Fantuan's Academia

Fantuan's Math Notes series

Notes on Mathematical Analysis

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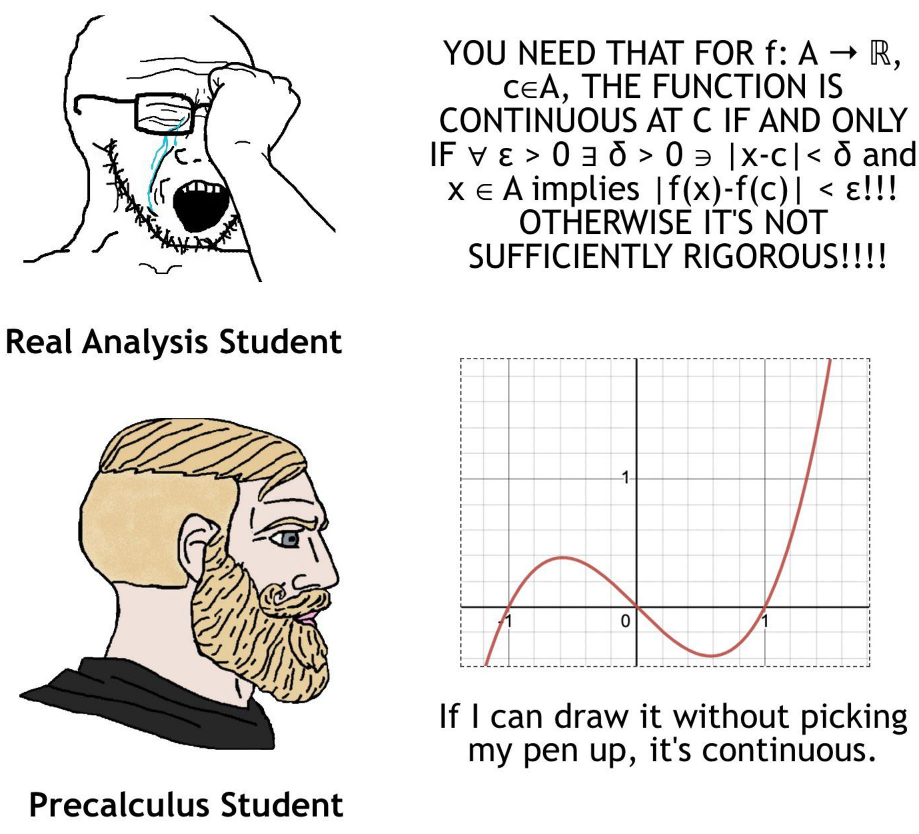


April 19, 2024

Contents

All the Sections with * are hard sections and can be skipped without losing coherence.

This note is referenced on **Understanding Analysis** by Stephen Abbott [1], **Principles of Mathematical Analysis** by Walter Rudin [2], **Analysis I** by Terence Tao [3], and MTH 117,118 notes of XJTLU.

Part I

Part I: The Real Line

Chapter 1

Real Numbers

1.1 Why Analysis?

Analysis, simply saying, is a course about 'rigorous calculus'. Somebody may ask then: "why we need another course about calculus?". Indeed, basic calculus concepts and various computing skills are introduced in Year I Calculus course. However, regarding calculus as a pure math object, it should maintain its full rigor. If we apply calculus in the real world problems without knowing where they came from and what is their constraints to be correctly applied, some pathological things will happen, as listed below.

Example 1.1.1. *Infinite Series*
Consider the divergent infinite series

$$S = 1 + 2 + 4 + 8 + 16 + \cdots \tag{1.1}$$

If we multiply it by 2,

$$2S = 2 + 4 + 8 + 16 + \cdots \tag{1.2}$$

Subtract (1.1) from (1.2), we will have the ridiculous result

$$S = -1$$

Example 1.1.2. *Interchanging Integrals*
We always change the order of double integral to make calculation easier. But, can we always do that in any cases? Consider

$$\int_0^\infty \int_0^1 \left(e^{-xy} - xye^{-xy}\right) \mathrm{d}y \, \mathrm{d}x$$

If we directly compute this, we can get,

$$\int_0^\infty \int_0^1 \left(e^{-xy} - xye^{-xy}\right) \mathrm{d}y \, \mathrm{d}x = \int_0^\infty \left[ye^{-xy}\right]_{y=0}^1 \mathrm{d}x = \int_0^\infty e^{-x} \, \mathrm{d}x = \left[-e^{-x}\right]_0^\infty = 1$$

However, if we change the order of integral

$$\int_0^1 \int_0^\infty \left(e^{-xy} - xye^{-xy}\right) \mathrm{d}x \, \mathrm{d}y = \int_0^1 \left[xe^{-xy}\right]_{x=0}^{\infty} \mathrm{d}y = \int_0^1 (0-0)\, \mathrm{d}y = 0$$

We arrive different answers!

Example 1.1.3. *Reordering Infinite Series*
Consider the alternating harmonic series

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

We know that this infinite series converges at some point. Therefore, nothing similar as Example 1.1.1 could happen here. However, if we do the following computation:

$$\begin{aligned} \frac{1}{2}S &= \quad \frac{1}{2} \quad - \frac{1}{4} \quad + \frac{1}{6} \quad - \frac{1}{8} \quad + \frac{1}{10} \quad - \frac{1}{12} \quad + \frac{1}{14} \quad - \frac{1}{16} + \cdots \\ S &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \frac{1}{14} + \frac{1}{15} - \frac{1}{16} + \cdots \end{aligned}$$

$$\frac{3}{2}S = \left(1 + \frac{1}{3}\right) - \frac{1}{2} + \left(\frac{1}{5} + \frac{1}{7}\right) - \frac{1}{4} + \left(\frac{1}{9} + \frac{1}{11}\right) - \frac{1}{6} + \left(\frac{1}{13} + \frac{1}{15}\right) - \frac{1}{8} + \cdots$$

We see that $\frac{3}{2}S$ is just a reordering of our initial infinite series (with two positive terms following one negative term)! Therefore, we just change the convergent point by simply reordering the infinite series.
This doesn't make sense! Since by intuition, reordering the terms in an algorithm will not change its result. However, you see here, the situation changes in the infinite case.

As showed above, we indeed need this course to make analysis as a much more rigorous math topic than Year I Calculus. To get started, we will first talk about real numbers.

1.2 From Rational to Irrational Numbers

The simplest number system we can call to our mind is the **Natural Numbers**

$$\mathbb{N} = \{0, 1, 2, 3, 4, \cdots\}$$

Obviously, this number system is based on counting. It is enough for the simple use of counting things. However, this number system is not closed under subtraction (i.e., one natural number subtracts another natural number may not result in a natural number). Therefore, we introduce the **Integer Numbers**

$$\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$$

This number system is still not complete since it is not closed under division. For example, $3 \div 5$ is not in the list. We further introduce the **Rational Numbers**

$$\mathbb{Q} = \left\{ \frac{p}{q} : q \neq 0, p, q \in \mathbb{Z} \right\}$$

Back to Pythagoras's era (500-400 BC), he only believes the existance of rational numbers, and so did his followers in Pythagoreanism, except for one: Hippasus of Metapontum. After Pythagoras announced his famous Pythagorean Theorem, Hippasus directly used this theorem to discover $\sqrt{2}$: an irrational number!

Consider a right-angled triangle with two right-angled edges of length 1. Then by Pythagorean Theorem, length of the hypotenuse z should satisfy

$$z^2 = 1^2 + 1^2 = 2$$

We denote this number as $\sqrt{2}$, for which the square of it is 2. It seems that we cannot write this number in the form of a rational number. And indeed, we can prove that it is not a rational number.

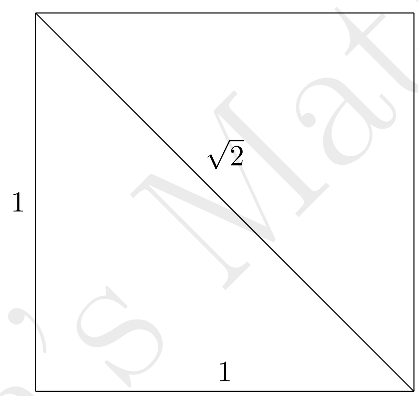


Figure 1.1: Pythagorean Theorem and $\sqrt{2}$

Proposition 1.2.1: Irrationality of $\sqrt{2}$

$\sqrt{2}$ is not a rational number.

Proof. We prove by contradiction. Suppose $\sqrt{2}$ is a rational number, then it can be written as

$$\sqrt{2} = \frac{p}{q}, \quad q \neq 0, p, q \in \mathbb{Z}, p, q \text{ are relatively prime}$$

Multiply by q on both sides and take square, we have

$$2q^2 = p^2$$

Because $2q^2$ is an even number (even number times a number must equal to an even number), p^2 is an even number. Therefore, p itself is an even number (if p is odd, then p^2 is odd, which is a contradiction). Hence, we can write p as

$$p = 2k, \quad k \in \mathbb{Z}$$

Substitute this in the previous equation, we have

$$2q^2 = 4k^2 \quad \Longrightarrow \quad q^2 = 2k^2$$

By the same arguement, we can also conclude that q is an even number. Then, p and q would have a common factor 2, which is a contradiction with our assumption that p, q are relatively prime. $\square$

Therefore, there is another kind of number except for rational numbers! This was a big shock for people during Pythagoras's time, and the discovery of $\sqrt{2}$ is called **'The First Mathematical Crisis'**. Since this discovery broke the belief of Pythagoreanism, Hippasus, who discovered this, was drowned at sea by Pythagoras's followers.

Fortunately, now we fully accept that there is 'irrational numbers'. Nobody would be sentenced to death for acknowledging the existence of irrational numbers. Rational and Irrational numbers together, are called **Real Numbers**, denoted by $\mathbb{R}$.

But, how we should construct real numbers from rational numbers? Could we construct a procedure just like what we did for extending integers to rational numbers? In section 1.7* we will introduce an elegant method, and another method would be introduced later in Chapter 2. Since these construction processes are hard, we should now temporarily just believe that there is indeed a set of numbers called real numbers. In next section we will state the axiom that real numbers should behave.

1.3 The Axiom of Completeness I: Supremum Property

One of the most important property of real number is: **It is complete**. The rigorous definition of completeness would be introduced later. Heuristically, completeness of real numbers means that 'All points on the real line are described by real numbers".

Consider rational numbers, they are 'almost everywhere' on the real line, i.e., there is no such a rational number a that is 'closest' to the rational number b. Indeed, suppose there is a b that is 'closest' to a, then, the rational number $\frac{a+b}{2}$ is 'closer' to a, which is a contradiction. This property is called **dense**, and will be introduced later.

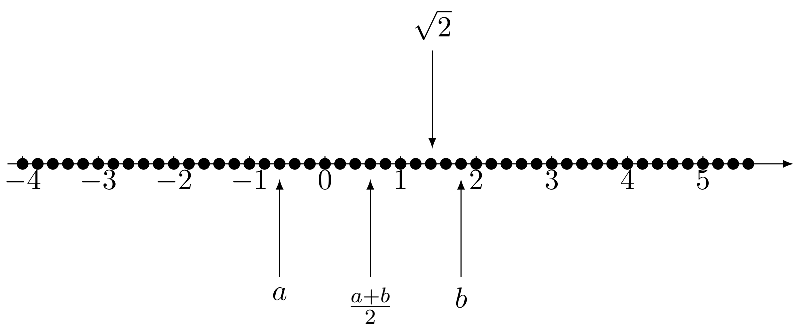


Figure 1.2: Rational Numbers $\mathbb{Q}$ is dense in Real Line $\mathbb{R}$

Even if $\mathbb{Q}$ is dense in $\mathbb{R}$, there are infinite many small 'holes' on the line that was not represented by any of the rational numbers. For example, the point at the distance of $\sqrt{2}$ from the origin, as showed in Figure 1.2. Completeness then means that these holes are exactly 'filled' by 'irrational numbers', so that each point on the line is represented by a unique real number.

To transform these discussions into mathematical language, we first introduce some simple definitions. In this whole note I will denote 'such that' by 's.t.' for simplicity.

Definition 1.3.1: Bounded Above/Bounded Below, Lower/Upper Bound

- A set $A \subseteq \mathbb{R}$ is **bounded above** if

$$\exists\, b \in \mathbb{R}, \text{ s.t. } \forall\, a \in A \implies a \leqslant b$$

The number b is called an **upper bound** for A.

- A set $A \subseteq \mathbb{R}$ is **bounded below** if

$$\exists\, l \in \mathbb{R}, \text{ s.t. } \forall\, a \in A \implies l \leqslant a$$

The number l is called an **lower bound** for A.

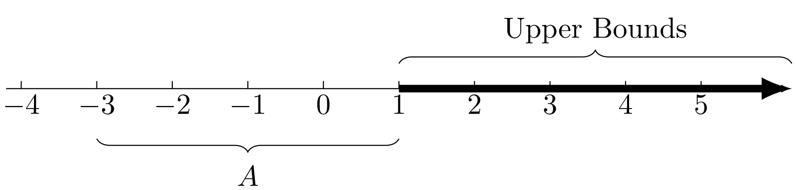


Figure 1.3: A set bounded above

Note that upper bound and lower bound of a set A may not be unique. In fact, if $A \subseteq \mathbb{R}$ is bounded above, upper bounds are always not unique. However, there would sometimes exist a 'least upper bound', which is the most important subject towards the construction of completeness axiom.

Definition 1.3.2: Supremum/Infimum

- A real number s is called the **supremum (least upper bound)** of a set $A \subseteq \mathbb{R}$ if

  1. s is an upper bound for A.
  2. For any upper bound b of A, we have $s \leqslant b$.

  This is denoted by $s = \sup A$.

- A real number l is called the **infimum (greatest lower bound)** of a set $A \subseteq \mathbb{R}$ if

  1. l is a lower bound for A.
  2. For any lower bound b of A, we have $b \leqslant l$.

  This is denoted by $l = \inf A$.

Note that although upper bound is not unique for sets, Supremum, if exists, is unique.

Proposition 1.3.3: Uniqueness of Supremum

A set $A \subseteq \mathbb{R}$ can have at most one supremum.

Proof. Suppose s_1, s_2 are suprema of a set A. Regard s_1 as an upper bound and s_2 as the supremum, we will arrive $s_2 \leqslant s_1$. Regard s_1 as the supremum and s_2 as an upper bound we will arrive $s_1 \leqslant s_2$. Therefore, $s_1 = s_2$. □

Now we should have all tools for the construction of our completeness theorem. This theorem would be seen as an **axiom**, i.e., no need to be proved and it is raised by nature, so that it is an inherent property of the set of real numbers. (In Section 1.7* we will use an elegant method to prove this axiom)

Axiom 1.3.4: Supremum Property

Every nonempty set of real numbers that is bounded above has a supremum.

Why this axiom expresses the completeness of real numbers? We consider a counterexample. Suppose we only have rational number system. Then consider the set $A = (0, \sqrt{2}) \cap \mathbb{Q}$. If the supremum s is less than $\sqrt{2}$, say $s = \sqrt{2} - \epsilon \in \mathbb{Q}$. Then there would be a number $k = \sqrt{2} - \frac{\epsilon}{2} \in A$, such that $k > s$, which is a contradiction to the definition of supremum. Similarly, we can derive that the supremum also cannot be larger than $\sqrt{2}$. Since $\sqrt{2} \notin \mathbb{Q}$, we conclude that this set A, in rational number system, has no supremum.

Therefore, the rational number system $\mathbb{Q}$ does not have this supremum property. It's only for real number system! Actually, this is the first **Axiom of Completeness for real numbers** in this note. In later chapter there would be more, and we will later on examine the relashionships between these Axiom of Completeness.

Note: We state the axiom of completeness only regarding to supremum. There is no need to assert that infimum exists as part of the axiom. To see this, let A be nonempty and bounded below, define B as

$$B = \{b \in \mathbb{R} : \, b \text{ is a lower bound for } A\}$$

Then we will get $\sup B = \inf A$, by the definition of supremum and infimum. For set A, we can then state the axiom of completeness with respect to the set B, i.e., with respect to the supremum.

To conclude this section, a characterization of supremum would be stated below. This is an EXTREMELY USEFUL TOOL since sometimes it is very difficult to work on supremum directly using its definition.

Proposition 1.3.5: Characterization of Supremum

Let $s \in \mathbb{R}$ and set $A \subseteq \mathbb{R}$. $s = \sup A$ if and only if

- s is an upper bound of A.
- $\forall \epsilon > 0, \exists\, a \in A,$ s.t. $s - \epsilon < a$.

Proof.
($\Longrightarrow$) Suppose $s = \sup A$. Then, s is indeed an upper bound by definition. Also, $s - \epsilon$ is not an upper bound for any $\epsilon > 0$, since $s - \epsilon < s$. Therefore, by definition, there exists $a \in A$, such that $s - \epsilon < a$.
($\Longleftarrow$) Suppose s satisfy the conditions stated in the proposition. Then by the second condition, any number smaller than s is not an upper bound. Therefore, s is the least upper bound. $\square$

1.4 Properties of real numbers

There are many applications of the Axiom of Completeness. We will first introduce the important **Archimedean Property**, which states how $\mathbb{N}$ behaves inside $\mathbb{R}$.

Theorem 1.4.1: Archimedean Property

- For any $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ satisfying that $n > x$.
- For any $y > 0$, there exists an $n \in \mathbb{N}$ satisfying that $\frac{1}{n} < y$

Proof.

1. To prove the first statement, we assume that there exists $x \in \mathbb{R}$ such that for all $n \in \mathbb{N}$ we have $n \leqslant x$. This is equivalent to say that, $\mathbb{N}$ is bounded above. By Supremum Property, supremum exists. Let $\alpha = \sup \mathbb{N}$. Then $\alpha + 1 \in \mathbb{N}$. This contradicts the definition of supremum since $\alpha + 1 > \alpha = \sup \mathbb{N}$. Thus, we arrive a contradiction.
2. The second statement follows from (1) by letting $x = 1/y$.

$\square$

Note: It seems that there is no need to prove the statement (1) in Archimedean Property. It just said that $\mathbb{N}$ is unbounded, and we know that as a common sense. However, it is worth noting that as a proper extension of $\mathbb{Q}$ (i.e., a set contains $\mathbb{Q}$ and not equal to $\mathbb{Q}$), the Archimedean Property is very unique for $\mathbb{R}$. Indeed, there **does exist** a proper extension of $\mathbb{Q}$ such that it is bounded (called the Extended-Real Numbers). Discussing this number system will go far out from the scope of this note. You should look for detailed explanation in my Real Analysis note.

Now we see how $\mathbb{Q}$ and $\mathbb{R}\backslash\mathbb{Q}$ behaves inside $\mathbb{R}$.

Proposition 1.4.2: $\mathbb{Q}$ is dense in $\mathbb{R}$

For any $a, b \in \mathbb{R}$, $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.

Proof. We need to produce $m, n \in \mathbb{Z}, n \neq 0$ such that $r = \frac{m}{n}$ and

$$a < \frac{m}{n} < b$$

The first thing we need to do is to choose sufficiently large n so that a ‘step length’ $\frac{1}{n}$ is less than the length of $b - a$, so that there would be some point of the form $\frac{m}{n}$ locating between the two points, as showed in the Figure 1.4.

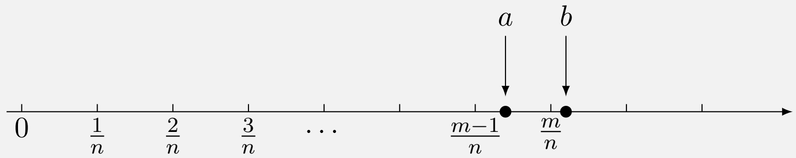


Figure 1.4: Choose sufficiently small step so that $\frac{m}{n}$ is between a and b

By Archimedean Property, we can actually take $n \in \mathbb{N}$ such that

$$\frac{1}{n} < b - a$$

Now, as showed in the picture, we need to choose $m \in \mathbb{Z}$ so that

$$\frac{m-1}{n} \leqslant a < \frac{m}{n} \quad \Longrightarrow \quad m - 1 \leqslant na < m$$

The only thing left is that to prove $\frac{m}{n} < b$. To do this, we see that

$$m \leqslant na + 1 < n\left(b - \frac{1}{n}\right) + 1 = nb$$

Therefore, the proposition is proved. □

We can use the same strategy to see that irrational number is also dense in $\mathbb{R}$. Before doing that, we need to prove some lemma about properties of operations in rational and irrational numbers.

Lemma 1.4.3: Operations in $\mathbb{Q}$ and $\mathbb{R}\backslash\mathbb{Q}$

1. If $a, b \in \mathbb{Q}$, then $a + b, ab \in \mathbb{Q}$.
2. If $a \in \mathbb{Q}, b \in \mathbb{R}\backslash\mathbb{Q}$, then $a + t, at(a \neq 0) \in \mathbb{R}\backslash\mathbb{Q}$

Proof.

1. Since $a, b \in \mathbb{Q}$, we have $a = \frac{m}{n}$, $b = \frac{r}{q}$ for $m, n, r, q \in \mathbb{Z}, n, q \neq 0$. Therefore,

$$a + b = \frac{mq + nr}{nq} \in \mathbb{Q}, \quad ab = \frac{mr}{nq} \in \mathbb{Q}$$

since $mq + nr, nq, mr \in \mathbb{Z}$.

2. Since $a \in \mathbb{Q}$, we have $a = \frac{m}{n}$ where $m, n \in \mathbb{Z}$, $n \neq 0$. Suppose $a + t \in \mathbb{Q}$, then we can write

$a+t=\frac{r}{q}, r,q\in\mathbb{Z}, q\neq 0$. Then,

$$t=(a+t)-a=\frac{r}{q}-\frac{m}{n}=\frac{rn-mq}{qn}\in\mathbb{Q}$$

which is a contradiction. Similarly, we can also see that $at\notin\mathbb{Q}$ (when $a\neq 0$). □

Proposition 1.4.4: $\mathbb{R}\backslash\mathbb{Q}$ is dense in $\mathbb{R}$

For any $a,b\in\mathbb{R}, a<b$, there exists a $t\in\mathbb{R}\backslash\mathbb{Q}$ such that $a<t<b$.

Proof. As in Proposition 1.4.2, we first choose $n\in\mathbb{N}$ such that

$$\frac{1}{n}<b-a$$

Then, we choose $m\in\mathbb{Z}$ such that

$$m+\sqrt{2}-1\leqslant na<m+\sqrt{2}$$

Obviously, we have $a<\frac{m+\sqrt{2}}{n}$. Also,

$$m+\sqrt{2}\leqslant na+1<n\left(b-\frac{1}{n}\right)+1=nb$$

Thus, $a<\frac{m+\sqrt{2}}{n}<b$. Since $m,\frac{1}{n}\in\mathbb{Q}$, $\sqrt{2}\in\mathbb{R}\backslash\mathbb{Q}$, by Lemma 1.4.3, we have $\frac{m+\sqrt{2}}{n}\in\mathbb{R}\backslash\mathbb{Q}$. The statement is proved. □

At this stage we have proved that $\sqrt{2}$ is not a rational number. But, we have not proved that it is a real number. Below we will prove this. Why we need to prove this obvious thing? Indeed, you will see from the prove below that the main theorem used is Supremum Property. Therefore, this inherent property of real numbers asserts that those irrational numbers we encounter often are all real numbers.

Proposition 1.4.5: $\sqrt{2}$ is a real number

There exists a real number $\alpha\in\mathbb{R}$ such that $\alpha^2=2$.

Proof. Consider the set

$$T=\{t\in\mathbb{R}:t^2<2\}$$

Set $\alpha=\sup T$.

- Suppose $\alpha^2 < 2$. Let $n \in \mathbb{N}$ be arbitrary, then

$$\left(\alpha + \frac{1}{n}\right)^2 = \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} < \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n} = \alpha^2 + \frac{2\alpha + 1}{n}$$

  If we choose $n \in \mathbb{N}$ such that

$$\frac{1}{n} < \frac{2 - \alpha^2}{2\alpha + 1}$$

  The existence of this n is promised by Archimedean Property. Note that $n > 0$ since we assume that $\alpha^2 < 2$. We would get

$$\left(\alpha + \frac{1}{n}\right)^2 < \alpha^2 + \frac{(2\alpha + 1)(2 - \alpha^2)}{2\alpha + 1} = 2$$

  Therefore, $\alpha + \frac{1}{n} \in T$, which means that α is not an upper bound, which is a contradiction to that $\alpha = \sup T$.

- Suppose $\alpha^2 > 2$. Similarly, we can write

$$\left(\alpha - \frac{1}{n}\right)^2 = \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} > \alpha^2 - \frac{2\alpha}{n}$$

  If we choose $n \in \mathbb{N}$ such that

$$\frac{1}{n} < \frac{\alpha^2 - 2}{2\alpha}$$

  Again, the existance is promised by Archimedean Property. Then we would have

$$\left(\alpha - \frac{1}{n}\right)^2 > \alpha^2 - \alpha^2 + 2 = 2$$

  This shows that $\alpha - \frac{1}{n}$ is an upper bound of T, which means that α is not the least upper bound, i.e., not the supremum. This is a contradiction to the assumption that $\alpha = \sup T$.

By Supremum Property, the supremum of T exists. Then, it can only be $\sqrt{2}$. Since the supremum of a set is a real number (which is promised in Supremum Property), we finally arrive that $\sqrt{2}$ is actually a real number. $\square$

1.5 The Axiom of Completeness II: Nested Interval Property

Another famous Axiom of Completeness of real numbers is called the **(Cantor) Nested Interval Property**. It says that for any nested closed interval sequence, the intersection of these intervals is not empty. Here is what all these words mean.

Axiom 1.5.1: Nested Interval Property

For each $n \in \mathbb{N}$, construct a closed interval $I_n = [a_n, b_n]$, where $a_n, b_n \in \mathbb{R}$. Assume $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$. Then, the intersection of these nested intervals

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

Figure 1.5: Nested Intervals

Actually, this axiom can be derived from Supremum Property, as showed below.

Proof. Consider the set

$$A = \{a_n : n \in \mathbb{N}\}$$

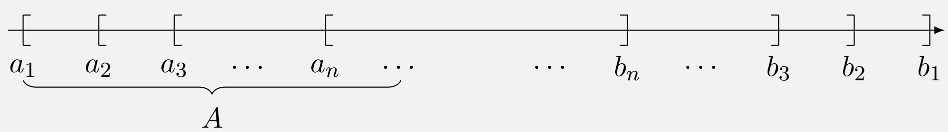


Figure 1.6: Nested Intervals with set A

We can see that each b_n is served as an upper bound for the set $[a_n, b_n]$, for every $n \in \mathbb{N}$. Set $x = \sup A$. Then $a_n \leqslant x$ for every n since x is the upper bound. Also, $b_n \geqslant x$ for every n since b_n are upper bounds and x is the supremum. Therefore we get,

$$\forall n \in \mathbb{N}, a_n \leqslant x \leqslant b_n$$

Hence, $x \in \bigcap_{n=1}^{\infty} I_n \neq \emptyset$. □

Can we, inversely, get Supremum Property from Nested Interval Property? The answer is no! Therefore, there is some 'strong' axioms and some 'weak' axioms. In this case, Supremum Property is 'strong' and Nested Interval Property is 'weak', since we can go from Supremum Property to Nested Interval, but not the reverse direction.

But, why we can't? In the 'proof' below I will show you a **circular reasoning**, indicating that we can only go from Nested Interval Property to Supremum Property if Archimedean Property exists.

Note: This is not a prove, but just a reasoning process.

Suppose Nested Interval Property is true. We want to prove Supremum Property. Let A be a nonempty set which is bounded above. Denote $\alpha = \sup A$. Choose $a \in A$ and b such that b is an upper bound of A. Then, consider the intervals $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$. Then, α is at least in one of these two intervals. Choose an interval that contains it. Continue bisecting this interval as we did at the last step. Again, the supremum would contain in one of the intervals. Choose that interval. $\cdots\cdots$

After n steps, the length of the chosen interval would be $\frac{b-a}{2^n}$. The only thing we are left to do is to prove that $\frac{b-a}{2^n}$ converges to 0 (The rigorous definition of limit would be introduced in Chapter 2, here you can just recall what you have learnt in Year I Calculus). However, the proof of this statement needs Archimedean Property, since we want for every $\epsilon > 0$, we can find a $n \in \mathbb{N}$ such that $\frac{b-a}{2^n} < \epsilon$. This is equivalent to what is said in the second statement of Archimedean Property. Recall how Archimedean Property is derived. Yes, it is derived from Supremum Property! So we cannot directly use Archimedean Property if we assume we don’t know Supremum Property and want to prove it. This is an example of **circular reasoning**, and in result, we have no way deriving Supremum Property from Nested Interval Property.

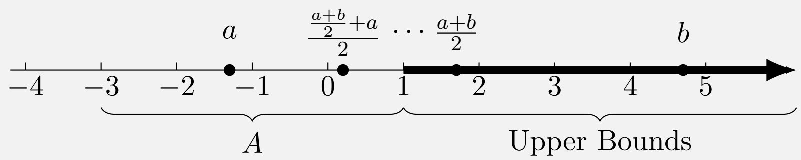


Figure 1.7: Bisecting intervals to approximate supremum

Therefore, we get the relationship between Supremum Property and Nested Interval Property. The relation is visually displayed below in Figure 1.8.

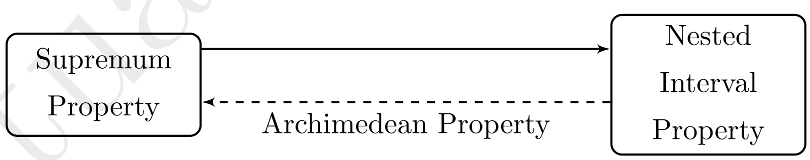


Figure 1.8: Relation between Axiom of Completeness

This graph would be further expanded when we encounter more and more Axiom of Completeness.

1.6 ‘Size of Infinity’: Cardinality

Does infinity also have different ‘sizes’? You may ask after seeing this section title. The answer is yes! We all know that there are infinitely many rational numbers and irrational numbers, and it seems that both of them are ‘almost everywhere’ on the real line. However, in this section we will introduce a surprising result: There are ‘much more’ irrational numbers than rational numbers!

Before we discuss this result, we need to first talk about the way of comparing two ‘infinite sizes’. Let’s start with finite one. To compare the number of elements in two finite sets, it is easy, just count them. For example, $A = \{1, 2, 3, 4\}$ and $B = \{5, 6, 7, 8\}$. They all have 4 elements, and naturally, have the same size. To extend this into infinite case, we need to use **bijective maps**.

Definition 1.6.1: Cardinality

Two sets A, B have the same **Cardinality** if there exists a bijective map $f : A \to B$ such that each element of A is mapped one-to-one and onto an element of B. Dented as $A \sim B$.

Then, the cardinality of a set just discribes the ‘size’ of that set. Let’s see some examples first.

Example 1.6.2: $\mathbb{N}$ ***has the same cardinality as the set of even numbers***

This is weird at first glance, since intuitively the set of even numbers is a proper subset of $\mathbb{N}$, and they could not have the same size. Denote the set of even numbers as

$$E = \{2, 4, 6, 8, \cdots\}$$

Then we can construct a bijective map $f : \mathbb{N} \to E$ by $f(n) = 2n, \forall\, n \in \mathbb{N}$.

$$\begin{array}{rcccccccc} \mathbb{N}: & 1 & 2 & 3 & 4 & \cdots & n & \cdots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots & \updownarrow & \cdots \\ E: & 2 & 4 & 6 & 8 & \cdots & 2n & \cdots \end{array}$$

Example 1.6.3: $\mathbb{N} \sim \mathbb{Z}$

We can construct a bijective map $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = \begin{cases} \dfrac{n-1}{2}, & \text{if } n \text{ is odd} \\ -\dfrac{n}{2}, & \text{if } n \text{ is even} \end{cases}$$

$$\begin{array}{rcccccccc} \mathbb{N}: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots \\ \mathbb{Z}: & 0 & -1 & 1 & -2 & 2 & -3 & 3 & \cdots \end{array}$$

In these two examples, we see that we correspond elements of some set with natural numbers $1, 2, 3, 4, \cdots$, just as we are counting them in some order. This is such an important case that we gave it a name.

Definition 1.6.4: Countable/Uncountable Set

A set is A is called

- **Finite** if the number of elements in A is finite.
- **Countable** if $A \sim \mathbb{N}$.
- **Uncountable** if it is infinite and not countable.

We can see from Example 1.6.2 and 1.6.3 that E and $\mathbb{Z}$ are countable sets. What does uncountable set looks like? **The next theorem is central for this section.** It says that $\mathbb{R}$ has a somewhat 'bigger' size than $\mathbb{Q}$.

Theorem 1.6.5: Countability of $\mathbb{Q}$, Uncountability of $\mathbb{R}$

1. $\mathbb{Q}$ is a countable set.
2. $\mathbb{R}$ is an uncountable set.

Proof.

1. There are two popular ways of proving that $\mathbb{Q}$ is countable. I will show both of them.
   METHOD I: Arrage all the rational numbers in an infinite matrix such that mth row and nth column corresponds to the number $\frac{n}{m}$. Then, assign natural numbers to them 'meanderingly', as showed below. If there is a number that has the same value with some number that has been assigned, we delete it. For example, $\frac{1}{1}$ is assigned 1, $\frac{1}{2}$ is assigned 2, $\frac{2}{1}$ is assigned 3, $\frac{3}{1}$ is assigned 4, $\frac{2}{2}$ is deleted since it has the same value with $\frac{1}{1}$, and $\frac{1}{3}$ is assigned 5...... Continuing this fashion, we will have a bijective map from positive rational numbers to natural numbers.

   With this, we can further map 0 to 0 and map negative rational numbers to negative integers.

Then, this whole map is a bijective map from $\mathbb{Q}$ to $\mathbb{Z}$. Since there also exists biject map from $\mathbb{Z}$ to $\mathbb{N}$, we have $\mathbb{N} \sim \mathbb{Q}$.

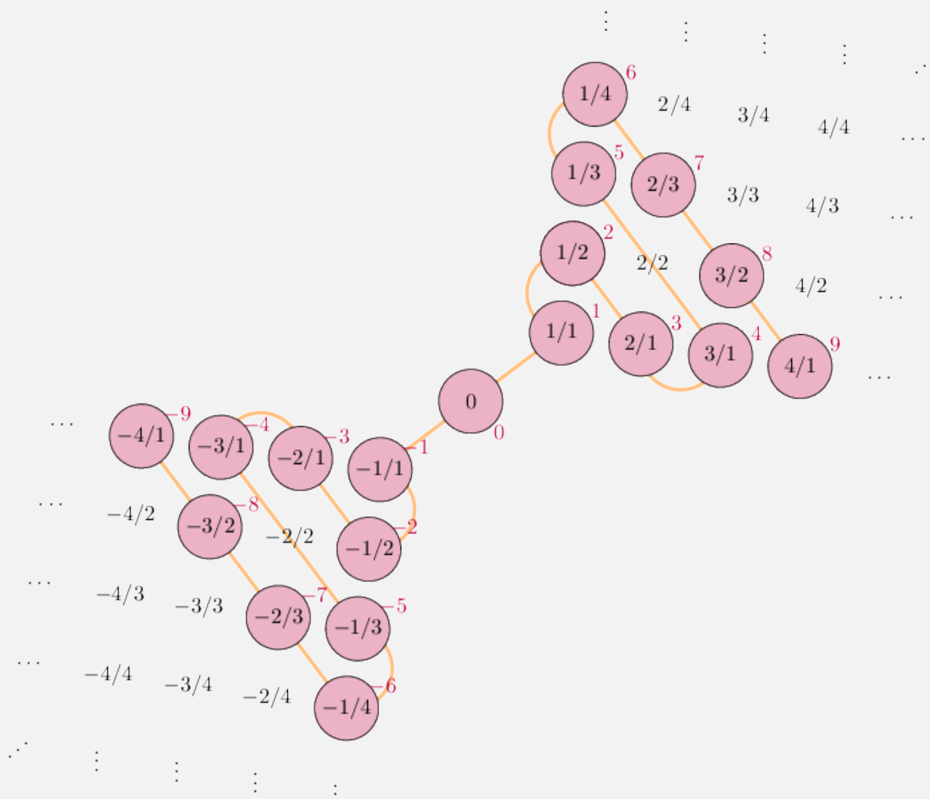
METHOD II: Set $A_1 = \{0\}$, and for all $n \geqslant 2$, set

$$A_n = \left\{ \pm\frac{p}{q} : p, q \in \mathbb{N} \text{ are relatively prime with } p + q = 0 \right\}$$

For example,

$$A_2 = \left\{ \frac{1}{1}, -\frac{1}{1} \right\}, \quad A_3 = \left\{ \frac{1}{2}, -\frac{1}{2}, \frac{2}{1}, -\frac{2}{1} \right\}, \quad A_4 = \left\{ \frac{1}{3}, -\frac{1}{3}, \frac{3}{1}, -\frac{3}{1} \right\}, \cdots$$

Each A_n is finite and every rational number appears in exactly one of these sets. Therefore, we can construct the bijective map by listing the elements in each A_n.



2. The main theorem used in this proof is the **Nested Interval Property**. We will prove by contradiction. Suppose there exists a bijective function $f : \mathbb{N} \to \mathbb{R}$. Then, each real number can be assigned to a natural number. Therefore, we can denote x_i as the real number being assigned to the natural number i, and write $\mathbb{R}$ as

$$\mathbb{R} = \{x_1, x_2, x_3, x_4, \cdots\}$$

Now, consider the set $[0, 9]$. We can divide it into 3 parts: $[0, 3] \cup [3, 6] \cup [6, 9]$. Then, x_1 can at most belong to two of them. Choose the interval that x_1 does not belong to, denote it as I_1.

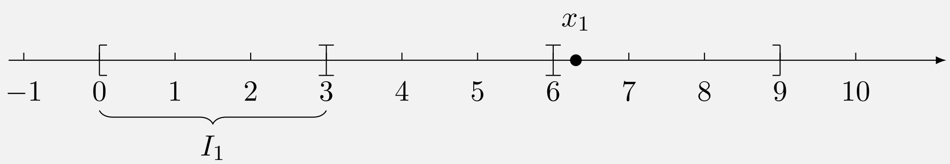


Figure 1.9: Construction Process of Nested Intervals, I

Then, we can divide I_1 into 3 equal parts just as the previous step. Again, x_2 can at most belong to one of these three intervals. Choose the one that x_2 does not belong to, and call it I_2. Continuing this fashion, for I_n, we divide it into 3 equal parts, and choose the interval that x_{n+1} does not belong to, call it $I_{n+1} \cdots$.

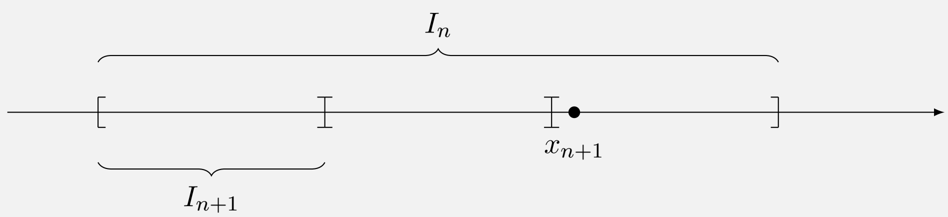


Figure 1.10: Construction Process of Nested Intervals, II

Using this procedure, we can produce nested intervals I_n such that

$$I_{n+1} \subseteq I_n, \forall\, n \in \mathbb{N}, \text{ and } \quad x_n \notin I_n$$

Therefore,

$$x_n \notin \bigcap_{n=1}^{\infty} I_n, \forall\, n \in \mathbb{N}$$

This shows that

$$\bigcap_{n=1}^{\infty} I_n = \emptyset$$

which is a contradiction to the Nested Interval Property. □

Therefore, $\mathbb{R}$ is a ‘bigger’ set than $\mathbb{N}$! There does exist uncountable sets. In examples before, we have seen some other sets that is countable. In the next few examples, we will see what does uncountable sets look like.

Example 1.6.6: $(-1,1) \sim \mathbb{R}$

Here we can construct the function $f : (-1,1) \to \mathbb{R}$ by

$$f(x) = \frac{x}{x^2-1}$$

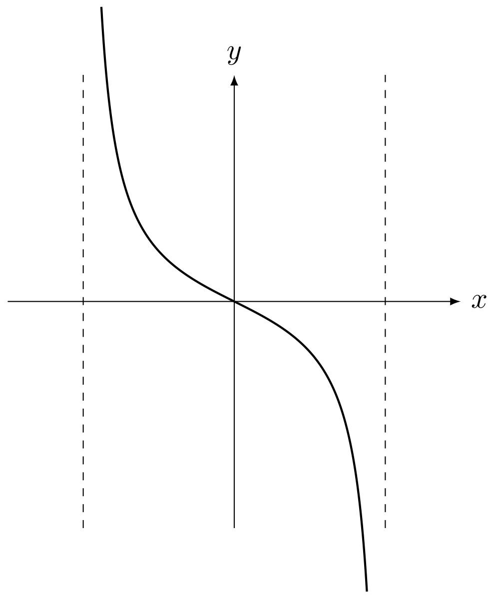


Figure 1.11: function $f(x) = \frac{x}{x^2-1}$

Example 1.6.7: $(a,b) \sim \mathbb{R}$

To extend the result from Example 1.6.6 to the case for every $a, b \in \mathbb{R}$, we can just do linear transformation on the function f in that example. Set

$$-1 < kx + c < 1, k > 0$$

We have

$$\frac{-1-c}{k} < x < \frac{1-c}{k}$$

Therefore, we can set

$$a = \frac{-1-c}{k}, \quad b = \frac{1-c}{k}$$

To get

$$k = \frac{2}{b-a}, \quad c = \frac{a+b}{a-b}$$

Thus

$$g(x) = f\left(\frac{2x}{b-a} + \frac{a+b}{a-b}\right)$$

will map (a, b) to the whole space $\mathbb{R}$.

1.7* The Dedekind Cuts: Construction from $\mathbb{Q}$ to $\mathbb{R}$

This section is a hard section and can be skipped without losing coherence.

Chapter 2

Infinite Sequences and Series

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