Exponential and Logarithmic Series

Exponential Series:

The sum of series $\left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\mathrm{to}\;\infty\right)$ is denoted by e which is an irrational number lying between 2 and 3.

$$\mathrm{e}=\lim_{n o\infty}\left(1+rac{1}{n}
ight)^n=\lim_{n o0}(1+n)^{rac{1}{n}}$$

$$e^x = \lim_{n o \infty} \left(1 + rac{1}{n}
ight)^{nx}$$

e = 2.718282(Approx)

Exponential series for all $x \in R$

i.
$$\mathrm{e^x}=\left(1+rac{x}{1!}+rac{x^2}{2!}+rac{x^3}{3!}+rac{x^4}{4!}+---\mathrm{toc}\,\infty
ight)$$
 $=\sum_{n=0}^\inftyrac{x^n}{n!}$

ii.
$$\mathrm{e}^{-\mathrm{x}}=\left(1-rac{x}{1!}+rac{x^2}{2!}-rac{x^3}{3!}+rac{x^4}{4!}+---\mathrm{to}\infty
ight)$$
 $=\sum_{n=0}^{\infty}(-1)^nrac{x^n}{n!}$

iii.
$$rac{\mathrm{e^x} + \mathrm{e^{-x}}}{2} = \left(1 + rac{x^2}{2!} + rac{x^4}{4!} + rac{x^6}{6!} - - - ext{to}\,\infty
ight)$$
 $= \sum_{n=0}^\infty rac{x^{2n}}{(2n)!}$

iv.
$$rac{\mathrm{e}^{\mathrm{x}}-\mathrm{e}^{-\mathrm{x}}}{2}=\left(x+rac{x^3}{3!}+rac{x^5}{5!}+--\ \mathrm{to}\ \infty
ight) \ =\sum_{n=0}^{\infty}rac{x^{2n+1}}{(2n+1)!}\ \mathrm{or}\ \sum_{n=1}^{\infty}rac{x^{2n-1}}{(2n-1)!}$$

v.
$$\mathrm{e} = \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + - - - - \cot \infty\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}$$

vi.
$$\mathrm{e}^{-1}=\left(1-rac{1}{1!}+rac{1}{2!}-rac{1}{3!}+---\mathrm{to}\,\infty
ight)$$
 $=\sum_{n=0}^{\infty}rac{(-1)^n}{n!}$

vii.
$$\dfrac{1}{2}ig(e+e^{-1}ig)=1+\dfrac{1}{2!}+\dfrac{1}{4!}+\dfrac{1}{6!}+----to\infty$$
 $=\sum_{n=0}^{\infty}\dfrac{1}{(2n)!}$

viii.
$$\dfrac{1}{2}ig(e-e^{-1}ig)=1+\dfrac{1}{1!}+\dfrac{1}{3!}+\dfrac{1}{5!}+----to\infty$$
 $=\sum_{n=0}^{\infty}\dfrac{1}{(2n+1)!}$ or $\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)!}$

ix.
$$\sum_{n=p}^{\infty} rac{1}{(n-k)!} = ext{ e for } p \leq k$$

(A Term containing factorial of a negative number is treated as zero)

$$\operatorname{x.} \sum_{n=1}^{\infty} \frac{1}{n!} = \operatorname{e} - 1$$

xi.
$$\sum_{n=0}^{\infty}rac{1}{(n+2)!}=\left(rac{1}{2!}+rac{1}{3!}+----\cot\infty
ight)$$
 $=e-2$

xii.
$$\sum_{n=1}^{\infty} rac{1}{(2n-1)!} = \left(rac{1}{1!} + rac{1}{3!} + rac{1}{5!} + - - - - - to\infty
ight)$$
 $= rac{\mathrm{e^x} - \mathrm{e^{-1}}}{2}$
 $= \sum_{n=0}^{\infty} rac{1}{(2n+1)!}$

xiii.
$$\mathrm{e}^{\mathrm{ax}}=1+rac{ax}{1!}+rac{(ax)^2}{2!}+rac{(ax)^3}{3!}+\ldots\ldots$$
 to ∞ $=\sum_{n=0}^\inftyrac{(ax)^n}{n!}$

Note:-

Expansion of a^x

$$egin{aligned} {
m a}^{
m x} &= 1 + rac{x}{1!} {
m log_e} \, a + rac{x^2}{2!} ({
m log_e} \, a)^2 + rac{x^3}{3!} ({
m log_e} \, a)^3 + - - \ \ {
m to} \ \infty \end{aligned}$$

Logarithmic Series:

$$\log_e(1+x) = x - rac{x^2}{2} + rac{x^3}{3} - rac{x^4}{4} + \dots to \, \infty (-1 < x \leq 1)$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}$$

$$\log_e(1-x) = -rac{x}{1} - rac{x^2}{2} - rac{x^3}{3} - rac{x^4}{4} + \dots to \, \infty (-1 \leq x < 1)$$

$$\log_e\!\left(rac{1+x}{1-x}
ight) = 2\left(rac{x}{1} + rac{x^3}{3} + rac{x^5}{5} + \ldots to\,\infty
ight)$$

Note:

i.
$$\log_e 2 = 1 \cdot rac{1}{2} + rac{1}{3} - rac{1}{4} + rac{1}{5} - \ldots \ldots$$
 to ∞

ii. The series expansion of $\log_e(1+x)$ is not valid for x=-1, because $\log_e(1-1)$ is not a finite quantity.