Fantuan's Academia

Fantuan's Math Notes series

Notes on Real Analysis

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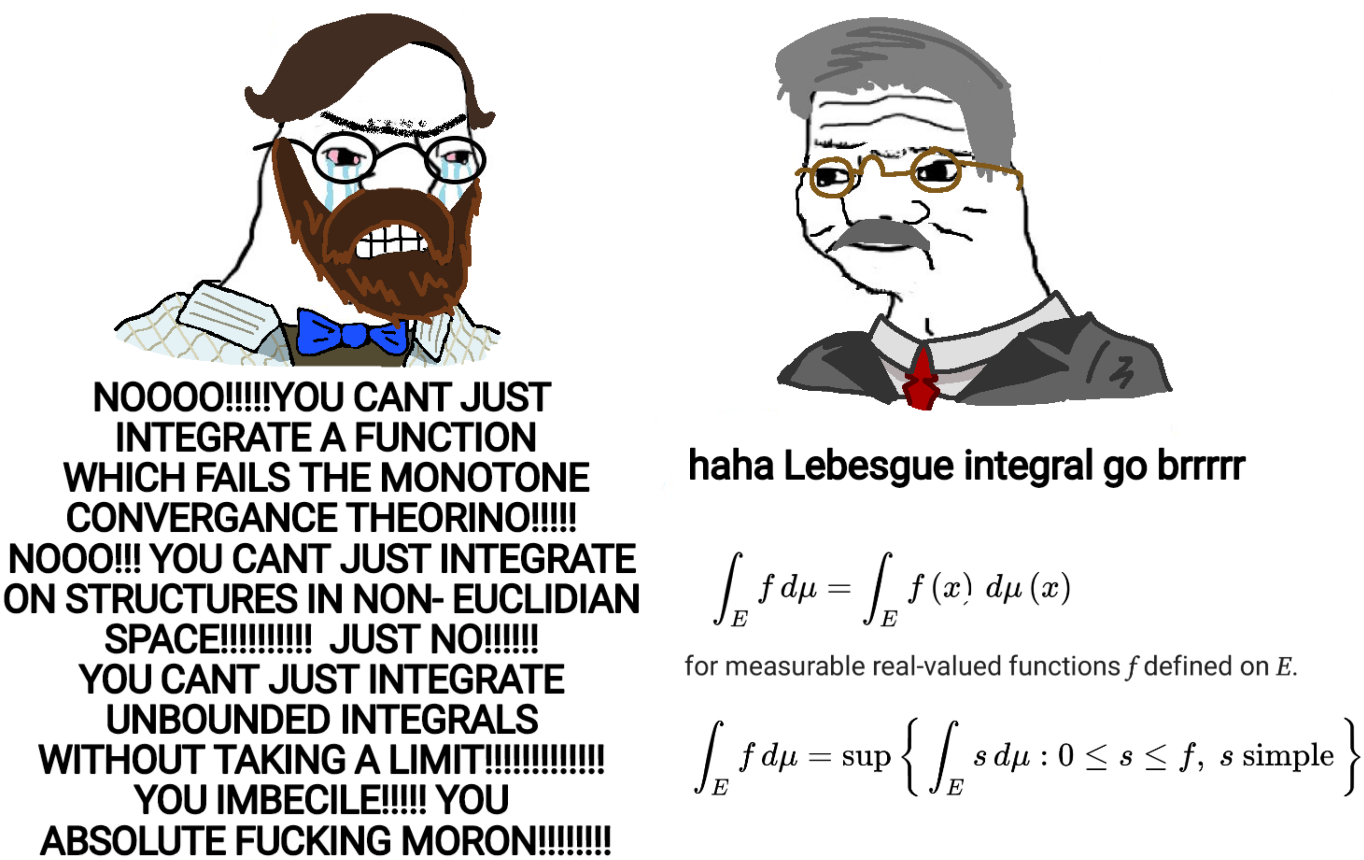

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Contents

All the Sections with * are hard sections and can be skipped without losing coherence.

This note is referenced on **Introduction to Real Analysis** by Christopher Heil [3], **Real Analysis: Modern Techniques and Applications** by Folland [2], **Real Analysis** by Stein [4], **Measure, Integration and Real Analysis** by Sheldon Axler[1] (the famous 'Linear Algebra Done Right' Author!), and **Real and Complex Analysis** by Walter Rudin[5].

Part I

PART I: Measure and Integration

Chapter 1

Lebesgue Measure

1.1 Why Real Analysis?

What is this note about if we already have the introductory mathematical analysis? The main reason is that **Riemann Integral** has many deficiencies so that we need a more rigorous approach to the integration theory to solve these problems.

Example 1.1.1: Some Functions are not Riemann Integrable

The **Dirichlet Function** $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

is not Riemann Integrable, since for any partition interval $[a,b] \subseteq [0,1]$ with $a < b$, we will have

$$\inf_{[a,b]} f = 0 \text{ and } \sup_{[a,b]} f = 1$$

The supremum of lower Riemann Sum and infimum of Upper Riemann Sum would never be equal to each other. Therefore, the integral does not exist. However, consider that $\mathbb{Q}$ is countable and $\mathbb{R}\backslash\mathbb{Q}$ is uncountable, we will reasonably guess that the 'area' under this curve should be 0 in some sense. So we need to fix this.

Example 1.1.2: Riemann Integral dos not preserve pointwise limit

Let $r_1, r_2, \cdots$ be a sequence that includes all rational numbers in $[0,1]$ exactly once. For $k \in \mathbb{N}$, define$f_k : [0,1] \to \mathbb{R}$ by

$$f_k(x) = \begin{cases} 1, & \text{if } x \in \{r_1, \cdots, r_k\} \\ 0, & \text{Otherwise} \end{cases}$$

Each f_k is Riemann Integrable, but the pointwise limit is the Dirichlet Function, which is not integrable!

You can see that Riemann Integral does not behave very well on pathological functions. In this note, we will fix these problems using a new integral method called the **Lebesgue Integral**, and derive new theories based on this.

The reason that this topic is called 'Real Analysis', is we mainly focus on the analysis of **Real-Valued Functions**.

Definition 1.1.3: Real-Valued Function

A **real-valued function** is a function $f : X \to \bar{\mathbb{R}}$, where X is an arbitrary set and $\bar{\mathbb{R}}$ is the extended real line $[-\infty, \infty]$.

We will also consider some **Complex-Valued Functions**, because its real and imaginary parts can be written as real-valued functions.

1.2 Exterior Measure

The first thing we need to do is to assign a 'size', or 'volume' to each set. You may think this is trivial. However, some pathological sets such us $\mathbb{R}\backslash\mathbb{Q}$, has no intuitive concept of 'volume'. To do this, we first start from very simple sets, then construct volume for each set based on this simple one. The basic element we use is **box**.

1.2.1 Box

Definition 1.2.1: Box

- A **box** in $\mathbb{R}$ is a Cartesian product of d finite closed intervals

$$Q = \prod_{i=1}^{d} [a_i, b_i] = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d], \quad a_j < b_j, \forall j$$

- The **volume** of the box is the product of lengths of each side,

$$|Q| = \prod_{i=1}^{d}(b_i - a_i) = (b_1 - a_1)(b_2 - a_2)\cdots(b_d - a_d)$$

- The **interior** of the box is the Cartesian Product

$$Q^\circ = \prod_{i=1}^{d}(a_i, b_i) = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_d, b_d), \quad a_j < b_j, \forall j$$

- The **boundary** of the box is $\partial Q = Q \backslash Q^\circ$.
- If the sides of a box have equal length, then we call it a **cube**.

The reason we use closed intervals is that, when constructing other volumes from boxes, we can 'overlap' the boundary of boxes, and the volumes can still be added up normally. With open intervals, we cannot easily do that. We give this kind of overlap a terminology.

Definition 1.2.2: Nonoverlapping Box

A collection of boxes $\{Q_k\}_{k\in I}$ is **nonoverlapping** if there interiors are disjoint. i.e.,

$$j \neq k \in I \quad \Longrightarrow \quad Q_j^\circ \cap Q_k^\circ = \emptyset$$

With boxes, we can start to construct the 'volume' of other sets. We first do the very simple case, which is a box constructed by union of finitely many nonoverlapping boxes. Intuitively, in this case the volume can just be added up. It is true, but the proof may not be as trivial as you might think.

Proposition 1.2.3: Union of Finitely Many Boxes

Let

$$Q = \prod_{i=1}^{d}[a_i, b_i]$$

be a box in $\mathbb{R}^d$. Suppose $\{Q_i\}_{i=1}^{n}$ are finitely many nonoverlapping boxes such that

$$Q = \bigcup_{i=1}^{n} Q_i$$

Then, the volume add up, i.e.,

$$|Q| = \sum_{i=1}^{n} |Q_i|$$

Proof. We begin by thinking the union of these boxes as a 'jigsaw-like' decomposition of the big box Q, as shown in left of Figure 1.1 below (for convenience, we draw the picture in $\mathbb{R}^2$ case).

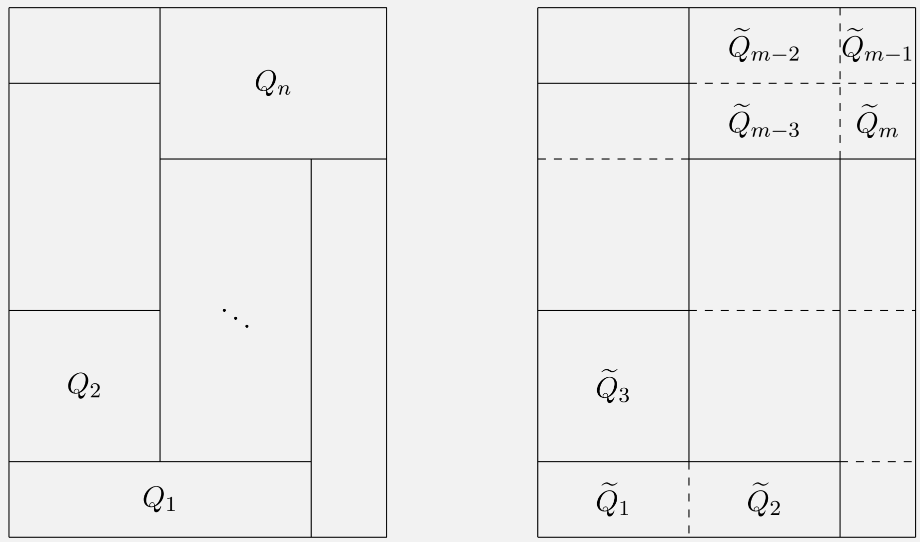

Figure 1.1: Jigsaw-like nonoverlapping union in $\mathbb{R}^2$. Left: Original Grid. Right: Grid after extension

Then, we can 'extend' these sides of boxes so that they decompose to many smaller 'grid-like' nonoverlapping boxes $\{\widetilde{Q}_i\}_{i=1}^m$. This forms a partition $J_1, J_2, \cdots, J_n$ such that

$$Q = \bigcup_{i=1}^m \widetilde{Q}_i, \quad \text{and} \quad Q_k = \bigcup_{j \in J_k} \widetilde{Q}_j$$

For example, in the figure above, $Q_1 = \widetilde{Q}_1 \cup \widetilde{Q}_2$, thus $J_1 = \{1, 2\}$. Note that $\{\widetilde{Q}_k\}$ are also nonoverlapping boxes. In this extended-grid case, adding volumes is just adding each sides and then times them all up, just as what we did in 2-dimensional case. For example, suppose in the figure above, $Q_1 = [0, 2] \times [0, 1]$ and $\widetilde{Q}_1 = [0, 1] \times [0, 1]$ and $\widetilde{Q}_2 = [1, 2] \times [0, 1]$. Then, their volume add up because $[0, 2] \times [0, 1] = ((1-0) + (2-1)) \times 1 = (1-0) \times (1-0) + (2-1) \times (1-0) = |\widetilde{Q}_1| + |\widetilde{Q}_2|$. Therefore,

$$|Q| = \sum_{j=1}^m |\widetilde{Q}_j| = \sum_{k=1}^n \sum_{j \in J_k} |\widetilde{Q}_j| = \sum_{i=1}^n |Q_i|$$

$\square$

With a little bit more thoughts, we can extend this result into overlapping case.

Proposition 1.2.4: Union of Overlapping Boxes

If $\{Q_i\}_{i=1}^n$ and Q are boxes, with $Q \subseteq \bigcup_{i=1}^n Q_i$. Then,

$$|Q| \leqslant \sum_{i=1}^{n} |Q_i|$$

With the same procedure as before, note that this time, Q_i are allowed to be overlapped, and Q is smaller than the union, we can easily get this result.

Now we start to construct open sets. We know that in one dimensional case, i.e., in $\mathbb{R}$, every open set can be written as union of countably many open intervals.

Proposition 1.2.5: Property of Open sets in $\mathbb{R}$

Every open subset G of $\mathbb{R}$ can be written uniquely as a countable union of disjoint open intervals.

Proof. For each $x \in G$, let I_x be the largest open interval containing x and contained in G. That is, if we denote

$$a_x = \inf\{a < x : (a, x) \subseteq G\} \quad \text{and} \quad b_x = \sup\{b > x : (x, b) \subseteq G)\}$$

We must have $a_x < x < b_x$. We let $I_x = (a_x, b_x)$, then by construction, we have $x \in I_x$ as well as $I_x \subseteq G$. Hence,

$$G = \bigcup_{x \in G} I_x$$

We will next prove that two intervals I_x and I_y is either equal or disjoint. Suppose these two intervals intersect. Then there union is also an open interval and is contained in G, and contains x. Since I_x is the maximal open interval that contains x, we must have $I_x \cup I_y \subseteq I_x$. Similarly, we can show that $I_x \cup I_y \subseteq I_y$. This can happen only if $I_x = I_y$. Therefore, any two distinct intervals in the collection $\mathcal{I} = \{I_x\}_{x \in G}$ must be disjoint.
Finally, we prove that there are only countably many such intervals. This is trivial since each interval must contain a rational number, and therefore exists a bijective map from all these intervals to a subset of rational numbers. The cardinality of $\mathcal{I}$ is at most countable. $\square$

The disappointing thing is that, this property does not carry further to $\mathbb{R}^d$ with $d \geqslant 2$. The analog of open intervals in larger dimension is the interior of boxes. This means that not every open set is the disjoint union of interior of boxes. We can see the following example.

Example 1.2.6: Set that cannot be constructed from Open Intervals

An open disc in $\mathbb{R}^2$ is not the disjoint union of interior of boxes.
Suppose there does exist a collection of disjoint interior of boxes $\{Q_k^\circ\}_{k\in I}$ such that the open disc D

$$D = \bigcup_{k\in I} Q_k^\circ$$

Then, for each $x \in D$, we have $x \in Q_k^\circ$ for some $k \in I$. Consider the boundary of this Q_k°. There must be some point on this boundary such that it is in D. If $y \in \partial Q_k^\circ = \partial Q_k$ and $y \in D$, then $y \notin Q_k^\circ$. Therefore it must be contained in another $Q_{k'}^\circ$. However, since $Q_{k'}^\circ$ is open, there exists a small open ball $B_\epsilon(y)$ with radius ϵ centered at y such that $B_\epsilon(y) \subseteq Q_{k'}^\circ$. However, $B_\epsilon(y)$ must intersect Q_k° since it is centered at a boundary, which means that Q_k and $Q_{k'}$ intersect, which is a contradiction.

Alternatively, can we construct open sets using boxes? First note that this cannot be done by finite union, since the finite union of boxes are closed sets. We must need inifinite sets to complete this. The next theorem shows that this can be done only using countably many boxes.

Theorem 1.2.7: Construct Open Set using Boxes

If G is a nonempty open subset of $\mathbb{R}^d$, then there exists countably many nonoverlapping cubes $\{Q_k\}_{k\in\mathbb{N}}$ such that

$$G = \bigcup_{k\in\mathbb{N}} Q_k$$

Proof. Let $Q = [0,1]^d$ be the unit cube and, for each $n \in \mathbb{N}$ and $k \in \mathbb{Z}^d$, set

$$Q_{n,k} = 2^{-n}Q + 2^{-n}k$$

This will generate a 'grid-like' cut of the whole space. This can be seen from Figure 1.2 below. Therefore, we fix any $n \in \mathbb{N}$, the collection $\{Q_{n,k}\}_{k\in\mathbb{R}^d}$ is a cover of $\mathbb{R}^d$ by nonoverlapping cubes that have sidelength 2^{-n}.

Let G be a nonempty open set, we will choose cubes from $\{Q_{n,k}\}$ to create a set of nonoverlapping cubes such that the union is G. We start by $n = 0$, and choose all cubes that is completely contained in G. We set

$$I_0 = \{k \in \mathbb{Z} : Q_{0,k} \subseteq G\}$$

Then, we choose $n = 1$, and choose all cubes such that it is completely contained in G but not contained in any $Q_{0,k}$ with $k \in I_0$. In a similar manner, we denote all the subscripts of these sets by I_1. Continuing this fashion, we will get smaller and smaller cubes that is contained in G, and the corresponding subscript set $I_0, I_1, I_2, \cdots$

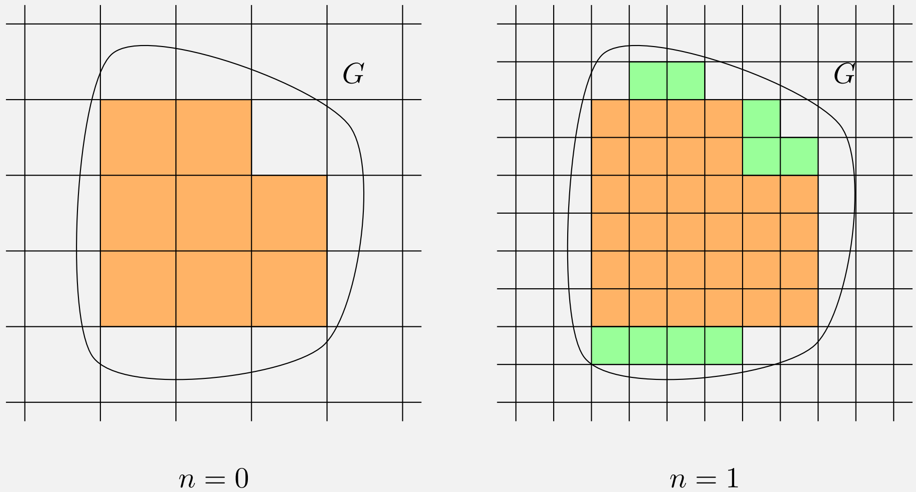

Figure 1.2: Decomposition of G into nonoverlapping cubes

Since each chosen cube is contained in G, we have the union must contain in G. Moreover, every point $x \in G$ must belong to at least one of these cubes. If not, then we can choose n large enough such that $x \in Q_{n,k} \in B_\epsilon(x)$ since 2^{-n} converges to zero and G is an open set. This derives a contradiction. Consequently,

$$G = \bigcup_{n\in\mathbb{N}} \bigcup_{k\in I_n} Q_{n,k}$$

which is a union of countably many cubes. $\square$

Note that we cannot yet say anything about the volume of this open set G. We have only examined the finite union cases, but not this infinite case. Moreover, the way of decomposing G into countably many cubes are not unique. We do not know yet whether the different approaches will lead to the same volume definition.

1.2.2 Exterior Lebesgue Measure

Now we are ready to define the 'volume' of any other kind of sets. From now on we will not call these sizes 'volume', we use the formal terminology **measure**. The idea is that we can approximate any set by a cover of it, and this cover can be union of boxes. We take the infimum of all the measures of these covers, so that we are approximating from outside. This is why it is called a *Exterior Measure*.

Definition 1.2.8: Exterior Measure

The **Exterior (Lebesgue) Measure** (or **Outer Measure**) of a set $E \subseteq \mathbb{R}^d$ is

$$|E|_e = \inf \left\{ \sum_k |Q_k| \right\}$$

where the infimum is taken over all countable collections of boxes $\{Q_k\}$ such that $E \subseteq \bigcup Q_k$.

Note: In the definition we are only using **countable** collections of boxes. The uncountable cases such as

$$E = \bigcup_{x \in E} \{x\}, \text{ where E is uncountable}$$

is not allowed.

By the definition of Infimum, we can get the first direct property of Exterior Measure:

Property 1.2.9: Infimum Property

- If $\{Q_k\}$ is any countable cover of E by boxes, then

$$|E|_e \leqslant \sum_k |Q_k|$$

- For any $\epsilon > 0$, there exists some countable cover $\{Q_k\}$ of E by boxes such that

$$\sum_k |Q_k| \leqslant |E|_e + \epsilon$$

Note that in both equations, $|E|_e$ can be infinite. It is easy to see that **All bounded sets have finite exterior measure**. By definition, if a set is bounded, then it is covered by some ball of finite radius. Use a box Q to cover this ball, we have,

$$|E|_e \leqslant |Q| < \infty$$

The converse is not true. **There is sets with finite exterior measure which is unbounded**.

Example 1.2.10: Unbounded Set with finite exterior measure

Consider the set

$$E = \bigcup_{k=1}^{\infty} [k, k + 2^{-k}]$$

E is unbounded, and $\{Q_k\}_{k \in \mathbb{N}}$ with $Q_k = [k, k + 2^{-k}]$ is a countable cover of it by boxes. Therefore, by Property 1.2.9,

$$|E|_e \leqslant \sum_k |Q_k| = 1 < \infty$$

Another good property of exterior Lebesgue measure is the **translation-invariance**.

Property 1.2.11: Translation-Invariant Property

For every set $E \subseteq \mathbb{R}^d$ and every vector $h \in \mathbb{R}^d$, we have

$$|E + h|_e = |E|_e$$

Proof. For each set $E \subseteq \mathbb{R}^d$ and its countable cover $\{Q_k\}$ by boxes, the collection $\{Q_k + h\}$ is then a countable cover of $E + h$ by boxes. Therefore, by Property 1.2.9,

$$|E + h|_e \leqslant \sum_k |Q_k + h| = \sum_k |Q_k|$$

This is true for every countable covering of E by boxes, thus taking infimum both sides we will have $|E + h|_e \leqslant |E|_e$. Symmetrically, we can also have $|E|_e \leqslant |E + h|_e$. □

This property is obvious. If you do a translation to an object without any deformation, its measure should not change. Also, if an object is completely contained in another one, its measure should not be larger than the containing object. This is called the **monotonicity**.

Property 1.2.12: Monotonic Property

If $A, B \subseteq \mathbb{R}^d$, then we have

$$A \subseteq B \quad \Longrightarrow \quad |A|_e \leqslant |B|_e$$

Proof. Suppose $A \subseteq B$, and $\{Q_k\}$ be a countable cover of B by boxes. Then, it is also a countable cover of A by boxes, so

$$|A|_e \leqslant \sum_k |Q_k|$$

Take infimum of both sides,

$$|A|_e \leqslant \inf \left\{ \sum_k |Q_k| \right\} = |B|_e$$

Since $\{Q_k\}$ is defined to be any countable cover of B by boxes. □

A very important finding is that **exterior measures of countable sets are zero**.

Property 1.2.13: Zero Measure of Countable Sets

If $E \subseteq \mathbb{R}^d$ is a countable set, then $|E|_e = 0$.

Proof. Since E is countable, we can write it as $E = \{x_k\}_{k\in\mathbb{N}}$. For each k, let Q_k be a box with volume $|Q_k| = \frac{\epsilon}{2^k}$ that contains k. Then, $\{Q_k\}$ is a cover of E. By property 1.2.9,

$$0 \leqslant |E|_e \leqslant \sum_k |Q_k| = \sum_k \frac{\epsilon}{2^k} = \epsilon$$

Since this is true for arbitrary $\epsilon > 0$, we conclude that $|E|_e = 0$. □

Finally, we will introduce the most important property of exterior measure, called **Countable Subadditivity**. It says that the exterior measure of union of countable sets is less or equal than the sum of exterior measure of those sets. We do not require these sets are disjoint here.

Theorem 1.2.14: Countable Subadditivity of Exterior Measure

If $E_1, E_2, E_3, \cdots$ are countably many sets in $\mathbb{R}^d$, then

$$\left|\bigcup_{k=1}^{\infty} E_k\right|_e \leqslant \sum_{k=1}^{\infty} |E_k|_e$$

Proof. This proof procedure is also very important.
If any particular set E_k has infinite exterior measure, then both sides of the equation are infinite, the inequality is trivial. So, suppose $|E_k|_e < \infty$ for every k. Fix $\epsilon > 0$. By Property 1.2.9, for each k we can find a countable covering $\{Q_j^{(k)}\}_j$ of E_k by boxes such that

$$\sum_j |Q_j^{(k)}| \leqslant |E_k|_e + \frac{\epsilon}{2^k}$$

Then, the collection $\{Q_j^{(k)}\}_{j,k}$ is a countable covering of $\cup_k E_k$ by boxes, since union of countably many countable set is still countable. Therefore, by Property 1.2.9, we have

$$\left|\bigcup_{k=1}^{\infty} E_k\right|_e \leqslant \sum_k \sum_j |Q_j^{(k)}| \leqslant \sum_k \left(|E_k|_e + \frac{\epsilon}{2^k}\right) = \sum_k |E_k|_e + \epsilon$$

Since this is true for arbitrary $\epsilon > 0$, we have our desired result. □

Note: Countable Subadditivity can only be used in countable union of sets! Things such as $E = \bigcup_{x\in E}\{x\}$, where E is uncountable, does not satisfy this property. Each $\{x\}$ has exterior measure 0, but E could have exterior measure larger than 0.

After seeing the properties, we are now interested in calculating some simple Lebesgue Exterior Measure. It is intuitive that for boxes, the exterior measure should be equal to its volume, and indeed it is true.

Property 1.2.15: Exterior Measure of a Box Equals its Volume

If Q is a box in $\mathbb{R}^d$, then

$$|Q|_e = |Q|$$

Proof. The proof may be much more lengthy than you would expect. Indeed, the definition of Exterior Measure is in a nonconstructive way, which makes it very difficult to compute the exact values.

($\Longrightarrow$) The "$\leqslant$" direction is easy. Note that Q is a cover of itself by box. Therefore, by the definition, $|Q|_e \leqslant |Q|$.

($\Longleftarrow$) The "$\geqslant$" direction is much more challenging. Let $\{Q_k\}$ be a countable covering of Q by boxes. Fix $\epsilon > 0$. For each $k \in \mathbb{N}$, let Q_k^* be a box that contains Q_k in its interior but is only slighly larger than Q_k such that

$$|Q_k^*| = (1+\epsilon)|Q_k|$$

This indeed can be done, for example, for $Q_k = \prod_{i=1}^d [a_i, b_i]$, choose $Q_k = \prod_{i=1}^d [a_i - \delta_k, b_i + \delta_k]$ to a achieve the ϵ larger volume. Since $Q_k \subseteq (Q_k^*)^\circ$ by our definition, the union of interiors of boxes Q_k^* forms an open covering of Q

$$Q \subseteq \bigcup_k Q_k \subseteq \bigcup_k (Q_k^*)^\circ$$

We know that Q is a box, thus it is compact in $\mathbb{R}^d$. By **Heine-Borel Theorem**, the covering $\bigcup_k (Q_k^*)^\circ$ must have a finite subcovering. That is, there exists some $N \in \mathbb{N}$ such that

$$Q \subseteq \bigcup_{k=1}^N (Q_k^*)^\circ \subseteq \bigcup_{k=1}^N Q_k^*$$

Then, by Proposition 1.2.4, we have

$$|Q| \leqslant \sum_{k=1}^N |Q_k^*| = (1+\epsilon)\sum_{i=1}^N |Q_k| \leqslant (1+\epsilon)\sum_{k=1}^\infty |Q_k|$$

Taking the infimum of both sides of this inequality, we have

$$|Q| \leqslant (1+\epsilon)|Q|_e$$

Since this is true for arbitrary $\epsilon > 0$, we finally have $|Q| \leqslant |Q|_e$. Combining two inequalities, we have desired $|Q| = |Q|_e$. □

With this, we can then show that the exterior measure of $\mathbb{R}$ is infinite.

Corollary 1.2.16

$|\mathbb{R}^d|_e = \infty$.

Proof. Let $Q_k = [-k, k]^d$. Then by Monotonicity (Property 1.2.12) and Property 1.2.15, we have

$$(2k)^d = |Q_k|_e \leqslant |\mathbb{R}^d|_e$$

Let $k \to \infty$, we have $|\mathbb{R}^d|_e = \infty$. □

It would be also intuitive that the exterior measure of open boxes should be also equal to their volume. You can think the boundary of a box in $\mathbb{R}^d$ as an object in $(d-1)$-dimension. For example, in $\mathbb{R}^2$, the boundary of a box is just a closed line, which is a 1-dimensional object, therefore does not count towards its volume.

Property 1.2.17: Exterior Measure of Interior and Boundary of a Box

If Q is a box in $\mathbb{R}^d$, then

$$|\partial Q|_e = 0, \quad \text{and} \quad |Q^\circ|_e = |Q|_e$$

Proof. For a box $Q = \prod_{i=1}^d [a_i, b_i]$, the edges are of the form

$$l_k = \{(k_1, k_2, \cdots, k_{n-1}, x, k_{n+1}, \cdots, k_d), k_i \in \{a_i, b_i\} : x \in [a_n, b_n]\}$$

For some $n \in \mathbb{N}, n \leqslant d$. Then we can just create box that covers l_k in the way of

$$Q_\epsilon^{(k)} = [k_1-\epsilon, k_1+\epsilon]\times[k_2-\epsilon, k_2+\epsilon]\times\cdots\times[k_{n-1}-\epsilon, k_{n-1}+\epsilon]\times[a_i, b_i]\times[k_{n+1}-\epsilon, k_{n+1}+\epsilon]\times\cdots\times[k_d-\epsilon, k_d+\epsilon]$$

such that $|Q_\epsilon^{(k)}| = (2\epsilon)^{d-1} \times (b_n - a_n)$. Then, by Property 1.2.9, we have $|l_k|_e \leqslant |Q_\epsilon^{(k)}|$. Since this is true for arbitrary $\epsilon > 0$, we have $|l_k| = 0$ for all k. Then, $\partial Q = \bigcup_{k=1}^N l_k$, where N is the number of edges, by Countable Subadditivity, we have

$$|\partial Q|_e \leqslant \sum_{k=1}^N |l_k|_e = 0$$

With this, we see that

$$|Q|_e = |Q^\circ \cup \partial Q|_e \leqslant |Q^\circ|_e + |\partial Q|_e = |Q^\circ|_e \leqslant |Q|_e$$

where the first inequality is by Countable Subadditivity, and the second inequality is by Monotonicity. Therefore, we have $|Q|_e = |Q^\circ|_e$. □

Note: For $d \geqslant 2$, the boundary of a box is then an uncountable set that has exterior measure 0.

Finally, another really important property of exterior measure is that it can always be approximated by a larger open set. This is called the **Outer Regular Property.**

Property 1.2.18: Outer Regularity

If $E \subseteq \mathbb{R}^d$ and $\epsilon > 0$, then there exists an open set $U \supseteq E$ such that $|E|_e \leqslant |U|_e \leqslant |E|_e + \epsilon$, i.e.,

$$|E|_e = \inf \{|U|_e : \ U \text{ open}, U \supseteq E\}$$

Proof. If $|E|_e = \infty$, then we can take $U = \mathbb{R}^d$. So, assume that $|E|_e < \infty$. By Property 1.2.9, there exists countably many boxes such that $E \subseteq \bigcup_k Q_k$ and

$$\sum_k |Q_k| \leqslant |E|_e + \frac{\epsilon}{2}$$

Now, take larger box Q_k^* that contains Q_k in its interior for each k, and satisfies

$$|Q_k^*| = |Q_k| + 2^{-k-1}\epsilon$$

Let $U = \bigcup (Q_k^*)^\circ$. Then, $E \subseteq U$ and U is open. We have

$$|E|_e \leqslant |U|_e \leqslant \sum_k |Q_k^*| \leqslant \sum_k |Q_k| + \frac{\epsilon}{2} \leqslant |E|_e + \epsilon$$

which is our desired result. □

Note: If $|E|_e$ is finite, we can actually refine this theorem by

$$|E|_e \leqslant |U|_e < |E|_e + \epsilon$$

since by this property, we can find U such that $|E|_e \leqslant |U|_e \leqslant |E|_e + \frac{\epsilon}{2}$, and since $|E|_e$ is finite, we have $|E|_e + \frac{\epsilon}{2} < |E|_e + \epsilon$.

1.3 Lebesgue Measure

Intuitively, we don't want our measure to be defined only having countable subadditivity, but also **Countable Additivity**, meaning that for countably many disjoint sets $E_1, E_2, E_3, \cdots$, there measure should add up, i.e.,

$$\left|\bigcup_{i=1}^{\infty} E_i\right|_e = \sum_{i=1}^{\infty} |E_i|_e, \quad E_i \text{ are disjoint}$$

However, exterior measure does not obtain this good property. There indeed are disjoint sets which does not have additive exterior measures. Since this breaks our intuition, and there is no way to define the 'real size' of this kind of sets according to the union, we call them **Lebesgue Nonmesurable**.

1.3.1 Nonmeasurable Set: An Example

The famous construction of nonmeasurable set uses the **Axiom of Choice** in Set Theory, which said that with a collection of disjoint nonempty sets, we can select one of the elements from each set to form a new set.

Axiom 1.3.1: The Axiom of Choice

If $\{X_\alpha\}_{\alpha\in I}$ is a nonempty collection of nonempty sets, then the set of maps

$$\prod_{\alpha\in I} X_\alpha = \left\{ f : I \to \bigcup_{\alpha\in I} X_\alpha \,\middle|\, \forall\, \alpha \in I \implies f(\alpha) \in X_\alpha \right\}$$

is nonempty. In particular, if $\{X_\alpha\}_{\alpha\in I}$ is a disjoint collection of nonempty sets, there is a set $Y = f(A) \subseteq \bigcup_{\alpha\in I} X_\alpha$ such that $Y \cap X_\alpha$ contains precisely one element from each $\alpha \in I$.

With this, we can start to construct the set. We will do this in $\mathbb{R}$, and this procedure can be extended to higher dimensions accordingly.

Proof. For $a \in [-1, 1]$, consider $\tilde{a}$, the set of all numbers in $[-1, 1]$ such that they differ from a by a rational number. i.e.,

$$\tilde{a} = \{c \in [-1, 1] : a - c \in \mathbb{Q}\}$$

This defines an **equivalence relation** (for the definition of equivalence relation, please see my Abstract Algebra Note). Therefore, if $a, b \in [-1, 1]$, then either $\tilde{a} \cap \tilde{b} = \emptyset$, or $\tilde{a} = \tilde{b}$. We do a small proof here for this statement. Suppose $\tilde{a} \cap \tilde{b} \neq \emptyset$, we need to prove that $\tilde{a} = \tilde{b}$. Suppose there exists $d \in \tilde{a} \cap \tilde{b}$, then $a - d$ and $b - d$ are rational numbers. Therefore, $a - d - (b - d) = a - b$ is then a rational number. According to the equation $a - c = (a - b) + (b - c)$, $a - c$ is a rational number if and only if $b - c$ is a rational number. This means that $\tilde{a} = \tilde{b}$.

Clearly $a \in \tilde{a}$ for each $a \in [-1,1]$. Thus,

$$[-1,1] = \bigcup_{a\in[-1,1]} \tilde{a}$$

By The **Axiom of Choice**, there exists a set V such that it contains exactly one element from each of the distinct sets in $\{\tilde{a} : a \in [-1,1]\}$.
Let $r_1, r_2, \cdots$ be a sequence of distinct rational numbers such that

$$[-2,2] \cap \mathbb{Q} = \{r_1, r_2, \cdots\}$$

Then,

$$[-1,1] \subseteq \bigcup_{k=1}^{\infty} (r_k + V)$$

This is because for any $a \in [-1,1]$, let v be the unique element of $V \cap \tilde{a}$. Then, we have $a - v \in \mathbb{Q}$, with $a, v \in [-1,1]$, which implies there exists $r_k \in [-2,2]$ such that $a = r_k + v$. Therefore, we have

$$2 = |[-1,1]|_e \leqslant \left| \bigcup_{k=1}^{\infty} (r_k + V) \right|_e \leqslant \sum_{k=1}^{\infty} |r_k + V|_e = \sum_{k=1}^{\infty} |V|_e \tag{1.1}$$

where the first equality comes from Property 1.2.15, the next inequality from monotonicity, the second inequality from countable subadditivity, and the final equality from translation-invariance. By this Equation 1.1, we can see that $|V|_e > 0$.
Moreover, we can see that

$$\bigcup_{k=1}^{\infty} (r_k + V) \subseteq [-3,3]$$

since $r_k \in [-2,2]$ for all k and $V \subseteq [-1,1]$. By monotonicity, we can have

$$\left| \bigcup_{k=1}^{\infty} (r_k + V) \right|_e \leqslant |[-3,3]| = 6 \tag{1.2}$$

However, we can take $n \in \mathbb{N}$ such that

$$\sum_{k=1}^{n} |r_k + V|_e = \sum_{k=1}^{n} |V|_e = n|V|_e > 6$$

since $|V|_e > 0$. Send n to infinity, we then have

$$\left| \bigcup_{k=1}^{\infty} (r_k + V) \right|_e \leqslant 6 < \sum_{k=1}^{n} |r_k + V|_e \tag{1.3}$$

Finally, we note that the sets $(r_k + V), k \in \mathbb{N}$ are disjoint. We do a short proof of this statement here.

Suppose $t \in (r_k + V) \cap (r_j + V)$, for $j \neq k$. Then, there exists $v_1, v_2 \in V$ such that $t = r_k + v_1 = r_j + v_2$. This implies that $v_1 - v_2 = r_j - r_k$. Since $r_j - r_k \in \mathbb{Q}$, we have $v_1 - v_2 \in \mathbb{Q}$. Our construction of V then shows that $v_1 = v_2$. This then implies $r_k = r_j$, which implies $j = k$.

Combining all the information above, we find a collection of disjoint sets $\{r_k + V\}_k$ such that $\left|\bigcup_{k=1}^{\infty}(r_k + V)\right|_e < \sum_{k=1}^{n} |r_k + V|_e$. This means that it does not follow countable additivity. $\square$

This shows that, the definition of exterior measure does not fully satisfy our expectation. We need a better way to define measures to obtain countable additivity. This is why we introduce **Lebesgue Measure**.

1.3.2 Lebesgue Measure and Countable Additivity

In this section we will refine the definition of Exterior Lebesgue Measure such that it obtains the countable additivity. We first go through what we want for a measure function to behave. To define a measure μ on $\mathbb{R}^d$, we want

(a) μ is a function from all set of subsets of $\mathbb{R}^d$ to $[0, \infty]$.

(b) $\mu(Q) = |Q|$ for every box $Q \subseteq \mathbb{R}^d$.

(c) $\mu(t + A) = \mu(A)$ for all $t \in \mathbb{R}^d$ and $A \subseteq \mathbb{R}^d$.

(d) $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$ for every disjoint sequence $A_1, A_2, \cdots$ of subsets of $\mathbb{R}^d$.

Unfortunately, it can be shown that these four properties cannot hold simultaneously.

Proposition 1.3.2: Incompatibility of Desired Properties of Measure

There does not exist a function μ that satisfies all four properties listed above.

Proof. Suppose there exists a function μ with all properties listed above. The strategy of proof is that, we will first prove all properties used in the construction of nonmesurable set in Subsection 1.3.1, then the procedure in that section can be repeated to show that there exists sets that does not follow countable additivity, which is a contradiction.

- Note first that $\mu(\emptyset) = 0$, since $\emptyset$ is just a cube with side length 0.

- We used **monotonicity** in the procedure. So we prove it here. Suppose $A \subseteq B$, we can then write

  $$\mu(B) = \mu(A) + \mu(B \backslash A) + \emptyset + \emptyset + \cdots \geqslant \mu(A)$$

  by Property (a) and (d), since A and $B \backslash A$ are disjoint.

- We used **countable subadditivity** in the procedure. So we prove it here. Let $A_1, A_2, A_3, \cdots$ be any sequence of subsets of $\mathbb{R}^d$. Then, $A_1, A_2\backslash A_1, A_3\backslash(A_1 \cup A_2), \cdots$ are disjoint and their union is $\bigcup_{k=1}^{\infty} A_k$. Therefore,

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu(A_1) + \mu(A_2\backslash A_1) + \mu(A_3\backslash(A_1 \cup A_2)) + \cdots \leqslant \sum_{k=1}^{\infty} \mu(A_k)$$

where the equality uses Property (d), and the inequality uses the monotonicity.

Then, μ has all the properties used in the construction of nonmeasurable set. Repeat that procedure, and we see that there exists sets that does not obtain countable additivity, which is a contradiction with property (d). □

Thus, we need to lose at least one property to achieve the desired countable additivity. (b) and (c) cannot be deleted since otherwise the definition of measure would be very counterintuitive and pathological. (d) cannot be deleted since this is our goal. Finally, the only one we can delete is (a). We relax the property (a) such that μ need not to have the domain of all set of subsets of $\mathbb{R}^d$. We choose some subset of these to achieve our goal. These sets we choose, are called **Lebesgue Measurable Sets**.

Consider the outer regularity of exterior measure, we can approximate a set E by some open set U that contains it. We can separate U as

$$U = E \cup (U\backslash E)$$

We know that, from countable subadditivity,

$$|U|_e \leqslant |E|_e + |U\backslash E|_e$$

However, we do not know that whether $|U|_e = |E|_e + |U\backslash E|_e$. We can try to limit our sets so that they satisfy this property, and then expect to derive countable additivity from this.

Definition 1.3.3: Lebesgue Measurable Set

A set $E \subseteq \mathbb{R}^d$ is **Lebesgue Measurable**, or simply **measurable**, if

$$\forall \epsilon > 0, \exists \text{ open } U \supseteq E \text{ such that } |U\backslash E|_e \leqslant \epsilon$$

If E is Lebesgue measurable, then its **Lebesgue measure** $|E|$ equals its exterior lebesgue measure, i.e., $|E| = |E|_e$.

We will soon verify that boxes are measurable, so our denotation $|E|$ here aligns with previous use. The collection of all Lebesgue measurable subsets of $\mathbb{R}^d$ is denoted by

$$\mathcal{L} = \mathcal{L}(\mathbb{R}^d) = \{E \subseteq \mathbb{R}^d : E \text{ is Lebesgue measurable}\}$$

Now let us examine some properties of Lebesgue measurable sets.

Property 1.3.4: Open Sets are Measurable

If $U \subseteq \mathbb{R}^d$ is open, then U is Lebesgue measurable.

Proof. If U is open, then U contains itself, and $|U \backslash U|_e = 0 < \epsilon$. □

Property 1.3.5: Zero-measure Sets are Measurable

If $Z \subseteq \mathbb{R}^d$ and $|Z|_e = 0$, then Z is measurable.

Proof. Fix $\epsilon > 0$, by outer regualrity, there is an open set $U \supseteq Z$ such that

$$|U|_e \leqslant |Z|_e + \epsilon = 0 + \epsilon = \epsilon$$

Since $U \backslash Z \subseteq U$, monoticity implies that

$$|U \backslash Z|_e \leqslant |U|_e \leqslant \epsilon$$

Therefore, Z is measurable. □

Next we prove a very important property considering a collection of measurable sets, but not a single set.

Property 1.3.6: Closure under Countable Unions

If $E_1, E_2, E_3, \cdots$ are measurable sets of $\mathbb{R}^d$, then there union $E = \bigcup_k E_k$ is also measurable.

Proof. Fix $\epsilon > 0$, since each E_k is measurable, there exists open sets $U_k \supseteq E_k$ such that

$$|U_k \backslash E_k|_e \leqslant \frac{\epsilon}{2^k}$$

Then, $U = \bigcup_k U_k$ is an open set, and $U \supseteq E$, with

$$U \backslash E = \left(\bigcup_k U_k\right) \backslash \left(\bigcup_k E_k\right) \subseteq \bigcup_k (U_k \backslash E_k)$$

Therefore,

$$|U\backslash E|_e \leqslant \left|\bigcup_k (U_k \backslash E_k)\right|_e \leqslant \sum_k |U_k \backslash E_k|_e \leqslant \sum_k \frac{\epsilon}{2^k} = \epsilon$$

where the first ineuqality comes from monotonicity, and the second inequality comes from countable subadditivity. Therefore, E is measurable. ☐

Now, we are ready to prove that boxes are measurable.

Property 1.3.7: Boxes are measurable

Every boxes in $\mathbb{R}^d$ is a Lebesgue measurable set.

Proof. Let Q be a box. Since Q° is open, we have Q° is measurable. Since ∂Q has exterior measure 0, we also have ∂Q is measurable. By the closure of measurable set under countable additivity, we have $Q = Q^\circ \cup \partial Q$ is measurable. ☐

Does any closed sets are measurable? The answer is not clear yet. To prove this, we first need some lemma. The next lemma says that no matter measurable or not, if two sets are some distance away, the exterior measure is then additive.

Lemma 1.3.8: Exterior Measure is additive for Distant Sets

Define the distance between two sets as

$$\operatorname{dist}(A, B) = \inf\{\|x - y\| : x \in A, y \in B\}$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^d$. Then, if $A, B \subseteq \mathbb{R}^d$ are nonempty, and $\operatorname{dist}(A, B) > 0$, then

$$|A + B|_e = |A|_e + |B|_e$$

Proof.
($\Longrightarrow$) By countable subadditivity, we have $|A + B|_e \leqslant |A|_e + |B|_e$. Now we need to prove the opposite direction.
($\Longleftarrow$) ☐

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