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Fantuan's Math Notes series

Notes on Real Analysis

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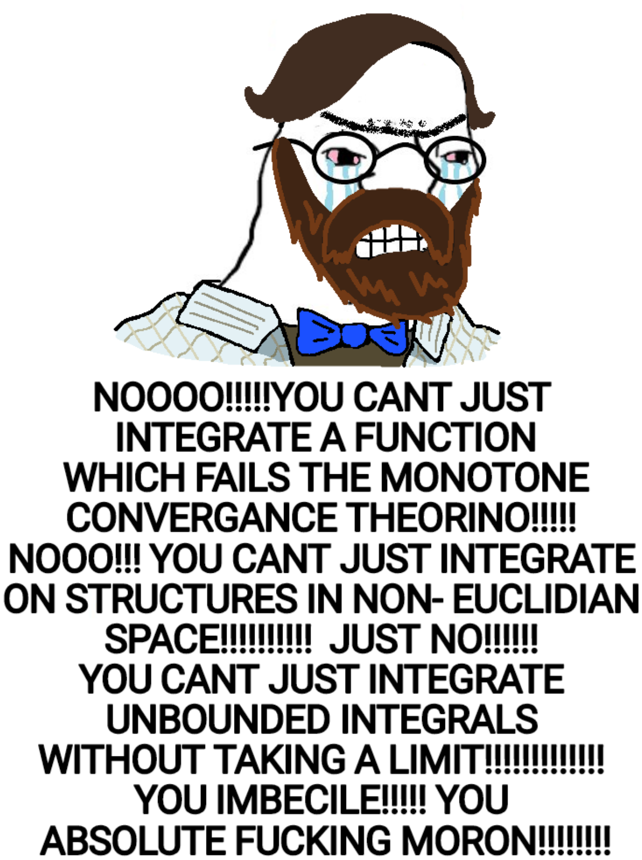

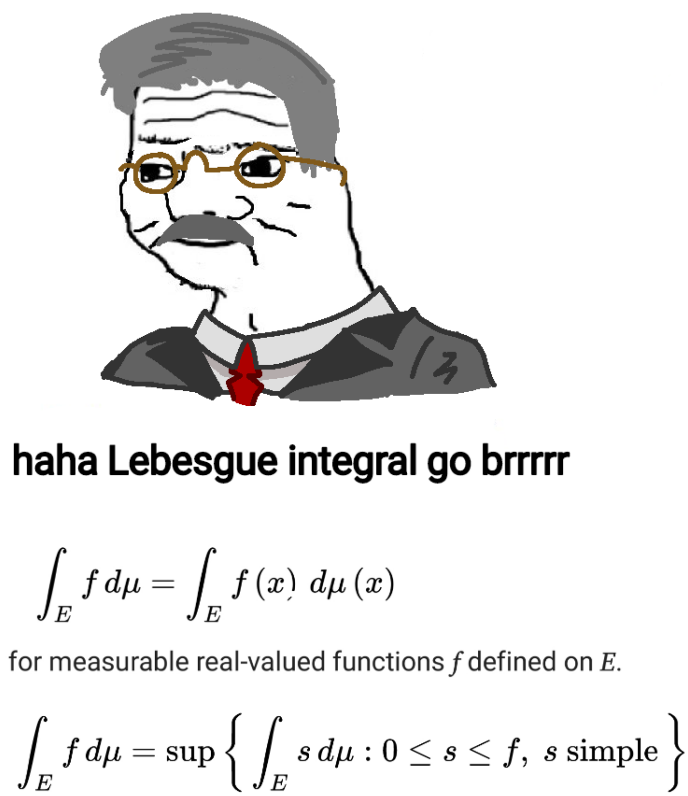

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Contents

All the Sections with * are hard sections and can be skipped without losing coherence.

This note is referenced on **Introduction to Real Analysis** by Christopher Heil [3], **Real Analysis: Modern Techniques and Applications** by Folland [2], **Real Analysis** by Stein [4], **Measure, Integration and Real Analysis** by Sheldon Axler[1] (the famous 'Linear Algebra Done Right' Author!), and **Real and Complex Analysis** by Walter Rudin[5].

Part I

PART I: Measure and Integration

Chapter 1

Lebesgue Measure

1.1 Why Real Analysis?

What is this note about if we already have the introductory mathematical analysis? The main reason is that **Riemann Integral** has many deficiencies so that we need a more rigorous approach to the integration theory to solve these problems.

Example 1.1.1: Some Functions are not Riemann Integrable

The **Dirichlet Function** $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

is not Riemann Integrable, since for any partition interval $[a,b] \subseteq [0,1]$ with $a < b$, we will have

$$\inf_{[a,b]} f = 0 \text{ and } \sup_{[a,b]} f = 1$$

The supremum of lower Riemann Sum and infimum of Upper Riemann Sum would never be equal to each other. Therefore, the integral does not exist. However, consider that $\mathbb{Q}$ is countable and $\mathbb{R} \backslash \mathbb{Q}$ is uncountable, we will reasonably guess that the 'area' under this curve should be 0 in some sense. So we need to fix this.

Example 1.1.2: Riemann Integral dos not preserve pointwise limit

Let $r_1, r_2, \cdots$ be a sequence that includes all rational numbers in $[0,1]$ exactly once. For $k \in \mathbb{N}$, define$f_k : [0,1] \to \mathbb{R}$ by

$$f_k(x) = \begin{cases} 1, & \text{if } x \in \{r_1, \cdots, r_k\} \\ 0, & \text{Otherwise} \end{cases}$$

Each f_k is Riemann Integrable, but the pointwise limit is the Dirichlet Function, which is not integrable!

You can see that Riemann Integral does not behave very well on pathological functions. In this note, we will fix these problems using a new integral method called the **Lebesgue Integral**, and derive new theories based on this.

The reason that this topic is called 'Real Analysis', is we mainly focus on the analysis of **Real-Valued Functions**.

Definition 1.1.3: Real-Valued Function

A **real-valued function** is a function $f : X \to \bar{\mathbb{R}}$, where X is an arbitrary set and $\bar{\mathbb{R}}$ is the extended real line $[-\infty, \infty]$.

We will also consider some **Complex-Valued Functions**, because its real and imaginary parts can be written as real-valued functions.

1.2 Exterior Measure

The first thing we need to do is to assign a 'size', or 'volume' to each set. You may think this is trivial. However, some pathological sets such us $\mathbb{R}\backslash\mathbb{Q}$, has no intuitive concept of 'volume'. To do this, we first start from very simple sets, then construct volume for each set based on this simple one. The basic element we use is **box**.

1.2.1 Box

Definition 1.2.1: Box

- A **box** in $\mathbb{R}$ is a Cartesian product of d finite closed intervals

$$Q = \prod_{i=1}^{d} [a_i, b_i] = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d], \quad a_j < b_j, \forall j$$

- The **volume** of the box is the product of lengths of each side,

$$|Q| = \prod_{i=1}^{d}(b_i - a_i) = (b_1 - a_1)(b_2 - a_2)\cdots(b_d - a_d)$$

- The **interior** of the box is the Cartesian Product

$$Q^\circ = \prod_{i=1}^{d}(a_i, b_i) = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_d, b_d), \quad a_j < b_j, \forall\, j$$

- The **boundary** of the box is $\partial Q = Q \backslash Q^\circ$.

- If the sides of a box have equal length, then we call it a **cube**.

The reason we use closed intervals is that, when constructing other volumes from boxes, we can 'overlap' the boundary of boxes, and the volumes can still be added up normally. With open intervals, we cannot easily do that. We give this kind of overlap a terminology.

Definition 1.2.2: Nonoverlapping Box

A collection of boxes $\{Q_k\}_{k \in I}$ is **nonoverlapping** if there interiors are disjoint. i.e.,

$$j \neq k \in I \quad \Longrightarrow \quad Q_j^\circ \cap Q_k^\circ = \emptyset$$

With boxes, we can start to construct the 'volume' of other sets. We first do the very simple case, which is a box constructed by union of finitely many nonoverlapping boxes. Intuitively, in this case the volume can just be added up. It is true, but the proof may not be as trivial as you might think.

Proposition 1.2.3: Union of Finitely Many Boxes

Let

$$Q = \prod_{i=1}^{d}[a_i, b_i]$$

be a box in $\mathbb{R}^d$. Suppose $\{Q_i\}_{i=1}^{n}$ are finitely many nonoverlapping boxes such that

$$Q = \bigcup_{i=1}^{n} Q_i$$

Then, the volume add up, i.e.,

$$|Q| = \sum_{i=1}^{n} |Q_i|$$

Proof. We begin by thinking the union of these boxes as a 'jigsaw-like' decomposition of the big box Q, as shown in left of Figure 1.1 below (for convenience, we draw the picture in $\mathbb{R}^2$ case).

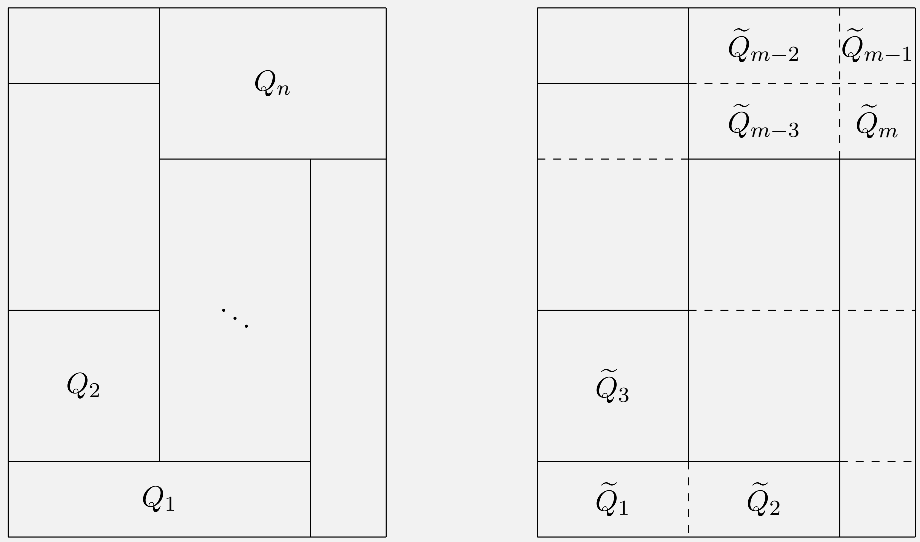

Figure 1.1: Jigsaw-like nonoverlapping union in $\mathbb{R}^2$. Left: Original Grid. Right: Grid after extension

Then, we can 'extend' these sides of boxes so that they decompose to many smaller 'grid-like' nonoverlapping boxes $\{\widetilde{Q}_i\}_{i=1}^m$. This forms a partition $J_1, J_2, \cdots, J_n$ such that

$$Q = \bigcup_{i=1}^{m} \widetilde{Q}_i, \quad \text{and} \quad Q_k = \bigcup_{j \in J_k} \widetilde{Q}_j$$

For example, in the figure above, $Q_1 = \widetilde{Q}_1 \cup \widetilde{Q}_2$, thus $J_1 = \{1, 2\}$. Note that $\{\widetilde{Q}_k\}$ are also nonoverlapping boxes. In this extended-grid case, adding volumes is just adding each sides and then times them all up, just as what we did in 2-dimensional case. For example, suppose in the figure above, $Q_1 = [0,2] \times [0,1]$ and $\widetilde{Q}_1 = [0,1] \times [0,1]$ and $\widetilde{Q}_2 = [1,2] \times [0,1]$. Then, their volume add up because $[0,2] \times [0,1] = ((1-0) + (2-1)) \times 1 = (1-0) \times (1-0) + (2-1) \times (1-0) = |\widetilde{Q}_1| + |\widetilde{Q}_2|$. Therefore,

$$|Q| = \sum_{j=1}^{m} |\widetilde{Q}_j| = \sum_{k=1}^{n} \sum_{j \in J_k} |\widetilde{Q}_j| = \sum_{i=1}^{n} |Q_i|$$

$\square$

With a little bit more thoughts, we can extend this result into overlapping case.

Proposition 1.2.4: Union of Overlapping Boxes

If $\{Q_i\}_{i=1}^n$ and Q are boxes, with $Q \subseteq \bigcup_{i=1}^n Q_i$. Then,

$$|Q| \leqslant \sum_{i=1}^{n} |Q_i|$$

With the same procedure as before, note that this time, Q_i are allowed to be overlapped, and Q is smaller than the union, we can easily get this result.

Now we start to construct open sets. We know that in one dimensional case, i.e., in $\mathbb{R}$, every open set can be written as union of countably many open intervals.

Proposition 1.2.5: Property of Open sets in $\mathbb{R}$

Every open subset G of $\mathbb{R}$ can be written uniquely as a countable union of disjoint open intervals.

Proof. For each $x \in G$, let I_x be the largest open interval containing x and contained in G. That is, if we denote

$$a_x = \inf\{a < x : (a, x) \subseteq G\} \quad \text{and} \quad b_x = \sup\{b > x : (x, b) \subseteq G)\}$$

We must have $a_x < x < b_x$. We let $I_x = (a_x, b_x)$, then by construction, we have $x \in I_x$ as well as $I_x \subseteq G$. Hence,

$$G = \bigcup_{x \in G} I_x$$

We will next prove that two intervals I_x and I_y is either equal or disjoint. Suppose these two intervals intersect. Then there union is also an open interval and is contained in G, and contains x. Since I_x is the maximal open interval that contains x, we must have $I_x \cup I_y \subseteq I_x$. Similarly, we can show that $I_x \cup I_y \subseteq I_y$. This can happen only if $I_x = I_y$. Therefore, any two distinct intervals in the collection $\mathcal{I} = \{I_x\}_{x \in G}$ must be disjoint.

Finally, we prove that there are only countably many such intervals. This is trivial since each interval must contain a rational number, and therefore exists a bijective map from all these intervals to a subset of rational numbers. The cardinality of $\mathcal{I}$ is at most countable. $\square$

The disappointing thing is that, this property does not carry further to $\mathbb{R}^d$ with $d \geqslant 2$. The analog of open intervals in larger dimension is the interior of boxes. This means that not every open set is the disjoint union of interior of boxes. We can see the following example.

Example 1.2.6: Set that cannot be constructed from Open Intervals

An open disc in $\mathbb{R}^2$ is not the disjoint union of interior of boxes.
Suppose there does exist a collection of disjoint interior of boxes $\{Q_k^\circ\}_{k\in I}$ such that the open disc D

$$D = \bigcup_{k\in I} Q_k^\circ$$

Then, for each $x \in D$, we have $x \in Q_k^\circ$ for some $k \in I$. Consider the boundary of this Q_k°. There must be some point on this boundary such that it is in D. If $y \in \partial Q_k^\circ = \partial Q_k$ and $y \in D$, then $y \notin Q_k^\circ$. Therefore it must be contained in another $Q_{k'}^\circ$. However, since $Q_{k'}^\circ$ is open, there exists a small open ball $B_\epsilon(y)$ with radius ϵ centered at y such that $B_\epsilon(y) \subseteq Q_{k'}^\circ$. However, $B_\epsilon(y)$ must intersect Q_k° since it is centered at a boundary, which means that Q_k and $Q_{k'}$ intersect, which is a contradiction.

Alternatively, can we construct open sets using boxes? First note that this cannot be done by finite union, since the finite union of boxes are closed sets. We must need inifinite sets to complete this. The next theorem shows that this can be done only using countably many boxes.

Theorem 1.2.7: Construct Open Set using Boxes

If G is a nonempty open subset of $\mathbb{R}^d$, then there exists countably many nonoverlapping cubes $\{Q_k\}_{k\in\mathbb{N}}$ such that

$$G = \bigcup_{k\in\mathbb{N}} Q_k$$

Proof. Let $Q = [0,1]^d$ be the unit cube and, for each $n \in \mathbb{N}$ and $k \in \mathbb{Z}^d$, set

$$Q_{n,k} = 2^{-n}Q + 2^{-n}k$$

This will generate a 'grid-like' cut of the whole space. This can be seen from Figure 1.2 below. Therefore, we fix any $n \in \mathbb{N}$, the collection $\{Q_{n,k}\}_{k\in\mathbb{R}^d}$ is a cover of $\mathbb{R}^d$ by nonoverlapping cubes that have sidelength 2^{-n}.

Let G be a nonempty open set, we will choose cubes from $\{Q_{n,k}\}$ to create a set of nonoverlapping cubes such that the union is G. We start by $n = 0$, and choose all cubes that is completely contained in G. We set

$$I_0 = \{k \in \mathbb{Z} : Q_{0,k} \subseteq G\}$$

Then, we choose $n = 1$, and choose all cubes such that it is completely contained in G but not contained in any $Q_{0,k}$ with $k \in I_0$. In a similar manner, we denote all the subscripts of these sets by I_1. Continuing this fashion, we will get smaller and smaller cubes that is contained in G, and the corresponding subscript set $I_0, I_1, I_2, \cdots$

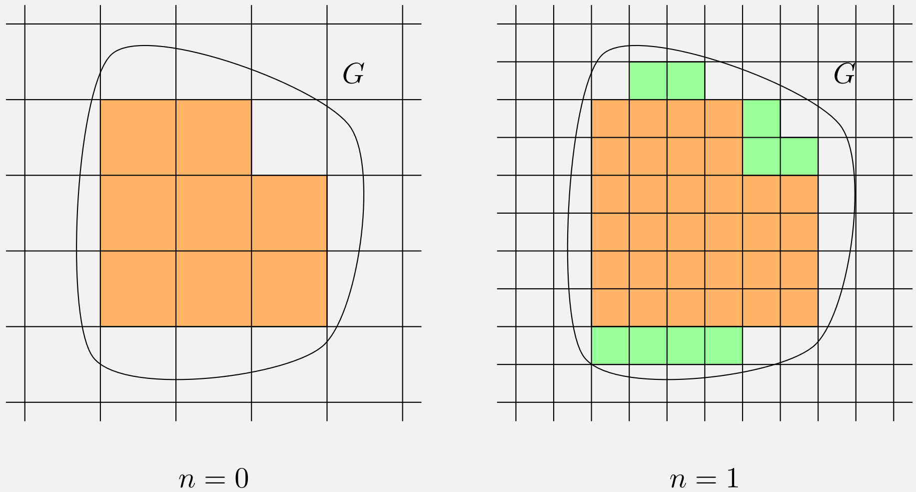

Figure 1.2: Decomposition of G into nonoverlapping cubes

Since each chosen cube is contained in G, we have the union must contain in G. Moreover, every point $x \in G$ must belong to at least one of these cubes. If not, then we can choose n large enough such that $x \in Q_{n,k} \in B_\epsilon(x)$ since 2^{-n} converges to zero and G is an open set. This derives a contradiction. Consequently,

$$G = \bigcup_{n\in\mathbb{N}} \bigcup_{k\in I_n} Q_{n,k}$$

which is a union of countably many cubes. □

Note that we cannot yet say anything about the volume of this open set G. We have only examined the finite union cases, but not this infinite case. Moreover, the way of decomposing G into countably many cubes are not unique. We do not know yet whether the different approaches will lead to the same volume definition.

1.2.2 Exterior Lebesgue Measure

Now we are ready to define the 'volume' of any other kind of sets. From now on we will not call these sizes 'volume', we use the formal terminology **measure**. The idea is that we can approximate any set by a cover of it, and this cover can be union of boxes. We take the infimum of all the measures of these covers, so that we are approximating from outside. This is why it is called a *Exterior Measure*.

Definition 1.2.8: Exterior Measure

The **Exterior (Lebesgue) Measure** (or **Outer Measure**) of a set $E \subseteq \mathbb{R}^d$ is

$$|E|_e = \inf \left\{ \sum_k |Q_k| \right\}$$

where the infimum is taken over all countable collections of boxes $\{Q_k\}$ such that $E \subseteq \bigcup Q_k$.

Note: In the definition we are only use **countable** collections of boxes. The uncountable cases such as

$$E = \bigcup_{x \in E} \{x\}, \text{ where E is uncountable}$$

is not allowed.

Bibliography

[1] Axler, S. (2020). *Measure, integration & real analysis.* Springer Nature.

[2] Folland, G. B. (1999). *Real analysis: modern techniques and their applications*, volume 40. John Wiley & Sons.

[3] Heil, C. (2019). *Introduction to real analysis*, volume 280. Springer.

[4] Stein, E. M. and Shakarchi, R. (2009). *Real analysis: measure theory, integration, and Hilbert spaces.* Princeton University Press.

[5] Walter, R. (1987). Real and complex analysis.