Exponential and Logarithmic Series

Exponential Series:

The sum of series $\left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+--+\text{to }\infty\right)$ is denoted by e which is an irrational number lying between 2 and 3.

$$\mathrm{e}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=\lim_{n\to 0}(1+n)^{\frac{1}{n}}$$

$$e^{x}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx}$$

$$\mathrm{e}=2.718282(\text{ Approx })$$

Exponential series for all x ∈ R

i. $\mathrm{e}^{\mathrm{x}}=\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+---\text{toc}\,\infty\right)$
$=\sum_{n=0}^{\infty}\frac{x^n}{n!}$

ii. $\mathrm{e}^{-\mathrm{x}}=\left(1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}+---\text{to}\infty\right)$
$=\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}$

iii. $\frac{\mathrm{e}^{\mathrm{x}}+\mathrm{e}^{-\mathrm{x}}}{2}=\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}---\text{to }\infty\right)$
$=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$

iv. $\frac{\mathrm{e}^{\mathrm{x}}-\mathrm{e}^{-\mathrm{x}}}{2}=\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+--\text{ to }\infty\right)$
$=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$ or $\sum_{n=1}^{\infty}\frac{x^{2n-1}}{(2n-1)!}$

v. $\mathrm{e}=\left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+-----\text{to}\,\infty\right)$
$=\sum_{n=0}^{\infty}\frac{1}{n!}$

vi. $\mathrm{e}^{-1}=\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+---\text{to}\,\infty\right)$
$=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}$

vii. $\frac{1}{2}\left(e+e^{-1}\right)=1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+----to\infty$
$=\sum_{n=0}^{\infty}\frac{1}{(2n)!}$

viii. $\frac{1}{2}\left(e-e^{-1}\right)=1+\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+-----to\infty$
$=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}$ or $\sum_{n=1}^{\infty}\frac{1}{(2n-1)!}$

ix. $\sum_{n=p}^{\infty} \frac{1}{(n-k)!} = \mathrm{e} \text{ for } p \leq k$
(A Term containing factorial of a negative number is treated as zero)

x. $\sum_{n=1}^{\infty} \frac{1}{n!} = \mathrm{e} - 1$

xi. $\sum_{n=0}^{\infty} \frac{1}{(n+2)!} = \left(\frac{1}{2!} + \frac{1}{3!} + ------- \text{toc} \infty\right)$
$= e - 2$

xii. $\sum_{n=1}^{\infty} \frac{1}{(2n-1)!} = \left(\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + ------ to \infty\right)$
$= \frac{\mathrm{e}^{\mathrm{x}} - \mathrm{e}^{-1}}{2}$
$= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}$

xiii. $\mathrm{e}^{\mathrm{ax}} = 1 + \frac{ax}{1!} + \frac{(ax)^2}{2!} + \frac{(ax)^3}{3!} + \ldots\ldots\ldots \text{ to } \infty$
$= \sum_{n=0}^{\infty} \frac{(ax)^n}{n!}$

Note:-

Expansion of a^x

$$\mathrm{a}^{\mathrm{x}} = 1 + \frac{x}{1!} \log_{\mathrm{e}} a + \frac{x^2}{2!} (\log_{\mathrm{e}} a)^2 + \frac{x^3}{3!} (\log_{\mathrm{e}} a)^3 + -- \text{ to } \infty$$

$(a > 0)$

Logarithmic Series:

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots to \infty (-1 < x \leq 1)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$

$$\log_e(1-x) = -\frac{x}{1} - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \ldots to \infty (-1 \leq x < 1)$$

$$\log_e\left(\frac{1+x}{1-x}\right) = 2\left(\frac{x}{1} + \frac{x^3}{3} + \frac{x^5}{5} + \ldots to \infty\right)$$

$(|x| < 1)$

Note:

i. $\log_e 2 = 1 \cdot \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots\ldots$ to ∞

ii. The series expansion of $\log_e(1+x)$ is not valid for $x = -1$, because $\log_e(1-1)$ is not a finite quantity.