Sequences and Series

Sequence

- A sequence of real numbers is a function (rule) whose domain is the set of all the natural numbers & whose range is a subset of $\mathbf{R}$. A sequence is a function on the set of positive integers. In other words, a function $f : \mathbf{Z}^+ \to \mathbf{R}$ is called a sequence.

  $R_f = \{f(x) : x \in Z\}$.

  For example
  i. $1, 2, 3, \cdots$ is a sequence whose n^{th} term is n.
  ii. $1, 4, 9, 16, \cdots$ is a sequence whose n^{th} term is n^2.

- A sequence with finite number of terms is called a **finite sequence**. Otherwise we mean an **infinite sequence**.

- A sequence $\{a_n\}$ such that $|a_n| \leq k$ for all n where k is a finite number is known as a **bounded sequence**.

- A sequence $\{a_n\}$ such that $a_n \leq a_{n+1}$ or $a_n \geq a_{n+1}$ for all n is called monotonic (increasing or decreasing) sequence.

Series

If $a_1, a_2, a_3, \ldots$ is a sequence, then the corresponding series is $a_1 + a_2 + a_3 + \ldots$

A series may or may not have a sum.

Arithmetic Sequence

- A sequence is said to be an arithmetic sequence if there is a uniform common difference between the consecutive terms.
  $a_1, a_2, a_3, \ldots$ is an A.S. then the common difference $(d) = a_2 - a_1 = a_3 - a_2 = a_{k+1} - a_k \; k \geq 1$

- The general term of an A.P or the n^{th} term of an A.P. $a, a+d, a+2d \ldots$ is $a + (n-1)d$
  i.e, $t_n = a + (n-1)d$
  where
  $a = 1^{st}$ term
  $d =$ common difference
  $n =$ number of terms

Arithmetic means (A.M.'s)

a. Let A be the arithmetic mean between a and b. Then $A = \dfrac{a+b}{2}$

b. Let $A_1, A_2, A_3, \cdots, A_n$ be the n A.M's between two numbers a & b.

   Then,

   $a, A_1, A_2 \cdots A_n, b$ are in A.P.

   So, $A_1 = a + d$
   $A_2 = a + 2d$
   $\ldots\ldots\ldots\ldots$
   $\ldots\ldots\ldots\ldots$
   $A_n = a + nd$ & so on.

   We have,
   $b = a + (n + 2 - 1)d$
   $$d = \frac{b-a}{(n+1)}$$

   where, n is number of means.

   Hence, common difference $d = \dfrac{b-a}{n+1}$

c. The sum of n A.M's between a and $b = n\left(\frac{a+b}{2}\right)$

Sum of an A.P.

Sum of the first n terms i.e. $S_n = \frac{n}{2}[2a + (n-1)d]$

If l denotes the last term, then

$$S_n = \frac{n}{2}[a + l]$$

Some properties of Arithmetic sequence

- If a fixed number is added to (or subtracted from) each term of an A.P., then the resulting terms are also in A.P.
- If each term of A.P. is multiplied (or divided) by a fixed non-zero number then the terms are also in A.P.
- If a, b, c are in A.P., then $b - a = c - b$

  $$\text{or } 2b = a + c$$
  $$\text{i.e } b = \frac{a+c}{2}$$

For practical purpose, we choose:

- any three numbers in A.P. are (a –d) , a, (a +d).
- any four numbers in A.P. are (a – 3d), (a – d) (a +d), (a + 3d)
- Five numbers in A.P. are (a – 2d), (a – d) , a, (a +d) (a + 2d) and so on.

Geometric Sequence

A sequence is said to be a geometric sequence if there is a uniform common ratio between any two consecutive terms in the series. If a is the first term and r the common ratio, then

n^{th} term of the series $t_n = ar^{n-1}$

By definition

$$r = \frac{t_2}{t_1} = \frac{t_3}{t_2} = \cdots \frac{t_n}{t_{n-1}}$$

i.e. $r = \frac{a_{k+1}}{a_k}; k \geq 1$

Sum up to the first n terms of G.P.

$S_n = \frac{a(1 - r^n)}{(1 - r)}, r \neq 1$

$S_n = \frac{lr - a}{r - 1}(r > 1)$ and $S_n = \frac{lr - a}{1 - r}(r < 1)$

For practical purpose, we choose,

three numbers $\frac{a}{r}$, a, ar are in G.P.,

four numbers $\frac{a}{r^3}$, $\frac{a}{r}$, ar, ar^3 are in G.P.,

five numbers $\frac{a}{r^2}$, $\frac{a}{r}$, a, ar, ar^2 are in G.P.

and so on.

Geometric means:

Geometric mean between any two numbers a and b is $\sqrt{ab}$

If a, b, c are any three numbers in G.P. then

$\frac{b}{a} = \frac{c}{b}$ i.e $b^2 = ac$

If $G_1, G_2, \cdots, G_n$ are the n Geometric means between a and b then

$a, G_1, G_2, \cdots, G_n, b$ are in G.P.

So, $G_1 = ar, G_2 = ar^2$ and so on.

i.e, $b = ar^{(n+2-1)}$

$$\therefore \left(\frac{b}{a}\right)^{\frac{1}{n+1}} = r$$

So, the common difference $(r) = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

Note:

If $a_1, a_2, \cdots$ are in A.P and $b_1, b_2, \cdots$ are in G.P., then

$a_1b_1, a_2b_2, \cdots$ is called an arithmetic geometric sequence.

Sum to infinity

a. Sum of an infinite geometric series with first term a and common ratio r such that $|r| < 1$ is given by:

$$S_\infty = \frac{a}{1-r}$$

b. Sum of an infinite Arithmetic Geometric (A.G) series with first term a and the common difference d of arithmetic series and common ratio r such that $|r| < 1$ of geometric series is given by,

$$S_\infty = \frac{a}{1-r} + \frac{d.r}{(1-r)^2}$$

Some properties of Geometric sequence

i. If each term of a G.S. is multiplied or divided by a non-zero constant number, then the resulting sequence is also in G.P.

If a, b, c, d .... are in G.S. and k is a non-zero constant number then ak, bk, ck, dk .... are in G.P.

Also,
$\frac{a}{k}, \frac{b}{k}, \frac{c}{k}, \frac{d}{k}$........... are in G.P.

ii. The reciprocals of a terms of a given G.P. forms a G.P.

i.e. a, ar, ar^2 ...., ar^{n-1} .... are in G.P., then

$\frac{1}{a}, \frac{1}{ar}, \frac{1}{ar^2}$ .... , $\frac{1}{ar^{n-1}}$ .... are also in G.P.

Common ratio of the new G.P. is $\frac{1}{r}$

iii. If each term of a G.S. is raised to a constant power, then the resulting sequence is also in G.P.

i.e. if a, b, c, d .... are in G.P.,then

a^k, b^k, c^k, d^k .... are in G.P.

Harmonic Sequence

If the terms of a given series are in arithmetic series then the series formed by taking the reciprocal of its each terms is called a harmonic series.

i. If a_1, a_2, a_3 ......, a_n are in A.P., then

$\frac{1}{a_1}$, $\frac{1}{a_2}$, $\frac{1}{a_3}$........ , $\frac{1}{a_n}$are in H.P.

ii. 1, 2, 3, 4 .... are in A.P.

So, 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ .... are in H.P.

iii. If a, b, c are any three numbers in H.P., then $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$are in A.P. So,

$\frac{1}{b}$ - $\frac{1}{a}$ = $\frac{1}{c}$- $\frac{1}{b}$

or, $\frac{2}{b}$= $\frac{1}{c}$+ $\frac{1}{a}$

$b = \frac{2ac}{a+c}$

iv. nth term of H.P. (General term)

$\frac{1}{a}$, $\frac{1}{a+d}$, $\frac{1}{a+2d}$........ $\frac{1}{a+(n-1)\,d}$

i.e. t_n = $\frac{1}{a+(n-1)\,d}$

v. H is the Harmonic mean between any two number a and b then

$H = \frac{2ab}{a+b}$

vi. H.M. between a_1, a_2 .... a_n is given by

$\frac{1}{H}$= $\frac{1}{n}$ $\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots\ldots\ldots+\frac{1}{a_n}\right)$

Properties of Harmonic sequence

If each term of H.S. is multiplied or divided by a constant quantity, then the resulting sequence is also in H.S.

i.e., If a, b, c, d .... are in H.S. and k is a constant (k 0),
then ka, kb, kc, kd ....are in H.P.

$\frac{a}{k}$, $\frac{b}{k}$, $\frac{c}{k}$, $\frac{d}{k}$.........are in H.P.

Sum to n terms

i. Sum of the 1^{st} n natural numbers

$1+2+3+\cdots+n = \frac{n(n+1)}{2}$

$\sum n = \frac{n(n+1)}{2}$

ii. Sum of the 1^{st} n even natural numbers

$2+4+6+\cdots$ to n terms $= n(n+1)$

iii. Sum of the 1^{st} n odd natural numbers $1+3+5+\cdots$ to n terms $= n^2$

iv. Sum of the squares of 1^{st} n natural numbers

$= 1^2$ +2^2 +3^2 + -------- +n^2

$= \frac{n(n+1)(2n+1)}{6}$

i.e. $\sum n^2 = \frac{n(n+1)(2n+1)}{6}$

v. Sum of the cubes of 1^{st} n natural numbers = 1^3 +2^3 + ------ +n^3

$= \left[\frac{n(n+1)}{2}\right]^2$

$[\text{sum of the} 1^{st} n \text{natural numbers}]^2$

i.e. $\sum n^3 = \left[\frac{n(n+1)}{2}\right]^2$

Relationship among AM, GM and HM

i. If A.M., G.M. & H. M. are the arithmetic mean, geometric mean & harmonic mean respectively between any two unequal positive numbers, then A.M. >G.M. > H.M.

ii. A.M. $\times$ H. M. = (G. M.)^2

Note

i. If A, G, H are the arithmetic mean, geometric mean and harmonic mean respectively between a and b, then

$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \begin{cases} A & when\, n = 0 \\ G & n = -\frac{1}{2} \\ H & n = -1 \end{cases}$$

ii. If S is the sum, P is the product & R is the sum of reciprocals of the n terms of a G.P. , then P^2 = $\left(\frac{S}{R}\right)^n$

iii. If any three numbers are in A.P. as well as in G.P. , then they are equal i.e. a, b, c are in A.P. as well as in G.P. then a = b = c

iv. For the sequence, a, (a + d)r, (a + 2d)r^2, --- we have
t$_n$ = [a + (n -1)d] r^{n-1}, n $\in N$ and

$$S_n = \frac{a}{1-r} + \frac{d \cdot r\left(1-r^{n-1}\right)}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{1-r}, n \in \mathbb{N}$$