Binomial theorem and Applications

Bionomial Expression:

A binomial is an algebraic expression of two terms which are connected by the operations '+' or '-'.

e.g. (x +2), (4x - 5y), (x + a) are the binomial expressions.

Binomial theorem

The binomial theorem for natural number states that x and a are real numbers and n is any positive integer then

$$(x+a)^n = x^n +^n c_1 x^{n-1} a +^n c_2 x^{n-2} a^2 + \ldots +^n c_r x^{n-r} a^r + \ldots + a^n$$
$$= \sum_{r=0}^{n} {}^n c_r x^{n-r} a^r$$

$$(1+x)^n = \sum_{r=0}^{n} c_r x^r$$

and $(1-x)^n = \sum_{r=0}^{n} (-1)^{rn} c_r x^r$

- The expression $(x + a)^n$ contains (n+1) terms after binomial expansion.
- Sum of the exponents in each term after binomial expansion is n.
- Powers of x are in descending order and powers of a are in ascending order.

Pascal's Triangle

The coefficients of various terms in $(a + b)^n$ for different values of n follows the pattern given below.

$(a + b)^n$	Cofficinets
$(a + b)^1$	1 1
$(a + b)^2$	1 2 1
$(a + b)^3$	1 3 3 1
$(a + b)^4$	1 4 6 4 1
$(a + b)^5$	1 5 10 10 5 1
$(a + b)^6$	1 6 15 20 15 6 1
$(a + b)^7$	1 7 21 35 35 21 7 1
$(a + b)^8$	1 8 28 56 70 56 28 8 1

General term

General term i.e. $(r + 1)^{th}$ term in the expansion of $(a + x)^n$ is

$$t_{r+1} =^n C_r a^{n-r} x^r$$

General term in the expansion of $(a - x)^n$ is

$$t_{r+1} = (-1)^r C(n, r) a^{n-r} x^r$$

Note:

1. For $1 \leq r \leq n, ^n C_r = \dfrac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}$
2. In particular, ${}^nC_0 =^n C_n = 1$
3. If $1 \leq r \leq n$, then ${}^nC_r =^n C_{n-r}$
4. If $1 \leq r \leq n$, then ${}^nC_r +^n C_{r-1} =^{n+1} C_r$

5. $^nC_r = n/r.^{n-1}C_{r-1} = \frac{n(n-1)}{r(r-1)} .^{n-2}C_{r-2}$ etc.

Middle term

If n is even :- There are (n+1) terms after binomial expansion which is odd. So, there is only one middle term and is obtained by $t_{n/2+1} = C(n, n/2)a^{n/2}x^{n/2}$ middle term = $\left(\frac{n}{2}+1\right)^{th}$ term.

If n is odd :- There are (n+1) terms after binomial expansion which is even. there are two middle terms.
middle terms are $\left(\frac{n+1}{2}\right)^{th}$ term and $\left(\frac{n+1}{2}+1\right)^{th}$ term.

Some special cases of binomial theorem

i. $(1-x)^{-1} = 1+x+x^2+x^3+x^4+\cdots\infty$
ii. $(1+x)^{-1} = 1-x+x^2-x^3+x^4+\cdots\infty$
iii. $(1-x)^{-2} = 1+2x+3x^2+4x^3+\cdots\infty$
iv. $(1+x)^{-2} = 1-2x+3x^2-4x^3+\cdots\infty$
v. $(1-x)^{-1/2} = 1+\frac{1}{2}x+\frac{1.3}{2.4}x^2+\cdots+\infty$
vi. $(1+x)^{-1/2} = 1-\frac{1}{2}x+\frac{1.3}{2.4}x^2+\cdots+\infty$

Note:

When n is other than positive integers, expansion of $(1+x)^n$ terminates and is valid only if |x|<1

$$(1+x)^n = 1+\frac{n}{1}x+\frac{n(n-1)}{1.2}x^2+-------+\infty$$

Binomial coefficients

The coefficients C(n, 0), C(n, 1) -----C(n, n) in the expansion of $(a+x)^n$ are known as Binomial coefficients

$$\mathrm{C}_0 =^{\mathrm{n}} \mathrm{C}_0 = 1$$

$$\mathrm{C}_1 =^{\mathrm{n}} \mathrm{C}_1 = \frac{n}{1}$$

$$\mathrm{C}_2 =^{\mathrm{n}} \mathrm{C}_2 = \frac{n(n-1)}{1.2}$$

$$\mathrm{C}_3 =^{\mathrm{n}} \mathrm{C}_3 = \frac{n(n-1)(n-2)}{1.2.3}$$

....
....
....

$$C_n =^n C_n = 1$$

Note:

i. Sum of the binomial coefficients = 2^n i.e.
$\mathrm{C}_0+\mathrm{C}_1+\mathrm{C}_2+\cdots+\mathrm{C}_\mathrm{n} = 2^\mathrm{n}$
$1+\mathrm{C}_1+\mathrm{C}_2+\cdots+\mathrm{C}_\mathrm{n} = 2^\mathrm{n}$
$\mathrm{C}_1+\mathrm{C}_2+\cdots\mathrm{C}_\mathrm{n} = 2^\mathrm{n}-1$

ii. Sum of odd Binomial coefficients = sum of even binomial coefficients = 2^{n-1}
i.e. $\mathrm{C}_0+\mathrm{C}_2+\mathrm{C}_4+\ldots = \mathrm{C}_1+\mathrm{C}_3+\mathrm{C}_5+\ldots = 2^{n-1}$

Note

Number of dissimilar terms in the expansion of $(x+y+z)^n, n \in N$ is given by

$$= \frac{(n+1)(n+2)}{2} \text{ or } \mathrm{C}(\mathrm{n}+\mathrm{m}-1, \mathrm{n})$$

No. of dissimilar terms in the expansion of $(x_1+x_2+\ldots\ldots\ldots\ldots+x_r)^n$ is $^{n+r-1}C_{r-1}$

Greatest term in the expansion of $(a+x)^n$

For positive a and x, the term t_{r+1} is the greatest term if r is the greatest possible value satisfying the equation $t_{r+1} \geq t_r$ i.e. $\frac{t_{r+1}}{t_r} \geq 1$

Greatest coefficient in the expansion of $(a+x)^n$

In the expansion of $(a + x)^n$, the coefficient of general term t_{r+1} is ^nC_r. If n is even, the ^nC_r is greatest when $r = n/2$ and if n is odd, then ^nC_r is greatest when $r = (n - 1)/2$ or $(n + 1)/2$