Polynomial Equations

Polynomial

An algebraic expression of the type

$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots\ldots\ldots + a_n x^n$

is called polynomial.

where a_0, a_1, a_2 .... a_n are constants.

Polynomial Equation

An equation of the form

$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a^n = 0$

where $(a_0 \neq 0)$ and $a_1, a_2, a_3, \ldots, a_n$ are constants is called a **polynomial equation of degree** n.

Here a_0 is known as **leading coefficient** and $a_0 x^n$ is the **leading term** of the polynomial.

If the coefficients $a_0, a_1, a_2, \ldots, a_n$ are real, then it is called **real polynomial.**

If the coefficients $a_0, a_1, a_2, \ldots, a_n$ are complex numbers then it is called **complex polynomial**.

Degree of polynomial

Degree of polynomial is the degree of the highest power of a variable in the given polynomial.

Example

The degree of the polynomial equation $P(x) = 3x^3 - 7x^2 + 5x - 1 = 0$ is 3

The degree of the equation $3x^2 - 5x + 4 = 0$ is 2

Roots or Zeros of the polynomial

If at $x = \alpha$, the value of polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ is zero, then α is a the root or zero of f(x).

That is, if $f(\alpha) = 0$, α is a root or zero of f(x).

Note

i. Number of roots of the equation depends upon the degree of the equation.

   i.e 2^{nd} degree equation has two roots and nth degree equation has n roots (real or imaginary, equal or distinct).

ii. A polynomial where all the coefficients are 0 is called **zero polynomial**.

   Degree of zero polynomial is not defined $(-\infty)$.

iii. Every polynomial of degree $n \geq 1$ has at least one real or imaginary root. (Fundamental theorem of Algebra)

iv. If all terms of an equation are +ve and there is no term of odd power of x, then all its roots are complex numbers.

Special types of polynomial equations are

i. Linear equation : $ax + b = 0\ (a \neq 0)$

ii. Quadratic equation : $ax^2 + bx + c = 0 (a \neq 0)$

iii. Cubic equation : $ax^3 + bx^2 + cx + d = 0 (a \neq 0)$

iv. Biquadratic equation : $ax^4 + bx^3 + cx^2 + dx + e = 0 (a \neq 0)$

Division Algorithm:

For any two polynomials $f(x)$ and $g(x) \neq 0$, there exist unique polynomials $Q(x)$ and $R(x)$ such that $f(x) = g(x)Q(x) + R(x)$ where $R(x) = 0$ or $\deg R(x) < \deg g(x)$

Remainder Theorem

If a polynomial $f(x)$ is divided by $(x - a)$ then the remainder $R = f(a)$.

Factor Theorem

$(x-a)$ is a factor of any polynomial function $f(x)$ if remainder $R = f(a) = 0$

If $f(a) = 0$ for polynomial $f(x)$, then a is a zero or root of polynomial $f(x)$.

Quadratic Equation

The equation of the form $ax^2 + bx + c = 0$, where a, b, c are constants and $a \neq 0$ is called a quadratic equation.

Degree of quadratic equation is two.

Quadratic equation cannot have more than two roots.

Methods of Factorization

Let $ax^2 + bx + c = 0, a \neq 0$ be the given quadratic equation and let the expression $ax^2 + bx + c = 0$ be capable of having $(dx + e)$ and $(fx + g)$ as two linear factors, where $d \neq 0$ and $f \neq 0$ then

$$ax^2 + bx + c = 0$$
$$(dx + e)(fx + g) = 0$$

Then,

Either,

$$dx + e = 0$$
$$x = \frac{-e}{d}$$

Or

$$fx + g = 0$$
$$x = \frac{-g}{f}$$

$\therefore$ the solution set $= \left\{ \frac{-e}{d}, \frac{-g}{f} \right\}$

Method of completing the square

Let the quadratic equation be:

$ax^2 + bx + c = 0, a, b, c \in R$ and $a \neq 0$

$$ax^2 + bx = -c$$
$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$
$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$
$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Roots of the quadratic equation

Let α and β be the roots of the quadratic equation $ax^2 + bx + c = 0$ then

x = $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$

where,

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$
$$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

i. Sum of the roots $(\alpha + \beta)$ = -b/a

= $\frac{\text{coefficient of } x}{\text{coefficient of } x^2}$

ii. product of the roots $(\alpha\beta)$ = c/a
$= \frac{\text{constant tem}}{\text{coefficient of } x^2}$

iii. If two roots of the equation are equal in magnitude but opposite in sign, then sum of the roots
$\alpha + (-\alpha) = 0$
$-b/a = 0$
$\text{i.e}, b = 0$

iv. α and β be the roots which are reciprocal to each other, then product of the roots
$\alpha \cdot (1/\alpha) = c/a$
$\text{i.e. } c = a$

Formation of quadratic equation

If α and β are the roots of the quadratic equation, then the equation may be
$(x-\alpha)(x-\beta) = 0$
$x^2 - (\alpha+\beta)x + \alpha\beta = 0$
$x^2 -$ (sum of the roots)$x +$product of the roots$= 0$

Nature of the roots:- If $a, b, c \in R$

Nature of the roots depends upon the discriminant, $D = b^2 - 4ac$

i. If $b^2 - 4ac > 0$, then the roots are real and unequal
ii. If $b^2 - 4ac < 0$, then the roots are imaginary and unequal
iii. If $b^2 - 4ac = 0$, then the roots are real and equal and the equal root is $-b/2a$.
iv. If a, b, c are rational and $(b^2 - 4ac)$ is a perfect square, then the roots are rational.
v. Irrational roots always occur in pair

If one root is p + $\sqrt{q}$,

then other root is p - $\sqrt{q}$.

vi. Complex roots are conjugate to each other:

If α = 2 +3i is one of the roots, then

β = 2 – 3i is the other root.

Some important results

If $f(x) = 0$ has roots α and β then

- f(1/x) = 0 has roots $\frac{1}{\alpha}, \frac{1}{\beta}$
- f($\sqrt{x}$) = 0 has roots α^2, β^2
- f(x $-$ a) = 0 has roots $\alpha + a$, $\beta + a$
- f(x/k) = 0 has roots kα, kβ
- f($-$x) = 0 has roots $-\alpha$, $-\beta$
- f$\left(\frac{x-q}{p}\right)$ = 0 has roots pα + q, (pβ+q)

  [as $\frac{xq}{p}$= α x = pα + q]

Note
If α, β, γ be the roots of the cubic equation $ax^3 + bx^2 + cx + d = 0$ then
(a) (sum of the roots $(\alpha + \beta + \gamma) = -b/a$
(b)$\alpha\beta + \beta\gamma + \gamma\alpha$= c/a
(c) $\alpha\beta\gamma = -d/a$

Reciprocal roots:

The equation whose roots are the reciprocal of the roots of the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ is:
$\frac{a}{x^4} + \frac{b}{x^3} + \frac{c}{x^2} + \frac{d}{x} + e = 0$
i.e. $ex^4 + dx^3 + cx^2 + bx + a = 0$

One Root Common

Let 'α' be the common root of the equation

$a_1x^2 + b_1x + c_1 = 0$

$a_2x^2 + b_2x + c_2 = 0$

then

$a_1\alpha^2 + b_1\alpha + c_1 = 0$

$a_2\alpha^2 + b_2\alpha + c_2 = 0$

By cross – multiplication:

$$\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{\alpha}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

taking 1st two terms

$$\alpha = \frac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1}$$

taking last two terms

$$\alpha = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

Hence,

$$\frac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1} = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

$(b_1c_2 - b_2c_1)(a_1b_2 - a_2b_1) = (c_1a_2 - c_2a_1)^2$

is the required condition for one root common.

Two Roots Common

Let α and β be the common roots of

$a_1x^2 + b_1x + c_1 = 0$

$a_2x^2 + b_2x + c_2 = 0$

then

sum of the roots (a+b) = $\frac{-b_1}{a_1} = \frac{-b_2}{a_2}$

$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2}$ ---------------(i)

product of the roots (ab) = $\frac{c_1}{a_1} = \frac{c_2}{a_2}$

$\therefore \frac{c_1}{c_2} = \frac{a_1}{a_2}$ ---------------(ii)

$\therefore$ The required condition is $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Some Important Results

Let a and b be the roots of the quadratic equation $ax^2 + b|x| + c = 0$.

(i) If a and b both are negative, no of real roots = 0

(ii) If a and b both are positive, no. of real roots = 4

(iii) If either a or b is negative and the other is positive. no. of real roots = 2

Equation	Root (a)	Root (b)	Number of real roots
$ax^2 + b\|x\| + c = 0$	+ve	+ve	no. of real roots =4
	-ve	-ve	no. of real roots = 0
	+ve	-ve	no. of real roots = 2
	-ve	+ve	no. of real roots = 2

Number of real roots

$x^2 + 5\|x\| + 4 = 0$	$\|x\| = -4$ $\|x\| = -1$	no. of real roots =0
$x^2 - 4\|x\| + 3 = 0$	$\|x\| = 3$ $\|x\| = 1$	no. of real roots =4 ±1, ±3
$x^2 + \|x\| - 12 = 0$	$\|x\| = 1$ $\|x\| = -4$	no. of real roots =2 ±1

Note

(i) If $ax^2 + bx + c = 0$ is satisfied by more than two distinct values of x, then **a=b=c=0**

(ii) The graph $y = f(x) = ax^2 + bx + c$, $a \neq 0$ is a parabola whose axis is parallel to y-axis

The parabola will be upward or downward according as a >0 or a<0.

(iii) If both the roots of the equation $ax^2+bx+c = 0$ are positive, then the sum of the roots and product of the roots are positive.

i.e. sum of the roots = -b/a >0

(i.e., b and a have opposite signs.)

and product of the roots = c/a > 0

(i.e. b and a have same sign.)

c and a have same sign but oppositive to that of b.

(iv) If a and c have same sign, then ac > 0.

Symmetric functions of the roots

If α, β are the roots of $ax^2 + bx + c = 0$, and if in any expression involving α, β we interchange α, β and expression does not change, then the expression is called a symmetric function of roots α, β. E.g., $\alpha + \beta, \alpha^2 + \beta^2, \alpha^3 + \beta^3, \alpha^2 + \alpha\beta + \beta^2$ etc. are symmetric functions of a, b.

1. If α, β are the roots of $ax^2 + bx + c = 0$, then $\alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}$ and $a\alpha^2 + b\alpha + c = 0$

   $\Rightarrow a\alpha + b + \frac{c}{\alpha} = 0$

   $\Rightarrow a\alpha + b = -\frac{c}{\alpha}$, similarly $a\beta + b = -\frac{c}{\beta}$
2. $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \frac{b^2 - 2ac}{a^2}$
3. $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$= \frac{b^2}{a^2} - 4\frac{c}{a}$

$\Rightarrow \alpha - \beta = \frac{\sqrt{b^2 - 4ac}}{a}$

Maximum and minimum values of quadratic function

- If a > 0, then the minimum value of the expression $= \frac{4ac - b^2}{4a}$ for $x = \frac{-b}{2a}$
- If a < 0, then the maximum value of the expression $= \frac{4ac - b^2}{4a}$ for $x = \frac{-b}{2a}$