Matrices and Determinant

Matrix

A matrix is an ordered system of numbers arranged in the formation of rows and columns and enclosed between round (or square) brackets. In other words, it is the rectangular array of numbers (real or complex) into ordered rows and ordered columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix}$$

is a matrix of order m×n

Rows and columns of a matrix

The horizontal lines are known as **rows** and the vertical lines are known as **columns.**

Order of a matrix:

The number of rows followed by the number of columns give the order or the size of the matrix.

If a matrix has m rows and n columns, then the order of the matrix = m $\times$ n (Read: m by n)

Sum and Difference of Two matrices

Let A = $(a_{ij})_{mxn}$ and B = $(b_{ij})_{mxn}$ be any two matrices of same order. Then, the sum of two matrices denoted by A +B is given by

A + B = $(a_{ij})_{mxn} + (b_{ij})_{mxn} = (a_{ij} + b_{ij})_{mxn}$

The difference between them is denoted by A – B is given by

A – B = $(a_{ij} - b_{ij})_{mxn}$

Note: Favourable condition for the addition of two matrices A and B is

order of A = order of B

Multiplication of Matrices:

Multiplication AB of two matrices A and B is defined if

Number of columns of A = Number of rows of B

If A = $(a_{ij})_{mxp}$ and B = $(b_{ij})_{pxq}$, then their multiplication AB is a matrix of size m×n and whose $(i, j)^{th}$ element is the sum of the product of the respective elements of the i^{th} row of A with the j^{th} column of B.

Scalar Multiplication of a Matrix

Let A = $(a_{ij})_{mxn}$ be any matrix and k be a scalar. Then the scalar multiplication kA is a matrix defined by

kA = $k(a_{ij})_{mxn} = (ka_{ij})_{mxn}$.

Equality of matrices

Two matrices A = $(a_{ij})_{mxn}$ and B = $(b_{ij})_{mxn}$ are said to be equal if they have same order and their respective elements are equal.

Transpose of a matrix:

Transpose of a matrix A, denoted by A^T or A', and is a matrix obtained by interchanging the respective rows and columns of the given matrix.

If A = $(a_{ij})_{mxn}$ be any matrix then its transpose $(A^T) = (a_{ji})_{nxm}$

Example:

$$A = \begin{pmatrix} 1 & 7 \\ 5 & 4 \end{pmatrix} \quad A^T = \begin{pmatrix} 1 & 5 \\ 7 & 4 \end{pmatrix}$$

Note:

If A is a matrix of order m x n, A^T is a matrix of order n x m.

Properties of transpose of a matrix:

i. $(A^T)^T = A$

ii. $(A + B)^T = A^T + B^T$

Where A and B are the matrices of same order

iii. $(AB)^T = B^T A^T$

iv. $(\alpha A)^T = \alpha A^T$, a is any scalar.

Adjoint of a matrix

The matrix obtained by taking the transpose of the matrix of co-factors of the elements & the given matrix is known as the adjoint of a matrix.

Inverse of a matrix

Inverse of matrix A denoted by A^{-1} and given by:

$A^{-1} = \frac{\text{adj. of A}}{|A|}$, $|A| \neq 0$ (non-singular matrix),

where adj A = transpose of matrix of co-factors of the elements of A

Thus, $AA^{-1} = I = A^{-1}A$

Note:

i. Inverse of the matrix A^{-1} is equals to A
$(A^{-1})^{-1} = A$.
ii. $(A^T)^{-1} = (A^{-1})^T$
iii. $(AB)^{-1} = B^{-1} A^{-1}$
iv. $(kA)^{-1} = A^{-1}$; $k \neq 0$

Singular matrix

A matrix A is said to be singular matrix if |A| = O. Inverse of a matrix does not exist when A is a singular matrix.

Invertible matrix

A matrix A is said to be invertible matrix if A^{-1} exists i.e. $|A| \neq$ O.

Types of Matrices

Row Matrix: -

A matrix having only one row is called a row matrix

e.g. (1), (2 3), (4 5 6) etc.

Column Matrix : -

A matrix having only one column is called a column matrix.

e.g. $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ etc.

Square matrix: -

A matrix having equal number of rows and columns is called square matrix.

e.g. $\begin{pmatrix} a & c \\ b & d \end{pmatrix}_{2\times 2} \begin{pmatrix} 1 & 3 & 5 \\ 4 & 0 & 1 \\ -6 & -2 & 3 \end{pmatrix}_{3\times 3}$

In case of square matrix $|A| = |A^T|$

Rectangular Matrix: -

A matrix in which number of rows is not equal to a number of columns is called rectangular matrix.

i.e. Number of columns of a matrix ≠ Number of rows of the matrix

e.g. $\begin{pmatrix} 1 & 5 & 7 \\ 9 & 2 & 4 \end{pmatrix}_{2\times 3}$

Null Matrix: -

If every element of a matrix is zero then it is called null matrix or zero matrix and is denoted by O.

Example:-

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_{2\times 2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{3\times 3}$$

Diagonal matrix

A square matrix having all the elements except the main diagonal elements are zero is called diagonal matrix.

If A = $[a_{ij}]_{m\times n}$ is a diagonal matrix then

$a_{ij} = 0$ for $i \neq j$

A = $\begin{bmatrix} a_{11} & o & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$ = diag (a_{11} a_{22} a_{33})

Note:

If A = diag [d_1, d_2......... d_n] then A^{-1} = diag [d_1^{-1}, d_2^{1} ....d_n^{-1}]

Example:-

$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{pmatrix}_{3\times 3}$, $\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}_{2\times 2}$ are diagonal matrices.

Scalar Matrix: -

Diagonal matrix having main diagonal elements equal is called scalar matrix.

Example:-

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}_{3\times 3}$$

Unit matrix or Identity matrix: -

Diagonal matrix having each element of principal diagonal is equal to unity(1) is called unit matrix or identity matrix. It is denoted by I or I_n

$$\mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2}, \quad \mathrm{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\times 3}$$

Sub - Matrix

A matrix obtained by omitting (removing) certain number of rows or columns of the given matrix is known as its sub-matrix.

Example:-

$$A = \begin{pmatrix} 1 & 4 & 3 \\ 2 & 5 & 6 \\ 7 & 1 & 4 \end{pmatrix}_{3\times 3} \quad B = \begin{pmatrix} 5 & 6 \\ 1 & 4 \end{pmatrix}_{2\times 2}$$

Then B is a sub-matrix of A.

Symmetric Matrix: -

A square matrix A = (a_{ij}) is said to be symmetric if $a_{ij} = a_{ji}$ for all i, j.

i.e. $A = A^T$

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 6 \\ 4 & 6 & 5 \end{pmatrix}_{3\times 3}$$

$a_{12} = a_{21} = 2$

$a_{23} = a_{32} = 6$

$a_{13} = a_{31} = 4$ etc.

Notes:

- A is a square matrix . Then the symmetric matrix corresponding to A is $\frac{1}{2}(A+A^T)$.
- For a square matrix $A = [a_{ij}]_{n \times n}$,
  - $A + A^T$ is a symmetric matrix.
  - $A - A^T$ is skew symmetric matrix.
  - AA^T and A^TA , A^2 are symmetric matrices.
- If A and B are symmetric matrices of same order, then AB is symmetric iff AB = BA i.e. A and B are commute under multiplication.
- All integral powers of a symmetric matrix are symmetric.

Skew-symmetric matrix

A square matrix A = (a_{ij}) is said to be skew symmetric matrix if $a_{ij}=-a_{ji}$ " i, j

i.e. $a_{12} = -a_{21}$

$a_{23} = -a_{32}$

$a_{31} = -a_{13}$ etc.

Examples

$$\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}_{2\times 2} \quad \begin{pmatrix} 0 & -2 & -3 \\ 2 & 0 & 5 \\ 3 & -5 & 0 \end{pmatrix}_{3\times 3}$$

Note:

- All diagonal elements are zero.
- For a skew symmetric matrix, $A = -A^T$.
- If A is a square matrix, then the skew symmetric matrix corresponding to A is $1/2(A - A^T)$.

Square matrix = symmetric matrix + skew symmetric matrix

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

Triangular matrix: -

- A square matrix having all the elements below the diagonal elements are zero is called **upper triangular matrix.**

Example:-

$$\begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}_{2\times 2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{pmatrix}_{3\times 3}$$

A square matrix A = (a_{ij}) is called upper triangular matrix if $a_{ij} = 0$ for all i>j

- A square matrix having all the elements above the diagonal elements are zero is called **lower triangular matrix.**

$$\begin{pmatrix} 3 & 0 \\ 5 & 4 \end{pmatrix}_{2\times 2} \begin{pmatrix} 6 & 0 & 0 \\ 2 & 5 & 0 \\ 1 & 4 & 7 \end{pmatrix}_{3\times 3} \quad \text{etc.}$$

A square matrix A = (a_{ij}) is called lower triangular matrix if $a_{ij} = 0$ for all i < j

Orthogonal matrix: -

A square matrix A is said to be orthogonal if $A^T A = I$

Example:-

1. i) I is an orthogonal matrix.

2. ii) The rotation matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is an orthogonal matrix.

Involutory matrix

A square matrix A is said to involutory matrix if $A^2 = I$

Idempotent matrix

A square matrix A is said to be idempotent if $A^2 = A$.

Nilpotent matrix

A square matrix A is said to be Nilpotent matrix if $A^m = 0$ for some positive integer m.

Note: A is called Nilpotent if $A^2 = O$

where m = 2

Hermitian matrix

A square matrix A = $(a_{ij})_{n´n}$ is said to be Hermitian matrix if a_{ij} = for all i, j = 1, 2, 3 …….. n

For example, $\begin{bmatrix} 1 & 3-2i \\ 3+2i & -5 \end{bmatrix}$

is a Hermitian matrix of order 2.

$\begin{bmatrix} 3 & 4-3i & -4 \\ 4+3i & 1 & 1-i \\ 4 & 1+i & 5 \end{bmatrix}$ is a Hermitian matrix of order 3.

Conjugate matrix

The matrix obtained by replacing each element of a matrix by its complex conjugate is called conjugate of the given matrix.

Let A be any matrix. Then its conjugate is denoted by . E.g. if

$$A = \begin{bmatrix} 1+i & 1 & 3+4i \\ 2+3i & 5+4i & 0 \\ 4-3i & 4+7i & 1+6i \end{bmatrix}$$
$$\bar{A} = \begin{bmatrix} 1-i & 1 & 3-4i \\ 2-3i & 5-4i & 0 \\ 4+3i & 4-7i & 1-6i \end{bmatrix}$$

Properties of Matrices

i. $A + B = B + A$ (commutative law of matrix addition)
ii. Matrix multiplication is not commutative in general i.e.
$AB \neq BA$
iii. $A + O = O + A = A$ (where O is the null matrix of same order as that of A)
iv. $A + (-A) = O = (-A) + A$
v. $(A + B) + C = A + (B + C)$ (Associative law of matrix addition)
vi. $A + B = A + C \Rightarrow B = C$ (cancellation Law)
vii. $(AB)C = A(BC)$ (Associative law of matrix multiplication)
viii. $k(AB) = (KA)B = A(KB)$
ix. $A - (B - C) = (A - B) + C$
x. $A(B + C) = AB + AC$ (Distributive law)
xi. $K(A + B) = KA + KB$
xii. $\lambda(A – B) = \lambda A – \lambda B$
where k and λ are scalars.
xiii. $(K_1 + K_2)A = K_1A + K_2A$
xiv. $AI = IA = A$ where I is the identity matrix.
xv. A and B are the inverses of each other and I is the unit matrix of same order, then
$AB = I = BA$
$(\therefore AA^{-1} = I = A^{-1}A)$
xvi. If $|A| \neq 0$, A^{-1} exists and $AB = AC \Rightarrow B = C$
xvii. $(A^T)^T = A$
xviii. $(A + B)^T = A^T + B^T$ where A and B are the matrices of same order.
xix. $(\alpha A)^T = \alpha A^T$, α is any scalar
xx. $(AB)^T = B^T A^T$
xxi. $(AB)^{-1} = B^{-1}A^{-1}$
xxii. $(A^{-1})^T = (A^T)^{-1}$
xxiii. $adj(AB) = (adj\,B)(adj\,A)$
xxiv. $A(adj\,A) = |A|I = (adjA)A$ where I is the unit matrix of same order as that of A
xxv. $adj(KI) = K^{n-1}I$, where n is the order of I.
xxvi. $(adj.\ A)' = adj\,A'$

xxvii. If A is an invertible matrix of order n, then $adj(adjA) = |A|^{n-2} \cdot A$

Note:-

(i) Additive inverse of matrix $A = \begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix}$ is

$$-A = \begin{pmatrix} -1 & -3 \\ -5 & -7 \end{pmatrix}$$

(ii) Additive identity of matrix A is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Multiplicative identity of matrix A is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Multiplicative inverse of a matrix

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 4 & -2 \\ 3 & 1 \end{pmatrix}$

- If $A = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$ then $A^n = \begin{pmatrix} 2^{n-1}x^n & 2^{n-1}x^n \\ 2^{n-1}x^n & 2^{n-1}x^n \end{pmatrix}$

where n is positive integer.

- If $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ then $A^n = \begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}$

Determinant

Determinant can be considered as a function. Each determinant has fixed value. Determinants of square matrices alone are defined. A determinant having n rows and n columns is known as determinant of order n.

Example:-

$$\mathrm{D} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Expanding along R_1

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Determinant of a square matrix of order 3:

If A is a square matrix then the determinant of matrix A is defined as the sum of product of the elements of any row or column and their corresponding co-factors. It is denoted by |A|.

If A = $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

|A| = $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$= a_{11} A_{11} + a_{21} A_{21} + a_{31} A_{31}$

Where A_{11}, A_{21}, A_{31} are the co-factors of a_{11}, a_{21}, a_{31} respectively.

Minors and co-factors:

(i) Minors of an element is the determinant left after removing the row and column in which the element belongs.

$$\mathrm{D} = \begin{vmatrix} \mathrm{a}_1 & \mathrm{a}_2 & \mathrm{a}_3 \\ \mathrm{b}_1 & \mathrm{b}_2 & \mathrm{b}_3 \\ \mathrm{c}_1 & \mathrm{c}_2 & \mathrm{c}_3 \end{vmatrix}$$

Minor of a_1 = M_{11} = $\begin{vmatrix} b_2 & b_3 \\ c_3 & c_3 \end{vmatrix}$

Minor of b_1 = M_{21} = $\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix}$

(ii) Co-factors of the element is the minor of the element with proper sign.

$A_{11} = \begin{vmatrix} b_2 & b_3 \\ c_3 & c_3 \end{vmatrix}$ $\quad A_{12} = -\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}$

(iii) The co-factor A_{ij} of the i^{th} row and j^{th} column of a_{ij} element is given by

$A_{ij} = (-1)^{i+j} M_{ij}$

The value of

$a_1A_1 + a_2A_2 + a_3A_3$

$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_3 \end{vmatrix}$

= D

Note:

- The value of $a_1A_1 + a_2A_2 + a_3A_3$

  $= a_1 \begin{vmatrix} b_2 & b_3 \\ c_3 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_3 \end{vmatrix}$

  = D

  $A_2A_2 + b_2B_2 + c_2C_2 = D$

  (A_1, A_2, A_3, ------------- are the cofactors)

- If the elements of the matrix are replaced by their co-factors, then the value of the determinant is $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$ = Determinant formed by co-factors of the corresponding elements = D^2.

Properties of Determinant

- If any two rows or columns of a determinant are identical, then the value of the determinant is zero\

  $$|A| = \begin{vmatrix} a & 1 & a \\ b & 1 & b \\ c & 1 & c \end{vmatrix}$$
  = 0 [$\because C_1 = C_3$]

- Interchanging the rows and columns of the determinant does not change the value of the determinant

  $$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

  Interchanging the rows and columns, we get

  $$|A^*| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

  then $|A| = |A^*|$

- Interchanging any two rows or columns changes the sign of the determinant

  $$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

  $$\Delta^1 = -\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$\Delta^1 = -\Delta$

- Determinant of unit matrix is 1

A = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

|A| = 1

- If all the elements in any row or column of a determinant are zero, then the value of the determinant is zero.

i.e, $\Delta = \begin{vmatrix} 0 & 0 & w^2 \\ 0 & w & 1 \\ 0 & w^2 & w \end{vmatrix}$

= 0

- If in any row or column of a determinant a multiple of another row or column is added the value of the determinant remains the same.

$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$\Delta^1 = \begin{vmatrix} a_1 + ka_3 & b_1 + kb_3 & c_1 + kc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

then expanding we get, $\Delta = \Delta^1$

- If each element of any row or column of a determinant is expressed as a sum of two terms then the determinant can be expressed as a sum of the determinants.

$\Delta = \begin{vmatrix} a_1 + \alpha & b_1 & c_1 \\ a_2 + \beta & b_2 & c_2 \\ a_3 + \gamma & b_3 & c_3 \end{vmatrix}$

$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha & b_1 & c_1 \\ \beta & b_2 & c_2 \\ \gamma & b_3 & c_3 \end{vmatrix}$

- If all the elements of any row or column are multiplied by a constant k, then the value of the determinant is multiplied by k.

|A| = $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

and $|A^*|$ = $\begin{vmatrix} ka_1 & b_1 & c_1 \\ ka_2 & b_2 & c_2 \\ ka_3 & b_3 & c_3 \end{vmatrix}$ = k|A|

Det (AB) = det (A) . det (B)

- For any square matrix A, $|A| = |A^T|$
- $|\lambda A| = \lambda^n |A|$, where n is the order of the square matrix A
- $|adj.\ A| = |A|^{n-1}$
- If the elements of a row or column are multiplied by cofactors of respective elements of another row or column, then the sum of all such product is zero.

If A = $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$,

Then, $a_1 A_2 + b_1 B_2 + c_1 C_2 = 0$