Sets and Relations

Set

A set is a well-defined collection of distinct objects. The objects of a set are called the elements or the members of the set. The sets are generally denoted by capital letters A, B, C … and the elements of the set are denoted by small letters a, b, c … The element of a set can be anything material or conceptual.

If x is an element of a set A, we write $x \in A$ which means that 'x belongs to A' or that 'x is an element of A.'

We write $x \notin A$ which means x does not belong to A or x is not an element of set A.

Representation of a set

(i) Listing method (Roaster method)

In this method, we write every element of the set in a row, separating every two elements by a comma and we enclose them by curly brackets or braces. This method is also called **tabular method**.

A = {c, a, n}

B = {2, 4, 6, 8 -----} etc.

This method is not convenient in case of larger sets.

(ii) Set Builder method (Rule method)

In this method, we write the property which gives us all the elements of the set.

A = {x : x is a vowel}

i.e. A = {a, e, i, o, u}

B = {x : x is an even positive integer less than 10}

i.e. B = {2, 4, 6, 8}

Types of sets:

(i) Universal set: -

If all the elements under consideration belongs to of a fixed set, then this fixed set is called a universal set. It is denoted by the symbol U or X.

(ii) Singleton set: -

A set having only one element is called a singleton set.

Examples: {a}, {1}, {the highest mountain on the earth}, {x : x = 0} are some singleton sets.

(iii) Finite set: -

A set having finite number of elements is called finite set.

Examples: A = {x : x is a set of vowels}

i.e. A = {a, e, i, o, u}

B = { x : x is a planet of the solar system}

C = {x : x set of odd numbers less than 30}

(iv) Infinite set: -

A set having infinite number of elements is called infinite set.

Example:

A = {1, 2, 3, 4 -----------}

B = {points in a line}

N, Q, R, Z are all infinite sets,

where; **N** = set of all the natural numbers,

Q = set of all the rational numbers,

R = set of all the real numbers,

Z = set of all the integers,

(v) Null set or void set : -

A set having no element is called a null, empty or void set. It is denoted by ϕ or { }

A = $\{x : x \neq x\} = \phi$

Null set is a finite set.

Example:

(i) set of point of intersection of parallel lines.

(ii) set of male students in Padma Kanya Campus

(iii) set of women prime ministers of Nepal

(iv) set of commerce students in Amrit Science College.

Note:-

$\phi \rightarrow$ empty set or set consisting of no elements.

$\{0\} \rightarrow$ set containing only one element 0. It is a singleton set.

(vi) Subset of a set: -

A set A is said to be a subset of set B if every element of A is also an element of B.

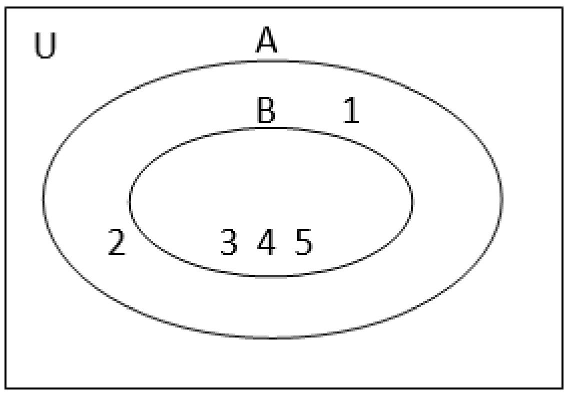
We write this relation as A $\subseteq$ B.

Examples:

(a) A = {1,. 2, 3, 4, 5}

B = {3, 4, 5}

Then B $\subset$ A



(b) P (set of all the prime numbers) $\subseteq$ N(set of all the Natural numbers) $\subseteq$ Z (set of integers) $\subseteq$ Q (set of all the rational numbers) $\subseteq$ R (set of all the real numbers)

i.e. P $\subseteq$ N $\subseteq$ Z $\subseteq$ Q $\subseteq$ R

Note: -

- Every set is a subset of itself A $\subseteq$ A.
- Empty set ϕ is a subset of every set.
- Every non-empty set has at least two subsets; i.e. ϕ and itself. They are called trivial subsets.
- If A is a subset of B, then B is a superset of A.
- A $\subseteq$ A
- A$\subseteq$B, B$\subseteq$A implies A = B
- A$\subseteq$B, B$\subseteq$C implies A$\subseteq$C
- A $\subset \phi$ implies A = ϕ

(vii) Proper subset : -

A Set A is said to be a proper subset of B if A $\subset$ B and A $\neq$ B.

We write this as A $\subset$ B

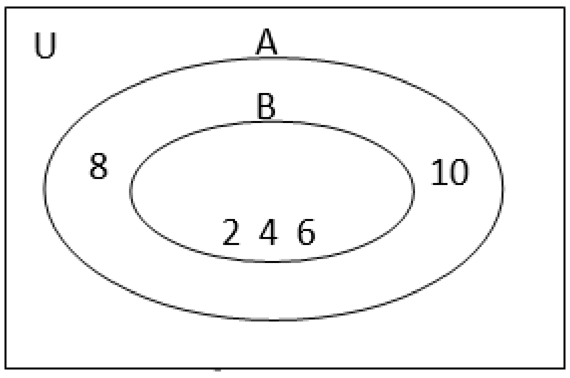
In this case set B is the superset of A.

e.g. A = {2, 4, 6, 8, 10}

B = { 2, 4, 6}

B $\subset$ A and A $\neq$ B

Hence B is proper subset of A.

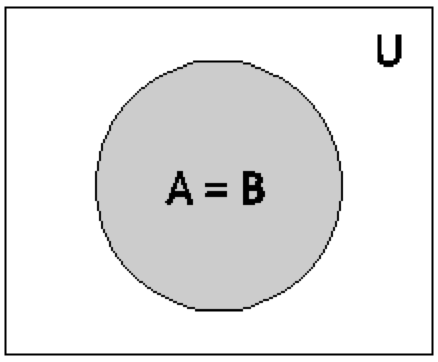


Equal sets:

Two sets A and B are said to be equal if every element of A is an element of B and every element of B is an element of A. i.e.

$A \subseteq B$ and $B \subseteq A$

then we write A=B



Examples:

(i) A = {3, 7, 2, 1}

B = {1, 2, 3, 7}

Then, A = B

(ii) A = {G, O, D}

B = {D, O, G}

then A = B

Two sets A and B are said to be comparable if $A \subseteq B$ or $B \subseteq A$.

(iii) A = {x: x is a prime number between 2 and 10}

B = {3,5,7}

then A = B

(ix) **Power Set: -**

Let A be any set. Then, the set of all the subsets of A is called the power set of A. It is denoted by 2^A or P(A).

Example: A = {1, 2}

P(A) = {ϕ, {1}, {2}, {1, 2}}

No. of subsets = $2^n = 2^2 = 4$

Example: Let B = {1, 2, 3}

Subsets containing one element : {1}, {2}, {3}

Subsets containing two elements: {1. 2}, {1, 3}, {2, 3}

Subsets containing three elements : {1, 2, 3}

Subset containing no element : ϕ

$\therefore$ 2^B = {f, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}}

(i) Number of elements in P(B)= $2^n = 2^3 = 8$

(ii) Number of proper subsets of B = $2^n - 1$

=8-1 = 7

(except the element {1, 2, 3})

(iii) Number of non empty proper subsets

$= 2^n - 2 = 8 - 2 = 6$

(except the elements ϕ and {1, 2, 3})

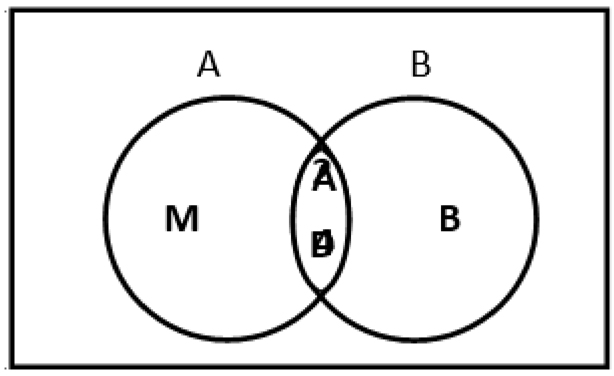
(x) **Intersecting sets: -**

Two sets A and B are said to be intersecting or overlapping if they have at least one element in common.

Example:

A = {M, A, P}

B = {B, A, D} are the intersecting sets.

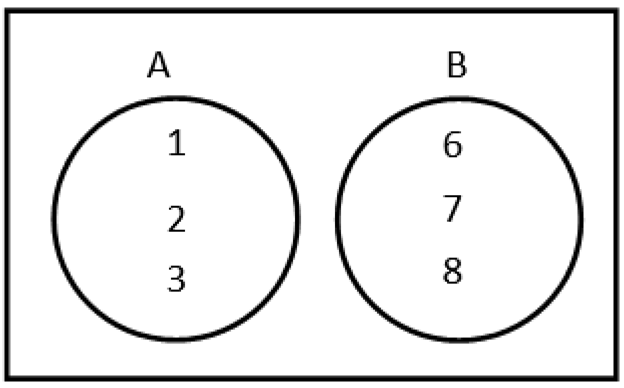


Two axes XOX' and YOY' intersect at the origin. Hence they are two intersecting sets.

(xi) **Disjoint sets: -**

Two sets are said to be disjoint sets if they have no element in common.

i.e $A \cap B = \phi$



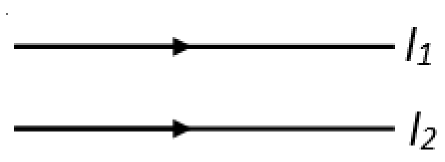
Examples:

(i) A = {1, 2, 3}

B = {6, 7, 8}

Then, A and B are disjoint sets.

(ii) Set of points on two parallel lines are disjoint sets.



(iii) Two sets A = {b, l,a, c,k}

B = {w, h, i ,t, e} are disjoint sets.

Equivalent sets

Two sets A and B are said to be equivalent sets if they have the same number of elements. In this case we write A~B and read as "A is equivalent to B".

Examples:

(i) Two sets

A = {n, i, n, e}

B = {p, i, n, e} are equivalent sets.

(ii) A = {set of angles of a $\triangle$}

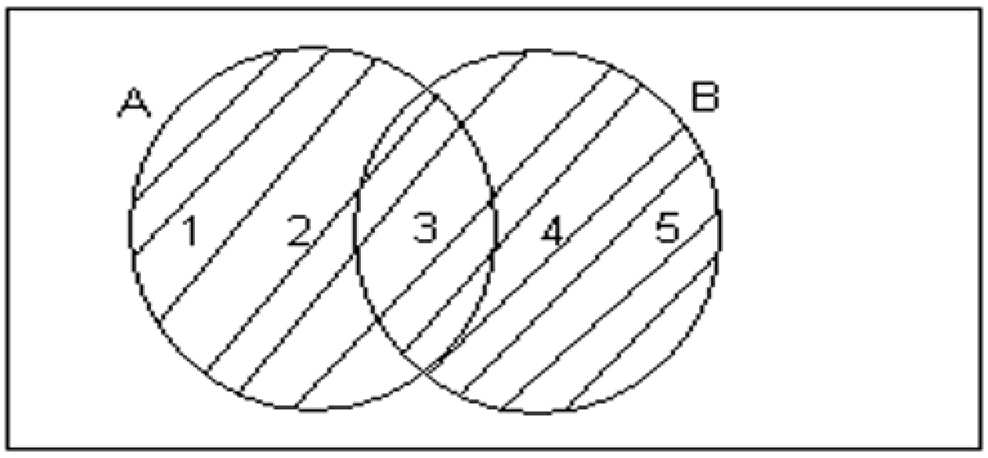
B = {set of sides of a △} are equivalent sets.

Set Operations

Union of sets:-

The union of two sets A and B is the set of all the elements which belong to either A or B or both.

i.e $A \cup B = \{x : x \in A \text{ or } x \in B\}$



$(A \cup B)$

Examples:

If A = {1, 2, 3}

& B = {3, 4, 5}, then

$A \cup B = \{1, 2, 3, 4, 5\}$

(i) A = {x:x is a prime number less than 9}

B = {2,4,6, 8}

i.e. A = {2,3,5,7}

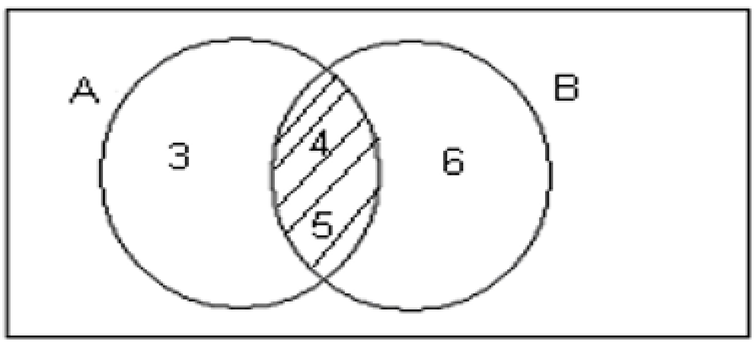
B = {2,4,6,8}

$A \cup B = \{2, 3, 4, 5, 6, 7, 8\}$

Intersection of Two sets: -

Intersection of two sets A and B is the set of all elements which belong to both A and B.

$A \cap B = \{x : x \in A \text{ and } x \in B\}$



$(A \cap B)$

Example:

A = {3, 4, 5}

B = {4, 5, 6}

$A \cap B = \{4, 5\}$

(i) A = {x: x is a set of integers}

B = {x : x is a natural no.}

i.e. A = {.....,-2, -1, 0, 1, 2, 3, ... }

B = {1, 2, 3, 4, ... } . Then,

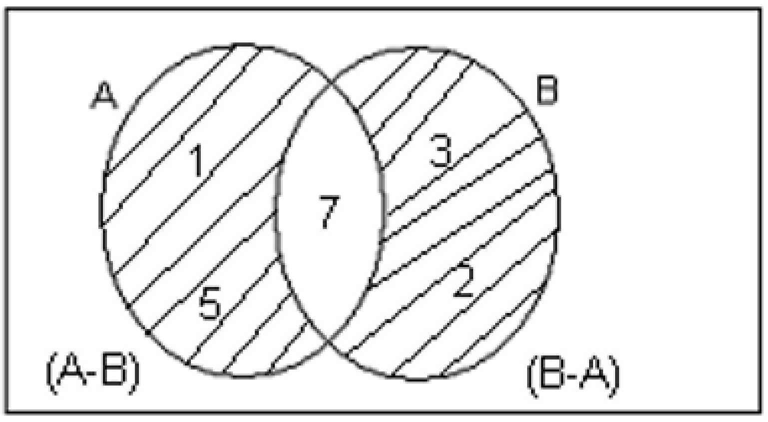
$A \cap B = \{1, 2, 3, 4, ... \}$

(set of natural numbers)

Difference of two sets: -

The difference of two sets A and B is the set of all the elements of A which do not belong to B.

i.e. $A - B = \{x : x \in A \text{ but } x \notin B\}$



Example:-

(i) $A = \{1, 7, 5\}$

$B = \{7, 3, 2\}$

Then, $A - B = \{1, 5\}$ and

$B - A = \{3, 2\}$

(ii) $A = \{x : x$ is a real number$\}$

$B = \{x : x$ is a rational number$\}$

$A - B = \{x : x$ is a irrational number$\}$

Complement of a set: -

The complement of a set A denoted by A^C or $\bar{A}$ or A' is the set of all the elements of the universal set U which do not belong to A.

i.e. $A^c = \{x : x \in U \text{ but } x \notin A\}$

Example:-

If $U = \{1, 2, 3, 4 \text{ ----- } 9, 10\}$

$A = \{3, 4, 5, 6, 7\}$

$A^c = U - A$

$= \{1, 2, 3, 4, \text{---------} 9, 10\} - \{3, 4, 5, 6, 7\}$

$= \{1, 2, 8, 9, 10\}$

Symmetric Difference: -

The symmetric difference of any two sets A and B is the union of the differences (A – B) and (B – A) i.e. this is the set of all non-common elements of A and B.

i.e. $A \Delta B = (A - B) \cup (B - A)$

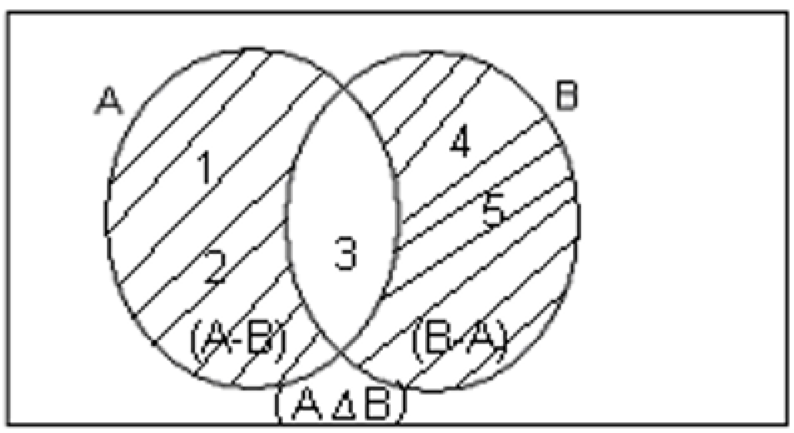
In other words:

$A \Delta B = (A \cup B) - (A \cap B)$

Example :

(i) $A = \{1, 2, 3\}$

$B = \{3, 4, 5\}$



A − B = {1, 2, 3} − {3, 4, 5}

= {1, 2}

B − A = {3, 4, 5} − {1, 2, 3}

= {4, 5}

A Δ B = (A − B) ∪ (B − A)

= {1, 2, 4, 5}

(ii) A = {s, a, d}

B = {b,a,d}

(A - B) = {s}

(B - A) = {b}

(A Δ B) = (A - B) ∪ (B − A)

= {s, b}

<u>Cardinal Number of a Set:-</u>

The number of distinct elements contained in a finite set A is called the cardinal number of the set. It is also called the order of the set and denoted by o(A) or n(A).

Example:-

V = {a, e, i, o, u}

Cardinal number of this set is n(V) = 5

If A = {1, 2, 3}, then

n(A) = 3

Notes:

For any three finite sets A, B and C.

- n (A∪B) = n(A) + n(B) − n (A∩B)
- n(A∪B) = n(A) + n(B) if n(A∩B) = 0
- n(A∪B∪C) = n(A) + n(B) + n(C) − n(A∩B) − n(B∩C) − n(C∩A) + n(A∩B∩C)
- n(A −B) = n(A) − n(A∩B) = n_o (A) for two sets A and B.
- n(A∪B) will be maximum if n(A∩B) is minimum.
- n(A∪B) will be minimum if n(A∩B) is maximum.

Algebraic Operations on Sets

- Idempotent Laws

A∪A = A

A ∩A = A

- Identity Laws

A∪ϕ = A   A∩ϕ = ϕ

A ∪ U = U   A∩U = A

- Commultative Laws

A∪B = B∪A

A∩B = B∩A

- Associative Laws

A∪(B∪C) = (A∪B)∪C

A∩(B∩C) = (A∩B)∩C

- Distributive Laws

A∪(B∩C) = (A∪B) ∩ (A∪C)

A∩(B∪C) = (A∩B) ∪ (A∩C)

- Complement Laws

A∪$\bar{A}$=U

A$\cap\bar{A}$=ϕ

$\bar{\bar{A}}$= A

$\bar{U}$ = ϕ and $\bar{\phi}$ = U

- De-Morgan's Law

$$(\overline{A \cup B}) = (\bar{A} \cap \bar{B})$$

$$(\overline{A \cap B}) = \bar{A} \cup \bar{B}$$

Field Axioms

Properties of Real Numbers:

For Addition

(i) For a, b $\in$**R** : a + b $\in$ **R** (Closure Property)

(ii) For a, b $\in$ **R** : a + b = b + a (Commutative property)

(iii) For a, b, c $\in$ **R** : (a + b) + c = a + (b + c) (Associative property)

(iv) 0 $\in$ **R** : a + 0 = 0 + a = a for all a $\in$ **R** (Existence of additive identity) where o is called additive identity.

(v) For each a $\in$ **R**, there exists an element -a $\in$ **R** such that a + (-a) = 0 (Existence of additive inverse)

where a and -a are additive inverses of each other.

For multiplication

(vi) For a, b $\in$ **R**, ab $\in$ **R** (Closure property)

(vii) For a, b $\in$ **R**, ab = ba (Commutative property)

(viii) For a, b, c $\in$ **R**, (ab)c = a(bc) (Associative property)

(ix) For a, b, c $\in$ **R**: a(b + c) = ab + ac (Distributive property)

(x) 1 $\in$ **R**: a.1 = a = 1.a (Existence of multiplicative identity)

Here 1 is called multiplicative identity.

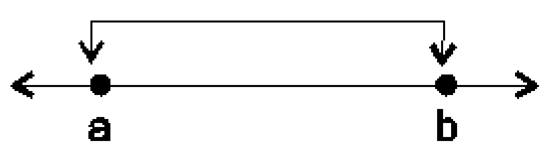
(xi) For each a $\in$ **R**, a $\neq$ 0 there exists an element a^{-1} $\in$ **R** such that a.a^{-1} = 1= a^{-1}.a. [Existence of multiplicative inverse]

where a and a^{-1} are inverse of each other.

Intervals

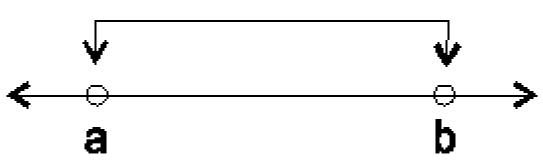
The set of all the real numbers between a and b (a, b $\in$ **R**, a < b) is called an interval with a and b as end points. The length of this interval is b-a and it mid-point is $\dfrac{a+b}{2}$.

(i) The interval containing both the end points is called a **closed interval**. If a and b are end points then the corresponding closed interval is:



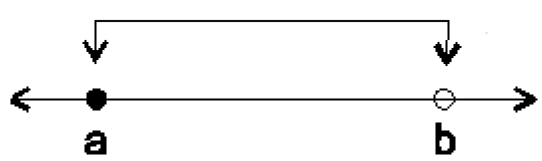
i.e. [a, b] = {x:a $\leq$ x $\leq$ b}

(ii) The interval containing all the points between the end points except the end points is called **open interval.**



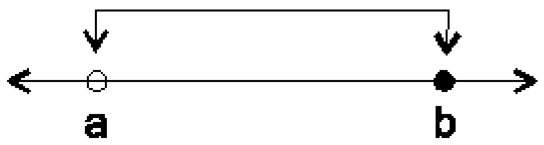
(a, b) = { x : a < x < b}

(iii) The interval [a, b) contains all the points between a and b and the point a. It is a **right open interval**.



i.e. [a, b) = {x: a $\leq$ x < b}

(iv) The set (a, b] contains all the points between a and b and the point b. It is called **left open interval**.



(a, b] = {x: a < x ≤ b}

(v) {x ∈R: x ≤ a} is denoted by (-∞, a]

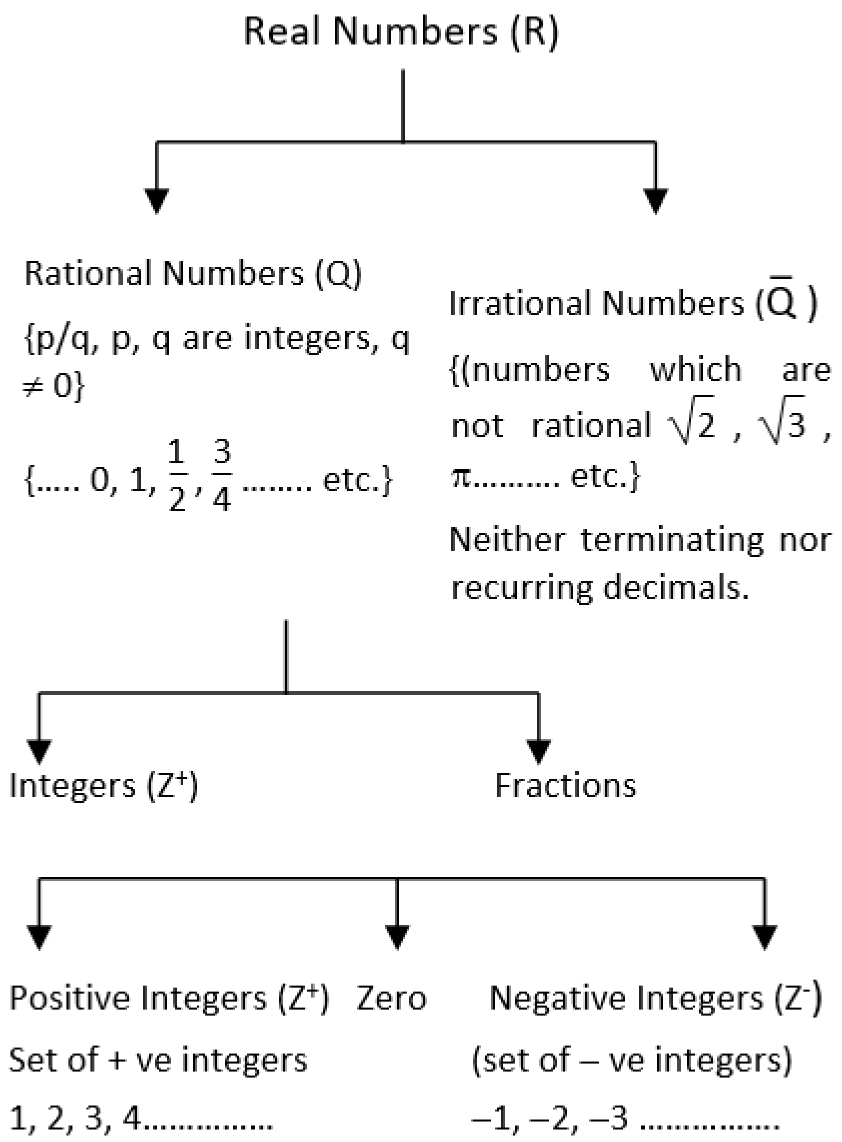
(vi) {x ∈ R: x < a} is denoted by (-∞, a)

(vii) {x ∈ R: x ≥ a} is denoted by [a, ∞)

(viii) {x ∈ R: x > a} is denoted by (a, ∞)

(ix) The set R of real numbers is written as (-∞, ∞)

Real Number System



- set of integers Z = {... -3, -2, -1, 0, 1, 2, 3 ...}
- set of all the natural numbers, N = {1, 2, 3, ...}
- set of all the negative integers, Z^- = {... -3, -2, -1 ...}
- set of all the positive integers, Z^+ = {1, 2, 3, ...}
- set of all the rational numbers, Q = {x : x = a/b, b ≠0}, where a and b are integers.
- set of all the irrational number $\overline{\mathbf{Q}} = \{\sqrt{2}, \sqrt{3}, \pi, e, \ldots\}$

Elementary properties of inequalities

(i) If a > b and b > c, then a > c. (Transitive property)

(ii) If a > b and c is any real number then a +c > b +c and a – c > b – c

(iii) If a > b and c > 0 i.e c is any positive real number, then ac > bc and a/c > b/c

(iv) If a > b > 0 then 0 < 1/a < 1/b

Absolute value

Absolute value of x is a non-negative real number denoted by |x| and given by

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \quad x < 0 \end{cases}$$

Note:

For any three real number x, y, z

- $|x + y| \leq |x| + |y|$
- $|x - y| \geq |x| - |y|$
- $|x + y| + |y - z| \geq |x - y|$
- For any positive real number a, $|x| < a$ if and only if $-a < x < a$.

  If d denotes a positive real number, then $|x - a| < \delta$ implies $a - \delta < x < a + \delta$.
- The length of the interval $(a-\delta, a+\delta)$ is 2δ.
- $|x|^2 = x^2$
- $|x| = \max\{x, -x\}$

Cartesian Product and Relation

Ordered Pair

A pair having one element as the first and the other as the second is called an ordered pair. An ordered pair having a as the first element and b as the second element is denoted by (a, b).

Two ordered pairs (a, b) and (c, d) are equal if a = c and b = d .

Cartesian Product

Let A and B be any two sets. Then, the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ is called Cartesian Product of A and B is denoted by $A \times B$.

In set builder notation, we have

$A \times B = \{(a, b): a \in A, b \in B\}$

Properties of Cartesian Product

(i) In general, $A \times B \neq B \times A$

(ii) If A contains m elements and Set B contains n elements then the number of elements in $A \times B$ is mn. i.e. $o(A \times B) = o(A).\ o(B)$

(iii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(iv) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(v) $A \times B$ = f if at least one of the set A or B is an empty set.

(vi) If A = B then $A \times B = B \times A$ where A and B are non-empty sets.

(vii) For any three sets A, B, C,

$A \times (B - C) = (A \times B) - (A \times C)$

(VIII)If A Ì B then $A \times C$ Ì $B \times C$ for any set C.

(ix) For any three sets A, B, C.

(a) $A \times (B' \cup C') = (A \times B) \cap (A \times C)$

(b) $A \times (B' \cap C') = (A \times B) \cup (A \times C)$

(x) Number of possible relations from set A to set B = 2^{mn},

where o(A) = m

o(B) = n

(xi) If $o(A \cap B) = n$ then $o((A \times B) \cap (B \times A)) = n^2$

(xii)The number of functions from a finite set A into a finite set B = $(n(B))^{n(A)}$

Relation

A relation from set A to B is a subset of the Cartesian product A x B. It is denoted by R. Then,

$$R \subset (A \times B)$$

$$R = \{(x, y) : x \in A, y \in B\} \subseteq (A \times B)$$

Inverse Relation

Let R be a relation from A to B. The relation R^{-1} from B to A is said to be inverse relation of R if

$R^{-1} = \{(y, x); y \in B, x \in A\} \subset B \times A$

Domain and Range

The set of all the 1^{st} elements of the ordered pairs (x, y) of the relation R is called the **domain of R.**

The set of all the 2^{nd} elements of R is called **the range of R.**

Note:

Domain of R $\subseteq$A and

range of R $\subseteq$ B.

Example:-

The domain, range and the inverse of the relation R = {(1, 3), (2, 4), (3, 5)} are:

Solution:-

Domain of R = {1, 2, 3}

Range of R = {3, 4, 5}

Inverse relation (R^{-1}) = {(3, 1), (4, 2), (5, 3)}

Example:-

- Let A = {1, 2, 4} and B = {1, 4, 16}. Then, relation R from A to B defined by $y = x^2$ is

Solution: -

A x B = {(1, 1),(1, 4),(1, 16), (2, 1), (2, 4),(2, 16), (4, 1), (4, 4), (4, 16)}

R = {x : $y = x^2$}

= {(1, 1), (2, 4), (4, 16)}

Example:-

- Let A = {1, 2, 3, 4} and R_1 = {(x, y) : x = y} on A then

R_1 = {(1, 1), (2, 2), (3, 3), (4, 4)}

Domain of R_1 = {1, 2, 3, 4}

Range of R_1 = {1, 2, 3, 4}

- If R_2 = {(x, y) : x +y $\leq$ 5} on A = {1, 2, 3, 4}

R_2 = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (4, 1)}

Some Special types of Relation

Let R be a relation on A.

Reflexive Relation:-

A relation R in a set A is called reflexive if (a, a) $\in$ R , for every a $\in$A.

Example: -

Let A = {1, 2, 4}

Then the relation

R = {(1, 1), (2, 2), (4, 4), (1, 4)} is reflexive.

Symmetric Relation:-

In the relation R if (x, y) $\in$ R. $\Rightarrow$ (y, x) $\in$ R, then the relation is known as a **symmetric relation.**

Example:

(i) If A = {1, 4, 5}

Then the relation {(4, 5), (5, 4), (1, 1), (1,4) (4, 1)} is symmetric.

(ii) R = {(4, 5), (5, 4), (5, 1)} is not symmetric.

Since (5, 1) $\in$ R but (1, 5) $\notin$ R

Anti – symmetric relation:-

In the relation, if (x, y) $\in$ R and (y, x)$\in$R implies x = y, then the relation is known as **anti-symmetric relation.**

Let A = {1, 2, 3, 4}

Then R = {(1, 1), (2, 2), (3, 3), (1, 2)} is anti-symmetric relation.

Transitive relation:-

In the relation R, if $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$, then the relation is known as a **transitive relation.**

Example:

Let A = {2, 4, 8}

then R = {(2, 4), (4, 8), (2, 8)} is transitive

In fact, 2 < 4 and 4 < 8 implies 2 < 8.

Equivalence relation:-

A relation R on a set A is called equivalence relation if it is reflexive, symmetric and transitive.

Note:

If R is a relation on a set A, then

(i) R is reflexive $\Rightarrow R^{-1}$ is reflexive

(ii) R is symmetric $\Rightarrow R^{-1}$ is symmetric

(iii) R is transitive $\Rightarrow R^{-1}$ is transitive