Basic Trigonometry

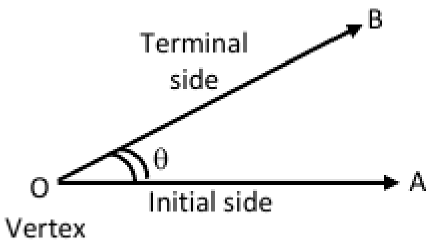
Trigonometry:

The literal meaning of "Trigonometry" is the science of measuring triangles. The modern definition of Trigonometry is the branch of mathematics which deals with the measurement of angles and the problems allied with angles.

Trigonometry is used in surveying, astronomy, navigation, physics, engineering etc. The periodic phenomena like simple harmonic vibration, wave equation appearing in physics or engineering are expressed as linear combination of sine or cosine.

Angle:

An angle is the amount of revolution of a revolving line with respect to a fixed line.



If the revolution is in clockwise direction, then the angle is negative and if the revolution is in anticlockwise direction, then the angle is positive.

Measurement of Angles:

There are three systems of measuring an angle, viz

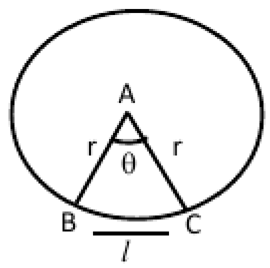
a. Sexagesimal system,
b. Circular system,
c. Centesimal system

a. **Sexagesimal system**
In this system, a right angle is divided into 90 equal parts and each part is called a degree, denoted by the symbol 10 . Each degree is divided into sixty equal parts and each part is calledminute denoted by the symbol 1′ and each minute is divided into sixty equal parts and each part is called a second denoted by the symbol 1′′. Thus,

1 right angle = 90 degrees (90^0)
1 degree (1^0) = 60 minuites (60 ′)
1 minuite (1′) = 60 seconds (60′′)

If degree measure of an angle is D then the angle $= D^o = \frac{D}{90} rt.angle$

b. **Circular system**
In this system an angle is measured in radians. A radian is an angle subtended at the centre of a circle by the arc whose length is equal to the radius of the circle.



θ is the angle subtended by the arc at the centre of the circle. l be the length of the arc and r be l the radius then $\theta = \frac{l}{r}$

A Constant Number π

The ratio of the circumference of a circle to the diameter of the circle is always the same. This ratio is constant for all circles and is denoted by the symbol π (pi). If C and D are the C circumference and diameter, then $\frac{C}{D} = \pi$

where, $C = 2\pi r$ and $D = 2r$

Relation between Degree and Radian

If θ is the angle at the centre opposite to an arc of length l
Then, $l = r\theta$

The angle subtended at the centre by its circumference $= 360^\circ = \frac{2\pi r}{r} = 2\pi$ radian

So, π radian $= 180^\circ = 2$ right angles

1 radian $= \frac{180^\circ}{\pi} = 57^\circ 17' 44.8''$ (approx)

c. Centisimal System

A right angle is divided into 100 equal parts and each part is called 1 grade and denoted by $1^g . 1^g$ is divided into 100 equal parts and each part is called minuite denoted by $1'$. One minuite is divided into 100 equal parts and each part is called second denoted by 1" . Thus, 1 right angle $= 100^g$

$1^{\mathrm{g}} = 100'$
$1' = 100''$
If grade measure of an angle is G then
the angle $\mathrm{G}^{\mathrm{g}} = \frac{\mathrm{G}}{100} \mathrm{rt.}$ angle

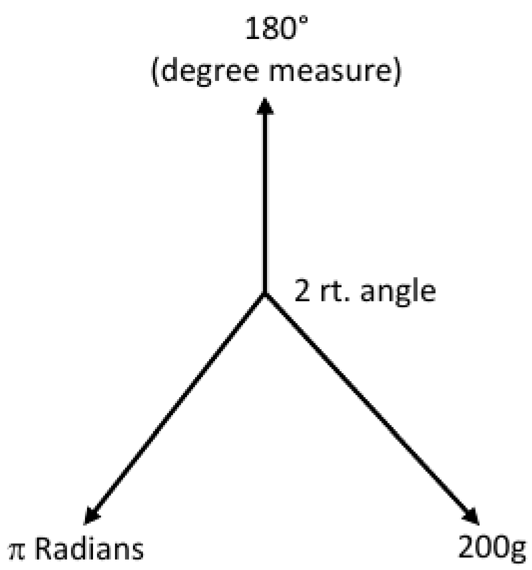
Important Note:

i. angle made in one complete revolution $= 360^\circ = 2\pi$ radians

ii. $180^\circ = \pi \mathrm{radian} = 200^{\mathrm{g}}$

iii. $90^\circ = 100^{\mathrm{g}}$

$$1^\circ = \left(\frac{10}{9}\right)^{\mathrm{g}}$$



$$1 \text{ radian} = \left(\frac{180}{\pi}\right)^\circ = \left(\frac{200}{\pi}\right)^g$$

iv. $100^{\mathrm{g}} = 90^\circ$

$$1^{\mathrm{g}} = \left(\frac{9}{10}\right)^\circ$$

v. 1 radian $= \left(\frac{180}{\pi}\right)^\circ = \left(\frac{200}{\pi}\right)^{\mathrm{g}}$

vi. Circular measure of angle of one radian is equals to one.

vii. Degree measure of an angle of π radian is 180°.

viii. 1 radian $= 57^\circ 17' 44.8(\text{ approx. })$

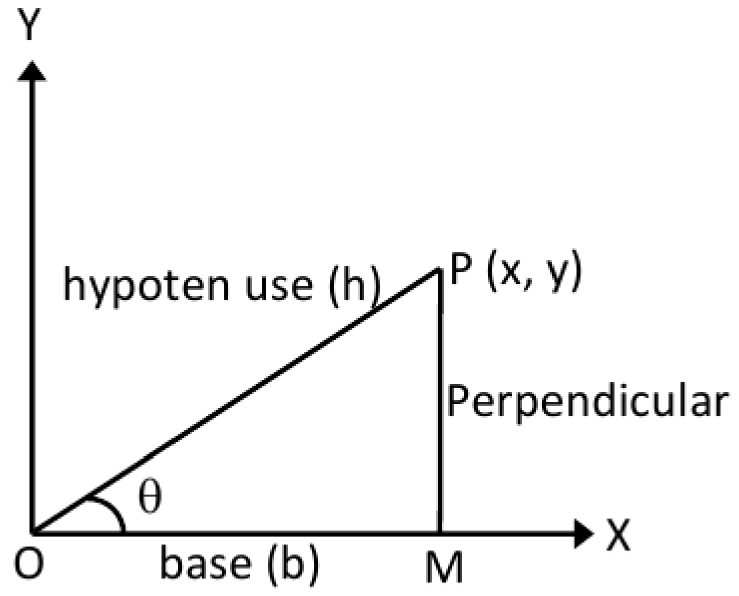
ix. π is a constant and defined as the ratio of the circumference of a circle to its diameter. i.e.

$$\frac{\text{circumference of a circle } (2\pi r)}{\text{Diameter of a circle } (2r)} = \pi$$

Note:- π is an irrational number and its value is $\pi = \frac{22}{7}$ or 3.14159

Trigonometric functions:

Trigonometric functions can be defined by the ratio of the sides in a right angled triangle. Trigonometric functions of an angle can be defined in terms of a circle and they are called circular functions.



$\sin\theta = \frac{PM}{OP} = \frac{\text{perpendicular}}{\text{hypoten use}} = \frac{P}{h}$

$\cos\theta = \frac{OM}{OP} = \frac{\text{base}}{\text{hypotenuse}} = \frac{b}{h}$

$\tan\theta = \frac{PM}{OM} = \frac{\text{perpendicular}}{\text{base}} = \frac{P}{b}$

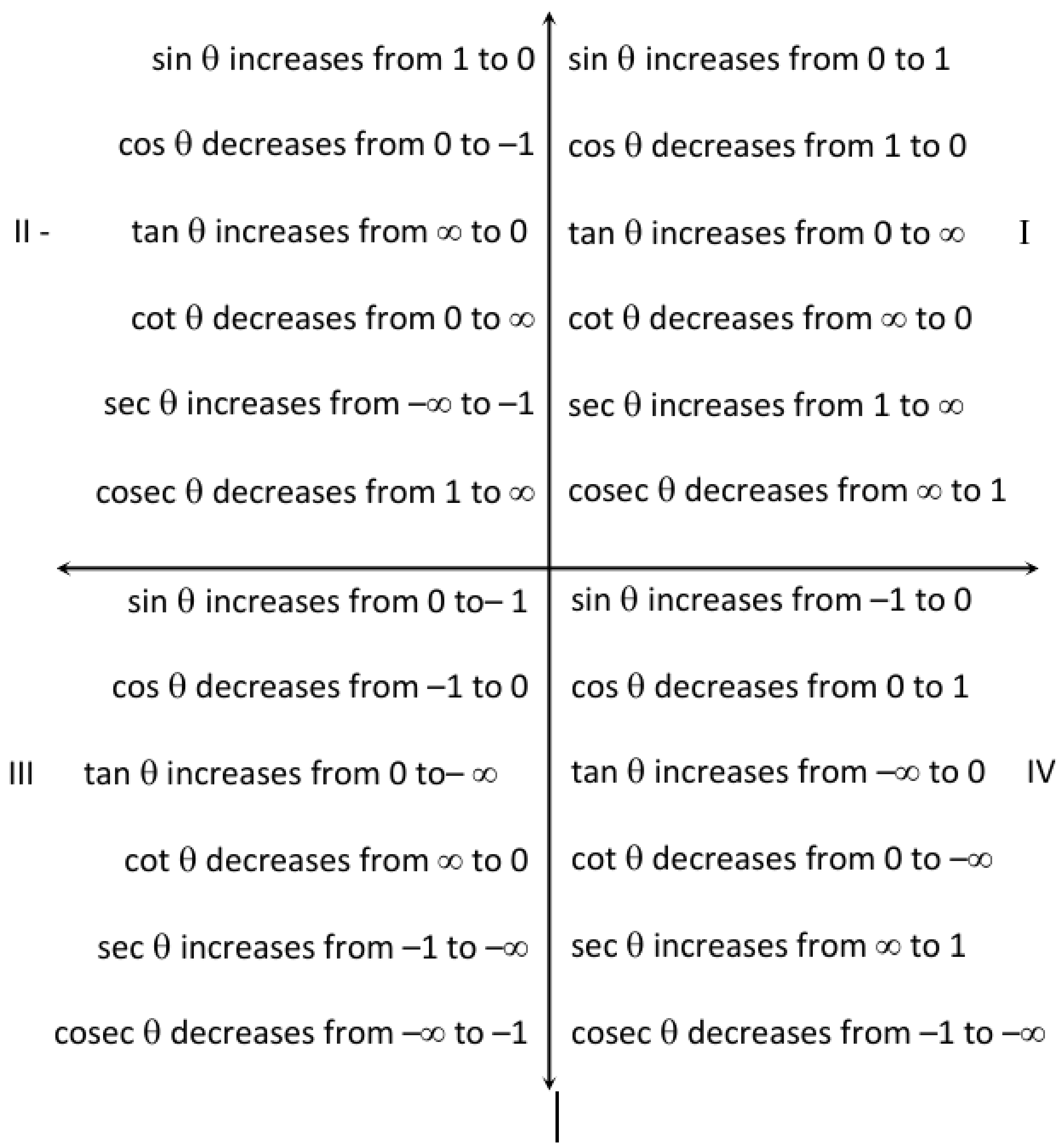
$\cot\theta = \frac{\mathrm{OM}}{\mathrm{PM}} = \frac{\text{base}}{\text{perpendicular}} = \frac{\mathrm{b}}{\mathrm{p}}$

$\sec\theta = \frac{\mathrm{OP}}{\mathrm{OM}} = \frac{\text{hypotenuse}}{\text{base}} = \frac{\mathrm{h}}{\mathrm{b}}$

$\csc\theta = \frac{\mathrm{OP}}{\mathrm{PM}} = \frac{\text{hypotenuse}}{\text{perpendicular}} = \frac{\mathrm{h}}{\mathrm{p}}$

$h^2 = p^2 + b^2$

Variation of various circular functions in different quadrants:

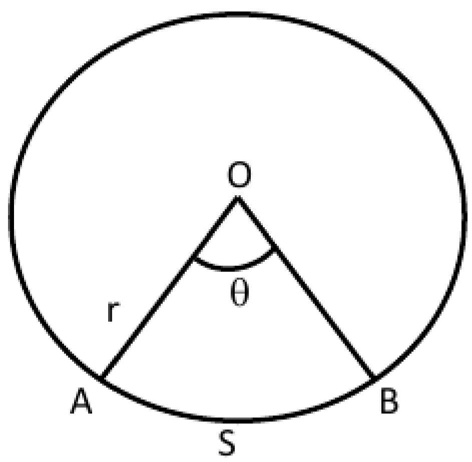


Types of angles:

i. an acute angle: $0^\circ \leq \theta < 90^\circ$
ii. an obtuse angle : $90^\circ < \theta < 180^\circ$
iii. a reflex angle: $180^\circ < \theta < 360^\circ$
iv. a right angle: $\theta = 90^\circ$
v. a straight angle : $\theta = 180^\circ$

Note:

i. sum of interior angles of a polygon of n sides = (n -2) × 180°
ii. All interior angles and exterior angles of a regular polygon are equal.
iii. Interior angle of a regular polygon $= \dfrac{(n-2) \times 180}{n}$, where $n =$ no. of sides
iv. Each interior angle = 180° - exterior angle
v. Each exterior angle $= \dfrac{360^\circ}{\mathrm{n}}$, where n = no. of sides.
vi. Sum of all exterior angle is 360°
vii. The hour hand of a clock rotate through an angle of 30° in one hour.
i.e. in 12 hrs = 360°
$$1\mathrm{hr} = \frac{360^\circ}{12} = 30^\circ$$
viii. The minute hand of a clock rotate through an angle of 6° in one minute.
i.e.in 60 minutes = 360°
ix. $\pi = \dfrac{\text{Circumference of a circle}}{\text{Diameter of a circle}}, \left(\dfrac{2\pi r}{2r} = \pi\right)$
x. $\theta = \dfrac{S}{r}$ (radian measure)



where
s→ length of an arc
θ→ angle subtended by the arc at the centre of the circle
r→ radius of the circle
area of sector $(\mathrm{A}) = \dfrac{1}{2} r^2 \theta = \dfrac{1}{2} \mathrm{rs}$

Trigonometric ratios:

a. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}, \cot\theta = \dfrac{\cos\theta}{\sin\theta}$
$\sin\theta \cdot \text{\cosec}\ \theta = 1$
$\cos\theta \cdot \sec\theta = 1$
$\tan\theta \cdot \cot\theta = 1$

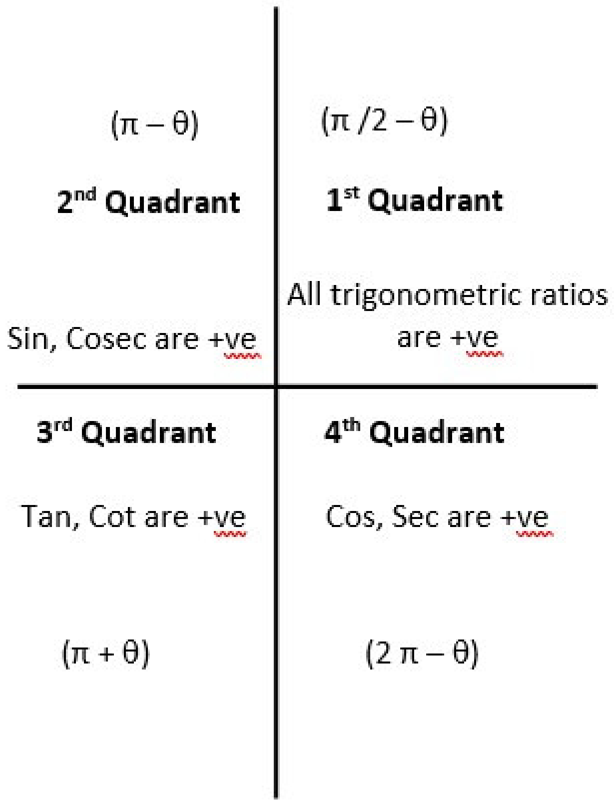
b. (i) $\sin^2\theta + \cos^2\theta = 1$
(ii) $\sec^2\theta - \tan^2\theta = 1$
(iii) $\text{\cosec}\ ^2\theta - \cot^2\theta = 1$

c. $|\sin\theta| \leq 1$
$|\cos\theta| \leq 1$
$|\text{\cosec}\ \theta| \geq 1$
$|\sec\theta| \geq 1$ for all real θ
$\tan\theta, \cot\theta \in \Re$

Table of values of Trigonometric function and different angles.

$\theta\rightarrow$	0^o	30^o	45^o	60^o	90^o	120^o	135^o	150^o	180^o
$\sin\theta$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0	$-1/2$	$-1/\sqrt{2}$	$-\sqrt{3}/2$	-1
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-1/\sqrt{3}$	0
$\mathrm{cosec}\theta$	∞	2	$\sqrt{2}$	$2/\sqrt{3}$	1	$-2/\sqrt{3}$	$\sqrt{2}$	$-\sqrt{3}/2$	∞
$\sec\theta$	1	$2/\sqrt{3}$	$\sqrt{2}$	2	∞	-2	$-\sqrt{2}$	$-2/\sqrt{3}$	-1
$\cot\theta$	∞	$\sqrt{3}$	1	$1/\sqrt{3}$	0	$-1/\sqrt{3}$	-1	$-\sqrt{3}$	∞

Quadrants:



Function	Quadrant			
	I	II	III	IV
sin θ	+	+	–	–
cos θ	+	–	–	+
tan θ	+	–	+	–
sec θ	+	–	–	+
cot θ	+	–	+	–
cosec θ	+	+	–	–

Quadrant Rule:

$0^0 < \theta < 90^o$	$90^0 < \theta < 180^o$	$180^0 < \theta < 270^o$	$270^0 < \theta < 360^o$

all trigonometric ratios are positive	sin & cosec positive others are negative	tan and cot positive others are negative	cos & sec positive others are negative

Angles in degrees	0°	30°	45°	60°	90°	120°	135°	150°	180°
Angles in circular measure	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π

Angles in degree	210°	240°	270°	300°	330°	360°
angles in circular measure	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π

Formulas for t-ratios:

I.		**II.**	
i.	sin (-A) = -sinA	i.	sin (90 ° +A) = cosA
ii.	cos (-A) = cosA	ii.	cos (90 ° + A) = -sinA
iii.	tan (-A) = -tanA	iii.	tan (90 ° + A)= -cotA
iv.	cot (-A) = -cotA	iv.	cot (90 ° + A)= -tanA
v.	sec (-A)= secA	v.	sec (90 ° +A)=-cosecA
vi.	cosec(-A) =-cosecA	vi.	cosec (90 ° + A)=secA
III.		**IV.**	
i.	sin(90°- A) = cosA	i.	sin (180 ° - A) = sinA
ii.	cos (90 ° - A)= sinA	ii.	cos (180 ° -A) = -cosA
iii.	tan (90 ° -A)= cotA	iii.	tan (180 ° - A)= -tanA
iv.	cot (90 ° - A)= tanA	iv.	cot (180 ° -A)= -cotA
v.	cosec (90 ° - A) = secA	v.	cosec (180 ° - A)= cosec A
vi.	sec (90 ° -A) = cosecA	vi	sec(180 ° + A)= -secA
V.		**VI.**	
i.	sin (180 ° + A) = -sinA	i.	sin (270 ° -A)= -cosA
ii.	cos (180 ° + A) = -cosA	ii.	cos (270 ° - A)= -sinA
iii.	tan (180 +A)= tanA	iii.	tan (270 ° - A)= cotA

iv.	cot (180° + A) = cotA	iv.	cot (270° -A)= tanA
v.	sec (180° +A)= -sec A	v.	sec (270° -A) =-cosecA
vi.	cosec(180° + A) = -cosecA	vi.	cosec (270° -A) = -secA
VII.		**VIII.**	**For n 0 I**
i.	sin (270° +A)= -cosA	i.	sin (n. 360° + θ)= sin θ
ii.	cos(270° +A)= sinA	ii.	cos (n.360° + θ)=cos θ
iii.	tan(270° +A)=-cotA	iii.	tan (n. 360° + θ)=tan θ
iv.	cot (270° +A)=-tanA	iv.	sec (n. 360° + θ)=sec θ
v.	sec (270° +A) = cosec A	v.	cosec(n.360°+θ)= cosec θ
vi.	cosec (270° + A) = -secA	vi.	cot (n.360° +θ) = cot θ

Results on compound angles

i. sin (A +B) = sinA . cosB + cosA. sinB
ii. sin (A – B) = sinA . cosB – cosA.sinB
iii. cos(A + B) = cosA cosB – sinA .sinB
iv. cos(A – B) = cosA . cosB + sinA.sinB
v. tan(A +B) = $\frac{\tan A+\tan B}{1-\tan A\cdot\tan B}$
vi. tan (A – B) = $\frac{\tan A-\tan B}{1+\tan A\cdot\tan B}$
vii. cot (A +B) = $\frac{\cot A\cdot\cot B-1}{\cot B+\cot A}$
viii. cot (A – B) = $\frac{\cot A\cdot\cot B+1}{\cot B-\cot A}$
ix. sin(A +B) . sin(A-B) = $\sin^2$A – $\sin^2$B = $\cos^2$B – $\cos^2$A
x. cos(A +B) . cos(A – B) = $\cos^2$A – $\sin^2$B = $\cos^2$B – $\sin^2$A

Multiple angles

i. $\sin 2\theta=2\sin\theta\cdot\cos\theta=\frac{2\tan\theta}{1+\tan^2\theta}$

ii. $\cos 2\theta=\begin{cases}\cos^2\theta-\sin^2\theta\\1-2\sin^2\theta\\2\cos^2\theta-1\end{cases}$

$=\frac{1-\tan^2\theta}{1+\tan^2\theta}$

iii. $\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$

iv. $2\cos^2 A=1+\cos 2A$

$\Rightarrow\cos^2 A=\frac{1+\cos 2A}{2}$

v. $2\sin^2 A=1-\cos 2A$

$\Rightarrow\sin^2 A=\frac{1-\cos 2A}{2}$

vi. $\sin 3A=3\sin A-4\sin^3 A$

vii. $\cos 3A=4\cos^3 A-3\cos A$

viii. $\tan 3A=\frac{3\tan A-\tan^3 A}{1-3\tan^2 A}$

Sub-multiple Angles

i. $\sin A=2\sin\frac{A}{2}\cdot\cos\frac{A}{2}=\frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$

ii. $\cos A=\cos^2\frac{A}{2}-\sin^2\frac{A}{2}$

$=1-2\sin^2\frac{A}{2}=2\cos^2\frac{A}{2}-1$

$=\frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$

iii. $\tan A=\frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$

iv. $2\cos^2\frac{A}{2}=1+\cos A$

v. $2\sin^2\frac{A}{2}=1-\cos A$

vi. $\sin A=3\sin\frac{A}{3}-4\sin^3\frac{A}{3}$

vii. $\cos A=4\cos^3\frac{A}{3}-3\cos\frac{A}{3}$

viii. $\tan A=\frac{3\tan\frac{A}{3}-\tan^3\frac{A}{3}}{1-3\tan^2\frac{A}{3}}$

Transformation Formulae

A. <u>Changing product into sum or difference</u>
   i. 2 sinA. cosB = sin(A +B) + sin(A −B)
   ii. 2 cosA . sinB = sin(A+B) − sin(A-B)
   iii. 2 cosA. cosB = cos(A +B) + cos(A-B)
   iv. 2sinA. sinB = cos(A -B) - cos(A + B)

B. <u>Changing sum or difference into product</u>.
   i. $\sin C+\sin D=2\sin\left(\frac{C+D}{2}\right)\cdot\cos\left(\frac{C-D}{2}\right)$
   ii. $\sin \mathrm{C}-\sin \mathrm{D}=2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$
   iii. $\cos C+\cos D=2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$
   iv. $\cos C-\cos D=2\sin\left(\frac{C+D}{2}\right)\cdot\sin\left(\frac{D-C}{2}\right)$

Trigonometrical Ratios for some Special Angles

θ	$7\frac{1}{2}°$	15°	$22\frac{1}{2}°$	18°	36°	75^0
$\sin\theta$	$\frac{\sqrt{4-\sqrt{2-6}}}{2\sqrt{2}}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{1}{2}\sqrt{2-\sqrt{2}}$	$\frac{\sqrt{5-1}}{4}$	$\frac{1}{4}\sqrt{10-2\sqrt{5}}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$
$\cos\theta$	$\frac{\sqrt{4+\sqrt{2+6}}}{2\sqrt{2}}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{1}{2}\sqrt{2+\sqrt{2}}$	$\frac{1}{4}\sqrt{10+2\sqrt{5}}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$
$\tan\theta$	$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-1}$	$2-\sqrt{3}$	$\sqrt{2}-1$	$\sqrt{25-\sqrt{10}\sqrt{5}}\,/\,5$	$\sqrt{5-2\sqrt{5}}$	$2+\sqrt{3}$

Some Important Values:

i. $\sin 15^\circ = \sin(45^\circ - 30^\circ) = \frac{\sqrt{3}-1}{2\sqrt{2}}$

ii. $\sin 75^\circ = \sin(45^\circ + 30^\circ) = \frac{\sqrt{3}+1}{2\sqrt{2}}$

iii. $\cos 75^\circ = \cos(45^\circ + 30^\circ) = \frac{\sqrt{3}-1}{2\sqrt{2}}$

iv. $\cos 15^\circ = \cos(45^\circ - 30^\circ) = \frac{\sqrt{3}+1}{2\sqrt{2}}$

Note:

$\sin 15^\circ = \cos 75^\circ$
$\cos 15^\circ = \sin 75^\circ$

v. $\sin 18^\circ = \frac{\sqrt{5}-1}{4} = \cos 72^\circ$

$\cos 18^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4} = \sin 72^\circ$

$\cos 36^\circ = \frac{\sqrt{5}+1}{4} = \sin 54^\circ$

Note:

If $\sin\theta + \operatorname{cosec}\theta = 2$ then $\sin^n\theta + \operatorname{cosec}^n\theta = 2, n$ being integer.

Important Notes

i. The maximum value of a $\sin x + b\cos x$ is $\sqrt{a^2+b^2}$ and the minimum value is $-\sqrt{a^2+b^2}$.

ii. If $x = \sec\theta + \tan\theta$ then $\frac{1}{x} = \sec\theta - \tan\theta$

iii. If $y = \csc\theta + \cot\theta$ then, $\frac{1}{y} = \csc\theta - \cot\theta$

a. Even function

A real valued function satisfying f(−x)=f(x) is known as **even function.**

Examples:
$\sin^2\theta, 2x^2-3, \sec\theta$
$\csc^2\theta, 2x^2-3, \cos\theta$
$\tan^4 x + x^2$
$\sin^2 x + x^4$ and so on

Note:-
Derivative of even function is odd and derivative of odd function is even.

b. **Odd function**

A real valued function satisfying
f(-x)=-f(x) is known as odd function.

Examples:
sinx + x^5, coscex, , x^3- 3x, $\sin^3$x + 3tanx, x + tanx

c. **Periodic function**

A function satisfying f(x +π) = f(x) where π is least positive number is called periodic function of the period π.

Examples:

i. $\sin x$, $\operatorname{cosec} x$, $\cos x$, $\sec x$ are periodic functions with common period 2π or 360°
ii. $\tan x$, & $\cot x$ are periodic functions with common period π or 180°

Note:

i. Periods of $\sin^n \theta, \cos^n \theta, \sec^n \theta$ and $\csc^n \theta = 2\pi$ if n is odd
$= \pi$ if n is even
ii. Periods of $\tan^n \theta, \cot^n \theta = \pi$
iii. Periods of $|\sin \theta|, |\cos \theta|, |\tan \theta|, |\cot \theta|, |\sec \theta|, |\operatorname{cosec} \theta| = \pi$
iv. $\text{Periods of} \quad \left.\begin{matrix} |\sin\theta| + |\cos\theta| \\ |\tan\theta| + |\cot\theta| \\ |\sec\theta| + |\cos ec\theta| \end{matrix}\right]$ all $\dfrac{\pi}{2}$