cdot\left(\lim_{\theta\to 0}\frac{-\sin\theta}{\cos\theta+1}\right)$$

$$=(1)\cdot\left(\frac{0}{1+1}\right)$$

$$=0$$

Now that we know those two special limits, we return to sine's derivative:

$$\frac{d}{dx}[\sin(x)]=\cos(x)\left(\lim_{h\to 0}\left(\frac{\sin(h)}{h}\right)\right)+\sin(x)\left(\lim_{h\to 0}\left(\frac{\cos(h)-1}{h}\right)\right)$$

$$=\cos(x)\cdot(1)+\sin(x)\cdot(0)$$

$$=\cos(x)$$

Derivative of Sine

$$\boxed{\frac{d}{dx}[\sin(x)]=\cos(x)}$$

Derivative of Cosine

$$\frac{d}{dx}[\cos(x)] = \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h}$$

addition trig identity
$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

$$= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h}$$

$$= \lim_{h\to 0} -\sin(x)\left(\frac{\sin(h)}{h}\right) + \cos(x)\left(\frac{\cos(h)-1}{h}\right)$$

← same two special limits appear!

$$= -\sin(x)\cdot(1) + \cos(x)\cdot(0)$$

$$= -\sin(x)$$

$$\boxed{\frac{d}{dx}[\cos(x)] = -\sin(x)}$$

Derivative of Tangent

$$\frac{d}{dx}[\tan(x)] = \frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right]$$

$$= \frac{\frac{d}{dx}[\sin(x)]\cos(x) - \sin(x)\cdot\frac{d}{dx}[\cos(x)]}{[\cos(x)]^2}$$

quotient rule!

$$= \frac{\cos(x)\cdot\cos(x) - \sin(x)\cdot(-\sin(x))}{\cos^2(x)}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

Pythagorean trig identity
$\cos^2(A) + \sin^2(A) = 1$

$$= \frac{1}{\cos^2(x)}$$

$$= \left(\frac{1}{\cos(x)}\right)^2$$

$$= \sec^2(x)$$

$$\boxed{\frac{d}{dx}[\tan(x)] = \sec^2(x)}$$

Derivative of Cosecant

$$\frac{d}{dx}[\csc(x)] = \frac{d}{dx}\left[\frac{1}{\sin(x)}\right]$$

$$= \frac{(0)\cdot\sin(x) - (1)\cdot\cos(x)}{\sin^2(x)}$$

$$= \frac{-\cos(x)}{\sin^2(x)}$$

$$= -\left(\frac{1}{\sin(x)}\right)\cdot\left(\frac{\cos(x)}{\sin(x)}\right)$$

$$= -\csc(x)\cdot\cot(x)$$

$$\boxed{\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)}$$

Derivative of Secant (quite similar to $\frac{d}{dx}[\csc(x)]$)

$$\boxed{\frac{d}{dx}[\sec(x)] = \sec(x)\cdot\tan(x)}$$

Derivative of Cotangent (quite similar to $\frac{d}{dx}[\tan(x)]$)

$$\boxed{\frac{d}{dx}[\cot(x)] = -\csc^2(x)}$$

Exercise 7.1. Find a formula for the n-th derivative of $\sin(x)$.

Start by computing $\frac{d}{dx}[\sin x]$, $\frac{d^2}{dx^2}[\sin x]$, $\frac{d^3}{dx^3}[\sin x]$,... see if you can spot the pattern.

THE CHAIN RULE

If g is differentiable at x and f is differentiable at $g(x)$, then $f \circ g(x)$ is differentiable at x and

$$\boxed{\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)}$$

Recall that $f \circ g(x) = f(g(x))$

Alternative setup for the Chain Rule:

Let $u = g(x)$, $y = f(u)$. Then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Some special examples of the Chain Rule:

Power Chain Rule:

for $n \in \mathbb{R}$,

$$\boxed{\frac{d}{dx}[(g(x))^n] = n[g(x)]^{n-1} \cdot g'(x)}$$

Example 7.2. Differentiate: $y = \sqrt{x^3 + 4e^x - \cos(x)}$

$$y = (x^3 + 4e^x - \cos(x))^{1/2}$$

$$y' = \frac{1}{2}(x^3 + 4e^x - \cos(x))^{-1/2} \cdot (3x^2 + 4e^x - (-\sin(x)))$$

$$= \frac{3x^2 + 4e^x + \sin(x)}{2\sqrt{x^3 + 4e^x - \cos(x)}}$$

Alternatively:

$u = g(x) = x^3 + 4e^x - \cos(x)$

$y = f(u) = u^{1/2}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{2}u^{-1/2} \cdot (3x^2 + 4e^x + \sin(x))$$

$$= \frac{3x^2 + 4e^x + \sin(x)}{2\sqrt{\underbrace{x^3 + 4e^x - \cos(x)}_{u}}}$$

Exponential Chain Rules:

$$\boxed{\frac{d}{dx}\left[e^{g(x)}\right] = e^{g(x)} \cdot g'(x)}$$

In particular, for a constant $k \in \mathbb{R}$, $\boxed{\frac{d}{dx}\left[e^{kx}\right] = e^{kx} \cdot (k) = k \cdot e^{kx}}$

For other bases $b>0, b \neq 1$,

$$\frac{d}{dx}\left[b^x\right] = \frac{d}{dx}\left[e^{\ln(b^x)}\right] = \frac{d}{dx}\left[e^{(\ln b)x}\right] = (\ln b) \cdot e^{(\ln b)x} = (\ln b) \cdot b^x$$

Example 7.3. Differentiate: $g(s) = -e^{s^3-s}$

$$g'(s) = -\left(e^{s^3-s}\right)\left(3s^2-1\right)$$

$$= e^{s^3-s} - 3s^2 e^{s^3-s}$$

$f(x) = e^{5x} + 3^x + 2^4$

$$f'(x) = \left(e^{5x}\right)(5) + (\ln 3) \cdot 3^x + 0$$

$$= 5e^x + (\ln 3)3^x$$

Trig Chain Rules:

$$\boxed{\frac{d}{dx}\left[\sin(g(x))\right] = \cos(g(x)) \cdot g'(x)}$$

$$\boxed{\frac{d}{dx}\left[\cos(g(x))\right] = -\sin(g(x)) \cdot g'(x)}$$

$$\boxed{\frac{d}{dx}\left[\tan(g(x))\right] = \sec^2(g(x)) \cdot g'(x)}$$

$$\boxed{\frac{d}{dx}\left[\sec(g(x))\right] = \sec(g(x)) \cdot \tan(g(x)) \cdot g'(x)}$$

$$\boxed{\frac{d}{dx}\left[\csc(g(x))\right] = -\csc(g(x)) \cdot \cot(g(x)) \cdot g'(x)}$$

$$\boxed{\frac{d}{dx}\left[\cot(g(x))\right] = -\csc^2(g(x)) \cdot g'(x)}$$

Example 7.4. Differentiate: $y = \sin\left(\frac{\pi}{x}\right)$ $= \sin(\pi x^{-1})$

$$\Rightarrow y' = \cos\left(\frac{\pi}{x}\right)\cdot(\pi)(-x^{-2})$$
$$= -\frac{\pi}{x^2}\cos\left(\frac{\pi}{x}\right)$$

Example 7.5. Differentiate: $f(\theta) = \left(\tan(5\theta+3) - \frac{\theta}{2}\right)^{100}$

$$f'(\theta) = 100\left(\tan(5\theta+3) - \frac{\theta}{2}\right)^{99}\cdot\frac{d}{d\theta}\Big[\overbrace{\tan(5\theta+3)}^{\text{derivative of this also uses Chain Rule}} - \frac{\theta}{2}\Big]$$
$$= 100\left(\tan(5\theta+3) - \frac{\theta}{2}\right)^{99}\cdot\left(\sec^2(5\theta+3)\cdot(5) - \frac{1}{2}\right)$$

Example 7.6. Differentiate: $y = \cos^2\theta$ $= (\cos\theta)^2$

$$\frac{dy}{d\theta} = 2(\cos\theta)^1\cdot\frac{d}{d\theta}[\cos\theta]$$
$$= 2\cos\theta\cdot(-\sin\theta)$$
$$= -2\cos\theta\sin\theta$$

Example 7.7. Find the slope of the tangent line to the graph of $y = e^{\sin x}$ at $x = 0$.

$y'(x) = (e^{\sin x})(\cos x)$

$$\Rightarrow y'(0) = (e^{\sin 0})(\cos 0)$$
$$= (e^0)(1)$$
$$= 1$$

$\therefore$ the slope of the tangent line at $x=0$ is $y'(0) = 1$

Study Guide

- **two special limits:** $\boxed{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}}$ $\boxed{\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta}}$
- **derivative rules for trig functions**:
  $\boxed{\frac{d}{dx}[\sin x] = \cos x}$ $\boxed{\frac{d}{dx}[\cos x] = -\sin x}$ $\boxed{\frac{d}{dx}[\tan x] = \sec^2 x}$ (and others!)
- **the chain rule:** $\boxed{\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)}$ (and more special cases of the chain rule!)