

# Fantuan's Academia

FANTUAN'S MATH NOTES SERIES

## Notes on Mathematical Analysis

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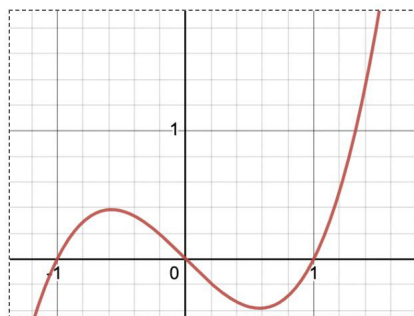


**Real Analysis Student**

YOU NEED THAT FOR  $f: A \rightarrow \mathbb{R}$ ,  
 $c \in A$ , THE FUNCTION IS  
CONTINUOUS AT  $c$  IF AND ONLY  
IF  $\forall \varepsilon > 0 \exists \delta > 0 \ni |x - c| < \delta$  and  
 $x \in A$  implies  $|f(x) - f(c)| < \varepsilon!!!$   
OTHERWISE IT'S NOT  
SUFFICIENTLY RIGOROUS!!!!



**Precalculus Student**



If I can draw it without picking  
my pen up, it's continuous.

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All the Sections with \* are hard sections and can be skipped without losing coherence.

This note is referenced on **Understanding Analysis** by Stephen Abbott [1], **Principles of Mathematical Analysis** by Walter Rudin [2], **Analysis I** by Terence Tao [3], and MTH 117,118 notes of XJTLU.



# Chapter 1

## Real Numbers

### 1.1 Why Analysis?

*Analysis*, simply saying, is a course about ‘rigorous calculus’. Somebody may ask then: “why we need another course about calculus?”. Indeed, basic calculus concepts and various computing skills are introduced in Year I Calculus course. However, regarding calculus as a pure math object, it should maintain its full rigor. If we apply calculus in the real world problems without knowing where they came from and what is their constraints to be correctly applied, some pathological things will happen, as listed below.

**Example 1.1.1.** *Infinite Series*

Consider the divergent infinite series

$$S = 1 + 2 + 4 + 8 + 16 + \cdots \quad (1.1)$$

If we multiply it by 2,

$$2S = 2 + 4 + 8 + 16 + \cdots \quad (1.2)$$

Subtract (1.1) from (1.2), we will have the ridiculous result

$$S = -1$$

**Example 1.1.2.** *Interchanging Integrals*

We always change the order of double integral to make calculation easier. But, can we always do that in any cases? Consider

$$\int_0^\infty \int_0^1 (e^{-xy} - xye^{-xy}) \, dy \, dx$$

If we directly compute this, we can get,

$$\int_0^\infty \int_0^1 (e^{-xy} - xye^{-xy}) \, dy \, dx = \int_0^\infty [ye^{-xy}]_{y=0}^1 \, dx = \int_0^\infty e^{-x} \, dx = [-e^{-x}]_0^\infty = 1$$

However, if we change the order of integral

$$\int_0^1 \int_0^\infty (e^{-xy} - xye^{-xy}) \, dx \, dy = \int_0^1 [xe^{-xy}]_{x=0}^\infty \, dy = \int_0^1 (0 - 0) \, dy = 0$$

We arrive different answers!

**Example 1.1.3. Reordering Infinite Series**

Consider the alternating harmonic series

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

We know that this infinite series converges at some point. Therefore, nothing similar as Example 1.1.1 could happen here. However, if we do the following computation:

$$\begin{aligned} \frac{1}{2}S &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} + \cdots \\ S &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \frac{1}{14} + \frac{1}{15} - \frac{1}{16} + \cdots \end{aligned}$$

$$\frac{3}{2}S = \left(1 + \frac{1}{3}\right) - \frac{1}{2} + \left(\frac{1}{5} + \frac{1}{7}\right) - \frac{1}{4} + \left(\frac{1}{9} + \frac{1}{11}\right) - \frac{1}{6} + \left(\frac{1}{13} + \frac{1}{15}\right) - \frac{1}{8} + \cdots$$

We see that  $\frac{3}{2}S$  is just a reordering of our initial infinite series (with two positive terms following one negative term)! Therefore, we just change the convergent point by simply reordering the infinite series.

This doesn't make sense! Since by intuition, reordering the terms in an algorithm will not change its result. However, you see here, the situation changes in the infinite case.

As showed above, we indeed need this course to make analysis as a much more rigorous math topic than Year I Calculus. To get started, we will first talk about real numbers.

## 1.2 From Rational to Irrational Numbers

The most simplest number system we can call to our mind is the **Natural Numbers**

$$\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$$

Obviously, this number system is based on counting. It is enough for the simple use of counting things. However, this number system is not closed under subtraction (i.e., one natural number subtracts another natural number may not result in a natural number). Therefore, we introduce the **Integer Numbers**

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

This number system is still not complete since it is not closed under division. For example,  $3 \div 5$  is not in the list. We further introduce the **Rational Numbers**

$$\mathbb{Q} = \left\{ \frac{p}{q} : q \neq 0, p, q \in \mathbb{Z} \right\}$$

Back to Pythagoras's era (500-400 BC), he only believes the existence of rational numbers, and so did his followers in Pythagoreanism, except for one: Hippasus of Metapontum. After Pythagoras announced his famous Pythagorean Theorem, Hippasus directly used this theorem to discover  $\sqrt{2}$ : an irrational number!

Consider a right-angled triangle with two right-angled edges of length 1. Then by Pythagorean Theorem, length of the hypotenuse  $z$  should satisfy

$$z^2 = 1^2 + 1^2 = 2$$

We denote this number as  $\sqrt{2}$ , for which the square of it is 2. It seems that we cannot write this number in the form of a rational number. And indeed, we can prove that it is not a rational number.

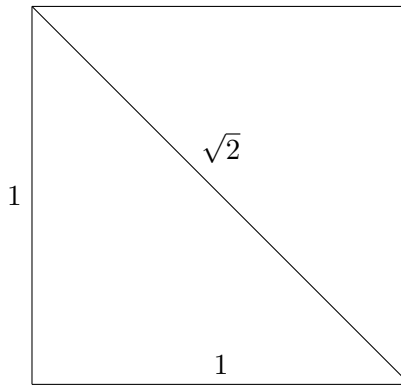


Figure 1.1: Pythagorean Theorem and  $\sqrt{2}$

**Proposition 1.2.1: Irrationality of  $\sqrt{2}$**

$\sqrt{2}$  is not a rational number.

*Proof.* We prove by contradiction. Suppose  $\sqrt{2}$  is a rational number, then it can be written as

$$\sqrt{2} = \frac{p}{q}, \quad q \neq 0, p, q \in \mathbb{Z}, p, q \text{ are relatively prime}$$

Multiply by  $q$  on both sides and take square, we have

$$2q^2 = p^2$$

Because  $2q^2$  is an even number (even number times a number must equal to an even number),  $p^2$  is an even number. Therefore,  $p$  itself is an even number (if  $p$  is odd, then  $p^2$  is odd, which is a contradiction). Hence, we can write  $p$  as

$$p = 2k, \quad k \in \mathbb{Z}$$

Substitute this in the previous equation, we have

$$2q^2 = 4k^2 \implies q^2 = 2k^2$$

By the same argument, we can also conclude that  $q$  is an even number. Then,  $p$  and  $q$  would have a common factor 2, which is a contradiction with our assumption that  $p, q$  are relatively prime.  $\square$

Therefore, there is another kind of number except for rational numbers! This was a big shock for people during Pythagoras's time, and the discovery of  $\sqrt{2}$  is called '**The First Mathematical Crisis**'. Since this discovery broke the belief of Pythagoreanism, Hippasus, who discovered this, was drowned at sea by Pythagoras's followers.

Fortunately, now we fully accept that there is 'irrational numbers'. Nobody would be sentenced to death for acknowledging the existence of irrational numbers. Rational and Irrational numbers together, are called **Real Numbers**, denoted by  $\mathbb{R}$ .

But, how we should construct real numbers from rational numbers? Could we construct a procedure just like what we did for extending integers to rational numbers? In section 1.5\* we will introduce an elegant method, and another method would be introduced later in Chapter 2. Since these construction processes are hard, we should now temporarily just believe that there is indeed a set of numbers called real numbers. In next section we will state the axiom that real numbers should behave.

### 1.3 The Axiom of Completeness I: Supremum Property

One of the most important property of real number is: **It is complete**. The rigorous definition of completeness would be introduced later. Heuristically, completeness of real numbers means that 'All points on the real line are described by real numbers'.

Consider rational numbers, they are 'almost everywhere' on the real line, i.e., there is no such a rational number  $a$  that is 'closest' to the rational number  $b$ . Indeed, suppose there is a  $b$  that is 'closest' to  $a$ , then, the rational number  $\frac{a+b}{2}$  is 'closer' to  $a$ , which is a contradiction. This property is called **dense**, and will be



introduced later.

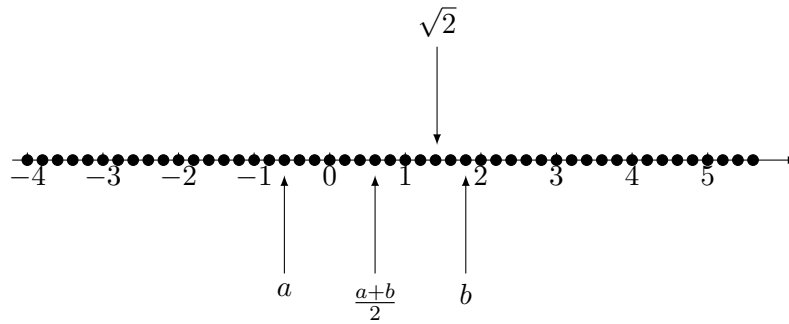


Figure 1.2: Rational Numbers  $\mathbb{Q}$  is dense in Real Line  $\mathbb{R}$

Even if  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , there are infinite many small ‘holes’ on the line that was not represented by any of the rational numbers. For example, the point at the distance of  $\sqrt{2}$  from the origin, as showed in Figure 1.2. Completeness then means that these holes are exactly ‘filled’ by ‘irrational numbers’, so that each point on the line is represented by a unique real number.

To transform these discussions into mathematical language, we first introduce some simple definitions. In this whole note I will denote ‘such that’ by ‘s.t.’ for simplicity.

#### Definition 1.3.1: Bounded Above/Bounded Below, Lower/Upper Bound

- A set  $A \subseteq \mathbb{R}$  is **bounded above** if

$$\exists b \in \mathbb{R}, \text{ s.t. } \forall a \in A \implies a \leq b$$

The number  $b$  is called an **upper bound** for  $A$ .

- A set  $A \subseteq \mathbb{R}$  is **bounded below** if

$$\exists l \in \mathbb{R}, \text{ s.t. } \forall a \in A \implies l \leq a$$

The number  $l$  is called an **lower bound** for  $A$ .

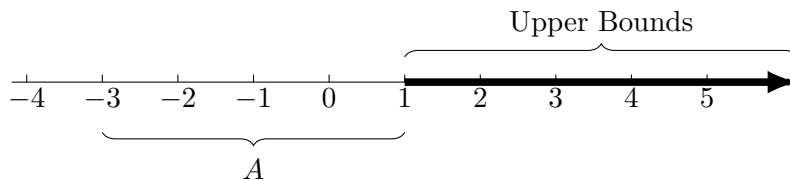


Figure 1.3: A set bounded above

Note that upper bound and lower bound of a set  $A$  may not be unique. In fact, if  $A \subseteq \mathbb{R}$  is bounded above, upper bounds are always not unique. However, there would sometimes exist a ‘least upper bound’, which is the most important subject towards the construction of completeness axiom.

### Definition 1.3.2: Supremum/Infimum

- A real number  $s$  is called the **supremum (least upper bound)** of a set  $A \subseteq \mathbb{R}$  if
  1.  $s$  is an upper bound for  $A$ .
  2. For any upper bound  $b$  of  $A$ , we have  $s \leq b$ .

This is denoted by  $s = \sup A$ .

- A real number  $l$  is called the **infimum (greatest lower bound)** of a set  $A \subseteq \mathbb{R}$  if
  1.  $l$  is a lower bound for  $A$ .
  2. For any lower bound  $b$  of  $A$ , we have  $b \leq l$ .

This is denoted by  $l = \inf A$ .

Note that although upper bound is not unique for sets, Supremum, if exists, is unique.

### Proposition 1.3.3: Uniqueness of Supremum

A set  $A \subseteq \mathbb{R}$  can have at most one supremum.

*Proof.* Suppose  $s_1, s_2$  are suprema of a set  $A$ . Regard  $s_1$  as an upper bound and  $s_2$  as the supremum, we will arrive  $s_2 \leq s_1$ . Regard  $s_1$  as the supremum and  $s_2$  as an upper bound we will arrive  $s_1 \leq s_2$ . Therefore,  $s_1 = s_2$ .  $\square$

Now we should have all tools for the construction of our completeness theorem. This theorem would be seen as an **axiom**, i.e., no need to be proved and it is raised by nature, so that it is an inherent property of the set of real numbers. (In Section 1.5\* we will use an elegant method to prove this axiom)

### Axiom 1.3.4: Supremum Property

Every nonempty set of real numbers that is bounded above has a supremum.

Why this axiom expresses the completeness of real numbers? We consider a counterexample. Suppose we only have rational number system. Then consider the set  $A = (0, \sqrt{2}) \cap \mathbb{Q}$ . If the supremum  $s$  is less than  $\sqrt{2}$ , say  $s = \sqrt{2} - \epsilon \in \mathbb{Q}$ . Then there would be a number  $k = \sqrt{2} - \frac{\epsilon}{2} \in A$ , such that  $k > s$ , which is a contradiction to the definition of supremum. Similarly, we can derive that the supremum also cannot be

larger than  $\sqrt{2}$ . Since  $\sqrt{2} \notin \mathbb{Q}$ , we conclude that this set  $A$ , in rational number system, has no supremum.

Therefore, the rational number system  $\mathbb{Q}$  does not have this supremum property. It's only for real number system! Actually, this is the first **Axiom of Completeness for real numbers** in this note. In later chapter there would be more, and we will later on examine the relationships between these Axiom of Completeness.

**Note:** We state the axiom of completeness only regarding to supremum. There is no need to assert that infimum exists as part of the axiom. To see this, let  $A$  be nonempty and bounded below, define  $B$  as

$$B = \{b \in \mathbb{R} : b \text{ is a lower bound for } A\}$$

Then we will get  $\sup B = \inf A$ , by the definition of supremum and infimum. For set  $A$ , we can then state the axiom of completeness with respect to the set  $B$ , i.e., with respect to the supremum.

To conclude this section, a characterization of supremum would be stated below. This is an **EXTREMELY USEFUL TOOL** since sometimes it is very difficult to work on supremum directly using its definition.

#### Proposition 1.3.5: Characterization of Supremum

Let  $s \in \mathbb{R}$  and set  $A \subseteq \mathbb{R}$ .  $s = \sup A$  if and only if

- $s$  is an upper bound of  $A$ .
- $\forall \epsilon > 0, \exists a \in A$ , s.t.  $s - \epsilon < a$ .

*Proof.*

( $\implies$ ) Suppose  $s = \sup A$ . Then,  $s$  is indeed an upper bound by definition. Also,  $s - \epsilon$  is not an upper bound for any  $\epsilon > 0$ , since  $s - \epsilon < s$ . Therefore, by definition, there exists  $a \in A$ , such that  $s - \epsilon < a$ .

( $\impliedby$ ) Suppose  $s$  satisfy the conditions stated in the proposition. Then by the second condition, any number smaller than  $s$  is not an upper bound. Therefore,  $s$  is the least upper bound.  $\square$

## 1.4 Properties of real numbers

There are many applications of the Axiom of Completeness. We will first introduce the important **Archimedean Property**, which states how  $\mathbb{N}$  behaves inside  $\mathbb{R}$ .

**Theorem 1.4.1: Archimedean Property**

- For any  $x \in \mathbb{R}$ , there exists an  $n \in \mathbb{N}$  satisfying that  $n > x$ .
- For any  $y > 0$ , there exists an  $n \in \mathbb{N}$  satisfying that  $\frac{1}{n} < y$

*Proof.*

1. To prove the first statement, we assume that there exists  $x \in \mathbb{R}$  such that for all  $n \in \mathbb{N}$  we have  $n \leq x$ . This is equivalent to say that,  $\mathbb{N}$  is bounded above. By Supremum Property, supremum exists. Let  $\alpha = \sup \mathbb{N}$ . Then  $\alpha + 1 \in \mathbb{N}$ . This contradicts the definition of supremum since  $\alpha + 1 > \alpha = \sup \mathbb{N}$ . Thus, we arrive a contradiction.
2. The second statement follows from (1) by letting  $x = 1/y$ .

□

**Note:** It seems that there is no need to prove the statement (1) in Archimedean Property. It just said that  $\mathbb{N}$  is unbounded, and we know that as a common sense. However, it is worth noting that as a proper extension of  $\mathbb{Q}$  (i.e., a set contains  $\mathbb{Q}$  and not equal to  $\mathbb{Q}$ ), the Archimedean Property is very unique for  $\mathbb{R}$ . Indeed, there **does exist** a proper extension of  $\mathbb{Q}$  such that it is bounded (called the Extended-Real Numbers). Discussing this number system will go far out from the scope of this note. You should look for detailed explanation in my Real Analysis note.

Now we see how  $\mathbb{Q}$  and  $\mathbb{R} \setminus \mathbb{Q}$  behaves inside  $\mathbb{R}$ .

**Proposition 1.4.2:  $\mathbb{Q}$  is dense in  $\mathbb{R}$** 

For any  $a, b \in \mathbb{R}$ ,  $a < b$ , there exists  $r \in \mathbb{Q}$  such that  $a < r < b$ .

**1.5\* The Dedekind Cuts: Construction from  $\mathbb{Q}$  to  $\mathbb{R}$** 

This section is a hard section and can be skipped without losing coherence.

## Chapter 2

# Infinite Sequences and Series



# Bibliography

- [1] Abbott, S. (2015). *Understanding analysis*. Springer.
- [2] Rudin, W. (1953). *Principles of mathematical analysis*.
- [3] Tao, T. (2006). *Analysis i*, volume 1. Springer.