Yen-Chun Chen
chenyenc@oregonstate.edu

Q1

For $y = \beta_0 + \beta_1 x \rightarrow \hat{\beta}_1 = \dfrac{S_{xy} - n\bar{x}\bar{y}}{S_{xx} - n\bar{x}^2}$, $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$

(a) If $\bar{x} = 0$,

$$\hat{\beta}_1 = \frac{S_{xy} - 0}{S_{xx} - 0} = \frac{S_{xy}}{S_{xx}} \quad \#$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \cdot 0 = \bar{y} \quad \#$$

(b) $\bar{x} = \cancel{\frac{1}{n}} \sum_{i=1}^{n} x_i = 0 \rightarrow \sum_{i=1}^{n} x_i = 0$

$$\rightarrow \overline{x^{(-j)}} = \frac{1}{n-1}\Big[\Big(\underbrace{\cancel{\sum_{i=1}^{n} x_i}}_{0}\Big) - x_j\Big] = \frac{-x_j}{n-1} \quad \#$$

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i \rightarrow \sum_{i=1}^{n} y_i = n\bar{y}$$

$$\overline{y^{(-j)}} = \frac{1}{n-1}\Big[\Big(\sum_{i=1}^{n} y_i\Big) - y_j\Big] = \frac{n \cdot \bar{y} - y_j}{n-1} \quad \#$$

(c) $\hat{\beta}_1 = \dfrac{S_{xy} - n\bar{x}\bar{y}}{S_{xx} - n\bar{x}^2}$, $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$

$$\hat{\beta}_1^{(-j)} = \frac{\sum_{i \neq j}^{n} x_i y_i - (n-1)\,\overline{x^{(-j)}}\;\overline{y^{(-j)}}}{\sum_{i \neq j}^{n} x_i^2 - (n-1)\,\overline{x^{(-j)}}^{\,2}} \quad \#$$

$$\hat{\beta}_0^{(-j)} = \overline{y^{(-j)}} - \hat{\beta}_1^{(-j)} \cdot \overline{x^{(-j)}}$$

$$= \frac{n\bar{y} - y_j}{n-1} + \hat{\beta}_1^{(-j)} \frac{x_j}{n-1} \quad \#$$

(d)

$$S_{xy} = \sum_{i=1}^{n} x_i y_i = \sum_{i\neq j}^{n} x_i y_i + x_j y_j$$

$$\rightarrow \begin{cases} \sum_{i\neq j}^{n} x_i y_i = S_{xy} - x_j y_j \\ \sum_{i\neq j}^{n} x_i^2 = S_{xx} - x_j^2 \end{cases}$$

$$\hat{\beta}_1^{(-j)} = \frac{\sum_{i\neq j}^{n} x_i y_i - (n-1)\,\overline{x^{(-j)}}\,\overline{y^{(-j)}}}{\sum_{i\neq j}^{n} x_i^2 - (n-1)\,\overline{x^{(-j)}}^2}$$

$$= \frac{S_{xy} - x_j y_j + \cancel{(n-1)}\cdot\frac{(+x_j)\cdot(n\bar{y}-y_j)}{\cancel{(n-1)}(n-1)}}{S_{xx} - x_j^2 - \cancel{(n-1)}\left(\frac{-x_j}{\cancel{n-1}}\right)^2}$$

$$= \frac{S_{xy} + \left(\frac{1}{n-1}\right)\left(-n x_j y_j + \cancel{x_j y_j} + n x_j \bar{y} - \cancel{x_j y_j}\right)}{S_{xx} + \left(\frac{1}{n-1}\right)\left(-n^2 x_j^2 + \cancel{x_j^2} - \cancel{x_j^2}\right)}$$

$$= \frac{S_{xy} + \left(\frac{n}{n-1}\right)(x_j\bar{y} - x_j y_j)}{S_{xx} + \left(\frac{n}{n-1}\right)(-x_j^2)} = \frac{S_{xy} - \left(\frac{n}{n-1}\right)x_j(y_j - \bar{y})}{S_{xx} - \left(\frac{n}{n-1}\right)x_j^2}$$

#

$$y_j - \hat{\beta}_0^{(-j)} = \frac{n y_j - \cancel{y_j}}{n-1} - \frac{n\bar{y} - \cancel{y_j}}{n-1} - \hat{\beta}_1^{(-j)}\frac{x_j}{n-1}$$

$$= \frac{n}{n-1}(y_j - \bar{y}) - \frac{1}{n-1}\hat{\beta}_1^{(-j)}\cdot x_j$$

#

(e)

$$y_j - \hat{y}_j = y_j - \hat{\beta}_0^{(-j)} - \hat{\beta}_1^{(-j)} \cdot x_j$$

$$= \frac{n(y_j - \bar{y})}{n-1} - \frac{1}{n-1}\hat{\beta}_1^{(-j)} x_j - \hat{\beta}_1^{(-j)} \cdot x_j$$

$$= \frac{n(y_j - \bar{y})}{n-1} - \frac{n}{n-1}\hat{\beta}_1^{(-j)} x_j = \frac{n}{n-1}\left(y_j - \bar{y} - \hat{\beta}_1^{(-j)} x_j \right)$$

$$= \frac{n}{n-1}\left(y_j - \bar{y} - \frac{(n-1) S_{xy} - n x_j (y_j - \bar{y})}{(n-1) S_{xx} - n x_j^2} x_j \right)$$

$$= \frac{(y_j - \bar{y})\left[n \cdot (n-1) S_{xx} - n^2 x_j^2 \right] - n \cdot \left[(n-1) S_{xy} - n x_j (y_j - \bar{y}) \right] x_j}{(n-1)\left[(n-1) S_{xx} - n x_j^2 \right]}$$

Given that $1 - h_j = \frac{(n-1) S_{xx} - n x_j^2}{n \cdot S_{xx}} \Rightarrow (n-1) S_{xx} - n x_j^2 = (1 - h_j)\, n \cdot S_{xx}$

$$\Rightarrow \frac{(y_j - \bar{y})\left[\cancel{n} \cdot (n-1) S_{xx} - n^{\cancel{2}} x_j^2 \right] - \cancel{n} \cdot \left[(n-1) S_{xy} - \cancel{n} x_j (y_j - \bar{y}) \right] x_j}{(n-1) \cdot \cancel{n} \cdot S_{xx} (1 - h_j)}$$

$$\Rightarrow \frac{(y_j - \bar{y})(n-1) S_{xx}}{(n-1) S_{xx} (1 - h_j)} - \frac{(y_j - \bar{y})\, n \cdot x_j^2}{(n-1) S_{xx} (1 - h_j)} - \frac{(n-1) S_{xy} x_j}{(n-1) S_{xx} (1 - h_j)} + \frac{(y_j - \bar{y}) x_j^2}{(n-1) S_{xx} (1 - h_j)}$$

$$\frac{-\beta_1 x_j}{1-h_j} = \frac{\beta_1 x_j \cdot n \cdot S_{xx}}{(n-1) S_{xx} - n \cdot x_j^2}$$

$$= \underbrace{\frac{n \cdot S_{xx} x_j}{(n-1) S_{xx} - n x_j^2}} \cdot \underbrace{\frac{(n-1) S_{xy} - n \cdot x_j \cdot (y_j - \bar{y})}{(n-1) S_{xx} - n \cdot x_j^2}}$$

$$= \frac{n(n-1) S_{xx} S_{xy} \cdot x_j - n^2 S_{xx} \cdot x_j^2 (y_j - \bar{y})}{\left[(n-1) S_{xx} - n x_j^2\right]^2}$$

Q3.

(a) $$RSS_{Ridge} = \sum_{i=1}^{n} (y_i - \beta - \beta x)^2 + \lambda \beta^2$$

$$\frac{d\,RSS}{d\beta} = \cancel{2} \sum_{i=1}^{n} (y_i - \beta - \beta x_i)(-1 - x_i) + \cancel{2}\lambda\beta = 0$$

$$\Rightarrow \lambda\beta = \sum (1+x_i) y_i - (1+x_i)\beta - (x_i + x_i^2)\beta$$

$$\Rightarrow \beta\left[\lambda + \sum_{i=1}^{n}(x_i + x_i^2 + 1 + x_i)\right] = \sum_{i=1}^{n} (1+x_i) y_i$$

$$\longrightarrow \hat{\beta}_R = \frac{\sum_{i=1}^{n} (1+x_i) y_i}{\lambda + \sum_{i=1}^{n} (x_i+1)^2} \quad \#$$

(b)

$$E[\varepsilon] = 0, \quad Var(\varepsilon) = \sigma^2$$

$$y = \beta + \beta x + \varepsilon \longrightarrow Var(y) = \sigma^2$$

$$Var(\hat{\beta}_R) = Var\left(\frac{\sum_{i=1}^{n} (1+x_i) y_i}{\lambda + \sum_{i=1}^{n} (x_i+1)^2}\right)$$

$$= \frac{\sum_{i=1}^{n} (1+x_i)^2 Var(y_i)}{\left[\lambda + \sum_{i=1}^{n} (1+x_i)^2\right]^2} = \frac{\sum_{i=1}^{n} (1+x_i)^2}{\left[\lambda + \sum_{i=1}^{n} (1+x_i)^2\right]^2} \sigma^2 \quad \#$$

(C) To show $Var(\hat{\beta}_R) \leq Var(\hat{\beta})$

$$\rightarrow \frac{\sum_{i=1}^{n}(1+x_i)^2}{\left[\lambda + \sum_{i=1}^{n}(1+x_i)^2\right]^2} \cancel{\sigma^2} \leq \frac{\cancel{\sigma^2}}{\sum_{i=1}^{n}(1+x_i)^2}$$

$$\rightarrow \left[\sum_{i=1}^{n}(1+x_i)^2\right]^2 \leq \left[\lambda + \sum_{i=1}^{n}(1+x_i)^2\right]^2$$

Since $\lambda > 0$, and $\sum_{i=1}^{n}(1+x_i)^2 \geq 0$

Thus, the equation $Var(\hat{\beta}_R) \leq Var(\hat{\beta})$ holds for all $\lambda > 0$

#