Complex Number

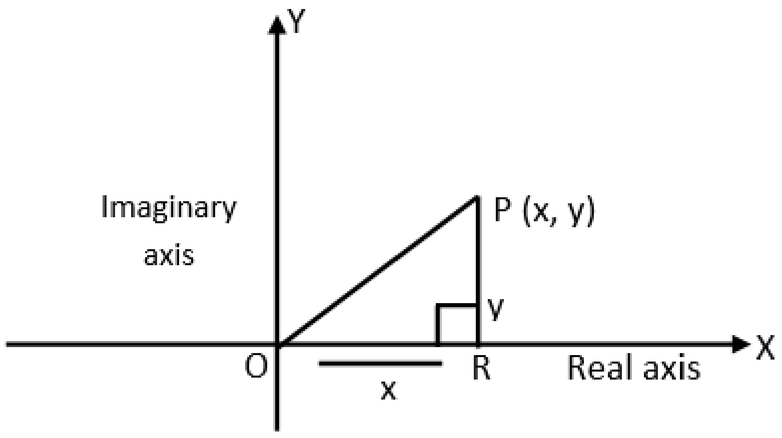
Introduction

$\sqrt{-1}$ is regarded as i(iota), called imaginary unit. For any a, b $\in \mathbb{R}$ (the set of real numbers), the number of the form z=a+ib is called a complex number. The complex number a+ib is also denoted by an ordered pair (a,b). a is called real part and b is called imaginary part of complex number z. We denote the set of all complex numbers by C. Then C=$\{$(a,b):a,b$\in \mathbb{R}\}$.

Every real number is a complex number

Every real number 'a' can be expressed as a complex number as a=a+i.0

Geometrical Representation of a complex number:

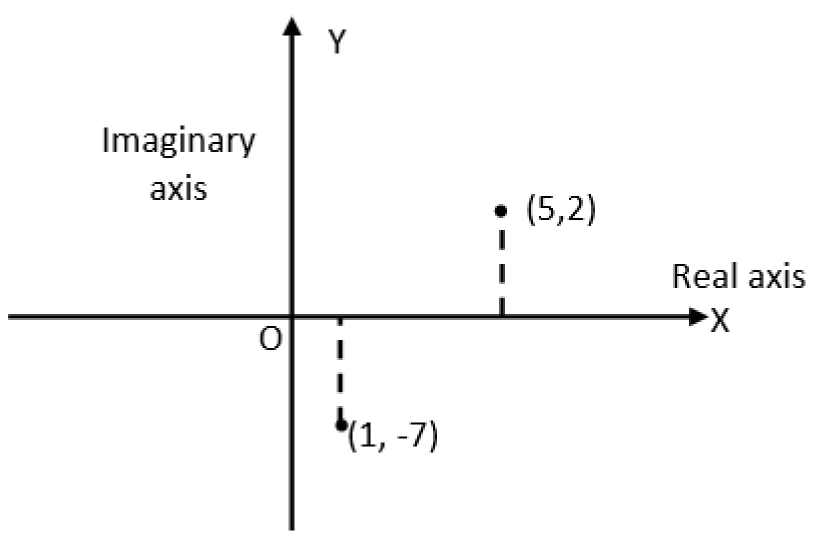


A complex number z = (x, y) is represented by the point P whose co-ordinates related to X and Y axes are x, and y respectively. X-axis is called **real axis** and y- axis is called **imaginary axis**.

Complex plane or Argand plane

A plane on which the complex numbers are represented is known as complex plane or Argand plane or Gauss plane.

Example:
z = (5 + 2i)
w = (1 – 7i) etc.
Thus, C = $\{$(a, b) : a, b$\in$ R$\}$



Operations of Complex Numbers

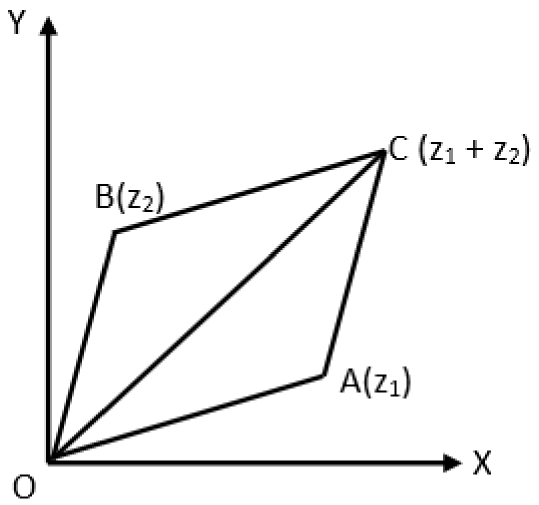
Equality of complex number

Any two complex number z = (a, b) and w = (c, d) are said to be equal if and only if
a = c and b = d
i.e real parts and imaginary parts are separately equal.

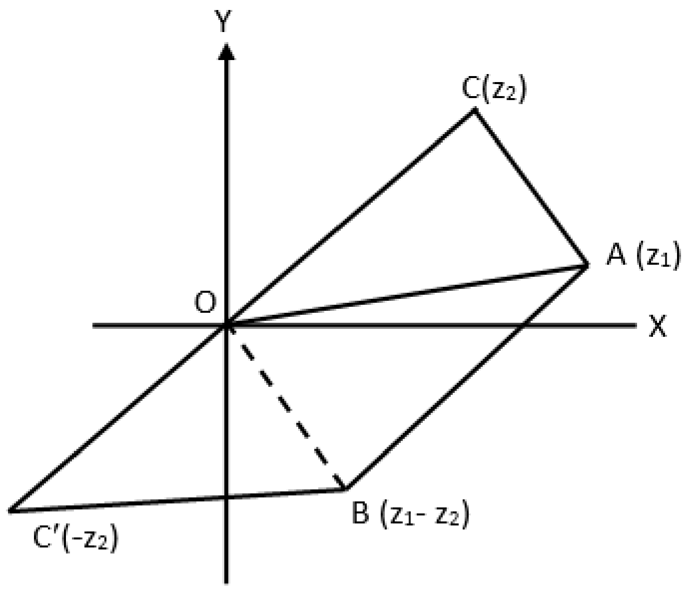
Addition of complex numbers

Let z = (a, b) and w = (c, d) be any two complex numbers. Then, the addition of two complex numbers denoted by z + w is given by,
z + w = (a, b) + (c, d)
= (a + c, b + d)
= (real part of z + real part of w, imaginary part of z + imaginary part of w)

Geometrically, sum of two complex numbers z_1 and z_2 is represented by the diagonal OC of the parallelogram OACB.



Similarly, the difference of two complex numbers z_1 and z_2 is represented by B of the parallelogram OABC'.



Example:-
z = (3, 4)
w = (2, 1)
z + w = (3 +2, 4 +1) = (5, 5)
and z - w= (3-2,4-1) = (1,3)

Multiplication of complex number

If z = (a, b) and w = (c, d) be any two complex numbers, then
z . w = (a, b) . (c, d) =(a+ib).(c+id)
= (ac − bd, ad + bc)

Example:

z = (2, 3) w = (2, 5)
then z . w = (2.2 – 3. 5, 2.5 + 2.3)= (−11, 16)

Division of complex numbers :

let z = a +ib

w = c + id be any two complex numbers, such that w$\neq$0. Then, their quotient $\frac{z}{w}$ is given by.

$$\frac{z}{w}=\frac{a+ib}{c+id}=\frac{(a+ib)(c-id)}{(c+id)(c-id)}$$
$$=\frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$$
$$=\left(\frac{ac+bd}{c^2+d^2}\right)+i\left(\frac{bc-ad}{c^2+d^2}\right)$$

Example:-

$$\frac{z}{w} = \frac{2+3i}{2+5i}$$

$$\frac{z}{w} = \frac{2+3i}{2+5i} \times \frac{2-5i}{2-5i}$$

$$= \frac{4-10i+6i-15i^2}{4-25i^2}$$

$$= \frac{4-4i+15}{29} = \frac{19-4i}{19}$$

$$\frac{z}{w} = \frac{19}{29} + i\left(-\frac{4}{19}\right)$$

Additive and Multiplicative inverses

i. Let z=(a,b) be any complex number. Then, the additive inverse of z is −z i.e.−z=(−a,-b)

ii. For the set of all the complex numbers, the additive identity is (0, 0).

iii. For the set of all the complex numbers, multiplicative identity is (1, 0).

iv. If z = (a, b) is any non-zero complex number, then the multiplicative inverse of z is 1/z and given by

$$\frac{1}{z} = \frac{1}{a+ib} = \frac{1}{a+ib} \cdot \frac{a-ib}{a-ib}$$
$$= \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$$

Conjugate of a complex number

If z=a+ib is a complex number, its conjugate, denoted by $\bar{z}$, is given by $\bar{z}$=a$-ib$.

Purely real and Purely imaginary

i. A complex number z = a +ib is said to be purely real if imaginary part b = 0 and

we have, $z = \bar{z}$

ii. A complex number is said to be purely imaginary if real part a = 0 and

we have, $z = -\bar{z}$

The imaginary Unit

'i' is called the imaginary unit of complex number
i = 0 + 1i = (0, 1)
The complex number (0, 1) is called the imaginary unit. We have,
i^2 = (0, 1) (0, 1) = (0, -1)= -1

Powers of i
$i^2 = -1$
$i^3 = i^2 \times i = (-1)\, i = -i$
$i^4 = i^2 \times i^2 = (-1) \times (-1) = 1$
$i^{4n} = (i^4)^n = (1)^n = 1$, where n is an integer
$i^{4n+2} = i^{4n} . i^2 = 1 \times -1 = -1$
$\frac{1}{i} = \frac{1}{i} . \frac{i}{i} = -i$

Some properties of complex numbers

Let $z_1 = (a, b)$, $z_2 = (c, d)$ and $z_3 = (e, f)$ be any three complex numbers. Then,

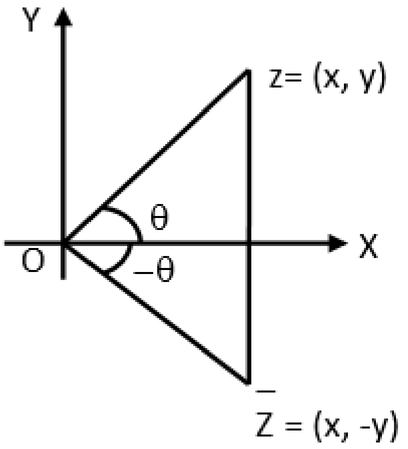
- **Commutative Law**
  $z_1 + z_2 = z_2 + z_1$

- **Associative law**
  $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$
  $z_1\,(z_2 z_3) = (z_1 z_2)\, z_3$

- **Distributive law**
  $z_1\,(z_2 + z_3) = z_1 z_2 + z_2 z_3$
  $(z_1 + z_2)\, z_3 = z_1 z_3 + z_2 z_3$

Conjugate of a Complex Number

Let z = a +ib be any complex number then its conjugate denoted by $zisgivenby$



$z = a – ib$

Example:-

z = 3 +4i
$\bar{z} = 3 - 4i$

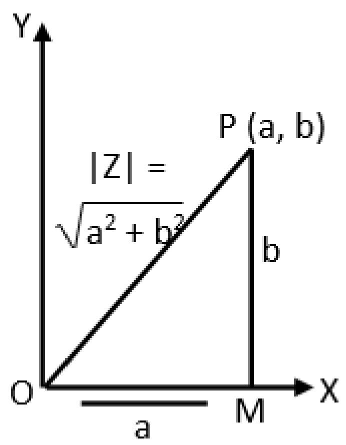
Properties of complex conjugate

- $(\bar{\bar{z}}) = z$
- $z + \bar{z} = 2\,\mathrm{Re}(z)$
- $z - \bar{z} = 2i\,\mathrm{Im}(z)$
- $\overline{z + w} = \bar{z} + \bar{w}$
- $\overline{zw} = \bar{z}\bar{w}$
- $\overline{z_1 \cdot z_2 \cdot z_3} = \overline{z_1} \cdot \overline{z_2} \cdot \overline{z_3}$
- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 + z_2 + z_3} = \overline{z_1} + \overline{z_2} + \overline{z_3}$
- $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$

Absolute value (Modulus) of a complex number

Let z = (a +ib) be any complex number. The absolute value of a complex number denoted by |z| is given by

$|z| = \sqrt{a^2 + b^2}$



Properties: -

i. The modulus of a complex number and its conjugate are same i.e.
   $|z| = |\bar{z}| = |-z|$
ii. $|z| = 0$ iff $z = 0$
iii. $-|z_1 \leq |Re(z_1) \leq |z_1|$
iv. $-|z_1| \leq Im(z_1) \leq |z_1|$
v. $|z_1 z_2| = |z_1| \,.\, |z_2|$
vi. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, Provided $z_2 \neq 0$
vii. $|z+w| \leq |z| + |w|$
viii. $|z - w| \geq |z| - |w|$
ix. $|z - w| \leq |z| + |w|$
x. $|z|^2 = z \cdot \bar{z}$
xi. $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$
xii. $|z_1 + z_2| = |z_1 - z_2| \Leftrightarrow arg(z_1) - \arg(z_2) = \pi/2$
xiii. $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow arg(z_1) = \arg(z_2)$

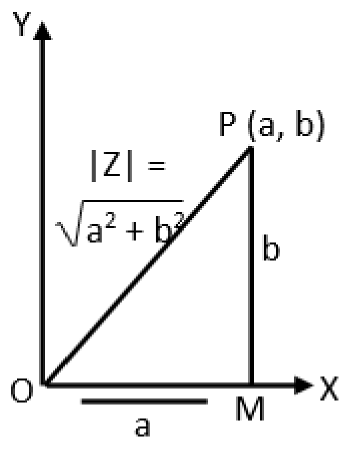
Different forms of Complex Number

Cartesian form:-

Complex number in the form z = (x +iy) is known as Cartesian form of complex number

Example:
$z=2+3i, 1+i$ etc.

Polar Form:-
Complex number in the form z = r(cos$\theta + isin\theta$) is known as polar form of complex number z = x + iy.
where, r^2 = x^2 + y^2
and tan$\theta = y/x$,
where θ is known as argument or amplitude.



Argument

i. If $x>0, y>0$ then $Arg(z)=\theta$ (1st quadrant)
ii. If $x<0, y>0$ then $Arg(z)=\pi-\theta$ (2nd quadrant)
iii. If $x<0, y<0$ then $Arg(z)=-(\pi-\theta)$ (3rd quadrant)
iv. If $x>0, y<0$ then $Arg(z)=-\theta$ (4th quadrant)
v. The modulus of $e^{i\theta} = |e^{i\theta}| = |\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$

Table of Arguments:

Complex numbers	Value of argument
i. $z=x, x>0$	0
ii. $z=-x, x<0$	π
iii. $z=iy, y>0$	$\frac{\pi}{2}$
iv. $z=-iy, y>0$	$-\frac{\pi}{2}$ or $\frac{3\pi}{2}$
v. $z_1 z_2$	$arg(z_1)+arg(z_2)$
vi. $\frac{z_1}{z_2}$	$arg(z_1)-arg(z_2)$
vii. z^n	$n\,arg(z)$
viii. $arg(\bar{z})$	$2\pi - arg(z)$

Note:

i. The principal value of argument of z denoted by arg (z) is given by:
arg(z) $=\tan^{-1}(y/x), z \neq 0 (-\pi < \theta \leq \pi)$
ii. The principal values of argument of z will be $\theta, \pi-\theta, -\pi+\theta$ and $-\theta$ according as the point 2 lies in the 1°, $2^{n+1}, 3^{nd}$ and 4^{th} quadrant.
iii. If arg(z) = 0 or π then z is purely real.
iv. If arg(z) = $\pi/2$ or $-\pi/2$ then z is purely imaginary.
v. Argument of the complex number 0 is not defined.
vi. $\mathrm{Arg}(2_1 z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$
vii. $\mathrm{Arg}\left(\frac{z_1}{z_2}\right) = \mathrm{Arg}(z_1) - \mathrm{Arg}(z_2)$
viii. $\mathrm{Arg}(\bar{z}) = 2\pi \cdot \mathrm{Arg}(z)$

Exponential Form

The complex number $(\cos\theta + \text{ isin }\theta)$ is denoted by e $\mathrm{e}^{\mathrm{i}\theta}$ i.e. $\mathrm{e}^{\mathrm{i}\theta} = (\cos\theta + \mathrm{i}\sin\theta)$. This representation is called exponential form.

Product and Quotient of Complex Numbers

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$z_1 z_2 = r_1 r_2 [(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

$\frac{z_1}{z_2} = \frac{r_1}{r_2}$[cos($\theta_1$ -θ_2) +isin(θ_1 - θ_2)];$z_2 0$

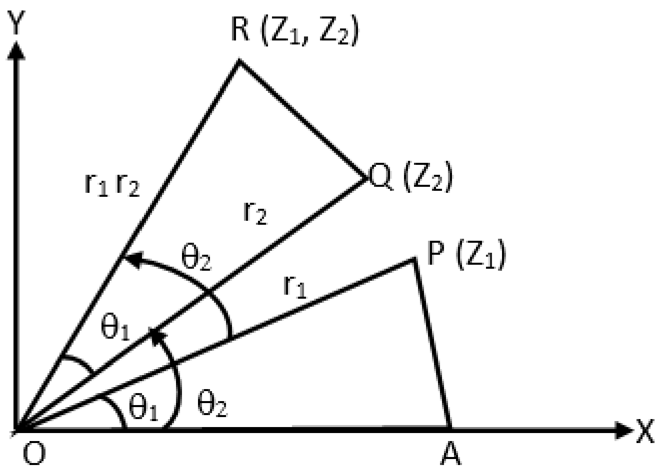
If there are three or more numbers z_1, z_2, z_3 etc in polar form then

$z_1 z_2 z_3 = r_1 r_2 r_3 [\cos(\theta_1+\theta_2+\theta_3) + i\sin(\theta_1+\theta_2) + \theta_3)]$

Similarly,

$\mathrm{z}_1 \mathrm{z}_2 \mathrm{z}_3 \ldots \mathrm{z}_n$
$= (\mathrm{r}_1 \cdot \mathrm{r}_2 \cdot \mathrm{r}_3 \ldots \mathrm{r}_n)[\cos(\theta_1 + \theta_2 + \ldots + \theta_n) + \mathrm{i}\sin(\theta_1+\theta_2+\ldots+\theta_n)]$

Geometrical Representation of $Z_1 Z_2$

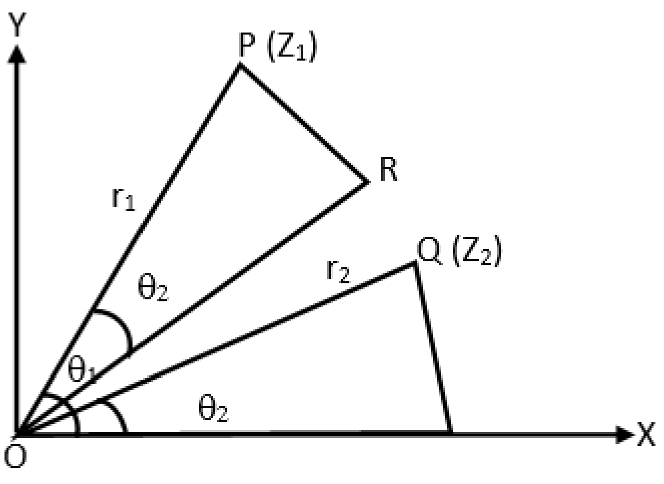


Let
$\mathrm{Z}_1 = \mathrm{r}_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)$
$\mathrm{Z}_2 = \mathrm{r}_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)$
$\mathrm{OP} = \mathrm{r}_1, \mathrm{OQ} = \mathrm{r}_2$

$\angle\mathrm{POX} = \theta_1$
$\angle\mathrm{QOX} = \theta_2$
$\mathrm{Z}_1\mathrm{Z}_2 = \mathrm{r}_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)\cdot\mathrm{r}_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)$
$= \mathrm{r}_1\mathrm{r}_2[\cos(\theta_1+\theta_2) + \mathrm{i}\sin(\theta_1+\theta_2)]$

Geometrical representation of $\frac{Z_1}{Z_2}$

Let Z_1 = r_1 (cosθ_1 + isinθ_1)
Z_2 = r_2 (cosθ_2 + isinθ_2)
OP = r_1, OQ = r_2
$\angle POX = \theta_1$ and $\angle QOX = \theta_2$
$\frac{Z_1}{Z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$
= $\frac{r_1}{r_2}$[cos($\theta_1 - \theta_2$) + isin($\theta_1 - \theta_2$)]



De Moivre's Theorem

For any integer n (positive or negative or zero),
[r(cos$\theta + i\sin\theta$)]^n = r^n (cosn$\theta + i\sin\theta$)

Note:

For any non-zero complex number z = r($\cos\theta + i\sin\theta$) and any positive integer n, the n distinct roots are given by:

$z_k = \sqrt[n]{r}\,(\cos\theta_k + i\sin\theta_k)$

where $\theta_k = \dfrac{\theta + k360^o}{n}$

k = 0, 1, 2, --------- (n-1)

Square roots of a complex number

Let z = a +ib be any complex number whose square roots is to be calculated.

Let $(x + iy)^2$ = a +ib

then $x^2 = \dfrac{\sqrt{a^2+b^2}+a}{2}$ and

$$y^2 = \frac{\sqrt{a^2+b^2}-a}{2}$$

where x, y and b must have same sign.

Logarithm of a Complex Number

Let, z = $re^{i\theta}$ be any complex number.

Then logz = log $re^{i\theta}$

= log r + log $e^{i\theta}$

= log r + iθ loge

= log r + iθ

∴ logz = log|z| + i amp (z)

Cube Roots of Unity

Let z = $\sqrt[3]{1}$

Cubing both sides,

$z^3 = 1$

$z^3 - 1 = 0$

$(z - 1)(z^2 + z + 1) = 0$

So, z = 1

or, $z^2 + z + 1 = 0$

we have z = $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = \dfrac{-1 \pm \sqrt{1-4}}{2 \cdot 1}$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1 \pm \sqrt{-1 \times 3}}{2}$$

$$= \frac{-1 \pm \sqrt{i^2 x 3}}{2}$$

$$= \frac{-1 \pm \sqrt{3}i}{2}$$

Taking positive and negative signs,

$$\omega = \frac{-1 + \sqrt{3}i}{2}$$

$$\omega^2 = \frac{-1 - \sqrt{3}i}{2}$$

Cube roots of unity are 1, $\dfrac{-1+\sqrt{3}i}{2}, \dfrac{-1-\sqrt{3}i}{2}$.

Properties of cube roots of unity

- Sum of the three cube roots of unity is zero i.e. 1 + ω +ω^2 = 0
- Product of imaginary cube roots of unity is 1 i.e ω . ω^2 = ω^3 = 1. Hence for any integer n ω^{3n} =1
- Cube roots of unity on the Argand plane forms equilateral triangle.
- Each imaginary cube roots of unity is square of the other.

  i.e. $(\omega)^2 = \omega^2$

  $(\omega^2)^2 = \omega$

  i. square roots of ω = $\pm\,\omega^2$

  ii. square roots of ω^2 = $\pm\,\omega$

Fourth roots of unity

- Fourth roots of unity are $\pm 1, \pm i$
- Sum of fourth roots of unity is 0,

  i.e. 1+i-1-i = 0

- Product of fourth roots of unity is -1
  i.e 1 x -1 x i x −i
  = i^2 = -1
- Fourth roots of unity on the Argand plane forms a square.
- n^{th} roots of unity on the Argand plane form a regular polygon having n sides.
- nth roots of unity are respectively :
  1, ω^2 .... ω^{n-1} $\omega \rightarrow$ which are in G.P. then
  a. sum of n, nth roots of unity is zero.
     (n > 1).
     1 + ω + ω^2 + ... +ω^{n-1} = 0
  b. the product of nth roots of unity is $(-1)^{n-1}$
- The area of triangle with vertices z, iz and z + iz is $\frac{1}{2}|z|^2$
- The sum of n, n^{th} roots of unity (n > 1) is zero.