System of Linear Equations

Equation

An equity relation is said to be an equation if it is true for centain values of the indeterminates.

Example:-

$y = x^2 - 3x + 4 = 0,\ y = x + 1 = 0$

Roots of the equation:

The value or the set of values which satisfies the given equation are known as the roots of the equation

Example:

$x^2 - 3x + 2 = 0$

$x^2 - 2x - x + 2 = 0$

$(x - 2)(x - 1) = 0$

Either,

x = 2

OR,

x = 1

1 and 2 are the roots of the equation $x^2 - 3x + 2 = 0$ as it is satisfied by these values.

Identity:-

An equity relation is said to be an identity if it is true for every value of the indeterminates.

Example:

$(x + 1)^2 = x^2 + 2x + 1$

This relation is an identity as it is satisfied by all the values of x.

Consistent and Inconsistent Systems

- **Consistent**

  System of equations is said to be consistent if it has a solution whether unique or infinite number of solution.

- **Inconsistent**

  If the system has no solution, then it is called inconsistent.

Trivial and Non-trivial solution:

A solution in which every variable has zero value is called trivial solution.

i.e. x = 0, y = 0, z = 0

A non-zero solution is called non-trivial solution.

For example

(0,0,0) is a trival solution of x+ 2y - z =0 and (1, 2, 5) is a non trivial solution.

Intersecting lines:

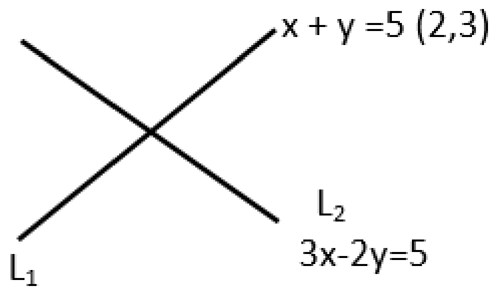
Any two geometrically intersecting lines have unique solution.

They are consistent and independent

Example:

x + y = 5

3x – 2y = 5



Parallel lines:-

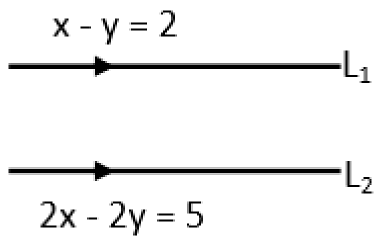
Any two geometrically parallel lines have no solution.

They are inconsistent and independent.

Example:

x − y = 2

2x − 2y = 5



Coincident lines:-

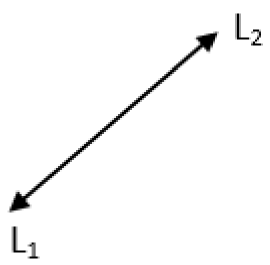
Any two geometrically coincident lines have infinite solution.

They are dependent and consistent.

Example:

3x − y = 5

6x − 2y = 10



If two equations have three variables then there are infinite solutions, in general.

Example:

x + 2y +z = 5

3x + 4y +z = 7 has infinite solution, where as

x + 2y + z = 5

2x + 4y + 2z = 7 has no solution.

The system of equations

x - 2y − z = -7

2x + y +z = 0

3x − 5y +8z = 13

has a unique solution i.e. x =-2, y=1 and z = 3

Solution of Simultaneous Linear Equations (Cramer's Rule)

(I) Consider a system of simultaneous linear equations: -

$a_1x + b_1y = c_1$

$a_2x + b_2y = c_2$

When

D = $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ = $a_1b_2 - a_2b_1$,

$D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ = $c_1b_2 - c_2b_1$ and

(Replacing 1st column by constant column)

$D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$ = $a_1c_2 - c_2a_1$,

(Replacing 2nd column by constant column),

then we get,

$x = \dfrac{D_1}{D}$ and $y = \dfrac{D_2}{D}$

(II) Consider a system of simultanceous linear equations: -

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

$a_{12}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$

When

$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$ (Replacing 1^{t} column by constant column)

$D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$ (Replacing 2^{nd} column by constant column)

$D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$ (Replacing 3^{rd} column by constant column)

Then we get,

$$x = \frac{D_1}{D}, y = \frac{D_2}{D}, z = \frac{D_3}{D}$$

Important Notes:-

- If $D \neq 0$ then the system has unique solution

$D \neq 0$

i.e. if $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$, the system is consistent and independent

- System has infinite solution if $D = D_1 = D_2 = 0$.

i.e. $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$, the system is dependent and consistent.

- When the system of linear equations is

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

$a_{12}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$

- These equations can be written in the matrix form:

A X = B where,

$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

X = $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

B = $\begin{bmatrix} \mathrm{b}_1 \\ \mathrm{b}_2 \\ \mathrm{b}_3 \end{bmatrix}$

- The matrix A is called the coefficient matrix.
- X is called the matrix of unknown.
- A set of values of variables x_1, x_2, x_3 which simultaneously satisfy these three equations is called a solution. A system may contain n equations in n variables.

Notes:-

For lines $a_1x + b_1y = c_1$

$a_2x + b_2y = c_2$

(i) If $\frac{\mathrm{a}_1}{\mathrm{a}_2} \neq \frac{\mathrm{b}_1}{\mathrm{b}_2}\left(\text{ or }\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0\right)$, the lines are consistent and independent.

It has unique solution.

(ii) If $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, (parallel line) the lines are inconsistent and independent.

It has no solution.

(iii) If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, (coincident lines) the lines are consistent and dependent.

It has infinite solutions.