Functions and Graphs

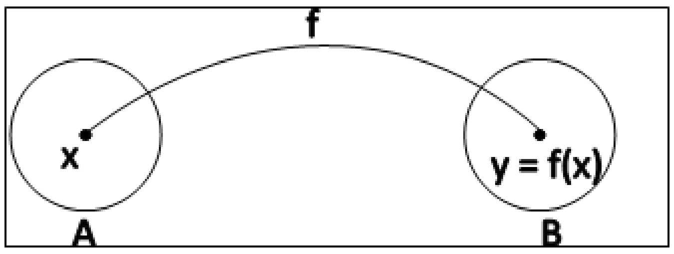
Relation between Sets

- A relation R from A to B is a subset of the Cartesian product A x B.
- The domain of a relation R is the set of all the first members of the pairs (x, y) of R
- The range of a relation R is the set of all the second members of the pairs (x, y) of R.

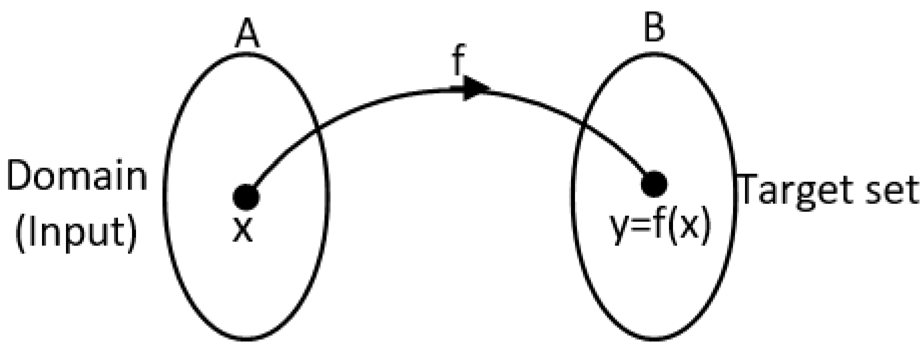
Function

Let A and B be any two non-empty sets and f is a relation from A to B. Then, the relation f is called a function from A to B if f assigns each element of A to a unique element in B.

If f is a function from A to B, then we write f: A $\rightarrow$ B



The set A is called the domain of the function f, and the set B is called the co-domain of f. If the element x of A corresponds to y ($\in$B) under the function f, then we say that y is the image of x under f and x is a pre-image of y under f and we write f(x) = y.



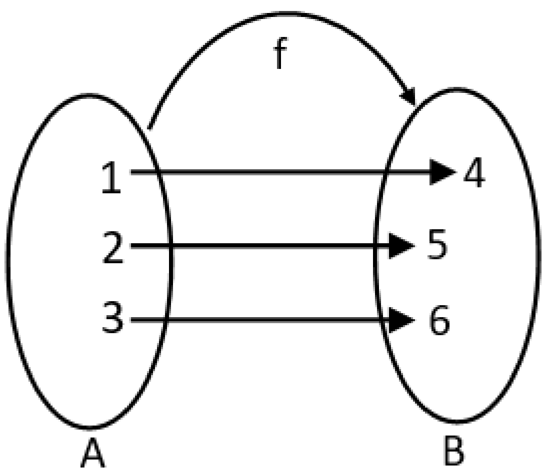
The subset of B containing the images of the elements of A is called the range of the function. The range of f is denoted by f(A).

In symbol, we write f(A) = {f(x) : x $\in$ A}.

If f : A $\rightarrow$ B is a function, then the subset {(x, f(x)):x $\in$ A} of A x B is called the graph of function f.

Example

(i)



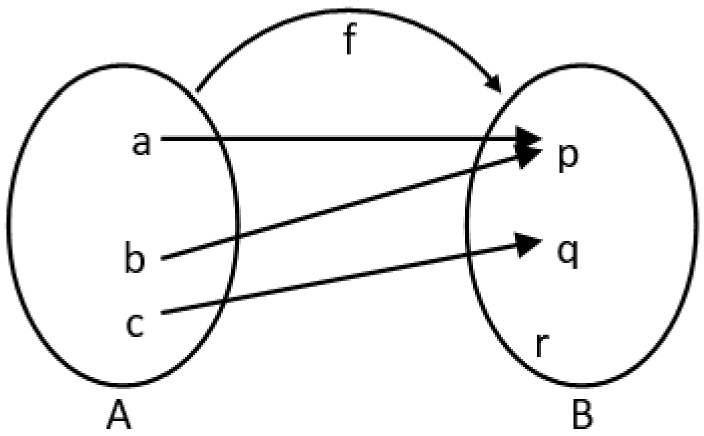
f(1) = 4

f(2) = 5

f(3) = 6

Each element in A has unique image in B. So, f is a function.

(ii)



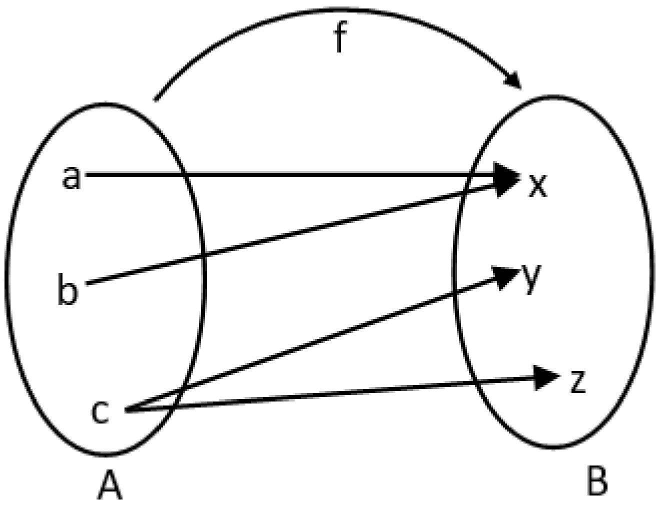
f(a) = p

f(b) = p

f(c) = q

Every element of A has unique image in B. So, f is **a function**.

(iii)



f is not a function as the element c in A is associated to two different elements y and z of B.

Note:

(i) The sine, consine and tangents of the angles are the functions of an angle.

i.e $f(\theta) = \sin\theta$

$f(\theta) = \cos\theta$

$f(\theta) = \tan\theta$ etc.

(ii) The area of a square is a function of its side i.e. $f(x) = x^2$

(iii) The economy of a country is a function of the number of good industries in the country.

(iv) The temperature of a region recorded in a day time is shown below.

Time(T)	5 AM	9 AM	1 PM	3 PM	5 PM	9 PM
Temp. (θ)	20°C	22°C	23°C	21°C	21°C	18°C

Set of all ordered pairs (T, θ) for different values of T is the example of a function.

Equal functions

Let f and g be two functions from the set A to the set B. The functions f and g are said to be equal functions if $f(x) = g(x)$ for all $x \in A$.

i.e.

(i) domain of f = domain of g

$D_f = D_g$

(ii) $f(x) = g(x)$ for all $x \in D_f$.

Domain and Range of Trigonometric functions

functions	Domain	Range

sinx	R	[-1,1]
cosx	R	[-1,1]
tanx	R-(2n+1)$\frac{\pi}{2}$,n $\in$ I	R
cotx	R-nπ, n$\in$I	R
secx	R-(2n+1)$\frac{\pi}{2}$,n $\in$ I	R-(-1,1)
cosecx	R-nπ , n $\in$ I	R-(-1,1)

Domain and Range of Inverse Trigonometric functions

Inverse circular functions	Domain	Range
y = $\sin^{-1}x$	[-1, 1]	[$-\pi/2, \pi/2$]
y = $\cos^{-1}x$	[-1,1]	[0, π]
y = $\tan^{-1}x$	($-\infty,\infty$)	($-\pi/2, \pi/2$)
y = $\cot^{-1}x$	R	(0, π)
y = $\text{cosec}^{-1}x$	R - (-1, 1)	[$-\pi/2, \pi/2$] - {0}
y = $\sec^{-1}x$	R - (-1, 1)	[0, π] - {$\pi/2$}

Note:

The number of functions from a finite set A into a finite set B $(n(B))^{n(A)}$

Types of Function

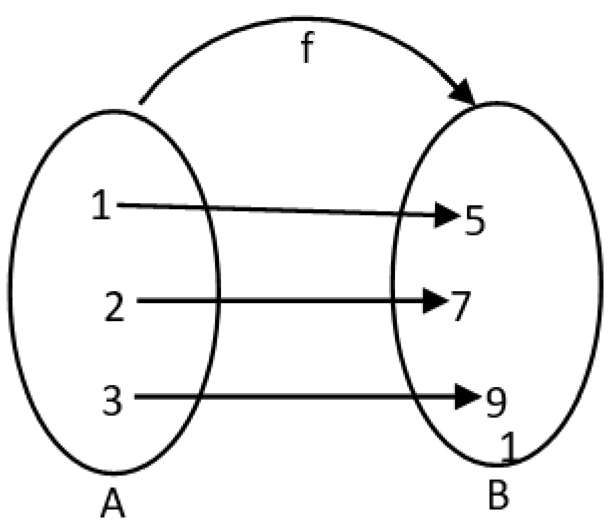
One to one function or injective mapping

A function f: A$\rightarrow$ B is said to be one to one function if different elements in A have different images in B.

In symbol, for $x_1, x_2 \in A$

$x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2)$

or, $f(x_1) = f(x_2) \quad \Rightarrow x_1 = x_2$

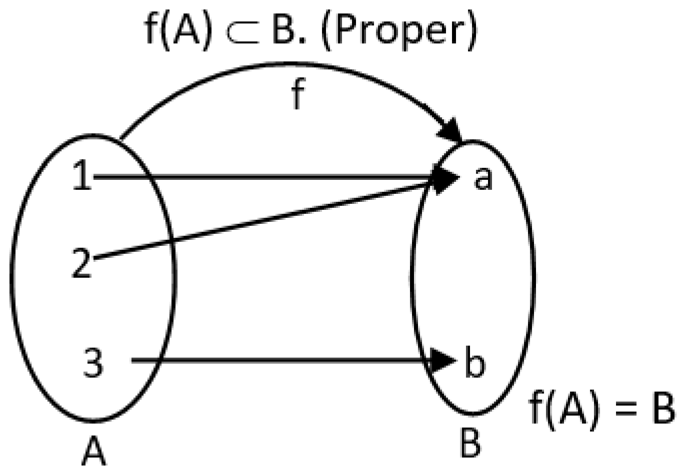


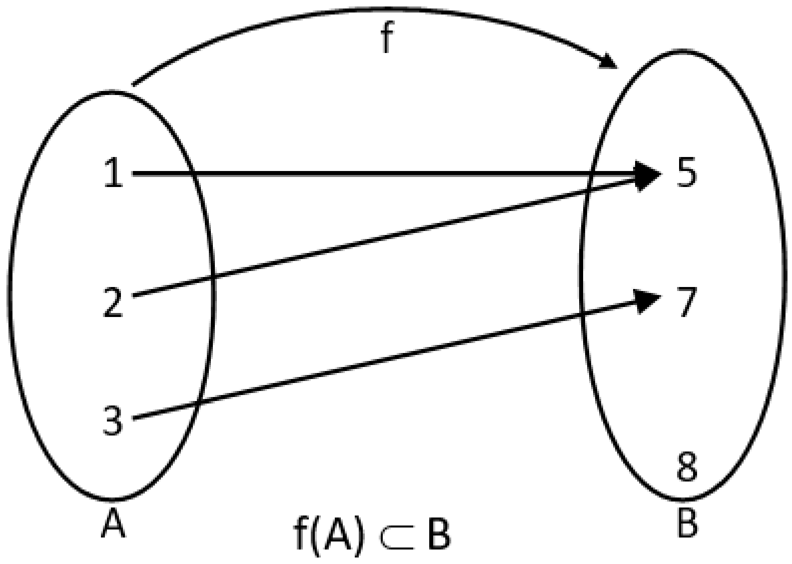
Onto – function or surjective function

A function f: A$\rightarrow$ B is said to be an onto function if every element in B has at least one pre-image in A, otherwise it is said to be into.

In this case, f(A) = B. For into functions:

$f(A) \subset B$. (Proper)





Examples:

(i) A constant function with singleton co-domain is an onto function.

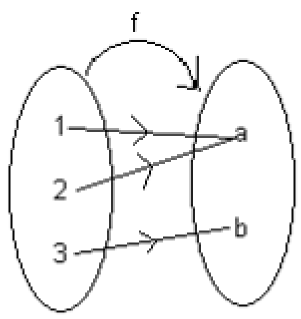
(ii) Every polynomial function $f:R \rightarrow R$ of degree odd is onto.

Bijective function

A function which is both one to one and onto i.e. both injective and surjective is known as a **bijective function.**

Many to one function

A function $f: A \rightarrow B$ is said to be many to one function if at least one element of B has more than one pre-images in A



Example:

A function $f: Z \rightarrow Z$ defined by $f(x) = |x|$ is many to one function.

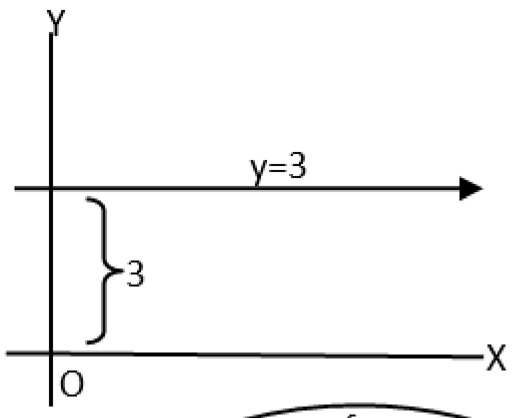
Constant function

A function $f : A \rightarrow B$ is a constant function if the range of the function is a **singleton set**. A function $f: R \rightarrow R$ defined by $f(x) = c$, for each $x \in R$ is a constant function;

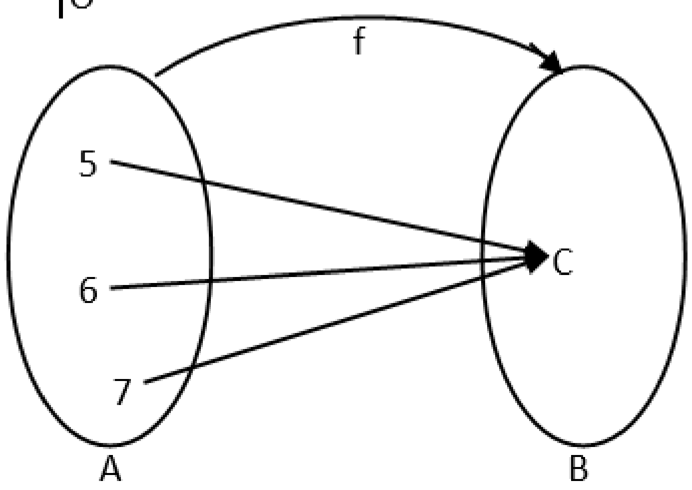
Example:

$f = \{(x, y) / x$ is a real number and $y = 3\}$

(i)

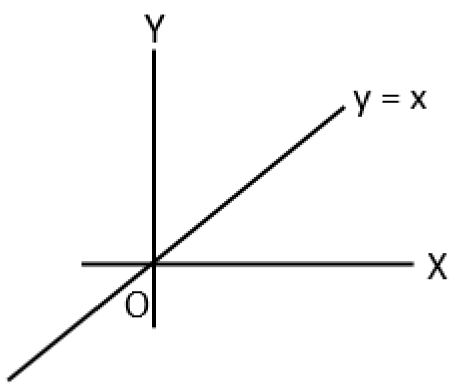


(ii)



Identity function

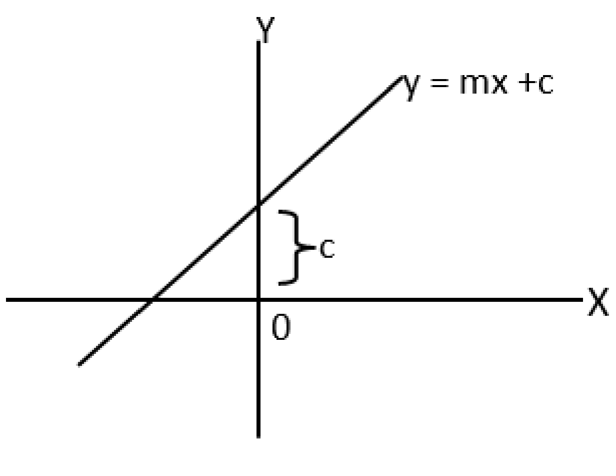
Let A be any set A function f: A$\rightarrow$ A is said to be identity function if y = f(x) = x for all x $\in$ A



Linear function

A function f :R$\rightarrow$R defined by f(x)=ax+b, a. b $\in$R where a and b are constants and a $\neq$0 is called linear function.

In other words, let A and B be any two sets, then a function f: A$\rightarrow$B defined by, y = f(x) = mx + c for x$\in$A where m and c are constants and m $\neq$ 0 is called linear function:

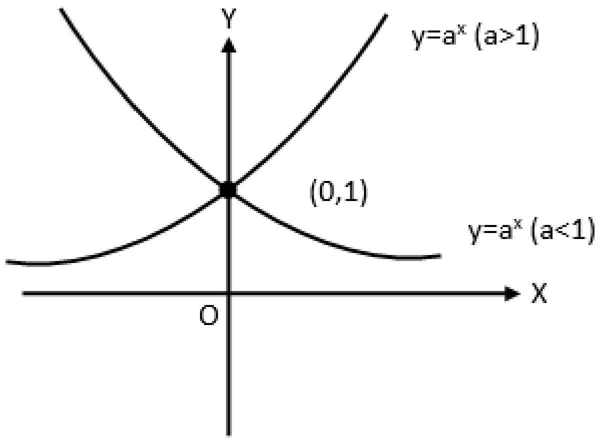


Exponential function

For every real number a > 0, (a $\neq$ 1) the exponential function f with base a is defined by;

y = f(x) = a^x, x$\in$R

e^x is a typical exponential function



where e = $\lim_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$

= $\lim_{n\to\infty}(1+h)^{\dfrac{1}{h}}$

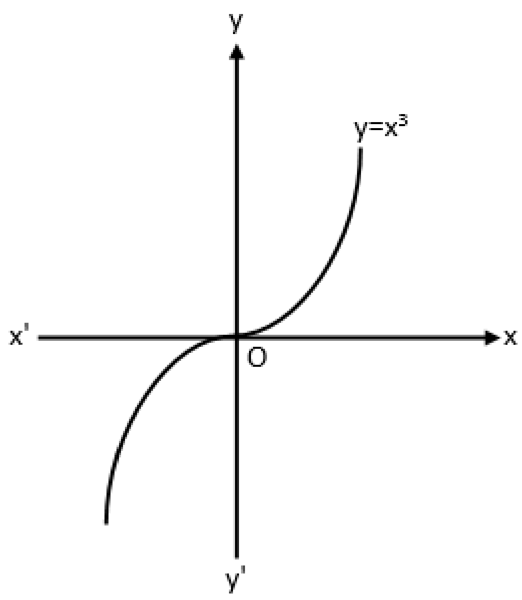
Quadratic function

A function f: A$\rightarrow$B is said to be quadratic if y = f(x) = ax^2 + bx +c (a≠0) for x $\in$ A and

a, b, c are constants.

Cubic function

A function f:R $\rightarrow$R defined by f(x) = ax^3 + bx^2 + cx + d where a, b, c, d are constants and a $\neq$0 is called a cubic function.



Polynomial function

A function f : A $\rightarrow$ B defined by

y = f(x) = a_0x^n +a_1 x^{n-1} + ---------- +a_n ($a_0 \neq 0$) and n is non-negative integer.

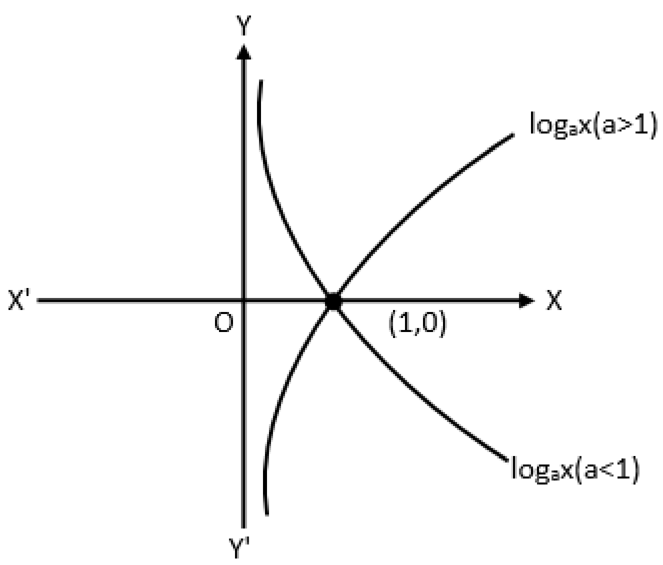
is called polynomial function of degree n, where a_0, a_1, a_2, ...a_n are constants.

Logarithmic function

The inverse of an exponential function is called a logarithmic function.

If y = a^x, a $\neq$ 1, a > 0 then its inverse is x = a^y

Or, $y=\log_a x$



Absolute value function

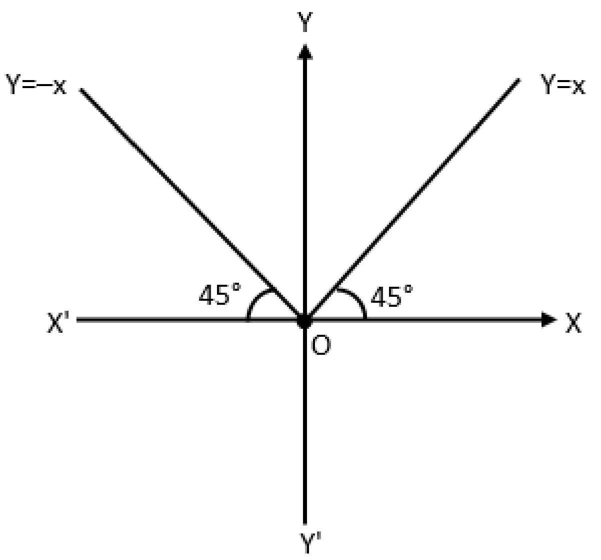
A function f(x) defined by f(x) = |x| is said to be an absolute value function and written as,

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & x < 0 \end{cases}$$

Domain (D_f) = R = $(-\infty,\infty)$

Range (R_f) = set of all non-negative real number s

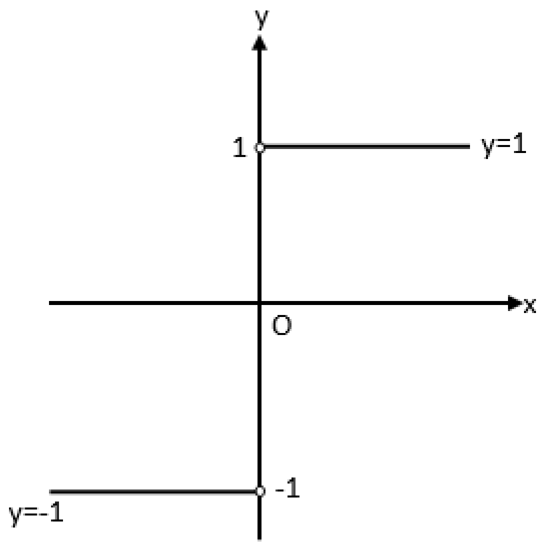
= $[0, \infty)$



Signum function

A function f: R $\rightarrow$ R is called signum function if

$$f(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

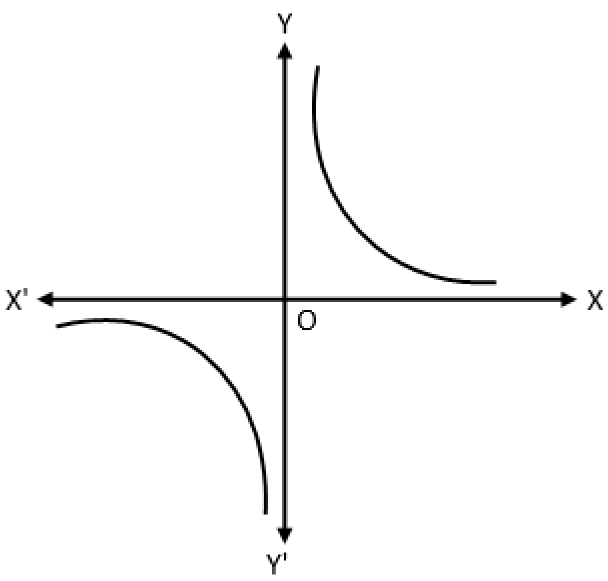


Domain of f(D_f) = R i.e. $(-\infty,\infty)$

Range of f = {1, -1, 0}

Reciprocal function

A function y = f(x) = , x $\neq$ 0 is called reciprocal function.



Domain of f (D_f) = R - {0}

Range of f(R_f) = R - {0}

Note: y = $\frac{1}{x}$ is a rectangular hyperbola xy = 1

Greatest integer function i.e. [x]

A function f: R → R defined by f(x) = [x] is the greatest integer function less or equal to x.

Greatest integer function [x] means the greatest integer not exceeding x.

Inverse function

If function f: A → B bijective, then there exists a function f^{-1}: B →A is called inverse function of. Then, $f_o f^{-1}$ is identity function on B and $f^{-1} f_o$ is identity function on A.

Inverse function of f exists if the function is one to one and onto.

Even and Odd functions

A function f : **R** → **R** is said to be (i) even if f(–x) = f(x) andn (ii) odd if f(–x) = –f(x) for all x ∈ **R**.

e.g. x^2, cosx, x^2 + |x| - $\sin^2$x etc. are even functions and x^3 - 3x, tanx + $\sin^3$x etc. are odd functions, whereas x^2 + x is neither even nor odd.

Composite function

Let f: A→B and g : B→C , then the composite of f and g denoted by gof is a function from A to C.

i.e. (gof) : A →C, defined by (gof) (x) = g(f(x))

for all x ∈ A.

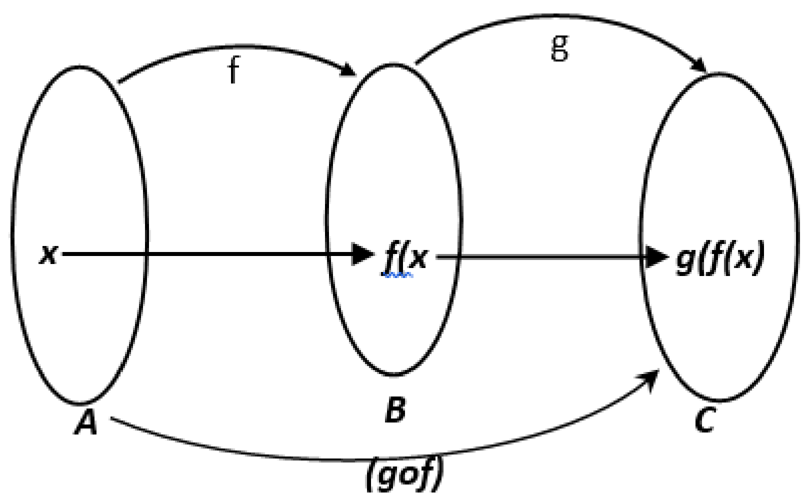


Table for Domain and Range

Function f(x)	Domain	Range
1. y = sinx	R	[-1, 1]
2. y = cosx	R	[-1, 1]
3. y = tanx	R – (2n +1)$\frac{\pi}{2}$	R
4. y = cotx	R - np	R
5. y = secx	R – (2n +1)$\frac{\pi}{2}$	[-∞,-1]∪[1,∞)
6. y = cosecx	R - np	(-∞,-1]∪[1,∞)
7. y = c	R	{c}

8. $y = x^2$	R	$[0, \infty)$
9. $y = \frac{1}{x^2}$	R – {0}	$(0, \infty)$
10. $y = \sqrt{a^2 - x^2}, a > 0$	[-a, a]	[0, a]
11. $y = \frac{1}{\sqrt{a^2 - x^2}}$	(-a, a)	$[1/a, \infty)$
12. y = |x|	R	$[0, \infty)$
13. $y = e^x$	R	$(0, \infty)$
14. $y = a^x$	R	$(0, \infty)$, $a \neq 1$
15. y = logx	$(0, \infty)$	R
16. y = [x]	R	I

Algebra of Real valued Functions

Let f: R $\rightarrow$ R and g : R $\rightarrow$ R be two real valued functions and k $\in$ R. Then,

(i) (f ± g) (x) = f(x) ± g(x)

(ii) (f + k) (x) = f(x) + k

(iii) (fg) (x) = f(x) g(x)

(iv) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$; $g(x) \neq 0$

(v) (|f|) (x) = |f(x)|

E.e. if f : R $\rightarrow$ R and g : R $\rightarrow$ R are given by f(x) = x + 3 and g(x) = x^2, then (i) (fg + 3)(x) = f(x) g(x) + 3 = (x + 3)x^2 + 3 = $x^3 + 3x^2 + 3$.

Logarithms

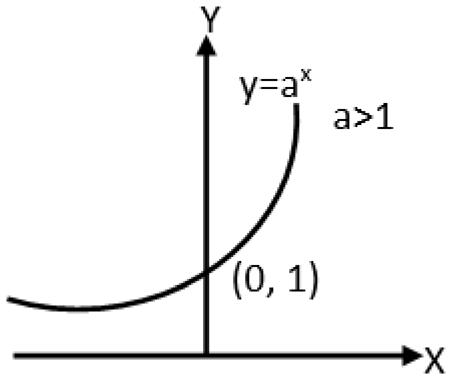
- Functions which are not algebraic are transcendental functions viz. exponential, logarithmic, trigonometric, inverse trigonometric functions.

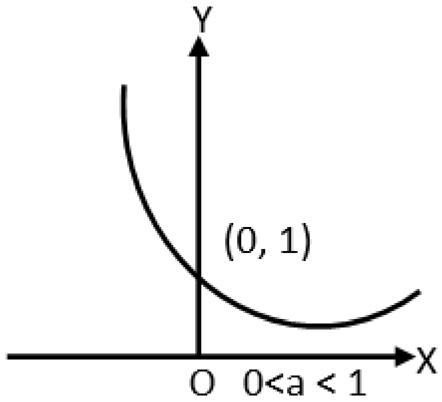
Exponential function

For every real number a >0, $a \neq 1$ the exponential function f with base a is defined by y = f(x) = a^x, x $\in$ **R**. It is bijective.

e^x is a typical exponential function,

where e = $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

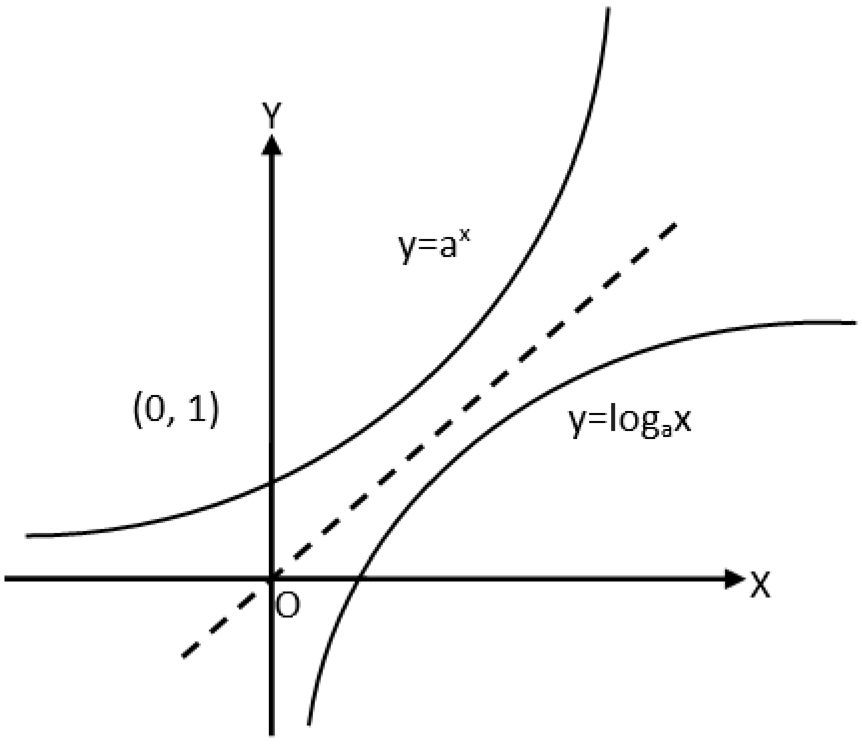




y = f(x) > 0, Domain = R, Range = (0, ∞)

Logarithmic function

Let y = a^x (a > 0, $a \neq 1$) be an exponential function. Then its inverse x = a^y is the logarthmic function to the base a. It is also written as y = $\log_a x$.



Common logarithms and natural logarithms

Logarithms to the base 10 are called common logarithms. $\log_{10}x$ is called common logarithm.

Logarithm to the base e is called **natural logarithm**. It is denoted by ln (x) or $\log_e x$.

Note

i. $2^3 = 8 \Leftrightarrow 3 = \log_2 8$
$4^3 = 64 \Leftrightarrow 3 = \log_4 64$
$5^2 = 25 \Leftrightarrow 2 = \log_5 25$
$10^1 = 10 \Leftrightarrow 1 = \log_{10} 10$

ii. $\log_{10}x = (0.4343)\, ln x$
$ln x = (2.3026) \log_{10}x$

Characteristic and mantissa

The integral part of a common logarithm is called characteristic and fractional part when expressed as decimal is called mantissa.

The characteristic of the logarithms of a positive number > 1 is less by one then the number of digits in its integral part and is non-negative.

Examples:

Characteristic of log 613 $\rightarrow$ (3 – 1) = 2

Characteristic of log 527 $\rightarrow$ 2

Characteristic of log 98.275 $\rightarrow$ (2 – 1) = 1

Characteristic of log 28.538 $\rightarrow$ 1

Characteristic of log 3349 $\rightarrow$ (4 – 1) = 3

Characteristic of log 6234 .78 $\rightarrow$ (4 – 1) = 3

The characteristic of the logarithms of a positive decimal fraction less than one is negative and one more than the number of consecutive zeros immediately after the decimal part.

Examples:

Characteristic of log (0.7892) $\rightarrow$ -1

Characteristic of log (0.632) $\rightarrow$ -1

Characteristic of log (0.0234) $\rightarrow$ -2

Characteristic of log (0.07892) $\rightarrow$ -2

Characteristic of log (0.00325) $\rightarrow$ -3

Characteristic of log (0.000382)$\rightarrow$ -4

Principal Properties of Logarithm

(We Always Take Base > 0, $\neq$1)

I.

i. $\log_a a = 1$, $\log_a 1 = 0$ ($a > 0, a \neq 1$)

ii. $\log_a b = \dfrac{1}{\log_b a}$ (base > 0 and $\neq$ 1)

iii. $\log_a b = \log_c b . \log_a c = \dfrac{\log_c b}{\log_c a}$

II. For x > 0, y> 0, base > 0, $\neq$1, we have :

i. $\log_a (xy) = \log_a x + \log_a y$
ii. $\log_a (x/y) = \log_a x - \log_a y$
iii. $\log_a (x^n) = n \log_a x$
iv. $\log_{an} (x) = 1/n \log_a x$
v. $\log_{an} (x^m) = \log_a x$.

III. More generally, we have:

i. $\log_a (xy) = \log_a |x| + \log_a |y|$, (xy >0)
ii. $\log_a (x/y) = \log_a |x| - \log_a |y|$, (xy > 0)
iii. $\log_a x^{2k} = 2k \log_a |x|$, (x $\neq$0, k any integer)
iv. $\log_{_a 2k}(x) = \dfrac{1}{2k} \log_{|a|} x$,(a $\neq$0,|a|$\neq$1,k$\neq$0,x>0)