Inverse Circular Functions

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The sine function sine: $R \to R$ given by $y = \sin x$ is not one-to-one and onto. In fact, $\sin\left(\frac{\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right)$ and there is no $x \in R$ for which sinx = 2. To make it bijective, we restrict the domain and co-domain as sine:

$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$.

Now, the function $y = \sin x$ is invertible and its inverse is $\mathrm{x} = \sin y$, It is also written as $\mathrm{y} = \sin^{-1}\mathrm{x}$, known as an inverse trigonometric function or inverse circular function. Similarly, other inverse circular functions, $\cos^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, $\sec^{-1} x$ and $\operatorname{cosec}^{-1} x$ can be defined.

Formulas for Inverse Functions

I.

i. $f\left(f^{-1}(x)\right) = x$

ii. $\sin^{-1}(\sin x) = x; \sin\left(\sin^{-1} y\right) = y; -\pi/2 \le x \le \pi/2$
$-1 \le y \le 1$

iii. $\cos^{-1}(\cos x) = x; \cos\left(\cos^{-1} y\right) = y; 0 \le x \le \pi-$
$1 \le y \le 1$

iv. $\tan^{-1}(\tan x) = x; \tan\left(\tan^{-1} y\right) = y; -\frac{\pi}{2} \le x \le \frac{\pi}{2}, y \in R$

v. $\cot^{-1}(\cot x) = x \;\; \cot(\cot^{-1} y) = y; \;\; 0 < x < \pi; x \ne \pi/2, y \le -1, y \ge 1$

vi. $\sec^{-1}(\sec x) = x; \sec\left(\sec^{-1} y\right) = y; 0 \le x \le \pi; x \ne \pi/2, y \le -1, y \ge 1$

vii. $\operatorname{cosec}^{-1}(\operatorname{cosec} x) = x; \operatorname{cosec}\left(\operatorname{cosec}^{-1} y\right) = y; -\frac{\pi}{2} \le x \le \frac{\pi}{2}, x \ne 0, y \le -1, y \ge 1$

II.

i. $\sin^{-1}(-x) = -\sin^{-1} x \quad -1 \le x \le 1$

ii. $\cos^{-1}(-x) = \pi - \cos^{-1} x \quad -1 \le x \le 1$

iii. $\tan^{-1}(-x) = \tan^{-1} x \quad -\infty < x < \infty$

iv. $\cot^{-1}(-x) = \pi - \cot^{-1}(x) \quad -\infty < x < \infty$

v. $\sec^{-1}(-x) = \pi - \sec^{-1}(x) \quad |x| \ge 1$

vi. $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}(x)|x| \ge 1$

III.

i. $\sin^{-1}\mathrm{x} = \operatorname{cosec}^{-1}\frac{1}{x}$

ii. $\operatorname{cosec}^{-1} x = \sin^{-1}\frac{1}{x}$

iii. $\cos^{-1} x = \sec^{-1}\frac{1}{x}$

iv. $\sec^{-1} x = \cos^{-1}\frac{1}{x}$

v. $\tan^{-1} x = \cot^{-1} \frac{1}{x}$

vi. $\cot^{-1} x = \tan^{-1} \frac{1}{x}$, for any numerical value x

IV.

i. $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

ii. $\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$

iii. $\cot^{-1} x + \tan^{-1} x = \frac{\pi}{2}$

iv. $\sin^{-1} x + \tan^{-1} \frac{\sqrt{1-x^2}}{x} = \frac{\pi}{2}$
for any numerical value x

V.

i.

$$\begin{aligned} \tan^{-1} x + \tan^{-1} y &= \tan^{-1}\left(\frac{x+y}{1-xy}\right) \text{ if } xy < 1 \\ &= \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right) \text{ if } xy > 1 \end{aligned}$$

ii. $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right) = \cot^{-1}\left(\frac{1+xy}{x-y}\right)$

iii. $\cot^{-1} x + \cot^{-1} y = \cot^{-1}\left(\frac{xy-1}{y+x}\right) = \tan^{-1} \frac{x+y}{xy-1}$

iv. $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}\{x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\}$

v. $\cos^{-1} \pm \cos^{-1} y = \cos^{-1}\{xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\}$

VI.

i. $2\tan^{-1} x = \sin^{-1} \frac{2x}{1+x^2} = \cos^{-1} \frac{1-x^2}{1+x^2} = \tan^{-1} \frac{2x}{1-x^2}$

ii. $2\sin^{-1} x = \sin^{-1}\{2x\sqrt{1-x^2}\}$

iii. $2\cos^{-1} x = \cos^{-1}(2x^2 - 1)$

VII.

i. $3\sin^{-1} x = \sin^{-1}(3x - 4x^3) 3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$

ii. $3\tan^{-1} x = \tan^{-1} \frac{3x - x^3}{1 - 3x^2}$

iii. $3\cot^{-1} x = \cot^{-1} \frac{x^3 - 3x}{3x^2 - 1}$

VIII.

i.

$$\begin{aligned} \sin^{-1} x &= \operatorname{cosec}^{-1} \frac{1}{x} \\ &= \cos^{-1} \sqrt{1-x^2} \\ &= \tan^{-1} \frac{x}{\sqrt{1-x^2}} \\ &= \cot^{-1} \frac{\sqrt{1-x^2}}{x} = \sec^{-1} \frac{1}{\sqrt{1-x^2}} \end{aligned}$$

ii. $\cos^{-1} x = \sin^{-1} \sqrt{1-x^2} = \tan^{-1} \frac{\sqrt{1-x^2}}{x} = \sec^{-1} \frac{1}{x}$

$= \operatorname{cosec}^{-1} \frac{1}{\sqrt{1-x^2}} = \cot^{-1} \frac{x}{\sqrt{1-x^2}}$

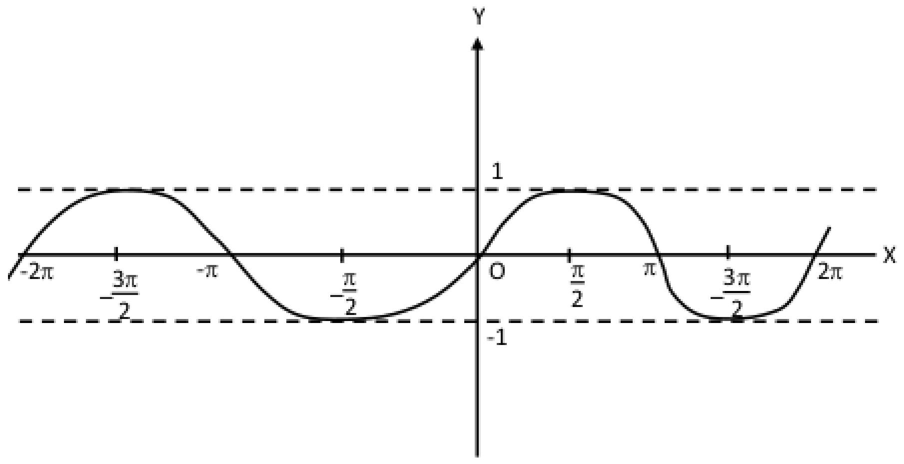
iii. $\tan^{-1} x = \sin^{1} \left(\frac{x}{\sqrt{1+x^2}} \right)$

$= \cos^{-1} \frac{1}{\sqrt{1+x^2}} = \sec^{-1} \sqrt{1+x^2} = \cot^{-1} \frac{1}{x}$

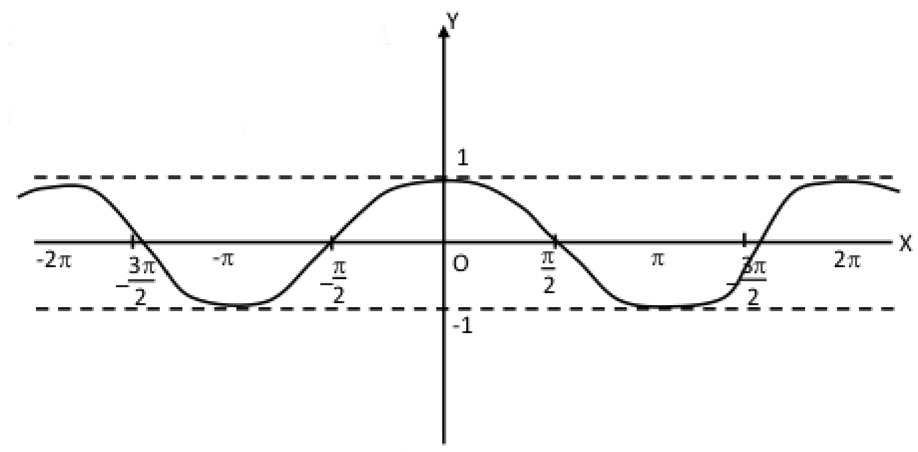
$= \operatorname{cosec}^{-1} \frac{\sqrt{1+x^2}}{x}$

Graphs of Trigonometric Functions

i. $y = \sin x$: It is defined $\forall x \in R$ and $-1 \leq \sin x \leq 1$ i.e., $y \in [-1, 1]$ and periodic with period 2π.

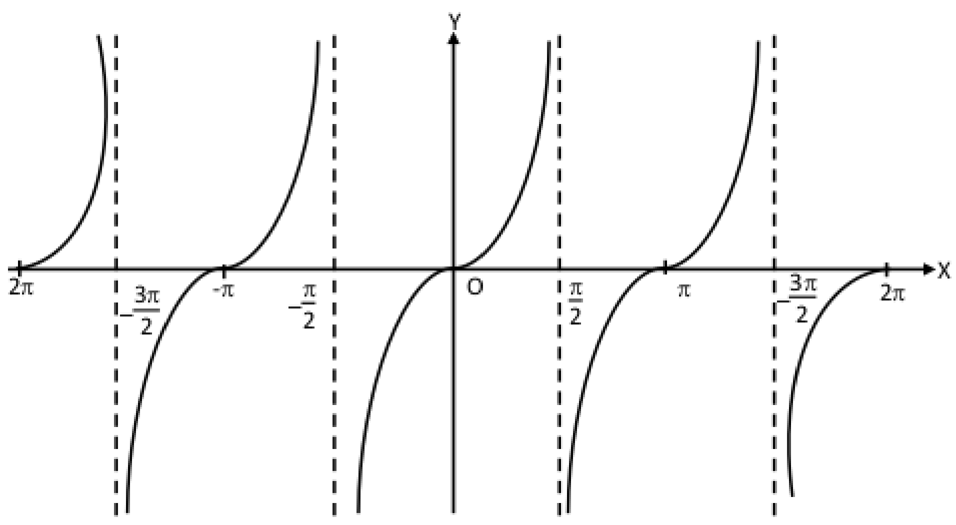


ii. $y = \cos x$: It is defined $\forall x \in R$ and $-1 \leq \cos x \leq 1$ i.e, $y \in [-1, 1]$ and period 2π

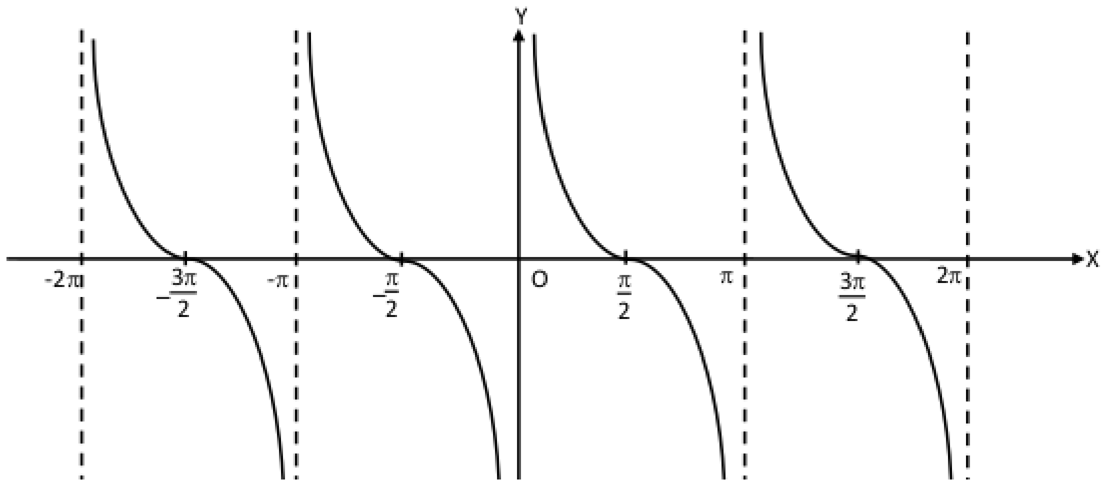


iii. $y = \tan x$: It is defined for $x \in R \left\{ (2n+1)\frac{\pi}{2}, n \in I \right\}$ because tanx is not defined if $\cos x = 0$

ie. $x = n\pi + \frac{\pi}{2} = (2n+1)\frac{\pi}{2}, n \in I$ and $-4 < \tan x < 4$ i.e., $y \in R$ and periodic with period π.

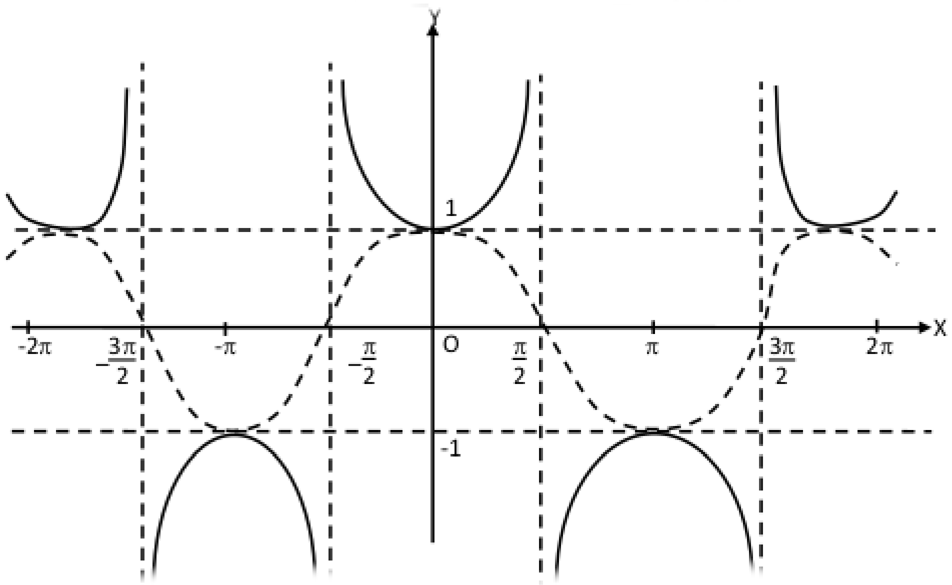


iv. $y = \cot x$: It is defined for $x \in R - \{n\pi, n \in I\}$ because $\cot x$ is not defined if $\sin x = 0$
i.e. $x = n\pi, n \in I$ and $-4 < \cot x < 4$ i.e., $y \in R$ and periodic with period π.



v. $y = \sec x$: It is defined $\forall x \in R - \left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$ because $secx$ is not defined if $\cos x = 0$

i.e. $x = n\pi + \frac{\pi}{2} = (2n+1)\frac{\pi}{2}$ and $|\sec x| \geq 1$ is i.e.,$\sec x \geq 1 \& \sec x \leq -1$ and periodic with period 2π.



vi. $y = \text{cosec}x$: it is defined $\forall x 0 R - \{n\pi, n \in \text{ If because cosec } x \text{ is not defined if } \sin x = 0$
i.e., $x = n\pi, n \in |$ and $|\text{cosec}x| \geq 1$ is $e.$, $\text{cosec}x \geq 1\&$ cosec $x \leq -1$ and periodic with period 2π

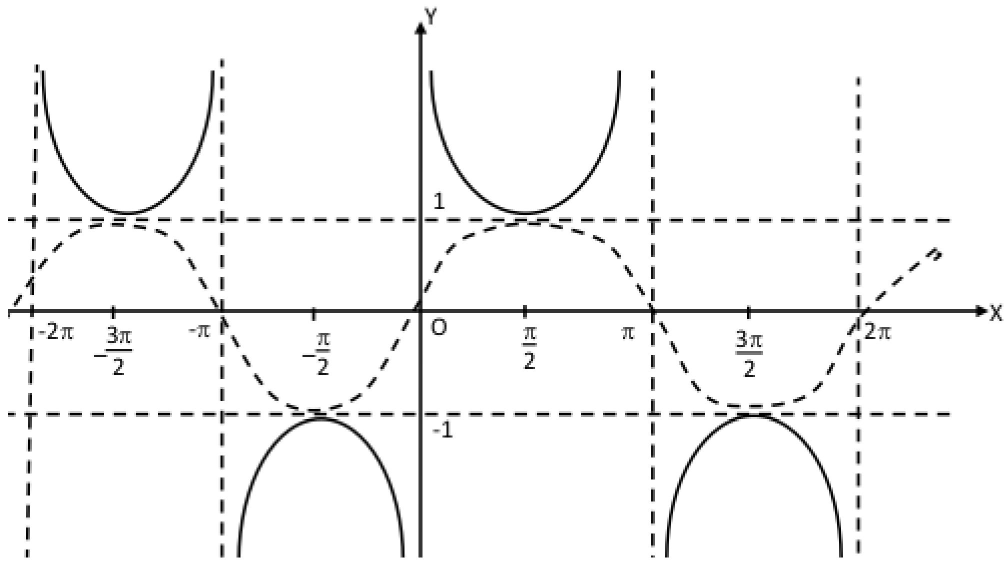


Table for Domain/Range

Function f(x)	Domain	Range
$y = \sin^{-1}x$	[-1, 1]	[-π/2, π/2]
$y = \cos^{-1}x$	[-1, 1]	[0. π]
$y = \tan^{-1}x$	(-∞, +∞)	(-π/2, π/2)
$y = \cot^{-1}x$	(-∞, +∞)	(0, π)
$y = \sec^{-1}x$	(-∞, -1] ∪[1, ∞)	[0, π] – {π/2}
$y = \text{cosec}^{-1}x$	(-∞, -1] ∪[1, ∞)	[-π/2, π/2] – {0}