Reciprocity and Inter-Reciprocity

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KCL

- The KCL (Kirchhoff's Current Law)
  - States that the sum of the currents leaving each node is zero
  - Define branch currents as ***I* =** $[i_1, i_2, \ldots, i_b]^T$

$$\boldsymbol{Ai = 0}$$

Where b is the number of branches in the circuit, ***0*** is the column vector of zeros

Define the incidence matrix A

$$a_{ij} = \begin{cases} +1 & \text{if branch } j \text{ leaves node } i \\ -1 & \text{if branch } j \text{ enters node } i \\ 0 & \text{if branch } j \text{ is not incident at node } i \end{cases}$$

KVL

- The KVL (Kirchhoff's Voltage Law):
  - States that the voltage across a branch is the difference of the node voltages at the terminals of the branch.
  - *Denote* $\boldsymbol{v}$ and $\boldsymbol{e}$ as the branch and node voltages, respectively. KVL gives

$$\boldsymbol{v} = \boldsymbol{A}^T . \boldsymbol{e}$$

Tellegen's Theorem

- Consider the total power developed in all branches of the circuit shown below

$$\sum_{k=1}^{\mathrm{N}} v_k i_k = \boldsymbol{v}^{\boldsymbol{T}}.\boldsymbol{i} = \left(\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{e}\right)^{\boldsymbol{T}}\boldsymbol{i} = \boldsymbol{e}^{\boldsymbol{T}}(\boldsymbol{A}\boldsymbol{i}) = 0$$

- Conservation of the power
  - In any network, the summation of power is zero.
- Applications
  - Impedance synthesis, sensitivity analysis and noise computation

Using Tellegen's Theorem in N-ports

- Consider two *N*-ports ***N*** and ***N'*** with the same ***A*** matrices. The previous equations give

$$\sum v_k i'_k + \boldsymbol{v}^{\boldsymbol{T}} \boldsymbol{i}' = \boldsymbol{Pp} + \boldsymbol{Pb} = 0$$

$$\sum v'_k i_k + \boldsymbol{v}'^{\boldsymbol{T}} \boldsymbol{i}' = \boldsymbol{Pp}' + \boldsymbol{Pb}' = 0$$

Here, ***Pp*** refers to the quantities *in the branches at the port* and ***Pb*** *r*efers to those *inside the N-port* in ***N.*** The primed quantities ***Pp'*** and ***Pb'*** *r*efer to the equivalent ones in ***N'.***

Using Tellegen's Theorem in N-ports

- If the inside structures of ***N*** and ***N'*** are such that ***Pb = Pb'*,** then ***Pp = Pp'*** also holds**.** The port voltages and currents satisfy this condition, or

$$\sum i_k v_k' = \sum i_k' v_k$$

where i_k and v_k are the port currents and voltages under one set of excitations, while i_k' and v_k' are those under a different one, f*or the same N-port.* Then, relations can be found between the transfer functions of the *N*-port!

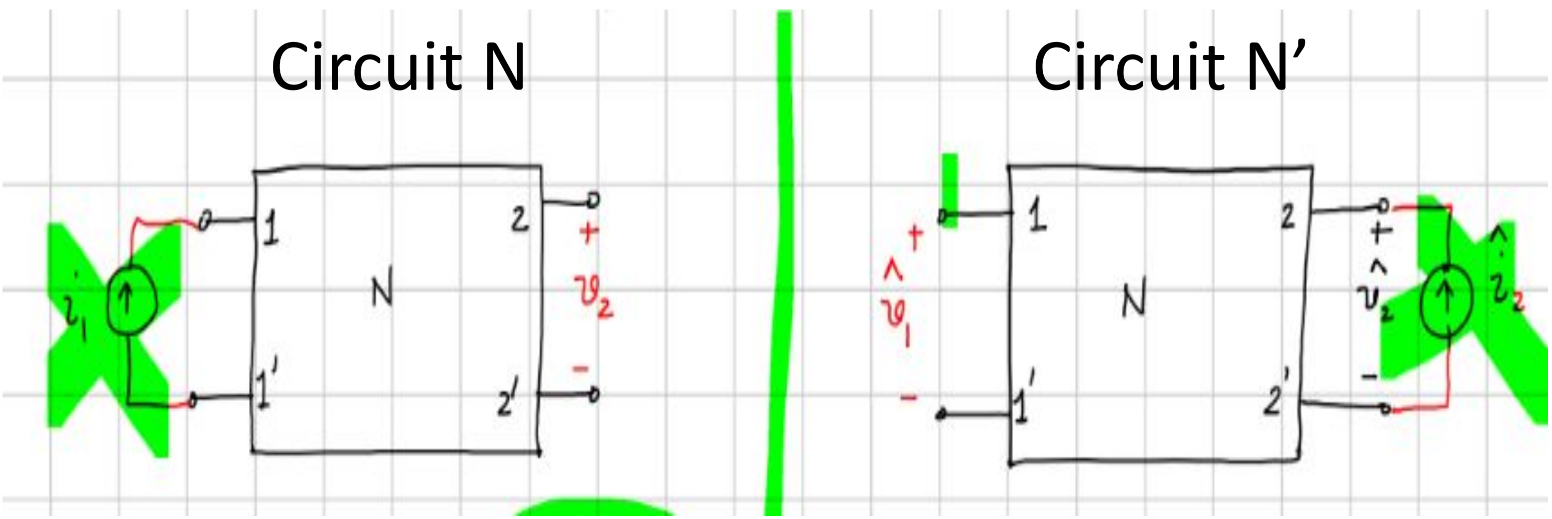
Such an N-port circuit is called *reciprocal.*

Two-Port Example-1

- Assume the following two circuits
  - contain only linear resistors, no controlled sources
  - have same internal branches, but different terminations
- Tellegen's Theorem gives

$$-v_2.i_2' + \sum v_k.i_k' = -v_2.i_2' + \sum i_k.R_k.i_k' = 0$$
$$v_1'.i_1 + \sum i_k.R_k.i_k' = 0$$

- *Subtracting the two gives:* $\boldsymbol{v_2.i_2' = v_1'.i_1}$ $\Rightarrow$ ***Reciprocal***
- *Example:*

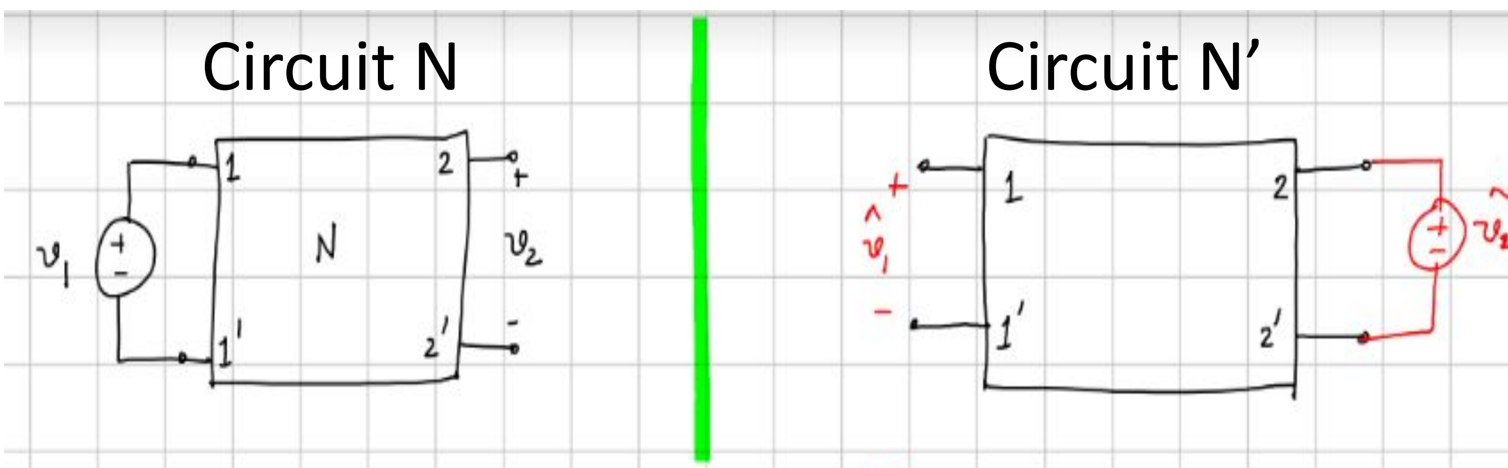


Two-Port Example-2

- *Thus, a two-port containing only resistors is reciprocal.*
- Next, the port relations of ***N*** and ***N'*** are found for another two-port

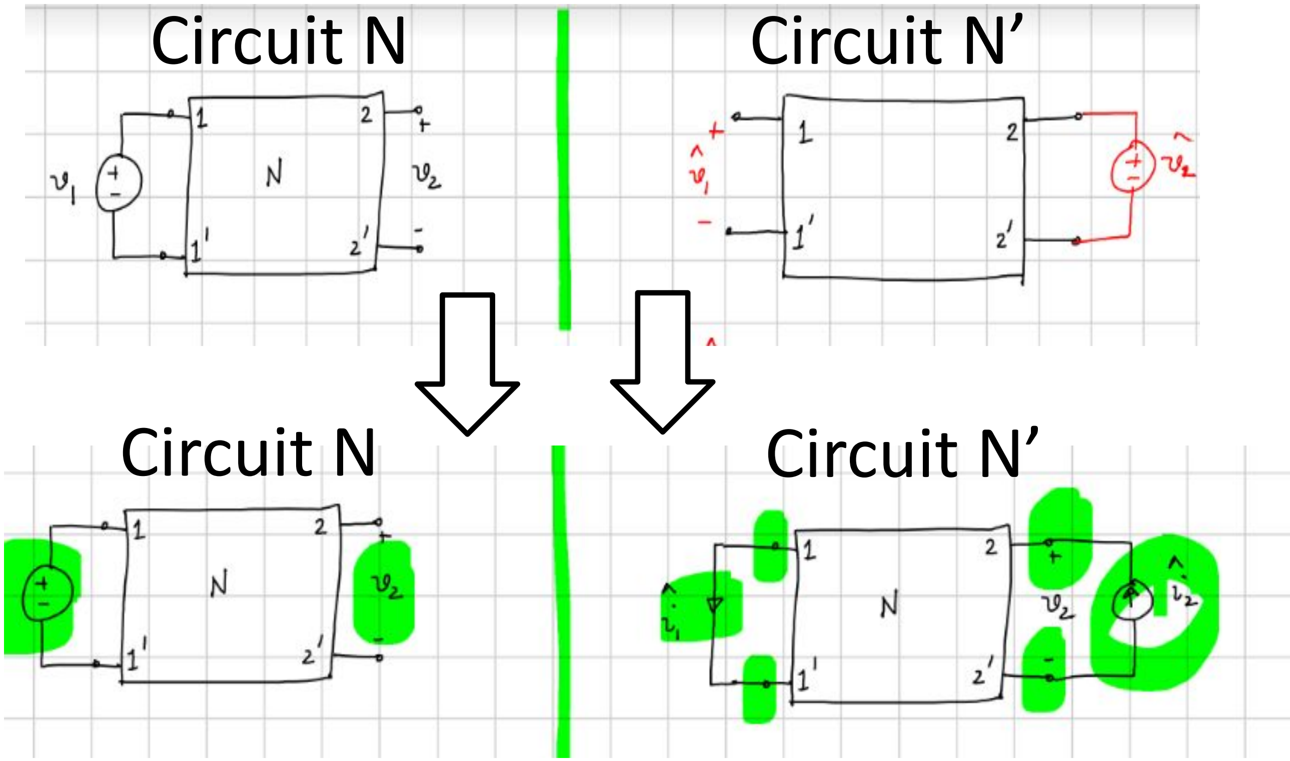
$$\begin{cases} \sum i_k v_k' = \sum i_k' v_k \\ i_2 = i_1' = 0 \end{cases} \Rightarrow \begin{cases} i_1 v_1' = i_2' v_2 \\ v_2 / i_1 = v_1' / i_2' \end{cases}$$

Note that $\boldsymbol{v_2 / v_1 \neq v_2' / v_1'}$***!***

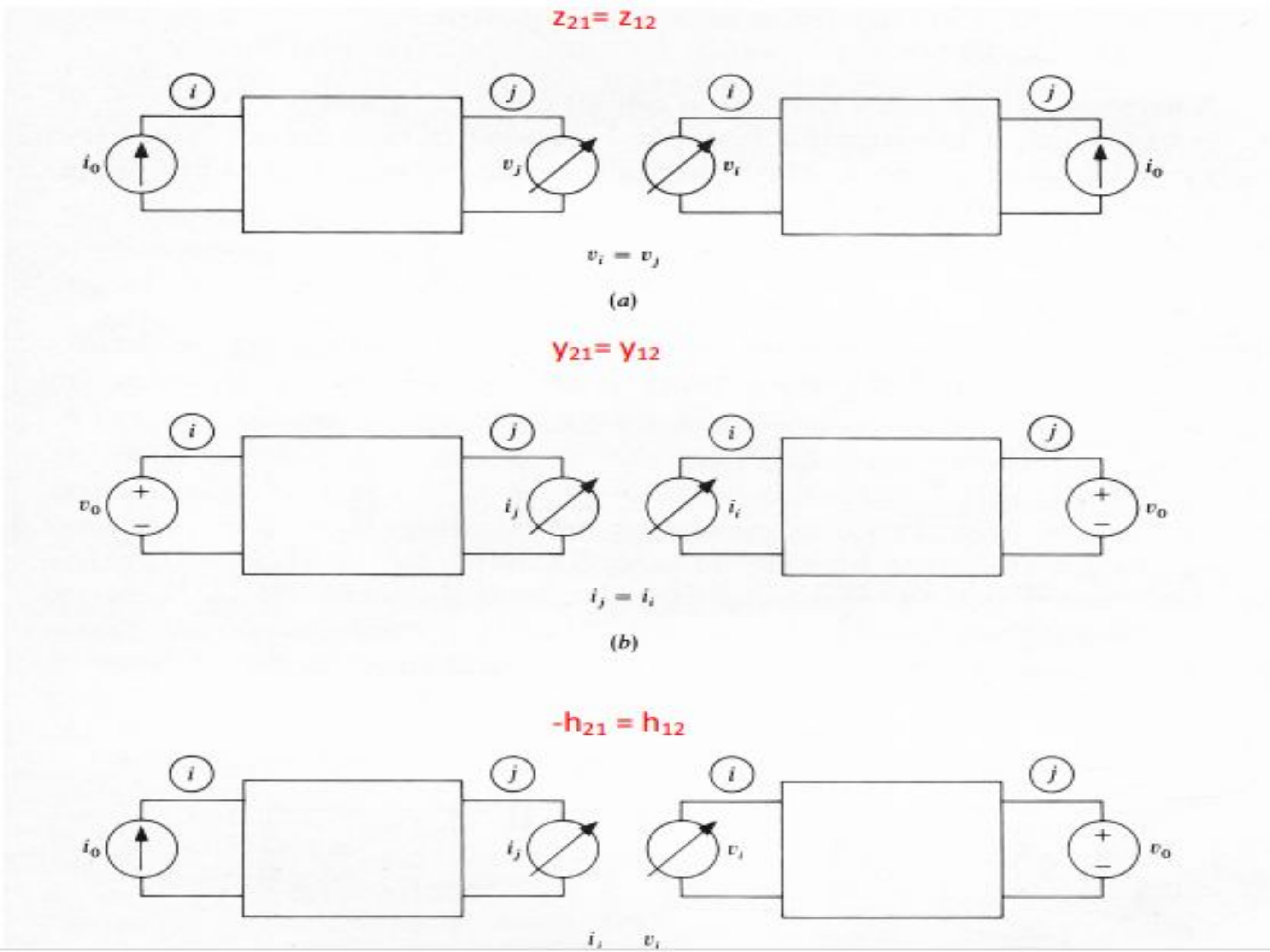


Two-Port Example-2

- The forward transfer of **N** equals the reverse transfer function of **N’** <u>only if the source impedances at the ports are the same</u>
- Thus, a voltage source in **N** should be replaced by a short circuit in **N**’, and a current source replaced by an open circuit:



Equal-impedance Termination



- Gains are equal. Valid also if two-ports are turned around.

Reciprocity In Dynamic N-Ports

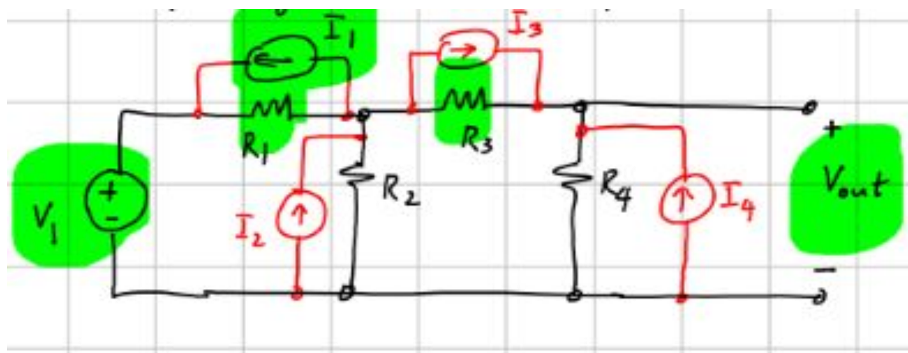
- Steady-state sine wave analysis represents voltages and currents by complex phasors $V(j\omega)$, $I(j\omega)$ in a linear circuit, and represents passive elements by their impedances $Z_k(j\omega)$.

- KCL, KVL and all previous derivations hold for the phasors.

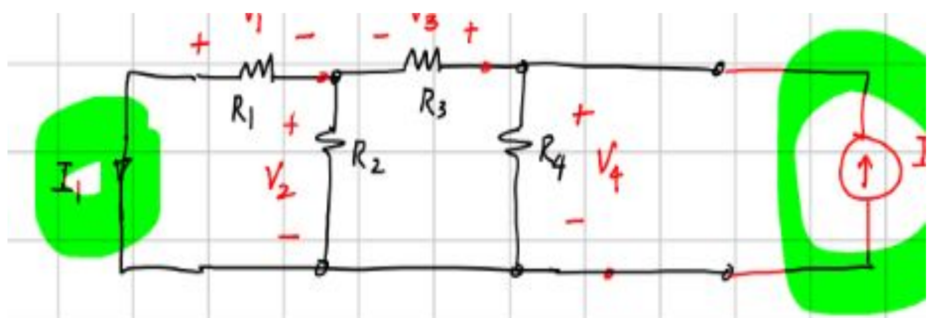
- Useful in the analysis of passive dynamic networks with multiple excitations, e.g., DACs (Digital-to-Analog Converters) design

Examples Using Reciprocity

- Below shows a circuit containing five independent sources
    - Analyze using superposition
        - Requires five analysis, each with one nonzero source



- Analyze using reciprocity
    - Requires only one circuit analysis



Analyzing Circuits Using Reciprocity

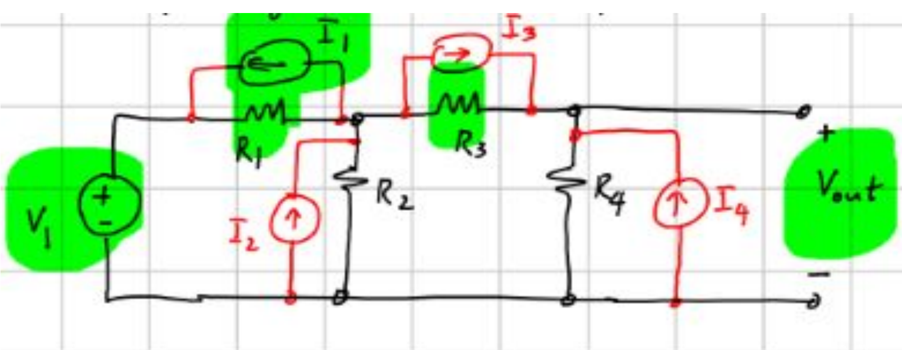
- The contribution of the voltage source V_1 to the output v_{out}

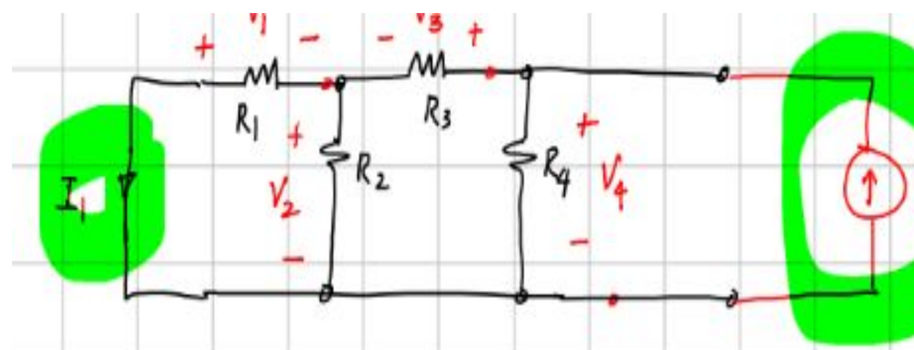
$$v_{out}/V_1 = i_1/I$$

- and those of the current sources I_k are

$$v_{out}/I_k = v_k/I\,, k = 1,2,\ldots,5$$

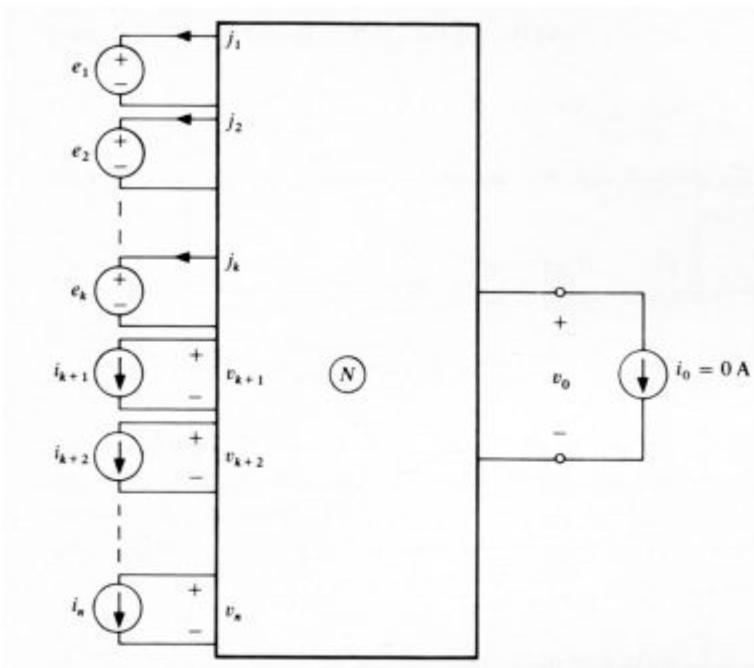
- The overall output voltage is the sum of these terms.





Using Reciprocity in Analysis

- Using $\sum i_k v_k' = \sum i_k' v_k$, you find that the output voltage *vo* is the weighted sum of the input source values. The weight of *ei is –ji’* , and that *ik* is *vk.* Also, *Zo* is the same, so the Thevenin equivalent is readily found. No MNA needed!



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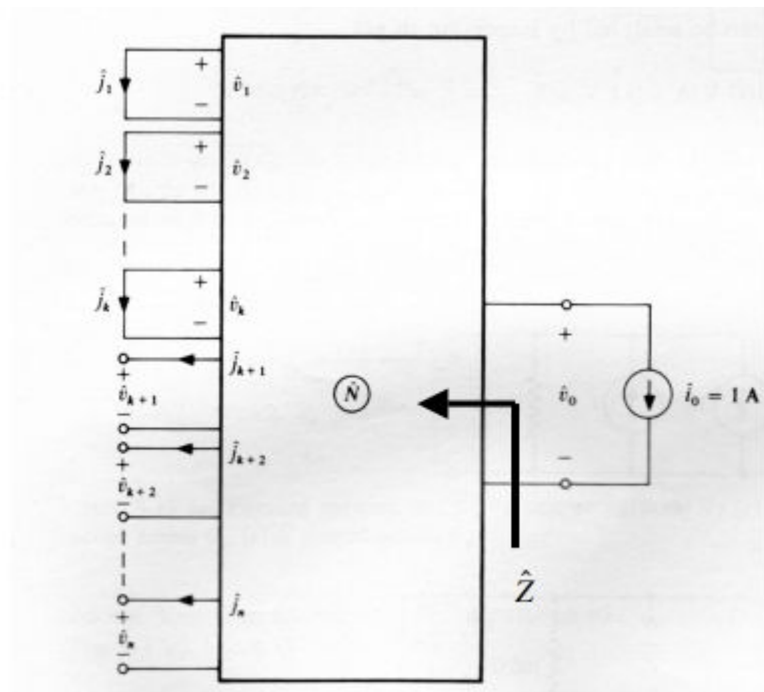


(b)

Generalized Method

- **To analyze a reciprocal circuit containing several independent sources:**

1) Redraw the circuit, setting the values of all independent sources to zero. Thus, voltage sources become short circuits, current sources open circuits in ***N'.***

2) Replace the output signal with an independent source. Replace voltage with a current source *I* ; replace current with a voltage source *V* (e.g., *I* = -1A, or *V*= 1V)

3) Analyze the transformed network. The desired result will be the weighted sum of the independent source values in the physical circuit, with the weight factor of each current source the voltage across its zero valued replica, and that of each voltage source the current flowing across its short-circuit replica.

Resistive N-Ports

- Assume an N-port circuit containing only resistors. To achieve ***Pb = Pb'*** for reciprocity, ***N'*** must satisfy

$$\boldsymbol{v_B^T \cdot G' \cdot v_B' = (v_B')^T \cdot G \cdot v_B}$$

where ***G*** is the branch admittance matrix connecting the internal branch voltages and currents in ***N*** by $\boldsymbol{i_B = G \cdot v_B}$, and similarly $\boldsymbol{i_B{}' = G' \cdot v_B{}'}$.

- Since the quantities on both sides are scalars, it is permissible to take the transpose of the right side. This gives

$$\boldsymbol{v_B^T \cdot G' \cdot v_B' = v_B^T \cdot G^T \cdot v_B' \;\; \Rightarrow G' = G^T}$$

- *Hence, to achieve* ***Pb = Pb'*** *,the branch admittance matrix* $\boldsymbol{G'}$ *of* $\boldsymbol{N'}$ *must be the transpose of* ***G****. If the circuit contains only resistors,* ***G*** *is symmetric, so* $\boldsymbol{G^T}$ *=* ***G****, and reciprocity is assured.*

Inter-Reciprocity and its Applications

- If the circuit contains controlled sources, ***N'*** cannot be the same as ***N.*** The reciprocity conditions fail for using the same N-port as ***N',*** since the ***G*** matrix will not be symmetric. ***N'*** must be chosen so as to restore $\boldsymbol{G}' = \boldsymbol{G}^{\boldsymbol{T}}$.

- *An adjoint network* is a modified version ***N'*** of the physical network ***N***, which is constructed such that the reciprocity condition on the port currents and voltages

$$i_1 v_1' + i_2 v_2' = i_1' v_1 + i_2' v_2$$

is restored for the two networks. By Tellegen's theorem, this requires the condition $\sum i_k v_k' = \sum i_k' v_k$ to be valid separately for both port and interior branches. The relation between ***N*** and ***N'*** is called *inter-reciprocity.*

Inter-Reciprocity in R-VCCS N-Port

- To satisfy ***Pb = Pb'*** , we must have

$$i_a v_a' + i_b v_b' = i_a' v_a + i_b' v_b$$

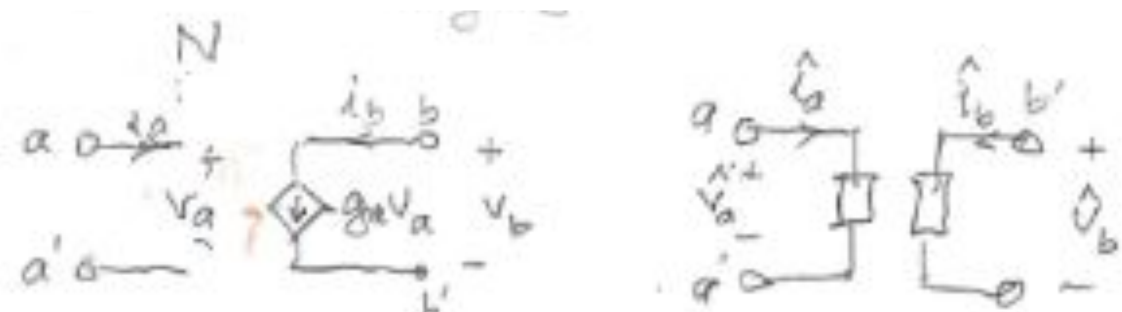
which can be used to find the two branches in N' corresponding to the VCVS

- For the VCCS, the branch relations are

$$i_a = 0 \;\; and \;\; i_b = g_m v_a$$

- The above two equations give the reciprocity condition which is

$$i_a' - g_m v_b' = -i_b'(v_b/v_a)$$



Inter-Reciprocity

- The equation derived must hold true for all values of v_b/v_a

$$i'_a - g_m v'_b = -i'_b(v_b/v_a)$$

- Only possible when

$$i'_b = 0 \;\; and \;\; i'_a = g_m v'_b$$

- The image in **N’** of the VCCS of circuit **N** is thus another VCCS but turned around!
- If there are several VCCS’s in **N,** then the adjoint network **N’** must contain such appropriately modified replicas (i.e., turned around VCCS stages) for all of them.

Modified Generalized Method

- **To analyze multi-source active networks using inter-reciprocity:**

1) Draw the adjoint circuit **N'**, setting the values of all previous independent sources to zero. Thus, voltage sources become short circuits, current sources open circuits.

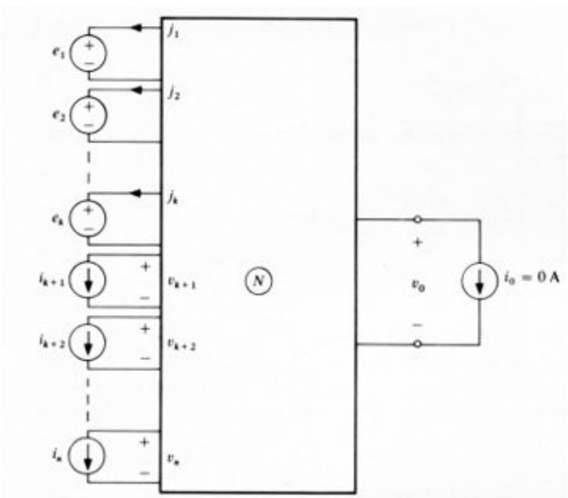
2) Replace the original output signal with an independent source. If it is voltage, replace it with a current source *I* ; if it is a voltage source, with a voltage source *V* (e.g., I = -1A, or V= 1V).

3) Analyze the *adjoint network* **N'**. The desired result for the output in **N** will be the weighted sum of the independent source values in **N** with the weight of each current source the voltage across its zero valued replica, and that of each voltage source the current flowing across its short-circuit replica.

Applications of Inter-Reciprocity

- Inter-reciprocity can also be used to efficiently obtain the Thevenin equivalent of a linear circuit with multiple independent and dependent sources.

- Obtain the output impedance Z of the network N by setting all independent sources to zero, and find the output voltage when the output port is excited by a -1A current source.



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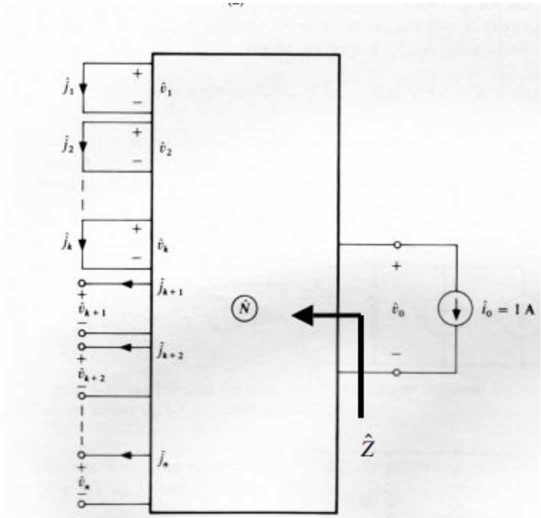


(b)

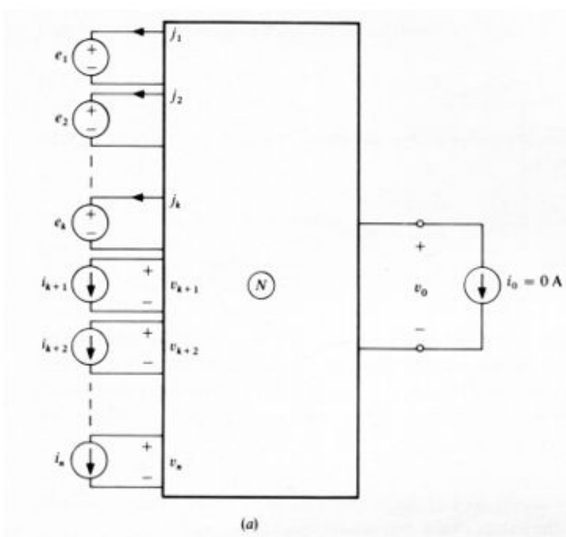
Applications of Inter-Reciprocity

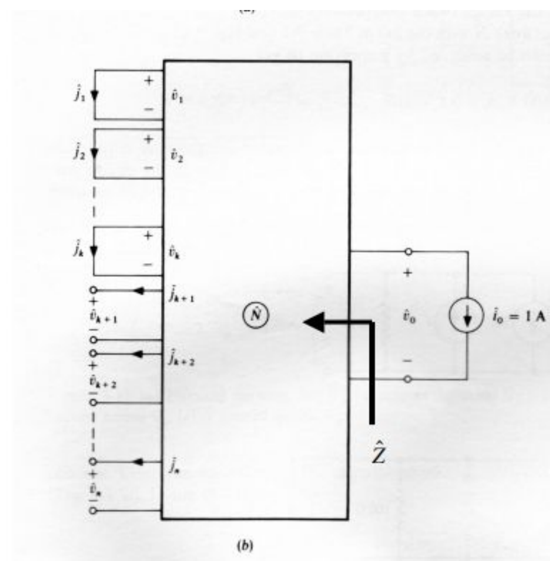
- By inter-reciprocity, we have

$$v_o' i_o = i_o' v_o$$

- Hence, the impedance in the Thevenin model numerically equals the output voltage of N'

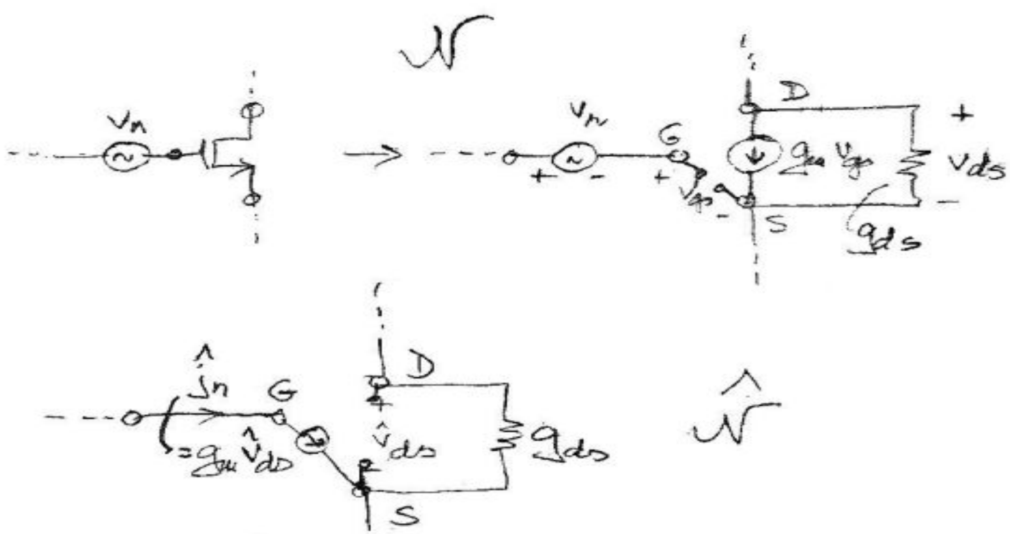
$$Z = Z' = -v_o'$$





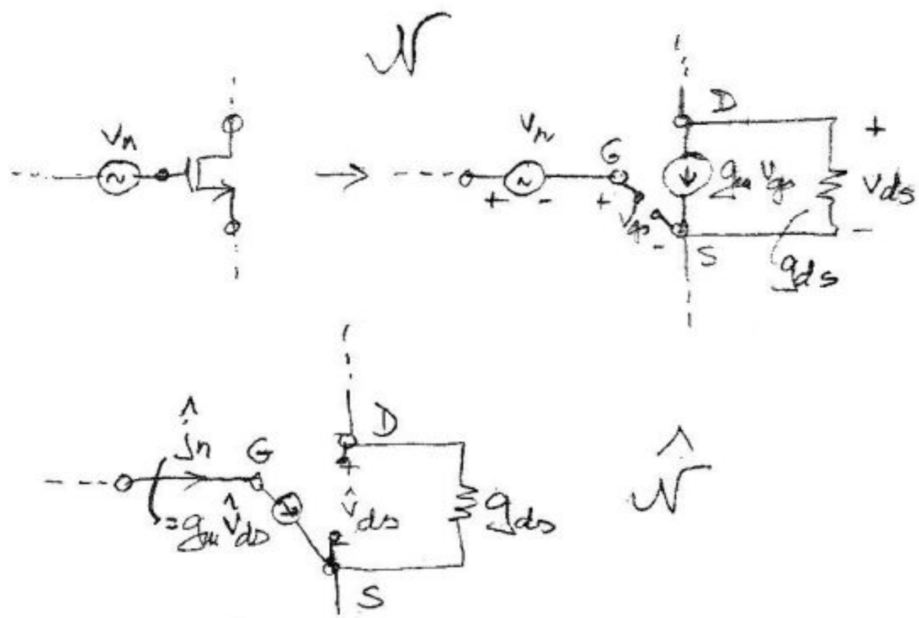
Noise Analysis Using Inter-Reciprocity

- Every transistor in CMOS IC is affected by thermal noise (random motion of the channel charge carriers) and by flicker noise or 1/f noise (trapping and releasing of charge carriers).
- This noise effect can be modeled by a single independent noise voltage source at the gate of the device.
- Inter-reciprocity can be applied to find the output noise.



Noise Analysis Using Inter-Reciprocity

- Model each transistor with a transconductance g_m and drain-to-source conductance g_{ds}.
- In the adjoint network, the transconductance is turned around.
- Using inter-reciprocity, the contribution of the noise voltage v_n to the output noise power will be $(j_n')^2 = (g_m v_{ds}')^2 v_n^2$.



Sensitivity Analysis Using Inter-Reciprocity

- The values of the circuit components will deviate from their theoretical ones when implementing circuits.
- The adjoint network is an efficient way to calculate the sensitivities of the key performance parameters, as suggested by Director and Rohrer [4][5].
- Assume a two-port containing resistors and VCCSs with all component values having a small error, changing $\boldsymbol{G}$ to $\boldsymbol{G} + \boldsymbol{\Delta G}$.
- Using the adjoint network $\boldsymbol{N'}$ with its transconductance matrix $\boldsymbol{G'} = \boldsymbol{G^T}$, we have for the internal branches the change $\boldsymbol{\Delta Pb} = \boldsymbol{v_B^T} \cdot \boldsymbol{\Delta G} \cdot \boldsymbol{v'_B}$. Based on this, the sensitivities of the output to conductances and transconductances can be found.

Process of Sensitivity Analysis

- Assume a voltage output v_{out} for the physical circuit N containing resistors and VCCSs. The process is as follows;

1) Construct the adjoint network ***N'***, in which the values of all independent sources of ***N*** are set to zero. Resistive branches remain unchanged, controlled sources replaced by their adjoint models. At the output port, place a current source I = 1 A.

2) Calculate the branch voltages v_k' across all resistors. According to the discussions above, the contribution of the incremental change ΔG_k of an admittance G_k to Δv_{out} will be $v_k'.v_k.\Delta G_k$. Thus, the output sensitivity to variations in G_k will be $\Delta v_{out}/\Delta G_k = v_k'.v_k$.

3) To find the sensitivity to changes in the transconductance G_{lm} of a VCCS between branches *m* and *l*, find the *controlling voltages* v_m and v_l' in N and N'. The sensitivity is $\Delta v_{out}/\Delta G_{lm} = v_l'.v_m$.

References

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[3] Bordewijk, J. L., "Inter-reciprocity applied to electric networks," Appl. Sci. Res., Sec **B**, pp. 1-74, 1956.

[4] Director, S. W. and Rohrer, R.A., "The generalized adjoint network and network sensitivities," IEEE Trans. Circuit Theory, Aug. 1969, pp. 318-323.

[5]] Director, S. W. and Rohrer, R.A., "Automated network design: the frequency domain case," IEEE Trans. Circuit Theory, Aug. 1969, pp. 330-337.