Chapter 1: BASIC CIRCUIT ANALYSIS

1.1Classification of Circuits

An electric circuit is *linear* if its response is proportional to the excitation. For such a circuit, superposition is applicable, i.e., the overall response is the sum of the responses due to the individual sources. Otherwise, the circuit is *nonlinear*. In practice, electronic circuits may behave approximately linearly, but only within a range of signals.

A circuit is *time-invariant* if its response to a given excitation is the same regardless of the time instant when the source is connected. It is *time-variant* otherwise.

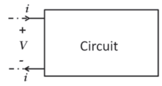


Figure 1.1: Associated directions.

A circuit is *passive* if it cannot provide more energy to the outside world than what was stored in it. Assume that it is connected to the outside world through two terminals **(Fig. 1.1).** Note that the reference directions of the voltage and currents are such that the reference current flows from the positive to the negative terminal in the circuit. These are *associated directions*, and will be assumed throughout this work. The power flowing into the network from the outside at time *t* is then *P(t) = v(t).i(t)*, and the energy delivered into it during the time period 0 to *t* is

$$E(t) = \int_0^t P(x)dx = \int_0^t v(x)i(x)dx \qquad (1.1)$$

For a passive circuit, *E(t)* is nonnegative. For *E(t)* < 0, the circuit is *active*. Strictly speaking, a device may only supply net electric energy to the outside if it converts chemical or solar energy into electric form, like a battery or solar cell. However, the adjective "active" is also often applied to circuits which are capable of converting the power from a fixed bias voltage or current into magnified signal, like a voltage amplifier. For a circuit with several terminal pairs the total energy delivered at all terminal pairs determines the active or passive property.

Terminal pairs where the current entering at one terminal is the same as the current leaving at the other are called *ports.* Thus, the structure in Fig. 1.1 is a *one-port* circuit.

Circuits may be *lossless* or *lossy*. The energy stored in a lossless circuit equals the energy supplied to it from the outside. For a lossy circuit, the stored energy is less than the energy input from outside. Thus, in a lossy network the electromagnetic energy is transformed into some other form of energy, such as heat or light.

A circuit is *memoryless* if it's response at a given time depends only on its excitation at the same time. This will happen if there are no energy storing elements in the circuit. Otherwise, the circuit is called *memoried*, or *dynamic*.

*Problem:***Fig.1.2**shows a simple circuit which illustrates some of the concepts discussed. A battery with a voltage *E* is switched at *t* = 0 to the series-connected *R* and *C*. (*R* may be the on-resistance of the switch.) The capacitor initially is uncharged. Find the energy stored in the capacitor after the charging is completed, and the energy is delivered by the battery.

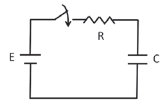


Figure1.2: Circuit example

Solution: the final energy stored in *C* is $C\frac{E^2}{2}$. The current flow during charging is $i = C\frac{dv}{dt}$, so the total energy E_B' supplied by the battery at the end of the charging process is

$$E_B'(t) = \int_0^\infty E \times C(dv/dt)\,dt = CE^2 \qquad (1.2)$$

This is twice the energy stored in *C.* Thus, half of the energy provided by the battery is lost in the process. It is dissipated in the switch resistance *R*. This energy is given by

$$E_R'(t) = \int_0^\infty i^2(t) \times R \times dt = \frac{CE^2}{2} \qquad (1.3)$$

as expected. It is interesting to note that the energy efficiency of the charging process is 50%, independent of the resistance of the switch. However, the speed of the charging process is increased when *R* is lowered.

1.2 Circuit Components

The circuit components discussed next include passive elements such as *R, L, C*, as well as active elements such as controlled sources and operational amplifiers. It will be assumed that all elements are ideal, i.e., *linear and time-invariant (LTI).* Then, their defining voltage-current relations are the following:

$$Resistor\ R{:}\ v = R.i\ \ and\ \ i = G.v, where\ G = \frac{1}{R} \qquad (1.4)$$

Eq. 1.4 is *Ohm's law*. Here, *R* is the resistance, and *G* the conductance of the resistor. Both have positive values. The power dissipated in the resistor is given by $P = v.i = R.i^2$. *P* is clearly nonnegative, and therefore so is the energy *E* which is its time integral. Hence, the resistor is a passive element.

$$Capacitor\ C{:}\ i = C\frac{dv}{dt}\ and\ v(t) = \frac{1}{C}\int_0^t i(x)dx + v(0) \qquad (1.5)$$

Here, $C > 0$ is the capacitance of the element. Finding the energy entering a capacitor from the time integral of the power $v \times i$shows that its value at time *t is* $E(t) = C\frac{v(t)^2}{2} > 0$. Thus, the capacitor is also a passive element.

$$Inductor\ L{:}\ v = L\frac{di}{dt}\ \ and\ \ i(t) = \frac{1}{L}\int_0^t v(x)dx + i(0) \qquad (1.6)$$

Here, $L > 0$is the inductance of the element. Integrating the power $v.i$gives $E(t) = L\frac{i(t)^2}{2} > 0$ for the energy which entered the inductor by time *t.* Thus, the inductor is also a passive element.

The energy entering the resistor leaves the circuit in the form of radiation of heat or an electromagnetic wave, and is lost for the circuit. However, the energy entering the capacitor is stored in the electric field between the plates; the energy of the inductor is stored in the magnetic field within and around the device. The stored energies can be redeemed by the circuit.

The voltages and currents in eqs. (1.4) – (1.6) are physical quantities. They are generally functions of time. As the equations indicate, the relations describing the operation of the circuit in which they are present are integro-differential ones. They can be reduced to algebraic equations, and thus their solution greatly simplified, by using Laplace transformation. Denoting

the transformed variables by capital letters, this changes the voltage-current relation of the capacitor into $I = sCV\ and\ V = 1/sC$. Similarly, for an inductor the relations become$V = sLI. and\ \ I = V/sL$. For the resistor, Ohm's law remains unchanged between *V* and *I.*

The ratio $Z = V/I$between the Laplace transformed quantities is called an *impedance,* its reciprocal $Y = I/V$ is called an *admittance.*

The Laplace-transformed variables are closely related to the phasors used in the steady-state sinusoid analysis. The equations used for Laplace transformed variables in terms of the variable *s* remain valid for phasors if *s* is replaced by *jω.*

The active components often encountered include the *independent and dependent (controlled) sources*. The voltage across an independent voltage source is defined by the designer, and it is independent of the current drawn from it. Similarly, the current provided by an independent current source is a given quantity, and is independent of the voltage across it.

A dependent voltage source provides a voltage v_j in branch *j* which is a function of another voltage v_i or currenti_i in the circuit. Usually, the function is linear, so the voltage is given by $v_j = \mu.v_i$ or by $v_j = r.i_j$. Similarly, a dependent current source has a value determined by the voltage or current in another branch. Usually, it is a linear dependence, so that the current is given by $i_j = g \times v_i$ or $i_j = \alpha \times i_i$. Fig. 1.3illustrates the symbols used for independent and dependent sources. The dependent sources are often called *voltage-controlled voltage source (VCVS), current-controlled voltage source (CCVS), voltage-controlled current source (VCCS),* or *current-controlled current source (CCCS),* depending on their defining relations.

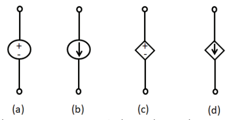


Figure1.3: Voltage and current sources: a. independent voltage source; b. independent current source; c. dependent voltage source; d. dependent current source.

Controlled sources are useful in high-level modelling of integrated devices. As an example,**Fig.1.4** illustrates a simple model for an N-channel MOS transistor which processes incrementally small signals at low frequencies. Its drain current is given by $i_d = g_m \times v_{gs} + v_{ds}/r_{ds}$.

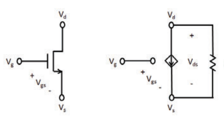


Figure1.4: NMOS transistor and its low-frequency model.

An important concept in analog integrated circuit design is that of the *operational amplifier* (**Fig. 1.5**). It is a voltage-controlled voltage source with an output voltage $v_{out} = A(v_+ - v_-)$.The ideal operational amplifier (opamp) has the following properties*: its voltage gain A is infinite and is independent of frequency; the input impedance is infinite, and the output impedance is zero.* To keep v_{out} finite, the opamp must operate in a negative feedback loop which keeps $v_+ - v_-$close to zero. A typical circuit which functions as a voltage amplifier with a gain of -2 is shown in **Fig. 1.6.**

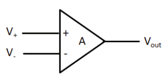


Figure1.5: Operational amplifier.

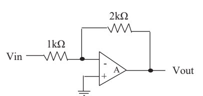


Figure1.6: Voltage amplifier with a gain of -2.

1.3 Kirchhoff's Laws and Nodal Analysis

The analysis of circuits is based *on Kirchhoff's current law (KCL) and voltage law (KVL). The KCL states that the sum of all currents leaving any node at any time is zero*. This is because charges ideally cannot accumulate at node. (In a real circuit, parasitic capacitances are present at all nodes. Their currents must be included to make this statement valid.) *The KVL states*

thatthesumof all branch voltages in a loop is zero. The Kirchhoff's laws are illustrated for a simple circuit in **Fig. 1.7.**

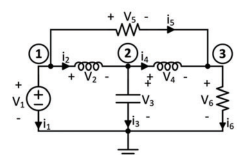


Figure1.7: Circuit analysis example.

Denoting the current in branch *k* by i_k, the KCL leads to the relations

$$\begin{gathered} i_1 + i_2 + i_5 = 0 \\ -i_2 + i_3 + i_4 = 0 \\ -i_4 - i_5 + i_6 = 0 \end{gathered} \qquad (1.7)$$

Denoting the branch voltages by $v_k$andthenode voltages by e_i, the KVL gives the equations

$$\begin{gathered} v_1 = e_1 \\ v_2 = e_1 - e_2 \\ v_3 = e_2 \\ v_4 = e_2 - e_3 \\ v_5 = e_1 - e_3 \\ v_6 = e_3 \end{gathered} \qquad (1.8)$$

It is efficient at this point to introduce vector and matrix notations. We define the *incidence matrix* ***A.*** Its element in row *I* and column *j* is

$$a_{ij} = \begin{cases} +1\ if\ branch\ j\ leaves\ node\ i \\ -1 if\ branch\ j\ enters\ node\ i \\ 0 if\ branch\ j\ is\ not\ incident\ at\ node\ i \end{cases} \qquad (1.9)$$

The branch currents can be collected into a column vector $\mathbf{I} = [i_1, i_2, \cdots, i_b]^T$. Here, *b* is the number of branches in the circuit.Then, the KCLs can be written in the simple form

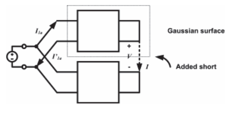
$$\boldsymbol{A}.\boldsymbol{i} = \boldsymbol{0} \tag{1.10}$$

Here, **0** is the column vector of zeros.

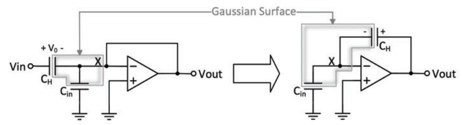
Similarly, collecting the branch and node voltages into column vectors, the KVL can be written in the concise form

$$\boldsymbol{v} = \boldsymbol{A}^T.\boldsymbol{e} \tag{1.11}$$

Kirchhoff's current law can be generalized. Let a part of the circuit enclosed in a closed surface, sometimes called a *Gaussian surface* (Fig.1.8). Within the surface lie *N* nodes. Since the KCL holds for all these, the sum of all currents leaving them is $\sum i$ = 0. Some of these currents flow between nodes *inside* the surface. These cancel in the summation, since they leave one node but enter another. Other branches connect to nodes which are *outside* the surface. The sum of these currents must therefore also be zero. Cutting these branches open will cause the circuit to be broken into parts. They are related to the *cutsets* of graph theory. Fig. 1.8(a) shows two parallel-connected two-ports. Using the Gaussian surface, it can be proven that the input currents I_{1a} andI_{1a}'are equalbefore the short circuit (broken line) is added between their outputs. As will be shown later, this shows that both stages function as two-ports In Fig. 1.8(b), a surface enclosing C_H andC_{in}shows that the the charge in the enclosed space remains unchanged when the circuit configuration is changed by switches (not shown). This allows a fast analysis of this switched-capacitor circuit.



(a)



(b)

Figure 1.8: Circuits with Gaussian surfaces.

In conclusion, *the sum of currents leaving a circuit within a closed surface is zero.* This is a useful generalization of the KCL. It can be applied in the analysis of both active and passive circuits. Note that the ground and bias nodes are not allowed to be within the surface, because their currents would violate the KCLs.

Kirchhoff's laws are based solely on the configuration (topology) of the circuit. Adding the relationships between branch voltages and branch currents, the circuit can be fully analyzed. Operating in the Laplace transform domain, let Y_{ii}denote the sum of all admittances connected to node *i*, and let Y_{ij}denote the sum of all admittances connected between nodes *i*and*j*. We can construct the *nodal admittance matrix* ***Y*** from these admittance parameters such that $Y_{ij}$is the element in row *I* andcolumn*j*. Then the relation

$$\boldsymbol{Ye} = \boldsymbol{I} \tag{1.12}$$

describes all KCLs for every node. Here, ***I*** is the column vector of the source currents entering the nodes.

Solving (1.12) for the node voltagese_i, and finding the branchvoltages from the KVL, the network can be fully analyzed. This *nodal analysis* is the basis of most computer-based circuit analysis programs.

The process is simplified if there are grounded voltage sources in the circuit, since the voltages at their floating nodes are then known. Ungrounded voltage sources, however, require a somewhat more involved process, called *modified nodal analysis (MNA)*. In MNA, in addition to the node voltages in ***e,*** the unknowns include the currents flowing through the floating voltage sources. Consider a circuit with *n* nodes and *s* floating voltage sources. Then we can write *n* KCLs

for the nodes, and *s* KVLs for the voltage sources. Thus, for a voltage source v_{s1} connected between nodes *a* and *b*, a KVL of the form $v_a - v_b = v_{s1}$ is added to the set of equation, and the current $i_{ab}$to the unknowns. The $n + s$ equations can then be solved for the *n* node voltages and the *s* currents through the voltage sources. Combined with the branch relations, this allows the complete analysis of the circuit.

Fig.1.9 shows an example of a circuit containing only resistors and current sources, which can be analyzed using nodal analysis.

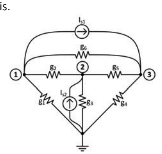


Figure 1.9. Circuit containing conductances and current sources.

The nodal equations are shown below. They are of the form ***G.v = I***$_s$, where ***G*** is the conductance matrix, ***v*** is the vector of node voltages, and***I***$_s$is the source vector:

$$\begin{bmatrix} g_2 + g_1 + g_6 & -g_2 & -g_6 \\ -g_2 & g_5 + g_2 + g_3 & -g_5 \\ -g_6 & -g_5 & g_6 + g_5 + g_4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -I_{s_1} \\ I_{s_2} \\ I_{s_1} \end{bmatrix} \tag{1.13}$$

Figure 1.10 shows a similar circuit containing two floating voltage sources. Here, direct nodal analysis cannot be used. Using the modified nodal analysis, the augmented nodal equations (1.14) result, which can be solved for the node voltages and the currents i_{s1} and i_{s3} flowingthrough the voltage sources.***G***is the matrix of the conductances, identical to the one in eq. (1.13).

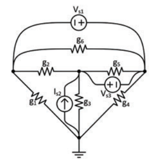


Figure 1.10. Circuit with two floating voltage sources.

$$\left[\begin{array}{ccc|cc} & & & -1 & 0 \\ & \boldsymbol{G} & & 0 & 1 \\ & & & 1 & -1 \\ \hline -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{array}\right] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \hline i_{s_1} \\ i_{s_3} \end{bmatrix} = \begin{bmatrix} 0 \\ I_s \\ 0 \\ \hline v_{s_1} \\ v_{s_1} \end{bmatrix} \tag{1.14}$$

Sometimes, a floating voltage source has an impedance in series. Then the branch can be transformed into a floating current source using Norton's equivalent circuit **(Fig. 1.11)**. This allows using simple node analysis of the circuit.

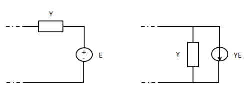
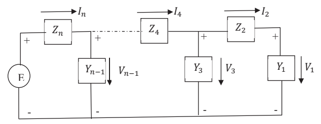


Figure 1.11: Norton's equivalence

*Example:***Fig.1.12**shows a *ladder* network, built from an alternation of series and shunt branches. Such circuits are used extensively in passive filters. They also play an important role in the design of analog-to-digital and digital-to-analog data converters, as well as in the modeling of interconnectsin an integrated circuit or printed circuit board. We assume that the circuit is linear and time-invariant.



Redraw with currents shown in wires and voltages with +/-.

Figure 1.12: Ladder network.

By applying KVL and KCL alternatively starting with the input branch and working forwards, the equations

$$\begin{gathered} E = I_n Z_n + V_{n-1} \\ 0 = -I_n + V_{n-1} Y + I_{n-1} \\ \cdots \\ 0 = -I_2 + V_1 I_1 \end{gathered} \tag{1.15}$$

result. This is a set of *n* linear equations in the *n* unknowns $V_1, I_2, V_3, I_4, \cdots, V_{n-1}, I_n$. The coefficient matrix has a special *tridiagonal* form, which allows a simple iterative algorithm for finding all unknowns [T.R. Bashkow, IRE Trans on Circuit Theory, June 1961].The solution process has a simple physical interpretation. Assume that the *output* voltage V_1 is known, and is equal to 1 volt, and that we need to find the *input* voltage $E_1$for this output voltage. Then, starting with the output branch, and working *backwards*, we can find all unknowns, one at a time. Thus, $I_2 = Y_1 I_2$. Next, we find$V_3 = I_2 Z_2 + 1$. Then, $I_4 = V_3 Y_3 + I_2$, etc. …, until $V_{n-1}$and finallyI_n are found. Then, the "unknown" input source is given by $E_1 = Z_n I_n$. At this point, we know the transfer function V_1/E_1. Since the circuit is linear, all actual voltages and currents can be obtained by multiplying the calculated values by E/E_1.

An important example of a ladder network is the *R-2R ladder,* shown in Fig. 1.13. Analysis using Bashkow's method gives V_1 = 1 , V_3 = 2, V_5 = 4, … Thus, the node voltages and the currents

flowing in the 2*R* resistors are both binary weighted. This property of the circuit makes it a useful stage in digital-to-analog converters (DACs) [*Carusone book].* Note that an even simpler way to analyze the *R*-2*R* ladder is to realize that the resistance seen to the *right* of each floating node is2*R,* while that seen to the *left* of the node is *R*. Thus, the circuit acts as a cascade of voltage and current dividers, scaling all quantities by powers of 2.

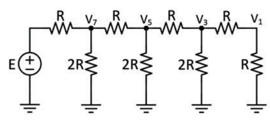


Figure 1.13: The R-2R ladder.

Example. In the hand analysis of linear active circuits, it is often efficient to perform the calculation working from the output towards input. As an example, consider the Riordan impedance convertor [ ] shown in Fig. 1.14.

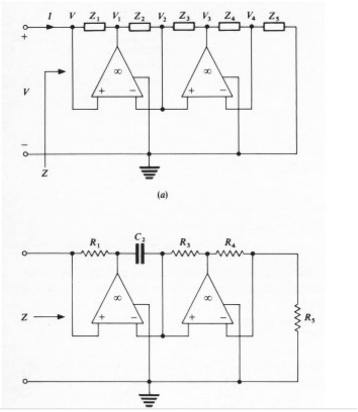


Fig. 1.14: The Riordan circuit.

Assuming ideal amplifiers and stable operation, the input voltages and currents of all amplifiers are zero. Hence, the voltage across the last impedance Z_5 is equal to the input voltage *V,* and $I_4 = I_5 = V/Z_5 I_4$. Therefore, $V_3 = V_5 + Z_4 I_4 = V[1 + Z_4/Z_5]$. Working back to the input port gives for the input impedance

$Z = \frac{Z_1 Z_3 Z_5}{Z_2 Z_4}$ (1.16)

Thus, theoretically this circuit can provide some interesting impedance functions. For example, if Z_2 and Z_4 are chosen as capacitors, while the other impedances as resistors, in the frequency domain the impedance is given by

$$Z(j\omega) = -\omega^2 R_1 C_2 R_3 C_4 R_5 \quad (1.17)$$

This is a *frequency-dependent negative resistor,* which has been proposed for the synthesis of active-RC filters [ ]. Note that the Riordan circuit is not suitable for integrated realization since the input terminals of the amplifiers are floating at signal voltages. This makes the suppression of common-mode signals at the output difficult.

1.4. Tellegen's Theorem

The development of the theory and applications of inter-reciprocity, to be discussed next, has an innate beauty. It is based on the simplest and most fundamental laws of circuit theory, Kirchhoff's voltage and current laws (KVLs and KCLs); and then in a few steps expands these basics to some sophisticated and highly useful results.

As discussed above, the KCL states that the voltage across each branch is the difference of the node voltages at the terminals of the branch, and KVL states that the sum of the currents leaving each node is zero. It is efficient to state these laws in vector and matrix notations. We define the *incidence matrix* ***A.*** Its element in row *I* and column *j* as

$$a_{ij} = \begin{cases} +1 \ if\ branch\ j\ leaves\ node\ i \\ -1 if\ branch\ j\ enters\ node\ i \\ 0 if\ branch\ j\ is\ not\ incident\ at\ node\ i \end{cases} \quad (1.18)$$

The branch currents can be collected into a column vector$\boldsymbol{I} = [i_1, i_2, \cdots, i_b]^T$. Here, *b* is the number of branches in the circuit.Then, the KCLs can be written in the simple form

$$\boldsymbol{A}.\boldsymbol{i} = \boldsymbol{0} \quad (1.19)$$

Here, **0**is the column vector of *b* zeros.

Similarly, collecting the branch and node voltages into column vectors, the KVL can be written in the concise form

$$\boldsymbol{v} = \boldsymbol{A}^T.\boldsymbol{e} \quad (1.20)$$

The first step in the development of the theory of inter-reciprocity was Tellegen's theory (ref.[1.1]). Consider the total power developed in all branches of a circuit, such as the one shown in Fig. 1.15. It is given by

$$\sum_{k=1}^{N} v_k j_k = \boldsymbol{v}^T \boldsymbol{j} = (\boldsymbol{A}^T \boldsymbol{e})^T \boldsymbol{j} = \boldsymbol{e}^T(\boldsymbol{A}\boldsymbol{j}) = \boldsymbol{0} \quad (1.21)$$

This equation has clear physical meaning. It formulates the conservation of power in the circuit: the generated power equals the dissipated power.

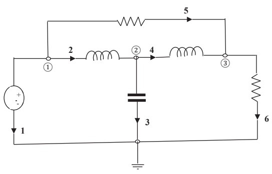


Figure 1.15: A circuit example.

Find next the related quantity

$$\sum_{k=1}^{N} v_k j'_k = \boldsymbol{v}^T \boldsymbol{j}' = (\boldsymbol{A}^T \boldsymbol{e})^T \boldsymbol{j}' = \boldsymbol{e}^T(\boldsymbol{A}\boldsymbol{j}') \quad (1.22)$$

Here, the branch voltages and currents were obtained from two *different* networks **N** and **N'.** Thus, the sum of their products has no clear physical interpretations. However, assume that **N** and **N'** have the same configurations, and therefore the same incidence matrix, so ***A=A'.*** for example, let **N** be the circuit of Fig. 1.15, while **N'** the one shown in Fig. 1.16. Then, the scalar product of their branch voltage and current vectors still satisfies

$$\sum_{k=1}^{N} v_k j'_k = \boldsymbol{v}^T \boldsymbol{j}' = (\boldsymbol{A}^T \boldsymbol{e})^T \boldsymbol{j}' = \boldsymbol{e}^T(\boldsymbol{A}\boldsymbol{j}') = \boldsymbol{0} \quad (1.23)$$

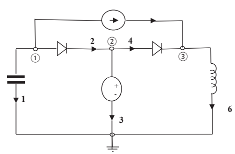


Figure 1.16: A different circuit with the same topology as that of Figure 1.15.

The equality of eq. (1.23) is known as *Tellegen's Theorem.* It plays an important role in the theory of impedance synthesis, and (as we shall see) also in the sensitivity analysis and noise computation of circuits.

At first glance, Tellegen's Theorem has no direct physical meaning. Note, howerever, that it is possible to construct a circuit ***N''*** with the same incidence matrix ***A***which contains the same branch voltages as ***N*** and the same branch currents as ***N'***. The new circuit ***N''*** can be constructed using a *tree* of the graph. A tree is a connected subgraph which includes all nodes but no loops. The branches not in the tree are called *links.* ***N''***is created so that itcontains voltage sources in its tree brancheswith values matching the corresponding branch voltages of ***N,*** and it haslink currents matching those of N'. Then Tellegen's theorem becomes the conservation of power for ***N''*** (ref, [1.2])**.**

1.5 Reciprocity and Its Application

A mere four years after the publication of Tellegen's theorem, J. L. Bordewijk (then a student of Professor Tellegen's at the Technical University of Delft) used his theorem to define and extend the concept of reciprocity. An *N*-port circuit is called *reciprocal* if its port (external) voltages and currents satisfy the relation $\sum i_k.v_k' = \sum i_k' v_k$. Here, the variables i_k and $v_k$are the port (external) currents and voltages and currents under one set of excitations, while i_k' and v_k' exist under a different one, f*or the same internal branches of the N-port.* The summations are for all *N* ports. By Tellegen's theorem, the same relation must then hold for the internal branches of the *N*-port [1.3].

The condition and its applications will next be analyzed for two-ports.

Consider the circuits shown in Fig. 1.17. Assume that the two-port contains only linear resistors, and no controlled sources. Regarding the circuit of Fig. 1.17(a) as ***N*** and that of Fig. 1.17(b) as ***N'***, Tellegen's theorem gives

$$-v_2 i_2' + \sum v_k i_k' = -v_2 i_2' + \sum i_k R_k i_k' = 0 \quad (1.24)$$

Here, the summations are for all *interior* branches inside the two-port. Interchanging the roles of the two circuits results in the relation

$$-v_1' i_1 + \sum v_k' i_k = -v_1' i_1 + \sum i_k R_k i_k' = 0 \quad (1.25)$$

Since the summations of the terms for the interior branches are equal, subtracting the two equations gives $v_2 i_2' = v_1' i_1$. Hence the resistive two-port is reciprocal. Also, we obtain

$$\frac{v_2}{i_1} = \frac{v_1'}{i_1'} \quad (1.26)$$

The two transfer functions (*transconductances)* are thus equal. Note that in this analysis **N** and **N'** contain the same internal branches, but under different terminations. Note also that assuming reciprocity for the two-port, the relation $\sum i_k v_k' = \sum i_k' v_k$gives directly$i_1 v_1' = i_2' v_2$,

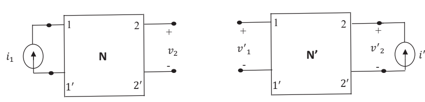


Figure 1.17: (a) Terminations for N; (b) Terminations for N'

Consider next the circuits shown in Fig. 1.18. Assuming that the two-port is reciprocal, combining $\sum i_k v_k' = \sum i_k' v_k$ with $i_2 = i_1' = 0$ gives $i_1 v_1' = i_2' v_2$ and $v_2/i_1 = v_1'/i_2'$. However, to use these relations for finding the relation between the transfer functions, the input impedance of **N**andthe output impedance of **N'** must also be found. Note in particular that $v_2/v_1$is *not* equal to v_2'/v_1'. In general, the forward transfer of **N e**quals the reverse transfer function of **N'**only if the source impedances at the ports are the same. This is valid for the circuits shownin Fig. 5. Analysis gives $v_2/v_1 = i_1'/v_1'$.

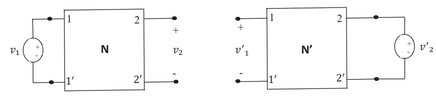


Figure 1.18: (a) Terminations for N; (b) Terminations for N'.Redraw without green lines.

The forward transfer of **N e**quals the reverse transfer function of **N'**only if the source impedances at the ports are the same. This is valid for the circuits shown in Figs. 1.17 and 51.19, but not for Fig. 1.18.

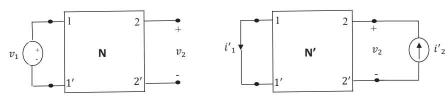


Figure 1.19: (a) Terminations for N; (b) Terminations for N'.Redraw without green lines.

The circuits which meet the equal-impedancetermination condition are shown in Fig. 1.20.

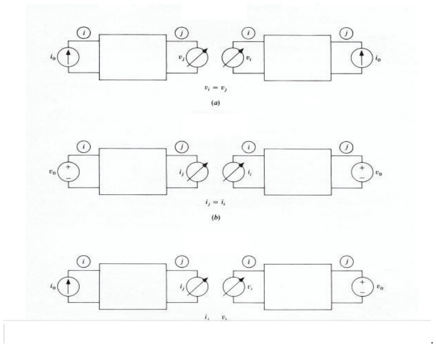


Figure 1.20. Reciprocity relations for passive linear circuits.

For the steady-state sine-wave analysis of a linear circuit, all voltages and currents can be represented by their complex phasors, *V(jω)* and *I(jω),* and the passive elements by their impedances $Z_k(j\omega)$. Since the KCL and KVL holds for the phasors, all derivations given above for the time-domain quantities can be replicated in terms of phasers, to give the same relations between the voltage and current phasors of reciprocal networks as the ones derived above in the time domain.

A useful application of reciprocity is in the analysis of passive networks with multiple excitations. Such circuits occur, e.g., in the design of digital-to-analog data converters (DACs). Consider the circuit of Fig. 1.21(a). It contains five independent sources. Hence, a straightforward approach to its analysis would be to use superposition. This would require five analyses, each with one of the sources active and all others set to zero. Alternatively, node analysis may be used. This would require solving simultaneous equations involving all current sources.

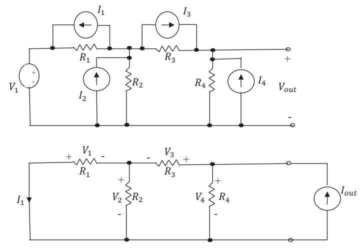


Figure 1.21: (a) Physical network **N;** (b) Auxiliary network **N'.**

It is much more efficienttouse reciprocity. The contribution of V_1 to the output v_{out}can be found from Fig. 1.21(b)using the relation

$$\frac{v_{out}}{V_1} = \frac{i_1}{I} \tag{1.27}$$

and those of the current sources $I_k$from the relations

$$\frac{v_{out}}{I_k} = \frac{v_k}{I} \qquad k = 1, 2, 3, 4, 5 \tag{1.28}$$

The overall output is the sum of the six terms shown in the two equations above. Note that this calculation requires only *one* circuit analysis, that of the circuit of Fig. 1.21(b).

Generalizing the method, the analysis requires the following steps:

1. Redraw the circuit, setting the values of all independent sources to zero. Thus, voltage sources become short circuits, current sources become open circuits.