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Fantuan's Math Notes series

Notes on Measure Theory

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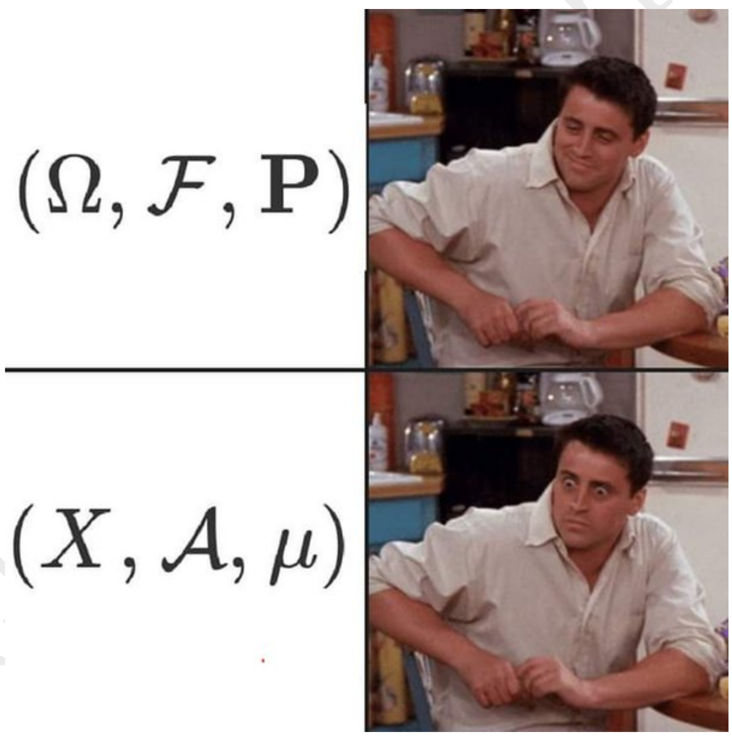

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Contents

All the Sections with * are hard sections and can be skipped without losing coherence.

This note is referenced on **Real Analysis: Modern Techniques and Applications** by Folland [3], **Measure Theory** by Cohn [2], **Real Analysis** by Stein [4], **Measure, Integration and Real Analysis** by Sheldon Axler[1] (the famous 'Linear Algebra Done Right' Author!), and **Real and Complex Analysis** by Walter Rudin[5].

Part I

PART I: Measure and Integration

Chapter 1

Measure

In this note we will construct **abstract measure theory**. In my *real analysis* note, we have seen that the only way to assign measures on space X with certain good properties is to restrict the domain to subsets of the power set $\mathcal{P}(X)$. In that note, we mainly restrict to Lebesgue measurable sets. We have seen that collection of Lebesgue measurable sets form a σ-algebra. We can start from this point of view, to construct a more general theory of measure.

1.1 σ-Algebra

1.1.1 Algebra and σ-Algebra

Definition 1.1.1: Algebra/Field

Let X be an arbitrary set. A collection $\mathcal{A}$ of subsets of X is an **algebra** (or a **field**) on X if

1. **Nonempty:** $\mathcal{A} \neq \emptyset$
2. **Closed under complement:** If $E \in \mathcal{A}$, then $E^c \in \mathcal{A}$
3. **Closed under finite union:** If $E_1, E_2, \cdots, E_n \in \mathcal{A}$, then $\bigcup_{i=1}^n E_i \in \mathcal{A}$

Note that an algebra has a few other properties:

(a) **Closed under finite intersection:** If $E_1, E_2, \cdots, E_n \in \mathcal{A}$, then $\bigcap_{i=1}^n E_i \in \mathcal{A}$

(b) $\emptyset \in \mathcal{A}$ and $X \in \mathcal{A}$

Proof.

(a) Note that

$$\bigcap_{i=1}^n E_i = \left(\bigcup_{i=1}^n E_i^c\right)^c$$

By De Morgan's Law. Since $\mathcal{A}$ is closed under finite union and complement, it follows that $\bigcap_{i=1}^{n} E_i \in \mathcal{A}$.

(b) Since $\mathcal{A}$ is nonempty, there exists a set $E \in \mathcal{A}$. Since $\mathcal{A}$ is closed under finite union, finite intersection and complement, we have

$$\emptyset = E \cap E^c \in \mathcal{A} \quad \text{and} \quad X = E \cup E^c \in \mathcal{A}$$

□

Definition 1.1.2: σ-Algebra/σ-Field

Let X be an arbitrary set. A collection $\mathcal{A}$ of subsets of X is a **σ-algebra** (or a **σ-field**) on X if

1. **Nonempty:** $\mathcal{A} \neq \emptyset$
2. **Closed under complement:** If $E \in \mathcal{A}$, then $E^c \in \mathcal{A}$
3. **Closed under countable union:** If $E_1, E_2, E_3, \cdots \in \mathcal{A}$, then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$

Note that a σ-algebra has a few other properties:

(a) **Closed under countable intersection:** If $E_1, E_2, E_3, \cdots \in \mathcal{A}$, then $\bigcap_{i=1}^{\infty} E_i \in \mathcal{A}$

(b) $\emptyset \in \mathcal{A}$ and $X \in \mathcal{A}$

(c) **Closed under finite union and intersection:** If $E_1, E_2, \cdots, E_n \in \mathcal{A}$, then $\bigcup_{i=1}^{n} E_i \in \mathcal{A}$ and $\bigcap_{i=1}^{n} E_i \in \mathcal{A}$

Proof.

(a) Note that

$$\bigcap_{i=1}^{\infty} E_i = \left(\bigcup_{i=1}^{\infty} E_i^c \right)^c$$

By De Morgan's Law. Since $\mathcal{A}$ is closed under countable union and complement, it follows that $\bigcap_{i=1}^{n} E_i \in \mathcal{A}$.

(b) Since $\mathcal{A}$ is nonempty, there exists a set $E \in \mathcal{A}$. Since $\mathcal{A}$ is closed under countable union, countable intersection and complement, we have

$$\emptyset = E \cap E^c \cap E \cap E \cap \cdots \in \mathcal{A} \quad \text{and} \quad X = E \cup E^c \cup E \cup E \cup \cdots \in \mathcal{A}$$

(c) The closedness under finite union and intersection can be naturally deduced by closedness under countable union and intersection, by letting $E_i = \emptyset$ for $i > n$ to prove closedness under finite union, and letting $E_i = X$ for $i > n$ to prove closedness under finite intersection.

$\square$

Finally, note that all σ-algebra are algebra. Here we see some examples of algebra and σ-algebra. We omit the verification here.

Example 1.1.3: Examples of Algebra and σ-Algebra

σ-algebra:

- Let X be a set, and $\mathcal{A}$ be the power set $\mathcal{P}(X)$, i.e., the collection of all subsets of X. Then, $\mathcal{A}$ is a σ-algebra on X.
- Let X be a set, and $\mathcal{A} = \{\emptyset, X\}$. Then $\mathcal{A}$ is a σ-algebra.
- Let X be a set, whatever countable or uncountable, and let $\mathcal{A}$ be the collection of all subsets E of X such that either E or E^c is countable. Then $\mathcal{A}$ is a σ-algebra.

Algebra but not a σ-algebra:

- Let X be an infinite set. Let $\mathcal{A}$ be the collection of all subsets E of X such that either E or E^c is finite. Then $\mathcal{A}$ is an algebra, but not a σ-algebra.
- Let $\mathcal{A}$ be the collection of all subsets of $\mathbb{R}$ that are unions of finitely many intervals of the form $(a, b]$, $(a, +\infty)$ or $(-\infty, b]$. Then, $\mathcal{A}$ is an algebra, but not a σ-algebra (for example, the open interval (a, b) is a countable union of these sets, but itself is not in $\mathcal{A}$).

Neither algebra nor σ-algebra:

- Let X be an infinite set. Let $\mathcal{A}$ be the collection of all finite subsets of X. Then it is not an algebra since X is not contained in $\mathcal{A}$.
- Let X be an uncountable set. Let $\mathcal{A}$ be the collection of all countable subsets of X. Then, it is not an algebra since X is not contained in $\mathcal{A}$.

There is a natural way to prove an algebra is actually a σ-algebra.

Proposition 1.1.4: Characterization of σ-algebra from algebra

Let X be a set. Let $\mathcal{A}$ be an algebra on X. Then $\mathcal{A}$ is a σ-algebra if either

1. $\mathcal{A}$ is closed under the union of increasing sequence of sets.
2. $\mathcal{A}$ is closed under the intersection of decreasing sequence of sets.

Proof. We only need to prove that it is closed under countable union.

1. Suppose (1) holds. Suppose $\{A_i\}_{i\in\mathbb{N}}$ is a sequence of sets that belongs to $\mathcal{A}$. Let
$$B_n = \bigcup_{i=1}^{n} A_i$$
Then the sequence $\{B_n\}$ is increasing. Since $\mathcal{A}$ is an algebra, each B_n belongs to $\mathcal{A}$. Note that $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$. Since $\mathcal{A}$ is closed under union of increasing sequence of set, $\bigcup_{i=1}^{\infty} B_i \in \mathcal{A}$. Therefore, $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$. Then, since $\{A_i\}$ is arbitrary, we have that $\mathcal{A}$ is closed under countable union.

2. Suppose (2) holds. It is enough to check whether the condition (1) holds. If $\{A_i\}$ is a sequence of increasing sets that belongs to $\mathcal{A}$. Then, $\{A_i^c\}$ is decreasing and thus $\bigcap_i A_i^c \in \mathcal{A}$ by condition (2). Since $\bigcup_i A_i = (\bigcap_i A_i^c)^c$, it follows that $\bigcup_i A_i$ also belongs to $\mathcal{A}$. Therefore, condition (1) holds, and $\mathcal{A}$ is a σ-algebra.

□

1.1.2 Generated σ-Algebra

We can construct new algebras from the existing ones. The most naive method is using intersection. The proof of the next proposition is extremely trivial.

Proposition 1.1.5: Intersection of σ-algebra

Let X be a set. Then the intersection of an arbitrary nonempty collection of σ-algebras on X is again a σ-algebra on X.

Proof. Let $\{\mathcal{A}_i\}_{i\in I}$ be a nonempty collection of σ-algebras on X, and let $\mathcal{A}$ be the intersection of the σ-algebras $\mathcal{A}_i$, $\mathcal{A} = \bigcap_{i\in I} \mathcal{A}_i$

- Since each $\mathcal{A}_i$ is a σ-algebra, $X \in \mathcal{A}_i$ for each $i \in I$. We have $X \in \mathcal{A}$, $\mathcal{A}$ is not empty.
- Suppose $E \in \mathcal{A}$. Then $E \in \mathcal{A}_i$ for each $i \in I$. Since each $\mathcal{A}_i$ is a σ-algebra, $E^c \in \mathcal{A}_i$ for each $i \in I$. Therefore, $E^c \in \mathcal{A}$. The collection $\mathcal{A}$ is thus closed under complement.
- Suppose $E_1, E_2, E_3, \dots \in \mathcal{A}$. Then, $E_1, E_2, E_3, \dots \in \mathcal{A}_i$ for each $i \in I$. Since each $\mathcal{A}_i$ is a σ-algebra, $\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}_i$ for each $i \in I$. Therefore, $\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$. The collection $\mathcal{A}$ is thus closed under countable union.

In conclusion, we have $\mathcal{A}$ is a σ-algebra. □

Note: The union of σ-algebras can fail to be a σ-algebra. For example, consider the set $X = \{1, 2, 3, 4\}$. The collection

$$\mathcal{A} = \{\emptyset, \{1\}, \{2, 3, 4\}, X\}, \quad \mathcal{B} = \{\emptyset, \{2\}, \{1, 3, 4\}, X\}$$

are then σ-algebra on X. The union

$$\mathcal{A} \cup \mathcal{B} = \{\emptyset, \{1\}, \{2\}, \{2, 3, 4\}, \{1, 3, 4\}, X\}$$

fail to be a σ-algebra since the union $\{1\} \cup \{2\} = \{1, 2\}$ does not belong to $\mathcal{A} \cup \mathcal{B}$.

The preceding proposition shows that, there exists a smallest σ-algebra that contains some subset of $\mathcal{P}(X)$.

Corollary 1.1.6: Smallest containing σ-algebra

Let X be a set. Let $\mathcal{E}$ be a collection of subsets of X. Then, there exists a unique smallest σ-algebra on X that contains $\mathcal{E}$. This smallest σ-algebra is exactly the intersection of all σ-algebra on X that contains $\mathcal{E}$.

Proof. Let $\{\mathcal{A}_i\}_{i \in I}$ be the collection of all σ-algebra on X that contains $\mathcal{E}$. Let $\mathcal{A} = \bigcap_{i \in I} \mathcal{A}_i$. By Proposition 1.1.5, $\mathcal{A}$ is a σ-algebra. It is naturally smaller than all σ-algebra that contains $\mathcal{E}$. □

This smallest σ-algebra is so important that we give it a name and a notation.

Definition 1.1.7: Generated σ-algebra

The smallest σ-algebra that contains $\mathcal{E}$ is called the σ-algebra **generated** by $\mathcal{E}$, it is denoted by $\sigma(\mathcal{E})$.

The next trivial lemma would be useful later.

Lemma 1.1.8

Let X be a set. Let $\mathcal{E}$ and $\mathcal{F}$ be two arbitrary subset of $\mathcal{P}(X)$. If $\mathcal{E} \subseteq \sigma(\mathcal{F})$, then $\sigma(\mathcal{E}) \subseteq \sigma(\mathcal{F})$.

Proof. Since $\sigma(\mathcal{F})$ is a σ-algebra that contains $\mathcal{E}$, it also contains the smallest σ-algebra that contains $\mathcal{E}$, i.e., $\sigma(\mathcal{E})$. □

1.1.3 Borel σ-Algebra

We use the preceding theory about generated σ-algebra to define an important family of σ-algebra.

Definition 1.1.9: Borel σ-Algebra/Borel Set

Let X be any topological space. The σ-algebra generated by the collection of all open sets (or equivalently, all closed sets) in X is called the **Borel σ-algebra** on X, denoted by $\mathcal{B}(X)$. Its elements are called **Borel sets**.

Note that the equivalence of the two definition using open sets and closed sets is because, a σ-algebra containing all open sets must contain all closed sets since it is closed under complement. The Borel σ-algebra on $\mathbb{R}$ would be in special importance. The next proposition shows other ways of defining $\mathcal{B}(\mathbb{R})$.

Proposition 1.1.10: Alternative Definitions of $\mathcal{B}(\mathbb{R})$

$\mathcal{B}(\mathbb{R})$ is generated by each of the following:

1. Open intervals: $\mathcal{E}_1 = \{(a, b) : a < b\}$.
2. Closed intervals: $\mathcal{E}_2 = \{[a, b] : a < b\}$.
3. Half-open intervals: $\mathcal{E}_3 = \{(a, b] : a < b\}$ or $\mathcal{E}_4 = \{[a, b) : a < b\}$.
4. Open-rays: $\mathcal{E}_5 = \{(a, +\infty) : a \in \mathbb{R}\}$ or $\mathcal{E}_6 = \{(-\infty, a) : a \in \mathbb{R}\}$
5. Closed-rays: $\mathcal{E}_7 = \{[a, +\infty) : a \in \mathbb{R}\}$ or $\mathcal{E}_8 = \{(-\infty, a] : a \in \mathbb{R}\}$

Proof. We will prove this iteratively. This proof is also trivial.

- Since open intervals are open sets, $\mathcal{E}_1$ is contained in the collection of all open sets. Therefore $\mathcal{B}(\mathbb{R}) \supseteq \mathcal{E}_1$. By Lemma 1.1.8, we have $\mathcal{B}(\mathbb{R}) \supseteq \sigma(\mathcal{E}_1)$.

- Since each closed interval can be written as countable intersection of open intervals, for example, by
$$[a, b] = \bigcap_{k=1}^{\infty} \left(a - \frac{1}{k}, b + \frac{1}{k}\right)$$
closed intervals are contained in the σ-algebra generated by $\mathcal{E}_1$. Therefore, $\sigma(\mathcal{E}_1) \supseteq \mathcal{E}_2$. By Lemma 1.1.8, we have $\sigma(\mathcal{E}_1) \supseteq \sigma(\mathcal{E}_2)$.

- Similarly, we can write half-open intervals as
$$(a, b] = \bigcap_{k=1}^{\infty} \left[a + \frac{1}{k}, b\right], \quad [a, b) = \bigcap_{k=1}^{\infty} \left(a - \frac{1}{k}, b - \frac{1}{k}\right]$$
Therefore, $\sigma(\mathcal{E}_2) \supseteq \mathcal{E}_3$ and $\sigma(\mathcal{E}_3) \supseteq \mathcal{E}_4$. By Lemma 1.1.8, we have $\sigma(\mathcal{E}_2) \supseteq \sigma(\mathcal{E}_3) \supseteq \sigma(\mathcal{E}_4)$.

- We can write sets in $\mathcal{E}_5$ and $\mathcal{E}_8$ as

$$(a, +\infty) = \bigcap_{k=1}^{\infty} \left[a + \frac{1}{k}, k\right), \quad (-\infty, a] = (a, +\infty)^c$$

Therefore, $\sigma(\mathcal{E}_4) \supseteq \mathcal{E}_5$ and $\sigma(\mathcal{E}_5) \supseteq \mathcal{E}_8$. By Lemma 1.1.8, we have $\sigma(\mathcal{E}_4) \supseteq \sigma(\mathcal{E}_5) \supseteq \sigma(\mathcal{E}_8)$.

- We can write sets in $\mathcal{E}_6$ and $\mathcal{E}_7$ as

$$(-\infty, a) = \bigcap_{k=1}^{\infty} \left(-\infty, a - \frac{1}{k}\right], \quad [a, \infty) = (-\infty, a)^c$$

Therefore, $\sigma(\mathcal{E}_8) \supseteq \mathcal{E}_6$ and $\sigma(\mathcal{E}_6) \supseteq \mathcal{E}_7$. By Lemma 1.1.8, we have $\sigma(\mathcal{E}_8) \supseteq \sigma(\mathcal{E}_6) \supseteq \sigma(\mathcal{E}_7)$.

- Finally, since each open interval can be written as

$$(a, b) = \left(\bigcap_{k=1}^{\infty} \left[a + \frac{1}{k}, \infty\right)\right) \cap [b, \infty)^c$$

and each open set in $\mathbb{R}$ can be written as countable union of disjoint open intervals. Therefore, all open sets are contained in the σ-algebra generated by $\mathcal{E}_7$. This means that $\sigma(\mathcal{E}_7) \supseteq \mathcal{B}(\mathbb{R})$.

In conclusion, we have, iteratively,

$$\mathcal{B}(\mathbb{R}) \supseteq \sigma(\mathcal{E}_1) \supseteq \sigma(\mathcal{E}_2) \supseteq \sigma(\mathcal{E}_3) \supseteq \sigma(\mathcal{E}_4) \supseteq \sigma(\mathcal{E}_5) \supseteq \sigma(\mathcal{E}_8) \supseteq \sigma(\mathcal{E}_6) \supseteq \sigma(\mathcal{E}_7) \supseteq \mathcal{B}(\mathbb{R})$$

All these σ-algebras are equivalent. □

There is a standard terminology for the level of hierarchy. A countable intersection of open sets is called a **G_δ-set**. A countable union of closed sets is called an **F_σ-set**. Similarly, we can define a countable union of G_δ-set as $G_{\delta\sigma}$-set, a countable intersection of F_σ-set as $F_{\sigma\delta}$-set, $\cdots$.

1.2 Measures

1.2.1 Definition and Examples

Definition 1.2.1: Measure

Let X be a set. Let $\mathcal{A}$ be a σ-algebra on X. A **measure** on $\mathcal{A}$ is a function $\mu : \mathcal{A} \to [0, +\infty]$ such that

1. $\mu(\emptyset) = 0$.

2. **Countable Additivity:** If $\{E_i\}_{i\in\mathbb{N}}$ is a sequence of disjoint sets in $\mathcal{A}$, then $\mu(\bigcup_i E_i) = \sum_i \mu(E_i)$

Relative to this terminology, there is another one with looser constraint.

Definition 1.2.2: Finite Additive Measure

Let X be a set. Let $\mathcal{A}$ be an algebra (not necessarily a σ-algebra) on X. A **finite additive measure** on $\mathcal{A}$ is a function $\mu : \mathcal{A} \to [0, +\infty]$ such that

1. $\mu(\emptyset) = 0$.
2. **Finite Additivity:** If $\{E_i\}_{i=1}^n$ are finitely many disjoint sets in $\mathcal{A}$, then $\mu(\bigcup_{i=1}^n E_i) = \sum_{i=1}^n \mu(E_i)$

Note that all measures are finitely additive (just let $E_i = \emptyset$ for $i > n$, and use the property that $\mu(\emptyset) = 0$). However, a finite additive measure may not be a measure.

Definition 1.2.3: Measurable Space/Measurable Set/Measure Space

Let X be a set. Let $\mathcal{A}$ be a σ-algebra on X. Let μ be a measure on $\mathcal{A}$.

- $(X, \mathcal{A})$ is called a **measurable space**. The sets in $\mathcal{A}$ are called **measurable sets**.
- $(X, \mathcal{A}, \mu)$ is called a **measure space**.

Now let's see some examples of measures.

Example 1.2.4: Examples of Measures

Measures:

- Let X be an arbitrary set. Let $\mathcal{A}$ be a σ-algebra on X. Define function $\mu : \mathcal{A} \to [0, +\infty]$ such that
$$\mu(E) = \begin{cases} n, & \text{if } |E| = n \\ +\infty, & \text{if } E \text{ is an infinite set} \end{cases}$$
Then μ is a measure. It is called the **counting measure** on $(X, \mathcal{A})$.
- Let X be a nonempty set. Let $\mathcal{A}$ be a σ-algebra on X. Let $x \in X$. Define function $\delta_x : \mathcal{A} \to [0, +\infty]$ such that
$$\delta_x(E) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{if } x \notin E \end{cases}$$
Then δ_x is a measure. It is called the **Dirac measure** or **point mass** at x.
- Let X be an arbitrary set. Let $\mathcal{A}$ be a σ-algebra on X. Define function $\mu : \mathcal{A} \to [0, +\infty]$ such

that

$$\mu(E) = \begin{cases} +\infty, & \text{if } E \neq \emptyset \\ 0, & \text{if } E = \emptyset \end{cases}$$

Then, μ is a measure.

- Let X be an uncountable set, let $\mathcal{A}$ be a σ-algebra that contains subsets E of X such that either E or E^c is countable. Define function $\mu : \mathcal{A} \to [0, +\infty]$ such that

$$\mu(E) = \begin{cases} 0, & \text{if } E \text{ is countable} \\ 1, & \text{if } E^c \text{ is countable} \end{cases}$$

Then μ is a measure. This is a little bit harder to see. First of all, $\mu(\emptyset) = 0$ since $\emptyset$ is countable. Further, for some set $E \in \mathcal{A}$, if E is uncountable, then E^c is countable. Thus all sets that is disjoint with E must be countable. Therefore, for disjoint sets $E_1, E_2, E_3, \cdots$ in this σ-algebra, there could be only one uncountable set. If there is, the union is uncountable and the measure is 1. If not, the measure is 0, and it is equal to the result of countable sum.

Finite Additive Measures:

- Let $X = \mathbb{Z}^+$, the set of all positive integers. Let $\mathcal{A}$ be the collection of subsets E of X such that either E or E^c is finite. Then $\mathcal{A}$ is an algebra, but not a σ-algebra. Define function $\mu : \mathcal{A} \to [0, +\infty]$ such that

$$\mu(E) = \begin{cases} 1, & \text{if } E \text{ is infinite} \\ 0, & \text{if } E \text{ is finite} \end{cases}$$

Then it is a finitely additive measure. However, it is not a measure.

Not finite additive measure nor measure:

- Let X be a set that has at least two elements. Let $\mathcal{A}$ be the collection of all subsets of X. Define function $\mu : \mathcal{A} \to [0, +\infty]$ such that

$$\mu(E) = \begin{cases} 1, & \text{if } E \neq \emptyset \\ 0, & \text{if } E = \emptyset \end{cases}$$

Then it is not a finite additive measure since for two nonempty disjoint sets E and F, $\mu(E \cup F) = 1 \neq \mu(E) + \mu(F) = 2$.

1.2.2 Properties of Measure

The basic properties of measures are summarized below:

Theorem 1.2.5: Properties of Measures

Let $(X, \mathcal{A}, \mu)$ be a measure space.

1. **Monotonicity:** If $E, F \in \mathcal{A}$, and $E \subseteq F$, then $\mu(E) \leqslant \mu(F)$.
2. **Countable Subadditivity:** For arbitrary $\{E_i\}_{i=1}^{\infty} \in \mathcal{A}$, we have
$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leqslant \sum_{i=1}^{\infty} \mu(E_i)$$
3. **Continuous from below:** For $\{E_n\}_{n=1}^{\infty} \in \mathcal{A}$ with $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$, we have
$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{i\to\infty} \mu(E_i)$$
4. **Continuous from above:** For $\{E_n\}_{n=1}^{\infty} \in \mathcal{A}$ with $E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$ and $\mu(E_1) < +\infty$, we have
$$\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{i\to\infty} \mu(E_i)$$

Proof.

1. If $E \subseteq F$, then $F = E \cup (F \backslash E)$. Since E and $F\backslash E$ are disjoint, by countable additivity, we have
$$\mu(F) = \mu(E) + \mu(F\backslash E) \geqslant \mu(E)$$
2. Let $F_1 = E_1$, and
$$F_k = E_k \backslash \left(\bigcup_{i=1}^{k-1} E_i\right), k > 1$$
Then, $\{F_k\}$ are disjoint, with $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i$ and $F_k \subseteq E_k$ for all k. Therefore, by countable additivity and monotonicity,
$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \mu\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} \mu(F_i) \leqslant \sum_{i=1}^{\infty} \mu(E_i)$$
3. Set $E_0 = \emptyset$ for convenience. Since the set $\{E_i\}$ is increasing, $\{E_i \backslash E_{i-1}\}$ are disjoint and

$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \mu\left(\bigcup_{i=1}^{\infty}(E_i \backslash E_{i-1})\right)$, by countable additivity,

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \mu\left(\bigcup_{i=1}^{\infty}(E_i \backslash E_{i-1})\right) = \sum_{i=1}^{\infty} \mu(E_i \backslash E_{i-1}) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(E_i \backslash E_{i-1}) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(E_n)$$

4. Let $F_i = E_1 \backslash E_i$. Then, $F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots$, and $\mu(E_1) = \mu(F_i) + \mu(E_i)$ by finite additivity. Also, $\bigcup_i^{\infty} F_i = E_1 \backslash (\bigcap_{i=1}^{\infty} E_i)$. By continuity from below, we have

$$\mu\left(E_1 \backslash \left(\bigcap_{i=1}^{\infty} E_i\right)\right) = \mu\left(\bigcup_{i=1}^{\infty} F_i\right) = \lim_{n\to\infty} \mu(F_n) = \lim_{n\to\infty} (\mu(E_1) - \mu(E_n)) = \mu(E_1) - \lim_{n\to\infty} \mu(E_n) \tag{1.1}$$

Note that, by countable additivity, we have

$$\mu\left(E_1 \backslash \left(\bigcap_{i=1}^{\infty} E_i\right)\right) + \mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \mu(E_1) \tag{1.2}$$

Combining the equation 1.1 and 1.2, we have

$$\mu(E_1) - \mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \mu(E_1) - \lim_{n\to\infty} \mu(E_n)$$

Since $\mu(E_1) < +\infty$, we can substract this term from both sides to get our expected result:

$$\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu(E_n)$$

$\square$

Note that the condition $\mu(E_1) < +\infty$ can be relaxed as $\mu(E_n) < +\infty$ for some $n \in \mathbb{N}$, since we can always omit the first n terms of the sequence and retain the same result. This assumption is necessary. Consider $X = \mathbb{R}$, and $\mathcal{A}$ be all Lebesgue measurable sets (this would be introduced later, or you can see my *real analysis* note). Let μ be the Lebesgue measure (also, this would be introduced later). The only thing we need to know is that this measure assign $\mu([a,b]) = b - a$, $\mu([a,+\infty]) = +\infty$ and $\mu(\mathbb{R}) = +\infty$. Consider a collection of decreasing set

$$E_n = [-n, n]^c, n \in \mathbb{N}$$

Then, each E_n has an infinite measure. However, the intersection is $\emptyset$, which has measure 0.

There is a way to prove a finite additive measure is indeed a measure. This needs to use the continuity from above and below.

Proposition 1.2.6: Characterization of measure from finite additive measure

Let $(X, \mathcal{A})$ be a measurable space. Let μ be a finite additive measure on $\mathcal{A}$. Then μ is a measure if one of the statement below is correct:

1. It is continuous from below
2. It is continuous from above and $\mu(X) < \infty$
3. $\lim_n \mu(E_n) = 0$ holds for each decreasing sequence $\{E_n\}$ of sets that belongs to $\mathcal{A}$ and satisfies $\bigcap_i E_i = \emptyset$.

Proof. We only need to verify the countable additivity of μ.

1. Let $\{A_i\}$ be an arbitrary sequence of disjoint sets in $\mathcal{A}$. Let $E_i = \bigcup_{k=1}^{i} A_k$. Then $\{E_i\}$ is increasing. Since μ is continuous from below, we have
$$\mu\left(\bigcup_i E_i\right) = \lim_{n\to\infty} \mu(E_n)$$
. Since μ is a finite additive measure, we also have
$$\mu(E_i) = \sum_{k=1}^{i} \mu(A_k)$$
Since $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} A_i$, we have
$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{i\to\infty} \mu(E_i) = \lim_{i\to\infty} \sum_{k=1}^{i} \mu(A_k) = \sum_{k=1}^{\infty} \mu(A_k)$$
2. Let $\{A_i\}$ be an arbitrary sequence of disjoint sets in $\mathcal{A}$. Let $E_i = \bigcup_{k=i}^{\infty} A_k$. Then, $\{E_i\}$ is decreasing. Since $\mu(X) < +\infty$, by monotonicity, for each $E \subseteq \mathcal{A}$, we have $\mu(E) < +\infty$. Therefore, $\mu(E_1) < +\infty$. Since μ is continuous from above, we have
$$\mu\left(\bigcap_i E_i\right) = \lim_{n\to\infty} \mu(E_n) \tag{1.3}$$
Since μ is a finite additive measure, we also have
$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(\left(\bigcup_{i=1}^{k-1} A_i\right) \cup E_k\right) = \sum_{i=1}^{k-1} \mu(A_i) + \mu(E_k)$$

Taking limit so that $k \to \infty$, we have

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) + \lim_{k\to\infty} \mu(E_k) \tag{1.4}$$

Combining Equation 1.3 and 1.4, we have

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) + \mu\left(\bigcap_{i} E_i\right) = \sum_{i=1}^{\infty} \mu(A_i) + \mu\left(\bigcap_{i=1}^{\infty}\bigcup_{k=i}^{\infty} A_k\right) = \sum_{i=1}^{\infty} \mu(A_i) + \mu\left(\limsup_{k\to\infty} A_k\right)$$

Since $\{A_k\}$ are disjoint, and the limit superior of A_k contains elements that is included in infinitely many A_k, we have $\mu\left(\limsup_k A_k\right) = \mu(\emptyset) = 0$. Therefore, we have our desired result

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

3. With the same setting as in (2), we have

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) + \lim_{k\to\infty} \mu(E_k)$$

Our assumption in (3) said that $\lim_{k\to\infty} \mu(E_k) = 0$. Hence $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$.

□

There is a standard terminology that represents the 'sizes' of measures.

Definition 1.2.7: Finite/σ-Finite/Semifinite

Let $(X, \mathcal{A}, \mu)$ be a measure space.

- If $\mu(X) < +\infty$, μ is called **finite**.
- If $X = \bigcup_{j=1}^{\infty} E_j$ where $E_j \in \mathcal{A}$ for all j, and $\mu(E_j) < +\infty$ for all j, μ is called **σ-finite**.
- If for each $E \in \mathcal{A}$ with $\mu(E) = +\infty$, there exists $F \in \mathcal{A}$ with $F \subseteq E$ such that $0 < \mu(F) < +\infty$, μ is called **semifinite**.

Note that there is a hierarchy here. **Every finite measure is σ-finite**. This is trivial. Also, **every σ-finite measure is semifinite**. This needs a little bit more work. Suppose μ is σ-finite. Then we can write $X = \bigcup_{j=1}^{\infty} E_j$. Now suppose $\mu(E) = +\infty$. Then E can be written as $E = \left(\bigcup_{j=1}^{\infty} E_j\right) \cap E = \bigcup_{j=1}^{\infty}(E_j \cap E)$.

Since $\mu(E_j) < +\infty$ and $E_j \cap E \subseteq E_j$, by monotonicity, we have $\mu(E_j \cap E) < +\infty$ for all j. If we set

$$A_j = E_j \cap E, \quad B_j = A_j \backslash \left(\bigcup_{k=1}^{j-1} A_j \right)$$

We have $\bigcup_i A_i = \bigcup_i B_i$, and B_i is disjoint. Also, by monotonicity, $\mu(B_j) < +\infty$ for all j. Therefore, by countable additivity,

$$+\infty = \mu(E) = \mu \left(\bigcup_{i=1}^{\infty} A_i \right) = \mu \left(\bigcup_{i=1}^{\infty} B_i \right) = \sum_{i=1}^{\infty} \mu(B_i)$$

With each $\mu(B_i) < +\infty$, to make $\sum_{i=1}^{\infty} \mu(B_i) = +\infty$, there must exist some i such that $0 < \mu(B_i) < +\infty$. Since this $B_i \subseteq E$, we have proved our result.

There are ways to construct new measures from known ones.

Proposition 1.2.8: Construction of new measure

Let $(X, \mathcal{A})$ be a measurable space.

- If $\mu_1, \mu_2, \cdots, \mu_n$ are measures on $(X, \mathcal{A})$, and $a_1, a_2, \cdots, a_n \in [0, +\infty)$, then

$$\sum_{i=1}^{n} a_i \mu_i$$

  is also a measure on $(X, \mathcal{A})$.

- If μ is a measure on $(X, \mathcal{A})$, and $A \in \mathcal{A}$, then **the restriction of μ on A**

$$\mu_A(E) = \mu(E \cap A)$$

  is also a measure on $(X, \mathcal{A})$.

Proof.

- First of all,

$$\left(\sum_{i=1}^{n} a_i \mu_i \right) (\emptyset) = \sum_{i=1}^{n} a_i (\mu_i(\emptyset)) = 0$$

  Moreover, If $\{E_j\}_{j \in \mathbb{N}}$ is a sequence of disjoint sets in $\mathcal{A}$, then

$$\left(\sum_{i=1}^{n} a_i \mu_i \right) \left(\bigcup_j E_j \right) = \sum_{i=1}^{n} a_i \mu_i \left(\bigcup_j E_j \right) = \sum_{i=1}^{n} a_i \left(\sum_{j=1}^{\infty} \mu_i(E_j) \right) = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_i \mu_i \right) (E_j)$$

- First of all,

$$\mu_A(\emptyset) = \mu(\emptyset \cap A) = \mu(\emptyset) = 0$$

Moreover, If $\{E_j\}_{j\in\mathbb{N}}$ is a sequence of disjoint sets in $\mathcal{A}$, then

$$\mu_A\left(\bigcup_i E_i\right) = \mu\left(\left(\bigcup_i E_i\right) \cap A\right) = \mu\left(\bigcup_i (E_i \cap A)\right) = \sum_i \mu(E_i \cap A) = \sum_i \mu_A(E_i)$$

Therefore, both of them are measures. □

1.2.3 Completion of measure

Definition 1.2.9: Null set

Let $(X, \mathcal{A}, \mu)$ be a measure space. A set $E \in \mathcal{A}$ such that $\mu(E) = 0$ is called a **null set**.

By subadditivity, we have any countable union of null sets is a null set, since

$$0 \leqslant \mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leqslant \sum_{i=1}^{\infty} \mu(E_i) = 0$$

Definition 1.2.10: Almost Everywhere

If a statement about points $x \in X$ is true except for x in some null set, we say that it is true **almost everywhere** (abbreviation **a.e.**).

If $\mu(E) = 0$ and $F \subseteq E$ with $F \in \mathcal{A}$, then $\mu(F) = 0$ by monotonicity. However, it needs not to be true that $F \in \mathcal{A}$.

Definition 1.2.11: Complete Measure

A measure whose domain includes all subsets of null sets is called **complete**.

Luckily, we can always enlarge domains of μ to make it complete.

Theorem 1.2.12: Completion of Measure

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $\mathcal{N} = \{N \in \mathcal{A} : \mu(N) = 0\}$ be the collection of null sets. Let

$$\bar{\mathcal{A}} = \{E \cup F : E \in \mathcal{A} \text{ and } F \subseteq N \text{ for some } N \in \mathcal{N}\}$$

Then, $\bar{\mathcal{A}}$ is a σ-algebra, and there exists a **unique** extension $\bar{\mu}$ of μ to a complete measure on $\bar{\mathcal{A}}$.

Proof.

- We first prove that $\bar{\mathcal{A}}$ is a σ-algebra.

  1. First, clearly $\bar{\mathcal{A}}$ is not empty.

  2. Second, suppose $E \cup F \in \bar{\mathcal{A}}$, where $E \in \mathcal{A}$ and $F \subseteq N \in \mathcal{N}$. If we replace F and N by $F \backslash E$ and $N \backslash E$ (this can be done since $N \backslash E = N \cap E^c$ is still in $\mathcal{A}$, thus in $\mathcal{N}$ by monotonicity), $E \cup F$ is the same set, so we can assume that $E \cap N = \emptyset$. In this case,

  $$E \cup F = (E \cup N) \cap (N^c \cup F)$$

  Therefore,
  $$(E \cup F)^c = (E \cup N)^c \cup (N \backslash F)$$

  Since on the right hand side, $(E \cup N)^c \in \mathcal{A}$, and $N \backslash F \subseteq N$, we have $(E \cup F)^c \in \bar{\mathcal{A}}$. Thus $\bar{\mathcal{A}}$ is closed under complement.

  3. Finally, since both $\mathcal{A}$ and $\mathcal{N}$ are closed under countable union, and sets in $\bar{\mathcal{A}}$ is just the union of sets in two, $\bar{\mathcal{A}}$ is also closed under countable union.

- Now we define the expected measure $\bar{\mu}$. For $E \cup F \in \bar{\mathcal{A}}$, we define

  $$\bar{\mu}(E \cup F) = \mu(E)$$

  We first need to show that this is well-defined. Suppose $E_1 \cup F_1 = E_2 \cup F_2$, where $F_j \subseteq N_j \in \mathcal{N}$. Then $E_1 \subseteq E_2 \cup N_2$, therefore

  $$\bar{\mu}(E_1 \cup F_1) = \mu(E_1) \leqslant \mu(E_2) + \mu(N_2) = \mu(E_2)$$

  Similarly, we have $E_2 \subseteq E_1 \cup N_1$, therefore,

  $$\bar{\mu}(E_2 \cup F_2) = \mu(E_2) \leqslant \mu(E_1) + \mu(N_1) = \mu(E_1)$$

  Therefore, $\bar{\mu}(E_1 \cup F_1) = \bar{\mu}(E_2 \cup F_2)$, the measure is well-defined.

- Now we need to show that $\bar{\mu}$ is indeed a measure.

  1. First of all, since $\bar{\mu}(E \cup F) = \mu(E)$, it is a function from $\bar{\mathcal{A}}$ to $[0, +\infty]$.

  2. Second, $\bar{\mu}(\emptyset) = \mu(\emptyset) = 0$.

  3. Third, let $\{\bar{E_j}\}_{i=1}^{\infty}$ be a disjoint collection of sets in $\bar{\mathcal{A}}$. Then,

  $$\bar{\mu}\left(\bigcup_{j=1}^{\infty} \bar{E_j}\right) = \bar{\mu}\left(\bigcup_{j=1}^{\infty} (E_j \cup F_j)\right) = \bar{\mu}\left(\bigcup_{j=1}^{\infty} E_j \cup \bigcup_{j=1}^{\infty} F_j\right)$$

Since $F_j \subseteq N_j \in \mathcal{N}$, and $N = \bigcup_{j=1}^{\infty} N_j$ is a null set. Since $\bigcup_{j=1}^{\infty} F_j \subseteq N$. Hence,

$$\bar{\mu}\left(\bigcup_{j=1}^{\infty} E_j \cup \bigcup_{j=1}^{\infty} F_j\right) = \mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j) = \sum_{j=1}^{\infty} \bar{\mu}(E_j \cup F_j) = \sum_{j=1}^{\infty} \bar{\mu}(\bar{E}_j)$$

Combining those two equations, $\bar{\mu}$ is countable additive.

- Let $\bar{E}$ be a null set in $\bar{\mathcal{A}}$. That is, if we write $\bar{E} = E \cup F$, then $\bar{\mu}(\bar{E}) = \mu(E) = 0$. Therefore, E is also a null set. Since F is a subset of a null set, $\bar{E}$ is a subset of a null set N. Let $\bar{F} \subseteq \bar{E}$. Then, immediately $\bar{F} \subseteq N$ and $\bar{F} \in \bar{\mathcal{A}}$ since we can write $\bar{F} = \emptyset \cup \bar{F}$. Therefore, the domain of $\bar{\mu}$ contains each subsets of null sets. $\bar{\mu}$ is complete.

- Now we see $\bar{\mu}$ is actually an extension. Since for all $E \in \mathcal{A}$, we can write $E = E \cup \emptyset$. Then $\bar{\mu}(E) = \mu(E)$ Therefore, μ and $\bar{\mu}$ take the same value on the domain $\mathcal{A}$. Therefore, $\bar{\mu}$ is an extension.

- Finally, we will show the uniqueness of $\bar{\mu}$. Suppose λ is another complete measure on $\bar{\mathcal{A}}$ such that it is an extension of μ. Suppose $\bar{E} = E \cup F \in \bar{\mathcal{A}}$, where $E \in \mathcal{A}$ and $F \subseteq N \in \mathcal{N}$. Similarly as before, we can take $E \cap N = \emptyset$. Then,

$$\mu(E) = \lambda(E) \leqslant \lambda(\bar{E}) \leqslant \lambda(E) + \lambda(N) = \mu(E) + \mu(N) = \mu(E)$$

Hence, $\lambda(\bar{E}) = \mu(E) = \bar{\mu}(\bar{E})$. Since $\bar{E}$ is arbitrary, $\lambda = \mu$.

□

1.3 Outer Measures

As stated in my *real analysis* note, we can always approximate the measure of a set from outside. In that note, we use this thought to produce Lebesgue Exterior measure. Here we generalize that thought to the abstract measure theory.

Definition 1.3.1: Outer Measure

An **outer measure** on a nonempty set X is a function $\mu^* : \mathcal{P}(X) \to [0, +\infty]$ such that

- $\mu^*(\emptyset) = 0$
- **Monotonicity:** For $A \subseteq B \subseteq X$, we have $\mu^*(A) \leqslant \mu^*(B)$
- **Countable Subadditivity:** For a sequence of set $\{E_i\}$ in X, we have $\mu^*(\bigcup_{i=1}^{\infty} E_i) \leqslant \sum_{i=1}^{\infty} \mu^*(E_i)$

Note that the domain of outer measure is $\mathcal{P}(X)$, which is the collection of all subsets of X, not an arbitrary σ-algebra.

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