

# Fantuan's Academia

FANTUAN'S MATH NOTES SERIES

## Notes on Mathematical Analysis

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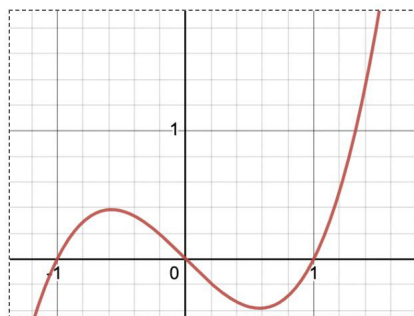


Real Analysis Student

YOU NEED THAT FOR  $f: A \rightarrow \mathbb{R}$ ,  
 $c \in A$ , THE FUNCTION IS  
CONTINUOUS AT  $c$  IF AND ONLY  
IF  $\forall \varepsilon > 0 \exists \delta > 0 \ni |x - c| < \delta$  and  
 $x \in A$  implies  $|f(x) - f(c)| < \varepsilon!!!$   
OTHERWISE IT'S NOT  
SUFFICIENTLY RIGOROUS!!!!



Precalculus Student



If I can draw it without picking  
my pen up, it's continuous.

May 23, 2024

Fantuan's Math Notes

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All the Sections with \* are hard sections and can be skipped without losing coherence.

This note is referenced on **Understanding Analysis** by Stephen Abbott [1], **Principles of Mathematical Analysis** by Walter Rudin [2], **Analysis I** by Terence Tao [3], and MTH 117,118 notes of XJTLU.

## Part I

### Part I: The Real Line



# Chapter 1

## Real Numbers

### 1.1 Why Analysis?

*Analysis*, simply saying, is a course about ‘rigorous calculus’. Somebody may ask then: “why we need another course about calculus?”. Indeed, basic calculus concepts and various computing skills are introduced in Year I Calculus course. However, regarding calculus as a pure math object, it should maintain its full rigor. If we apply calculus in the real world problems without knowing where they came from and what is their constraints to be correctly applied, some pathological things will happen, as listed below.

**Example 1.1.1.** *Infinite Series*

Consider the divergent infinite series

$$S = 1 + 2 + 4 + 8 + 16 + \cdots \quad (1.1)$$

If we multiply it by 2,

$$2S = 2 + 4 + 8 + 16 + \cdots \quad (1.2)$$

Subtract (1.1) from (1.2), we will have the ridiculous result

$$S = -1$$

**Example 1.1.2.** *Interchanging Integrals*

We always change the order of double integral to make calculation easier. But, can we always do that in any cases? Consider

$$\int_0^\infty \int_0^1 (e^{-xy} - xye^{-xy}) \, dy \, dx$$

If we directly compute this, we can get,

$$\int_0^\infty \int_0^1 (e^{-xy} - xye^{-xy}) \, dy \, dx = \int_0^\infty [ye^{-xy}]_{y=0}^1 \, dx = \int_0^\infty e^{-x} \, dx = [-e^{-x}]_0^\infty = 1$$

However, if we change the order of integral

$$\int_0^1 \int_0^\infty (e^{-xy} - xye^{-xy}) \, dx \, dy = \int_0^1 [xe^{-xy}]_{x=0}^\infty \, dy = \int_0^1 (0 - 0) \, dy = 0$$

We arrive different answers!

**Example 1.1.3. Reordering Infinite Series**

Consider the alternating harmonic series

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

We know that this infinite series converges at some point. Therefore, nothing similar as Example 1.1.1 could happen here. However, if we do the following computation:

$$\begin{aligned} \frac{1}{2}S &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} + \cdots \\ S &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \frac{1}{14} + \frac{1}{15} - \frac{1}{16} + \cdots \end{aligned}$$

$$\frac{3}{2}S = \left(1 + \frac{1}{3}\right) - \frac{1}{2} + \left(\frac{1}{5} + \frac{1}{7}\right) - \frac{1}{4} + \left(\frac{1}{9} + \frac{1}{11}\right) - \frac{1}{6} + \left(\frac{1}{13} + \frac{1}{15}\right) - \frac{1}{8} + \cdots$$

We see that  $\frac{3}{2}S$  is just a reordering of our initial infinite series (with two positive terms following one negative term)! Therefore, we just change the convergent point by simply reordering the infinite series.

This doesn't make sense! Since by intuition, reordering the terms in an algorithm will not change its result. However, you see here, the situation changes in the infinite case.

As showed above, we indeed need this course to make analysis as a much more rigorous math topic than Year I Calculus. To get started, we will first talk about real numbers.

## 1.2 From Rational to Irrational Numbers

The simplest number system we can call to our mind is the **Natural Numbers**

$$\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$$

Obviously, this number system is based on counting. It is enough for the simple use of counting things. However, this number system is not closed under subtraction (i.e., one natural number subtracts another natural number may not result in a natural number). Therefore, we introduce the **Integer Numbers**

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$



This number system is still not complete since it is not closed under division. For example,  $3 \div 5$  is not in the list. We further introduce the **Rational Numbers**

$$\mathbb{Q} = \left\{ \frac{p}{q} : q \neq 0, p, q \in \mathbb{Z} \right\}$$

Back to Pythagoras's era (500-400 BC), he only believes the existence of rational numbers, and so did his followers in Pythagoreanism, except for one: Hippasus of Metapontum. After Pythagoras announced his famous Pythagorean Theorem, Hippasus directly used this theorem to discover  $\sqrt{2}$ : an irrational number!

Consider a right-angled triangle with two right-angled edges of length 1. Then by Pythagorean Theorem, length of the hypotenuse  $z$  should satisfy

$$z^2 = 1^2 + 1^2 = 2$$

We denote this number as  $\sqrt{2}$ , for which the square of it is 2. It seems that we cannot write this number in the form of a rational number. And indeed, we can prove that it is not a rational number.

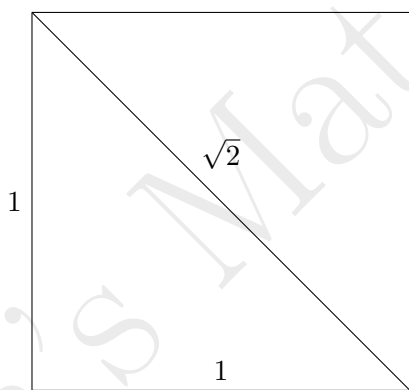


Figure 1.1: Pythagorean Theorem and  $\sqrt{2}$

#### Proposition 1.2.1: Irrationality of $\sqrt{2}$

$\sqrt{2}$  is not a rational number.

*Proof.* We prove by contradiction. Suppose  $\sqrt{2}$  is a rational number, then it can be written as

$$\sqrt{2} = \frac{p}{q}, \quad q \neq 0, p, q \in \mathbb{Z}, p, q \text{ are relatively prime}$$

Multiply by  $q$  on both sides and take square, we have

$$2q^2 = p^2$$

Because  $2q^2$  is an even number (even number times a number must equal to an even number),  $p^2$  is an even number. Therefore,  $p$  itself is an even number (if  $p$  is odd, then  $p^2$  is odd, which is a contradiction). Hence, we can write  $p$  as

$$p = 2k, \quad k \in \mathbb{Z}$$

Substitute this in the previous equation, we have

$$2q^2 = 4k^2 \implies q^2 = 2k^2$$

By the same argument, we can also conclude that  $q$  is an even number. Then,  $p$  and  $q$  would have a common factor 2, which is a contradiction with our assumption that  $p, q$  are relatively prime.  $\square$

Therefore, there is another kind of number except for rational numbers! This was a big shock for people during Pythagoras's time, and the discovery of  $\sqrt{2}$  is called '**The First Mathematical Crisis**'. Since this discovery broke the belief of Pythagoreanism, Hippasus, who discovered this, was drowned at sea by Pythagoras's followers.

Fortunately, now we fully accept that there is 'irrational numbers'. Nobody would be sentenced to death for acknowledging the existence of irrational numbers. Rational and Irrational numbers together, are called **Real Numbers**, denoted by  $\mathbb{R}$ .

But, how we should construct real numbers from rational numbers? Could we construct a procedure just like what we did for extending integers to rational numbers? In section 1.8\* we will introduce an elegant method, and another method would be introduced later in Chapter 2. Since these construction processes are hard, we should now temporarily just believe that there is indeed a set of numbers called real numbers. In next section we will state the axiom that real numbers should behave.

### 1.3 Axiom of Completeness I: Supremum Property

One of the most important property of real number is: **It is complete**. The rigorous definition of completeness would be introduced later. Heuristically, completeness of real numbers means that 'All points on the real line are described by real numbers'.

Consider rational numbers, they are 'almost everywhere' on the real line, i.e., there is no such a rational number  $a$  that is 'closest' to the rational number  $b$ . Indeed, suppose there is a  $b$  that is 'closest' to  $a$ , then, the rational number  $\frac{a+b}{2}$  is 'closer' to  $a$ , which is a contradiction. This property is called **dense**, and will be

introduced later.

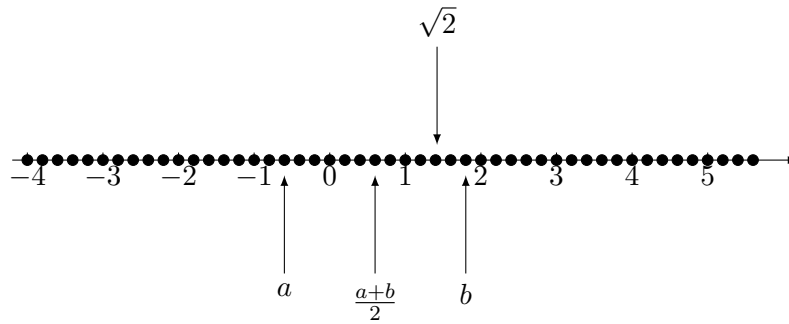


Figure 1.2: Rational Numbers  $\mathbb{Q}$  is dense in Real Line  $\mathbb{R}$

Even if  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , there are infinite many small ‘holes’ on the line that was not represented by any of the rational numbers. For example, the point at the distance of  $\sqrt{2}$  from the origin, as showed in Figure 1.2. Completeness then means that these holes are exactly ‘filled’ by ‘irrational numbers’, so that each point on the line is represented by a unique real number.

To transform these discussions into mathematical language, we first introduce some simple definitions. In this whole note I will denote ‘such that’ by ‘s.t.’ for simplicity.

#### Definition 1.3.1: Bounded Above/Bounded Below, Lower/Upper Bound

- A set  $A \subseteq \mathbb{R}$  is **bounded above** if

$$\exists b \in \mathbb{R}, \text{ s.t. } \forall a \in A \implies a \leq b$$

The number  $b$  is called an **upper bound** for  $A$ .

- A set  $A \subseteq \mathbb{R}$  is **bounded below** if

$$\exists l \in \mathbb{R}, \text{ s.t. } \forall a \in A \implies l \leq a$$

The number  $l$  is called an **lower bound** for  $A$ .

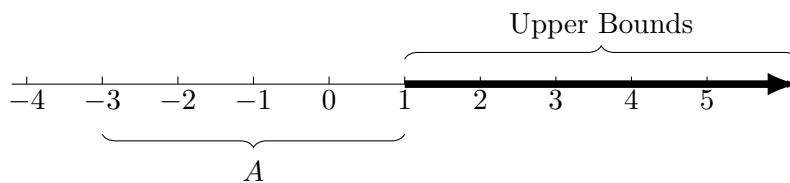


Figure 1.3: A set bounded above

Note that upper bound and lower bound of a set  $A$  may not be unique. In fact, if  $A \subseteq \mathbb{R}$  is bounded above, upper bounds are always not unique. However, there would sometimes exist a ‘least upper bound’, which is the most important subject towards the construction of completeness axiom.

### Definition 1.3.2: Supremum/Infimum

- A real number  $s$  is called the **supremum (least upper bound)** of a set  $A \subseteq \mathbb{R}$  if
  1.  $s$  is an upper bound for  $A$ .
  2. For any upper bound  $b$  of  $A$ , we have  $s \leq b$ .

This is denoted by  $s = \sup A$ . If  $s \in A$ , it is also the **maximum** of  $A$ .

- A real number  $l$  is called the **infimum (greatest lower bound)** of a set  $A \subseteq \mathbb{R}$  if
  1.  $l$  is a lower bound for  $A$ .
  2. For any lower bound  $b$  of  $A$ , we have  $b \leq l$ .

This is denoted by  $l = \inf A$ . If  $l \in A$ , it is also the **minimum** of  $A$ .

Note that although upper bound is not unique for sets, Supremum, if exists, is unique.

### Proposition 1.3.3: Uniqueness of Supremum

A set  $A \subseteq \mathbb{R}$  can have at most one supremum.

*Proof.* Suppose  $s_1, s_2$  are suprema of a set  $A$ . Regard  $s_1$  as an upper bound and  $s_2$  as the supremum, we will arrive  $s_2 \leq s_1$ . Regard  $s_1$  as the supremum and  $s_2$  as an upper bound we will arrive  $s_1 \leq s_2$ . Therefore,  $s_1 = s_2$ .  $\square$

Now we should have all tools for the construction of our completeness theorem. This theorem would be seen as an **axiom**, i.e., no need to be proved and it is raised by nature, so that it is an inherent property of the set of real numbers. (In Section 1.8\* we will use an elegant method to prove this axiom)

### Axiom 1.3.4: Supremum Property

Every nonempty set of real numbers that is bounded above has a supremum.

Why this axiom expresses the completeness of real numbers? We consider a counterexample. Suppose we only have rational number system. Then consider the set  $A = (0, \sqrt{2}) \cap \mathbb{Q}$ . If the supremum  $s$  is less than  $\sqrt{2}$ , say  $s = \sqrt{2} - \epsilon \in \mathbb{Q}$ . Then there would be a number  $k = \sqrt{2} - \frac{\epsilon}{2} \in A$ , such that  $k > s$ , which is a contradiction to the definition of supremum. Similarly, we can derive that the supremum also cannot be

larger than  $\sqrt{2}$ . Since  $\sqrt{2} \notin \mathbb{Q}$ , we conclude that this set  $A$ , in rational number system, has no supremum.

Therefore, the rational number system  $\mathbb{Q}$  does not have this supremum property. It's only for real number system! Actually, this is the first **Axiom of Completeness for real numbers** in this note. In later chapter there would be more, and we will later on examine the relationships between these Axiom of Completeness.

**Note:** We state the axiom of completeness only regarding to supremum. There is no need to assert that infimum exists as part of the axiom. To see this, let  $A$  be nonempty and bounded below, define  $B$  as

$$B = \{b \in \mathbb{R} : b \text{ is a lower bound for } A\}$$

Then we will get  $\sup B = \inf A$ , by the definition of supremum and infimum. For set  $A$ , we can then state the axiom of completeness with respect to the set  $B$ , i.e., with respect to the supremum.

To conclude this section, a characterization of supremum would be stated below. This is an EXTREMELY USEFUL TOOL since sometimes it is very difficult to work on supremum directly using its definition.

#### Proposition 1.3.5: Characterization of Supremum

Let  $s \in \mathbb{R}$  and set  $A \subseteq \mathbb{R}$ .  $s = \sup A$  if and only if

- $s$  is an upper bound of  $A$ .
- $\forall \epsilon > 0, \exists a \in A$ , s.t.  $s - \epsilon < a$ .

*Proof.*

( $\implies$ ) Suppose  $s = \sup A$ . Then,  $s$  is indeed an upper bound by definition. Also,  $s - \epsilon$  is not an upper bound for any  $\epsilon > 0$ , since  $s - \epsilon < s$ . Therefore, by definition, there exists  $a \in A$ , such that  $s - \epsilon < a$ .

( $\impliedby$ ) Suppose  $s$  satisfy the conditions stated in the proposition. Then by the second condition, any number smaller than  $s$  is not an upper bound. Therefore,  $s$  is the least upper bound.  $\square$

## 1.4 Properties of real numbers

There are many applications of the Axiom of Completeness. We will first introduce the important **Archimedean Property**, which states how  $\mathbb{N}$  behaves inside  $\mathbb{R}$ .

**Theorem 1.4.1: Archimedean Property**

- For any  $x \in \mathbb{R}$ , there exists an  $n \in \mathbb{N}$  satisfying that  $n > x$ .
- For any  $y > 0$ , there exists an  $n \in \mathbb{N}$  satisfying that  $\frac{1}{n} < y$

*Proof.*

1. To prove the first statement, we assume that there exists  $x \in \mathbb{R}$  such that for all  $n \in \mathbb{N}$  we have  $n \leq x$ . This is equivalent to say that,  $\mathbb{N}$  is bounded above. By Supremum Property, supremum exists. Let  $\alpha = \sup \mathbb{N}$ . Then  $\alpha + 1 \in \mathbb{N}$ . This contradicts the definition of supremum since  $\alpha + 1 > \alpha = \sup \mathbb{N}$ . Thus, we arrive a contradiction.
2. The second statement follows from (1) by letting  $x = 1/y$ .

□

**Note:** It seems that there is no need to prove the statement (1) in Archimedean Property. It just said that  $\mathbb{N}$  is unbounded, and we know that as a common sense. However, it is worth noting that as a proper extension of  $\mathbb{Q}$  (i.e., a set contains  $\mathbb{Q}$  and not equal to  $\mathbb{Q}$ ), the Archimedean Property is very unique for  $\mathbb{R}$ . Indeed, there **does exist** a proper extension of  $\mathbb{Q}$  such that it is bounded (called the Extended-Real Numbers). Discussing this number system will go far out from the scope of this note. You should look for detailed explanation in my Real Analysis note. Now we see how  $\mathbb{Q}$  and  $\mathbb{R} \setminus \mathbb{Q}$  behaves inside  $\mathbb{R}$ .

**Proposition 1.4.2:  $\mathbb{Q}$  is dense in  $\mathbb{R}$** 

For any  $a, b \in \mathbb{R}$ ,  $a < b$ , there exists  $r \in \mathbb{Q}$  such that  $a < r < b$ .

*Proof.* We need to produce  $m, n \in \mathbb{Z}, n \neq 0$  such that  $r = \frac{m}{n}$  and

$$a < \frac{m}{n} < b$$

The first thing we need to do is to choose sufficiently large  $n$  so that a ‘step length’  $\frac{1}{n}$  is less than the length of  $b - a$ , so that there would be some point of the form  $\frac{m}{n}$  locating between the two points, as showed in the Figure 1.4.

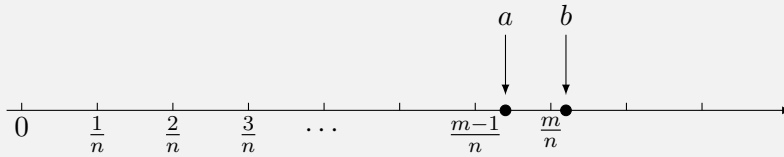


Figure 1.4: Choose sufficiently small step so that  $\frac{m}{n}$  is between  $a$  and  $b$

By Archimedean Property, we can actually take  $n \in \mathbb{N}$  such that

$$\frac{1}{n} < b - a$$

Now, as showed in the picture, we need to choose  $m \in \mathbb{Z}$  so that

$$\frac{m-1}{n} \leq a < \frac{m}{n} \implies m-1 \leq na < m$$

The only thing left is that to prove  $\frac{m}{n} < b$ . To do this, we see that

$$m \leq na + 1 < n \left( b - \frac{1}{n} \right) + 1 = nb$$

Therefore, the proposition is proved. □

We can use the same strategy to see that irrational number is also dense in  $\mathbb{R}$ . Before doing that, we need to prove some lemma about properties of operations in rational and irrational numbers.

#### Lemma 1.4.3: Operations in $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$

1. If  $a, b \in \mathbb{Q}$ , then  $a + b, ab \in \mathbb{Q}$ .
2. If  $a \in \mathbb{Q}, b \in \mathbb{R} \setminus \mathbb{Q}$ , then  $a + t, at (a \neq 0) \in \mathbb{R} \setminus \mathbb{Q}$

*Proof.* 1. Since  $a, b \in \mathbb{Q}$ , we have  $a = \frac{m}{n}, b = \frac{r}{q}$  for  $m, n, r, q \in \mathbb{Z}, n, q \neq 0$ . Therefore,

$$a + b = \frac{mq + nr}{nq} \in \mathbb{Q}, \quad ab = \frac{mr}{nq} \in \mathbb{Q}$$

since  $mq + nr, nq, mr \in \mathbb{Z}$ .

2. Since  $a \in \mathbb{Q}$ , we have  $a = \frac{m}{n}$  where  $m, n \in \mathbb{Z}, n \neq 0$ . Suppose  $a + t \in \mathbb{Q}$ , then we can write

$a + t = \frac{r}{q}$ ,  $r, q \in \mathbb{Z}$ ,  $q \neq 0$ . Then,

$$t = (a + t) - a = \frac{r}{q} - \frac{m}{n} = \frac{rn - mq}{qn} \in \mathbb{Q}$$

which is a contradiction. Similarly, we can also see that  $at \notin \mathbb{Q}$  (when  $a \neq 0$ ).  $\square$

**Proposition 1.4.4:**  $\mathbb{R} \setminus \mathbb{Q}$  is dense in  $\mathbb{R}$

For any  $a, b \in \mathbb{R}$ ,  $a < b$ , there exists a  $t \in \mathbb{R} \setminus \mathbb{Q}$  such that  $a < t < b$ .

*Proof.* As in Proposition 1.4.2, we first choose  $n \in \mathbb{N}$  such that

$$\frac{1}{n} < b - a$$

Then, we choose  $m \in \mathbb{Z}$  such that

$$m + \sqrt{2} - 1 \leq na < m + \sqrt{2}$$

Obviously, we have  $a < \frac{m+\sqrt{2}}{n}$ . Also,

$$m + \sqrt{2} \leq na + 1 < n \left( b - \frac{1}{n} \right) + 1 = nb$$

Thus,  $a < \frac{m+\sqrt{2}}{n} < b$ . Since  $m, \frac{1}{n} \in \mathbb{Q}$ ,  $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$ , by Lemma 1.4.3, we have  $\frac{m+\sqrt{2}}{n} \in \mathbb{R} \setminus \mathbb{Q}$ . The statement is proved.  $\square$

At this stage we have proved that  $\sqrt{2}$  is not a rational number. But, we have not proved that it is a real number. Below we will prove this. Why we need to prove this obvious thing? Indeed, you will see from the prove below that the main theorem used is Supremum Property. Therefore, this inherent property of real numbers asserts that those irrational numbers we encounter often are all real numbers.

**Proposition 1.4.5:**  $\sqrt{2}$  is a real number

There exists a real number  $\alpha \in \mathbb{R}$  such that  $\alpha^2 = 2$ .

*Proof.* Consider the set

$$T = \{t \in \mathbb{R} : t^2 < 2\}$$

Set  $\alpha = \sup T$ .



- Suppose  $\alpha^2 < 2$ . Let  $n \in \mathbb{N}$  be arbitrary, then

$$\left(\alpha + \frac{1}{n}\right)^2 = \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} < \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n} = \alpha^2 + \frac{2\alpha + 1}{n}$$

If we choose  $n \in \mathbb{N}$  such that

$$\frac{1}{n} < \frac{2 - \alpha^2}{2\alpha + 1}$$

The existence of this  $n$  is promised by Archimedean Property. Note that  $n > 0$  since we assume that  $\alpha^2 < 2$ . We would get

$$\left(\alpha + \frac{1}{n}\right)^2 < \alpha^2 + \frac{(2\alpha + 1)(2 - \alpha^2)}{2\alpha + 1} = 2$$

Therefore,  $\alpha + \frac{1}{n} \in T$ , which means that  $\alpha$  is not an upper bound, which is a contradiction to that  $\alpha = \sup T$ .

- Suppose  $\alpha^2 > 2$ . Similarly, we can write

$$\left(\alpha - \frac{1}{n}\right)^2 = \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} > \alpha^2 - \frac{2\alpha}{n}$$

If we choose  $n \in \mathbb{N}$  such that

$$\frac{1}{n} < \frac{\alpha^2 - 2}{2\alpha}$$

Again, the existence is promised by Archimedean Property. Then we would have

$$\left(\alpha - \frac{1}{n}\right)^2 > \alpha^2 - \alpha^2 + 2 = 2$$

This shows that  $\alpha - \frac{1}{n}$  is an upper bound of  $T$ , which means that  $\alpha$  is not the least upper bound, i.e., not the supremum. This is a contradiction to the assumption that  $\alpha = \sup T$ .

By Supremum Property, the supremum of  $T$  exists. Then, it can only be  $\sqrt{2}$ . Since the supremum of a set is a real number (which is promised in Supremum Property), we finally arrive that  $\sqrt{2}$  is actually a real number.  $\square$

## 1.5 Axiom of Completeness II: Nested Interval Property

Another famous Axiom of Completeness of real numbers is called the **(Cantor) Nested Interval Property**. It says that for any nested closed interval sequence, the intersection of these intervals is not empty. Here is what all these words mean.

**Axiom 1.5.1: Nested Interval Property**

For each  $n \in \mathbb{N}$ , construct a closed interval  $I_n = [a_n, b_n]$ , where  $a_n, b_n \in \mathbb{R}$ . Assume  $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ . Then, the intersection of these nested intervals

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

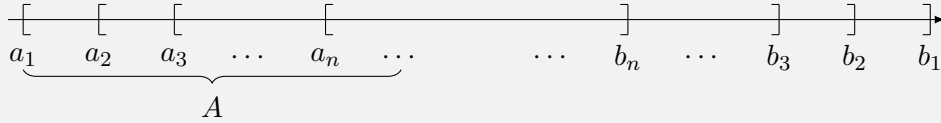


Figure 1.5: Nested Intervals

Actually, this axiom can be derived from Supremum Property, as showed below.

*Proof.* Consider the set

$$A = \{a_n : n \in \mathbb{N}\}$$

Figure 1.6: Nested Intervals with set  $A$ 

We can see that each  $b_n$  is served as an upper bound for the set  $[a_n, b_n]$ , for every  $n \in \mathbb{N}$ . Set  $x = \sup A$ . Then  $a_n \leq x$  for every  $n$  since  $x$  is the upper bound. Also,  $b_n \geq x$  for every  $n$  since  $b_n$  are upper bounds and  $x$  is the supremum. Therefore we get,

$$\forall n \in \mathbb{N}, a_n \leq x \leq b_n$$

Hence,  $x \in \bigcap_{n=1}^{\infty} I_n \neq \emptyset$ . □

Can we, inversely, get Supremum Property from Nested Interval Property? The answer is no! Therefore, there is some ‘strong’ axioms and some ‘weak’ axioms. In this case, Supremum Property is ‘strong’ and Nested Interval Property is ‘weak’, since we can go from Supremum Property to Nested Interval, but not the reverse direction.

But, why we can’t? In the ‘proof’ below I will show you a **circular reasoning**, indicating that we can only go from Nested Interval Property to Supremum Property if Archimedean Property exists.

**Note:** This is not a prove, but just a reasoning process.

Suppose Nested Interval Property is true. We want to prove Supremum Property. Let  $A$  be a nonempty set which is bounded above. Denote  $\alpha = \sup A$ . Choose  $a \in A$  and  $b$  such that  $b$  is an upper bound of  $A$ . Then, consider the intervals  $[a, \frac{a+b}{2}]$  and  $[\frac{a+b}{2}, b]$ . Then,  $\alpha$  is at least in one of these two intervals. Choose an interval that contains it. Continue bisecting this interval as we did at the last step. Again, the supremum would contain in one of the intervals. Choose that interval. ....

After  $n$  steps, the length of the chosen interval would be  $\frac{b-a}{2^n}$ . The only thing we are left to do is to prove that  $\frac{b-a}{2^n}$  converges to 0 (The rigorous definition of limit would be introduced in Chapter 2, here you can just recall what you have learnt in Year I Calculus). However, the proof of this statement needs Archimedean Property, since we want for every  $\epsilon > 0$ , we can find a  $n \in \mathbb{N}$  such that  $\frac{b-a}{2^n} < \epsilon$ . This is equivalent to what is said in the second statement of Archimedean Property. Recall how Archimedean Property is derived. Yes, it is derived from Supremum Property! So we cannot directly use Archimedean Property if we assume we don’t know Supremum Property and want to prove it. This is an example of **circular reasoning**, and in result, we have no way deriving Supremum Property from Nested Interval Property.

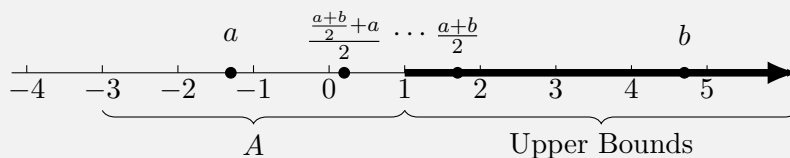


Figure 1.7: Bisection intervals to approximate supremum

Therefore, we get the relationship between Supremum Property and Nested Interval Property. The relation is visually displayed below in Figure 1.8.



Figure 1.8: Relation between Axiom of Completeness

This graph would be further expanded when we encounter more and more Axiom of Completeness.

## 1.6 ‘Size of Infinity’: Cardinality

Does infinity also have different ‘sizes’? You may ask after seeing this section title. The answer is yes! We all know that there are infinitely many rational numbers and irrational numbers, and it seems that both of

them are ‘almost everywhere’ on the real line. However, in this section we will introduce a surprising result: There are ‘much more’ irrational numbers than rational numbers!

Before we discuss this result, we need to first talk about the way of comparing two ‘infinite sizes’. Let’s start with finite one. To compare the number of elements in two finite sets, it is easy, just count them. For example,  $A = \{1, 2, 3, 4\}$  and  $B = \{5, 6, 7, 8\}$ . They all have 4 elements, and naturally, have the same size. To extend this into infinite case, we need to use **bijective maps**.

### Definition 1.6.1: Cardinality

Two sets  $A, B$  have the same **Cardinality** if there exists a bijective map  $f : A \rightarrow B$  such that each element of  $A$  is mapped one-to-one and onto an element of  $B$ . Dented as  $A \sim B$ .

Then, the cardinality of a set just describes the ‘size’ of that set. Let’s see some examples first.

### Example 1.6.2: $\mathbb{N}$ has the same cardinality as the set of even numbers

This is weird at first glance, since intuitively the set of even numbers is a proper subset of  $\mathbb{N}$ , and they could not have the same size. Denote the set of even numbers as

$$E = \{2, 4, 6, 8, \dots\}$$

Then we can construct a bijective map  $f : \mathbb{N} \rightarrow E$  by  $f(n) = 2n, \forall n \in \mathbb{N}$ .

$$\begin{array}{ccccccc} \mathbb{N} : & 1 & 2 & 3 & 4 & \dots & n & \dots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \dots & \updownarrow & \dots \\ E : & 2 & 4 & 6 & 8 & \dots & 2n & \dots \end{array}$$

### Example 1.6.3: $\mathbb{N} \sim \mathbb{Z}$

We can construct a bijective map  $f : \mathbb{N} \rightarrow \mathbb{Z}$  by

$$f(n) = \begin{cases} \frac{n-1}{2}, & \text{if } n \text{ is odd} \\ -\frac{n}{2}, & \text{if } n \text{ is even} \end{cases}$$

$$\begin{array}{ccccccc} \mathbb{N} : & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \dots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \dots \\ \mathbb{Z} : & 0 & -1 & 1 & -2 & 2 & -3 & 3 & \dots \end{array}$$

In the two examples above, we see that we correspond elements of some set with natural numbers  $1, 2, 3, 4, \dots$ , just as we are counting them in some order. This is such an important case that we gave

it a name.

#### Definition 1.6.4: Countable/Uncountable Set

A set  $A$  is called

- **Finite** if the number of elements in  $A$  is finite.
- **Countable** if  $A \sim \mathbb{N}$ .
- **Uncountable** if it is infinite and not countable.

We can see from Example 1.6.2 and 1.6.3 that  $E$  and  $\mathbb{Z}$  are countable sets. What does uncountable set looks like? **The next theorem is central for this section.** It says that  $\mathbb{R}$  has a somewhat 'bigger' size than  $\mathbb{Q}$ .

#### Theorem 1.6.5: Countability of $\mathbb{Q}$ , Uncountability of $\mathbb{R}$

1.  $\mathbb{Q}$  is a countable set.
2.  $\mathbb{R}$  is an uncountable set.

*Proof.*

1. There are two popular ways of proving that  $\mathbb{Q}$  is countable. I will show both of them.

**METHOD I:** Arrange all the rational numbers in an infinite matrix such that  $m$ th row and  $n$ th column corresponds to the number  $\frac{n}{m}$ . Then, assign natural numbers to them 'meanderingly', as showed below. If there is a number that has the same value with some number that has been assigned, we delete it. For example,  $\frac{1}{1}$  is assigned 1,  $\frac{1}{2}$  is assigned 2,  $\frac{2}{1}$  is assigned 3,  $\frac{3}{1}$  is assigned 4,  $\frac{2}{2}$  is deleted since it has the same value with  $\frac{1}{1}$ , and  $\frac{1}{3}$  is assigned 5..... Continuing this fashion, we will have a bijective map from positive rational numbers to natural numbers.



With this, we can further map 0 to 0 and map negative rational numbers to negative integers. Then, this whole map is a bijective map from  $\mathbb{Q}$  to  $\mathbb{Z}$ . Since there also exists biject map from  $\mathbb{Z}$  to  $\mathbb{N}$ , we have  $\mathbb{N} \sim \mathbb{Q}$ .

**METHOD II:** Set  $A_1 = \{0\}$ , and for all  $n \geq 2$ , set

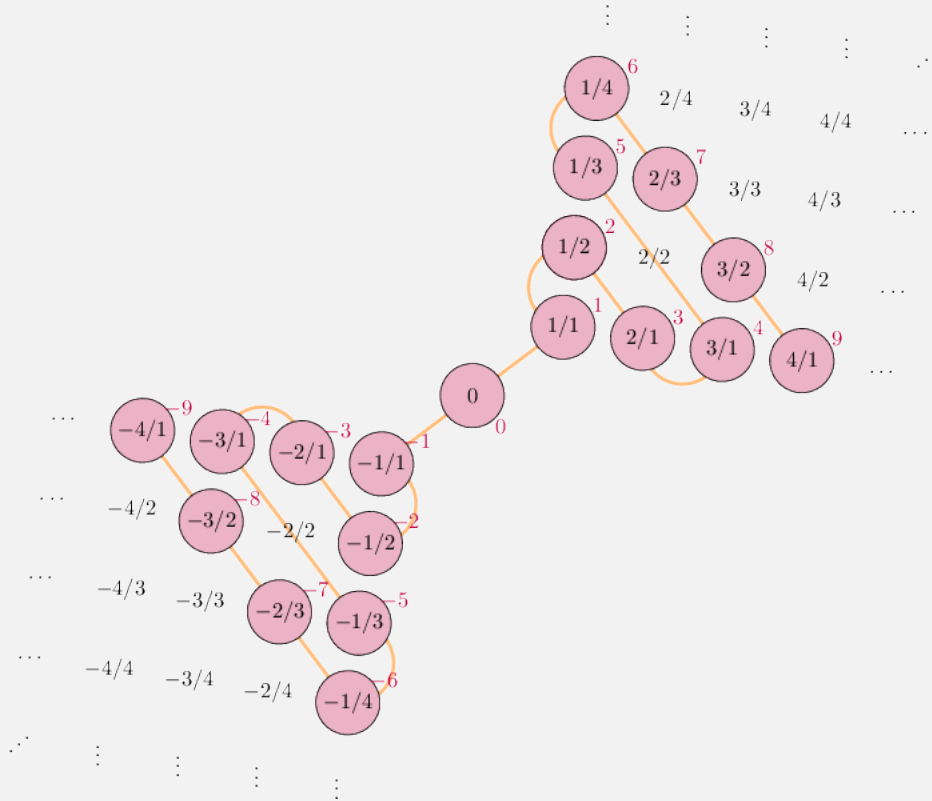
$$A_n = \left\{ \pm \frac{p}{q} : p, q \in \mathbb{N} \text{ are relatively prime with } p + q = n \right\}$$

For example,

$$A_2 = \left\{ \frac{1}{1}, -\frac{1}{1} \right\}, \quad A_3 = \left\{ \frac{1}{2}, -\frac{1}{2}, \frac{2}{1}, -\frac{2}{1} \right\}, \quad A_4 = \left\{ \frac{1}{3}, -\frac{1}{3}, \frac{3}{1}, -\frac{3}{1} \right\}, \dots$$

**Each  $A_n$  is finite and every rational number appears in exactly one of these sets.**

Therefore, we can construct the bijective map by listing the elements in each  $A_n$ .



- The main theorem used in this proof is the **Nested Interval Property**. We will prove by contradiction. Suppose there exists a bijective function  $f : \mathbb{N} \rightarrow \mathbb{R}$ . Then, each real number can be assigned to a natural number. Therefore, we can denote  $x_i$  as the real number being assigned

to the natural number  $i$ , and write  $\mathbb{R}$  as

$$\mathbb{R} = \{x_1, x_2, x_3, x_4, \dots\}$$

Now, consider the set  $[0, 9]$ . We can divide it into 3 parts:  $[0, 3] \cup [3, 6] \cup [6, 9]$ . Then,  $x_1$  can at most belong to two of them. Choose the interval that  $x_1$  does not belong to, denote it as  $I_1$ .

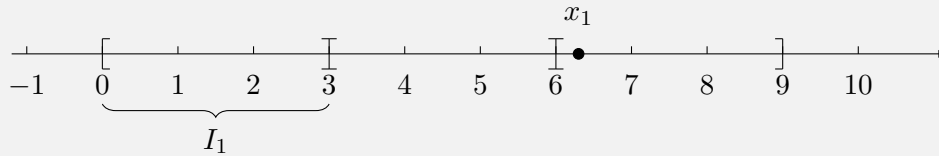


Figure 1.9: Construction Process of Nested Intervals, I

Then, we can divide  $I_1$  into 3 equal parts just as the previous step. Again,  $x_2$  can at most belong to one of these three intervals. Choose the one that  $x_2$  does not belong to, and call it  $I_2$ . Continuing this fashion, for  $I_n$ , we divide it into 3 equal parts, and choose the interval that  $x_{n+1}$  does not belong to, call it  $I_{n+1} \dots$

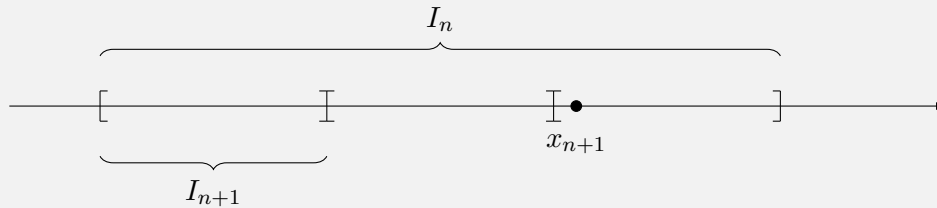


Figure 1.10: Construction Process of Nested Intervals, II

Using this procedure, we can produce nested intervals  $I_n$  such that

$$I_{n+1} \subseteq I_n, \forall n \in \mathbb{N}, \text{ and } x_n \notin I_n$$

Therefore,

$$x_n \notin \bigcap_{n=1}^{\infty} I_n, \forall n \in \mathbb{N}$$

This shows that

$$\bigcap_{n=1}^{\infty} I_n = \emptyset$$

which is a contradiction to the Nested Interval Property. □

Therefore,  $\mathbb{R}$  is a ‘bigger’ set than  $\mathbb{N}$ ! There does exist uncountable sets. In examples before, we have seen some other sets that is countable. In the next few examples, we will see what does uncountable sets look like.

**Example 1.6.6:**  $(-1, 1) \sim \mathbb{R}$

Here we can construct the function  $f : (-1, 1) \rightarrow \mathbb{R}$  by

$$f(x) = \frac{x}{x^2 - 1}$$

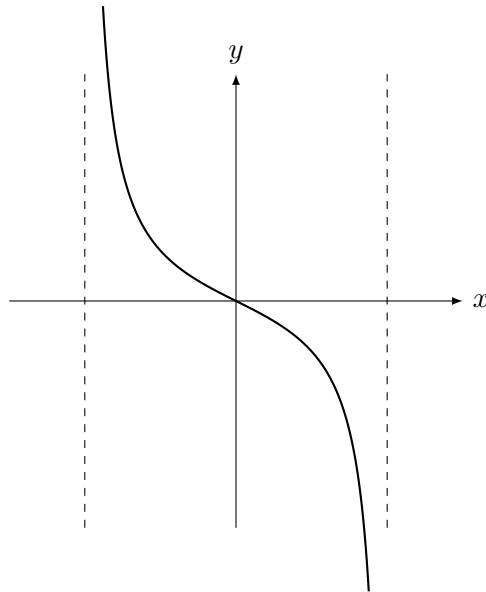


Figure 1.11: function  $f(x) = \frac{x}{x^2-1}$

**Example 1.6.7:**  $(a, b) \sim \mathbb{R}$

To extend the result from Example 1.6.6 to the case for every  $a, b \in \mathbb{R}$ , we can just do linear transformation on the function  $f$  in that example. Set

$$-1 < kx + c < 1, k > 0$$

We have

$$\frac{-1 - c}{k} < x < \frac{1 - c}{k}$$

Therefore, we can set

$$a = \frac{-1 - c}{k}, \quad b = \frac{1 - c}{k}$$



To get

$$k = \frac{2}{b-a}, \quad c = \frac{a+b}{a-b}$$

Thus

$$g(x) = f\left(\frac{2x}{b-a} + \frac{a+b}{a-b}\right)$$

will map  $(a, b)$  to the whole space  $\mathbb{R}$ .

#### Example 1.6.8: $(a, \infty) \sim \mathbb{R}$

We can construct the function  $f : (a, \infty) \rightarrow \mathbb{R}$  by

$$f(x) = \log(x - a)$$

#### Example 1.6.9: $[0, 1) \sim (0, 1)$

This one is interesting. It seems very easy. However, if you try that, it is not. Let us consider one famous paradox in mathematics: **Hilbert's paradox of the Grand Hotel**. There is a hotel with countably infinitely many rooms in the hotel, and the hotel is full. Then, some travellers come and want a room. The reception just send a call to everybody in the room: For person who live in room  $n$ , just move to the room  $n + 1$ , and the room 1 would be available for this traveller. Weird! Since we seem to create a new empty room from nowhere.

This problem is just the same. We need to move some countable set accordingly to make a new room for this new comer 0. Therefore, we can construct the bijective function  $f : [0, 1) \rightarrow (0, 1)$  like

$$f(x) = \begin{cases} 1/2, & \text{if } x = 0 \\ 1/4, & \text{if } x = \frac{1}{2} \\ 1/8, & \text{if } x = \frac{1}{4} \\ 1/16, & \text{if } x = \frac{1}{8} \\ \dots & \\ x, & \text{otherwise} \end{cases}$$

Now we should be familiar with countable sets and uncountable sets. In Chapter 1.7\* we will further go into n-dimensions, to see that  $\mathbb{R}$  are also uncountable sets, of the same cardinality with  $\mathbb{R}$ .

Finally, to end this section, we will see that whether the set of irrational numbers is countable or uncountable. Before doing that, let's prove some theorems about the properties of countable sets.

**Theorem 1.6.10: Subsets and Unions of countable sets**

1. If  $A \subseteq B$ ,  $B$  is countable, then,  $A$  is either countable or finite.
2. If  $A_1, A_2, \dots, A_m$  are countable, then,

$$\bigcup_{n=1}^m A_n$$

is countable.

3. If  $A_n$  is countable for each  $n \in \mathbb{N}$ , then,

$$\bigcup_{n=1}^{\infty} A_n$$

is countable.

*Proof.*

1. Let  $B$  be a countable set. Then, there exists bijective map  $f : \mathbb{N} \rightarrow B$ . Let  $A \subseteq B$  be an infinite subset of  $B$ . We must show that  $A$  is countable.

We now start to define a bijective map from  $\mathbb{N}$  to  $A$ . Let

$$n_1 = \min\{n \in \mathbb{N} : f(n) \in A\}, \text{ Set } g(1) = f(n_1)$$

$$n_2 = \min\{n \in \mathbb{N} \setminus \{1, 2, \dots, n_1\} : f(n) \in A\}, \text{ Set } g(2) = f(n_2)$$

$$n_3 = \min\{n \in \mathbb{N} \setminus \{1, 2, \dots, n_2\} : f(n) \in A\}, \text{ Set } g(3) = f(n_3)$$

$$\vdots$$

Inductively, as we can easily verify, that this function is bijective from  $\mathbb{N}$  to  $A$ .

2. We first prove that this is true for two countable sets,  $A_1, A_2$ . Let  $B_2 = A_2 \setminus A_1$ . Then,  $A_1$  and  $B_2$  is disjoint, and  $A_1 \cup A_2 = A_1 \cup B_2$ . Therefore, we only need to prove that  $A_1 \cup B_2$  is countable. By statement 1 above, we can see that  $B_2 = A_2 \setminus A_1 \subseteq A_2$  is countable or finite. First suppose that  $B_2$  is countable. Then, we can write both sets in the enumerated form

$$A_1 = \{a_1, a_2, a_3, \dots\}$$

$$B_2 = \{b_1, b_2, b_3, \dots\}$$

Since they are disjoint, we can write their union as

$$A_1 \cup B_2 = \{a_1, b_1, a_2, b_2, a_3, b_3, \dots\}$$

A bijective map then can be constructed as  $f : \mathbb{N} \rightarrow A_1 \cup B_2$  such that

$$f(n) = \begin{cases} a_{\frac{n}{2}}, & \text{if } n \text{ is even} \\ b_{\frac{n+1}{2}}, & \text{if } n \text{ is odd} \end{cases}$$

Now suppose  $B_2$  is finite. This time

$$A_1 = \{a_1, a_2, a_3, \dots\}$$

$$B_2 = \{b_1, b_2, b_3, \dots, b_m\}$$

Since they are disjoint, we can write their union as

$$A_1 \cup B_2 = \{b_1, b_2, b_3, \dots, b_m, a_1, a_2, a_3, \dots\}$$

A bijective map then can be constructed as  $f : \mathbb{N} \rightarrow A_1 \cup B_2$  such that

$$f(n) = \begin{cases} b_n, & \text{if } n \leq m \\ a_{n-m}, & \text{if } n > m \end{cases}$$

We have proved that  $A_1 \cup A_2$  is countable. Now if  $A_3$  is countable, we can directly see that,

$$A_1 \cup A_2 \cup A_3 = (A_1 \cup A_2) \cup A_3$$

is countable. Inductively, we have

$$\bigcup_{n=1}^m A_n$$

is countable if  $A_1, \dots, A_m$  are countable sets.

3. Suppose  $A_n$  is countable for all  $n \in \mathbb{N}$ . Then each set can be written as

$$A_n = \{a_{n1}, a_{n2}, a_{n3}, \dots\}$$

We can arrange the elements into a infinite matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ \vdots & \vdots & \vdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Then, we can assign each element a natural number just as what we did in the proof of Theorem 1.6.5. This constructs a bijective map. □

Here is an interesting fact derived from the theorem: We can have a countable collection of disjoint open intervals, such as  $(0, 1), (1, 2), (2, 3), \dots$ . However, **we cannot have an uncountable collection of disjoint open intervals**. Suppose for contradiction, there is such a collection. Each open interval must contain a rational number (since  $\mathbb{Q}$  is dense in  $\mathbb{R}$ ). Then, for each interval, we can randomly select a rational number that is contained in it. This constructs a biject map from these intervals to a subset of  $\mathbb{Q}$ . By the first statement in the previous theorem, the number of intervals must be countable.

Let's go back to the topic. Naturally from the theorem proved above, we can have the result:

**Corollary 1.6.11:  $\mathbb{R} \setminus \mathbb{Q}$  is uncountable**

The set of irrational numbers,  $\mathbb{R} \setminus \mathbb{Q}$ , is uncountable.

*Proof.* Suppose  $\mathbb{R} \setminus \mathbb{Q}$  is countable. Then, by Theorem 1.6.10, the union  $\mathbb{R} = \mathbb{R} \setminus \mathbb{Q} \cup \mathbb{Q}$  is countable since  $\mathbb{Q}$  is countable, which contradicts the fact that  $\mathbb{R}$  is uncountable. □

There are 'more' irrational numbers than rational numbers! Even if they are all dense in the real line. Indeed, this kind of pattern is what you will always encounter down the road of mathematics study. The 'pathological' mathematical objects are far more than the well-behaved ones. For example, there are more discontinuous functions than continuous ones. There are more transcendental numbers than algebraic numbers.....

## 1.7\* Aleph Numbers and Continuum Hypothesis

This section is a hard section and can be skipped without losing coherence.

### 1.7.1\* Cantor's Diagonalization Method

In 1891, Cantor offered another elegant proof of the fact that  $\mathbb{R}$  is uncountable. This method is called **Cantor's Diagonalization Method**. It uses the **decimal representations** for real numbers.

**Definition 1.7.1: Decimal Representation**

A decimal representation of a non-negative real number  $r$  is its expression as a sequence of symbols consisting of decimal digits traditionally written with a single separator:

$$r = b_k b_{k-1} \cdots b_0 . a_1 a_2 a_3 \cdots, \quad b_i, a_j \in \{0, 1, 2, 3, \dots, 9\}, i \in \{1, 2, 3, \dots, k\}, j \in \mathbb{N}$$

and it represents the infinite sum

$$r = \sum_{i=0}^k b_i 10^i + \sum_{i=1}^{\infty} \frac{a_i}{10^i}$$

For the rigorous definition of Infinite sum, see Chapter 2. **Here is a problem, each real number has at least one decimal representation. Some real number has two decimal representation. The mapping from decimal representation to real numbers is not bijective.** To solve this problem, we observe that **a real number has two such representations if and only if one has a trailing infinite sequence of 0, and the other has a trailing infinite sequence of 9.** To make the mapping into a bijection, we will ban the use of decimal representations with a trailing infinite sequence of 9. Now we state the Cantor's proof.

**Proof. CANTOR'S DIAGONALIZATION METHOD**

We have already seen that  $(0, 1) \sim \mathbb{R}$ . If we can prove that  $(0, 1)$  is uncountable, we are done.

We prove by contradiction. Suppose there exists a bijective function  $f : \mathbb{N} \rightarrow (0, 1)$ . Then, for each  $m \in \mathbb{N}$ ,  $f(m)$  is a real number that has decimal representation

$$f(m) = .a_{m1}a_{m2}a_{m3} \cdots$$

where  $a_{mn} \in \{0, 1, \dots, 9\}$ . The bijective correspondence is summarized below as a table

$\mathbb{N}$	$(0, 1)$						
1	$\longleftrightarrow$	$f(1)$	=	. <b>a</b> <sub>11</sub>	$a_{12}$	$a_{13}$	$a_{14} \cdots$
2	$\longleftrightarrow$	$f(2)$	=	. $a_{21}$	<b>a</b> <sub>22</sub>	$a_{23}$	$a_{24} \cdots$
3	$\longleftrightarrow$	$f(3)$	=	. $a_{31}$	$a_{32}$	<b>a</b> <sub>33</sub>	$a_{34} \cdots$
4	$\longleftrightarrow$	$f(4)$	=	. $a_{41}$	$a_{42}$	$a_{43}$	<b>a</b> <sub>44</sub> $\cdots$
$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$

Then, every decimal representation of numbers in  $(0, 1)$  would appear somewhere in the table. However,

we can define a real number  $x \in (0, 1)$  with decimal representation of  $x = .b_1b_2b_3 \cdots$  such that

$$b_n = \begin{cases} 1, & \text{if } a_{nn} \neq 1 \\ 2, & \text{if } a_{nn} = 1 \end{cases}$$

Then,  $x \neq f(1)$  because  $a_{11} \neq b_1$ .  $x \neq f(2)$  because  $a_{22} \neq b_2$ .  $x \neq f(3)$  because  $a_{33} \neq b_3 \cdots$ . Therefore, we have for each  $n \in \mathbb{N}$ ,  $x \neq f(n)$ , which is a contradiction.  $\square$

**Note:** This proof cannot be used on  $\mathbb{Q}$  since, as we know from junior or high school, that rational numbers have infinite-repeating decimals, and the construction for  $x$  would lead to a real number.

This kind of method can be used not only on the proof of the fact that  $\mathbb{R}$  is an uncountable set. Let's see another example.

#### Example 1.7.2: Set of infinite 0 – 1 sequence is uncountable

Let

$$S = \{(a_1, a_2, a_3, \cdots) : a_n = 0 \text{ or } 1\}$$

$S$  is uncountable.

*Proof.* Suppose it is countable. We can then write elements in  $S$  as  $S = \{x_1, x_2, x_3, \cdots\}$ ,  $x_n = (a_{n1}, a_{n2}, a_{n3}, \cdots)$ , where  $a_{nm} = 0$  or  $1$ . Then, we can construct a similar table as in Cantor's Diagonalization Method. Now consider an infinite sequence

$$b_n = \begin{cases} 0, & \text{if } a_{nn} = 1 \\ 1, & \text{if } a_{nn} = 0 \end{cases}$$

It is not in the list, which is a contradiction.  $\square$

### 1.7.2\* Schröder-Bernstein Theorem, Cardinality of Real Space $\mathbb{R}^n$

Sometimes it is very difficult to find a bijective function between two sets  $A$  and  $B$  with the same cardinality. However, it is almost always easy to find an injective function from  $A$  to  $B$ , and another injective function from  $B$  to  $A$ . Does this imply that there exists a bijective function? Yes! And it is called the **Schröder-Bernstein Theorem**.

#### Theorem 1.7.3: Schröder-Bernstein Theorem

Suppose there exists injective function  $f : X \rightarrow Y$  and another injective function  $g : Y \rightarrow X$ . Then, there exists a bijective function from  $X$  to  $Y$ , hence  $X \sim Y$ .

*Proof.* There are many versions of proof of this theorem. The most famous one is presented by a 19-year old student, Bernstein, who was in Cantor's Seminar. Almost simultaneously, Schröder presents another proof.

To make this proof clear and well-structured, I will separate them into several STEPS. The basic idea is to partition  $X$  and  $Y$  into components

$$X = A \cup A' \text{ and } Y = B \cup B'$$

with  $A \cap A' = \emptyset$  and  $B \cap B' = \emptyset$ , in such way that  $f$  maps  $A$  surjectively onto  $B$ , and  $g$  maps  $B'$  surjectively onto  $A'$

**STEP I:** Set  $A_1 = X \setminus g(Y)$ . If  $A_1 = \emptyset$  then  $g(Y) = X$ ,  $g$  is bijective, then we are done. So, assume  $A_1 \neq \emptyset$ . Inductively define a sequence of sets by letting  $A_{n+1} = g(f(A_n))$ . We will show that  $\{A_n : n \in \mathbb{N}\}$  is pairwise disjoint collection of subsets of  $X$ .

We first show that  $A_1 \cap A_k = \emptyset$ . This is obvious since  $A_1 = X \setminus g(Y)$  and  $A_k = g(f(k-1)) \in g(Y)$ .

Now we turn to prove more general case that  $A_j \cap A_k = \emptyset$ . Define  $h(x) = g(f(x))$ . Because both  $f$  and  $g$  are injective, we have  $h$  is injective as well. Note that

$$h(A \cap B) = h(A) \cap h(B)$$

This can be proved by the following details:

( $\implies$ ) Suppose  $x \in h(A \cap B)$ , since  $h$  is injective, there exists unique  $y \in A \cap B$  such that  $h(y) = x$ . Since  $y \in A$  and  $y \in B$ , we have  $x = h(y) \in h(A) \cap h(B)$ . This shows that  $h(A \cap B) \subseteq h(A) \cap h(B)$ .

( $\impliedby$ ) Suppose  $x \in h(A) \cap h(B)$ , then  $x \in h(A)$  and  $x \in h(B)$ . Since  $h$  is injective, there exists unique  $y \in X$  such that  $x = h(y)$ . This means that  $y \in A$  and  $y \in B$ . Thus,  $y \in A \cap B$ ,  $x = h(y)$ . We conclude that  $x \in h(A \cap B)$ . Therefore,  $h(A \cap B) \supseteq h(A) \cap h(B)$ .

Denote  $h^2 = h \circ h$ , and inductively denote  $h^k$  as the  $k$ th composition of  $h$ . Note that  $h^k$  is injective. Therefore, we have

$$A_{j+1} \cap A_{k+1} = h^k(A_{j-k}) \cap h^k(A_1) = h^k(A_{j-k} \cap A_1) = h^k(\emptyset) = \emptyset, \quad j, k \in \mathbb{N}$$

**STEP II:** Now we prove that  $\{f(A_n) : n \in \mathbb{N}\}$  is a pairwise disjoint collection of subsets of  $Y$ . This is easy. Since  $f$  is injective, we have  $f(A_j) \cap f(A_k) = f(A_j \cap A_k) = f(\emptyset) = \emptyset$ .

**STEP III:** Now we let  $A = \bigcup_{n=1}^{\infty} A_n$  and  $B = \bigcup_{n=1}^{\infty} f(A_n)$ . Note that since  $A_n$  are pairwise disjoint,

$$f(A) = f\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} f(A_n) = B$$

Therefore,  $f$  maps  $A$  surjectively onto  $B$ .

**STEP IV:** We finally show that for  $A' = X \setminus A$  and  $B' = X \setminus B$ , we have  $g$  maps  $B'$  surjectively onto  $A'$ . To prove this, we need to show that  $g(B') = A'$ . We prove both directions by contradiction.

( $\implies$ ) To prove  $g(B') \subseteq A'$ , suppose for contradiction that there exists  $b' \in B'$  such that  $g(b') \in A$ . Since  $A_1 \cap g(Y) = \emptyset$ , we must have  $g(b') \notin A_1$ , thus  $g(b') \in A \setminus A_1$ . Note that

$$g(B) = g\left(\bigcup_{n=1}^{\infty} f(A_n)\right) = \bigcup_{n=1}^{\infty} g(f(A_n)) = \bigcup_{n=1}^{\infty} A_{n+1} = A \setminus A_1$$

We have  $g(b') \in g(B)$ . This means that there exists  $b \in B$  such that  $b' \neq b$  (this is because  $B \cap B' = \emptyset$ ), and  $g(b') = g(b)$ , which is a contradiction to the injectivity of  $g$ .

( $\impliedby$ ) To prove  $g(B') \supseteq A'$ , suppose for contradiction that there exists  $a' \in A'$  such that  $a' \notin g(B')$ . Immediately, because  $A' \in g(Y)$ , we have  $a' \in g(B)$  (since  $a' \notin g(B')$ ). This will contradict the fact that  $a' \in A'$  since  $g(B) = A \setminus A_1 \subseteq A$ .

**STEP V:** Now we know that  $f : A \rightarrow B$  and  $g : B' \rightarrow A'$  are bijective functions. Define

$$l(x) = \begin{cases} f(x), & \text{if } x \in A \\ g^{-1}(x), & \text{if } x \in A' \end{cases}$$

This is a bijective function from  $X$  to  $Y$ . □

You see from this proof another fact in mathematics: Some theorems that seem to be easily structured can be very hard to prove.

Let's see some applications of this theorem. The very important one is that we can use Schröder-Bernstein Theorem to prove that  $\mathbb{R}^n$  has the same cardinality with  $\mathbb{R}$ .

#### Example 1.7.4: $\mathbb{R}^n \sim \mathbb{R}$

Let us first consider the interval  $(0, 1)$ . Let

$$S = \{(x, y) : 0 < x, y < 1\}$$

The function  $f : (0, 1) \rightarrow S$ , where  $f(x) = (x, x)$  maps  $(0, 1)$  injectively into  $S$ , but not surjective. Now we want to find an injective function from  $S$  to  $(0, 1)$ . Recall the decimal representation of real numbers. We construct the map in the following way: Let  $(x, y) \in S$ , such that

$$x = .a_1a_2a_3\cdots, \quad y = .b_1b_2b_3\cdots$$

Let

$$z = .a_1b_1a_2b_2a_3b_3\cdots$$



We define  $g : S \rightarrow (0, 1)$  by  $g(x, y) = z$ . This is then an injective function, because the decimal representation is unique for each real number. By Schröder-Bernstein Theorem, there exists a bijective map from  $(0, 1)$  to  $S$ , thus  $(0, 1) \sim S$ .

Now, if we define the function  $h : S \rightarrow \mathbb{R}^2$  by

$$h(x, y) = \left( \frac{x}{x^2 - 1}, \frac{y}{y^2 - 1} \right)$$

We see from Example 1.6.6 that this function is bijective. Therefore, we have

$$\mathbb{R} \sim (0, 1) \sim S \sim \mathbb{R}^2$$

Inductively, we can also have

$$\mathbb{R} \sim \mathbb{R}^n$$

This is actually a surprising result. We have  $\mathbb{Q}$  is countable and  $\mathbb{R}$  is uncountable, since we can think  $\mathbb{Q}$  as a set of ‘countable points’, but  $\mathbb{R}$  as a complete real ‘line’. This jump from ‘1-dimensional’ to ‘2-dimensional’ space intuitively explained why  $\mathbb{R}$  is a much bigger set. However, this intuition does not work for dimensions more than this. Hyper-Eulidean Spaces in all dimensions have the same cardinality! Then, a question would raise naturally: Would there be a set, that is even larger than  $\mathbb{R}$ ? The answer is yes, and will be discussed in the next subsection.

### 1.7.3\* Cantor’s Theorem

In the same paper where Cantor published his Diagonalization Method, he also stated the proof of **Cantor’s Theorem**, which says that **the power set of a set is strictly ‘larger’ than the original set**.

For those who don’t know what is a power set, the power set of  $A$  is the collection of all subsets of  $A$ . For example,  $A = \{1, 2, 3\}$ , the power set is then  $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$ . In finite case, it is easy to see that for a finite set with  $n$  elements, the power set of it has  $2^n$  elements. Therefore, in finite case, it is obvious that there is no surjective function from original set to its power set. However, what is surprising, is that this is also true in infinite case.

#### Theorem 1.7.5: Cantor’s Theorem

Given any set  $A$ , there does not exist a function  $f : A \rightarrow \mathcal{P}(A)$  that is surjective.

*Proof.* Assume for contradiction, that  $f : A \rightarrow \mathcal{P}(A)$  is surjective. Note that for each element  $a \in A$ ,  $f(a)$  is a *subset* of  $A$ . (This proof is super tortuous, please follow very carefully!)

Surjective means that for each  $y \in \mathcal{P}(A)$ , there exists  $a \in A$ , such that  $f(a) = y$ . To arrive a contradiction, we will produce a set  $B \subseteq A$  such that it is not equal to  $f(a)$  for any  $a \in A$ .

For each  $a \in A$ , consider  $f(a) \subseteq A$ . We will conclude all  $a$  in a set  $B$  such that  $f(a)$  does not contain  $a$ . In precise,

$$B = \{a \in A : a \notin f(a)\}$$

Now, because  $f$  is surjective, there must be some  $a' \in A$  such that  $f(a') = B$ .

- If  $a' \in B$ , then  $a' \notin f(a')$  by the definition of  $B$ . Since  $f(a') = B$ , we have  $a' \notin B$ , which is a contradiction.
- If  $a' \notin B$ , then  $a' \in f(a')$  by the definition of  $B$ . Since  $f(a') = B$ , we have  $a' \in B$ , which is a contradiction.

Therefore, we have a contradiction,  $f$  cannot be surjective. □

This theorem will directly mean that, the cardinality of  $\mathcal{P}(\mathbb{R})$  is larger than the cardinality of  $\mathbb{R}$ . We find an even larger set than  $\mathbb{R}$ ! Moreover, we will have a full spectrum of cardinalities, such that  $\mathcal{P}(\mathcal{P}(\mathbb{R}))$  is larger than  $\mathcal{P}(\mathbb{R})$ , and  $\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{R})))$  is larger than  $\mathcal{P}(\mathcal{P}(\mathbb{R})) \cdots$  and there does not exist a ‘largest set’. Thus, statement like ‘Let  $U$  be a set of all possible things’ would become a paradox, because we can immediately find a larger set by constructing the power set of  $U$ .

To see the most important application of this theorem, let’s first prove that  $\mathcal{P}(\mathbb{N}) \sim \mathbb{R}$ .

#### Example 1.7.6: $\mathcal{P}(\mathbb{N}) \sim \mathbb{R}$

Define  $f : \mathcal{P}(\mathbb{N}) \rightarrow S$ , where

$$S = \{(a_1, a_2, a_3, \dots) : a_n = 0 \text{ or } 1\}$$

by  $f(A) = (a_1, a_2, a_3, \dots)$  where  $a_i = 0$  if  $i \notin A$  and  $a_i = 1$  if  $i \in A$ , for each  $i \in \mathbb{N}$ . This is a bijective map, thus  $\mathcal{P}(\mathbb{N}) \sim S$ . Recall Example 1.7.2, we have proved that  $S \sim \mathbb{R}$ . Therefore,  $\mathcal{P}(\mathbb{N}) \sim \mathbb{R}$ .

The cardinality of natural numbers and real numbers are so important, so they have a fancy name.

**Definition 1.7.7: Aleph Number**

1. The cardinality of natural number is called **Aleph 0**, and is denoted as  $\aleph_0$ .
2. The cardinality of real numbers is called **Aleph 1**, and is denoted as  $\aleph_1$ , where  $2^{\aleph_0} = \aleph_1$ .

The more rigorous definition of aleph numbers would be stated in my Set Theory note. A question would naturally arise: Is there any other aleph numbers between  $\aleph_0$  and  $\aleph_1$ ? Cantor published a hypothesis that there is no such aleph number.

**Conjecture 1.7.8: Continuum Hypothesis**

There does not exist an aleph number  $c$  such that

$$\aleph_0 < c < \aleph_1$$

This connected back to the time of '**The Third Mathematical Crisis**', when Bertrand Russell (1872-1970) stated his famous '**Russell's Paradox**'. It says that

$$\text{Let } R = \{x : x \notin x\}, \text{ then } R \in R \iff R \notin R$$

A lively example of this is called **Barber Paradox**. The barber is the "one who shaves all those, and those only, who do not shave themselves". The question is, does the barber shave himself? If the barber shaves himself, then himself becomes the group of people who shaves themselves, so he cannot shave himself. If he doesn't, then he must shave himself because he shaves all those who do not shave themselves. A statement cannot be simultaneously right and wrong!

After few decades, there was a system of axiom constructed in set theory, called 'ZFC system', excluded the situation which Russell presented, and ended the Third Mathematical Crisis. Is that the end? No! **Kurt Gödel** (1906-1978) then stated his famous **Incompleteness Theorem**, which said that even in ZFC system, mathematics is not complete, i.e., there exists a theorem that we cannot prove it right or wrong.

What does this story relate to Continuum Hypothesis? Indeed, Continuum Hypothesis is one of the theorem that is 'undecidable', where it can be accepted or rejected without making any logical contradictions. Whether continuum Hypothesis is true or not, will be never known to us. To intuitively explain why this happens, you can go back to see the proof of Cantor's Theorem, which arrives the contradiction in the way like 'If  $a \in B$ , then  $a \notin B$  and if  $a \notin B$ , then  $a \in B$ ', which is a similar kind of paradox!

## 1.8\* The Dedekind Cuts: Completion from $\mathbb{Q}$ to $\mathbb{R}$

This section is a hard section and can be skipped without losing coherence.

We refer the Supremum as an ‘axiom’, meaning that there is nothing to be proved. The real numbers were defined simply as an extension of the rational numbers in which bounded sets have supremum. No attempt was made to demonstrate that such extension is possible. Now, in this advanced section, we will actually prove that such extension exist.

### 1.8.1\* Cut

We begin this chapter pretended that we don’t know there exists a thing called ‘real number’, and we assume we know all the familiar addition, multiplication and order rule for rational numbers. The goal is to extend rational numbers into a larger set so that Supremum Property holds.

#### Definition 1.8.1: Cut

A subset  $A$  of the rational numbers is called a **cut** if

1.  $A \neq \emptyset$  and  $A \neq \mathbb{Q}$ .
2. If  $r \in A$ , then for all  $q \in \mathbb{Q}$  such that  $q < r$ , we have  $q \in A$ .
3.  $A$  does not have a maximum. i.e., if  $r \in A$ , then there exists  $s \in A$  with  $r < s$ .

This is the main tool used in constructing real numbers in this chapter. Let’s see some examples of cut.

#### Example 1.8.2: Examples of Cuts

1. Fix  $r \in \mathbb{Q}$ . The set  $C_r = \{t \in \mathbb{Q} : t < r\}$  is a cut.
2.  $T = \{t \in \mathbb{Q} : t^2 < 2 \text{ or } t < 0\}$  is a cut.
3.  $U = \{t \in \mathbb{Q} : t^2 \leq 2 \text{ or } t < 0\}$  is a cut. The proof of statement 3 in the definition of cut for  $U$  can follow the pattern in proof of Proposition 1.4.5.
4. **Counterexample:**  $S = \{t \in \mathbb{Q} : t \leq 2\}$  is not a cut since it has maximum 2.

All the verification of 3 properties of cut above are easy, and thus are omitted here. Now we define our goal: The set of real numbers.

#### Definition 1.8.3: Real Number in Dedekind Cut Sense

The real numbers  $\mathbb{R}$  is the set of all cuts in  $\mathbb{Q}$ .

You may ask: what? we define real numbers as sets! This looks very weird at first. The most intuitive (but heuristic) explanation to this weird definition is that, we can construct a bijection from each cut to each real number such that for a cut  $A$ , it is mapped to a real number on the ‘cut point’. For example, for the cut  $T$  above, it is mapped to the real number  $\sqrt{2}$ . (**Note:** Since we assume at first we don’t know real numbers, including  $\sqrt{2}$ , this explanation is just heuristic one.) This bijection is called a **isomorphism** if and only if the algebraic structure on  $\mathbb{R}$  we defined above is the same as the set of real numbers in our common sense. Then the two sets are called **isomorphic**. If two sets are isomorphic, they are essentially the same, with just different notations.

Now we discuss exactly which algebraic structures  $\mathbb{R}$  should obtain. Before that, we need to know what is a ‘structure’.

### 1.8.2\* Field and Ordering

#### Definition 1.8.4: (Binary) Operation

Given a set  $F$ , an **operation** on  $F$  is a function  $f : F \times F \rightarrow F$ .

For example, the ‘addition’ operation on rational numbers takes  $(2, 3) \in \mathbb{Q} \times \mathbb{Q}$  to the element  $5 \in \mathbb{Q}$ . The ‘multiplication’ operation on rational numbers takes  $(2, 3) \in \mathbb{Q} \times \mathbb{Q}$  to the element  $6 \in \mathbb{Q}$ . With this, we can define what is a field.

#### Definition 1.8.5: Field

A triple  $(F, +, \times)$ , where  $F$  is a set and  $+, \times$  are two arbitrary operations, is a **field** if

- Commutativity:  $x + y = y + x$  and  $x \times y = y \times x$ ,  $\forall x, y \in F$ .
- Associativity:  $(x + y) + z = x + (y + z)$  and  $(x \times y) \times z = x \times (y \times z)$ ,  $\forall x, y, z \in F$ .
- Identity: There exists  $0 \in F$  and  $1 \in F$  such that  $x + 0 = x$  and  $x \times 1 = x$  for all  $x \in F$ .
- Inverse: Given  $x \in F$ , there exists  $-x \in F$  such that  $x + (-x) = 0$ . If  $x \neq 0$ , there exists an element  $x^{-1} \in F$  such that  $x \times x^{-1} = 1$ .
- Distributive Property:  $x \times (y + z) = x \times y + x \times z$ ,  $\forall x, y, z \in F$ .

**Note:** If you are the first time encountering these definitions, note that the  $+$  and  $\times$  notation need not to represent addition and multiplication. As long as there are two operations on the set that satisfies these 5 properties correspondingly, it is a field. For example, sometimes in functional spaces, the function composition  $\circ$  would take the position of  $\times$ .

**Example 1.8.6: Examples of Fields**

$\mathbb{Q}$  is a field,  $\mathbb{N}$  and  $\mathbb{Z}$  are not field, as you can verify.

**Definition 1.8.7: (Binary) Relation**

A **relation** on  $F$  is a subset of  $F \times F$ .

This definition is very abstract. However, the following definition would give a strict example of this.

**Definition 1.8.8: Ordering/Ordered Field**

An **Ordering** on a set  $F$  is a relation, represented by  $\leq$ , with properties

- At least one of the  $x \leq y$  and  $y \leq x$  is true,  $\forall x, y \in F$ .
- If  $x \leq y$  and  $y \leq x$ , then  $x = y$ ,  $\forall x, y \in F$ .
- If  $x \leq y$  and  $y \leq z$ , then  $x \leq z$ ,  $\forall x, y, z \in F$ .

A field  $F$  is called an **ordered field** if  $F$  is endowed with an ordering such that

- If  $y \leq z$ , then  $x + y \leq x + z$ ,  $\forall x, y, z \in F$ .
- If  $0 \leq x$ ,  $0 \leq y$ , then  $0 \leq x \times y$ ,  $\forall x, y \in F$ .

Since  $\mathbb{Q}$  is a field, and has an ordering on it, our goal now is to construct addition, multiplication, and ordering on  $\mathbb{R}$ , such that it is ordered, and it is a field.

**1.8.3\* Algebra on  $\mathbb{R}$** 

We first define an ordering on  $\mathbb{R}$ .

**Definition 1.8.9: Ordering on  $\mathbb{R}$** 

Let  $A, B \in \mathbb{R}$  be two cuts. Define  $A \leq B$  to mean  $A \subseteq B$ .

*Proof.* Now we need to prove this definition satisfies the 3 properties of an ordering.

1. Suppose  $A \not\subseteq B$ . We need to prove that  $B \subseteq A$ . If  $A \not\subseteq B$ , there exists  $a \in A$  such that  $a \notin B$ . This means that  $\forall b \in B, b < a$ . Then, by the second property in definition of cut, we have  $\forall b \in B, b \in A$ . This means that  $B \subseteq A$ .
2. If  $A \subseteq B$  and  $B \subseteq A$ , then  $A = B$  by definition of equality of set.
3. If  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$  by the property of set.

Therefore, this definition satisfy the properties of ordering.  $\square$

Further, we define an addition on  $\mathbb{R}$ .

#### Definition 1.8.10: Addition on $\mathbb{R}$

Let  $A, B \in \mathbb{R}$  be cuts. Define

$$A + B = \{a + b : a \in A, b \in B\}$$

We first need to prove that  $A + B \in \mathbb{R}$ , i.e.,  $A + B$  is indeed a cut.

*Proof.* We need to go through the three properties of cuts.

- Since  $A, B$  are cuts, by property 1 of cut,  $A, B \neq \emptyset$ . Therefore, we can find  $a \in A$  and  $b \in B$ , then  $a + b \in A + B$ . Thus  $A + B \neq \emptyset$ .

Since  $A, B$  are cuts, by property 1 of cut,  $A, B \neq \mathbb{Q}$ . Therefore, we can find  $a' \notin A$  and  $b' \notin B$ , then for all  $a \in A$  and  $b \in B$ , we have  $a < a', b < b'$ . Therefore, for all  $a + b \in A + B$ , we will have  $a + b < a' + b'$ . Thus  $a' + b' \notin A + B$ . We have  $A + B \neq \mathbb{Q}$ .

- To prove property 2 of cut, let  $a + b \in A + B$  be arbitrary and let  $s \in \mathbb{Q}$  satisfy  $s < a + b$ . Then,  $s - b < a$ , which implies that  $s - b \in A$  because  $A$  is a cut. Then,

$$s = (s - b) + b \in A + B$$

Since  $s$  is arbitrary, we have for every  $s \in \mathbb{Q}$  that  $s < a + b$ , we also have  $s \in A + B$ .

- To prove property 3, since  $A, B$  are cuts, for  $a \in A, b \in B$ , we can find  $a' \in A, b' \in B$  such that  $a < a', b < b'$ . Then, for each  $a + b \in A + B$ , we can find  $a' + b' \in A + B$  such that  $a + b < a' + b'$ .

Therefore,  $A + B$  is indeed a cut.  $\square$

Then, we need to prove that this definition satisfies the related properties in the field, and also the ordering field property 1.

*Proof.*

- We first prove the **Commutativity**. Obviously,

$$A + B = \{a + b : a \in A, b \in B\} = \{b + a : a \in A, b \in B\} = B + A$$

- Then, we prove the **Associativity**. Obviously,

$$(A+B)+C = \{(a+b)+c : a \in A, b \in B, c \in C\} = \{a+(b+c) : a \in A, b \in B, c \in C\} = A+(B+C)$$

- Next, we prove the existence of **Identity**, define

$$O = \{p \in \mathbb{Q} : p < 0\}$$

We want to prove that this is served as the identity.

( $\Rightarrow$ ) Let  $a+o \in A+O$  where  $a \in A$  and  $o \in O$ . Then  $o < 0$ . Therefore,  $a+o < a$ . By property of cut,  $a+o \in A$ . Thus  $A+O \subseteq A$ .

( $\Leftarrow$ ) Let  $a \in A$ . Then by property of cut, we can find  $a < a' \in A$ . Define  $s = a - a' < 0$ . We have  $s \in O$ . Then,  $a = s + a' \in A+O$ . Thus,  $A \subseteq A+O$ .

In conclusion, we have  $A = A+O$ , proving that  $O$  is the identity.

- Now, we prove the existence of **Inverse**. This is a little bit more difficult than the identity, since the normal definition of  $-A$  would not be a cut. We alternatively, define

$$-A = \{r \in \mathbb{Q} : \text{there exists } t \notin A \text{ with } t < -r\}$$

This is just like a ‘reflection’ of the ‘cut point’ with respect to the origin.

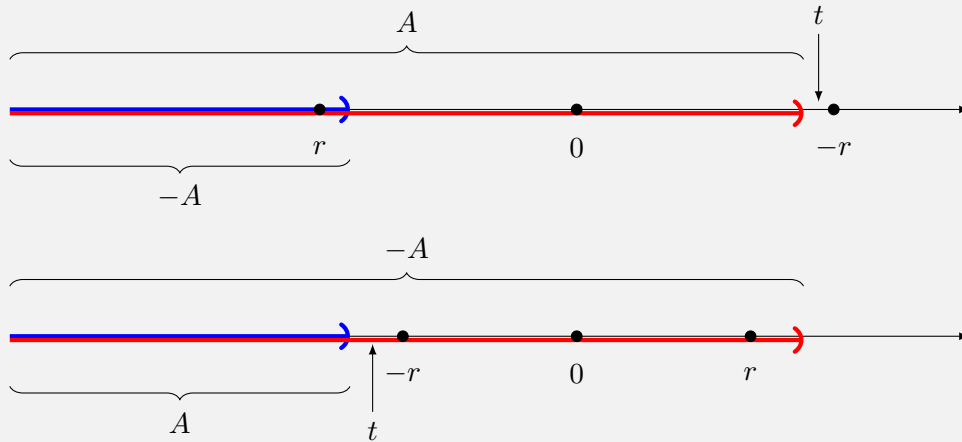


Figure 1.12:  $A$  and its inverse

We first need to prove that  $-A$  is indeed a cut.

1. To prove the first property of cut, since  $A$  is a cut, we can find  $t \notin A$ , and since  $\mathbb{Q}$  is unbounded, we can find some  $-r \in \mathbb{Q}$  such that  $t < -r$ . Thus,  $-A \neq \emptyset$ .



Let  $a \in A$ . Then for all  $t \notin A$ , we have  $t > a$ . Thus  $-a \notin -A$  since if it is, then  $t < -(-a) = a$ , which is a contradiction. Therefore,  $-A \neq \mathbb{Q}$ .

2. To prove the second property of cut, let  $r \in -A$  and let  $q < r$ . Then,  $t < -r < -q$  for the  $t \notin A$  such that  $t < -r$ . Hence, by definition of  $-A$ ,  $q \in -A$ .
3. To prove the third property of cut, let  $r \in -A$  and let  $t \notin A$  such that  $t < -r$ . Let  $q = \frac{t-r}{2}$ , we have  $t < q < -r$ . Thus,  $-q \in -A$  and  $-q > r$ . There is no maximum.

After this, we need to prove that this indeed defines an inverse.

( $\implies$ ) If  $a \in A$  and  $r \in -A$ , then there exists  $t \notin A$  with  $t < -r$ . Since  $t \notin A$ , we have  $t > a$ . Then,  $a + r < a - t < 0$ , hence  $a + r \in O$ . This shows that  $A + (-A) \subseteq O$ .

( $\impliedby$ ) Now, let  $o \in O$ . We would like to find  $a \in A$  and  $r \in -A$  satisfying  $o \leq a + r$ . This would imply  $O \subseteq A + (-A)$ . Since  $o \in \mathbb{Q}$ , and  $o < 0$ , let  $o = -p/q$  where  $p, q \in \mathbb{N}$ ,  $q \neq 0$ . We first prove the lemma:

For any cut  $A$  and  $n \in \mathbb{N}$ , we can find  $z \in A$  where

$$\frac{z}{n} \in A \quad \text{and} \quad \frac{z+1}{n} \notin A$$

To do this, start with  $a \in A$  and  $a' \notin A$ . Find  $N, M \in \mathbb{Z}$  such that

$$\frac{N}{n} < a \quad \text{and} \quad \frac{M}{n} > a'$$

Clearly  $N/n \in A$  and  $M/n \notin A$ . Therefore, there must exist a transition point  $z$  such that the lemma holds.

Now if we let  $a = \frac{n}{2q} \in A$  and  $\frac{n+1}{2q} \notin A$ , then  $r = -\frac{n+2}{2q} \notin A$ , and

$$a + r = -\frac{1}{q} \geq -\frac{p}{q} = o$$

Therefore,  $O \subseteq A + (-A)$ . In conclusion,  $A + (-A) = O$ .

- Finally, we prove the **first property of ordering field**.

Let  $Y \subseteq Z$ . Let  $x \in X$  and  $y \in Y$ . Since  $Y \subseteq Z$ , we have  $y \in Z$  either. This implies  $x + y \in X + Z$ . Thus,  $X + Y \subseteq X + Z$ .  $\square$

**Note:** We cannot simply define the inverse as

$$-A = \{r \in \mathbb{Q} : -r \notin A\}$$

which would be the first thought of most people. The counterexample is that for  $A = \{t \in \mathbb{Q} : t < -2\}$ , we

will have  $-A = \{t \in \mathbb{Q} : t \leq 2\}$ , which is not a cut. Therefore, we need the additional ‘ $t$ ’ in the definition of inverse to avoid this situation.

Now, we define a multiplication in  $\mathbb{R}$ . This is also difficult because negative set times negative set would be positive. Therefore, we first consider the case of  $A \geq O$  and  $B \geq O$ .

**Definition 1.8.11: Multiplication on  $\mathbb{R}$ , positive case**

Given  $A \geq O$  and  $B \geq O$ , define the product

$$AB = \{ab : a \in A, b \in B \text{ with } a, b \geq 0\} \cup O$$

Similarly, we need to first prove that the definition indeed results in a cut.

*Proof.*

- choose  $-1 \in O$ . Then,  $-1 \in AB$ , we have  $AB \neq \emptyset$ . Since  $A, B$  are cuts, choose  $a' \notin A, b' \notin B$ , then for all  $a \in A, b \in B$ , we have  $a' > a, b' > b$ . Also,  $a', b' > 0$  since otherwise it will be in  $O$ . Thus, for all  $r \in AB$ ,  $a'b' > r$ . This indicates that  $AB \neq \mathbb{Q}$ .
- Suppose  $r \in AB$ , let  $q < r$ . If  $r < 0$  then  $q < r < 0$ , we have  $q \in AB$ . If  $r > 0$ ,  $r = ab$  for some  $a \in A$  and  $b \in B$  with  $a, b \geq 0$ . If  $q < 0$ , then obviously  $q \in AB$ . If  $q > 0$ , we have  $\frac{q}{b} < \frac{r}{b} = a$ , thus  $\frac{q}{b} \in A$ . This would indicate that  $q = \frac{q}{b}b \in AB$ .
- Let  $r \in AB$ . If  $r < 0$  then obviously  $\frac{r}{2} \in AB$  and  $r < \frac{r}{2} \in AB$ . If  $r > 0$  then,  $r = ab$  for some  $a \in A, b \in B, a, b \geq 0$ . Since  $A, B$  are cuts, we could find  $a' \in A$  and  $b' \in B$  with  $a' > a$  and  $b' > b$ . Then  $a'b' \in AB$  and  $a'b' > ab$ . There is no maximum.

Therefore,  $AB$  is indeed a cut. □

After defining this, we can accordingly define the negative cut cases such that

$$AB = \begin{cases} -[A(-B)], & \text{if } A \geq O \text{ and } B < O \\ -[(-A)B], & \text{if } A < O \text{ and } B \geq O \\ (-A)(-B), & \text{if } A < O \text{ and } B < O \end{cases}$$

We can accordingly, prove that these are cuts, and prove the commutativity, associativity, distributive property, and the second property of the ordering field. However, we will not do it here since it will be way tedious. The proving pattern is just like that for addition. Just note that the multiplicative identity is

$$I = \{t \in \mathbb{Q} : t < 1\}$$

and the multiplicative inverse for the positive cut  $A$  (defined in Definition 1.8.11) is defined as

$$A^{-1} = \left\{ a \in \mathbb{Q} : \text{there exists } t \notin A \text{ with } t < \frac{1}{a} \right\} \cup \{0\} \cup O$$

### 1.8.4\* Rediscover Supremum Property

After proving that  $\mathbb{R}$  satisfies all the **Ordered Field Property**, we finally, prove the **Supremum Property**, which we see as an axiom throughout the first chapter.

Note that now, since we define real numbers as cuts, ‘a set of real numbers’ would be a collection of cuts (set of sets). For these collections we will denote them using the calligraphy font, such as  $\mathcal{A}$ . Obviously, for a set  $\mathcal{A}$  which is nonempty and bounded above, the desired supremum would be the union of all cuts  $A \in \mathcal{A}$

$$S = \bigcup_{A \in \mathcal{A}} A$$

Now we prove that  $S$  is indeed a cut.

*Proof.*

- Let  $A \in \mathcal{A}$ , and  $a \in A$ . Then  $a \in S$ . Thus  $S \neq \emptyset$ . Now let  $B$  be a bound on  $\mathcal{A}$ , let  $b' \notin B$ . Since for any  $s \in S$ , we will have  $s \in A$  for some  $A \in \mathcal{A}$ , and  $A \leq B$  for all  $A$ , we have  $s \in B$ . This will indicate that  $b' > s$ . Thus,  $b' \notin S$ ,  $S \neq \mathbb{Q}$ .
- Consider arbitrary  $s \in S$  with  $s \in A \in \mathcal{A}$ . Then, for any  $q < s$  we have  $q \in A$ , thus  $q \in S$ .
- Continuing the proof of property 2, we can also find a  $a \in A$  such that  $s < a$ . There is no maximum.

Therefore,  $S$  is indeed a cut. □

Finally, we prove that  $S$  is the supremum for  $\mathcal{A}$ .

*Proof.* First,  $S$  is indeed an upper bound since for all  $a \in A$ , we have  $a \in S$ . Second, let  $B$  be an upper bound for  $\mathcal{A}$ . Now for any  $s \in S$  with  $s \in A \in \mathcal{A}$ , we have  $A \leq B$ , so  $s \in B$ . Therefore,  $S \leq B$ ,  $S$  is indeed a supremum! □

We are not finished yet. Since this construction of  $\mathbb{R}$  as a set of cuts, is a completely different notation from that of rational numbers. We are doing an extension, meaning that  $\mathbb{Q}$  should be a subfield of  $\mathbb{R}$ .

However, note that if we write all rational numbers as ‘rational cuts’

$$Q = \{t \in \mathbb{Q} : t < r, r \in \mathbb{Q}\}$$

This would be an **isomorphism** from rational numbers to rational cuts, and indeed, we can easily verify that the set of all rational cuts is an ordered field (just re-prove all the ordering field properties before, using the sets of all rational cuts).

In conclusion, our result is:

**Theorem 1.8.12: Extension of  $\mathbb{Q}$  to  $\mathbb{R}$**

There exists an ordered field in which every nonempty set that is bounded above has a supremum. In addition, this field contains  $\mathbb{Q}$  as a subfield.

## Chapter 2

# Infinite Sequences and Series

### 2.1 Limit of a Sequence

The first important mathematical object we need to analysis in this chapter is **sequences**.

#### Definition 2.1.1: Sequence

A **sequence** is a function whose domain is  $\mathbb{N}$ .

Considering the definition, we can reasonably write a sequence in the form of  $(a_1, a_2, a_3, \dots)$  where  $a_i$  is the element that  $i \in \mathbb{N}$  maps to. We always denote it as  $(a_n)_{n \in \mathbb{N}}$ , or simply,  $(a_n)$ .

**Note:** Sometimes sequences will start not with  $x_1$ , but with  $x_{n_0}$  where  $n_0 > 1$ ,  $n_0 \in \mathbb{N}$ . This does not matter much, because we are only interested in how the sequence behaves at the infinite ‘tail’, i.e., the **limit**.

#### Definition 2.1.2: Convergence of a Sequence

A sequence  $(a_n)$  **converges** to a real number  $a$  if,

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \text{ s.t. } n > N \implies |a_n - a| < \epsilon$$

We denote this as  $\lim_{n \rightarrow \infty} a_n = a$ .

This is **the** most important mathematical language in analysis, called  $\epsilon$ - $\delta$  language, or  $\epsilon$ - $N$  in this case. To fully understand the meaning of this definition, we first fix an  $\epsilon$ , and by the definition, there exists a point, where all of the terms in the sequence after this point should be in the  $\epsilon$ -range centered at  $a$ .

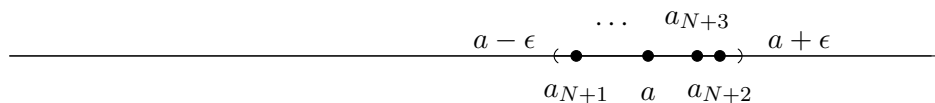


Figure 2.1: Definition of convergence

The critical point is that, we can choose all  $\epsilon$ , regardless how small it is, we can always find  $N$  such that, all the points after are ‘approaching’ the limit  $a$ , and they can never jump out this  $\epsilon$  range.

### Definition 2.1.3: Divergence of a Sequence

A sequence that does not converge is said to **diverge**.

**Note:** We cannot identify divergence by the negation of the definition of convergence, i.e.,

$$\exists \epsilon > 0, \text{ s.t. } \forall N \in \mathbb{N}, \text{ s.t. } \exists n > N \implies |a_n - a| \geq \epsilon$$

since by this statement, the sequence may converge, just not converges to the point  $a$ .

We can easily see that the limit of a sequence must be unique.

### Proposition 2.1.4: Uniqueness of Limit

The limit of a sequence is unique.

*Proof.* Suppose  $a, a'$  are limits of a sequence  $(a_n)$ . Then, by definition,

$$\text{Fix } \epsilon > 0, \exists N_1 \in \mathbb{N}, \text{ s.t. } n > N_1 \implies |a_n - a| < \frac{\epsilon}{2}$$

$$\text{Fix } \epsilon > 0, \exists N_2 \in \mathbb{N}, \text{ s.t. } n > N_2 \implies |a_n - a'| < \frac{\epsilon}{2}$$

Therefore, for current fixed  $\epsilon$  and  $n > \max\{N_1, N_2\}$ , we have

$$|a - a'| = |a - a_n + a_n - a'| \leq |a - a_n| + |a_n - a'| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

This is true for arbitrary  $\epsilon$ . Therefore,  $a = a'$ . □

The proof of convergence using definition can generally follow these steps:

- ‘Let  $\epsilon > 0$  be arbitrary’.
- Demonstrate a choice for  $N \in \mathbb{N}$ . This step usually requires some work on draft paper, to see which  $N$  is suitable. Note that  $N$  may (and commonly will) depend on  $\epsilon$ .
- Assume  $n > N$ , show that  $|a_n - a| < \epsilon$ .

**Example 2.1.5: Prove convergence using definition**

Prove that

$$\lim \frac{2n^2}{n^3 + 3} = 0$$

**Things that will appear on your draft paper:**

$$\left| \frac{2n^2}{n^3 + 3} - 0 \right| = \frac{2n^2}{n^3 + 3} < \frac{2n^2}{n^3} = \frac{2}{n} < \epsilon$$

it seems that  $N = \frac{2}{\epsilon}$  would be a good choice.

**Things that will appear on your answer sheet:**

*Proof.* Let  $\epsilon$  be arbitrary. Let  $N = \frac{2}{\epsilon}$ . Assume  $n > N$ . Then,

$$\left| \frac{2n^2}{n^3 + 3} - 0 \right| = \frac{2n^2}{n^3 + 3} < \frac{2n^2}{n^3} = \frac{2}{n} < \frac{2}{2} \epsilon = \epsilon$$

Therefore, by definition, the sequence converges to 0. □

## 2.2 Properties of Limit

We first see that every convergent sequence are bounded. To rigorously state this, we need to define what is ‘bounded’.

**Definition 2.2.1: Bounded Sequence**

A sequence  $(x_n)$  is bounded if there exists  $M > 0$  such that  $|x_n| \leq M$  for all  $n \in \mathbb{N}$ .

**Proposition 2.2.2: Boundedness of Convergence Sequence**

Every convergent sequence is bounded.

*Proof.* Suppose  $(x_n)$  converges to  $l$ . Then, fix  $\epsilon = 1$ , we have

$$\exists N \in \mathbb{N}, \text{ s.t. } n \geq N \implies |x_n - l| < 1$$

This means that for all  $n \geq N$

$$|x_n| \leq |x_n - l| + |l| < |l| + 1$$

For the terms before  $N$ , since there are only finite terms, there must be a maximum. Let

$$M = \max\{|x_1|, |x_2|, \dots, |x_{N-1}|, |l| + 1\}$$

We can conclude that  $|x_n| \leq M$  for all  $n \in \mathbb{N}$ . □

Next, we state **Algebraic Limit Theorem**, which shows that limit of sequences behave very well under addition, multiplication and division.

### Proposition 2.2.3: Algebraic Limit Theorem

Let  $\lim a_n = a$  and  $\lim b_n = b$ . Then,

1.  $\lim(ca_n) = ca, \forall c \in \mathbb{R}$
2.  $\lim(a_n + b_n) = a + b$
3.  $\lim(a_nb_n) = ab$
4.  $\lim(a_n/b_n) = a/b$ , provided  $b \neq 0$

*Proof.* 1. When  $c = 0$ , it is trivial. So suppose  $c \neq 0$ . Since  $\lim a_n = a$ , we have

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \text{ s.t. } n > N \implies |a_n - a| < \frac{\epsilon}{|c|}$$

Then we have for  $n > N$ ,

$$|ca_n - ca| = |c||a_n - a| < |c| \frac{\epsilon}{|c|} = \epsilon$$

which shows our desired result.

2. Since  $\lim a_n = a$  and  $\lim b_n = b$ , we have

$$\forall \epsilon > 0, \exists N_1 \in \mathbb{N}, \text{ s.t. } n > N_1 \implies |a_n - a| < \frac{\epsilon}{2}$$

$$\forall \epsilon > 0, \exists N_2 \in \mathbb{N}, \text{ s.t. } n > N_2 \implies |b_n - b| < \frac{\epsilon}{2}$$

Let  $N = \max\{N_1, N_2\}$ . Then, for  $n > N$ , we will have

$$|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

3. This is a little bit harder. The goal is to find an  $N$  such that for all  $n > N$  we have  $|a_nb_n - ab| < \epsilon$ . Note that

$$|a_nb_n - ab| = |a_nb_n - ab_n + ab_n - ab| \leq |a_nb_n - ab_n| + |ab_n - ab| = |b_n||a_n - a| + |a||b_n - b|$$



Since  $b_n$  converges, by Proposition 2.2.2, it is bounded. Therefore,  $|b_n| \leq M$  for some  $M$  and for all  $n \in \mathbb{N}$ . Then if we choose  $N_1$  and  $N_2$  such that

$$\forall \epsilon > 0, \exists N_1 \in \mathbb{N}, \text{ s.t. } n > N_1 \implies |a_n - a| < \frac{\epsilon}{2M}$$

$$\forall \epsilon > 0, \exists N_2 \in \mathbb{N}, \text{ s.t. } n > N_2 \implies |b_n - b| < \frac{\epsilon}{2|a|}$$

and let  $N = \max\{N_1, N_2\}$ . Then for all  $n > N$ , we have

$$|a_n b_n - ab| \leq |b_n| |a_n - a| + |a| |b_n - b| < M \frac{\epsilon}{2M} + |a| \frac{\epsilon}{2|a|} = \epsilon$$

4. We can prove this statement if only if we can prove

$$(b_n) \longrightarrow b \quad \text{implies} \quad \left( \frac{1}{b_n} \right) \longrightarrow \frac{1}{b}$$

since we can then get the desired result from (3). The goal is to find  $N$  such that for all  $n > N$  we have  $\left| \frac{1}{b_n} - \frac{1}{b} \right| < \epsilon$ . Note that

$$\left| \frac{1}{b_n} - \frac{1}{b} \right| = \frac{|b - b_n|}{|b| |b_n|}$$

We know  $|b - b_n|$ , so we need to control the size of  $\frac{1}{|b| |b_n|}$ . **This is a very important trick.** Now we are not concerning about the upper bound of  $b_n$ , we are concerning the lower bound. The trick is to use the convergence relation between  $(b_n)$  and  $b$  to construct desired inequality. Since  $(b_n) \rightarrow b$ , we can fix  $\epsilon = \frac{|b|}{2}$ , then

$$\forall \epsilon > 0, \exists N_1 \in \mathbb{N}, \text{ s.t. } n > N_1 \implies |b_n - b| < \frac{|b|}{2}$$

Further simplify this relationship, we have for all  $n > N_1$ ,

$$|b_n| = |(b_n - b) + b| > ||b_n - b| - |b|| = |b| - |b_n - b| > \frac{|b|}{2}$$

where the first inequality is by inverse triangular inequality. Therefore, if we choose  $N_2$  such that

$$\forall \epsilon > 0, \exists N_2 \in \mathbb{N}, \text{ s.t. } n > N_2 \implies |b_n - b| < \frac{\epsilon |b|^2}{2}$$

and let  $N = \max\{N_1, N_2\}$ . Then for all  $n > N$  we have

$$\left| \frac{1}{b_n} - \frac{1}{b} \right| = \frac{|b - b_n|}{|b| |b_n|} < \frac{\epsilon |b|^2}{2} \frac{1}{|b| \frac{|b|}{2}} = \epsilon$$

This shows the desired result. □

Next, we show **Order Limit Theorem**, which shows that the limit preserves the order of two related elements.

#### Proposition 2.2.4: Order Limit Theorem

Let  $\lim a_n = a$  and  $\lim b_n = b$ . Then,

1. If  $a_n \leq b_n$  for all  $n \in \mathbb{N}$ , then  $a \leq b$
2. If  $c \in \mathbb{R}$ , and  $a_n \leq c$  for all  $n \in \mathbb{N}$ , then  $a \leq c$ . Similarly, if  $b_n \geq c$  for all  $n \in \mathbb{N}$ , then  $b \geq c$

*Proof.* 1. For every  $\epsilon > 0$  we can find  $N \in \mathbb{N}$  such that for all  $n > N$ , we have

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \text{ s.t. } n > N \implies a_n - a > -\epsilon$$

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \text{ s.t. } n > N \implies b_n - b < \epsilon$$

Therefore, we can get

$$a - b \leq a - b + (b_n - a_n) = (b_n - b) - (a_n - a) < \epsilon - (-\epsilon) = 2\epsilon$$

where the first inequality hold because  $a_n \leq b_n$ . Therefore,

$$b > a - 2\epsilon$$

Since this holds for all  $\epsilon > 0$ , we have  $b \geq a$ .

2. Take  $a_n = c$  or  $b_n = c$ , we can prove the second argument. □

The **most useful** corollary of Proposition 2.2.4 is the famous **Squeeze Theorem**.

#### Corollary 2.2.5: Squeeze Theorem

If  $x_n \leq y_n \leq z_n$  for all  $n \in \mathbb{N}$ , and  $\lim x_n = \lim z_n = l$ , then  $\lim y_n = l$ .

## 2.3 Completeness and Convergence

### 2.3.1 Axiom of Completeness III: The Monotone Convergence Theorem

Here we consider the third form of Axiom of Completeness of Real Number: Monotone Convergence Theorem. To state this, we first define what is a monotone sequence.

#### Definition 2.3.1: Monotone Sequence

A sequence  $(a_n)$  is **increasing** if  $a_n \leq a_{n+1}$  for all  $n \in \mathbb{N}$  and **decreasing** if  $a_n \geq a_{n+1}$  for all  $n \in \mathbb{N}$ . A sequence is **monotone** if it is increasing or decreasing.

Now we state the theorem.

#### Theorem 2.3.2: Monotone Convergence Theorem (MCT)

If a sequence is monotone and bounded, then it converges. Specifically, if it is increasing, then it converges to the supremum of elements. If it is decreasing, then it converges to the infimum of elements.

*Proof.* Let  $(a_n)$  be monotone and bounded. Assume  $(a_n)$  is increasing, and the decreasing case can be handled similarly. We let

$$s = \sup\{a_n : n \in \mathbb{N}\}$$

We will then prove that  $\lim a_n = s$ . Let  $\epsilon > 0$ . Because  $s$  is the least upper bound,  $s - \epsilon$  then, is not the upper bound. Then, there exists a point  $s_N$  in the sequence such that  $s - \epsilon < a_N$ . Since  $a_n$  is increasing, we have if  $n \geq N$ , we have  $a_N \leq a_n$ . Hence,

$$s - \epsilon < a_N < a_n \leq s < s + \epsilon$$

for all  $n > N$ , as desired. □

Actually, we could have used the MCT in place of Supremum Property as our starting axiom for building a proper theory of real numbers. Intuitively, for a nonempty set which is bounded above, there must exist a increasing bounded sequence in it, and it converges by MCT. The limit is then the supremum. One of the proof is stated below.

*Proof.* The idea is that, we prove Archimedean Property and Nested Interval Property using MCT without the use of Supremum Property, so that we can go from MCT to Nested Interval Property, finally to Supremum Property.

Since the sequence  $(1/n)$  is monotone and bounded, by MCT, it converges. It can then only converge

to 0 by Order Limit Theorem. This shows that we can find  $n$  such that  $|1/n - 0| \leq \epsilon$  for any  $\epsilon$ , which is the Archimedean Property.

Consider an arbitrary collection of nested intervals  $\{I_n\}_n$ . For  $I_n = [a_n, b_n]$ , we have  $\{a_n : n \in \mathbb{N}\}$  is bounded above by  $b_1$ , bounded below by  $a_1$ , and it is monotone. Similarly,  $\{b_n : n \in \mathbb{N}\}$  is bounded below by  $a_1$ , bounded above by  $b_1$ , and it is monotone. Therefore, by MCT, both  $(a_n)$  and  $(b_n)$  converges, say  $a_n \rightarrow a$  and  $b_n \rightarrow b$ . By order limit theorem,  $a \leq b$  since for all  $n$ ,  $a_n \leq b_n$ . Therefore,  $a \in I_n$  for all  $n$ , and thus  $a \in \bigcap_{i=1}^{\infty} I_n$  and thus  $\bigcap_{i=1}^{\infty} I_n \neq \emptyset$ .  $\square$

Below we extend the Relation graph 1.8.

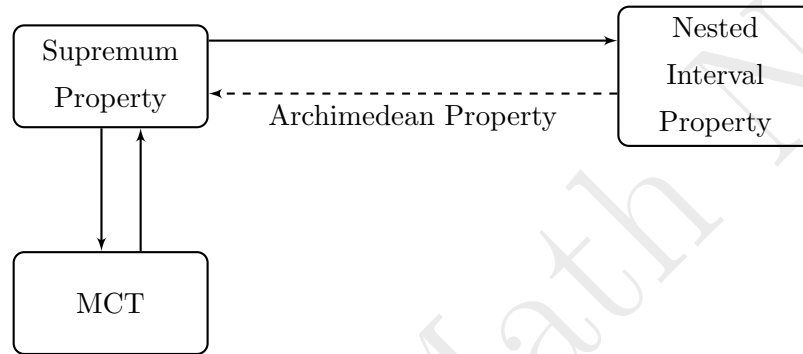


Figure 2.2: Relation between Axiom of Completeness

### 2.3.2 Axiom of Completeness IV: Bolzano-Weierstrass Theorem

A very important terminology in analysis is **subsequence**. A sequence can be divergent with some of its subsequence converges.

#### Definition 2.3.3: Subsequence

Let  $(a_n)$  be a sequence of real numbers, and let  $n_1 < n_2 < n_3 < \dots$  be an increasing sequence of natural numbers. Then the sequence

$$(a_{n_1}, a_{n_2}, a_{n_3}, \dots)$$

is called a **subsequence** of  $(a_n)$ , denoted by  $(a_{n_k})$ , where  $k \in \mathbb{N}$ .

Obviously, from intuition, if a sequence converges, then its subsequences also converge.

#### Proposition 2.3.4: Convergence of Subsequence

Subsequences of a convergent sequence converge to the same limit as the original sequence.

*Proof.* Assume  $(a_n) \rightarrow a$ , let  $(a_{n_k})$  be a subsequence. Given  $\epsilon > 0$ , there exists  $N$  such that  $|a_n - a| < \epsilon$  whenever  $n \geq N$ . Because  $n_k \geq k$  for all  $k$ , the same  $N$  will suffice for the subsequence.  $\square$

Note that not all sequences contain a convergent subsequence. Consider  $(a_n) = (1, 2, 3, 4, \dots)$ , there is no subsequence contained in it. However, for bounded sequence, the situation is changed.

### Theorem 2.3.5: Bolzano-Weierstrass Theorem

Every bounded sequence contains a convergent subsequence.

*Proof.* Let  $(a_n)$  be a bounded sequence so that there exists  $M > 0$  such that  $|a_n| \leq M$  for all  $n \in \mathbb{N}$ . Bisect the closed interval  $[-M, M]$  into two parts  $[-M, 0]$  and  $[0, M]$ . Now, it must be that at least one of these closed intervals contains an infinite number of the terms in the sequence  $(a_n)$ . Select the half for which this is the case and label that interval as  $I_1$ . Then, let  $a_{n_1}$  be some term in the sequence  $(a_n)$  satisfying  $a_{n_1} \in I_1$ .

Now, bisect  $I_1$  in the same way, choose  $I_2$  as the interval with infinite terms in it. Since there are infinite terms, we can choose an  $a_{n_2}$  from the original sequence such that  $n_2 > n_1$  and  $a_{n_2} \in I_2 \dots$ . Continuing this fashion, we can form a nested interval  $\{I_n\}$  and select  $n_1 < n_2 < n_3 < \dots$  so that each  $a_{n_k} \in I_k$ .

Now we argue that  $(a_{n_k})$  is convergent. We first need to choose the candidate for the limit. By Nested Interval Property, there exists at least one  $x \in \bigcap_{k=1}^{\infty} I_k$ . We choose this  $x$  as the candidate.

Let  $\epsilon > 0$ . By construction, the length of  $I_k$  is  $M(1/2)^{k-1}$ , which converges to 0, as you can verify. Choose  $N$  so that  $k \geq N$  implies that the length of  $I_k$  is less than  $\epsilon$ . Because  $a_{n_k}$  and  $x$  are both in  $I_k$ , it follows that  $|a_{n_k} - x| < \epsilon$ .  $\square$

As you can see, we again use Archimedean Property (to show that  $M(1/2)^{k-1}$  converges to 0) and Supremum Property in the proof. This suggests that it can be seen as another Axiom of Completeness of real numbers. We do not show the proof of Bolzano-Weierstrass to Supremum Property here, since showing each connection within axioms would be extremely lengthy. Below we extend the relation graph again.

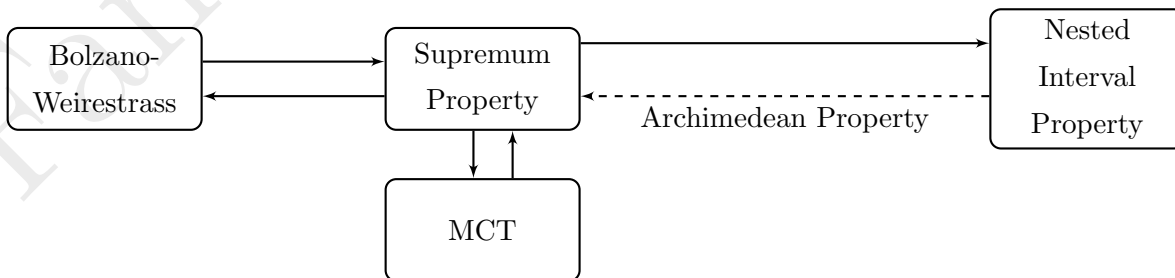


Figure 2.3: Relation between Axiom of Completeness

### 2.3.3 Axiom of Completeness V: Cauchy Criterion

When we consider the convergence of a sequence, we must first guess what is the limit first. However, there is a way to state the convergence without having any explicit knowledge of what the limit might be.

#### Definition 2.3.6: Cauchy Sequence

A sequence  $(a_n)$  is called a **Cauchy Sequence** if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \text{ s.t. } m, n > N \implies |a_n - a_m| < \epsilon$$

The definition just said that, when indices becomes larger, the distance between terms in sequence are getting closer and closer. It is clear that convergent sequences also have this property.

#### Proposition 2.3.7: Convergent Sequences are Cauchy

Every convergent sequence is a Cauchy sequence.

*Proof.* Suppose  $(a_n)$  is a convergent sequence. Since it is convergent, we have

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \text{ s.t. } m, n > N \implies |a_n - a| < \frac{\epsilon}{2} \quad \text{and} \quad |a_m - a| < \frac{\epsilon}{2}$$

Then, by triangular inequality, we have

$$|a_n - a_m| = |a_n - a + a - a_m| \leq |a_n - a| + |a_m - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

This shows that it is a Cauchy sequence. □

Is the inverse also true? The answer is yes. Before proving that, we first need a lemma.

#### Lemma 2.3.8: Boundedness of Cauchy sequence

Cauchy sequences are bounded.

*Proof.* Let  $\epsilon = 1$ . there exists  $N \in \mathbb{N}$  such that  $|x_m - x_n| < 1$  for all  $m, n > N$ . Therefore, let  $m = N + 1$ , we must have  $|x_n| = |x_n - x_{N+1} + x_{N+1}| < |x_{N+1}| + 1$  for all  $n > N$ , by triangular inequality. Take

$$M = \max\{|x_1|, |x_2|, |x_3|, \dots, |x_N|, |x_{N+1}| + 1\}$$

$(x_n)$  is then bounded by  $M$ . □

Now we are ready to prove the inverse direction. The idea is to use Bolzano-Weirestrass Theorem to find

a convergent subsequence. Then, we can resort to the limit of this subsequence, and prove that the whole sequence converges to this limit.

### Theorem 2.3.9: Cauchy Criterion

In  $\mathbb{R}$ , every Cauchy sequence converges.

*Proof.* Suppose  $(x_n)$  is Cauchy. By Lemma 2.3.8, it is bounded. By Bolzano-Weierstrass Theorem, it contains a convergent subsequence  $(x_{n_k})$ . Set

$$x = \lim x_{n_k}$$

Now we want to prove that the whole sequence converges to this limit. Let  $\epsilon > 0$ . Since  $(x_n)$  is Cauchy, there exists  $N_1$  such that

$$|x_n - x_m| < \frac{\epsilon}{2}$$

for all  $n, m > N_1$ . We also know that  $(x_{n_k}) \rightarrow x$ , so choose  $n_K > N_2$ , we will have

$$|x_{n_K} - x| < \frac{\epsilon}{2}$$

Now if we choose  $n > \max\{N_1, N_2\}$ , by triangular inequality,

$$|x_n - x| = |x_n - x_{n_K} + x_{n_K} - x| \leq |x_n - x_{n_K}| + |x_{n_K} - x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which shows that it converges to  $x$ . □

Note that this Cauchy Criterion may fail if it is not on  $\mathbb{R}$ . Consider a sequence on the set of rational numbers,  $\mathbb{Q}$ , and the field we are interested in is only the set of rational numbers. Let  $(a_n)$  be the sequence such that the  $n$ th term is the  $n$ th decimal approximation of  $\pi$ , i.e.,

$$a_1 = 3.1, \quad a_2 = 3.14, \quad a_3 = 3.141, \quad a_4 = 3.1415, \dots$$

Then, all terms in this sequence are rational numbers, but it converges to  $\pi$ , a rational number, which is not in the rational field. We then cannot say this sequence ‘converges’ if we only consider  $\mathbb{Q}$  as our field.

Therefore, Cauchy criterion implicitly reflects the completeness of real numbers, and it is indeed another Axiom of Completeness. **It is so important that it would be stated as a ‘defining property’ of (sequentially) compact sets.** However, this axiom, as Nested Interval Property, needs Archimedean Property to achieve Supremum Property. Thus, it is a ‘weak’ axiom.

Here we attach Cauchy criterion to our axiom relation graph.

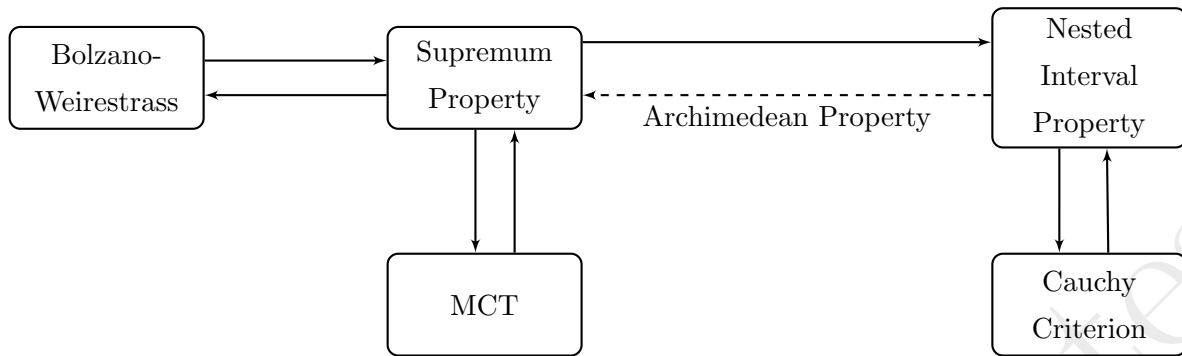


Figure 2.4: Relation between Axiom of Completeness

## 2.4 Convergence of Infinite Series

We are now transfer our sight from sequences to series. An **infinite series** is just the sum of all terms in a sequence.

### Definition 2.4.1: Infinite Series

Let  $(b_n)$  be a sequence. An **infinite series** is an expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \cdots$$

Notice that the result of an infinite series can be infinity. For example, for sequence  $(a_n) = (1, 1, 1, \dots)$ , its corresponding infinite series is then  $\sum_{n=1}^{\infty} 1 = \infty \times 1 = \infty$ . Therefore, a series can be convergent or divergent. Below we rigorously define the convergence of a series.

### Definition 2.4.2: Convergence of Series

For a series  $\sum_{n=1}^{\infty} b_n$ , we define the corresponding sequence of **partial sum**  $(s_m)$  as

$$s_m = \sum_{n=1}^m b_n$$

and say that series converges to  $x$  if the sequence  $(s_m)$  converges to  $x$ . We write  $\sum_{n=1}^{\infty} b_n = x$  in this case.

Now we see some classic examples of infinite series.



**Example 2.4.3: Harmonic Series**

The famous **harmonic series** is defined as

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

It is a well-known divergent series. Consider the corresponding partial sum

$$s_m = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}$$

Notice the pattern that

$$s_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 2$$

$$s_8 = 1 + \frac{1}{2} + \cdots + \frac{1}{8} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = \frac{5}{2}$$

$$\vdots$$

$$\begin{aligned} s_{2^k} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{k-1}+1} + \frac{1}{2^{k-1}+2} + \cdots + \frac{1}{2^k}\right) \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^k} + \frac{1}{2^k} + \cdots + \frac{1}{2^k}\right) \\ &= 1 + \underbrace{\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}}_{k \text{ terms total}} = 1 + \frac{k}{2} \end{aligned}$$

Therefore,  $s_m$  is not bounded, and the series does not converge.

**Example 2.4.4: Basel problem**

The Basel problem asks for the sum of the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Though we can not solve the precise sum yet, we can give an approximation of the upper bound.

Consider the partial sum

$$s_m = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{m^2}$$

Note that

$$s_m = 1 + \frac{1}{2} + \frac{1}{3 \times 3} + \frac{1}{4 \times 4} + \cdots + \frac{1}{m \times m}$$

$$\begin{aligned}
&< 1 + \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots + \frac{1}{m(m-1)} \\
&= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{m-1} - \frac{1}{m}\right) \\
&= 1 + 1 - \frac{1}{m} < 2
\end{aligned}$$

Therefore, 2 is an upper bound for the partial sum. By Monotone Convergence Theorem, the partial sum converges to some limit less than 2. We will show later that this sum equals  $\frac{\pi^2}{6}$ , which is a surprising result.

#### Example 2.4.5: Geometric Series

A series is called **geometric** if it is of the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots$$

Here we consider the case where  $|r| < 1$ . Since

$$(1-r)(1+r+r^2+r^3+\cdots+r^{m-1}) = 1-r^m$$

We can write the partial sum as

$$s_m = a + ar + ar^2 + \cdots + ar^{m-1} = \frac{a(1-r^m)}{1-r}$$

when  $m \rightarrow \infty$ , we can see that this partial sum converges to

$$\sum_{k=0}^{\infty} ar^k = \lim_{m \rightarrow \infty} s_m = \frac{a}{1-r}$$

Now we examine some properties of infinite series. We first note that since the convergence of infinite series is defined using convergence of sequences, we can immediately translate some result of sequence into statements about series.

#### Corollary 2.4.6: Algebraic Limit Theorem for Series

If  $\sum_{k=1}^{\infty} a_k = A$  and  $\sum_{k=1}^{\infty} b_k = B$ , then

1.  $\sum_{k=1}^{\infty} ca_k = cA, c \in \mathbb{R}$
2.  $\sum_{k=1}^{\infty} (a_k + b_k) = A + B$

The reason that another two statements in Algebraic Limit Theorem is not transformed is, when dealing with  $\lim s_n q_n$ , where  $s_n, q_n$  are partial sums of  $a_n$  and  $b_n$  respectively, it does not equal to  $\sum_{k=1}^{\infty} a_k b_k$ , so it is of no practical use.

Moreover, since in  $\mathbb{R}$ , Cauchy sequences are convergent, we can use the definition of Cauchy sequence to restate the definition of convergence of series.

#### Proposition 2.4.7: Cauchy Criterion for Series

The series  $\sum_{k=1}^{\infty} a_k$  converges if and only if,

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \quad \text{s.t.} \quad n > m > N \implies |a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon$$

*Proof.* Observe that

$$|s_n - s_m| = |a_{m+1} + a_{m+2} + \cdots + a_n|$$

Thus, apply Cauchy criterion, we can get the result. □

For a series to be convergent, its corresponding sequence must have a extremely small ‘tail’, so that we are adding smaller and smaller terms to make the size controlled not to be large.

#### Proposition 2.4.8: Tail of convergent series

If the series  $\sum_{k=1}^{\infty} a_k$  converges, then  $(a_k) \rightarrow 0$ .

*Proof.* Consider the special case  $n = m + 1$  in the Cauchy Criterion for series, we have

$$|a_n| < \epsilon$$

for all  $n > N$ . Therefore,  $(a_n) \rightarrow 0$ . □

**Note:** The converse is not true. Consider the harmonic series, its corresponding sequence  $(1/n)$  converges to 0, but the series itself diverges.

## 2.5 Convergence Tests for Series

### 2.5.1 Comparison Test

Sometimes it is very difficult to calculate the exact value of a infinite series. However, we have abundant tools to judge whether a series is convergent or not. One famous test is the **Comparison Test**, it uses another series as measuring stick to determine the convergency of another series.

**Proposition 2.5.1: Comparison Test**

Assume  $(a_k)$ ,  $(b_k)$  are sequences satisfying  $0 \leq a_k \leq b_k$  for all  $k \in \mathbb{N}$ ,

1. If  $\sum_{k=1}^{\infty} b_k$  converges, then  $\sum_{k=1}^{\infty} a_k$  converges.
2. If  $\sum_{k=1}^{\infty} a_k$  diverges, then  $\sum_{k=1}^{\infty} b_k$  diverges.

*Proof.*

1. If  $\sum_{k=1}^{\infty} b_k$  converges, then by Cauchy criterion, for all  $n > m > N$  where  $N \in \mathbb{N}$  we have

$$|b_{m+1} + b_{m+2} + \cdots + b_n| < \epsilon$$

Notice that

$$|a_{m+1} + a_{m+2} + \cdots + a_n| \leq |b_{m+1} + b_{m+2} + \cdots + b_n| < \epsilon$$

We can conclude that  $\sum_{k=1}^{\infty} a_n$  also converges.

2. This is just the contrapositive of (1). □

**Note:** The comparison test requires that the terms of series must be positive. These are called **positive infinite series**, and they are generally easier to manipulate than some arbitrary series.

Also, when we consider the limit of sequences and series, we are not very much interested in the first few terms, but only the tail of the sequence/series. Therefore, the condition in Comparison test can be relaxed to  $0 \leq a_k \leq b_k$  for all  $k > N$  with  $N \in \mathbb{N}$ .

A simple but important application of comparison tests is shown below.

**Corollary 2.5.2: Application of Comparison Test**

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if  $p > 1$ .

*Proof.* We consider the geometric series

$$\sum_{n=1}^{\infty} p^n$$

We have shown that it is convergent with  $p > 1$ . Choose large enough  $n$ , we will eventually have

$$\frac{1}{n^p} < \frac{1}{p^n}$$

Indeed, we can take  $n$  large enough so that  $\log n/n < \log p/p$  since the sequence  $\log n/n$  converges to 0, and the result follows. Then, by comparison test with relaxed condition, since geometric series converges, we have that  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  also converges.

Now take  $p \leq 1$ . Since  $1/n^p \geq 1/n$ , by comparison test and the fact that harmonic series diverges, we have  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  diverges.  $\square$

## 2.5.2 Absolute and Conditional Convergence

To deal with more general series, we need to first define two terminologies.

### Definition 2.5.3: Absolute and Conditional Convergence

- If  $\sum_{n=1}^{\infty} |a_n|$  converges, then we say that the original series  $\sum_{n=1}^{\infty} a_n$  **converges absolutely**.
- If the series  $\sum_{n=1}^{\infty} a_n$  converges but  $\sum_{n=1}^{\infty} |a_n|$  does not converge, we say that the original sequence  $\sum_{n=1}^{\infty} a_n$  **converges conditionally**.

It may be easily noted that if a series converges absolutely, then it must converge.

### Proposition 2.5.4: Absolute Convergence Test

If the series  $\sum_{n=1}^{\infty} |a_n|$  converges, then  $\sum_{n=1}^{\infty} a_n$  converges as well.

*Proof.* Since  $\sum_{n=1}^{\infty} |a_n|$  converges, by Cauchy criterion, given  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that

$$|a_{m+1}| + |a_{m+2}| + \cdots + |a_n| < \epsilon$$

for all  $n > m > N$ . By triangular inequality,

$$|a_{m+1} + a_{m+2} + \cdots + a_n| \leq |a_{m+1}| + |a_{m+2}| + \cdots + |a_n| < \epsilon$$

Therefore, the original series  $\sum_{n=1}^{\infty} a_n$  also converges.  $\square$

**Note:** The converse is not true. Below we will show that the alternating **harmonic series**

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

indeed converges. However, taking absolute value each term, we have the divergent harmonic series.

How should we prove that the alternating harmonic series converges? The next proposition gives a more general result.

### Proposition 2.5.5: Alternating Series Test

Let  $(a_n)$  be a sequence satisfying

1.  $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$
2.  $(a_n) \rightarrow 0$

Then, the alternating series  $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$  converges.

*Proof.* Let  $N \in \mathbb{N}$  be even and let  $n > N$ . Because the series is alternating and the terms is decreasing, we have

$$s_N \leq s_n \leq s_{N+1}$$

Since  $(a_n)$  converges to 0, we can make  $|s_{N+1} - s_N| = |a_N|$  be arbitrarily small when we increase  $N$ . Set  $N$  large enough so that  $|a_N| < \frac{\epsilon}{2}$ , we have

$$|s_m - s_n| \leq |s_m - s_N| + |s_N - s_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all  $n > m > N$ . Therefore,  $(s_n)$  is Cauchy, thus it converges. □

### 2.5.3 Root and Ratio Test

Before diving into the root test, we first introduce two extremely important concepts in analysis, the **limit superior** and the **limit inferior**.

#### Definition 2.5.6: Limit Superior/Inferior

- The **limit superior** of a sequence  $(x_n)$  is defined as

$$\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left( \sup_{m \geq n} x_m \right) = \inf_{n \geq 0} \left( \sup_{m \geq n} x_m \right)$$

- The **limit inferior** of a sequence  $(x_n)$  is defined as

$$\liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left( \inf_{m \geq n} x_m \right) = \sup_{n \geq 0} \left( \inf_{m \geq n} x_m \right)$$

The reason for the two definitions above  $\lim_{n \rightarrow \infty} (\sup_{m \geq n} x_m)$  and  $\inf_{n \geq 0} (\sup_{m \geq n} x_m)$  are equal is, the sequence  $(\sup_{m \geq n} x_m)_{n \in \mathbb{N}}$  is actually decreasing, as you should notice. Therefore, the inferior is equal to the limit.

Intuitively, limit superior and limit inferior is just the ‘largest’ and the ‘smallest’ number that some subsequence will converge to. For example, a sequence

$$a_n = \left( \frac{2}{3}, -\frac{2}{3}, 0, \frac{3}{4}, -\frac{3}{4}, 0, \frac{4}{5}, -\frac{4}{5}, 0, \frac{5}{6}, -\frac{5}{6}, 0, \dots \right)$$

There are three obvious convergent subsequences,

$$a_{3k-2} = \left( \frac{k+1}{k+2} \right)$$

$$a_{3k-1} = \left( -\frac{k+1}{k+2} \right)$$

$$a_{3k} = (0)$$

they will converge to 1, -1 and 0, respectively. Then  $\limsup_{n \rightarrow \infty} a_n = 1$  and  $\liminf_{n \rightarrow \infty} a_n = -1$ . This can also be verified using the definition.

Limit superior and limit inferior has many greater properties that limit does not generally have.

#### Proposition 2.5.7: Properties of Limit Superior/Inferior

1.  $\limsup$  and  $\liminf$  will always exist for bounded sequence.
2. For every bounded sequence,  $\liminf a_n \leq \limsup a_n$ , the equality is attained if and only if  $\lim a_n$  exists, and in this case  $\liminf a_n = \limsup a_n = \lim a_n$ .

*Proof.* 1. Since  $\sup_{m \geq n} a_m$  and  $\inf_{m \geq n} a_m$  are monotone sequences, by monotone convergence theorem, both of them will converge.

2. Since  $\inf_{m \geq n} a_m < \sup_{m \geq n} a_m$  for all  $n$ , by order limit theorem, we have  $\liminf a_n \leq \limsup a_n$ . Moreover, since  $\inf_{m \geq n} a_m \leq a_n \leq \sup_{m \geq n} a_m$ , by squeeze theorem, if  $\liminf a_n = \limsup a_n$ , then  $\lim a_n$  must exist. On the other hand, if  $\lim a_n$  exists, by cauchy criterion,

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \text{ s.t. } m, n > N \implies |a_n - a_m| < \epsilon$$

Then by the definition of superior and inferior, we should have

$$\left| \inf_{m \geq n} a_m - \sup_{m \geq n} a_m \right| \leq \epsilon$$

for all  $n > N$ . Since  $\epsilon > 0$  is arbitrary, we have  $\liminf a_n = \limsup a_n = \lim a_n$ .  $\square$

With these definitions set up, we can state and prove the famous root and ratio tests for convergence of series.

### Theorem 2.5.8: Root Test

Let  $\sum_{n=1}^{\infty} a_n$  be a series of real numbers, let  $\alpha = \limsup_{n \rightarrow \infty} |a_n|^{1/n}$ .

- (a) If  $\alpha < 1$ , then the series  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent.
- (b) If  $\alpha > 1$ , then the series  $\sum_{n=1}^{\infty} a_n$  is not convergent.
- (c) If  $\alpha = 1$ , we cannot assert any conclusion.

*Proof.* (a) Suppose  $\alpha < 1$ . We must have  $\alpha > 0$  since  $|a_n|^{1/n} \geq 0$  for all  $n$ . Then, we can find  $\epsilon > 0$  such that  $0 < \alpha + \epsilon < 1$ . Since

$$\alpha = \limsup_{n \rightarrow \infty} |a_n|^{1/n} = \inf_{n \geq 0} \left( \sup_{m \geq n} |a_m|^{1/m} \right)$$

By the definition of superior and inferior, and the fact that  $\sup_{m \geq n} |a_m|^{1/m}$  is decreasing, there must exist  $N \in \mathbb{N}$  such that

$$|a_n|^{1/n} \leq \alpha + \epsilon \implies |a_n| \leq (\alpha + \epsilon)^n$$

for all  $n \geq N$ . But from the geometric series we have that

$$\sum_{n=N}^{\infty} (\alpha + \epsilon)^n$$

is absolutely convergent, since  $0 < \alpha + \epsilon < 1$ . Thus by the comparison test, we see that  $\sum_{n=N}^{\infty} a_n$  is absolutely convergent, and thus  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent since first few terms will not influence the convergence of series.

- (b) Now suppose that  $\alpha > 1$ . Then again, By the definition of superior and inferior, and the fact that  $\sup_{m \geq n} |a_m|^{1/m}$  is decreasing, there exists  $N \in \mathbb{N}$  such that

$$|a_n|^{1/n} > 1 \implies |a_n| > 1$$

By the contrapositive of proposition 2.4.8, since  $(a_n)$  does not converge to 0, the corresponding series then does not converge.  $\square$



The root test is phrased using limit superior, but of course if  $|a_n|^{\frac{1}{n}}$  converges we can state it using just limit. Now we state and prove the **Ratio Test**.

### Theorem 2.5.9: Ratio Test

Let  $\sum_{n=1}^{\infty} a_n$  be a series of real numbers such that  $a_n \neq 0$ , let

$$r = \limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

- (a) If  $r < 1$ , then the series  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent.
- (b) If  $r > 1$ , then the series  $\sum_{n=1}^{\infty} a_n$  is not convergent.
- (c) If  $r = 1$ , we cannot assert any conclusion.

*Proof.* (a) Suppose  $r < 1$ . Let  $r'$  satisfies  $r < r' < 1$ . We have

$$r = \limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left( \sup_{m \geq n} \left| \frac{a_{m+1}}{a_m} \right| \right) < 1$$

By the definition of superior, and the fact that  $\sup_{m \geq n} \left| \frac{a_{m+1}}{a_m} \right|$  is decreasing, we can find  $N \in \mathbb{N}$  such that  $n \geq N$  implies

$$\left| \frac{a_{n+1}}{a_n} \right| \leq r' \implies |a_{n+1}| \leq |a_n| r'$$

Since the geometric series  $|a_N| \sum_{n=N}^{\infty} (r')^n$  converges, by comparison test,

$$|a_{N+1}| \leq r' |a_N|$$

$$|a_{N+2}| \leq r' |a_{N+1}| \leq (r')^2 |a_N|$$

$$|a_{N+3}| \leq (r')^3 |a_N|$$

$$\vdots$$

$$|a_{N+n}| \leq (r')^n |a_N|$$

we have that  $\sum_{n=N}^{\infty} |a_n|$  also converges. Therefore,  $\sum_{n=1}^{\infty} |a_n|$  is convergent, the original sequence is absolutely convergent.

- (b) If  $r > 1$ , By the definition of superior, and the fact that  $\sup_{m \geq n} \left| \frac{a_{m+1}}{a_m} \right|$  is decreasing, we can find

$N \in \mathbb{N}$  such that  $n \geq N$  implies

$$\left| \frac{a_{n+1}}{a_n} \right| \geq 1 \implies |a_{n+1}| \geq |a_n|$$

for all  $n > N$ . Since  $a_n \neq 0$  and  $|a_n| > 0$  for all  $n$ , we conclude that  $(a_n)$  cannot converge to 0. Therefore, the corresponding series does not converge.  $\square$

### 2.5.4\* Cauchy Condensation Test, Abel's Test and Dirichlet's Test

To end this section, we introduce another three tests for convergence of series. Based on the proof process of divergency of harmonic series, we state the following general argument.

#### Theorem 2.5.10: Cauchy Condensation Test

Suppose  $(b_n)$  is decreasing and satisfies  $(b_n) \geq 0$  for all  $n \in \mathbb{N}$ . Then, the series  $\sum_{n=1}^{\infty} b_n$  converges if and only if the series

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + 8b_8 + 16b_{16} + \cdots$$

is convergent.

*Proof.*

- ( $\implies$ ) First assume that  $\sum_{n=0}^{\infty} 2^n b_{2^n}$  converges. Then, by Proposition 2.2.2, the partial sum

$$t_k = b_1 + 2b_2 + \cdots + 2^k b_{2^k}$$

is bounded. That is, there exists  $M > 0$  such that  $t_k \leq M$  for all  $k \in \mathbb{N}$ . We want to prove that  $\sum_{n=1}^{\infty} b_n$  converges. Since  $b_n \geq 0$ , we know that the partial sum  $s_m$  for this sequence is increasing. By Monotone Convergence Theorem, we only need to prove  $s_m = b_1 + b_2 + \cdots + b_m$  is bounded.

Fix  $m$  and let  $k$  be large enough to ensure  $m \leq 2^{k+1} - 1$ . Then,  $s_m \leq s_{2^{k+1}-1}$  and

$$\begin{aligned} s_{2^{k+1}-1} &= b_1 + (b_2 + b_3) + (b_4 + b_5 + b_6 + b_7) + \cdots + (b_{2^k} + \cdots + b_{2^{k+1}-1}) \\ &\leq b_1 + (b_2 + b_2) + (b_4 + b_4 + b_4 + b_4) + \cdots + \underbrace{(b_{2^k} + \cdots + b_{2^k})}_{2^k \text{ terms}} = t_k \end{aligned}$$

Therefore,  $s_m \leq t_k \leq M$ . the partial sum is bounded. By monotone convergence theroem, we have that  $\sum_{n=1}^{\infty} b_n$  is convergent.

- ( $\impliedby$ ) We now prove that if  $\sum_{n=1}^{\infty} b_n$  converges, then  $\sum_{n=0}^{\infty} 2^n b_{2^n}$  converges. Instead of proving

this, we prove its contrapositive. Suppose  $\sum_{n=0}^{\infty} 2^n b_{2^n}$  diverges. Since the partial sum  $t_k$  is increasing, by monotone convergence theorem, it is unbounded (otherwise it is convergent). Therefore, for any  $2M > 0$  we can find  $k \in \mathbb{N}$  such that  $t_k > 2M$ . Therefore, with the fact that  $(b_n)$  is decreasing,

$$\begin{aligned} s_{2^k} &= b_1 + b_2 + \cdots + b_{2^k} \geq b_1 + b_2 + (b_4 + b_4) + (b_8 + b_8 + b_8 + b_8) + \cdots + \underbrace{(b_{2^k} + \cdots + b_{2^k})}_{2^{k-1} \text{ terms}} \\ &= b_1 + b_2 + 2b_4 + 4b_8 + \cdots + 2^{k-1}b_{2^k} = \frac{1}{2}t_k + \frac{1}{2}b_1 > M \end{aligned}$$

Since  $M > 0$  is arbitrary, we have that  $s_{2^k}$  is unbounded. Therefore, the series  $\sum_{n=1}^{\infty} b_n$  is not convergent.  $\square$

There are two more tests that is very useful in judging whether a series of the form  $\sum_{n=1}^{\infty} f(n)g(n)$  converges or not. For example, we may want to judge whether

$$\sum_{n=1}^{\infty} \frac{\cos(n)}{n}$$

converges or not. Then, the terms can be separated into  $\frac{1}{n}$  and  $\cos(n)$ , and these theorems can be applied.

### Theorem 2.5.11: Abel's Test

Suppose  $\sum_{n=1}^{\infty} a_n$  converges, and  $(b_n)$  is monotone and bounded. Then, the series

$$\sum_{n=1}^{\infty} a_n b_n$$

is convergent.

*Proof.* Since  $(b_n)$  is monotone and bounded, by monotone convergence theorem, it converges. Suppose  $(b_n) \rightarrow b$ . Let  $a_0 = 0$ ,  $s_m = \sum_{n=0}^m a_n$ . Then,  $a_m = s_m - s_{m-1}$  for all  $m \geq 1$ . We can first consider the finite sum, and decompose the target sequence as

$$\begin{aligned} \sum_{n=1}^N a_n b_n &= \sum_{n=1}^N b_n (s_n - s_{n-1}) \\ &= b_1 s_1 + b_2 s_2 - b_2 s_1 + b_3 s_3 - b_3 s_2 + \cdots + b_N s_N - b_N s_{N-1} \\ &= s_1(b_1 - b_2) + s_2(b_2 - b_3) + \cdots + s_{N-1}(b_{N-1} - b_N) + s_N b_N \end{aligned}$$

$$= \left( \sum_{n=1}^{N-1} s_n(b_n - b_{n+1}) \right) + s_N b_N$$

Since  $(s_N)$  is convergent, and  $(b_N)$  is also convergent, by algebraic limit theorem, we have  $(s_N b_N)$  must converge. Moreover, since  $s_N$  is convergent, it is bounded by some  $M > 0$ , we see that the sum is telescoping

$$\sum_{n=1}^{N-1} s_n(b_n - b_{n+1}) \leq M \sum_{n=1}^{N-1} (b_n - b_{n+1}) = M(b_1 - b_N)$$

Since  $b_N$  is convergent, by algebraic limit theorem,  $\sum_{n=1}^{N-1} s_n(b_n - b_{n+1})$  also converges. These will finally show that

$$\lim_{n \rightarrow \infty} \sum_{n=1}^N a_n b_n = \sum_{n=1}^{\infty} a_n b_n = \left( \sum_{n=1}^{N-1} s_n(b_n - b_{n+1}) \right) + s_N b_N$$

is convergent. □

Now we change our mind a little bit. In the previous theorem we get the convergence from the convergence of  $\sum_{n=1}^{\infty} a_n$ . Now we relax this condition mildly, to only require the partial sums to be bounded. As a compensation, we strengthen the another assumption so that  $(b_n)$  is not only monotone and bounded, but also converges to 0. **Note how the term  $s_N b_N$  is not necessarily convergent if we relax the assumption of  $a_n$  in Abel's test, or relax assumption of  $b_n$  in Dirichlet's Test.**

### Theorem 2.5.12: Dirichlet's Test

Suppose that for two sequences  $(a_n)$  and  $(b_n)$ , we have  $\sum_{n=1}^N a_n$  is bounded for all  $N$  (but the series does not necessarily need to be convergent), i.e.,

$$\left| \sum_{n=1}^N a_n \right| \leq M \quad \text{for all } N \in \mathbb{N}$$

and  $(b_n)$  is monotone so that  $\lim b_n = 0$ . Then, the series

$$\sum_{n=1}^{\infty} a_n b_n$$

is convergent.

*Proof.* Similarly with the last proof, we can write the series as

$$\sum_{n=1}^N a_n b_n = \left( \sum_{n=1}^{N-1} s_n(b_n - b_{n+1}) \right) + s_N b_N$$

Since  $s_N$  is bounded by  $M$ , and  $b_N$  converges to 0, we have

$$\lim_{N \rightarrow \infty} s_N b_N \leq M \lim_{N \rightarrow \infty} b_N = 0$$

is convergent. For the telescoping term, we still have

$$\sum_{n=1}^{N-1} s_n(b_n - b_{n+1}) \leq M \sum_{n=1}^{N-1} (b_n - b_{n+1}) = M(b_1 - b_N)$$

and it converges to  $M(b_1 - 0) = Mb_1$ . Therefore, the whole partial sum  $\sum_{n=1}^N a_n b_n$  converges.  $\square$

## 2.6 Rearrangement of Series

We see in Example 1.1.3 of the opening section of this whole note that some series will change its value after reordering the terms. To refresh your memory, we will restate it here.

### Example 2.6.1. Reordering Infinite Series

Consider the alternating harmonic series

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

We know that this infinite series converges at some point. Therefore, nothing similar as Example 1.1.1 could happen here. However, if we do the following computation:

$$\begin{aligned} \frac{1}{2}S &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} + \cdots \\ S &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \frac{1}{14} + \frac{1}{15} - \frac{1}{16} + \cdots \end{aligned}$$

$$\frac{3}{2}S = \left(1 + \frac{1}{3}\right) - \frac{1}{2} + \left(\frac{1}{5} + \frac{1}{7}\right) - \frac{1}{4} + \left(\frac{1}{9} + \frac{1}{11}\right) - \frac{1}{6} + \left(\frac{1}{13} + \frac{1}{15}\right) - \frac{1}{8} + \cdots$$

We see that  $\frac{3}{2}S$  is just a reordering of our initial infinite series (with two positive terms following one negative term)! Therefore, we just change the convergent point by simply reordering the infinite series.

We will rigorously define what is a rearrangement first.

### Definition 2.6.1: Rearrangement

Let  $\sum_{n=1}^{\infty} a_n$  be a series. A series  $\sum_{n=1}^{\infty} b_n$  is a **rearrangement** of  $\sum_{n=1}^{\infty} a_n$  if there exists a bijective

function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that

$$b_{f(k)} = a_k$$

for all  $k \in \mathbb{N}$ .

Now we are ready to see why this happens. It can be seen that for the alternating harmonic series, only taking the sum of positive terms or only take the sum of negative terms will both lead to a divergent series. This means that, we can reach ‘any far’ on the real line by choosing terms in this series and add them. **This only happens when the convergence is conditional.**

### Theorem 2.6.2: Rearrangement Criterion

If a series converges absolutely, then any rearrangement of this series converges to the same limit.

*Proof.* Assume  $\sum_{k=1}^{\infty} a_k$  converges absolutely to  $A$ , and let  $\sum_{n=1}^{\infty} b_k$  be a rearrangement of  $\sum_{n=1}^{\infty} a_k$ . Denote  $s_n, t_m$  as the partial sum of  $(a_k)$  and  $(b_k)$ , respectively. We want to show that  $(t_m) \rightarrow A$ . Let  $\epsilon > 0$ . Since  $(s_n) \rightarrow A$ , we can have  $N_1 \in \mathbb{N}$  so that

$$|s_n - A| < \frac{\epsilon}{2}$$

for all  $n \geq N_1$ . Because the convergence is absolute, we can choose  $N_2 \in \mathbb{N}$  so that

$$\sum_{k=m+1}^n |a_k| < \frac{\epsilon}{2}$$

for all  $n > m \geq N_2$ . Take  $N = \max\{N_1, N_2\}$ . We know that the finite set of terms  $\{a_1, a_2, a_3, \dots, a_N\}$  must all appear in the rearranged series since rearrangement is a bijection, and we want to move far enough out in the series  $\sum_{n=1}^{\infty} b_n$  so that we have included all these terms. Thus, choose

$$M = \max\{f(k) : 1 \leq k \leq N\}$$

It should be now evident that if  $m \geq M$ , then  $(t_m - s_N)$  consists of a finite set of terms, and the absolute values of which appear in the tail  $\sum_{k=m+1}^{\infty} |a_k|$ . Thus the choice of  $N_2$  guarantees that  $|t_m - s_N| < \frac{\epsilon}{2}$ , and

$$|t_m - A| = |t_m - s_N + s_N - A| \leq |t_m - s_N| + |s_N - A| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all  $m > M$ , which shows that  $(t_n) \rightarrow A$ . □

## 2.7\* Cauchy Sequence Method: Completion from $\mathbb{Q}$ to $\mathbb{R}$

**Note:** This session is just an introduction to this topic. There is no full detail since it needs tedious proof procedure just like the Dedekind cut, and it needs many prerequisites from abstract algebra.

To end this whole session, we mention another different way from Dedekind Cut, to construct real numbers from rational numbers. The main tool we use here is the Cauchy Sequences.

Here is the intuitive idea. We know that all convergent sequences are Cauchy. However, the inverse would only be true in complete spaces such as  $\mathbb{R}$ . This is because some Cauchy sequence will converge to a point that is outside of the space (an example is given in the corresponding Cauchy sequence chapter). In our case, some Cauchy sequence in rational field will converge to a point in real number field, and we cannot say that is convergent in this sense. However, if we define these points using the ‘limiting property’ of Cauchy sequence, i.e., two points in the sequence is getting closer and closer, we can actually define those points using Cauchy sequences!

Before we formally start to prove that, we need to first introduce some terminologies that is maybe too early to appear. The definition of limit and convergence depends on the definition of ‘distance’ between points. For example, on the real line, we measure the distance between two points  $x$  and  $y$  by calculating the absolute value  $|x - y|$ . In any other space, if we can define such a distance function on some general space  $X$  to be  $d : X \times X \rightarrow [0, \infty)$ , which takes elements in  $X$  and take values on nonnegative real numbers, we can also perform convergence analysis on it. To make this distance function ‘regular’ (i.e., not pathological) enough, we assume

- $d(x, x) = 0, x \in X$
- $d(x, y) = d(y, x), x, y \in X$
- $d(x, z) \leq d(x, y) + d(y, z), x, y, z \in X$

This is just the idea of **Metric Space**, which will formally appear in PART III of this note. A distance function is called a **metric**.

The second terminology is **isometry**. Intuitively, two spaces are isometry if they have exactly the same topological properties, but just with different notations. We define formally, an **isometry**  $\phi : X \rightarrow \tilde{X}$ , is a function from a space to another space such that  $\tilde{d}(\phi(x), \phi(y)) = d(x, y)$ , where  $d$  and  $\tilde{d}$  are metrics on  $X$  and  $\tilde{X}$ , respectively. Two spaces are **isometric** if there exists such a isometry between them.

Finally, we will introduce the **equivalence relation**, which will often appear in the context of abstract algebra. A **binary relation**  $R$  on a set  $X$  is a subset of  $X \times X$ . Two elements  $x, y \in X$  are equivalent if

$(x, y) \in R$ , denoted by  $x \sim y$ . For example, we can define a relation ' $\geq$ ' on the set  $\mathbb{R}$  so that it is the set  $\{(x, y) : y \geq x, x, y \in \mathbb{R}\}$ . Then, an **equivalence relation** will have properties

- **Reflexive:**  $x \sim x, \forall x \in X$
- **Symmetric:**  $x \sim y \Leftrightarrow y \sim x, \forall x, y \in X$
- **Transitive:**  $x \sim y, y \sim z \Rightarrow x \sim z, \forall x, y, z \in X$

An **equivalence class**  $[x]$  is then a set of all equivalence element to some  $x$ , i.e.,  $[x] = \{y \in X : x \sim y\}$ .

With all these three tools, we can state our result:

**Theorem 2.7.1: Cauchy Sequence construction of  $\mathbb{Q}$  to  $\mathbb{R}$**

There exists a complete space  $\mathbb{R}$  with a properly defining metric such that it has a subspace  $W$  which is isometric with  $\mathbb{Q}$ , and is dense in  $\mathbb{Q}$ . The space  $\mathbb{R}$  is unique up to isometry.

By 'unique up to isometry', we mean that if two spaces satisfy the conditions, and they are isometric, we see them as the same. Now we prove the theorem. It is actually way more simple and concise than Dedekind Method.

Define the equivalence relation: Let  $(x_n), (x'_n)$  be rational Cauchy sequences. Define  $(x_n) \sim (x'_n)$  if

$$\lim_{n \rightarrow \infty} |x_n - x'_n| = 0$$

Let  $\mathbb{R}$  be defined as the set of all equivalence classes of Cauchy sequences. Then, as in the procedure of Dedekind Cut, we can prove all the properties of real numbers, including the supremum property.

**Note:** Here we construct  $\mathbb{R}$  using the metric  $|x - y|$ . For other metrics, we can construct other complete spaces called the **p-adic numbers**.



## Chapter 3

# Topology on the Real Line

Here we move on to another interesting topic of mathematical analysis, the **point set topology**, which sees sets as geometric objects and analyze their properties such as closedness, compactness, separability and connectedness, etc. In this chapter, we mainly focus on the real line. We consider the real line as a geometric object, and consider those topological properties on it.

### 3.1 Open and Closed Sets

#### 3.1.1 Open Sets

To define what is an open set, we first introduce the interior point.

##### Definition 3.1.1: Interior Point

A point  $a$  is an **interior point** of a set  $A$  if

$$\exists \epsilon > 0, \text{ s.t. } (a - \epsilon, a + \epsilon) \subseteq A$$

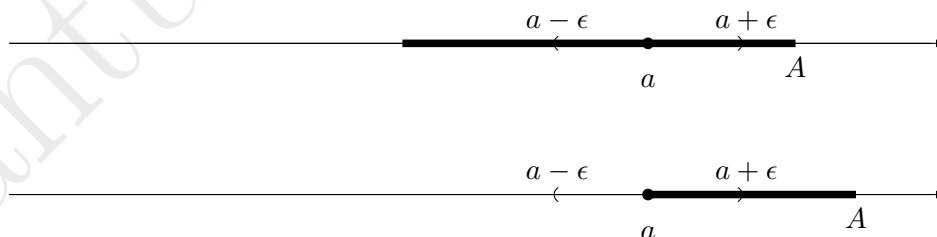


Figure 3.1: Up: Interior Point Down: Not Interior Point

As drawn in the figure, if the point is in the ‘interior’ of the set, there would always be a small  $\epsilon$  region that is completely contained in the set. However, if the point is at the boundary, then half of the  $\epsilon$  region

would always be out of the set. This explains why it is called interior point.

### Definition 3.1.2: Open Set

A set  $G$  is **open** if all points  $a \in G$  are interior points.

#### Example 3.1.1.

- $\mathbb{R}$  is an open set. For any point  $a \in \mathbb{R}$ , we are free to choose  $\epsilon$  and always  $(a - \epsilon, a + \epsilon) \subseteq \mathbb{R}$ .
- Open intervals are open sets. Consider open interval  $(a, b)$ . For any point  $x \in (a, b)$ , we can take  $\epsilon = \min\{x - a, b - x\}$ , i.e., we choose the smallest distance from  $x$  to the boundary of this open interval. Then  $(x - \epsilon, x + \epsilon) \subseteq (a, b)$ .
- Closed intervals are not open set. Consider closed interval  $[a, b]$ . Choose  $a \in [a, b]$ , for all  $\epsilon > 0$ , the set  $(a - \epsilon, a + \epsilon)$  will always fall out the interval.

An important result is that union of open intervals is open. The next theorem states this fact. It is very important so that it is later introduced as the defining property of a general topological space.

### Theorem 3.1.3: Topological property of open set

1. The union of **arbitrary collection** of open sets is open.
2. The intersection of a **finite collection** of open sets is open.

*Proof.*

1. Let  $\{G_\lambda : \lambda \in I\}$  be a collection of open sets and let  $G = \bigcup_{\lambda \in I} G_\lambda$ . Let  $a \in G$ . Then, we have  $a \in G_{\lambda'}$  for some specific  $\lambda' \in I$ . Since  $G_{\lambda'}$  is open, there exists  $\epsilon > 0$  such that

$$(a - \epsilon, a + \epsilon) \subseteq G_{\lambda'} \subseteq G$$

Therefore,  $G$  is open.

2. Let  $\{G_1, G_2, G_3, \dots, G_n\}$  be finite collection of open sets. If  $a \in \bigcap_{k=1}^n G_k$ , then  $a \in G_k$  for all  $k$ . Since all  $G_k$  are open sets, we can find a  $\epsilon_k$  for each set  $G_k$  such that  $(a - \epsilon_k, a + \epsilon_k) \subseteq G_k$ . Take  $\epsilon = \min\{\epsilon_1, \epsilon_2, \dots, \epsilon_n\}$ . It follows that  $(a - \epsilon, a + \epsilon) \subseteq (a - \epsilon_k, a + \epsilon_k) \subseteq G_k$  for all  $k$  and thus  $(a - \epsilon, a + \epsilon) \subseteq \bigcap_{k=1}^n G_k$ .

This follows that  $\bigcap_{k=1}^n G_k$  is open. □

Note that the intersection must be finite, since in the case such as

$$G = \bigcap_{n=1}^{\infty} \left( -\frac{1}{n}, \frac{1}{n} \right) = \{0\}$$

is not open. This can also be shown in the proving process that we are allowed to choose the minimum of all the  $\epsilon_k$ , which does not always exist in infinite case.

### 3.1.2 Closed Sets

For closed point, there is also a kind of point that is related to.

#### Definition 3.1.4: Limit Point

A point  $x$  is a **limit point (accumulation point)** of a set  $A$  if

$$\forall \epsilon > 0, (a - \epsilon, a + \epsilon) \cap (A \setminus \{a\}) \neq \emptyset$$

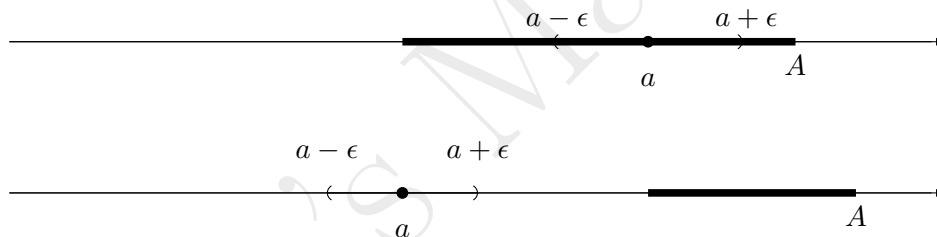


Figure 3.2: Up: Limit Point Down: Not Limit Point

As showed in the figure, if the point has other points ‘accompanied closely’ to the point  $a$ , then it is a cumulative point. If it is completely isolated, it is not.

The reason that it is called limit point, is that this point can be a limit of a sequence in this set.

#### Proposition 3.1.5: Limit Point Characterization

A point  $x$  is a limit point of  $A$  if and only if  $x = \lim a_n$  for some  $(a_n) \in A$  satisfying  $a_n \neq x$  for all  $n \in \mathbb{N}$ .

*Proof.*

□

### 3.2 Compactness and Axiom of Completeness VI: Heine-Borel Theorem

### 3.3 Perfect Sets and Connected Sets

### 3.4 The Cantor Set

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## Chapter 4

# Continuity of Functions on $\mathbb{R}$

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## Part II

# PART II: Calculus on the Real Line





## Part III

# PART III: Metric Space



## Part IV

# PART IV: Calculus on the Real Space



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