Q3.

(a) $$RSS_{Ridge} = \sum_{i=1}^{n} (y_i - \beta - \beta x)^2 + \lambda \beta^2$$

$$\frac{d\,RSS}{d\beta} = \cancel{2} \sum_{i=1}^{n} (y_i - \beta - \beta x_i)(-1 - x_i) + \cancel{2}\lambda\beta = 0$$

$$\Rightarrow \lambda\beta = \sum (1+x_i)\, y_i - (1+x_i)\beta - (x_i + x_i^2)\beta$$

$$\Rightarrow \beta\left[\lambda + \sum_{i=1}^{n}(x_i + x_i^2 + 1 + x_i)\right] = \sum_{i=1}^{n} (1+x_i)\, y_i$$

$$\longrightarrow \hat{\beta}_R = \frac{\sum_{i=1}^{n} (1+x_i)\, y_i}{\lambda + \sum_{i=1}^{n} (x_i+1)^2} \quad \#$$

(b)

$$E[\varepsilon] = 0, \quad Var(\varepsilon) = \sigma^2$$

$$y = \beta + \beta x + \varepsilon \longrightarrow Var(y) = \sigma^2$$

$$Var(\hat{\beta}_R) = Var\left(\frac{\sum_{i=1}^{n} (1+x_i)\, y_i}{\lambda + \sum_{i=1}^{n} (x_i+1)^2}\right)$$

$$= \frac{\sum_{i=1}^{n} (1+x_i)^2\, Var(y_i)}{\left[\lambda + \sum_{i=1}^{n} (1+x_i)^2\right]^2} = \frac{\sum_{i=1}^{n} (1+x_i)^2}{\left[\lambda + \sum_{i=1}^{n} (1+x_i)^2\right]^2}\,\sigma^2 \quad \#$$

(C) To show $Var(\hat{\beta}_R) \leq Var(\hat{\beta})$

$$\rightarrow \frac{\sum_{i=1}^{n}(1+x_i)^2}{\left[\lambda + \sum_{i=1}^{n}(1+x_i)^2\right]^2} \cancel{\sigma^2} \leq \frac{\cancel{\sigma^2}}{\sum_{i=1}^{n}(1+x_i)^2}$$

$$\rightarrow \left[\sum_{i=1}^{n}(1+x_i)^2\right]^2 \leq \left[\lambda + \sum_{i=1}^{n}(1+x_i)^2\right]^2$$

Since $\lambda > 0$, and $\sum_{i=1}^{n}(1+x_i)^2 \geq 0$

Thus, the equation $Var(\hat{\beta}_R) \leq Var(\hat{\beta})$ holds for all $\lambda > 0$

#