Limits And Continuity

Limit of a function

A function f(x) is said to have the limit A at a point $x = a$ if for a given small positive number ϵ, there exists another number δ such that
$|f(x) - A| < \epsilon$
whenever $0 < |x - a| < \delta$
Symbolically, $\lim_{x \to a} f(x) = A$

Indeterminate form:-

If a function in x takes any of the forms $\frac{0}{0}, \frac{\infty}{\infty}, (\infty - \infty), 0 \times \infty, 0^{\infty}, 1^{\infty}, \infty^0, 0^0$ etc. for certain value of x, then it is said to be in indeterminate form.

Infinitesimal

A function f(x) is said to be an infinitesimal as $x \rightarrow$ a if $\lim\limits_{x \to a} f(x) = 0$
(i.e. limit of a function is zero)

Example

The function $f(x) = \frac{1}{x^2}$ is an infinitesimal as $x \rightarrow \infty$

i.e. $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} \frac{1}{x^2}$
$= 0$

Right hand limit

If x approaches 'a' from the right i.e. from larger values of x, then the limit of $f(x)$ is called the right hand limit and is written as $\lim_{x \to a^+} f(x)$ or $f(a^+)$

Note

Limit of a function exists at a point $x = a$ iff **right hand liit=left hand limit** at that point,

i.e if $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x)$, then,

$\lim\limits_{x \to a} f(x)$ exists.

L-Hospital's Rule

Let $f(x)$ and $g(x)$ be two functions such that $f(a) = 0$ and $g(a) = 0$.

Then, $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}$

ptovided that $g'(a) \neq 0$.

$f'(x)$ and $g'(x)$ represent differentiation of $f(x)$ and $g(x)$ respectively.

Note: *This rule is applicable for* $\frac{0}{0}$ *and* $\frac{\infty}{\infty}$ *form.*

Formulas

i. $\lim\limits_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

ii. $\lim\limits_{\theta \to 0} \frac{\sin \theta}{\theta} = 1, \theta$ is measured in radians.

iii. $\lim\limits_{\theta \to 0} \frac{\sin \theta^{\circ}}{\theta} = \frac{\pi}{180}$

iv. $\lim_{\theta \to 0} \cos \theta = 1$

v. $\lim_{\theta \to 0} \frac{\tan \theta}{\theta} = 1$

vi. $\lim_{\theta \to \infty} \frac{\sin \theta}{\theta} = 0$

vii. $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$

viii. $\lim_{h \to \infty} \left(1 + \frac{a}{h}\right)^h = a$

ix. $\lim_{x \to 0} \left(\frac{e^x - 1}{x}\right) = 1$

x. $\lim_{x \to 0} \frac{\log(1 + x)}{x} = 1$

xi. $\lim_{x \to 0} (1 + x)^{y_x} = e$

xii. $x \to 0 \frac{\log_a(1 - x)}{x} = -\log_a e$

xiii. $\lim_{x \to 0} \frac{\log_a(1 + x)}{x} = \log_a e$

xiv. $\lim_{x \to 0^+} (\sin x)^x = 1$

xv. $\lim_{\theta \to 0} \theta \cdot \sin \frac{1}{\theta} = 0$

xvi. $\lim_{x \to 0} \frac{\sin^{-1} x}{x} = 1$

and

$\lim_{x \to 0} \frac{\tan^{-1} x}{x} = 1$

NOTE:

i. $\lim_{x \to 0} \sin \frac{1}{x}, \cos \frac{1}{x}, \tan \frac{1}{x}$ do not exist.

ii. $\lim_{x \to \infty} e^{1/x} = 0$

iii. $\lim_{x \to 0^+} e^{1/x} \to \infty$

iv. $\lim_{x \to 0} \frac{|x - a|}{x - a}, \lim_{x \to a} \frac{1}{x - a}$ do not exist.

Some important results

i. $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

ii. $\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

iii. $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, where $\lim_{x \to a} g(x) \neq 0$

iv. $\lim_{x \to a}[cf(x)] = c.\lim_{x \to a} f(x)$

v. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$

vi. $\lim_{x \to a} \log(f(x)) = \log\left[\begin{matrix}\lim \\ x \to a\end{matrix} f(x)\right]$

vii. $\lim_{x \to a} e^{f(x)} = e^{\lim_{x \to a} f^{(x)}}$

Continuity of a function

A function is said to be continuous at point x = a if the limiting value of the function is equals to functional value at the same point:

$\text{Limiting Value} = \text{Functional Value}$

i.e, Right hand limit = left hand limit = functional value

i.e. $\lim_{x \to a} f(x) = f(a)$

i.e. $f(a+) = f(a-) = f(a)$

$f(x)$ is said to be continuous at $x = a$ if $\lim_{h \to 0} f(a+h) = \lim_{h \to 0} f(a-h) = f(a)$

NOTE:

A function $f(x)$ defined in the neighborhood of the point $x = a$ is said to be continuous at $x = a$ if

$\lim_{x \to a} f(x) = f(a)$ provided that

(i) $f(a)$ is defined.

(ii) $\lim_{x \to a} f(x)$ exists.

Discontinuity of a function

A function $f(x)$ is said to be discontinuous at point $x = a$ if any of the following criteria is not satisfied. i.e,

$\lim_{x \to a} f(x)$ does not exist.

i.e, $\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$

$\lim_{x \to a} f(x)$ exists but is not equal to $f(a)$

i.e, $\lim_{x \to a} f(x) \neq f(a)$

(Limiting value $\neq$ functional value)

functional value doesn't exist at $x = a$

There are two types of discontinuities i.e,

i. Removable discontinuity

ii. Ordinary discontinuity

Removable discontinuity

If $\lim_{x \to a} f(x)$ exists but is not equal to $f(a)$, i.e, $\lim_{x \to a} f(x) \neq f(a)$, then $f(x)$ is said to have a removable discontinuity at $x = a$

Ordinary discontinuity

If $\lim_{x \to a} f(x)$ dosen't exist i.e, $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$ then $f(x)$ is said to have an ordinary discontinuity or a jump discontinuity at $x = a$.

Properties of continuous functions

i. A function $f(x)$ is said to be continuous function in a set if it is continuous at every point of the set.
ii. A function $f(x)$ is said to be continuous in open interval (a, b) if it is continuous at every point on (a, b).
iii. A function $f(x)$ is said to be continuous in closed interval $[a, b]$ if it is continuous at every point of open interval (a, b) and also at point a from the right and point b from the left.
iv. A continuous function on closed interval has a maximum and a minimum on the interval.
v. $\sin x, \cos x$ are continuous for all real values of x.
vi. If $f,\ g$ are continuous at $x = a$, then $|f|, f \pm g, fg, f/g(g \neq 0)$ are also continuous at $x = a$.

NOTE

i. Every constant function $f(x) = c$ (for all x) is continuous everywhere.

ii. The identity function $f(x) = x$ is continuous everywhere.
iii. The modulus function $f(x) = |x|$ is continuous every where.
iv. The exponential function $f(x) = a^x$ is continuous everywhere for $a > 0, x \in R$
v. The polynomial function $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots\ldots$ is continuous everywhere, $x \in R$
vi. Point function (domain and range consists one value only) is not a continuous function.