Definitions

- **Network** – Any structure containing interconnected elements.
- **Circuit** – Usually physical structure constructed from electrical components.

(A) **Linear Network**: response proportional to excitation. Superposition applies:

$$\textit{If}\quad e_1(t) \to w_1(t) \quad \textit{and} \quad e_2(t) \to w_2(t)$$

Then

$$k_1 \cdot e_1(t) + k_2 \cdot e_2(t) \to k_1 \cdot w_1(t) + k_2 \cdot w_2(t)$$

(B) **Time-Invariant Network**: $e(t) \to w(t)$ relation the same if $t \to t + t_1$. Time varying otherwise.

(C) **Passive Network**: EM energy delivered always non-negative. Specifically:

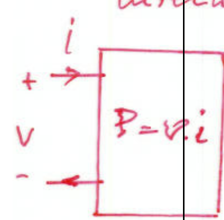
$$E(t) = \int_{-\infty}^{t} v(x)i(x)dx \geq 0$$

or

$$E(t) = \int_{t_0}^{t} v(x)i(x)dx + E(t_0) \geq 0$$

This must be true for any voltage and its resulting current for all t

Associated directions:



In general

$$\sum_j \int v_j i_j dx \geq 0$$

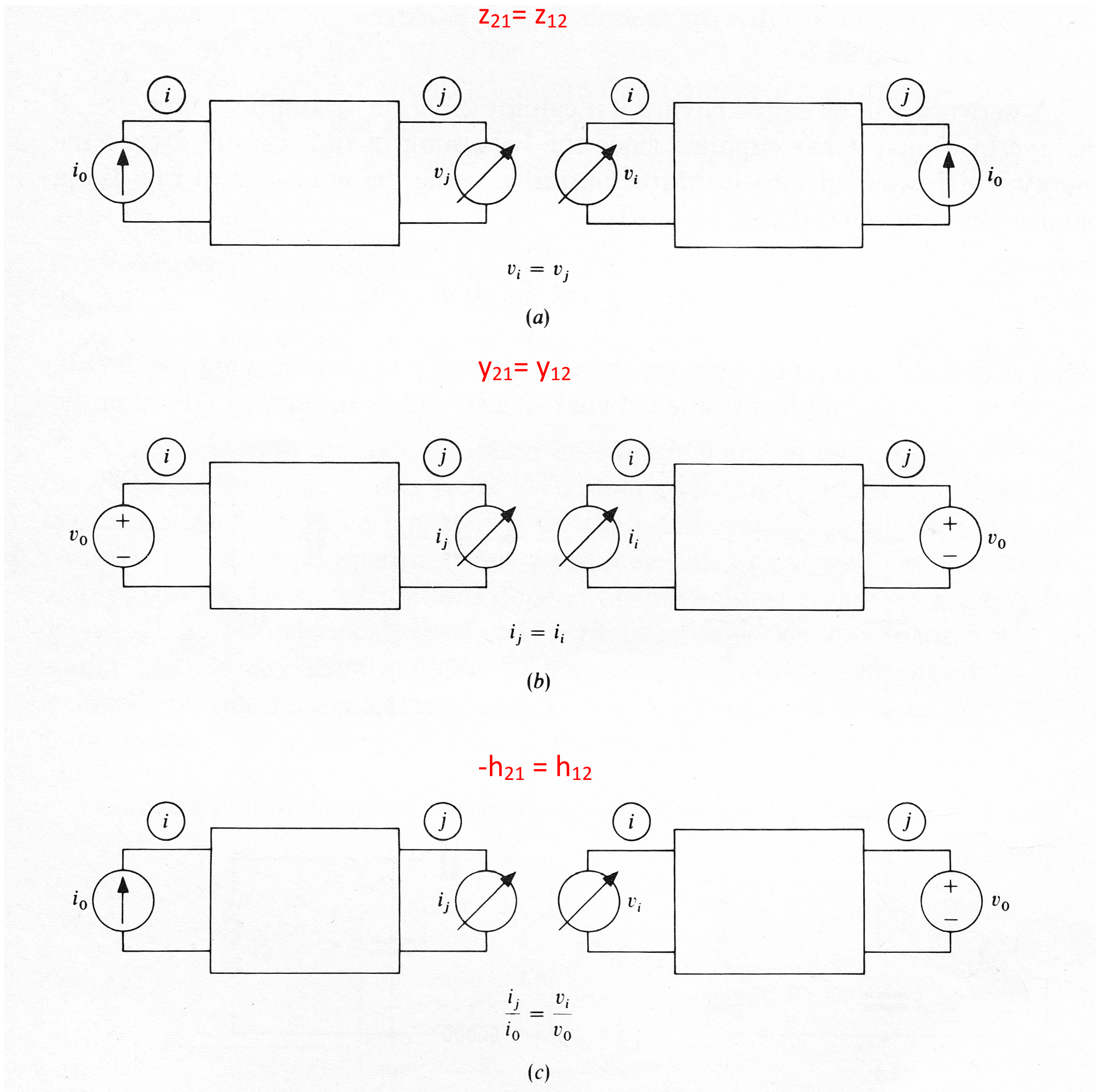
Otherwise, active.

(D) **Lossless Circuit**: input energy is always equal to the energy stored in the network. Otherwise, lossy.

(E) **Distributed Network**: must use Maxwell's equation to analyze. Examples: transmission lines, high speed VLSI circuits.

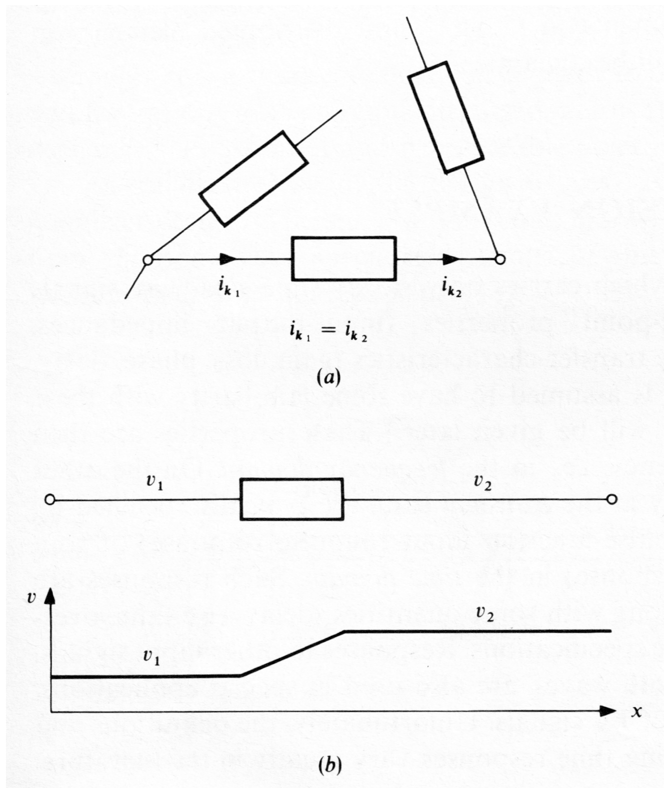
(F) **Memoryless or Resistivity Circuit**: no energy storing elements. Response depends only on instantaneous excitation. Otherwise, dynamic or memoried circuit.

(G) **Reciprocity**: response remains the same if excitation and response locations are interchanged. Specifically:



Otherwise, non-reciprocal.

(H) **Lumped Network**: physical dimensions can be considered zero. In reality, much smaller than the wavelength of the signal.



(I) **Continuous-Time Circuit**: the signals can take on any value at any time.

(J) **Sampled-Data Circuit**: the signals have a known value only at some discrete time instances. Digital, analog circuits.

An ideal RLC circuit is linear, time-invariant, passive, lossy, reciprocal, lumped, dynamic continuous-time network.

(A) **Ideal R, L, C**:

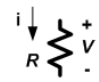
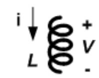
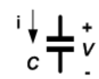
Element	Parameter	Voltage-Current Relationship: Direct	Voltage-Current Relationship: Inverse	Symbol
Resistor	Resistance ***R*** Conductance ***G***	$v = Ri$	$i = \frac{1}{R} v = Gv$	
Inductor	Inductance ***L*** Inverse Inductance ***T***	$v = L \frac{di}{dt}$	$i(t) = \frac{1}{L} \int_0^t v(x)dx + i(0)$	
Capacitor	Capacitance ***C*** Elastance ***D***	$i = C \frac{dv}{dt}$	$v(t) = \frac{1}{C} \int_0^t i(x)dx + v(0)$	

Table 1

Each passive.

Assuming standard references, the energy delivered to each of the elements starting at a time when the current and voltage were zero will be:

$$E_R(t) = \int_{-\infty}^{t} Ri^2(x)dx \geq 0 \tag{67}$$

$$E_L(t) = \int_{-\infty}^{t} L \frac{di(x)}{dx} i(x)dx = \int_0^{i(t)} Li' di' = \frac{1}{2} Li^2(t) \geq 0 \tag{68}$$

$$E_C(t) = \int_{-\infty}^{t} C \frac{dv(x)}{dx} v(x)dx = \int_0^{v(t)} Cv' dv' = \frac{1}{2} Cv^2(t) \geq 0 \tag{69}$$

(B) **Ideal Transformer**:

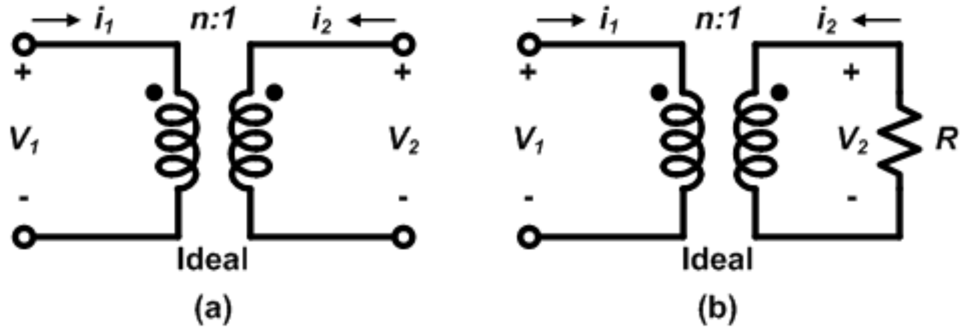


Fig. 6 An ideal transformer

Defined in terms of the following v-i relationships:

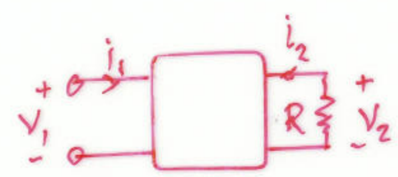
Memoryless

$$v_1 = nv_2 \tag{70a}$$

$$i_2 = -ni_1 \tag{70b}$$

or

$$\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0 & n \\ -n & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ v_2 \end{bmatrix} \tag{70c}$$



$$v_1 = nv_2 = -nRi_2 = (n^2R)i_1 \tag{71}$$

At the input terminals, then, the equivalent resistance is n^2R. Observe that the total energy delivered to the ideal transformer from connections made at its terminals will be

$$E(t) = \int_{-\infty}^{t} (v_1(x)i_1(x) + v_2(x)i_2(x))dx = 0$$
$$P = 0 \tag{72}$$

Lossless, memoryless!

The right-hand side results when the v-i relations of the ideal transformer are inserted in the middle. Thus, the device is passive; it transmits, but neither stores nor dissipates energy.

Memoryless!

(C) **Physical Transformer**:

L_1: primary self-inductance

M: mutual inductance

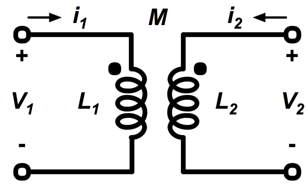


Fig. 7 A transformer

The diagram is almost the same except that the diagram of the ideal transformer shows the turns ratio directly on it. The transformer is characterized by the following v-i relationships for the reference shown in Fig. 7:

$$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \tag{73a}$$

And

$$v_2 = M \frac{di_1}{dt} + L_2 \frac{di_2}{dt} \tag{73b}$$

Thus it is characterized by three parameters: the two self-inductances L_1 and L_2, and the mutual inductance M. The total energy delivered to the transformer from external sources is

$$\begin{aligned} E(t) &= \int_{-\infty}^{t} [v_1(x)i_1(x) + v_2(x)i_2(x)]dx \\ &= \int_0^{i_1} L_1 i_1' di_1' + \int_0^{i_1,i_2} M d(i_1' i_2') + \int_0^{i_2} L_2 i_2' di_2' \\ &= \frac{1}{2}(L_1 i_1^2 + 2M i_1 i_2 + L_2 i_2^2) \geq 0 \end{aligned} \tag{74}$$

It is easy to show that the last line will be non-negative if

$$\frac{M^2}{L_1 L_2} = k^2 \leq 1 \tag{75}$$

Since physical considerations require the transformer to be passive, this condition must apply. The quantity ***k*** is called the *coefficient of coupling*. Its maximum value is unity for a closely-coupled transformer.

A transformer for which the coupling coefficient takes on its maximum value $k = 1$ is called a *perfect*, or *perfectly coupled*, transformer. A perfect transformer is not the same thing as an ideal transformer. To find the difference, turn to the transformer equations (73) and insert the perfect-transformer condition $M = \sqrt{L_1 L_2}$; then take the ratio v_1/v_2. The result will be

$$\frac{v_1}{v_2} = \frac{L_1 \dfrac{di_1}{dt} + \sqrt{L_1 L_2}\,\dfrac{di_2}{dt}}{\sqrt{L_1 L_2}\,\dfrac{di_1}{dt} + L_2 \dfrac{di_2}{dt}} = \sqrt{L_1/L_2}\,. \tag{76}$$

This expression is identical with $v_1 = n v_2$ for the ideal transformer† if

$$n = \sqrt{L_1/L_2}\,. \tag{77}$$

Next let us consider the current ratio. Since (73) involve the derivatives of the currents, it will be necessary to integrate. The result of inserting the perfect-transformer condition $M = \sqrt{L_1 L_2}$ and the value $n = \sqrt{L_1/L_2}$, and integrating (73*a*) from 0 to t will yield, after rearranging,

$$i_1(t) = -\frac{1}{n}\, i_2(t) + \left\{ \frac{1}{L_1} \int_0^t v_1(x)\, dx + \left[i_1(0) + \frac{1}{n}\, i_2(0) \right] \right\}. \tag{78}$$

This is to be compared with $i_1 = -i_2/n$ for the ideal transformer. The form of the expression in brackets suggests the *v-i* equation for an inductor. The diagram shown in Fig. 8 satisfies both (78) and (76). It shows how a perfect transformer is related to an ideal transformer. If, in a perfect transformer, L_1 and L_2 are permitted to approach infinity, but in such a way that their ratio remains constant, the result will be an ideal transformer.

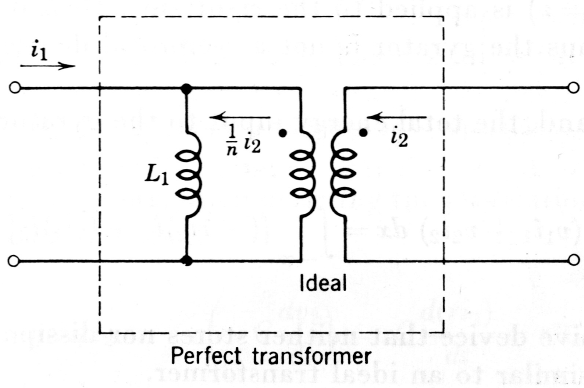


Fig. 8. Relationship between a perfect and an ideal transformer.

Losssless, memoried element.

(D) **The Gyrator**:

$i_1 = i_2$

Definitions:

- **Port**: Two terminals, both input leads always carrying the same current.
- **Gyrator**: A two port network requiring active components for realization.

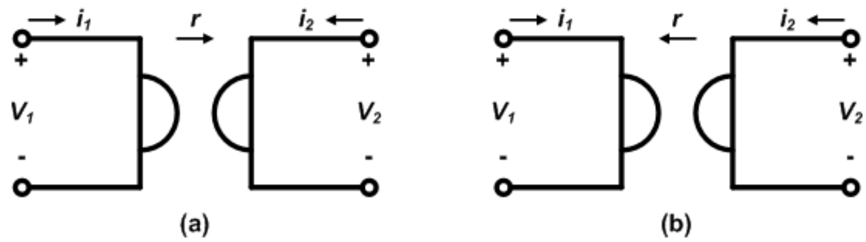


Fig. 9 A gyrator

阻抗 complex number

Often used to transform (convert) impedance into a different kind. Generally,

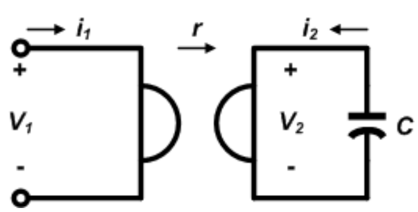
$$Z_{in} = \frac{r^2}{Z_{load}}, \quad in \quad s - domain$$

For Fig. 9(a)
$$V_1 = -ri_2 \qquad V_2 = ri_1 \quad \text{or} \quad \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 & -r \\ r & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \tag{79a}$$

For Fig. 9(b)
$$V_1 = ri_2 \qquad V_2 = -ri_1 \quad \text{or} \quad \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 & r \\ -r & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \tag{79b}$$

$P = IV$

$$E(t) = \int_{-\infty}^{t} (v_1 i_1 + v_2 i_2)dx = \int_{-\infty}^{t} [(-ri_2)i_1 + (ri_1)i_2]dx = 0 \tag{80}$$



Lossless, but

Nonreciprocal

Fig. 11 Gyrator terminated in a capacitor C

$i_2 = -C\dfrac{dv_2}{dt}$. Therefore, upon inserting the v-i relations associated with the gyrator, we observe that

$$v_1 = -ri_2 = -r\left(-C\frac{dv_2}{dt}\right) = rC\frac{d(ri_1)}{dt} = r^2C\frac{di_1}{dt} = L\frac{di_1}{dt} \tag{82}$$

(The first one is more practical, using transconductors)

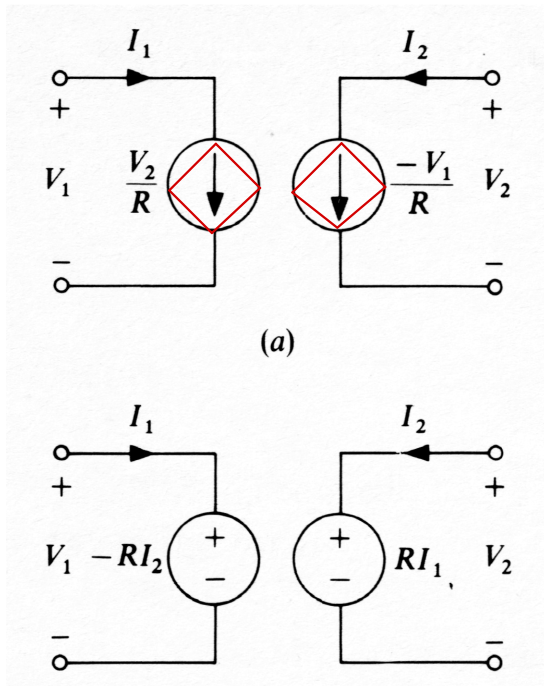


Figure 7-18 Ideal gyrator circuit

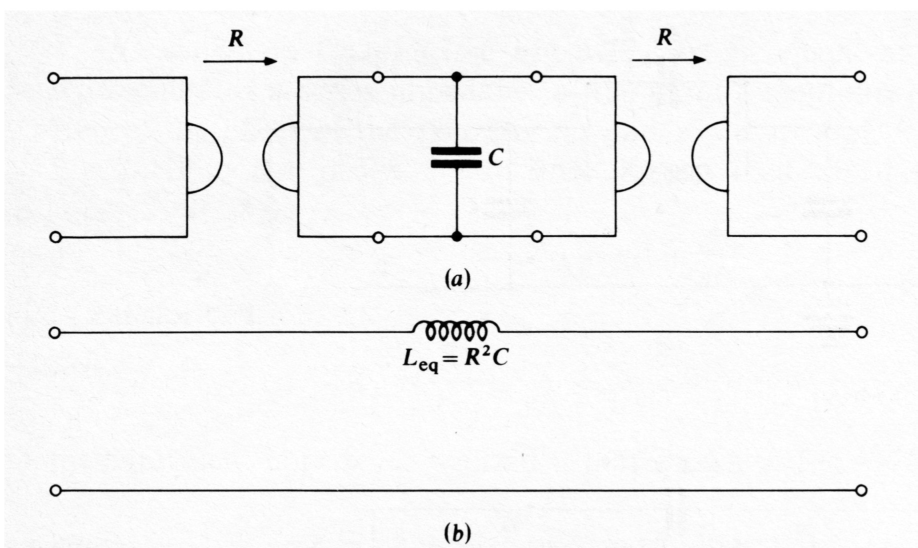


Figure 7-24 Floating-inductor simulation using gyrator

The Riordan circuit using two op-amps:

Riordan GIC/GII: general impedance converter or inverter

Laplace Domain

$V(s)$

$I(s)$

$Z = \frac{V(s)}{I(s)}$

$S = jW \rightarrow$ phasor

$V = 3\sin(2Wt)$

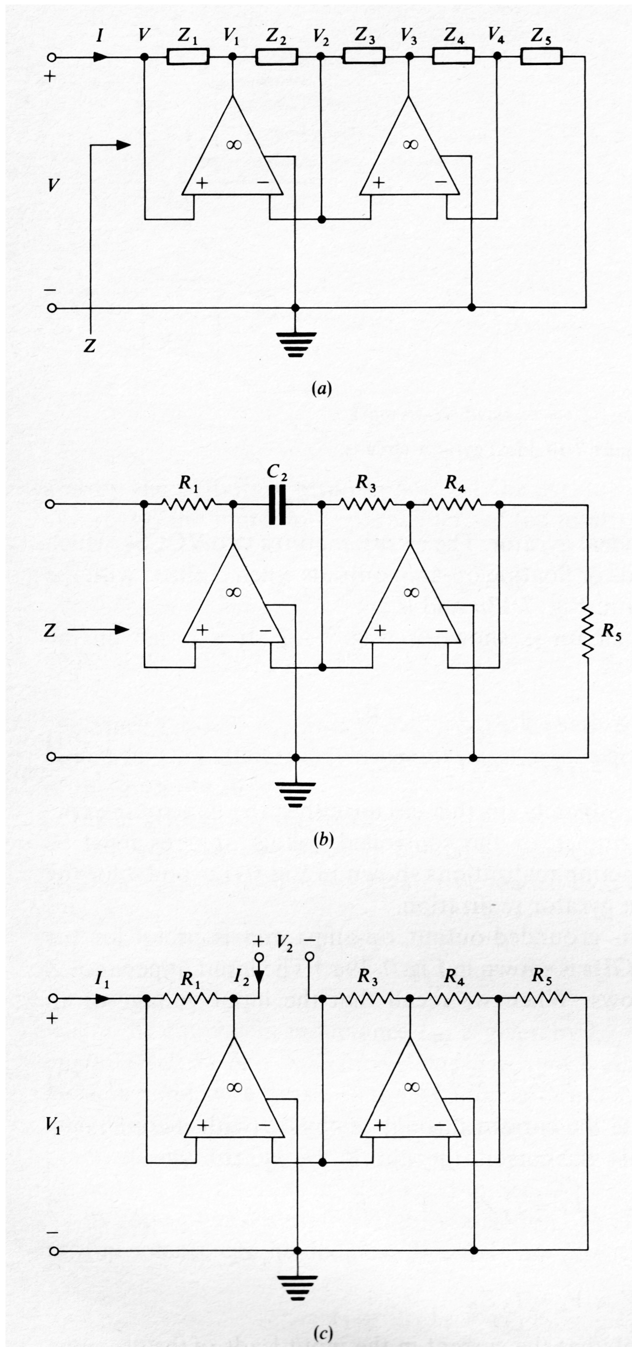
$i = \cos(2Wt)$

$Z \neq 3\tan(2Wt)$

What is $Z(jW)$?

Homework 4

GIC



Gyrator

Figure 7-19 The Riordan circuit: (a) basic circuit;

(b) used as an inductor; (c) used as a gyrator

A circuit which uses two grounded-output op-amps and is useful for the realization of either GICs or GIIs is shown in Fig. 7-19*a*.† The input impedance Z can easily be found, as follows. When we recall that the input voltage of an op-amp is very nearly zero,

$$V \approx V_2 \approx V_4 \tag{7-62}$$

is obtained. Also, if we denote the current through Z_1 by I_1 (with the reference direction pointing left to right), the current through Z_2 by I_2, etc., clearly

$$\begin{aligned} I_1 &\approx I & V - V_1 &= I_1 Z_1 \approx V_2 - V_1 = -I_2 Z_2 \\ I_3 &\approx I_2 & V_2 - V_3 &= I_3 Z_3 \approx V_4 - V_3 = -I_4 Z_4 \\ I_5 &\approx I_4 & V &\approx V_4 = I_5 Z_5 \end{aligned} \tag{7-63}$$

Here we assumed, as usual, that the current in the input leads of the op-amps is zero.

Working backward in (7-63) leads to

$$V \approx I_5 Z_5 \approx I_4 Z_5 \approx -I_3 \frac{Z_3}{Z_4} Z_5 \approx -I_2 \frac{Z_3}{Z_4} Z_5 \approx I_1 \frac{Z_1}{Z_2}\frac{Z_3}{Z_4} Z_5 \approx I \frac{Z_1 Z_3 Z_5}{Z_2 Z_4} \tag{7-64}$$

Hence
$$\boxed{Z = \frac{V}{I} \approx \frac{Z_1 Z_3 Z_5}{Z_2 Z_4}} \tag{7-65}$$

If Z_5 is regarded as a load impedance, the circuit behaves like a GIC; (7-46) takes the form

$$Z(s) = f(s) Z_5(s) \qquad f(s) \equiv \frac{Z_1(s) Z_3(s)}{Z_2(s) Z_4(s)} \tag{7-66}$$

If, for example, $Z_1 = R_1$, $Z_2 = 1/sC_2$, $Z_3 = R_3$, $Z_4 = R_4$, and $Z_5 = R_5$ (Fig. 7-19*b*), then $f(s) = R_1 R_3 / [(1/sC_2) R_4]$ and

$$Z = \frac{R_1 R_3}{(1/sC_2) R_4} R_5 = s \frac{R_1 C_2 R_3 R_5}{R_4} \tag{7-67}$$

Hence, the input impedance is that of an *inductor*, with an equivalent inductance value $L_{eq} = R_1 C_2 R_3 R_5 / R_4$.

As (7-67) suggests, and as can be directly verified from (7-65), the two-port formed by regarding the terminals of Z_2 as an output port is a *gyrator* if all other impedances are purely resistive (Fig. 7-19*c*). More generally, if the terminals of Z_5 (or Z_1 or Z_3) constitute the output port, the circuit of Fig. 7-19*a* is a GIC; if the terminals of Z_2 (or Z_4) form the output port, the resulting two-port is a GII.

Assume now that we choose Z_2 and Z_4 as capacitive and Z_1, Z_3, and Z_5 as resistive impedances. Then (7-65) gives, for $s = j\omega$,

$$Z(j\omega) = \frac{R_1 R_3 R_5}{(1/j\omega C_2)(1/j\omega C_4)} = -\omega^2 R_1 C_2 R_3 C_4 R_5 \tag{7-68}$$

We note that $Z(j\omega)$ is pure real, negative, and a function of ω. Such an impedance† is called a *frequency-dependent negative resistance* (FDNR). A slightly different form of FDNR can be obtained, e.g., by choosing Z_1 and Z_3 as capacitors and Z_2, Z_4, and Z_5 as resistors. Then

$$Z(j\omega) = -\frac{R_5}{C_1 R_2 C_3 R_4} \frac{1}{\omega^2} \tag{7-69}$$

As we shall see later, FDNRs are very useful for the design of active filters.

Graph Theory, Topological Analysis

- Topological Analysis: General, systematic, suited for CAD.
- Graph: Nodes and directed branches, describes the topology of the circuit, ref. direction.
- Tree: Connected subgraph containing all nodes but no loops.
- Branches in tree: twigs.
- Branches not in tree: links
- Links: Cotree

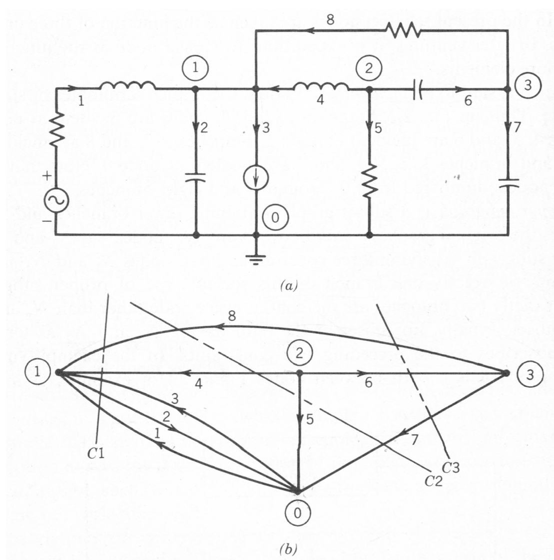


Fig. 2.2 (a) Linear circuit (b) corresponding linear directed graph

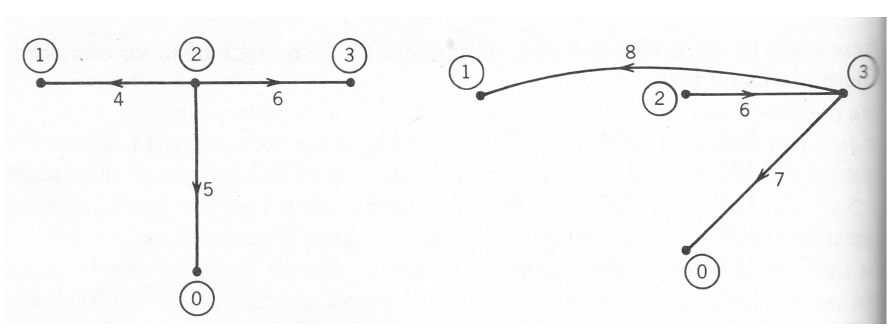


Fig. 2.3 Two of the 32 trees in the graph of Fig. 2.2b

<u>Incidence Matrix A</u>: Describes connectivity between nodes and branches.

Rule:

$$a_{ij} = \begin{cases} +1, & \text{if branch } j \text{ is directed away from node } i \\ -1, & \text{if branch } j \text{ is directed toward node } i \\ 0, & \text{if branch } j \text{ is not incident with node } i \end{cases}$$

As an example, the node-to-branch incidence matrix for the graph of Fig. 2.2b is

$$A_A = \begin{array}{c} \begin{array}{cccccccc} (1) & (2) & (3) & (4) & (5) & (6) & (7) & (8) \end{array} \\ \left[\begin{array}{cccccccc} -1 & 1 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \\ 1 & -1 & -1 & 0 & -1 & 0 & -1 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ (1) \\ (2) \\ (3) \\ (0) \end{array}$$

Augmented incidence matrix: contains reference node (0).

<u>Row: Nodes; Column Branches</u>

One row may be omitted, since sum of entries in each column is zero. (Reference node omitted.)

Resulting matrix: $\underline{A}$. # of non reference nodes N ≤ # of branches B → rank of $\underline{A}$ ≤ N.

Partitioned incidence matrix: Choose a tree, put its twigs in the first N columns of $\underline{A}$. Then

$$\underline{A} = [\underline{A}_t \mid \underline{A}_c] \quad \text{tree} \mid \text{cotree}$$

It can be shown that $\det\{A_t\} = \pm 1$; and that $\det\{\underline{A}\underline{A}_t\}$ = # of trees.

This proves that rank $\underline{A}$ = N! Largest singular submatrix N x N.

$\underline{M}\underline{x} = \underline{y}$, $\quad \underline{x} = \underline{M}^{-1}\underline{y}$

EXAMPLE 2. In the graph of Fig. 2.2*b*, select the tree defined by branches {2, 6, 8}. Then, using (2.3), A is written

$$A = \begin{array}{c} \begin{array}{cccccccc} (2) & (6) & (8) & (1) & (3) & (4) & (5) & (7) \end{array} \\ \left[\begin{array}{ccc|ccccc} 1 & 0 & -1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ (1) \\ (2) \\ (3) \end{array}$$

for which A_t is seen to be

$$A_t = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Graph Definitions

These trees can be found by systematically listing possible combinations of the three branches. These are listed below.

123	234	345	456	567	678
124	235	346	457	568	
125	236	347	458		
126	237	348			
127	238				
128					
134	145	156	167	178	
135	146	157	168		
136	147	158			
137	148				
138					
245	256	267	278	356	
246	257	268		357	
247	258			358	
248					
367	378	467	478	578	
368			468		

Each entry in the list must now be scrutinized to see if it contains all nodes and no loops.

TABLE 2.1
Trees and Cotrees for Graph of Fig. 2.2*b*

Trees		Cotrees	
345	246	12578	13578
347	246	12568	13568
348	248	12567	13567
456	256	12378	13478
457	257	12368	13468
458	258	12367	13467
568	267	12347	13458
678	268	12345	13457
146	356	23578	12478
147	357	23568	12468
148	358	23567	12467
156	367	23478	12458
157	368	23468	12457
158	467	23467	12358
167	478	23458	12356
168	578	23457	12346

Branch-to-Node Voltage Transformation: (KVL)

Branch Voltage: $V^t = [v_1 v_2 .. v_b]$

Node Voltage: $E^t = [e_1 e_2 .. e_n]$

By KVL, if branch k goes from node I to node j, so $a_{ik} = 1$ and $a_{jk} = -1$, then
$$V_k = e_i - e_j = a_{jk}e_i + a_{jk}e_j = [k^{th} \ \ column \ \ of \ \ A]^t \cdot \underline{E} = a_{ik}e_i + ... + a_{Nk}e_N$$

In general, $\underline{V} = \underline{A^t E}$.

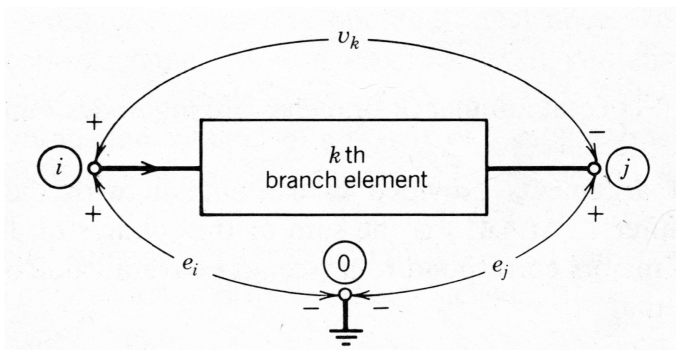


Fig. 2.4 Schematic for definition of branch and node voltages

Branch voltages expressed in terms of node voltages → there are fewer!
Purpose: formulate smallest set of linear equations before solving them.

The KCL in Topological Formulation

The KCL says that the sum of currents leaving any node is zero. Since $a_{ij} = 1(-1)$ means branch j leaves (enters) node i, the KCL for node i means

$$\sum_j a_{ij} \cdot i_j = 0 \text{ or } [i^{th} \text{ row of } \underline{A}]\underline{I}=0,\ i=1, \ldots, N. \text{ Hence, } \underline{AI} = 0.$$

Choose a tree, and partition $\underline{A}$ and $\underline{I}$ so that $\underline{A} = \{\underline{A}_t \mid \underline{A}_c\}$ and $\underline{I}^t = \{\underline{I}_t \mid \underline{I}_c\}$. Then $\underline{A}_t\underline{I}_t + \underline{A}_c\underline{I}_c = 0$ and $\underline{I}_t = -(\underline{A}_t)^{-1}\underline{A}_c\underline{I}_c$. This gives the twig currents from $\underline{A}$ and the link current. Note that $\underline{A}_t$ cannot be singular.

Example:

EXAMPLE 4. With reference to Example 2,

$$A_t = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

whence

$$A_t^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Then,

$$A_t^{-1}A_c = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

With

$$I_t^T = \begin{bmatrix} i_2 & i_6 & i_8 \end{bmatrix}$$

$$I_c^T = \begin{bmatrix} i_1 & i_3 & i_4 & i_5 & i_7 \end{bmatrix}$$

(2.16) produces

$$i_2 = i_1 - i_3 - i_5 - i_7$$

$$i_6 = -i_4 - i_5$$

$$i_8 = -i_4 - i_5 - i_7$$

Twig currents can be found from link current. Fewer twigs than links.

These equations corroborate completely the branch current relationships exemplified in the circuit of Fig. 2.2a. (p.12)

Generalized Branch Relations

General branch for lumped linear network contains a (single) element b_k which may be an R, L, C, and dependent sources, as well as a voltage and a current source which may include the representation of initial energy stored in b_k:

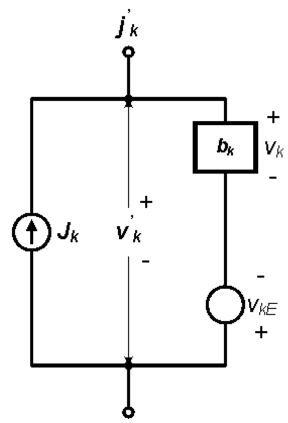


Fig. 3.1 Generalized schematic representation of *k* th branch in linear circuit

Since $i_k' = i_k - J_k$ and $v_k' = v_k - V_{kE}$, for the branch vectors $I' = I - J$ and $V' = V = V_E$ hold.

Nodal Analysis

Combining the branch relatons with the KVL ($\underline{V}' = \underline{A}^t\underline{E}$) and KCL ($\underline{AI}' = \underline{0}$) the matrix relations

$$\text{(1)} \qquad \underline{V} = \underline{V_E} + \underline{A^t E}$$
$$\text{(2)} \qquad \underline{AI} = \underline{AJ}$$

Result 2 equations, 3 unknown vectors: $\underline{V}$, $\underline{I}$, $\underline{E}$.

Let the V-I relations of the b_k elements be described by the matrix relation $\underline{I} = \underline{YV}$, where the diagonal element y_{ii} of $\underline{Y}$ represents the internal admittance of b_i in branch $\underline{I}$, and the off-diagonal one $y_{kl} = \dfrac{i_k}{v_l}$ represents a dependent I source in the branch k controlled by branch V_2. Combining (1), (2) and $\underline{I} = \underline{YV}$, and eliminating $\underline{V}$ and $\underline{I}$, in the Laplace domain, the nodal equations $\underline{Y}_N(s)\underline{E}(s) = \underline{J}_N(s)$ result, where $\underline{Y}_N(s) = \underline{A}\underline{Y}(s)\underline{A}^t$ is the N x N <u>nodal admittance matrix</u>, and $\underline{J}_N(s) = \underline{A}[\underline{J}(s) - \underline{Y}(s)V_E(s)]$ the equivalent <u>nodal current excitation vector</u>. (Due to independent sources J_k and V_{ke}.)

Node analysis parameters

Branch element voltage, currents	$v_k,\ i_k \ \rightarrow \ \underline{V},\ \underline{I}$
Branch voltages, currents	$v_k',\ i_k' \ \rightarrow \ \underline{V}',\ \underline{I}'$
Source voltages, currents	$V_{KE},\ J_K \ \rightarrow \ \underline{V_E},\ \underline{J}$
Branch admittances, branch admittance matrix	$y_{ij} \ \rightarrow \ \underline{Y}$
Nodal admittances, nodal admittance matrix	$y_{ijN} \ \rightarrow \ \underline{Y_N}$
Nodal current excitations, n. c. e. vector	$J_{iN} \ \rightarrow \ \underline{J_N}$
Node voltages	$e_i \ \rightarrow \ \underline{E}$

Example:

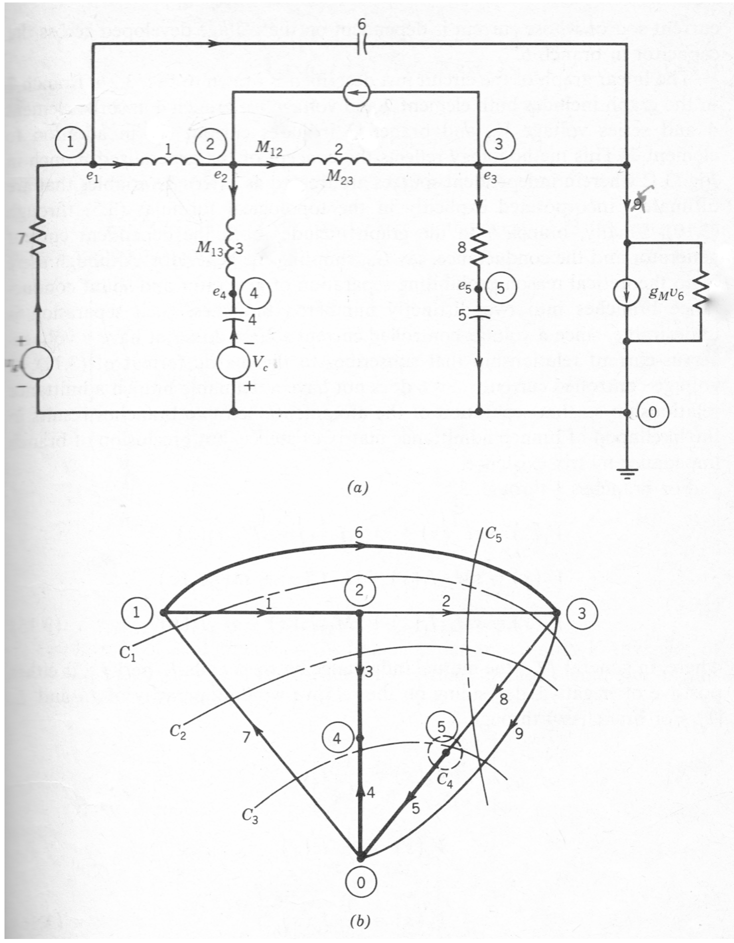


Fig. 3.2 (a) Circuit used to exemplify $\overline{Z}$ and $\overline{Y}$ matrices (b) Graph of circuit

EXAMPLE 2. For the graph of Fig. 3.2b,

$$\begin{array}{cc} & \begin{array}{ccccccccc} (1) & (2) & (3) & (4) & (5) & (6) & (7) & (8) & (9) \end{array} \\ A = & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{array}{c} (1) \\ (2) \\ (3) \\ (4) \\ (5) \end{array} \end{array}$$

In the interest of mathematical simplicity, let all $M_{ij} = 0$. Then from (3.29), (3.34), (3.35), and (3.26),

$$\begin{array}{ccc} & & \begin{array}{ccccccccc} (1) & (2) & (3) & (4) & (5) & (6) & (7) & (8) & (9) \end{array} \\ Y(s) = & \begin{array}{c} (1) \\ (2) \\ (3) \\ (4) \\ (5) \\ (6) \\ (7) \\ (8) \\ (9) \end{array} & \begin{bmatrix} \Gamma_1/s & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Gamma_2/s & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Gamma_3/s & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & sC_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & sC_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & sC_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & G_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_8 & 0 \\ 0 & 0 & 0 & 0 & 0 & g_M & 0 & 0 & G_9 \end{bmatrix} \end{array}$$

It follows that

$$AY(s) = \begin{bmatrix} \Gamma_1/s & 0 & 0 & 0 & 0 & sC_6 & -G_7 & 0 & 0 \\ -\Gamma_1/s & \Gamma_2/s & \Gamma_3/s & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\Gamma_2/s & 0 & 0 & 0 & (g_M - sC_6) & 0 & G_8 & G_9 \\ 0 & 0 & -\Gamma_3/s & -sC_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & sC_5 & 0 & 0 & -G_8 & 0 \end{bmatrix}$$

and since

$$A^T = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$Y_N(s) = A\,Y(s)A^T$ is as submitted below:

$$Y_N(s) = \begin{bmatrix} (G_7 + sC_6 + \Gamma_1/s) & -\Gamma_1/s & -sC_6 & 0 & 0 \\ -\Gamma_1/s & (\Gamma_1 + \Gamma_2 + \Gamma_3)/s & -\Gamma_2/s & -\Gamma_3/s & 0 \\ (g_M - sC_6) & -\Gamma_2/s & (G_8 + G_9 - g_M + sC_6 + \Gamma_2/s) & 0 & -G_8 \\ 0 & -\Gamma_3/s & 0 & (sC_4 + \Gamma_3/s) & 0 \\ 0 & 0 & -G_8 & 0 & (G_8 + sC_5) \end{bmatrix}$$

The nodal current vector is obtained through use of (3.43). With

$$J^T(s) = \left[0\; I_g(s)\; 0\,0\,0\,0\,0\,0\,0\right]$$

and

$$V_E^T(s) = \left[0\,0\,0 - V_c(s)\; 0\,0\; V_g(s)\; 0\,0\right]$$

$$Y(s)V_E(s) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -sC_4V_c(s) \\ 0 \\ 0 \\ G_7V_g(s) \\ 0 \\ 0 \end{bmatrix}$$

and, thus,

$$J(s) - Y(s)V_E(s) = \begin{bmatrix} 0 \\ I_g(s) \\ 0 \\ sC_4V_c(s) \\ 0 \\ 0 \\ -G_7V_g(s) \\ 0 \\ 0 \end{bmatrix}$$

Finally,

$$J_N(s) = \begin{bmatrix} G_7V_g(s) \\ I_g(s) \\ -I_g(s) \\ -sC_4V_c(s) \\ 0 \end{bmatrix}$$

It is understood that

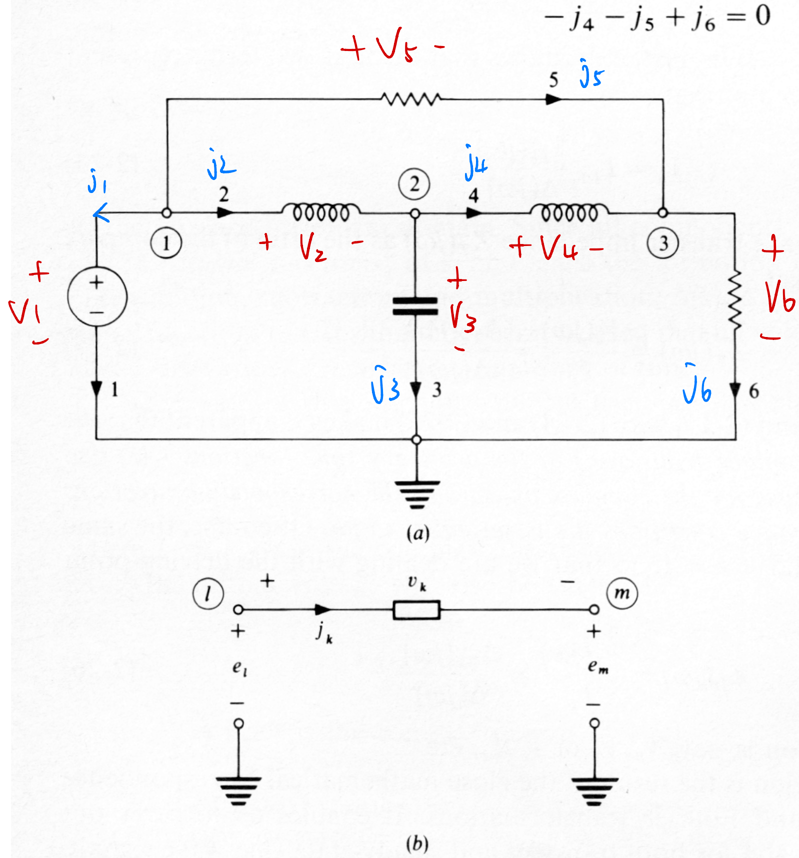
$$J_N(s) = Y_N(s)E(s)$$

2-2 TELLEGEN'S THEOREM

Next, an important law of circuit theory, *Tellegen's theorem*, will be introduced. This basic theorem will help us understand the fundamental properties of physically realizable impedance functions discussed later in this chapter. It will also be used much later in the book (in Chaps. 9 and 10) in connection with the sensitivity analysis of circuits.

Consider the circuit shown in Fig. 2-4*a*. With the notations of Fig. 2-4*b*, the Kirchhoff current laws (KCL) give

$$\begin{aligned} j_1 + j_2 \qquad\qquad + j_5 \qquad &= 0 \\ -j_2 + j_3 + j_4 \qquad\qquad &= 0 \\ -j_4 - j_5 + j_6 &= 0 \end{aligned} \tag{2-27}$$



3 nodes

6 branches

Figure 2-4 (*a*) Linear circuit; (*b*) notations used in the analysis

$$A = \begin{array}{c} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{l} \rightarrow \text{Node 1} \\ \rightarrow \text{Node 2} \\ \rightarrow \text{Node 3} \end{array}$$

Note the use of *associated directions*[1] for v_k and j_k. Equations (2-27) can also be written in the form†

$$\mathbf{Aj} = \mathbf{0} \tag{2-28}$$

where **A** is the *incidence matrix*

$$\mathbf{A} = \begin{array}{c} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \tag{2-29}$$

The elements of **A** are given by

$$a_{ij} = \begin{cases} +1 & \text{if branch } j \text{ leaves node } i \\ -1 & \text{if branch } j \text{ enters node } i \\ 0 & \text{if branch } j \text{ is not incident with node } i \end{cases} \tag{2-30}$$

Similarly, from the Kirchhoff voltage law (KVL)

$$\begin{aligned} v_1 &= e_1 \\ v_2 &= e_1 - e_2 \\ v_3 &= \phantom{e_1 - {}} e_2 \\ v_4 &= \phantom{e_1 - {}} e_2 - e_3 \\ v_5 &= e_1 \phantom{{} - e_2} - e_3 \\ v_6 &= \phantom{e_1 - e_2 - {}} e_3 \end{aligned} \tag{2-31}$$

or, using (2-29),

$$\mathbf{v} = \mathbf{A}^T\mathbf{e} \qquad \text{KVL} \tag{2-32}$$

where $\mathbf{A}^T$ denotes the transpose of **A**. Equations (2-28) and (2-32) are, of course, the general matrix forms of the KCL and KVL, respectively.[1]

Since all relations (2-27) to (2-32) involve only additions and subtractions, they remain valid if we perform any linear operations on the j_i, v_k, and e_l. For example, they hold also for the Laplace transforms $J_i(s)$, $V_k(s)$, and $E_l(s)$ or for the phasors J_i, V_k, and E_l, etc.

Let the branch power $v_k j_k$ be summed for all N branches of the circuit. Then, by (2-32) and (2-28),

$$V^T = \begin{bmatrix} V_1 \\ \vdots \\ V_6 \end{bmatrix}$$

$$\sum_{k=1}^{N} v_k j_k = \mathbf{v}^T\mathbf{j} = (\mathbf{A}^T\mathbf{e})^T\mathbf{j} = \mathbf{e}^T(\mathbf{Aj}) = 0 \tag{2-33}$$

Here the familiar rules $(\mathbf{Ae})^T = \mathbf{e}^T\mathbf{A}^T$, $(\mathbf{A}^T)^T = \mathbf{A}$, and $\mathbf{e}^T\mathbf{0} = 0$ of vector algebra have been used.

Equation (2-33) is equivalent, of course, to the conservation of power in the circuit, and thus its emergence from Kirchhoff's laws is not too surprising.

† Here, and in the rest of the book, **x** denotes a column vector and **A** denotes a matrix.

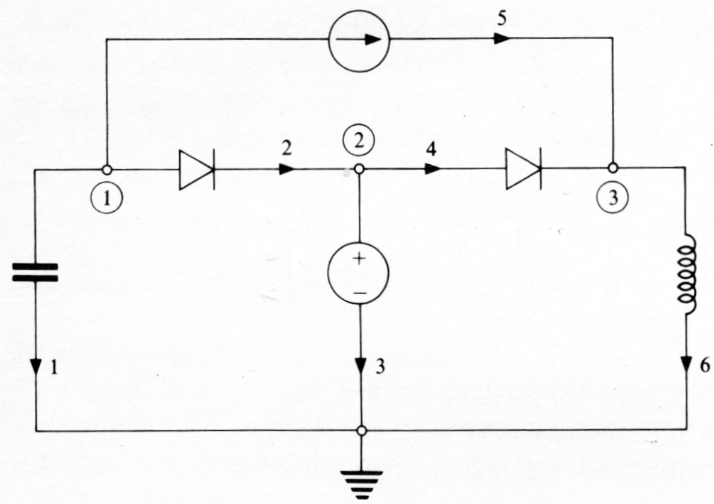


Figure 2-5 Circuit with the same configuration as that of Fig. 2-4 but with different elements.

Consider, however, the circuit of Fig. 2-5, which has the same topological configuration, same reference directions and numbering, and hence the same **A** as the circuit of Fig. 2-4. Hence, all the equations (2-27) to (2-32) remain valid for this circuit as well, with **A** remaining the same. Let the electric quantities of the circuit of Fig. 2-5 be $\mathbf{j}'$, $\mathbf{v}'$, and $\mathbf{e}'$. Then

$$\mathbf{A}\mathbf{j}' = \mathbf{0} \tag{2-34}$$

and

$$\mathbf{v}' = \mathbf{A}^T\mathbf{e}' \tag{2-35}$$

hold.

Next, let the physically meaningless quantity $\sum_{k=1}^{N} v_k j'_k$ be found:

$$\sum_{k=1}^{N} v_k j'_k = \mathbf{v}^T\mathbf{j}' = (\mathbf{A}^T\mathbf{e})^T\mathbf{j}' = \mathbf{e}^T(\mathbf{A}\mathbf{j}') = 0 \tag{2-36}$$

where (2-32) and (2-34) were used. While the left-hand side of (2-36) does have the dimension of watts, it does not correspond to physical power since v_k and j'_k exist in two different circuits.

An entirely analogous derivation gives

$$\sum_{k=1}^{N} v'_k j_k = \mathbf{v}'^T\mathbf{j} = 0 \tag{2-37}$$

Equations (2-36) and (2-37) are general forms and (2-33) is a special form of *Tellegen's theorem*. The general forms have great significance (as will be shown in Chaps. 9 and 10) in the calculation of circuit sensitivities.

Consider now a linear passive RLCM one-port (Fig. 2-6). By (2-32), using Laplace transformation,

$$\mathbf{V}(s) = \mathbf{A}^T\mathbf{E}(s) \tag{2-38}$$

Physical proof:

Construct a new network NN with the same topology. Choose a tree. Put voltage sources in the tree branches, whose value equal the corresponding branch voltages of N', and put current sources in the links whose values equal the corresponding link of N. Thus, conservation of power gives Tellegen's theorem.

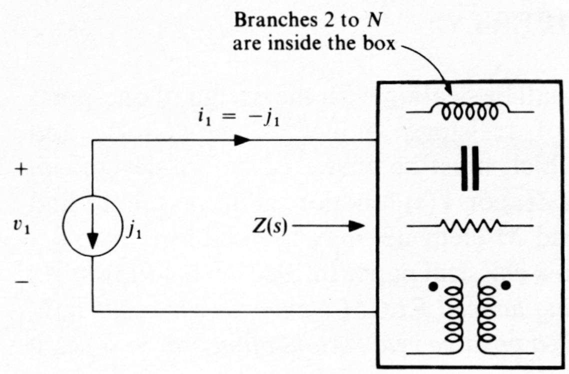


Figure 2-6 *RLCM* one-port.

and from (2-28), using Laplace transformation and taking the complex conjugate,

$$\mathbf{AJ}^*(s) = \mathbf{0} \tag{2-39}$$

Hence

$$\mathbf{V}^T\mathbf{J}^* = \mathbf{E}^T\mathbf{AJ}^* = 0 \tag{2-40}$$

or

$$\sum_{k=1}^{N} V_k(s)J_k^*(s) = 0 \tag{2-41}$$

From Fig. 2-6 and Eq. (2-41), using $i_1 = -j_1$, we get

$$-V_1(s)J_1^*(s) = V_1(s)I_1^*(s) = \sum_{k=2}^{N} V_k J_k^* \tag{2-42}$$

Note that branches 2 to N are *inside* the one-port.

Defining the *impedance* of the one-port as the ratio of $V_1(s)$ and $I_1(s)$,

$$Z(s) \triangleq \frac{V_1(s)}{I_1(s)} = \frac{V_1(s)I_1^*(s)}{I_1(s)I_1^*(s)} = \frac{V_1(s)I_1^*(s)}{|I_1(s)|^2}$$

leads to

$$Z(s) = \frac{1}{|I_1(s)|^2} \sum_{k=2}^{N} V_k(s)J_k^*(s) \tag{2-43}$$

It is easy to show, using an entirely analogous derivation, that the dual relation

$$Y(s) \triangleq \frac{I_1(s)}{V_1(s)} = \frac{1}{|V_1(s)|^2} \sum_{\substack{\text{all internal}\\ \text{branches}}} V_k^*(s)J_k(s) \tag{2-44}$$

holds for the input *admittance* $Y(s)$ of the one-port.

Equations (2-43) and (2-44) are fundamental to the analysis and design of one-ports, as will be shown in the next section.

Network Functions

<u>Driving-Point and Transfer Functions for Linear Networks</u>

Node equations: $\underline{Y}_n(s)\underline{E}(s) = \underline{J}_n(s)$ can (in principle) be solved using Cramer's Rule, so that the node voltages are given by

$$\underline{E}(s) = \underline{Y}_n^{-1}(s)\underline{J}_n(s) \qquad \text{(a)}$$

$$\underline{Y}_n\underline{E} = \underline{J}_n = \underline{A}(\underline{J} - \underline{YV}_e) \qquad \text{(b)}$$

Where the ij element of the inverse matrix is $\frac{\Delta_{ij}}{\Delta}$, Δ being the determinant of Y_n and Δ_{ij} its ij cofactor (signed subdeterminant). For a lumped linear circuit, Δ and the Δ_{ij} are all real and rational is s.

It follows from equation (a) that all response voltages and currents are weighted sums of the excitations, which enter $\underline{J}_n(s)$.

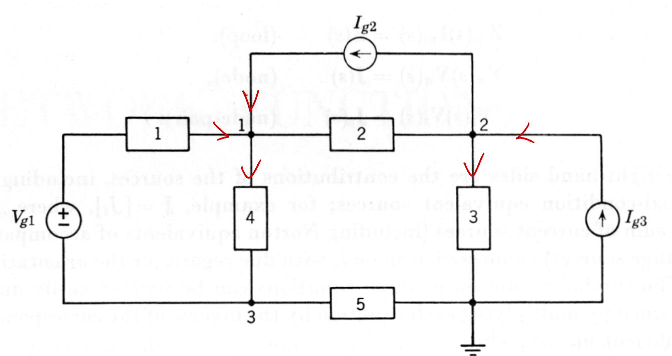


Fig. 1. Illustration for equivalent sources.

example:

Lumped Linear Network

$$\mathbf{J} = \begin{bmatrix} J_1 \\ J_2 \\ J_3 \end{bmatrix} = \begin{bmatrix} I_{g2} + Y_1V_{g1} \\ -I_{g2} + I_{g3} \\ -Y_1V_{g1} \end{bmatrix}.$$

Nodal current

Excitation vector

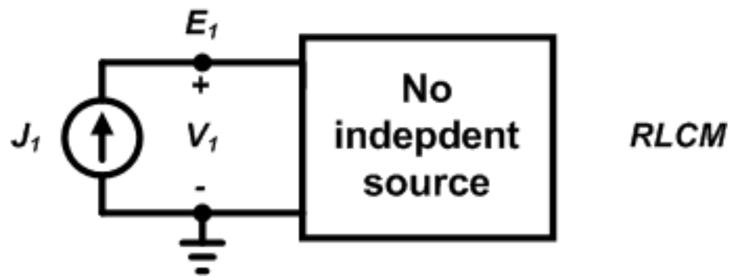
$$V_k(s) = \left(\frac{Y_1\Delta_{1k} - Y_1\Delta_{3k}}{\Delta}\right)V_{g1} + \left(\frac{\Delta_{1k} - \Delta_{2k}}{\Delta}\right)I_{g2} + \left(\frac{\Delta_{2k}}{\Delta}\right)I_{g3}. \qquad k = 1, 2, 3$$

$$\underline{E}(s) = \begin{bmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} \\ \Delta_{21} & \Delta_{22} & \Delta_{23} \\ \Delta_{31} & \Delta_{32} & \Delta_{33} \end{bmatrix} \frac{\underline{J}(s)}{\Delta}$$

Superposition

Driving-Point Functions (Immittances)

One-port circuit:



Driving-point impedance:

$$Z(s) = \frac{V_1(s)}{I_1(s)}$$

Driving-point admittance:

$$Y(s) = \frac{I_1(s)}{V_1(s)} = \frac{1}{Z(s)}$$

From (a), $Z(s) = \dfrac{\Delta_{11}(s)}{\Delta(s)}$; a real rational function of s.

The positive real (PR) property:

Z(s) is a PR function of s if $\Re e \;\; Z(s) \geq 0$ for $\Re e \;\; s \geq 0$

Brune's Theorem:

Any real rational PR Z(s) can be realized using physical RLCM elements (R, L, C all ≥ 0), and vice versa, any such physical impedance must satisfy the real rational PR conditions.

Proof of the PR property:

Consider a circuit containing only R, L, and C elements. For an R, $V = R \cdot J$, so $V \cdot J^* = R \cdot |J|^2$. Similarly, for an L, $V \cdot J^* = sL \cdot |J|^2$, and for a C, $V \cdot J^* = \dfrac{1}{sC} \cdot |J|^2$. Hence, for the complete network, substituting into Tellegen's Theorem with J_1 = 1A.

$$Z(s) = \sum_k V_k(s) J_k^*(s) = F_o(s) + \frac{1}{s} V_o(s) + sT_o(s)$$

Where,

$$F_o(s) = \sum_k R_k |J_k|^2, \;\; V_o(s) = \sum_k \frac{|J_k|^2}{C_k}, \;\; and \;\; T_o(s) = \sum_k L_k |J_k|^2$$

For a physical circuit, all R_k, L_k, C_k are non-negative real numbers, and hence so are F_o, V_o, and T_o, for any s.

Next let

$$s = \sigma + j\omega$$

$$\frac{1}{\sigma + j\omega} = \frac{\sigma - j\omega}{\sigma^2 + \omega^2} \Rightarrow \frac{\sigma}{\sigma^2 + \omega^2} \geq 0$$

Where $\sigma \geq 0$. Then both $\Re e(s)$ and $\Re e(\frac{1}{s}) = \frac{\sigma}{\sigma^2 + \omega^2}$ are non-negative, and hence so is $\Re e Z(s)$.

The above can be extended to transformers.

The proof of the sufficiency of the real rational PR conditions is based on a synthesis, which always leads to physical element values. It uses resistors, capacitors and closely coupled transformers.

Brune synthesis

$$\frac{s^2 + \ldots\ s + 3}{s^5 + \ldots\ 2s + 1}$$

Closely coupled “physical transformer”

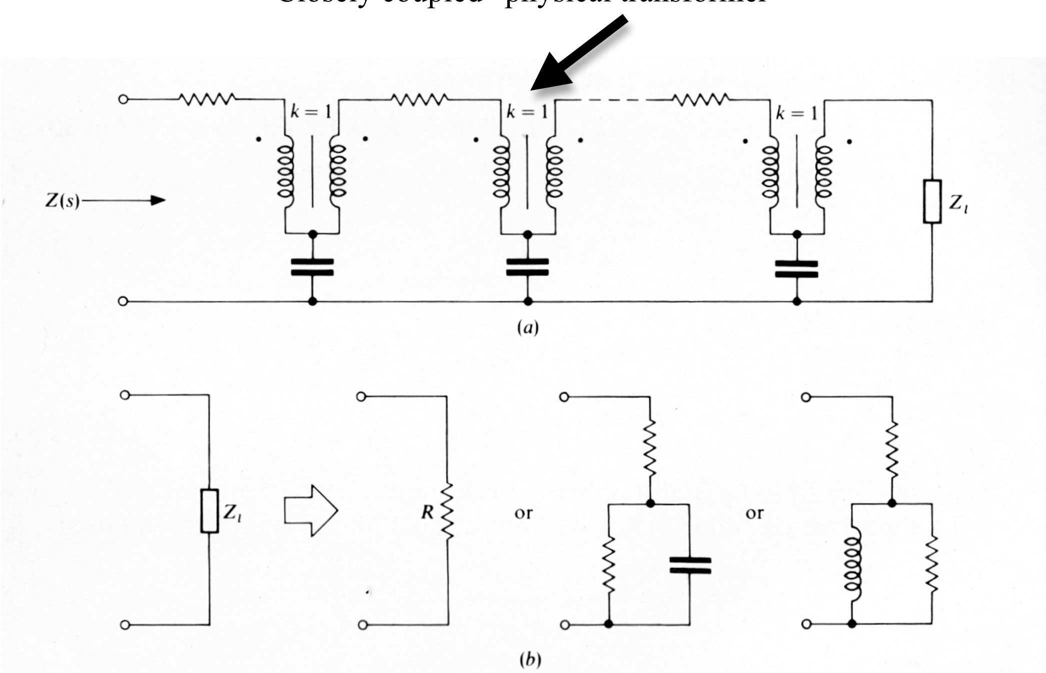


Figure 4-12 (*a*) Brune realization of an impedance $Z(s)$; (*b*) realizations for the last impedance Z_l.

Node Analysis Summary

$\underline{A}$	Incidence matrix, all analysis in s domain

Kirchhoff's Laws:	
$\underline{V} = \underline{A}^t \underline{E}$	$\underline{V}: branch \;\; voltage \;\; vector$ $\underline{E}: node \;\; voltage \;\; vector$
$\underline{A}\underline{I} = \underline{0}$	$\underline{I}: branch \;\; current \;\; vector$ $\underline{0}: zero \;\; vector$

Branch Relations:	
$\underline{I}' = \underline{I} - \underline{J}$	$\underline{I}': branch \;\; current \;\; vector$ $\underline{I}: element \;\; current \;\; vector$ $\underline{J}: souce \;\; current \;\; vector$
$\underline{V}' = \underline{V} - \underline{V_E}$	$\underline{V}': branch \;\; voltage \;\; vector$ $V: element \;\; voltage \;\; vector$ $\underline{V_E}: souce \;\; voltage \;\; vector$
$\underline{I} = \underline{Y}\underline{V}$	$\underline{Y}: branch \;\; admit\tan ce \;\; matrix$

Combining relations:	
$\underline{Y_N} = \underline{A}\underline{Y}\underline{A}^t$	$\underline{Y_N}: node \;\; admit\tan ce \;\; matrix$
$\underline{J_N} = \underline{A}[\underline{J} - \underline{Y}\underline{V}]$	$\underline{J_N}: node \;\; current \;\; excitation \;\; vector$
$\underline{Y_N}\underline{E} = \underline{J_N}$	$generaizednode \;\; node \;\; equations$

Transfer Function for Two-Port Networks

Two-port networks are constrained to have the same currents at each port: $I_1' = I_1$, $I_2' = I_2$
Hence, there are 4 unknowns: I_1, V_1, I_2 and V_2. The source and load provide 2 relations between these, so the two-port must give 2 more for an analyzable linear circuit.

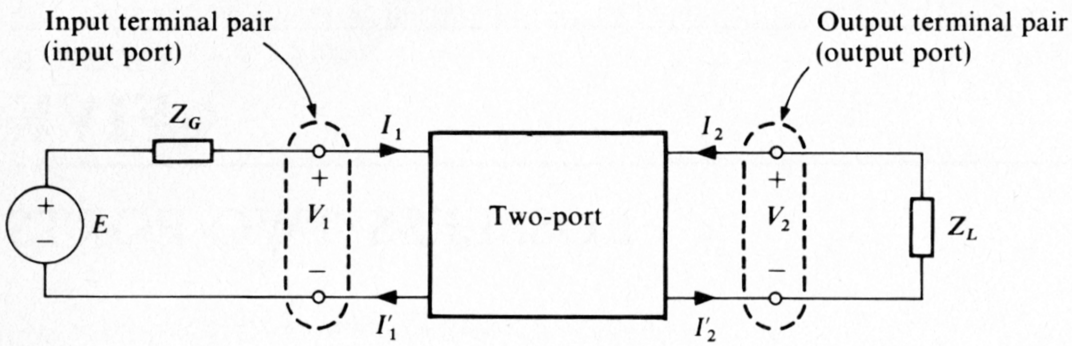


Figure 5-1 Two-port network.

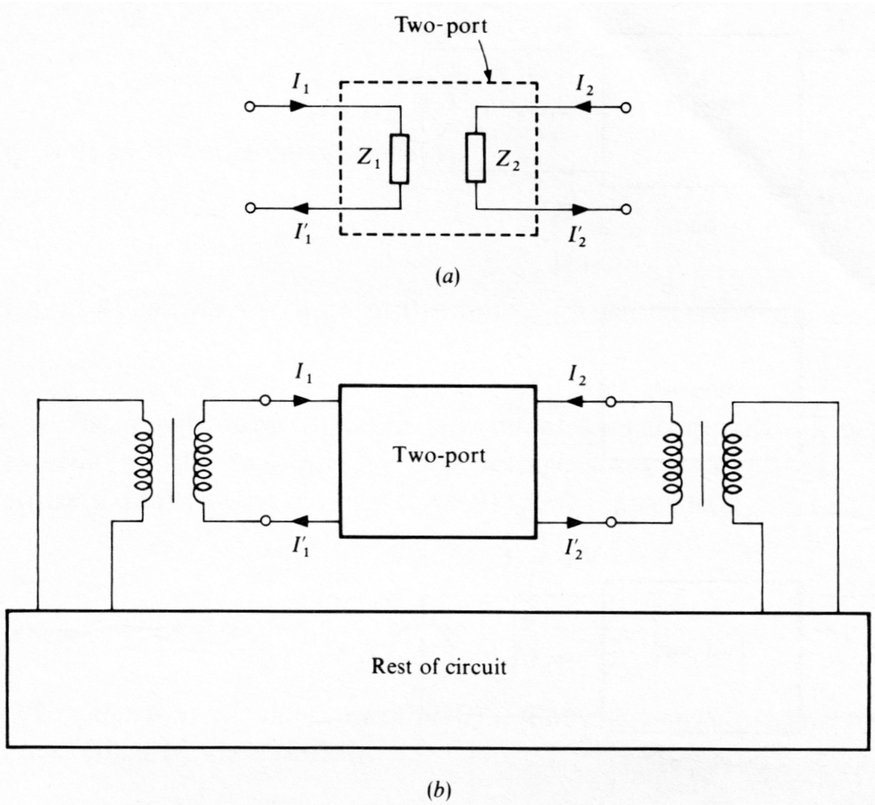


Figure 5-3 Circuits whose structure assures the satisfaction of the two-port conditions. (*a*) The internal structure which forces (5-1) to hold. (*b*) The transformers at the two ports.

3.3 TWO-PORT NETWORKS

At this point it is possible to proceed by treating the general multiport network and discussing sets of equations relating the port variables. After this is done, the results can be applied to the special case of a two-port. An alternative approach is to treat the simplest multiport (namely, the two-port) first. This might be done because of the importance of the two-port in its own right, and because treating the simplest case first can lead to insights into the general case that will not be obvious without experience with the simplest case. We shall take this second approach.

A two-port network is illustrated in Fig. 7. Because of the application

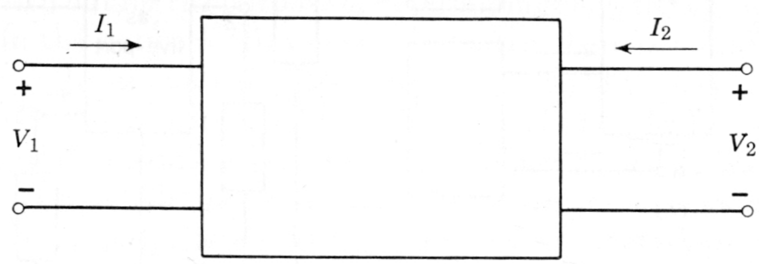


Fig. 7. Two-port network.

of two-ports as transmission networks, one of the ports—normally the port labeled 1—is called the *input*; the other, port 2, is called the *output*. The port variables are two port currents and two port voltages, with the standard references shown in Fig. 7. (In some of the literature the reference for I_2 is taken opposite to our reference. When comparing any formulas in other publications, verify the references of the port parameters.) External networks that may be connected at the input and output are called the *terminations*. We shall deal throughout with the transformed variables and shall assume the two-port to be initially relaxed and to contain no independent sources.

The discussion that follows may appear somewhat unmotivated, since in restricting ourselves to analysis we have lost much of the motivation for finding various ways of describing the behavior of two-port networks. The need for these various schemes arises from the demands made by the many applications of two-ports. The usefulness of the different methods of description comes clearly into evidence when the problem is one of synthesizing or designing networks—filters, matching networks, wave-shaping networks, and a host of others. A method of description that is convenient for a power system, say, may be less so for a filter network, and may be completely unsuited for a transistor amplifier. For this reason we shall describe many alternative, but equivalent, ways of describing two-port behavior.

In the problem of synthesizing a network for a specific application, it is often very convenient to break down a complicated problem into several parts. The pieces of the overall network are designed separately and then put together in a manner consistent with the original decomposition. In order to carry out this procedure it is necessary to know how the description of the behavior of the overall network is related to the behavior of the components. For this reason we shall spend some time on the problem of interconnecting two-ports.

Many of the results obtained in this section require a considerable amount of algebraic manipulation that is quite straightforward. We shall not attempt to carry through all the steps, but shall merely outline the desired procedure, leaving to you the task of filling in the omitted steps.

OPEN- AND SHORT-CIRCUIT PARAMETERS

To describe the relationships among the port voltages and currents of a linear multiport requires as many linear equations as there are ports. Thus for a two-port two linear equations are required among the four variables. However, which two variables are considered "independent" and which "dependent" is a matter of choice and convenience in a given application. To return briefly to the general case, in an n-port, there will be $2n$ voltage-and-current variables. The number of ways in which these $2n$ variables can be arranged in two groups of n each equals the number of ways in which $2n$ things can be taken n at a time; namely, $(2n)!/(n!)^2$. For a two-port this number is 6.

One set of equations results when the two-port currents are expressed in terms of the two-port voltages:

$$\begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix}. \tag{10}$$

It is a simple matter to obtain interpretations for these parameters by letting each of the voltages be zero in turn. It follows from the equation that

$$\begin{aligned} y_{11}(s) &= \left.\frac{I_1(s)}{V_1(s)}\right|_{V_2=0}, & y_{12}(s) &= \left.\frac{I_1(s)}{V_2(s)}\right|_{V_1=0}, \\ y_{21}(s) &= \left.\frac{I_2(s)}{V_1(s)}\right|_{V_2=0}, & y_{22}(s) &= \left.\frac{I_2(s)}{V_2(s)}\right|_{V_1=0}. \end{aligned} \tag{11}$$

Dimensionally, each parameter is an admittance. Setting a port voltage to zero means short-circuiting the port. Hence the y's (for which the lower case letter y will be reserved) are called the ***short-circuit admittance parameters*** (the y-parameters for short). The matrix of y's is designated $\mathbf{Y}_{sc}$ and is called ***the short-circuit admittance matrix***. The terms y_{11} and y_{22} are the short-circuit driving-point admittances at the two ports, and y_{21} and y_{12} are short-circuit transfer admittances. In particular, y_{21} is the *forward* transfer admittance—that is, the ratio of a current response in port 2 to a voltage excitation in port 1—and y_{12} is the *reverse* transfer admittance.

A second set of relationships can be written by expressing the port voltages in terms of the port currents:

$$\begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix}. \tag{12}$$

This time interpretations are obtained by letting each current be zero in turn. Then

$$\begin{aligned} z_{11}(s) &= \left.\frac{V_1(s)}{I_1(s)}\right|_{I_2=0}, & z_{12}(s) &= \left.\frac{V_1(s)}{I_2(s)}\right|_{I_1=0}, \\ z_{21}(s) &= \left.\frac{V_2(s)}{I_1(s)}\right|_{I_2=0}, & z_{22}(s) &= \left.\frac{V_2(s)}{I_2(s)}\right|_{I_1=0}. \end{aligned} \tag{13}$$

Dimensionally, each parameter is an impedance. Setting a port current equal to zero means open-circuiting the port. Hence the z's (for which the lower case letter z will be reserved) are called the *open-circuit impedance parameters* (the z-parameters for short). The matrix of z's is designated $\mathbf{Z}_{oc}$ and is called the *open-circuit impedance matrix*. The elements z_{11} and z_{22} are the driving-point impedances at the two ports, and z_{21} and z_{12} are the transfer impedances; z_{21} is the *forward* transfer impedance, and z_{12} is the *reverse* transfer impedance.

It should be clear from (10) and (12) that the $\mathbf{Y}_{sc}$ and $\mathbf{Z}_{oc}$ matrices are inverses of each other; for example,

$$\mathbf{Y}_{sc} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \mathbf{Z}_{oc}{}^{-1} = \frac{1}{\det \mathbf{Z}_{oc}} \begin{bmatrix} z_{22} & -z_{12} \\ -z_{21} & z_{11} \end{bmatrix}. \tag{14}$$

From this it follows that

$$\det \mathbf{Y}_{sc} = \frac{1}{\det \mathbf{Z}_{oc}}. \tag{15}$$

Demonstration of this is left as an exercise.

The results developed so far apply whether the network is passive or active, reciprocal or nonreciprocal. Now consider the two transfer functions y_{21} and y_{12}. If the network is reciprocal, according to the definition in Section 1.4, they will be equal. So also will z_{12} and z_{21}; that is, for a reciprocal network

$$y_{12} = y_{21}, \qquad z_{12} = z_{21}, \tag{16}$$

which means that both $\mathbf{Y}_{sc}$ and $\mathbf{Z}_{oc}$ are symmetrical for reciprocal networks.

HYBRID PARAMETERS

The z and y representations are two of the ways in which the relationships among the port variables can be expressed. They express the two voltages in terms of the two currents, and vice versa. Two other sets of equations can be obtained by expressing a current and voltage from opposite ports in terms of the other voltage and current. Thus

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix} \tag{17}$$

and

$$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}. \tag{18}$$

The interpretations of these parameters can be easily determined from the preceding equations to be the following:

$$\begin{aligned}
h_{11} &= \left.\frac{V_1(s)}{I_1(s)}\right|_{V_2=0}, & h_{12} &= \left.\frac{V_1(s)}{V_2(s)}\right|_{I_1=0}, \\
h_{21} &= \left.\frac{I_2(s)}{I_1(s)}\right|_{V_2=0}, & h_{22} &= \left.\frac{I_2(s)}{V_2(s)}\right|_{I_1=0}, \\
g_{11} &= \left.\frac{I_1(s)}{V_1(s)}\right|_{I_2=0}, & g_{12} &= \left.\frac{I_1(s)}{I_2(s)}\right|_{V_1=0}, \\
g_{21} &= \left.\frac{V_2(s)}{V_1(s)}\right|_{I_2=0}, & g_{22} &= \left.\frac{V_2(s)}{I_2(s)}\right|_{V_1=0}.
\end{aligned} \tag{19}$$

Thus we see that the h- and g-parameters are interpreted under a mixed set of terminal conditions, some of them under open-circuit and some under short-circuit conditions. They are called the hybrid h- and hybrid g-parameters. From these interpretations we see that h_{11} and g_{22} are impedances, whereas h_{22} and g_{11} are admittances. They are related to the z's and y's by

$$h_{11} = \frac{1}{y_{11}}, \qquad g_{11} = \frac{1}{z_{11}},$$
$$h_{22} = \frac{1}{z_{22}}, \qquad g_{22} = \frac{1}{y_{22}}. \tag{20}$$

The transfer g's and h's are dimensionless. The quantity h_{21} is the *forward short-circuit current gain*, and g_{12} is the *reverse short-circuit current gain*. The other two are voltage ratios: g_{21} is the *forward open-circuit voltage gain*, whereas h_{12} is the *reverse open-circuit voltage gain*. We shall use **H** and **G** to represent the corresponding matrices.

By direct computation we find the following relations among the transfer parameters:

$$h_{12} = \frac{-z_{12}}{z_{21}} h_{21}, \tag{21a}$$

$$g_{12} = \frac{-y_{12}}{y_{21}} g_{21}. \tag{21b}$$

In the special case of reciprocal networks these expressions simplify to $h_{12} = -h_{21}$ and $g_{12} = -g_{21}$. In words this means that for reciprocal networks the open-circuit voltage gain for transmission in one direction through the two-port equals the negative of the short-circuit current gain for transmission in the opposite direction.

Just as $\mathbf{Z}_{oc}$ and $\mathbf{Y}_{sc}$ are each the other's inverse, so also **H** and **G** are each the other's inverse. Thus

$$\mathbf{G}(s) = \mathbf{H}^{-1}(s), \qquad \det \mathbf{G} = \frac{1}{\det \mathbf{H}}. \tag{22}$$

You should verify this.

Chain Parameters (T parameters)

CHAIN PARAMETERS

The remaining two sets of equations relating the port variables express the voltage and current at one port in terms of the voltage and current at the other. These were, in fact, historically the first set used—in the analysis of transmission lines. One of these equations is

$$\begin{bmatrix} V_1(s) \\ I_1(s) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2(s) \\ -I_2(s) \end{bmatrix}. \tag{23}$$

They are called the *chain*, or $ABCD$, parameters. The first name comes from the fact that they are the natural ones to use in a *cascade*, or *tandem*, or *chain* connection typical of a transmission system. Note the negative sign in $-I_2$, which is a consequence of the choice of reference for I_2.

Note that we are using the historical symbols for these parameters rather than using, say, a_{ij} for i and j equal 1 and 2, to make the system of notation uniform for all the parameters. We are also not introducing further notation to define the inverse parameters obtained by inverting (23), simply to avoid further proliferation of symbols.

The determinant of the chain matrix can be computed in terms of z's and y's. It is found to be

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = AD - BC = \frac{z_{12}}{z_{21}} = \frac{y_{12}}{y_{21}}, \tag{24}$$

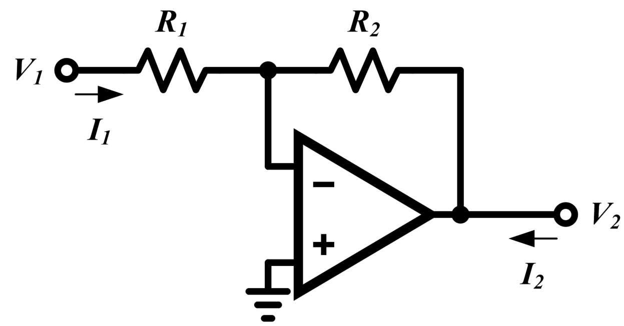
which is equal to 1 for reciprocal two-ports.

The preceding discussion is rather detailed and can become tedious if one loses sight of the objective of developing methods of representing the external behavior of two-ports by giving various relationships among the port voltages and currents. Each of these sets of relationships finds useful applications. For future reference we shall tabulate the interrelationships among the various parameters. The result is given in Table 1. Note that these relationships are valid for a general nonreciprocal two-port.

TRANSMISSION ZEROS

There is an important observation that can be made concerning the locations of the zeros of the various transfer functions. This can be seen most readily, perhaps, by looking at one of the columns in Table 1; for

Example:



$$V_1 = AV_2 - BI_2$$
$$I_1 = CV_2 - DI_2$$

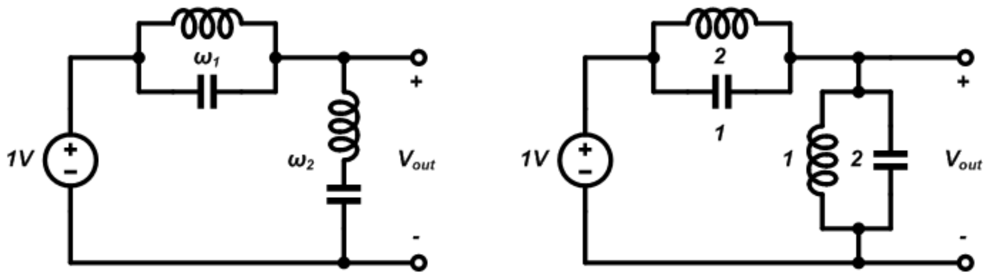
$$V_2 = -\frac{R_2}{R_1}V_1 \quad \rightarrow \quad A = -\frac{R_1}{R_2}, \;\; B = 0$$

$$I_1 = \frac{V_1}{R_1} = A\frac{V_2}{R_1} = -\frac{V_2}{R_2} \quad \rightarrow \quad C = -\frac{1}{R_2}, \;\; D = 0$$

$$AD - BC = 0 - 0 \neq 1$$

$$\begin{bmatrix} -\dfrac{R_1}{R_2} & 0 \\ -\dfrac{1}{R_2} & 0 \end{bmatrix} \quad \Rightarrow \quad non-reciprocal$$

A transmission zero is a value of s (or frequency) for which the output signal equals zero even though the input was nonzero. Example: ladder network



Transmission zeros
$\pm j\omega_1, \pm j\omega_2$

No transmission zeros

Example

As an illustrative example of the computation of two-port parameters, consider the network shown in Fig. 8, which can be considered as a

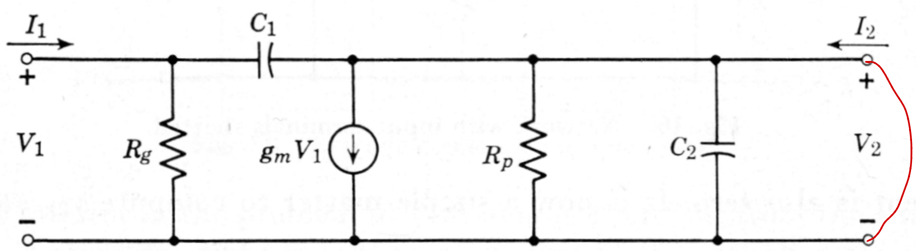


Fig. 8. Example for calculating two-port parameters.

model for a vacuum triode under certain conditions. (The capacitances are the grid-to-plate and plate-to-cathode capacitances.) Let us compute the y-parameters for this network. The simplest procedure is to use the interpretations in (11). If the output terminals are short-circuited, the resulting network will take the form shown in Fig. 9. As far as the input terminals are concerned, the controlled source has no effect. Hence y_{11} is the admittance of the parallel combination of R_g and C_1:

$$y_{11}(s) = \frac{1}{R_g} + sC_1.$$

Parallel branches across ports → y_{ij} simplest !

$\frac{1}{R_g} + sC_1$

$I_2 = g_m V_1 - \frac{V_1}{\frac{1}{sC_1}}$

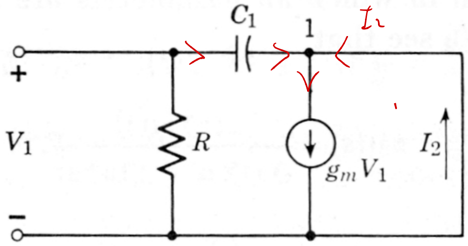


Fig. 9. Network with output terminals shorted.

To find y_{21}, assume that a voltage source with transform $V_1(s)$ is applied at the input terminals. By applying Kirchhoff's current law at the node labeled 1 in Fig. 9, we find that $I_2 = g_m V_1 - sC_1 V_1$. Hence y_{21} becomes

$$y_{21} = \left.\frac{I_2(s)}{V_1(s)}\right|_{V_2=0} = g_m - sC_1.$$

Now short-circuit the input terminals of the original network. The result will take the form in Fig. 10. Since V_1 is zero, the dependent source

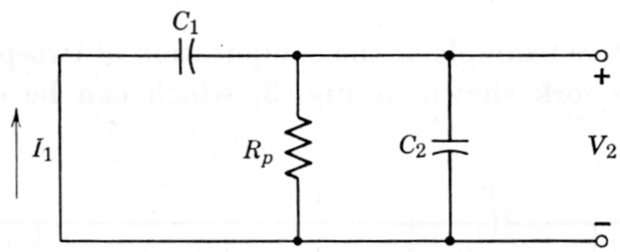


Fig. 10. Network with input terminals shorted.

current is also zero. It is now a simple matter to compute y_{22} and y_{12}:

$$y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = s(C_1 + C_2) + \frac{1}{R_p},$$

$$y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = -sC_1.$$

We see that y_{12} is different from y_{21}, as it should be, because of the presence of the controlled source.

If the y-parameters are known, any of the other sets of parameters can be computed by using Table 1. Note that even under the conditions that C_1 and C_2 are zero and R_g infinite, the y-parameters exist, but the z-parameters do not (z_{11}, z_{22}, and z_{21} become infinite).

How to find a new set of two-port parameters from a given one:

Example: Find the <u>H</u> parameters from the <u>Z</u> ones. We want

$$V_1 = h_{11}I_1 + h_{12}V_2$$
$$I_2 = h_{21}I_1 + h_{22}V_2$$

From second <u>Z</u> equation:

$$V_2 = z_{21}I_1 + z_{22}I_2$$
$$I_2 = \left(\frac{I}{h_{22}}\right)V_2 - \left(\frac{z_{21}}{h_{22}}\right)I_1 \leftarrow$$

Plug I_2 in first equation:

$$V_1 = z_{11}I_1 + z_{12}I_2 = z_{11}I_1 + \left(\frac{z_{12}}{z_{22}}\right)(V_2 - z_{21}.I_1)$$
$$V_1 = \underbrace{\frac{z_{12}/z_{22}}{}}_{h_{12}}V_2 + \underbrace{\frac{z_{11} - z_{12}z_{21}/z_{22}}{}}_{h_{11}} \leftarrow$$

Comparing (A) and (B) with the H equations shows that $h_{11} = \det Z / z_{22}$, $h_{12} = z_{12} / z_{22}$,

$h_{21} = -{}^{z_{21}}\!/\!_{z_{22}}$, and $h_{22} = {}^{1}\!/\!_{z_{22}}$

See Table 1 on p. 166 in B&B for a complete set of formulas. In general, rewrite original

Equations:

$$V_1 = z_{11}I_1 + z_{12}I_2 \rightarrow V_1\text{-}z_{12}I_2 = (z_{11}I_1) = a_1$$
$$V_2 = z_{21}I_1 + z_{22}I_2 \rightarrow 0 + z_{22}I_2 = (v_2 - z_{21}I_1) = a_2$$

Solve for V_1 and I_2 in terms of I_1, V_2.

Z ; Y , ABCD parameters Converters

Table 1

	Open-Circuit Impedance Parameters	Short-Circuit Admittance Parameters	Chain Parameters	Hybrid h-Parameters	Hybrid g-Parameters
z	$\begin{matrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{matrix}$	$\begin{matrix} \frac{y_{22}}{\lvert y\rvert} & \frac{-y_{12}}{\lvert y\rvert} \\ \frac{-y_{21}}{\lvert y\rvert} & \frac{y_{11}}{\lvert y\rvert} \end{matrix}$	$\begin{matrix} \frac{A}{C} & \frac{AD-BC}{C} \\ \frac{1}{C} & \frac{D}{C} \end{matrix}$	$\begin{matrix} \frac{\lvert h\rvert}{h_{22}} & \frac{h_{12}}{h_{22}} \\ \frac{-h_{21}}{h_{22}} & \frac{1}{h_{22}} \end{matrix}$	$\begin{matrix} \frac{1}{g_{11}} & \frac{-g_{12}}{g_{11}} \\ \frac{g_{21}}{g_{11}} & \frac{\lvert g\rvert}{g_{11}} \end{matrix}$
y	$\begin{matrix} \frac{z_{22}}{\lvert z\rvert} & \frac{-z_{12}}{\lvert z\rvert} \\ \frac{-z_{21}}{\lvert z\rvert} & \frac{z_{11}}{\lvert z\rvert} \end{matrix}$	$\begin{matrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{matrix}$	$\begin{matrix} \frac{D}{B} & \frac{-(AD-BC)}{B} \\ \frac{-1}{B} & \frac{A}{B} \end{matrix}$	$\begin{matrix} \frac{1}{h_{11}} & \frac{-h_{12}}{h_{11}} \\ \frac{h_{21}}{h_{11}} & \frac{\lvert h\rvert}{h_{11}} \end{matrix}$	$\begin{matrix} \frac{\lvert g\rvert}{g_{22}} & \frac{g_{12}}{g_{22}} \\ \frac{-g_{21}}{g_{22}} & \frac{1}{g_{22}} \end{matrix}$
$ABCD$	$\begin{matrix} \frac{z_{11}}{z_{21}} & \frac{\lvert z\rvert}{z_{21}} \\ \frac{1}{z_{21}} & \frac{z_{22}}{z_{21}} \end{matrix}$	$\begin{matrix} \frac{-y_{22}}{y_{21}} & \frac{-1}{y_{21}} \\ \frac{-\lvert y\rvert}{y_{21}} & \frac{-y_{11}}{y_{21}} \end{matrix}$	$\begin{matrix} A & B \\ C & D \end{matrix}$	$\begin{matrix} \frac{-\lvert h\rvert}{h_{21}} & \frac{-h_{11}}{h_{21}} \\ \frac{-h_{22}}{h_{21}} & \frac{-1}{h_{21}} \end{matrix}$	$\begin{matrix} \frac{1}{g_{21}} & \frac{g_{22}}{g_{21}} \\ \frac{g_{11}}{g_{21}} & \frac{\lvert g\rvert}{g_{21}} \end{matrix}$
h	$\begin{matrix} \frac{\lvert z\rvert}{z_{22}} & \frac{z_{12}}{z_{22}} \\ \frac{-z_{21}}{z_{22}} & \frac{1}{z_{22}} \end{matrix}$	$\begin{matrix} \frac{1}{y_{11}} & \frac{-y_{12}}{y_{11}} \\ \frac{y_{21}}{y_{11}} & \frac{\lvert y\rvert}{y_{11}} \end{matrix}$	$\begin{matrix} \frac{B}{D} & \frac{AD-BC}{D} \\ \frac{-1}{D} & \frac{C}{D} \end{matrix}$	$\begin{matrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{matrix}$	$\begin{matrix} \frac{g_{22}}{\lvert g\rvert} & \frac{-g_{12}}{\lvert g\rvert} \\ \frac{-g_{21}}{\lvert g\rvert} & \frac{g_{11}}{\lvert g\rvert} \end{matrix}$
g	$\begin{matrix} \frac{1}{z_{11}} & \frac{-z_{12}}{z_{11}} \\ \frac{z_{21}}{z_{11}} & \frac{\lvert z\rvert}{z_{11}} \end{matrix}$	$\begin{matrix} \frac{\lvert y\rvert}{y_{22}} & \frac{y_{12}}{y_{22}} \\ \frac{-y_{21}}{y_{22}} & \frac{1}{y_{22}} \end{matrix}$	$\begin{matrix} \frac{C}{A} & \frac{-(AD-BC)}{A} \\ \frac{1}{A} & \frac{B}{A} \end{matrix}$	$\begin{matrix} \frac{h_{22}}{\lvert h\rvert} & \frac{-h_{12}}{\lvert h\rvert} \\ \frac{-h_{21}}{\lvert h\rvert} & \frac{h_{11}}{\lvert h\rvert} \end{matrix}$	$\begin{matrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{matrix}$

Cascade Connection

3.4 INTERCONNECTION OF TWO-PORT NETWORKS

A given two-port network having some degree of complexity can be viewed as being constructed from simpler two-port networks whose ports are interconnected in certain ways. Conversely, a two-port network that is to be built can be designed by combining simple two-port structures as building blocks. From the designer's standpoint it is much easier to design simple blocks and to interconnect them than to design a complex network in one piece. A further practical reason for this approach is that it is much easier to shield smaller units and thus reduce parasitic capacitances to ground.

CASCADE CONNECTION

There are a number of ways in which two-ports can be interconnected. In the simplest interconnection of 2 two-ports, called the *cascade*, or tandem-connection, one port of each network is involved. Two two-ports are said to be connected *in cascade* if the output port of one is the input port of the second, as shown in Fig. 11.

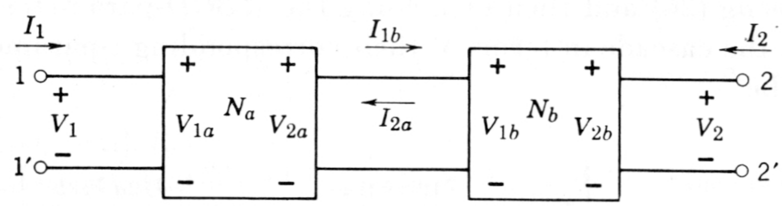


Fig. 11. Cascade connection of two-ports.

Our interest in the problem of "interconnection" is, from the analysis point of view, to study how the parameters of the overall network are related to the parameters of the individual building blocks. The tandem combination is most conveniently studied by means of the *ABCD*-parameters. From the references in the figure we see that

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} V_{1a} \\ I_{1a} \end{bmatrix}, \qquad \begin{bmatrix} V_{2a} \\ -I_{2a} \end{bmatrix} = \begin{bmatrix} V_{1b} \\ I_{1b} \end{bmatrix}, \qquad \begin{bmatrix} V_{2b} \\ -I_{2b} \end{bmatrix} = \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}.$$

Hence for the *ABCD* system of equations of the network N_b we can write

$$\begin{bmatrix} V_{2a} \\ -I_{2a} \end{bmatrix} = \begin{bmatrix} V_{1b} \\ I_{1b} \end{bmatrix} = \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}.$$

Furthermore, if we write the $ABCD$ system of equations for the network N_a and substitute in the last equation, we get

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} V_{2a} \\ -I_{2a} \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}.$$

Thus the ABCD-matrix of two-ports in cascade is equal to the product of the ABCD-matrices of the individual networks; that is,

$$\boxed{\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix}.} \tag{26}$$

Once the relationships between the parameters of the overall two-port and those of the components are known for any one set of parameters, it is merely algebraic computation to get the relationships for any other set; for example, the open-circuit parameters of the overall two-port can be found in terms of those for each of the two cascaded ones by expressing the z-parameters in terms of the $ABCD$-parameters for the overall network, using (26) and then expressing the $ABCD$-parameters for each network in the cascade in terms of their corresponding z-parameters. The result will be

$$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = \begin{bmatrix} z_{11a} - \dfrac{z_{12a} z_{21a}}{z_{22a} + z_{11b}} & \dfrac{z_{12a} z_{12b}}{z_{22a} + z_{11b}} \\ \dfrac{z_{21a} z_{21b}}{z_{22a} + z_{11b}} & z_{22b} - \dfrac{z_{12b} z_{21b}}{z_{22a} + z_{11b}} \end{bmatrix}. \tag{27}$$

The details of this computation are left to you.

A word of caution is necessary. When it is desired to determine some specific parameter of an overall two-port in terms of parameters of the components in the interconnection, it may be simpler to use a direct analysis than to rely on relationships such as those in Table 1. As an example, suppose it is desired to find the expression for z_{21} in Fig. 11. The term z_{21} is the ratio of open-circuit output voltage to input current: $z_{21} = V_2/I_1$. Suppose a current source I_1 is applied; looking into the output terminals of N_a, let the network be replaced by its Thévenin equivalent. The result is shown in Fig. 12. By definition, $z_{21b} = V_2/I_{1b}$

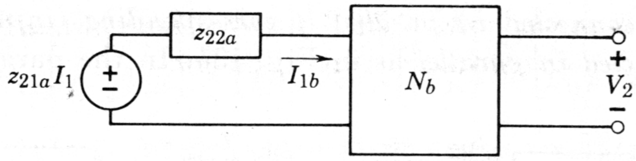


Fig. 12. Replacement of network N_a by its Thévenin equivalent.

with the output terminals open. Now I_{1b} can easily be found from the network in Fig. 12 to be

$$I_{1b} = \frac{z_{21a} I_1}{z_{22a} + z_{11b}}.$$

Hence

$$z_{21b} = \frac{V_2}{I_{1b}} = \frac{V_2}{\dfrac{z_{21a} I_1}{z_{22a} + z_{11b}}} = \left(\frac{z_{22a} + z_{11b}}{z_{21a}}\right)\frac{V_2}{I_1}.$$

Finally,

$$z_{21} = \frac{V_2}{I_1} = \frac{z_{21a} z_{21b}}{z_{22a} + z_{11b}}, \tag{28}$$

which agrees with (27).

An important feature of cascaded two-ports is observed from the expressions for the transfer impedances in (27). The zeros of z_{21} are the zeros of z_{21a} and z_{21b}. (A similar relationship holds for z_{12}.) Thus the transmission zeros of the overall cascade consist of the transmission zeros of each of the component two-ports. This is the basis of some important methods of network synthesis. It permits individual two-ports to be designed to achieve certain transmission zeros before they are connected together. It also permits independent adjustment and tuning of elements within each two-port to achieve the desired null without influencing the adjustment of the cascaded two-ports.

PARALLEL AND SERIES CONNECTIONS

Now let us turn to other interconnections of two-ports, which, unlike the cascade connection, involve both ports. Two possibilities that immediately come to mind are parallel and series connections. Two *two-ports* *are said to be connected in parallel if corresponding (input and output) ports are connected in parallel* as in Fig. 13*a*. In the parallel connection

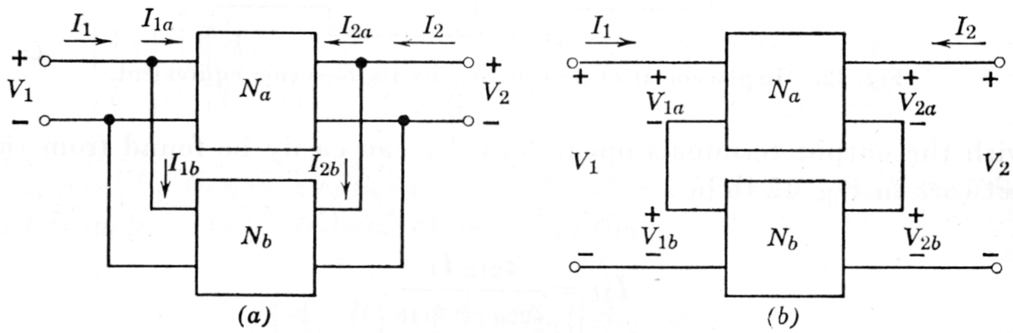


Fig. 13. Parallel and series connections of two-ports.

the input and output voltages of the component two-ports are forced to be the same, whereas the overall port currents equal the sums of the corresponding component port currents. This statement assumes that the port relationships of the individual two-ports are not altered when the connection is made. In this case the overall port relationship can be written as

$$\boxed{\begin{aligned}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} I_{1a} \\ I_{2a} \end{bmatrix} + \begin{bmatrix} I_{1b} \\ I_{2b} \end{bmatrix} &= \begin{bmatrix} y_{11a} & y_{12a} \\ y_{21a} & y_{22a} \end{bmatrix}\begin{bmatrix} V_{1a} \\ V_{2a} \end{bmatrix} + \begin{bmatrix} y_{11b} & y_{12b} \\ y_{21b} & y_{22b} \end{bmatrix}\begin{bmatrix} V_{1b} \\ V_{2b} \end{bmatrix} \\ &= \begin{bmatrix} y_{11a}+y_{11b} & y_{12a}+y_{12b} \\ y_{21a}+y_{21b} & y_{22a}+y_{22b} \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}.\end{aligned}}$$

That is, *the short-circuit admittance matrix of two-ports connected in parallel equals the sum of the short-circuit admittance matrices of the component two-ports*:

$$\mathbf{Y}_{sc} = \mathbf{Y}_{sca} + \mathbf{Y}_{scb}. \tag{30}$$

The dual of the parallel connection is the series connection. Two *two-ports are connected in series if corresponding ports (input and output) are connected in series*, as shown in Fig. 13*b*. In this connection the input and output port currents are forced to be the same, whereas the overall port voltages equal the sums of the corresponding port voltages of the individual two-ports. Again, it is assumed that the port relationships of the individual two-ports are not altered when the connection is made. In this case the overall port relationship can be written as

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_{1a} \\ V_{2a} \end{bmatrix} + \begin{bmatrix} V_{1b} \\ V_{2b} \end{bmatrix} = \begin{bmatrix} z_{11a} & z_{12a} \\ z_{21a} & z_{22a} \end{bmatrix} \begin{bmatrix} I_{1a} \\ I_{2a} \end{bmatrix} + \begin{bmatrix} z_{11b} & z_{12b} \\ z_{21b} & z_{22b} \end{bmatrix} \begin{bmatrix} I_{1b} \\ I_{2b} \end{bmatrix}$$

$$= \begin{bmatrix} z_{11a} + z_{11b} & z_{12a} + z_{12b} \\ z_{21a} + z_{21b} & z_{22a} + z_{22b} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}. \tag{31}$$

That is, ***the open-circuit impedance matrix of two-ports connected in series equals the sum of the open-circuit impedance matrices of the component two-ports***:

$$\mathbf{Z}_{oc} = \mathbf{Z}_{oca} + \mathbf{Z}_{ocb}\,. \tag{32}$$

Of these two—parallel and series connections—the parallel connection is more useful and finds wider application in synthesis. One reason for this is the practical one that permits two common-terminal (grounded) two-ports to be connected in parallel, the result being a common-terminal two-port. An example of this is the *parallel-ladders network* (of which the twin-tee null network is a special case) shown in Fig. 14.

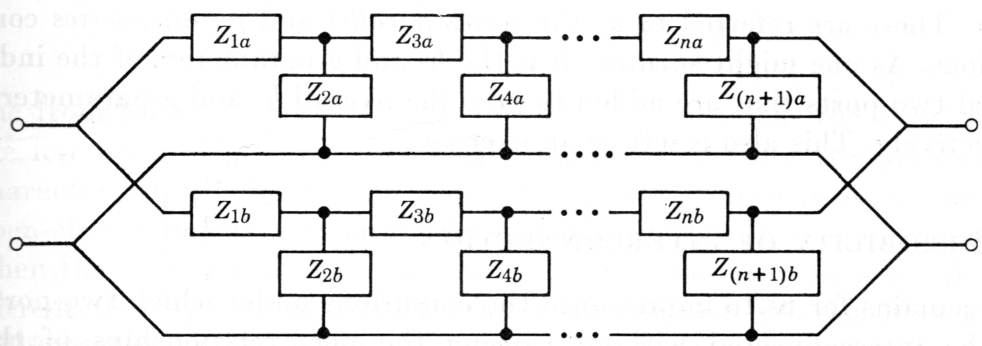


Fig. 14. Parallel-ladders network.

On the other hand, the series connection of two common-terminal two-ports is not a common-terminal two-port unless one of them is a tee network. Consider two grounded two-ports connected in series, as in Fig. 15*a*. It is clear that this is inadmissible, since the ground terminal of N_a will short out parts of N_b, thus violating the condition that the individual two-ports be unaltered by the interconnection. The situation is remedied by making the common terminals of both two-ports common to each other, as in Fig. 15*b*. In this case the resulting two-port is not a common-terminal one. If one of the component two-ports is a tee, the series connection takes the form shown in Fig. 15*c*. This can be redrawn, as in Fig. 15*d*, as a common-terminal two-port. That the last two networks have the same *z*-parameters is left for you to demonstrate.

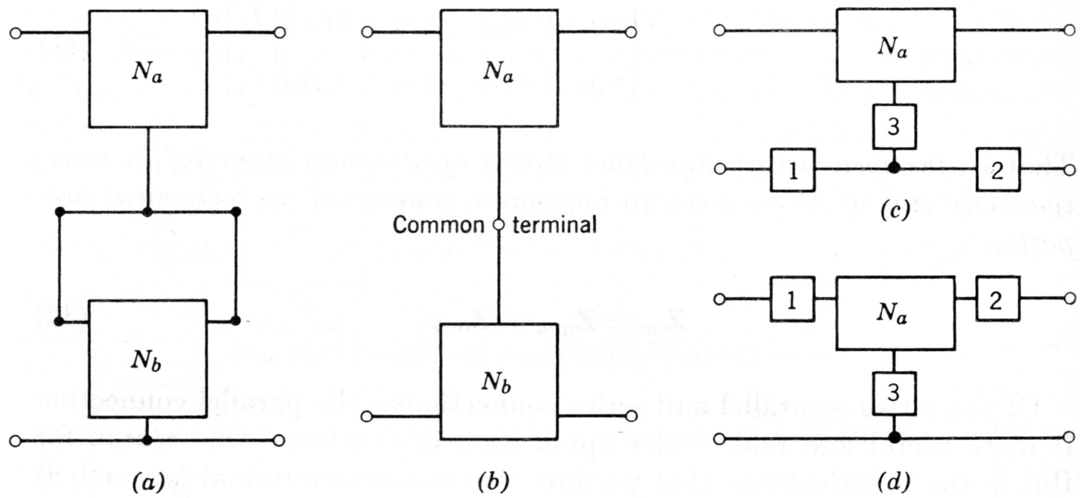


Fig. 15. Series connection of common-terminal two-ports.

Variations of the series and parallel types of interconnections are possible by connecting the ports in series at one end and in parallel at the other. These are referred to as the *series-parallel* and *parallel-series* connections. As one might surmise, it is the *h*- and *g*-parameters of the individual two-ports that are added to give the overall *h*- and *g*-parameters, respectively. This also is left as an exercise.

PERMISSIBILITY OF INTERCONNECTION

It remains for us to inquire into the conditions under which two-ports can be interconnected without causing the port relationships of the individual two-ports to be disturbed by the connection. For the parallel connection, consider Fig. 16. A pair of ports, one from each two-port, is

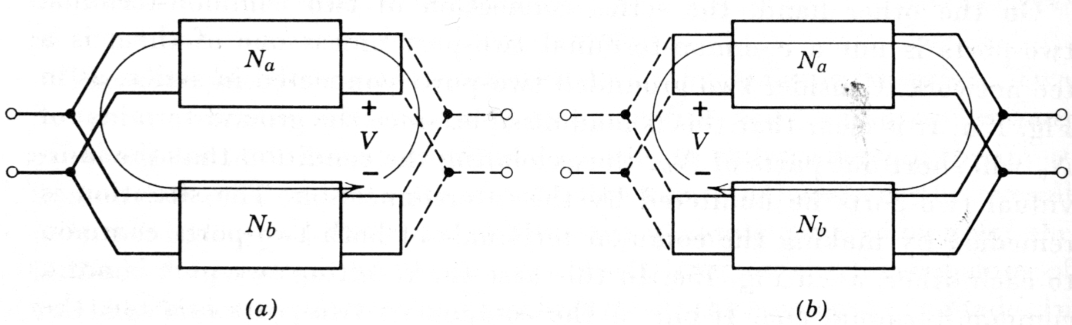
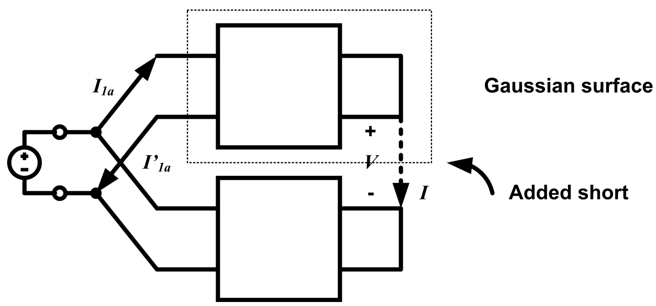
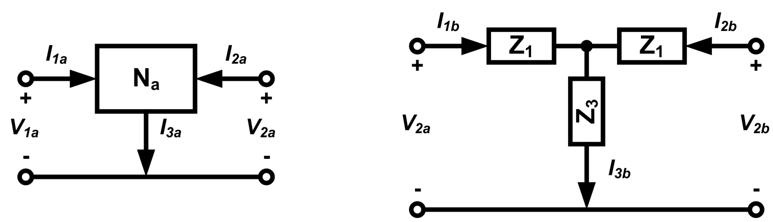


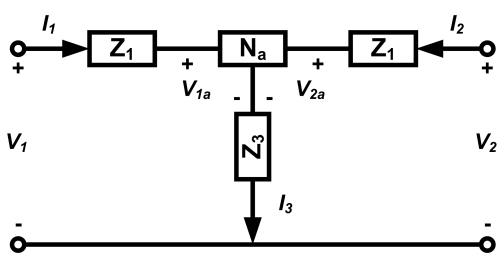
Fig. 16. Test for parallel-connected two-ports.



Before added short, $I_{1a} = I'_{1a}$ guaranteed; afterwards, if V=0, I also = 0, so $I_{1a} = I'_{1a}$ still holds. Added short needed when finding y_{11} and y_{21}.



$$Z_b = \begin{bmatrix} Z_1 + Z_3 & Z_3 \\ Z_3 & Z_2 + Z_3 \end{bmatrix}$$



For $I_1 = 1$, $I_2 = 0$, $I_3 = I_1 = 1$:

$$Z_{11} = V_1 = Z_1 + V_{1a} + Z_3 = Z_{11a} + Z_{11b}$$
$$Z_{21} = V_2 = V_{21} + Z_3 = Z_{21a} + Z_{21b}$$

Same result for $I_2 = 1$, $I_1 = 0$.

connected in parallel, whereas the other ports are individually short-circuited. The short circuits are employed because the parameters characterizing the individual two-ports and the overall two-port are the short-circuit admittance parameters. If the voltage V shown in Fig. 16 is nonzero, then when the second ports are connected there will be a circulating current, as suggested in the diagram. Hence the condition that the current leaving one terminal of a port be equal to the current entering the other terminal of each individual two-port is violated, and the port relationships of the individual two-ports are altered.

For the case of the series connection, consider Fig. 17. A pair of ports,

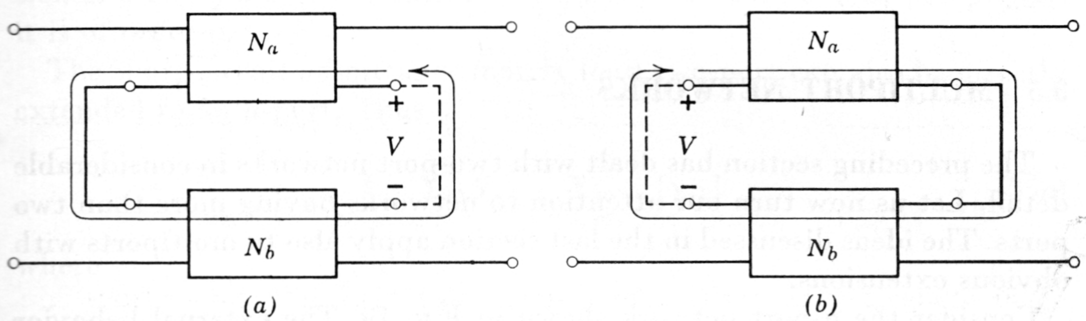


Fig. 17. Test for series-connected two-ports.

one from each two-port, is connected in series, whereas the other ports are left open. The open circuits are employed because the parameters characterizing the individual two-ports and the overall two-port are the open-circuit impedance parameters. If the voltage V is nonzero, then when the second ports are connected in series there will be a circulating current, as suggested in the diagram. Again, the port relationships of the individual two-ports will be modified by the connection, and hence the addition of impedance parameters will not be valid for the overall network.

Obvious modifications of these tests apply to the series-parallel and parallel-series connections. The preceding discussion of the conditions under which the overall parameters for interconnected two-ports can be obtained by adding the component two-port parameters has been in rather skeletal form. We leave to you the task of supplying details.

When it is discovered that a particular interconnection cannot be made because circulating currents will be introduced, there is a way of stopping such currents and thus permitting the connection to be made. The approach is simply to put an isolating ideal transformer of 1 : 1 turns ratio at one of the ports, as illustrated in Fig. 18 for the case of the parallel connection.

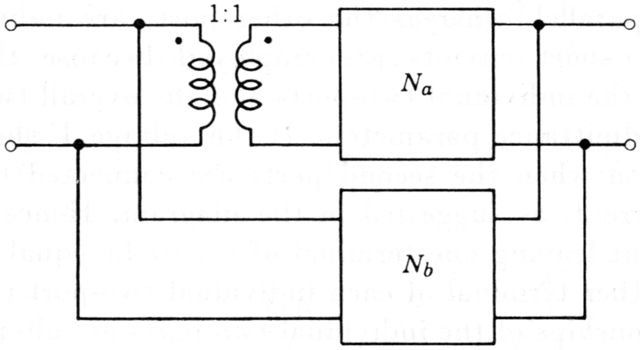


Fig. 18. Isolating transformer to permit interconnection.

3.5 MULTIPORT NETWORKS

The preceding section has dealt with two-port networks in considerable detail. Let us now turn our attention to networks having more than two ports. The ideas discussed in the last section apply also to multiports with obvious extensions.

Consider the n-port network shown in Fig. 19. The external behavior

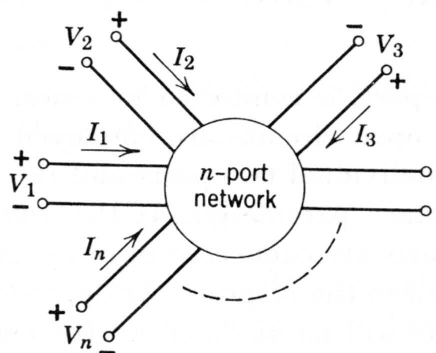


Fig. 19. Multiport network.

of this network is completely described by giving the relationships among the port voltages and currents. One such relationship expresses all the port voltages in terms of the port currents:

$$\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{nn} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_n \end{bmatrix} \tag{33a}$$

or

$$\mathbf{V} = \mathbf{Z}_{oc}\mathbf{I}. \tag{33b}$$

By direct observation, it is seen that the parameters can be interpreted as

$$z_{jk} = \left.\frac{V_j}{I_k}\right|_{\substack{\text{all other}\\ \text{currents} = 0}} \tag{34}$$

which is simply the extension of the open-circuit impedance representation of a two-port. The matrix $\mathbf{Z}_{oc}$ is the same as that in (12) except that it is of order n.

The short-circuit admittance matrix for a two-port can also be directly extended to an n-port. Thus

$$\mathbf{I} = \mathbf{Y}_{sc}\mathbf{V}, \qquad \mathbf{Y}_{sc} = [y_{jk}] \tag{35a}$$

where

$$y_{jk} = \left.\frac{I_j}{V_k}\right|_{\substack{\text{all other}\\ \text{voltages} = 0}} \tag{35b}$$

If we now think of extending the hybrid representations of a two-port, we encounter some problems. In a hybrid representation the variables are mixed voltage and current. For a network of more than two ports, how are the "independent" and "dependent" variables to be chosen? In a three-port network, for example, the following three choices can be made:

$$\begin{bmatrix} V_1 \\ V_2 \\ I_3 \end{bmatrix} = \mathbf{M}_1 \begin{bmatrix} I_1 \\ I_2 \\ V_3 \end{bmatrix}, \qquad \begin{bmatrix} V_1 \\ I_2 \\ V_3 \end{bmatrix} = \mathbf{M}_2 \begin{bmatrix} I_1 \\ V_2 \\ I_3 \end{bmatrix}, \qquad \begin{bmatrix} I_1 \\ V_2 \\ V_3 \end{bmatrix} = \mathbf{M}_3 \begin{bmatrix} V_1 \\ I_2 \\ I_3 \end{bmatrix}$$

as well as their inverses. In these choices each vector contains exactly one variable from each port. It would also be possible to make such selections as

$$\begin{bmatrix} V_1 \\ V_2 \\ I_2 \end{bmatrix} = \mathbf{M}_4 \begin{bmatrix} I_1 \\ I_3 \\ V_3 \end{bmatrix}$$

Just as in the case of two-ports, it is possible to interconnect multi-ports. Two multi-ports are said to be connected in parallel if their ports are connected in parallel in pairs. It is not, in face, necessary for the two multi-ports to have same number of ports. The ports are connected in parallel in pairs until we run out of ports. It does not matter whether we run out for both networks at the same time or earlier for one network. Similarly, two multi-ports are said to be connected in series if their ports are connected in series of pairs. Again, the two multi-ports need not have the same number of ports.

As in the case of two-ports, the overall y-matrix for two n-ports connected in parallel equals the sum of the y-matrices of the individual n-ports. Similarly, the overall z-matrix of two n-ports connected in a series equals the sum of the z-matrices of the individual n-ports. This assumes, of course, that the interconnection does not alter the parameters of the individual n-ports.

Back to the chain matrix:

$$V_1 = AV_2 - BI_2 \qquad I_1 = CV_2 - DI_2 \tag{5-24}$$

A, B, C, and D are called the *chain parameters;* their matrix

$$\mathbf{T} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{5-25}$$

$$A = \left(\frac{V_1}{V_2}\right)_{I_2=0} \qquad B = \left(\frac{V_1}{-I_2}\right)_{V_2=0} \qquad C = \left(\frac{I_1}{V_2}\right)_{I_2=0} \qquad D = \left(\frac{I_1}{-I_2}\right)_{V_2=0} \tag{5-27}$$

Hence, **T** can be found from the schemes shown in Fig. 5-11.

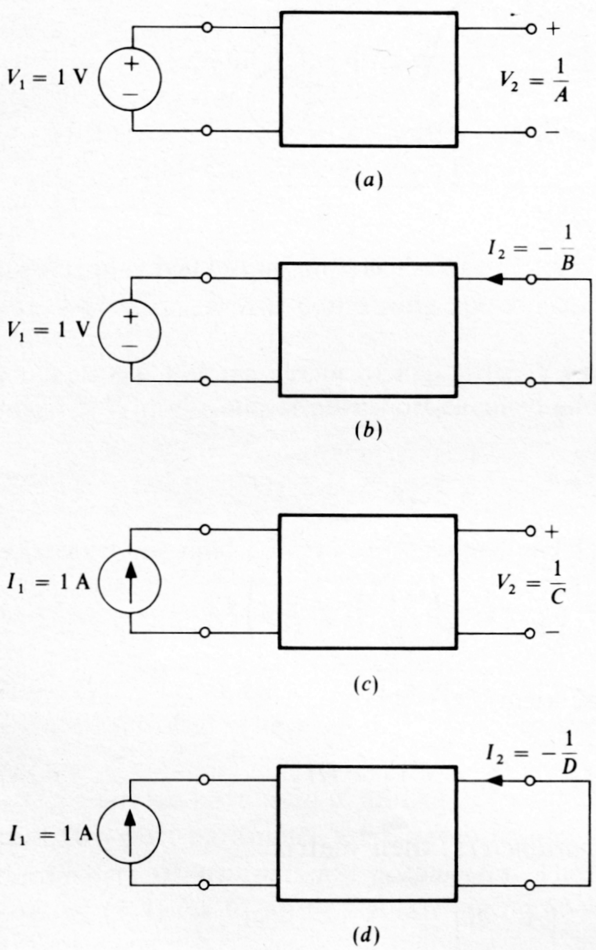
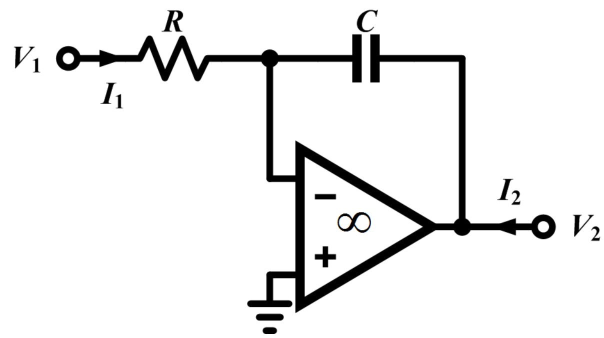


Figure 5-11 Schemes for calculating the chain parameters of a two-port.

Integrator:



$V_2 = -\frac{I_1}{C} = \frac{-V_1}{RC}$

$$V_2 = \frac{-V_1}{sRC} \qquad I_1 = \frac{V_1}{R}$$

$$V_1 = -sRCV_2 + 0.I_2$$
$$I_1 = V_1 / R = -sCV_2 + 0.I_2$$
$$T = \begin{bmatrix} -sRC & 0 \\ -sC & 0 \end{bmatrix} = sC\begin{bmatrix} R & 0 \\ 1 & 0 \end{bmatrix}$$

Cascaded integrators:

$$T_1T_2 = s^2C_0^2\begin{bmatrix} R^2 & 0 \\ R & 0 \end{bmatrix} = (-sRC_0)^2\begin{bmatrix} 1 & 0 \\ 1/R & 0 \end{bmatrix}$$

NOTE: B=D=0 means buffered two-port!

Example 5-3 Find **Y** and **Z** for the circuit of Fig. 5-7*a*.

From Fig. 5-7*b* (where we have chosen for simplicity $V_1 = 1$ V), by inspection

$$y_{11} \triangleq \frac{I_1^1}{V_1} = I_1^1 = sC_1 + \frac{1}{R} \qquad y_{21} \triangleq \frac{I_2^1}{V_1} = I_2^1 = -\frac{1}{R}$$

and from Fig. 5-7*c*, with $V_2 = 1$ V,

$$y_{12} \triangleq \frac{I_1^2}{V_2} = I_1^2 = -\frac{1}{R} \qquad y_{22} \triangleq \frac{I_2^2}{V_2} = I_2^2 = sC_2 + \frac{1}{R}$$

Hence

$$\mathbf{Y} = \begin{bmatrix} sC_1 + \dfrac{1}{R} & -\dfrac{1}{R} \\ -\dfrac{1}{R} & sC_2 + \dfrac{1}{R} \end{bmatrix}$$

Therefore

$$\Delta_Y = \left(sC_1 + \frac{1}{R}\right)\left(sC_2 + \frac{1}{R}\right) - \frac{1}{R^2} = s^2 C_1 C_2 + \frac{s(C_1 + C_2)}{R}$$

and, by (5-17),

$$\mathbf{Z} = \begin{bmatrix} \dfrac{sC_2 + 1/R}{\Delta_Y} & \dfrac{1/R}{\Delta_Y} \\ \dfrac{1/R}{\Delta_Y} & \dfrac{sC_1 + 1/R}{\Delta_Y} \end{bmatrix}$$

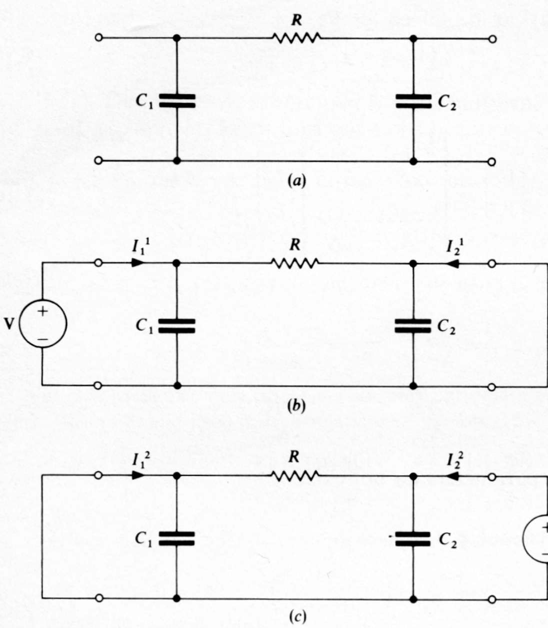


Figure 5-7 Simple *RC* two-port and the calculation of its admittance parameters.

Of course, **Z** can also be obtained directly, using the method illustrated in Fig. 5-4. For our example, this gives

$$z_{11} = \frac{V_1^1}{I_1} = \frac{1}{sC_1 + \dfrac{1}{R + 1/sC_2}}$$

$$z_{11} = \frac{sRC_2 + 1}{s^2 RC_1 C_2 + s(C_1 + C_2)} = \frac{sC_2 + 1/R}{s^2 C_1 C_2 + s(C_1 + C_2)/R}$$

Example 5-4 Find **Z** and **Y** for the two-port shown in Fig. 5-8.

With an input current $I_1 = 1$ A, from (5-2) and Fig. 5-4*b*,

$$z_{11} = V_1^1 = \frac{sL_1 R_1}{sL_1 + R_1} + \frac{R_2/sC}{R_2 + 1/sC} = \frac{s}{s+1} + \frac{1/s}{1 + 1/s} = 1$$

and

$$z_{21} = V_2^1 = \frac{R_2/sC}{R_2 + 1/sC} = \frac{1/s}{1 + 1/s} = \frac{1}{s+1}$$

Similarly, from (5-3) and Fig. 5-4*c*,

$$z_{12} = V_1^2 = \frac{R_2/sC}{R_2 + 1/sC} = \frac{1}{s+1}$$

and

$$z_{22} = V_2^2 = \frac{sL_2 R_3}{sL_2 + R_3} + \frac{R_2/sC}{R_2 + 1/sC} = 1$$

Hence,

$$\Delta_Z = 1 - \frac{1}{(s+1)^2} = \frac{s^2 + 2s}{s^2 + 2s + 1}$$

and by (5-19)

$$y_{11} = \frac{z_{22}}{\Delta_Z} = \frac{s^2 + 2s + 1}{s^2 + 2s}$$

$$y_{12} = y_{21} = -\frac{z_{12}}{\Delta_Z} = -\frac{s+1}{s^2 + 2s}$$

$$y_{22} = \frac{z_{11}}{\Delta_Z} = \frac{s^2 + 2s + 1}{s^2 + 2s}$$

Again, y_{11} will be checked using the scheme of Fig. 5-5*b*. For $V_1 = 1$ V,

$$y_{11} = I_1^1 = \frac{1}{\dfrac{sL_1 R_1}{sL_1 + R_1} + \dfrac{1}{1/R_2 + sC + 1/R_3 + 1/sL_2}}$$

$$= \frac{1}{\dfrac{s}{s+1} + \dfrac{1}{2 + s + 1/s}} = \frac{s^2 + 2s + 1}{s^2 + 2s}$$

as before.

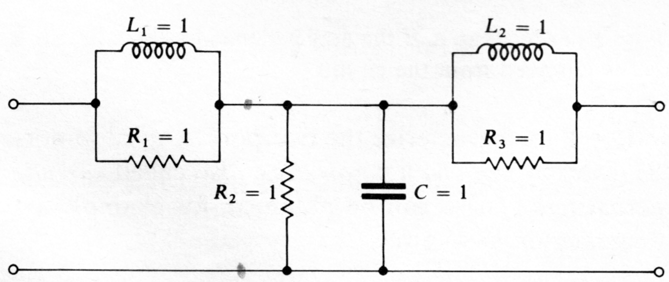


Figure 5-8 *RLC* two-port example.

Symmetric, reciprocal constant-resistance twoport.

Example 5-5 Find the z_{ij} for the two-port shown in Fig. 5-9.
From Fig. 5-4, we obtain

$$z_{11} = Z_1 + \frac{Z_2(Z_3 + Z_4)}{Z_2 + Z_3 + Z_4}$$

$$z_{12} = z_{21} = \frac{Z_2 Z_4}{Z_2 + Z_3 + Z_4}$$

$$z_{22} = \frac{Z_4(Z_2 + Z_3)}{Z_2 + Z_3 + Z_4}$$

The reader should fill in the details.

Hitherto, all two-port examples contained reciprocal circuits. Hence, by (5-5), $z_{12} = z_{21}$ held, and so did

$$y_{12} = y_{21} \tag{5-21}$$

Equation (5-21) can be obtained either from Eqs. (5-5) and (5-19) or from Fig. 1-6*b*. For circuits containing active elements, (5-5) and (5-21) need not hold.

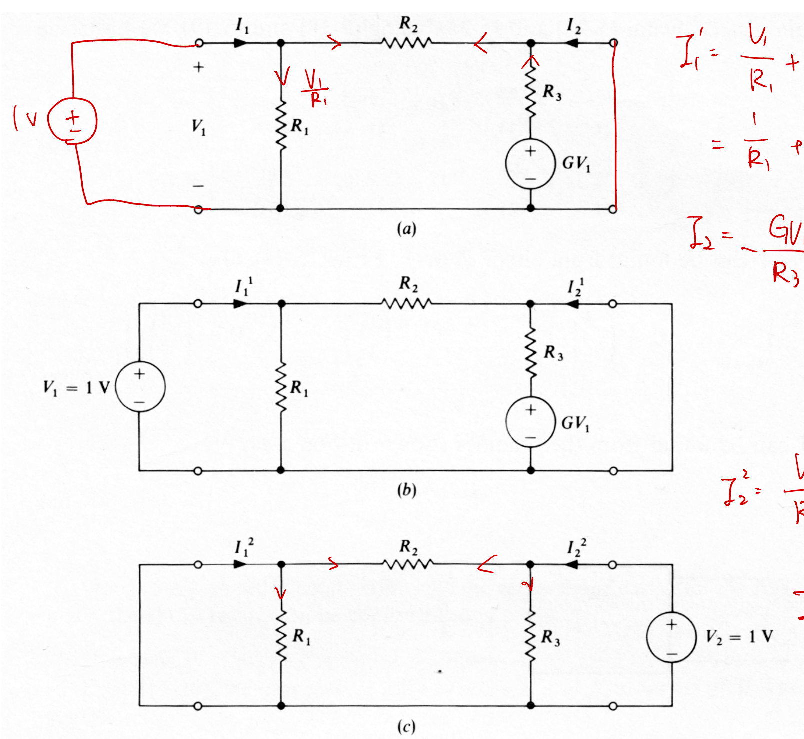
Example 5-6 For the circuit of Fig. 5-10*a*, **Y** can readily be found using the scheme of Fig. 5-5. With the output port short-circuited and $V_1 = 1$ V (Fig. 5-10*b*),

$$y_{11} = I_1^1 = \frac{1}{R_1} + \frac{1}{R_2} \qquad y_{21} = I_2^1 = -\frac{G}{R_3} - \frac{1}{R_2}$$

whereas if the input port is short-circuited and $V_2 = 1$ V (Fig. 5-10*c*),

$$y_{12} = I_1^2 = -\frac{1}{R_2} \qquad \text{and} \qquad y_{22} = \frac{1}{R_3} + \frac{1}{R_2}$$

Here, $y_{12} \neq y_{21}$, except if $G = 0$ or $R_3 \to \infty$, that is, if the active element (which here is a voltage-controlled voltage source) is removed from the circuit.



$$I_1' = \frac{V_1}{R_1} + \frac{V_1}{R_2} = \frac{1}{R_1} + \frac{1}{R_2}$$

$$I_2 = -\frac{GV_1}{R_3} - \frac{V_1}{R_2}$$

$$I_2^2 = \frac{V_2}{R_2} + \frac{V_2}{R_3}$$

$$I_1^2 = \frac{-1}{R_2}$$

Figure 5-10 (*a*) Nonreciprocal two-port; (*b*) and (*c*) calculation of **Y** for the two-port.

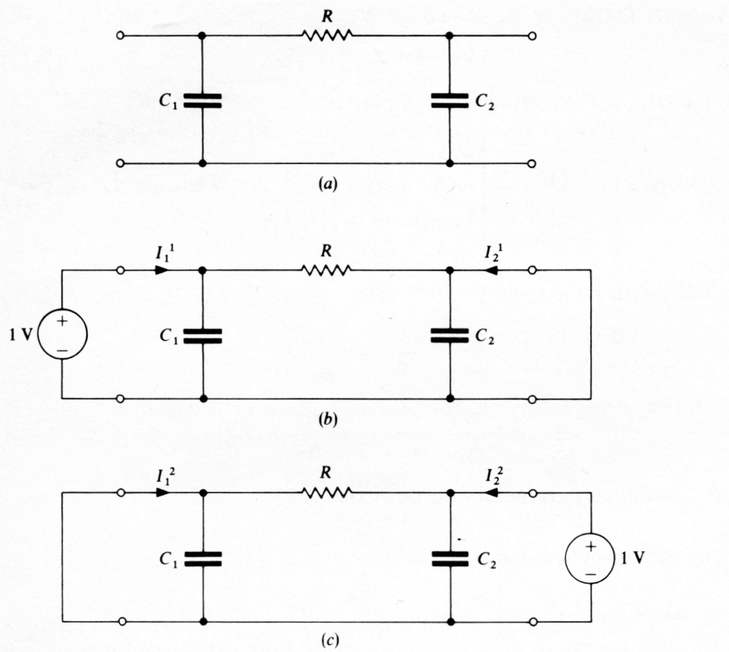


Figure 5-7 Simple *RC* two-port and the calculation of its admittance parameters.

Example 5-7 For the two-port of Fig. 5-7*a*, using Fig. 5-11, we find

$$A = \left(\frac{1}{V_2}\right)_{\substack{V_1 = 1\text{ V}\\ I_2 = 0}} = \left(\frac{1/sC_2}{R + 1/sC_2}\right)^{-1} = sRC_2 + 1$$

$$B = \left(-\frac{1}{I_2}\right)_{\substack{V_1 = 1\text{ V}\\ V_2 = 0}} = R$$

$$C = \left(\frac{1}{V_2}\right)_{\substack{I_1 = 1\text{ A}\\ I_2 = 0}} = \left(\frac{1}{V_1 \dfrac{1/sC_2}{R + 1/sC_2}}\right)_{\substack{I_1 = 1\text{ A}\\ I_2 = 0}}$$

$$= \frac{1}{\dfrac{(1/sC_1)(R + 1/sC_2)}{(1/sC_1 + R + 1/sC_2)} \dfrac{1/sC_2}{R + 1/sC_2}} = s^2RC_1C_2 + s(C_1 + C_2)$$

and $$D = \left(-\frac{1}{I_2}\right)_{\substack{I_1 = 1\text{ A}\\ V_2 = 0}} = \left(\frac{1/R}{sC_1 + 1/R}\right)^{-1} = sRC_1 + 1$$

When (5-26) is used in conjunction with the values found earlier for the **Z** and **Y** of this circuit, the above results can be confirmed.

Check:

Example 5-8 For the circuit of Fig. 5-7*a*,

$$\Delta_T = (sRC_2 + 1)(sRC_1 + 1) - R(s^2RC_1C_2 + sC_1 + sC_2) = 1$$

as expected.

Example 5-9 For the circuit of Fig. 5-10*a*, the chain parameters can be found from the admittance parameters calculated earlier, for example:

$$A = -\frac{y_{22}}{y_{21}} = -\frac{1/R_3 + 1/R_2}{-G/R_3 - 1/R_2} = \frac{R_2 + R_3}{GR_2 + R_3}$$

$$B = -\frac{1}{y_{21}} = -\frac{1}{-G/R_3 - 1/R_2} = \frac{R_2 R_3}{GR_2 + R_3}$$

$$C = -\frac{y_{11}y_{22} - y_{12}y_{21}}{y_{21}} = -\frac{y_{11}y_{22}}{y_{21}} + y_{12}$$

$$= -\frac{(1/R_1 + 1/R_2)(1/R_3 + 1/R_2)}{-G/R_3 - 1/R_2} - \frac{1}{R_2}$$

$$= \frac{(R_1 + R_2)(R_2 + R_3)}{(GR_2 + R_3)R_1 R_2} - \frac{1}{R_2}$$

$$D = -\frac{y_{11}}{y_{21}} = -\frac{1/R_1 + 1/R_2}{-G/R_3 - 1/R_2} = \frac{(R_1 + R_2)R_3}{R_1(GR_2 + R_3)}$$

Hence

$$AD - BC = \frac{\dfrac{(R_2 + R_3)(R_1 + R_2)R_3}{R_1} - R_2 R_3 \left[\dfrac{(R_1 + R_2)(R_2 + R_3)}{R_1 R_2} - \dfrac{GR_2 + R_3}{R_2}\right]}{(GR_2 + R_3)^2}$$

or, after simplifications,

$$AD - BC = \frac{R_3}{GR_2 + R_3}$$

Hence, $AD - BC = 1$ holds only if $R_3 \to \infty$ or $G \to 0$, that is, only if the controlled source is removed from the circuit.

Since Eq. (5-4) can be rearranged six different ways with two parameters on the left-hand side and two on the right-hand side, we can define six different sets of two-port parameters. The reader should consult Ref. 1, table 17-1, for a listing of these parameters and for the formulas needed to convert from one set to another.

Transfer Functions for Terminated Twoports

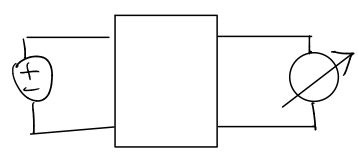
Transfer Function: Possible output/known input (voltage or current. For a two-port, we may have

Voltage ratio, voltage gain: $A_v(s) = V_s(s) / E(s)$

Transfer Admittance: $Y_T(s) = I_2(S) / E(s)$

Transfer Impedance: $Z_T(s) = V_2(s) / I(s)$

Current Ratio, current gain: $A_I(s) = I_2(s) / I(s)$



5-5 TRANSFER FUNCTIONS

When the two-port is excited by a generator and terminated by a load, as illustrated in Fig. 5-1, the signal-transfer properties of the *complete* circuit can be described by an appropriately chosen *transfer function.* We define the transfer function as the ratio of an output variable (voltage or current) to a known input quantity (generator voltage or current).† Since most practical generator and load impedances are essentially resistive, we shall restrict our discussions to resistor-terminated reactance two-ports.

In the simplest (and least useful) situation, both terminations are zero or infinite. Then we have one of the four configurations depicted in Fig. 5-18. These circuits are called *unterminated* (*unloaded*) two-ports. The proper choice of a transfer function for any of these circuits is obvious and unique. For example, for the circuit of Fig. 5-18*a*, the output variable must be V_2 since $I_2 \equiv 0$; the known input quantity is the generator voltage E. Hence, we must choose the *voltage ratio* A_V, defined by

$$A_V(s) \triangleq \frac{V_2(s)}{E(s)} \tag{5-84}$$

[The reader should keep in mind the dual interpretation of the variable s. Thus, for $s = j\omega$, $A_V(j\omega)$ may represent the ratio of the steady-state sine-wave voltage phasors at the output and input. In general, however, $A_V(s)$ is the ratio of the Laplace-transformed output signal $v_2(t)$ and generator signal $e(t)$ for a two-port initially free of stored energy.]

Similarly, for the circuit in Fig. 5-18*b* the transfer function must be the *transfer admittance*

$$Y_T(s) \triangleq \frac{I_2(s)}{E(s)} \tag{5-85}$$

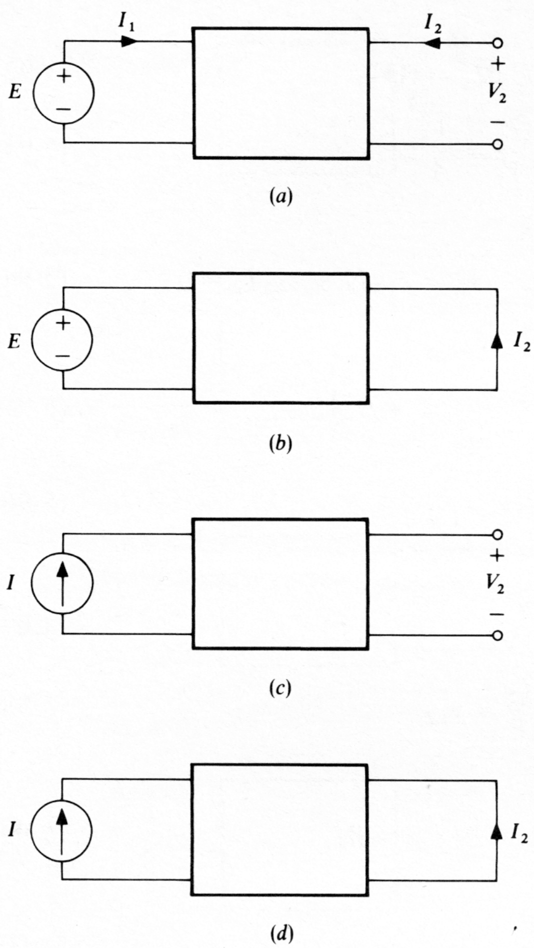
For the circuit of Fig. 5-18*c*, the transfer function is the *transfer impedance*

$$Z_T(s) \triangleq \frac{V_2(s)}{I(s)} \tag{5-86}$$

† Here all quantities are assumed to be functions of s, not t.

$V_1 = Z_{11} I_1 + \cancel{Z_{12} I_2} = Z_{11} I_1$

$V_2 = Z_{12} I_1 + \cancel{Z_{22} I_2} = Z_{12} I_1$



open

short

Figure 5-18 Unterminated two-ports.

Finally, for the circuit of Fig. 5-18*d*, the transfer function is the *current ratio*

$$A_I(s) \triangleq \frac{I_2(s)}{I(s)} \tag{5-87}$$

The transfer functions can readily be calculated from the two-port parameters **Z** or **Y** or **T**. For example, for the circuit of Fig. 5-18*a*, from (5-9)

$$V_1 = z_{11} I_1 + z_{12} I_2 = z_{11} I_1 \qquad V_2 = z_{12} I_1 + z_{22} I_2 = z_{12} I_1 \tag{5-88}$$

Hence

$$A_V = \frac{V_2}{E} = \frac{V_2}{V_1} = \frac{z_{12}}{z_{11}} \tag{5-89}$$

Alternatively, from (5-10)

$$I_2 = y_{12} V_1 + y_{22} V_2 = 0 \tag{5-90}$$

which gives

$$A_V = \frac{V_2}{V_1} = -\frac{y_{12}}{y_{22}} \tag{5-91}$$

Or, from (5-24),

$$V_1 = AV_2 - BI_2 = AV_2 \tag{5-92}$$

so that

$$A_V = \frac{V_2}{V_1} = \frac{1}{A} \tag{5-93}$$

For the circuit of Fig. 5-18*b*, from (5-9),

$$V_1 = z_{11}I_1 + z_{12}I_2 = E \qquad V_2 = z_{12}I_1 + z_{22}I_2 = 0 \tag{5-94}$$

Solving (5-94) for I_2 gives

$$I_2 = -\frac{Ez_{12}}{z_{11}z_{22} - z_{12}^2} = -E\frac{z_{12}}{\Delta_Z} \tag{5-95}$$

so that

$$Y_T \triangleq \frac{I_2}{E} = -\frac{z_{12}}{\Delta_Z} \tag{5-96}$$

Alternatively, from (5-10)

$$I_2 = y_{12}V_1 + y_{22}V_2 = y_{12}E \tag{5-97}$$

so that

$$Y_T = \frac{I_2}{E} = y_{12} \tag{5-98}$$

Or, from (5-24),

$$V_1 = E = AV_2 - BI_2 = -BI_2 \tag{5-99}$$

which gives

$$Y_T = \frac{I_2}{E} = -\frac{1}{B} \tag{5-100}$$

directly.

Exactly analogous manipulations give

$$Z_T = \frac{V_2}{I} \triangleq z_{12} = -\frac{y_{12}}{\Delta_Y} = \frac{1}{C} \tag{5-101}$$

for the circuit of Fig. 5-18*c* and

$$A_I = -\frac{z_{12}}{z_{22}} = \frac{y_{12}}{y_{11}} = -\frac{1}{D} \tag{5-102}$$

for the circuit of Fig. 5-18*d*.

If the two-port has a single resistive termination, it is called a *singly terminated* or *singly loaded* two-port. Four possible circuits for such a two-port are illustrated in Fig. 5-19. Notice that two other possible circuits exist which may be obtained by replacing the generator and its internal impedance R by its Norton equivalent in the circuit of Fig. 5-19*b* or by its Thevenin equivalent in Fig. 5-19*d*. Their transfer functions differ only by a factor R from those of Fig. 5-19*b* and *d*, and hence they do not merit separate treatment.

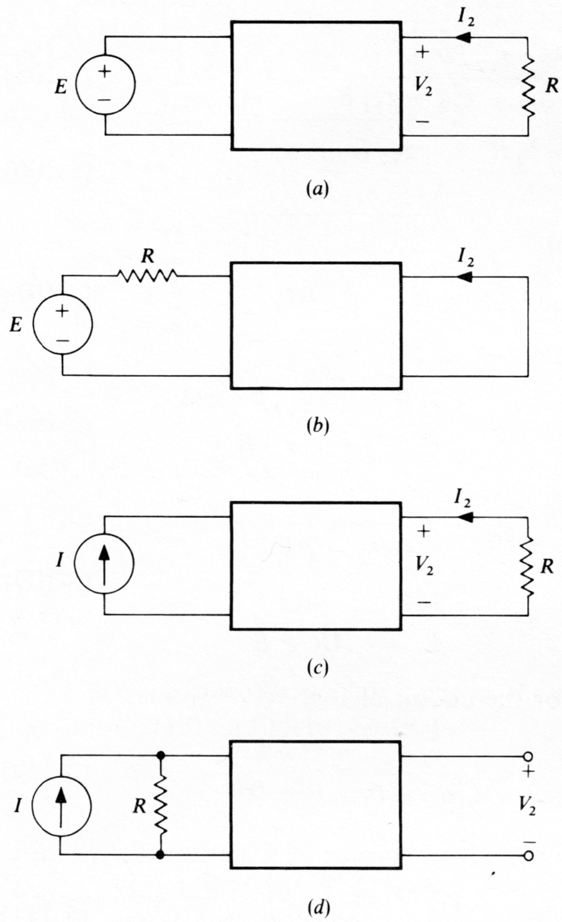


Figure 5-19 Singly terminated two-ports.

For the circuit of Fig. 5-19*a*, we can choose $A_V = V_2/E$ as the transfer function. (Again, a trivially different choice is to select $Y_T \triangleq I_2/E$; since $I_2 = -V_2/R$, here $Y_T = -A_V/R$.)

For the circuit of Fig. 5-19*b*, the transfer function may be selected as $Y_T = I_2/E$ [or as $A_I = I_2/I = I_2/(E/R) = RY_T$ if the Norton generator model is substituted].

For the circuit of Fig. 5-19*c*, we can use $A_I = I_2/I$ or, as trivial variant, $Z_T = V_2/I = -I_2R/I = -RA_I$. For the circuit of Fig. 5-19*d*, the transfer function can be $Z_T = V_2/I$ or if the Thevenin equivalent is used for the generator, $A_V = V_2/E = V_2/(IR) = Z_T/R$.

The transfer functions of the singly loaded two-port can also easily be found in terms of the two-port parameters and R, as will be shown next. For the circuit of Fig. 5-19*a*, combining the branch relations

$$V_1 = E \qquad V_2 = -RI_2 \tag{5-103}$$

and the two-port relations (5-9), we get

$$z_{11}I_1 + z_{12}I_2 = E \qquad z_{12}I_1 + (z_{22} + R)I_2 = 0 \tag{5-104}$$

which gives

$$I_2 = \frac{z_{12}E}{-z_{11}z_{22} + z_{12}^2 - z_{11}R} \tag{5-105}$$

so that

$$A_V = \frac{V_2}{E} = \frac{-I_2 R}{E} = \frac{z_{12}R}{\Delta_Z + z_{11}R} \tag{5-106}$$

Alternatively, from (5-103) and (5-10)

$$I_2 = y_{12}V_1 + y_{22}V_2 = y_{12}E - y_{22}RI_2 \tag{5-107}$$

which gives

$$I_2 = \frac{y_{12}E}{1 + y_{22}R} \qquad A_V = \frac{-I_2 R}{E} = \frac{-y_{12}R}{1 + y_{22}R} \tag{5-108}$$

Finally, from (5-24), using (5-103), we have

$$\begin{gathered} V_1 = E = AV_2 - BI_2 = -ARI_2 - BI_2 \\ I_2 = \frac{-E}{AR + B} \qquad A_V = \frac{-I_2 R}{E} = \frac{R}{AR + B} \end{gathered} \tag{5-109}$$

Similar calculations performed for the circuit of Fig. 5-19*b* give

$$Y_T \triangleq \frac{I_2}{E} = \frac{-z_{12}}{\Delta_Z + z_{22}R} = \frac{y_{12}}{1 + y_{11}R} = \frac{-1}{B + DR} \tag{5-110}$$

For the circuit of Fig. 5-19*c*,

$$A_I \triangleq \frac{I_2}{I} = \frac{-z_{12}}{z_{22} + R} = \frac{y_{12}}{\Delta_Y R + y_{11}} = \frac{-1}{CR + D} \tag{5-111}$$

Finally, for the circuit of Fig. 5-19*d*,

$$Z_T \triangleq \frac{V_2}{I} = \frac{z_{12}R}{z_{11} + R} = \frac{-y_{22}R}{\Delta_Y R + y_{22}} = \frac{R}{A + RC} \tag{5-112}$$

The most important and most widely used circuit is the doubly terminated (or doubly loaded) reactance two-port, illustrated in Fig. 5-20. Depending on whether Thevenin or Norton model is used for the generator and whether V_2 or I_2 is used as output variable, any one of the four transfer functions A_V, A_I, Z_T, and Y_T can be used to describe the transmission properties of the circuit. If the circuit of Fig. 5-20*a* is chosen, for example, i.e., a Thevenin generator model, and V_2 as output variable, A_V is the proper transfer function. Now the branch relations are

$$V_1 = E - R_G I_1 \qquad V_2 = -I_2 R_L \tag{5-113}$$

They can be combined with the two-port relations (5-9) to give

$$(z_{11} + R_G)I_1 + z_{12}I_2 = E \qquad z_{12}I_1 + (z_{22} + R_L)I_2 = 0 \tag{5-114}$$

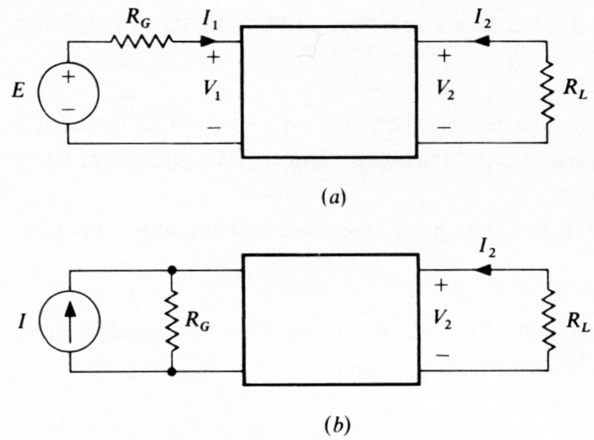


Figure 5-20 Doubly terminated two-ports.

$V_1 = E - R_G I_1 \quad V_2 = -I_2 R_L$

Solving (5-114) for I_2 gives

$$I_2 = \frac{-z_{12}E}{\Delta_Z + z_{11}R_L + z_{22}R_G + R_G R_L} \tag{5-115}$$

Hence
$$A_V = \frac{V_2}{E} = \frac{-I_2 R_L}{E} = \frac{z_{12}R_L}{\Delta_Z + z_{11}R_L + z_{22}R_G + R_G R_L} \tag{5-116}$$

Carrying out the calculations in terms of the y_{ij}, that is, combining and solving (5-113) and (5-10), gives

$$A_V = \frac{-y_{12}R_L}{\Delta_Y R_G R_L + y_{11}R_G + y_{22}R_L + 1} \tag{5-117}$$

Finally, to express A_V in terms of the chain parameters, we combine and solve Eqs. (5-113) and (5-24). This results in

2023 (B)

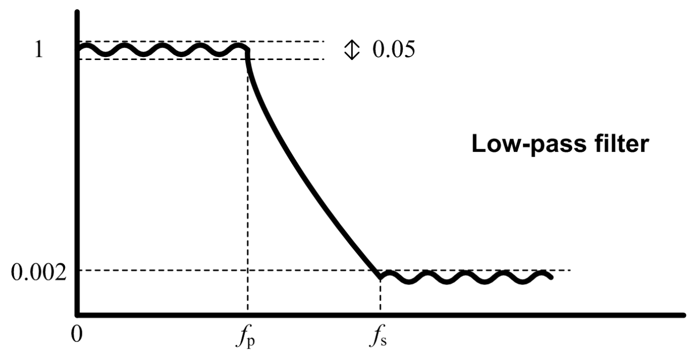
$$\boxed{A_V = \frac{R_L}{AR_L + B + CR_G R_L + DR_G}} \tag{5-118}$$

As will be shown in Chap. 6, the design of a doubly terminated two-port is expediently performed using a different transfer function $H(s)$, which is related to $A_V(s)$ by

$$H(s) \triangleq \frac{1}{2}\sqrt{\frac{R_L}{R_G}}\frac{E}{V_2} = \frac{\sqrt{R_L/R_G}}{2A_V} \tag{5-119}$$

Design of passive two-ports: filter, equalizer (gain or phase).

Delay line



Specs → Transfer functions → Z_{oc}, Y_{sc}, T, … → circuit

Example:

Calculate $Z_T = V_2 / I$ for the circuit shown using its z_{ij} parameters.

Solution:

Since $Z_T = z_{12}R / (z_{11} + R)$, we first find

$$z_{11} = \left(2s^2+1\right)/s\left(s^2+2\right) \approx 1/2s, s \to 0$$
$$z_{12} = \left(V_2/I_1\right) \text{ for } I_2 = 0 \to z_{12} = 1/(s^3 + 2s),\ s \to \infty$$

Substituting gives

$$Z_T = 1/\left(s^3 + 2s^2 + 2s + 1\right)$$

Checks: for s=0, $Z_T(s) = 1$; true from circuit diagram. For $s \to \infty$, $Z_T(s) \to s^{-3}$, also obvious from circuit.

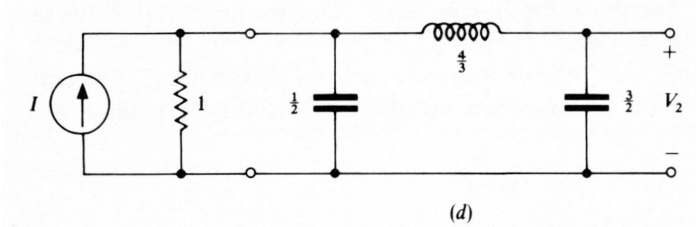


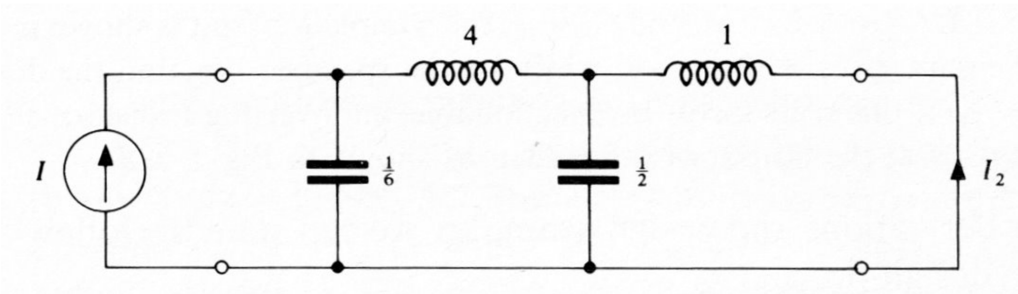
Figure 5-25 Reactance-ladder realization example.

<u>Example:</u>

Find the z_{ij} and AI for the circuit shown. The results are:

$$z_{12} = \frac{3}{s(s^2+2)} \qquad z_{22} = \frac{(s^2+1)(s^2+3)}{s(s^2+2)}$$

$$A_I \equiv \frac{I_2}{I} = -\frac{z_{12}}{z_{22}} = \frac{-3}{(s^2+1)(s^2+3)}$$



(c)

Again, testing for z=0 shows that $z_{22} = -z_{12} = 1/(2s/3)$ and AI=-1 are correct, as are $z_{22} \to s$, $z_{12} \to \frac{3}{s^3}$ and AI $\to \frac{-3}{s^4}$