Circular Motion

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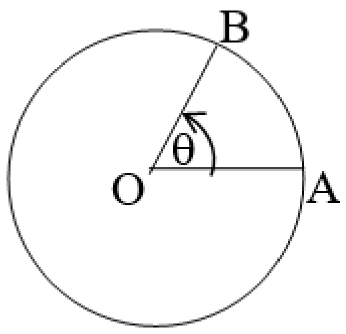
Motion of a body along the circumference of circular path is called circular motion.

Angular Displacement (θ):

Angle made by two positions of a body at centre of circular path is called angular displacement. It is measured in radian.



A and B are two positions of a body on the circumference of circular path then $\angle$ AOB is called angular displacement.

Angular velocity (ω):

The rate of change in angular displacement of body is called angular velocity. It is measured in radian/s.

$\therefore$ θ be the angular displacement in time t then

Angular velocity $(\omega)=\dfrac{\theta}{t}$

For non uniform motion $\Delta\theta$ be the small angular displacement in small time $\Delta\mathrm{t}$ then angular velocity at any instant of time is

Angular velocity $(\omega)=\lim_{\Delta t\to 0}\dfrac{\Delta\theta}{\Delta t}$

Relationship between linear and angular velocities:

v be the linear velocity and ω be the angular velocity of body along the circumference of circular path or radius r then

$$\theta=\frac{s}{r}$$
$$\Rightarrow s=r\theta$$
$$\Rightarrow \frac{\Delta s}{\Delta t}=r\frac{\Delta\theta}{\Delta t}$$
$$\therefore v=\mathrm{r}\omega$$

Again, $\dfrac{\Delta v}{\Delta t}=r\dfrac{\Delta\omega}{\Delta t}$

$$\therefore a=\mathrm{r}\alpha$$

Frequency (f):

The number of revolutions made by body in one second

$\therefore$ Angular velocity $(\omega)=2\pi f$

Time period (T):

The time taken by body to complete one revolution is called time period.

$\therefore$ Angular velocity $(\omega)=\dfrac{2\pi}{T}$

Now, $f=\dfrac{1}{T}$

Centripetal Force

When a body is moving along the circumference of circular path with uniform speed then the velocity of body is changing due to change in direction. As the velocity of body changes, then body is said accelerating and this acceleration is directed towards centre is called centripetal acceleration. A constant force is needed to maintain this acceleration towards centre is called centripetal force.

Centripetal acceleration $(a) = \frac{v^2}{r} = r\omega^2$

For a body of mass m,

Force $(F) = \frac{mv^2}{r} = mr\omega^2$

The force of magnitude equal to $\frac{mv^2}{r}$ should be applied on a body moving in circular path. As the force towards centre disappear, body starts to move along the direction of instantaneous velocity.

If velocity of body is variable:

When a body is moving along the circumference of circular path with changing speed then the centripetal acceleration is $a_c = \frac{v^2}{r}$ and tangential acceleration a_t at any position then

Net acceleration $(a) = \sqrt{a_c^2 + a_t^2}$

No work is done by centripetal force since the force and displacement are perpendicular to each other. Hence energy is not spent by body moving in circular path.

Centrifugal force (F):

As the body moves along the circumference of circular path with uniform speed then the velocity of body changes towards centre of circular path. Due to change in velocity of body towards centre, the body tends to move away from centre of circular path. This force by which body is thrown away from centre of circular path is called centrifugal force. The magnitude of centrifugal force is also equal to that of centripetal force.

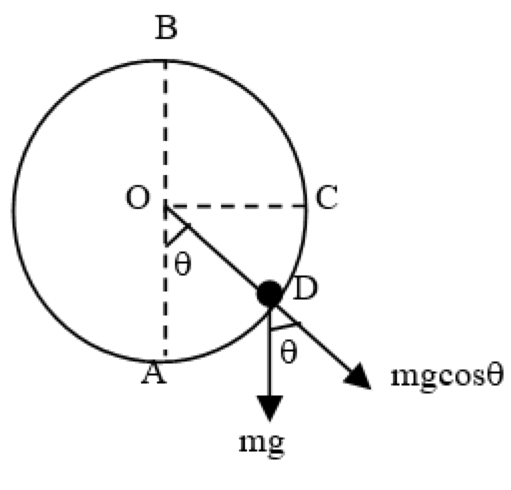
$\therefore$ Centrifugal force $(F) = \frac{mv^2}{r} = mr\omega^2$

Motion of a body in a vertical circle

When a body is revolved in a vertical circle of radius r with uniform speed then tension on the string varies from point to point. At any position D at an angle θ with vertical,

$$\mathrm{T} - \mathrm{mg}\cos\theta = \frac{mv^2}{r}$$

$$\therefore \mathrm{T} = \frac{mv^2}{r} + \mathrm{mg}\cos\theta$$



- Case 1

  At A, lowest point of vertical circle $\theta = 0^\circ$ so $T = \frac{mv^2}{r} + mg$ i.e. Tension become maximum.

- Case 2

  At C, horizontal position of vertical circle $\theta = 90^\circ$ so $T = \frac{mv^2}{r}$

- Case 3

  At B, highest point of vertical circle $\theta = 90^\circ$ so $T = \frac{mv^2}{r}$ i.e. Tension become least.

Minimum speed of body in a vertical circle

When body is revolved in a vertical circle with minimum speed then

- Speed will be least at highest point if $T = 0$ so $0 = \frac{mv^2}{r} - mg$

  $\frac{mv^2}{r} = mg$

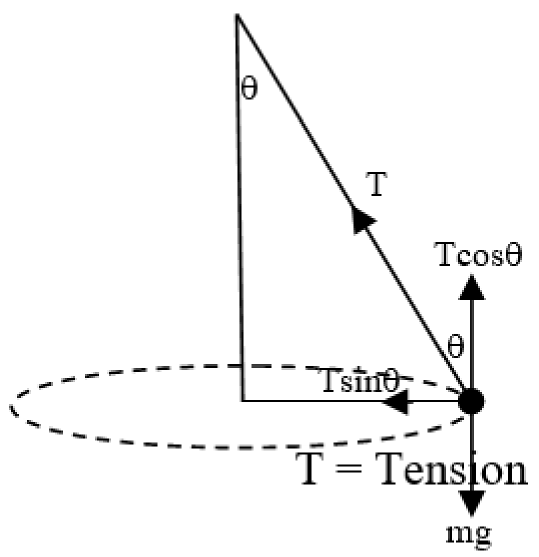
  $v = \sqrt{gr}$ is least speed at highest point.

- The body with least speed at greatest point reaches to horizontal position then body will have least speed for that position so
  $v^2 = u^2 - 2gr$
  $gr = u^2 - 2gr$
  $u = \sqrt{3gr}$
  $\therefore$ The minimum speed at horizontal position of vertical circle is $v = \sqrt{3gr}$

- The minimum speed of body at lowest point of vertical circle to loop the circle is $v = \sqrt{5gr}$

Motion of a body in horizontal circle (Conical pendulum)

A body of mass m is revolved in a horizontal circle of radius 'r' with speed 'v' by a string inclined at an angle θ with vertical than.



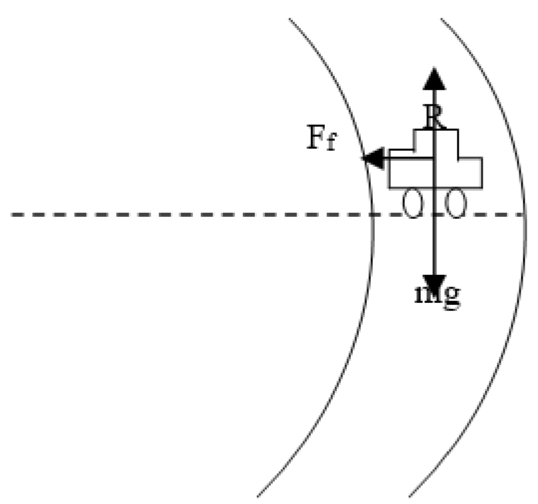
$\mathrm{T}\cos\theta = \mathrm{mg}$ ....(i)
$\mathrm{T}\sin\theta = \frac{mv^2}{r}$ ....(ii)

i.e. $\tan\theta = \frac{v^2}{rg}$

$\therefore$ Time period (T) $= \frac{2\pi r}{v} = 2\pi\sqrt{\frac{l\cos\theta}{g}} = 2\pi\sqrt{\frac{h}{g}}$

Motion of a vehicle along labeled circular path

When a vehicle of mass 'm' is moving in a lebelled circular track of radius r with speed 'v' then frictional force between tyre and road provide the necessary centripetal force so,



$\mathrm{F_f} \geq \frac{mv^2}{r}$

$\Rightarrow \mu\mathrm{mg} \geq \frac{mv^2}{r}$

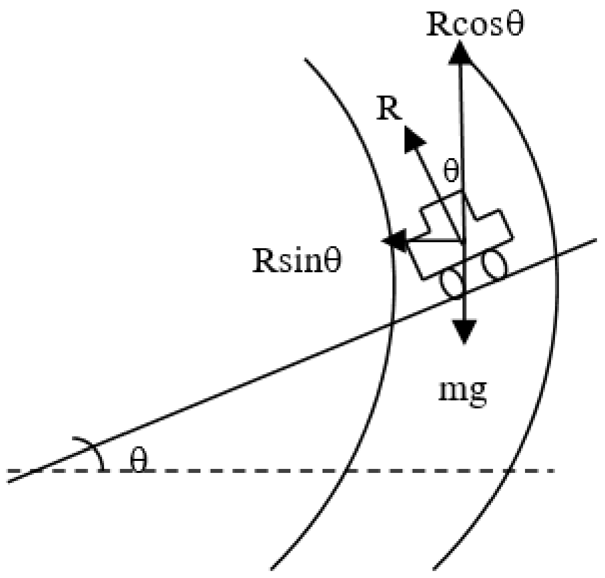
$\Rightarrow \mu\mathrm{rg} \geq \mathrm{v}^2$

$\therefore V_{max} = \sqrt{\mu rg}$ is the maximum permissible speed.

Banking of track

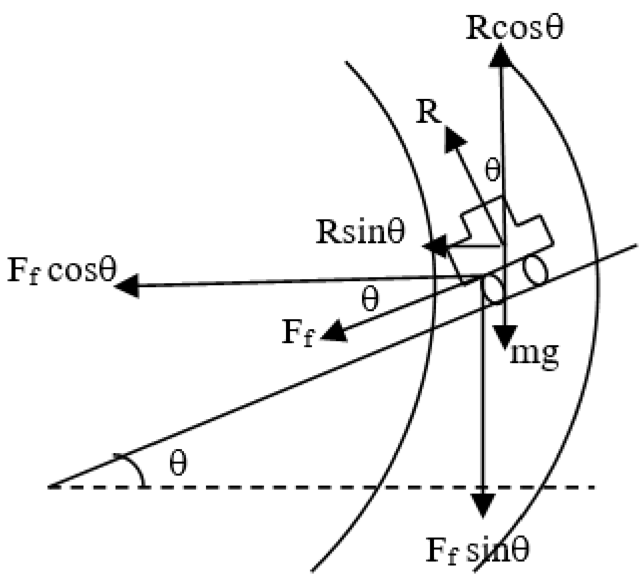
Roads are raised away from centre of circular path to increase the permissible speed of vehicle in circular path. θ be the inclination of road then,

$$\mathrm{R}\cos\theta = \mathrm{mg}$$
$$\mathrm{R}\sin\theta = \frac{mv^2}{r}$$
$$\tan\theta = \frac{v^2}{r\mathrm{g}}$$



R = Reaction of banked track on vehicle.

When the frictional force is also accounted then,



$$\mathrm{F_f}\cdot\cos\theta + \mathrm{R}\sin\theta = \frac{mv^2}{r}\ldots\ldots(\mathrm{i})$$

For vertical equilibrium,

$$\mathrm{F_f}\sin\theta + \mathrm{mg} = \mathrm{R}\cos\theta\ldots\ldots(\mathrm{iii})$$

Here,

$\mathrm{F_f} = \mu_\mathrm{s}\mathrm{R}$ then

$\mu_\mathrm{s}\mathrm{R}\cos\theta + \mathrm{R}\sin\theta = \frac{mv^2}{r}$ ....(iii)

And

$\mathrm{R}\cos\theta - \mu_\mathrm{s}\mathrm{R}\sin\theta = \mathrm{mg}$ .....(iv)

Dividing (iii) and (iv),

$$\frac{v^2}{rg} = \frac{\mu_s\cos\theta + \sin\theta}{\cos\theta - \mu_s\sin\theta}$$

$\Rightarrow v = \sqrt{rg\frac{(\mu_s + \tan\theta)}{(1-\mu_s\tan\theta)}}$ is the maximum safe speed

- If the outer edge of path is raised at 'h' at a distance d between inner & outer wheel then

$$\tan\theta = \frac{v^2}{\mathrm{rg}} = \frac{\mathrm{h}}{\mathrm{d}}$$

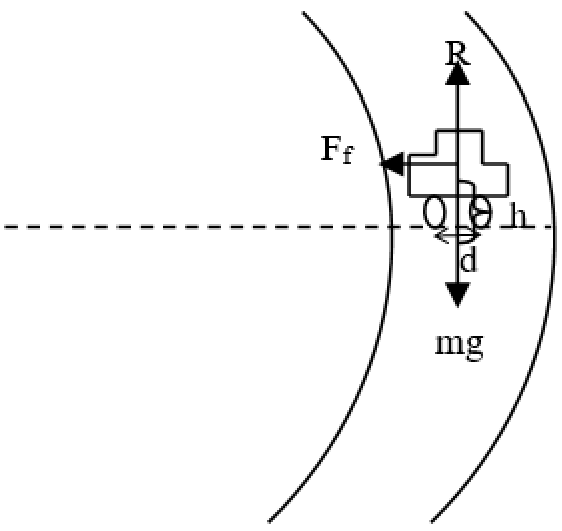
$\therefore \mathrm{h}=\frac{v^2 \mathrm{d}}{\mathrm{rg}}$ is the height between inner and outer wheel.

Maximum velocity for skidding and overturning

A vehicle is moving in a circular track then R_1 and R_2 be the action on the inner and outer wheel of vehicle then

$$\mathrm{F_f} \geq \frac{mv^2}{r}$$

$$\mu \mathrm{mg} \geq \frac{mv^2}{r}$$
$$\Rightarrow \mu r \mathrm{g} \geq \mathrm{v}^2$$
$$\Rightarrow \mathrm{v}^2 \leq \mu \mathrm{rg}$$
$$\Rightarrow \mathrm{v} \leq \sqrt{\mu r g}$$



For maximum speed for no skidding

$$\mathrm{V}_{\max}=\sqrt{\mu r g}$$

For no overturning h be the height from ground in which 'd' the distance between inner & outer wheels of vehicle then taking moment about G.

$$\mathrm{R}_2 . \frac{d}{2}=\mathrm{R}_1 \frac{d}{2}+\left(\mu \mathrm{R}_1+\mu \mathrm{R}_2\right) \mathrm{h}$$

$$\Rightarrow \mathrm{R}_2 \cdot \frac{d}{2}=\mathrm{R}_1 \frac{d}{2}+\mu \mathrm{mgh}$$

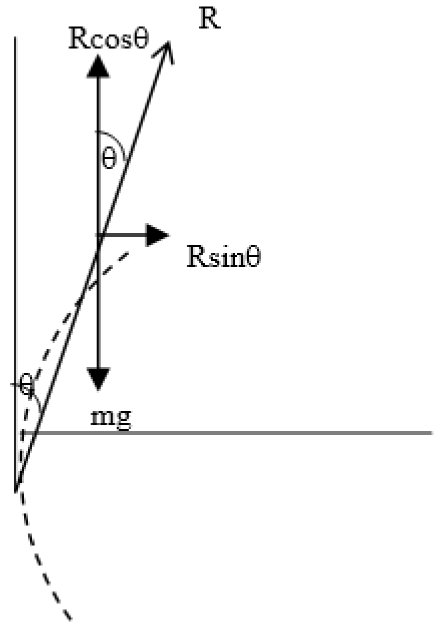
$$\Rightarrow \mathrm{R}_2 . \frac{d}{2}=\mathrm{R}_1 \frac{d}{2}+\frac{mv^2}{r} \mathrm{h}$$

Car will overturn if reaction R_1 on inner wheel is zero so

$$\mathrm{R}_1=0$$
$$\mathrm{R}_2 \frac{d}{2} \geq \frac{mv^2}{r} \mathrm{h}$$
$$\mu \mathrm{g} \frac{d}{2} \geq \frac{mv^2}{r} \mathrm{h}$$

$v=\sqrt{\frac{rgd}{2h}}$ is maximum speed for no overturning

Motion of cyclist in circular path cyclist in cline himself at certain angle with vertical to provide the necessary centripetal force so,



$\mathrm{R}\sin\theta = \dfrac{mv^2}{r}$ ....(i)

$\mathrm{R}\cos\theta = \mathrm{mg}$

$\therefore \tan\theta = \dfrac{v^2}{rg}$

$\theta = \tan^{-1}\left(\dfrac{v^2}{\mathrm{rg}}\right)$

$\therefore v = \sqrt{\mu rg}$ is the speed for no overturning.