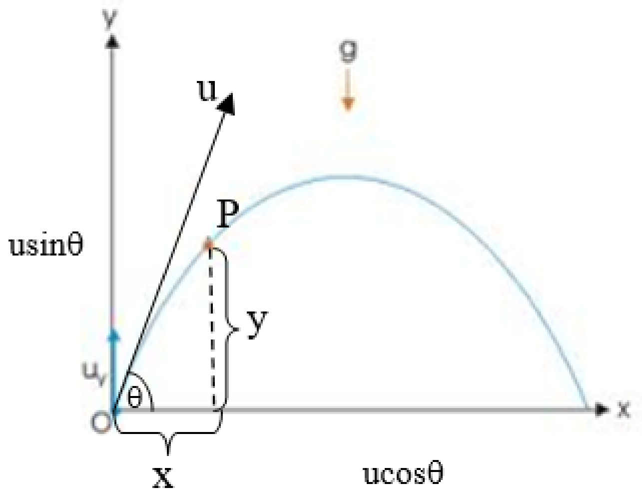
Projectile Motion

Introduction to Projectile Motion

Body projected in space having two dimensional motion is which one component of velocity is accelerating and another component of velocity remain constant is called projectile Angle made by projectile with horizontal is called angle of projection and path along which projectile move is called trajectory of projectile.

A body is projected with velocity 'u' from ground at an angle θ with horizontal then vertical component u sinθ is accelerating and horizontal component u cosθ remain constant throughout the motion.



For vertical displacement,

$$y = u\sin\theta t - \frac{1}{2}gt^2 \cdots\cdots(i)$$

For horizontal displacement,

$$x = u\cos\theta t$$
$$\Rightarrow t = \frac{x}{u\cos\theta} \ldots\ldots(ii)$$

From equation (i) and (ii),

$$y = u\sin\theta\frac{x}{u\cos\theta} - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)$$

$$\therefore \mathrm{y} = x\tan\theta - \frac{\mathrm{g}}{2\mathrm{u}^2\cos^2\theta}$$

is similar to $y = ax + bx^2$ so the path of projectile is parabolic.

Time of flight (T)

Time for which projectile remain is space. After time of flight h = 0 so,

$$\mathrm{h} = \mathrm{u}\sin\theta\mathrm{T} - \frac{1}{2}\mathrm{gT}^2$$
$$0 = \mathrm{u}\sin\theta\mathrm{T} - \frac{1}{2}\mathrm{gT}^2$$
$$\frac{1}{2}\mathrm{gT}^2 = \mathrm{u}\sin\theta.\mathrm{t}$$

$$\mathrm{T} = \frac{2\mathrm{u}\sin\theta}{\mathrm{g}}$$

For time of ascent V = 0 so,

$$\mathrm{V} = \mathrm{u}\sin\theta - \mathrm{gt}$$
$$\mathrm{t} = \frac{\mathrm{u}\sin\theta}{\mathrm{g}}$$

For time of descent $\mathrm{t} = \frac{\mathrm{u}\sin\theta}{\mathrm{g}}$

Hence, Time of ascent = Time of descent

Maximum Height Reached (H)

Projectile acquire vertical height unless vertical component of velocity become zero so At maximum height, V_y = 0 so,

$$\mathrm{V_y^2} = (\mathrm{u}\sin\theta)^2 - 2\mathrm{gH}$$
$$0 = \mathrm{u}^2\sin^2\theta - 2\mathrm{gH}$$
$$\mathrm{H} = \frac{\mathrm{u}^2\sin^2\theta}{2\mathrm{g}} = \frac{u_y^2}{2g}$$

Range of projectile (R)

Horizontal distance traveled by projectile during time of flight is called horizontal range.

$$\begin{aligned}\therefore \mathrm{Range(R)} &= \mathrm{u}\cos\theta.\mathrm{T} \\ &= \mathrm{u}\cos\theta\cdot\frac{2\mathrm{u}\sin\theta}{\mathrm{g}} \\ &= \frac{\mathrm{u}^2\sin 2\theta}{\mathrm{g}}\end{aligned}$$

$$\therefore \mathrm{R} = \frac{\mathrm{u}^2\sin 2\theta}{\mathrm{g}} = \frac{2\mathrm{u_x u_y}}{\mathrm{g}}$$

For maximum range $\sin 2\theta = 1 = \sin 90^\circ$

$$2\theta = 90^\circ$$
$$\Rightarrow \theta = 45^\circ$$

$$\therefore \mathrm{R_{max}} = \frac{\mathrm{u}^2}{\mathrm{g}}$$

Hence projectile projected at certain velocity will acquired maximum range if the angle of projection is 45°.

Maximum height for the projectile with maximum range

When a projectile is fired at an angle of 45° then

$$\mathrm{R_{max}} = \frac{\mathrm{u}^2}{\mathrm{g}}$$

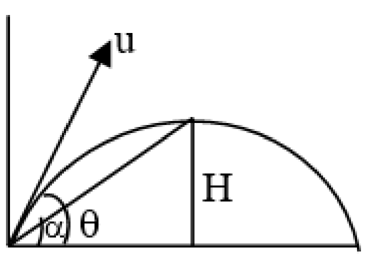
For maximum height,

$$\mathrm{H} = \frac{\mathrm{u}^2\sin^2\theta}{2\mathrm{g}} = \frac{\mathrm{u}^2}{2\mathrm{g}}\sin^2 45^\circ = \frac{\mathrm{u}^2}{2\mathrm{g}}\times\frac{1}{2} = \frac{1}{4}\mathrm{R_{max}}$$

$$\therefore \mathrm{R_{max}} = 4\mathrm{H}$$

The relation between angle of projection and angle of elevation at maximum height is

$$\tan\alpha = \frac{\mathrm{H}}{\mathrm{R}/2} = \frac{2\mathrm{H}}{\mathrm{R}} = \frac{2\mathrm{H}}{4\mathrm{H}\cot\theta} = \frac{\tan\theta}{2}$$



Two Angle of Projection for same R

A projectile is fired from ground with velocity u at an angle of θ with horizontal then

Range $(R_1 = \frac{\mathrm{u}^2\sin 2\theta}{\mathrm{g}}$

Another angle of projection for same range with same velocity will be

$$\begin{aligned}(R_1 &= \frac{\mathrm{u}^2\sin 2\theta}{\mathrm{g}} \\ &= \frac{\mathrm{u}^2}{\mathrm{g}}\sin(180^\circ - 2\theta) \\ &= \frac{\mathrm{u}^2}{\mathrm{g}}\sin 2\,(90^\circ - \theta)\end{aligned}$$

$$= \frac{u^2}{g}\sin 2\theta' \text{ where } \theta' = (90^\circ - \theta)$$

$\therefore \theta$ and $(90^\circ - \theta)$ are the two angle of projection for a projectile with same range with same velocity.

- T_1 & T_2 be the time of flight in two cases then
  $$\frac{T_1}{T_2} = \frac{u\sin\theta}{u\sin(90^\circ - \theta)} = \tan\theta$$
- H_1 & H_2 be the maximum height reached in two cases then
  $$\frac{H_1}{H_2} = \frac{u^2\sin^2\theta}{u^2\sin^2(90^\circ - \theta)} = \tan^2\theta$$

  and $H_1 + H_2 = \dfrac{u^2}{2g}$

- $$T_1 \times T_2 = \frac{2u\sin\theta}{g} \times \frac{2u\sin(90^\circ - \theta)}{g} = \frac{2u^2\sin 2\theta}{g^2} = \frac{2R}{g}$$

Velocity and Direction of Projectile at any height

The horizontal component of projectile remain constant throughout the motion but vertical component is accelerating. At any point P, at horizontal displacement x and vertical displacement y

Horizontal velocity $= u\cos\theta$

Vertical velocity V_y is given by

$$V_y^2 = (u\sin\theta)^2 - 2gy$$
$$\Rightarrow V_y = \sqrt{u^2\sin^2\theta - 2gy}$$

$\therefore$ **Resultant** velocity $(V) = \sqrt{V_x^2 + V_y^2} = \sqrt{u^2\cos^2\theta + u^2\sin^2\theta - 2gy} = \sqrt{u^2 - 2gy}$

For **direction**, let α be the direction of resultant with horizontal,

$$\tan\alpha = \frac{V_y}{V_x} = \frac{\sqrt{u^2\sin^2\theta - 2gy}}{u\cos\theta}$$

KE of projectile of mass m is

$KE = \frac{1}{2}mv^2 = \frac{1}{2}m(V_x^2 + V_y^2)$

- $$\frac{R}{H} = 4\cot\theta$$

- $$\frac{H}{T^2} = \frac{g}{8} \Rightarrow H = \frac{gT^2}{8}$$

- While moving from ground to maximum height
  - Change in speed
    $$\begin{aligned}(\Delta v) &= u - u\cos\theta \\ &= u(1-\cos\theta) \\ &= 2u\sin^2\theta/2\end{aligned}$$
  - Change in velocity
    $$\begin{aligned}(\overrightarrow{\Delta v}) &= (u\cos\theta\hat{i}) - (u\cos\theta\hat{i} + u\sin\theta\hat{j}) \\ &= -u\sin\theta\hat{j} = u\sin\theta\text{downward}\end{aligned}$$
  - Change in KE
    $$\begin{aligned}(\Delta KE) &= \frac{1}{2}mu^2 - \frac{1}{2}mu^2\cos^2\theta \\ &= \frac{1}{2}mu^2\left(1-\cos^2\theta\right)\end{aligned}$$
- While moving from ground to returning to ground
  - Change in speed
    $(\Delta v) = 0$

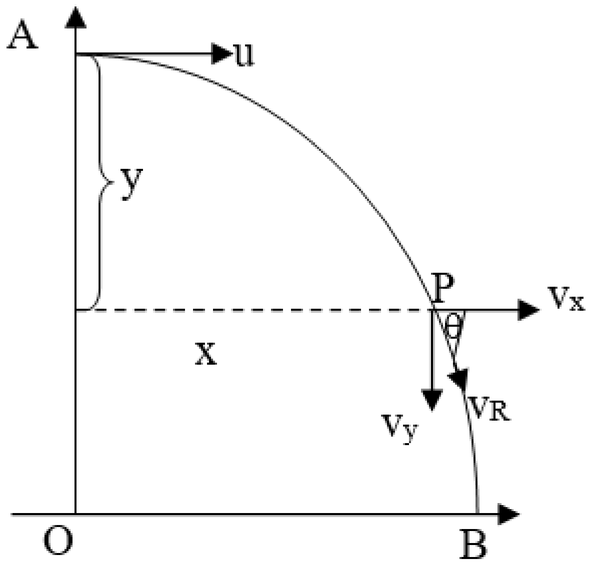
- Change in velocity

$$\overrightarrow{(\Delta v)} = (\mathrm{u}\cos\theta\hat{\mathrm{i}} - \mathrm{u}\sin\theta\hat{\mathrm{j}}) - (\mathrm{u}\cos\theta\hat{\mathrm{i}} + \mathrm{u}\sin\theta\hat{\mathrm{j}})$$
$$= -2\mathrm{u}\sin\theta\hat{\mathrm{j}} = 2\mathrm{u}\sin\theta\,\mathrm{downward}$$

Motion of a projectile thrown horizontally

A body is thrown horizontally form top of tower of height 'h' with velocity 'u' then it reaches to point P at a vertical displacement y and horizontal displacement '*x*' then.

For vertical motion,



$$y = \frac{1}{2}gt^2 \cdots\cdots (i)$$

For horizontal motion,

$$t = \frac{x}{u} \cdots\cdots (ii)$$

From equation (i) and equation (ii)

$$\mathrm{y} = \frac{\mathrm{g}}{2}\frac{x^2}{\mathrm{u}^2}$$

$\Rightarrow y = \dfrac{g}{2u^2} \cdot x^2$ is the equation of parabola.

For time of flight

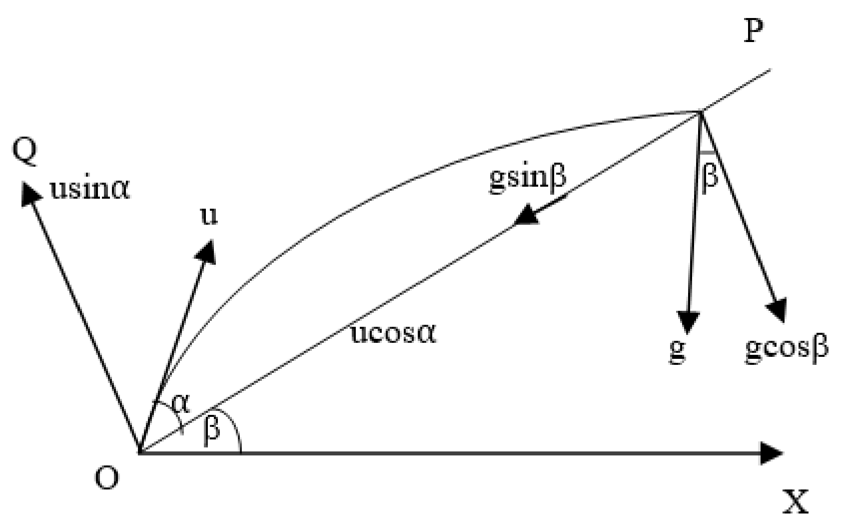
$$\mathrm{h} = \frac{1}{2}\mathrm{gT}^2$$

For horizontal range

Range $(\mathrm{R}) = \mathrm{u} \cdot \mathrm{T} = \mathrm{u}\sqrt{\dfrac{2\mathrm{h}}{\mathrm{g}}}$

Projectile on an inclined plane

An inclined plane OP which is inclined at an angle β with horizontal. From foot of inclined plane O a body is thrown with velocity u at an angle α with plane then $u\cos\alpha$ be the component of velocity along the plane and $u\sin\alpha$ be the component perpendicular to plane then



Along OP,

$$x = u\cos\alpha t - \frac{1}{2}g\sin\beta t^2$$

Along OQ,

$$y = u\sin\alpha t - \frac{1}{2}g\cos\beta t^2$$

For time of flight

After time of flight displacement along OQ become O so,

$$0 = \mathrm{u}\sin\alpha\mathrm{T} - \frac{1}{2}\mathrm{g}\cos\beta\mathrm{T}^2$$

$$\Rightarrow \frac{1}{2}g\cos\beta T^2 = u\sin\alpha T$$

$$\Rightarrow \mathrm{T} = \frac{2\mathrm{u}\sin\alpha}{\mathrm{g}\cos\beta}$$

For Range: The distance travelled along inclined plane is range so,

Range $(\mathrm{R}) = \mathrm{u}\cos\alpha\mathrm{T} - \frac{1}{2}\mathrm{g}\sin\beta\mathrm{T}^2$

$$= u\cos\alpha\cdot\frac{2u\sin\alpha}{g\cos\beta} - \frac{g\sin\beta}{2}\left(\frac{2u\sin\alpha}{g\cos\beta}\right)^2$$

$$= \frac{2u^2\sin\alpha\cdot\cos\alpha}{g\cos\beta} - \frac{1}{2}g\sin\beta\frac{4u^2\sin^2\alpha}{g^2\cos^2\beta}$$

$$= \frac{2\mathrm{u}^2\sin\alpha\cdot\cos\alpha}{\mathrm{g}\cos\beta} - \frac{2\mathrm{u}^2\sin^2\alpha\sin\beta}{\mathrm{g}\cos^2\beta}$$

$$= \frac{2u^2\sin\alpha}{\mathrm{g}\cos^2\beta}(\cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta)$$

$$= \frac{2u^2\sin\alpha}{\mathrm{g}\cos^2\beta}\cos(\alpha+\beta)$$

$$\therefore \mathrm{R} = \frac{2\mathrm{u}^2\sin\alpha\cos(\alpha+\beta)}{\mathrm{g}\cos^2\beta}$$