Vectors and Scalars

Physical Quantities

Scalars

The physical quantities which need only magnitude to describe it completely are called scalar quantities. Scalar quantities are added algebraically. Examples: mass, temperature, length, energy etc.

Vectors

The physical quantities which need both magnitudes as well as direction to describe it completely are called vector quantities. The vector quantities follow law of vector addition i.e. the magnitude and direction varies with the direction of each vector. Examples velocity, force displacement momentum, electric field, magnetic moment etc.

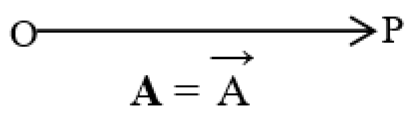
Some physical quantities having magnitude as well as direction are called scalar quantities since these quantities do not follow the law of vector addition. These quantities are added algebraically. Eg. Electric current, pressure, large angle etc.

- Vectors associated with rotation about axis are called axial vector examples Torque, angular velocity, angular momentum etc.

- Vectors which start from a point and terminate at another point is called polar vector. The direction of polar vector reverse if coordinate axis is reversed.

Representation of Vector

A vector quantity is represented by a straight line with arrow head. The length of straight line give the magnitude of vector.



and arrow head give the direction of vector. The starting point 0 is called origin or tail of vector while the end point P is called head or terminus of vector. The vector is written by bold faced letter $\mathbf{A}$ or $\vec{A}$.

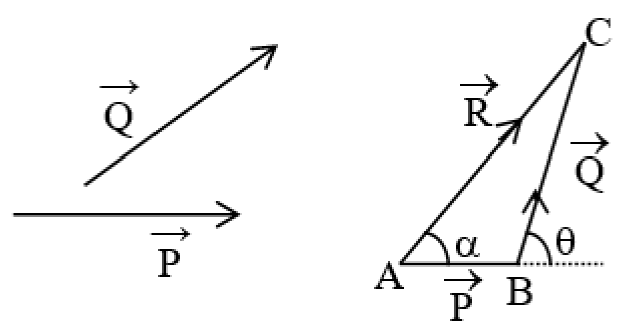
Addition of vectors

Vectors are added by following laws

Two vectors are added as follows:

- **Triangle law of vector addition:**

When two vectors are represented in magnitude and direction by two sides of a triangle taken in order then their resultant is represented in magnitude and direction by third side of same triangle in opposite order.

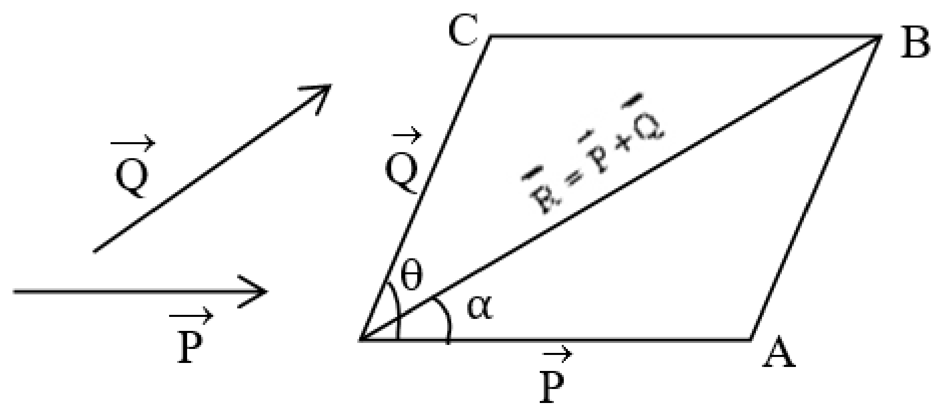


$\vec{P}$ and $\vec{Q}$ be two vector quantities in which $\vec{P}$ is represented in magnitude and direction by side AB and tail of $\vec{Q}$ is placed on head of $\vec{P}$ and $\vec{Q}$ is represented in magnitude and direction by side BC of triangle ABC then, their resultant is represented in magnitude and direction by third side AC of same triangle in opposite order.

$\therefore \vec{R} = \vec{P} + \vec{Q}$

- **Parallelogram law of vector addition:**

Two vectors are represented in magnitude and direction by two adjacent sides of a parallelogram then the resultant is represented in magnitude and direction by diagonal of same parallelogram from same vertex.



$\vec{P}$ and $\vec{Q}$ be two vector quantities acting at an angle q in which $\vec{P}$ is represented by side OA and $\vec{Q}$ is represented by adjacent side OC in magnitude and direction then the resultant is represented in magnitude and direction by diagonal OB of parallelogram OABC.

$\therefore \vec{R} = \vec{P} + \vec{Q}$

Parallelogram law of vector addition is equivalent to triangle law of vector addition.

- **Magnitude** of resultant $\vec{R}$ is $\mathrm{R} = \sqrt{\mathrm{P}^2 + 2\mathrm{PQ}\cos\theta + \mathrm{Q}^2}$
- **Direction** of resultant $\vec{R}$:
  Let α be the direction of resultant with $\vec{P}$, then
  $$\tan\alpha = \frac{Q\sin\theta}{P + Q\cos\theta}$$
  Let β be the direction of resultant with $\vec{Q}$, then
  $$\tan\beta = \frac{P\sin\theta}{Q + P\cos\theta}$$

Case I:

If θ=0° i.e. two vectors are in same direction then

$$\mathrm{R} = \sqrt{\mathrm{P}^2 + 2\mathrm{PQ} + \mathrm{Q}^2} = \sqrt{(\mathrm{P}+\mathrm{Q})^2} = \mathrm{P} + \mathrm{Q}$$

$\therefore R = P + Q$ i.e. the resultant becomes maximum.

Case II:

If θ=180° i.e. two vectors are in opposite direction then $R = \sqrt{P^2 - 2PQ + Q^2} = \sqrt{(P-Q)^2} = P - Q$

$\therefore R = P - Q$ i.e. the resultant becomes minimum.

Case III:

If θ=90° i.e. two vectors are are perpendicular to each other then $R = \sqrt{P^2 + Q^2}$

For direction with $\vec{P}$,

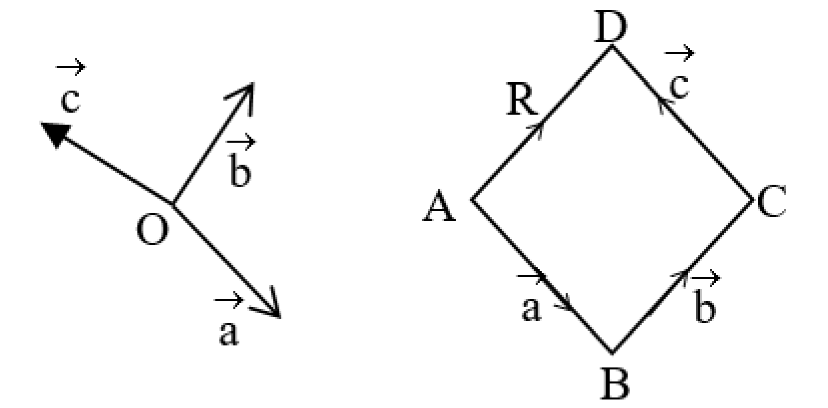
$\tan\alpha = \dfrac{Q}{P}$ or $\alpha = \tan^{-1}\left(\dfrac{Q}{P}\right)$

For direction with $\vec{Q}$,

$\tan\beta = \dfrac{P}{Q}$ or $\beta = \tan^{-1}\left(\dfrac{P}{Q}\right)$

- **Polygon law of vector addition:**

When 3 or more than 3 vectors acting on a body are represented in magnitude and direction by sides of a polygon taken in order then their resultant is represented in magnitude and direction by the side enclosing the polygon in opposite order $\vec{a}, \vec{b}, \vec{c}$ be the three vectors acting on a body at 0 are represented in magnitude and direction by side AB, BC, CD respectively of a polygon taken in order then the resultant is represented in magnitude and direction by side AD enclosing the polygon in opposite order.

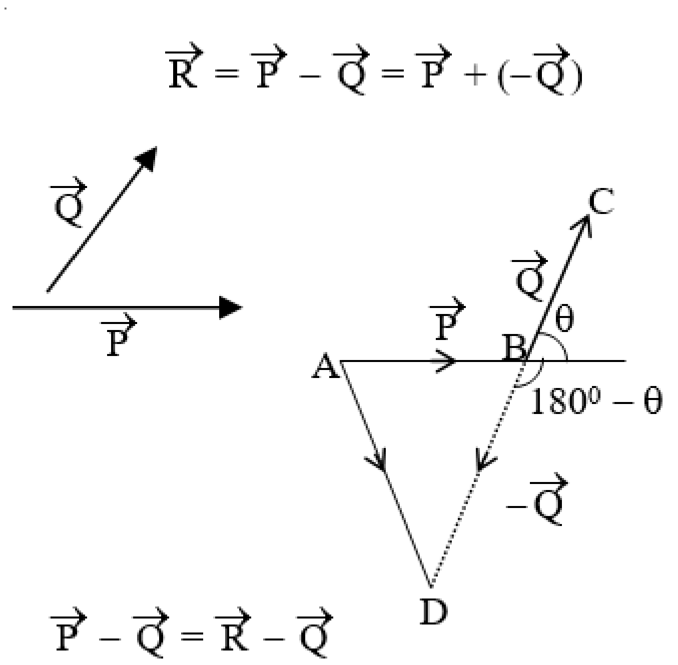


$$\overrightarrow{\mathrm{R}} = \overrightarrow{\mathrm{a}} + \overrightarrow{\mathrm{b}} + \overrightarrow{\mathrm{c}} = \overrightarrow{\mathrm{AB}} + \overrightarrow{\mathrm{BC}} + \overrightarrow{\mathrm{CD}} = \overrightarrow{\mathrm{AC}} + \overrightarrow{\mathrm{CD}} = \overrightarrow{\mathrm{AD}}$$

Subtraction of Vectors

The sum of a vector quantity and –ve of another vector quantity is called substraction of vectors.

Let $\vec{P}$ and $\vec{Q}$ be two vector quantities acting at θ then substraction of $\vec{Q}$ from $\vec{P}$ is



The resultant $\vec{P} - \vec{Q}$ is given by the third side AD in opposite order of triangle ABD made by $\vec{P}$ and $-\vec{Q}$.

- **Magnitude of difference**

$$\mathrm{R} = \sqrt{\mathrm{P}^2 + 2\mathrm{PQ}\cos(180^\circ - \theta) + \mathrm{Q}^2}$$

$$= \sqrt{\mathrm{P}^2 - 2\mathrm{PQ}\cos\theta + \mathrm{Q}^2}$$

$$\therefore R = \sqrt{\mathrm{P}^2 - 2\mathrm{PQ}\cos\theta + \mathrm{Q}^2}$$

- **Direction of difference**

Let α be the direction of resultant with $\vec{P}$, then

$$\tan\alpha = \frac{Q\sin(180^\circ - \theta)}{\mathrm{P} + \mathrm{Q}\cos(180^\circ - \theta)}$$

$$= \frac{Q\sin\theta}{P - Q\cos\theta}$$

$\alpha = \tan^{-1}\left(\dfrac{Q\sin\theta}{P - Q\cos\theta}\right)$ is the direction of $\vec{P} - \vec{Q}$.

Components of vector

The process of splitting a vector is parts is called resolution of vector and each parts are called components of vector. The sum of components of a vector is equal to vector itself.

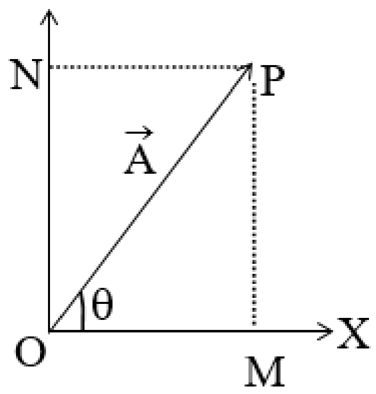
Resolution of a vector in two dimension:

$\overrightarrow{OP} = \vec{A}$ be a vector in which O is origin. OX and OY are rectangular coordinate system then $\overrightarrow{OM} = A\cos\theta\hat{i}$ and $\overrightarrow{ON} = A\sin\theta\hat{j}$ are the rectangular components of $\vec{A}$ where

$\hat{i} =$ unit vector along OX

$\hat{j} =$ unit vector along OY

And θ is angle made by $\vec{A}$ with OX.



Now,

$\overrightarrow{\mathrm{OP}} = \overrightarrow{\mathrm{OM}} + \overrightarrow{\mathrm{ON}}$

$\therefore$ Magnitude of $\overrightarrow{\mathrm{OP}} = \sqrt{\mathrm{OM}^2 + \mathrm{ON}^2}$

Resolution of a vector in three dimension:

Consider $\vec{A}$ be a vector whose tail is at origin in which OX, OY and OZ be axis in space.

$\overrightarrow{A_{\mathrm{x}}} = A\cos\alpha\hat{i}, \overrightarrow{\mathrm{A_y}} = A\cos\beta\hat{j}$ and $\overrightarrow{\mathrm{A_z}} = \mathrm{A}\cos\gamma\hat{k}$ where,

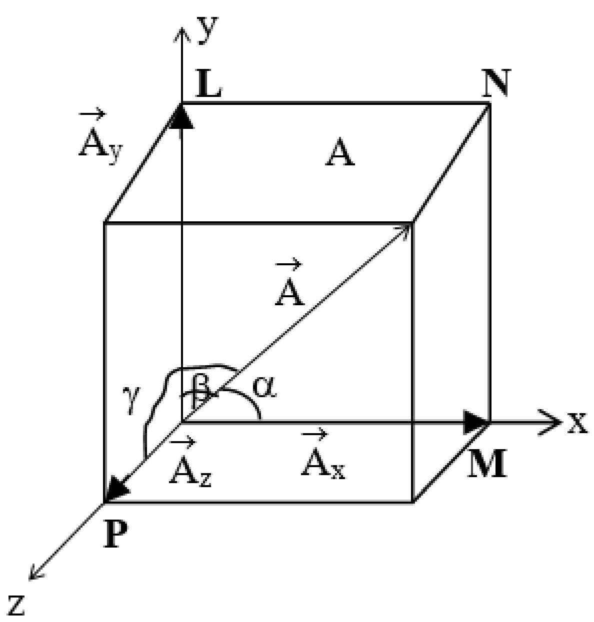
$\hat{i} =$ unit vector along OX

$\hat{j} =$ Unit vector along OY

$\hat{k} =$ Unit vector along OZ

$\therefore \overrightarrow{\mathrm{A}} = \overrightarrow{\mathrm{A_x}} + \overrightarrow{\mathrm{A_y}} + \overrightarrow{\mathrm{A_z}}$

$\therefore$ Magnitude of $\overrightarrow{\mathrm{A}} = \sqrt{\mathrm{A_x^2} + \mathrm{A_y^2} + \mathrm{A_z^2}}$



Addition of more than two vectors

In plane:

Let $\overrightarrow{\mathrm{A}}, \overrightarrow{\mathrm{B}}, \overrightarrow{\mathrm{C}}$,............ be the vectors whose components along OX are $\overrightarrow{A_{\mathrm{x}}}, \overrightarrow{\mathrm{B_x}}, \overrightarrow{\mathrm{C_x}}$,....... and components along OY are $\overrightarrow{A_{\mathrm{y}}}, \overrightarrow{\mathrm{B_y}}, \overrightarrow{\mathrm{C_y}}$,.......... then

Resultant along OX is

$\overrightarrow{\mathrm{R_x}} = \overrightarrow{\mathrm{A_x}} + \overrightarrow{\mathrm{B_x}} + \overrightarrow{\mathrm{C_x}} + \ldots\ldots$

Resultant along OY is

$\overrightarrow{\mathrm{R_y}} = \overrightarrow{\mathrm{A_y}} + \overrightarrow{\mathrm{B_y}} + \overrightarrow{\mathrm{C_y}} + \ldots\ldots$

$\therefore$ **Magnitude of Resultant** $R=\sqrt{R_x^2+R_y^2}$

Direction of resultant $\vec{A}$ with Ox is α then

$$\tan\alpha=\frac{R_y}{R_x}$$

$$\therefore \alpha=\tan^{-1}\left(\frac{\mathrm{R_y}}{\mathrm{R_x}}\right)$$

In space:

Let $\overrightarrow{\mathrm{A}}, \overrightarrow{\mathrm{B}}, \overrightarrow{\mathrm{C}}$,............ be the vectors whose components along OX are $\overrightarrow{A_{\mathrm{x}}}, \overrightarrow{\mathrm{B_x}}, \overrightarrow{\mathrm{C_x}}$,......., along OY are $\overrightarrow{A_{\mathrm{y}}}, \overrightarrow{\mathrm{B_y}}, \overrightarrow{\mathrm{C_y}}$,.......... and components along OZ are $\overrightarrow{A_{\mathrm{z}}}, \overrightarrow{\mathrm{B_z}}, \overrightarrow{\mathrm{C_z}}$,......., then

Resultant along OX

$$\overrightarrow{\mathrm{R_x}}=\overrightarrow{\mathrm{A_x}}+\overrightarrow{\mathrm{B_x}}+\overrightarrow{\mathrm{C_x}}+\ldots\ldots\ldots$$

Resultant along OY

$$\overrightarrow{\mathrm{R_y}}=\overrightarrow{\mathrm{A_y}}+\overrightarrow{\mathrm{B_y}}+\overrightarrow{\mathrm{C_y}}+\ldots\ldots\ldots$$

Resultant along OZ

$$\overrightarrow{\mathrm{R_z}}=\overrightarrow{\mathrm{A_z}}+\overrightarrow{\mathrm{B_z}}+\overrightarrow{\mathrm{C_z}}+\ldots\ldots\ldots$$

Magnitude of resultant ,

$$\mathrm{R}=\sqrt{\mathrm{R_X^2}+\mathrm{R_Y^2}+\mathrm{R_Z^2}}$$

Direction of resultant with OX is α then

$$\cos\alpha=\frac{R_x}{R}$$

Direction of resultant with OY is β then

$$\cos\beta=\frac{R_y}{R}$$

Direction of resultant with OZ is γ then

$$\cos\gamma=\frac{R_z}{R}$$

Multiplication of Vectors

Vectors can be multiplies in two ways:

Scalar product or dot product:

When two vector quantities are multiplied by a dot then the result is a scalar quantity is called scalar product or dot product of two vectors.

If $\vec{A}$ and $\vec{B}$ be two vectors then scalar product $vecA\cdot\vec{B}=|\vec{A}||\vec{B}|\cos\theta$

where $\theta=$ Angle between $\vec{A}$ and $\vec{B}$

Properties of scalar product:

- Commutative law

  $$\overrightarrow{\mathrm{A}}\cdot\overrightarrow{\mathrm{B}}=\overrightarrow{\mathrm{B}}\cdot\overrightarrow{\mathrm{A}}$$

- If two vectors $\vec{A}$ and $\vec{B}$ are perpendicular to each other then $\vec{A}\cdot\vec{B}=0$
- Scalar product of two equal vectors is equal to the square of magnitude.

  $$\therefore \overrightarrow{\mathrm{A}}\cdot\overrightarrow{\mathrm{A}}=\mathrm{A}^2$$
  $$\therefore \hat{i}\cdot\hat{i}=\hat{j}\cdot\hat{j}=\hat{k}\cdot\hat{k}=1$$

- In plane $\overrightarrow{\mathrm{A}}=\mathrm{A_x}\hat{\mathrm{i}}+\mathrm{A_y}\hat{\mathrm{j}}$ and $\overrightarrow{\mathrm{B}}=\mathrm{B_x}\hat{\mathrm{i}}+\mathrm{B_y}\hat{\mathrm{j}}$ are two vectors then
  $\overrightarrow{\mathrm{A}}.\overrightarrow{\mathrm{B}}=\mathrm{A_x B_x}+\mathrm{A_y B_y}$

  In space $\overrightarrow{\mathrm{A}}=\mathrm{A_x}\hat{i}+\mathrm{A_{yj}}\hat{j}+\mathrm{A_z}\hat{k}$ and $\overrightarrow{\mathrm{B}}=\mathrm{B_x}\hat{i}+\mathrm{B_y}\hat{j}+\mathrm{B_z}\hat{k}$ be two vectors then
  $\overrightarrow{\mathrm{A}}\cdot\overrightarrow{\mathrm{B}}=\mathrm{A_x B_x}+\mathrm{A_{yB_y}}+\mathrm{A_z B_z}$

Vector product or cross product:

When two vector quantities are multiplied by cross then the result is a vector quantity whose direction is perpendicular to plane containing both vectors.

Let $\vec{A}$ and $\vec{B}$ be two vector quantities then

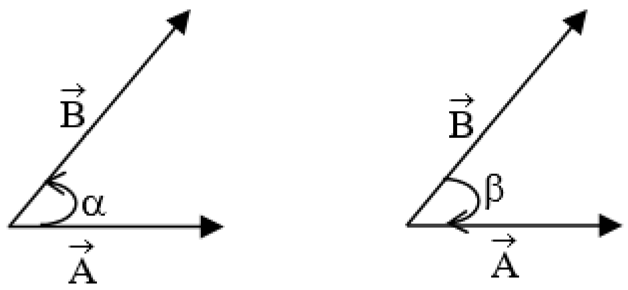
Cross product $=\overrightarrow{\mathrm{A}}\times\overrightarrow{\mathrm{B}}=|\overrightarrow{\mathrm{A}}||\overrightarrow{\mathrm{B}}|\sin\theta\hat{\mathrm{n}}$

where θ = Angle between $\vec{A}$ and $\vec{B}$

$\hat{n}$ = Unit vector perpendicular to plane containing $\vec{A}$ and $\vec{B}$.

<u>Characteristics of vector product</u>:

- Vector product does not obey commutative law.
  $\overrightarrow{\mathrm{A}}\times\overrightarrow{\mathrm{B}}\neq\overrightarrow{\mathrm{B}}\times\overrightarrow{\mathrm{A}}$



  $\overrightarrow{\mathrm{A}}\times\overrightarrow{\mathrm{B}}$ is a vector perpendicularly upwards.

  $\overrightarrow{\mathrm{B}}\times\overrightarrow{\mathrm{A}}$ is a vector perpendicularly upwards.

  $\therefore\overrightarrow{\mathrm{A}}\times\overrightarrow{\mathrm{B}}=-(\overrightarrow{\mathrm{B}}\times\overrightarrow{\mathrm{A}})$
- Vector product of collinear vectors is 0.
- Vector product of a vector with itself is 0. i.e. $\overrightarrow{\mathrm{A}}\times\overrightarrow{\mathrm{A}}=0$.
- If $\hat{i},\hat{j}$ and $\hat{k}$ be the unit vectors along OX, OY and OZ respectively
  $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=0$ and
  $\hat{i}\times\hat{j}=\hat{k},\hat{j}\times\hat{k}=\hat{i},\hat{k}\times\hat{i}=\hat{j}$
- If $\overrightarrow{\mathrm{A}}=\mathrm{A_x}\hat{i}+\mathrm{A_y}\hat{j}$ and $\overrightarrow{\mathrm{B}}=\mathrm{B_x}\hat{i}+\mathrm{B_y}\hat{j}$ be the vectors in plane then
  $\overrightarrow{\mathrm{A}}\times\overrightarrow{\mathrm{B}}=(\mathrm{A_x B_y}-\mathrm{B_x A_y})\hat{k}$
- In space, $\overrightarrow{\mathrm{A}}=\mathrm{A_x}\hat{i}+\mathrm{A_y}\hat{j}+\mathrm{A}_z\hat{k}$ and $\overrightarrow{\mathrm{B}}=\mathrm{B_x}\hat{i}+\mathrm{B_y}\hat{j}+\mathrm{B}_z\hat{k}$ then
  $\overrightarrow{\mathrm{A}}\times\overrightarrow{\mathrm{B}}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\\mathrm{A_x}&\mathrm{A_y}&\mathrm{A_z}\\\mathrm{B_x}&\mathrm{B_y}&\mathrm{B_z}\end{vmatrix}=$ Determinant
- If three vectors having position vectors $\overrightarrow{\mathrm{A}},\overrightarrow{\mathrm{B}},\overrightarrow{\mathrm{C}}$ will be collinear if
  $\overrightarrow{\mathrm{A}}\times\overrightarrow{\mathrm{B}}+\overrightarrow{\mathrm{B}}\times\overrightarrow{\mathrm{C}}+\overrightarrow{\mathrm{C}}\times\overrightarrow{\mathrm{A}}=0$

Some examples of product of vectors

- Work done is scalar product of Force $(\vec{F})$ and displacement $(\vec{S})$.

  So, $\mathrm{W}=\overrightarrow{\mathrm{F}}\cdot\overrightarrow{\mathrm{S}}$
- Moment of force or torque is the vector product of position vector and force.

Torque $(\vec{\tau}) = \overrightarrow{\mathrm{r}} \times \overrightarrow{\mathrm{F}}$

- Angular momentum is the moment of momentum

  $\therefore$ Angular momentum $(\overrightarrow{\mathrm{L}}) = \overrightarrow{\mathrm{r}} \times \overrightarrow{\mathrm{P}} = \vec{r} \times m\vec{v}$

- When $\vec{A}, \vec{B}$ and $\vec{C}$ be the three vectors then $\overrightarrow{\mathrm{A}} \cdot \overrightarrow{\mathrm{B}} \times \overrightarrow{\mathrm{C}}$ is a scalar quantity.

  $\overrightarrow{\mathrm{A}} \times (\overrightarrow{\mathrm{B}} \times \overrightarrow{\mathrm{C}}) =$ is a vector quantity.