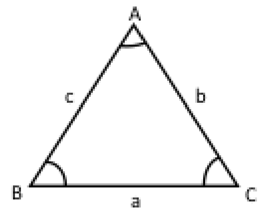
Properties of Triangles

Introduction

In a $\Delta\mathrm{ABC}$, there are three angles $\mathrm{A}, \mathrm{B}, \mathrm{C}$ and the three sides $\mathrm{a}=\mathrm{BC}, \mathrm{b}=\mathrm{CA}$, and $\mathrm{c}=AB$. The semi-perimeter of $\Delta\mathrm{ABC}$ is $\frac{\mathrm{AB}+\mathrm{BC}+\mathrm{CA}}{2}$, which is represented by s. and given by, $s=\frac{\mathrm{a}+\mathrm{b}+\mathrm{c}}{2}$

The area of $\Delta\mathrm{ABC}$ is denoted by Δ. Here we shall study many relations among the sides $\mathrm{a}, \mathrm{b}, \mathrm{c}$ and angles $\mathrm{A}, \mathrm{B}, \mathrm{C}$ of a triangle.

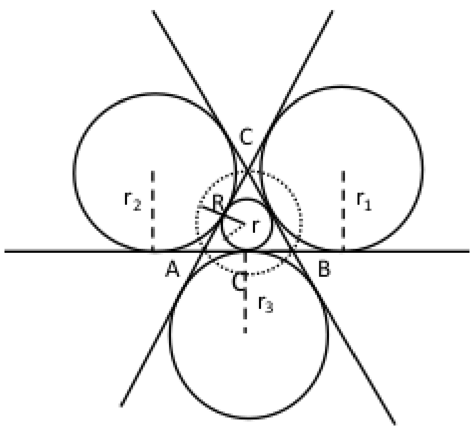


Results of conditional identity

If $A+B+C=\pi$ then

I. $\sin 2A+\sin 2B+\sin 2C=4\sin A\cdot\sin B\cdot\sin C$
II. $\sin A+\sin B+\sin C=4\cos A/2\cdot\cos B/2\cdot\cos C/2$
III. $\cos A+\cos B+\cos C=1+4\sin A/2\cdot\sin B/2\cdot\sin C/2$
IV. $\sin^2 A+\sin^2 B+\sin^2 C=2+2\cos A\cdot\cos B\cdot\cos C$
V. $\cos^2 A+\cos^2 B+\cos^2 C=1-2\cos A\cdot\cos B\cdot\cos C$
VI. $\cos^2 A/2+\cos^2 B/2+\cos^2 C/2=2+2\sin A/2\cdot\sin B/2\cdot\sin C/2$
VII. $\tan A+\tan B+\tan C=\tan A\cdot\tan B\cdot\tan C$
VIII. $\tan A/2\cdot\tan B/2+\tan B/2\cdot\tan C/2+\tan C/2\cdot\tan A/2=1$
IX. $\cot A\cdot\cot B+\cot B\cdot\cot C+\cot A=1$

Geometrical configuration



Some important Results

$A+B+C=\pi$
$A+B=(\pi-C)$
$\therefore \sin(A+B)=\sin(\pi-C)=\sin C$

Similarly $\sin(\mathrm{B}+\mathrm{C})=\sin\mathrm{A}$

$\sin(\mathrm{C}+\mathrm{A})=\sin\mathrm{B}$
$\cos(\mathrm{A}+\mathrm{B})=\cos(\pi-\mathrm{C})=-\cos\mathrm{C}$

$\mathrm{A}/2+\mathrm{B}/2+\mathrm{C}/2=\pi/2$

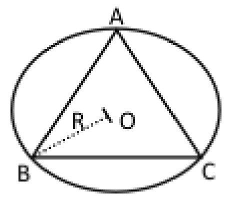
$\therefore \sin(A/2+B/2)=\sin(\pi/2-C/2)=\cos C/2$
$\cos(A/2+B/2)=\cos(\pi/2-C/2)=\sin C/2$
$\tan(A/2+B/2)=\cot C/2\&$ so on

Sine law

In any triangle ABC

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$$

where R is the radius of circum-circle or circum radius.



$$a = 2R\sin A \Rightarrow \sin A = a/2R$$
$$b = 2R\sin B \Rightarrow \sin B = b/2R$$
$$c = 2R\sin C \Rightarrow \sin C = c/2R$$

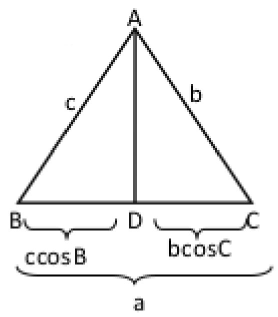
$$\Delta = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$$

Cosine law

- $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$
- $\cos B = \dfrac{c^2+a^2-b^2}{2ca}$
- $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

Projection Law

In $\Delta \mathrm{ABC}$
(i) $a\cos B + b\cos A = c$
(ii) $c\cos A + a\cos C = b$
(iii) $b\cos C + c\cos B = a$



Tangent Law

- $\tan\left(\dfrac{B-C}{2}\right) = \left(\dfrac{b-c}{b+c}\right)\cot A/2$
- $\tan\left(\dfrac{C-A}{2}\right) = \left(\dfrac{c-a}{c+a}\right)\cot B/2$
- $\tan\left(\dfrac{A-B}{2}\right) = \left(\dfrac{a-b}{a+b}\right)\cot C/2$

Area of a triangle

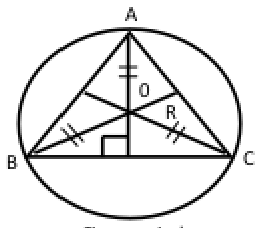
- $\Delta = \dfrac{1}{2}b\cdot h = \dfrac{1}{2}$ base $\times$ height
- $\Delta = \dfrac{\sqrt{3}}{4}a^2(\text{ Equilateral triangle only })$
- $\Delta = \dfrac{1}{2}bc\sin A = \dfrac{1}{2}ca\sin B = \dfrac{1}{2}ab\sin C$
- $\Delta = \dfrac{abc}{4R}$
- $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \dfrac{a+b+c}{2}$

- $\Delta = \dfrac{1}{4}\sqrt{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4}$

Circles Associated to a Triangle

Circum Circle

The circle which passes through the vertices of a triangle is called the **circum-circle** and the radius is called **circum-radius**.

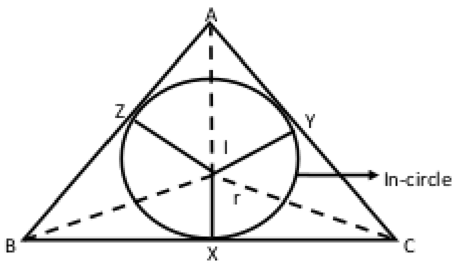


Circum-circle

Circum-centre: It is the point of intersection of the right bisectors of the sides of the triangle. In the figure, O is the circum-centre.

In-circle

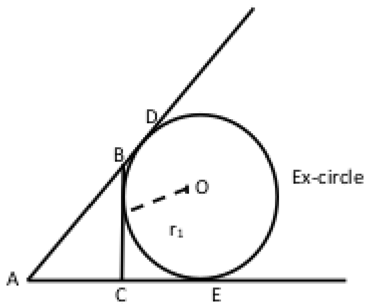
The circle which is inscribed in a triangle and touches the sides of a triangle internally is called in-circle and the radius is in-radius.



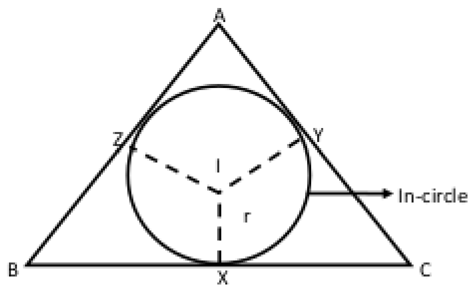
In centre: It is the point of intersection of the internal bisectors of the angles of the triangle. In the figure, I is the incentre.

Ex-cribed circle

The circle which touches only one side of a triangle and other two produced sides of a triangle is called **ex-cribed circle**. There are three ex-circles and so three ex-radii r_1, r_2 and r_3.



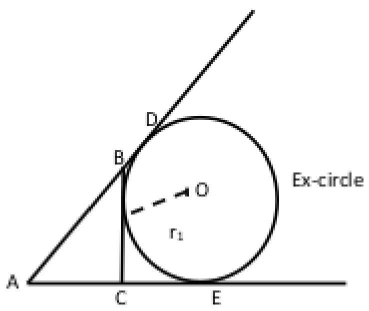
Formula for In-radius (r)



$r = \dfrac{\Delta}{s}$
$r = 4R \sin A/2 \cdot \sin B/2 \cdot \sin C/2$
$r = (s-a)\tan A/2 = (s-b)\tan B/2$
$= (s-c)\tan C/2$

Formula for Ex-radius (r_1, r_2, r_3)

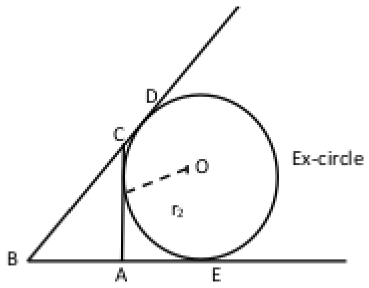
(A) $r_1 \to$ radius of Ex-circle opposite to angle A



- $r_1 = \dfrac{\Delta}{(s-a)}$

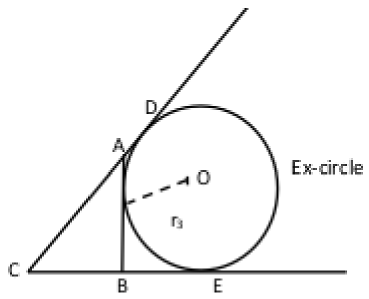
- $r_1 = 4R \sin A/2 \cdot \cos B/2 \cdot \cos C/2$
- $r_1 = s \tan A/2$
- $r_1 = a \sec A/2 \cdot \cos B/2 \cdot \cos C/2$

(B) $r_2 \rightarrow$ radius of Ex-circle opposite to angle B



- $r_2 = \dfrac{\Delta}{(s-b)}$
- $r_2 = 4R \cos A/2 \cdot \sin B/2 \cdot \cos C/2$
- $r_2 = s \tan B/2$
- $r_2 = b \cos A/2 \cdot \sec B/2 \cdot \cos C/2$

(C) $r_3 \rightarrow$ radius of Ex-circle opposite to angle C



- $r_3 = \dfrac{\Delta}{(s-c)}$
- $r_3 = 4R \cos A/2 \cdot \cos B/2 \cdot \sin C/2$
- $r_3 = s \tan C/2$
- $r_3 = c \cos A/2 \cdot \cos B/2 \cdot \sec C/2$

Half angle formula

i. $\sin A/2 = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$

ii. $\cos A/2 = \sqrt{\dfrac{s(s-a)}{bc}}$

iii. $\sin B/2 = \sqrt{\dfrac{(s-a)(s-c)}{ac}}$

iv. $\cos B/2 = \sqrt{\dfrac{s(s-b)}{a}}$

v. $\sin C/2 = \sqrt{\dfrac{(s-b)(s-c)}{ab}}$

vi. $\cos C/2 = \sqrt{\dfrac{s(s-c)}{ab}}$

vii. $\tan A/2 = \sqrt{\dfrac{s(s-a)}{s(s-a)}}$

viii. $\cot A/2 = \sqrt{\dfrac{s(s-a)}{(s-b)(s-c)}}$

ix. $\tan B/2 = \sqrt{\dfrac{(s-b)(s-c)}{s(s-b)}}$

x. $\cot B/2 = \sqrt{\dfrac{s(s-b)}{(s-a)(s-c)}}$

xi. $\tan C/2 = \sqrt{\dfrac{(s-b)(s-c)}{s(s-c)}}$

xii. $\cot C/2 = \sqrt{\dfrac{s(s-c)}{(s-a)(s-b)}}$

Important Results

I.

i. $r_1 + r_2 + r_3 - r = 4R$

ii. $\mathrm{rr_1r_2r_3} = \Delta^2$

iii. $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} = \dfrac{1}{r}$

iv. $\sin A + \sin B + \sin C = s/R$

v. $\cos A + \cos B + \cos C = 1 + r/R$

vi. $\operatorname{cosec} A = \dfrac{1}{\sin A} = \dfrac{2R}{a} = \dfrac{2abc}{a \cdot 4\Delta} = \dfrac{bc}{2\Delta}$ etc

vii.

$$\tan A = \frac{\sin A}{\cos A} = \frac{\frac{a}{2R}}{\frac{b^2+c^2-a^2}{2bc}}$$

$$= \frac{abc}{R} \cdot \frac{1}{b^2+c^2-a^2} = \frac{4\Delta}{b^2+c^2-a^2}$$

II.

i. If $\sin A, \sin B, \sin C$ are in A.P. / G.P. / H.P. then a, b, c are in A.P./ G.P./ H.P. respectively.

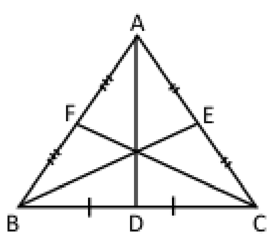
ii. If a, b, c are in A.P., then r_1, r_2, r_3 are in HP

iii. If $\cot A/2, \cot B/2, \cot C/2$ are in A.P., then a, b, c are in A.P.

iv. If $\cot A, \cot B, \cot C$ are in A.P. then a^2, b^2, c^2 are in AP

v. If $\sin A, \sin B,$ sinC are in H.P., then $(1-\cos A), (1-\cos B), (1-\cos C)$ are in H.P.

NOTE: In ΔABC if $AD,\ BE,\ CF$ are medians, then:

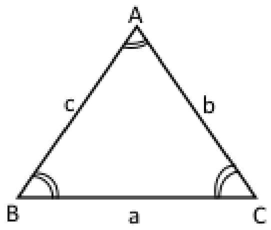


$$
\begin{aligned}
\mathrm{AD} &= \frac{1}{2}\sqrt{\mathrm{b}^2+\mathrm{c}^2+2\mathrm{bc}\cos\mathrm{A}} \\
&= \frac{1}{2}\sqrt{2b^2+2c^2-a^2} \\
\mathrm{BE} &= \frac{1}{2}\sqrt{\mathrm{c}^2+\mathrm{a}^2+2\mathrm{ca}\cos\mathrm{B}} \\
&= \frac{1}{2}\sqrt{2c^2+2a^2-b^2} \\
\mathrm{CF} &= \frac{1}{2}\sqrt{\mathrm{a}^2+\mathrm{b}^2+2\mathrm{ab}\cos\mathrm{C}} \\
&= \frac{1}{2}\sqrt{2a^2+2b^2-c^2}
\end{aligned}
$$

Solution of Triangles

The process of finding the unknown elements of a triangle from the known elements is known as finding the **solution of triangle.**
In solving a triangle, the various cases arises:

i. When three angles are given:



If three angles of a triangle are given, we use sine law to find the ratio of the sides.

ie. $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$

ii. When three sides are given:

If three sides of a Δ are given, then we use cosine law to find the angles of a triangle.

i.e.

$$\cos A=\frac{b^2+c^2-a^2}{2bc}$$

$$\cos B=\frac{c^2+a^2-b^2}{2ca}$$

$$\cos C=\frac{a^2+b^2-c^2}{2ab}$$

iii.When two angles and one side are given:

If two angles and one side are given, then by using the relation

$A+B+C=180^\circ$

3^{rd} angle can be determined.

Then, to find other two sides, we use the formula.

i. $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$

ii. $A+B+C=180^\circ$

and $\Delta=\dfrac{1}{2}ab\sin C, \dfrac{1}{2}bc\sin A, \dfrac{1}{2}$ casin B

iv. When two sides a, b and included angle C are given:

If two sides a, b and included angle C are given, then we can use cosine law and sine law to find the remaining parts.

We can also use the following relations:

i. $\Delta = \frac{1}{2}$ absinc

ii. $A + B + C = 180^\circ$

iii. $\tan\left(\frac{A-B}{2}\right) = \left(\frac{a-b}{a+b}\right)\cot\frac{C}{2} \quad \left[\frac{B+C}{2} = 90^\circ - \frac{A}{2}\right]$

iv. $c = \frac{a\sin C}{\sin A} \quad \left[\frac{a}{\sin A} = \frac{c}{\sin C}\right]$

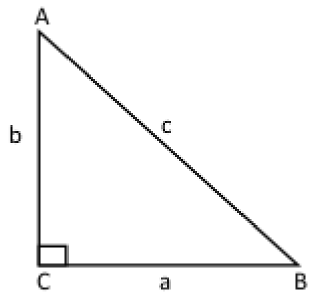
v. When two sides and an opposite angle are given

If two sides and an opposite angle are given, then we can use sine law to find the remaining parts:

Solution of a rt.angled Δ

I. When two sides are given

If two sides of a rt. angle ΔABC are given, then the remaining parts can be find as follows:



Given	To be calculated
i. a, b C = 90°	tanA = $\frac{a}{b}$, B = 90° - A c = $\frac{a}{\sin A}$ (sine law)
ii. a, c C = 90°	sinA = $\frac{a}{c}$, B = 90° - A b = c cosA, or b = $\sqrt{c^2 - a^2}$

When a side and an acute angle are given

If a side and an acute angle are given then the remaining parts can be find as follows:

Given	To be calculated
i. a, A	B = 90° - A C = $\frac{a}{\sin A}$
ii. C, A	B = 90° - A, a = CsinA

NOTE: Number of solutions:

1. If $\sin A > 1$, no angle A can be determined and in this case, there is no solution.
2. If $\sin A = 1,$ then $A = 90^\circ$ (rt. angled Δ) and only one solution is possible.
3. If $\sin A < 1$, two angles can be determined, an acute angle A and obtuse angle $(180^\circ - A)$ Thus we have possibility of two solutions.