# Introduction to State Space Models

## The State Space Approach to Modeling Systems

A system model is necessary before any analysis or design of control system can be done. Mathematical models of systems can be found by the application of one or more fundamental laws peculiar to the physical nature of the system or component. Alternatively, experimental methods may be used. Two approaches can be used when modeling control systems. These are the classical control theory and the modern control theory.

The classical theory of modeling systems involves representing a system using a differential equation that relates the system output to the system input. The differential equation can be transformed into an algebraic expression in the *s*-domain (by Laplace Transforms). This method is more applicable to SISO systems. The classical control theory was covered in the previous Control Engineering course.

The modern control theory, which is also known as the state space or state variable approach, involves use of a set of first order differential equations and matrix algebra. The system behavior is represented by several variables called states. The models are then in the form of a set of first order differential equations. Each differential equation relates the first derivative of a state to the states and the inputs to the system. To work on the models compactly, matrix algebra is exploited. Though the state space approach is applicable to SISO systems, it finds important applications in multivariable systems. State space approach is the subject of part of this course on Control Engineering.

## *Illustrative Examples*

The following examples illustrate the concept of states as well as how state equations can be obtained from the Physical Laws governing a system.

### Example 1.1

Consider the system shown in figure 1.1.

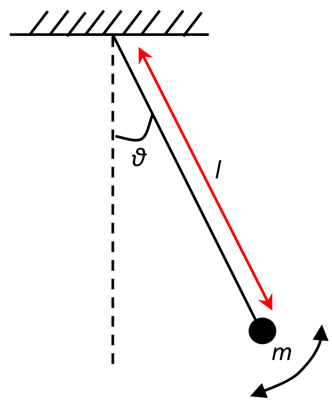



Figure 1.1

Suppose the states for the system are chosen as the angular position, $\theta(t)$ and angular velocity of the pendulum $\dot{\theta}(t)$. Let the states be denoted $x_1$ and $x_2$ respectively. Taking first derivatives of the states:

$$\dot{x}_1(t) = \dot{\theta}(t) = x_2(t)$$
$$\dot{x}_2(t) = \ddot{x}(t) = -\frac{g}{l}x_2(t)$$

Thus, basically state equations as they are known express relations between the first derivatives of the states with the states.

In the example, the first expression comes out from the fact that the first derivative of the first state is the angular velocity which is the second state. The second expression is obtained by applying fundamental Laws governing the operation of the pendulum so as to get the expression for the first derivative of the second state in terms of the states.

### Example 1.2

Consider the circuit shown in figure 1.2.

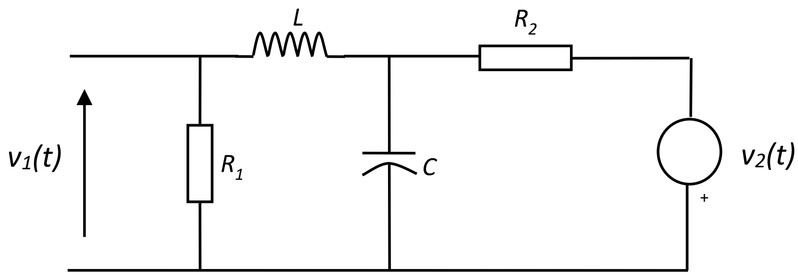



Figure 1.5

Let the states be $x_1$ for current through inductor and $x_2$ for voltage across the capacitor (reasons for this choice will be highlighted in later topics). $v_1$ and $v_2$ are inputs to the circuit and $y$ is the output which is voltage across $R_2$. Using the fundamental laws of circuit analysis, the state equations are derived as follows:
From loop 1,

$$v_{R1} - v_L - v_C = 0$$
$$v_1 - v_L - v_C = 0$$
$$v_1 - L\frac{di_L}{dt} - v_C = 0$$
$$v_1 - L\dot{x}_1 - x_2 = 0$$

Hence, $\dot{x}_1 = -\frac{x_2}{L} + \frac{v_1}{L}$

From loop 2,

$$v_C - v_{R2} - v_2 = 0$$
$$v_C - (i_L - i_C)R_2 + v_2 = 0$$
$$v_C - (i_L - C\frac{dv_C}{dt})R_2 - v_2 = 0$$
$$x_2 - (x_1 - C\dot{x}_2)R_2 - v_2 = 0$$

Hence, $\dot{x}_2 = \frac{x_1}{R_2C} - \frac{x_2}{R_2C} + \frac{v_2}{R_2C}$

The output is given by

$$v_{R2} = v_C - v_2$$

Hence, $y = x_2 - v_2$

The equations can be expressed in a compact form as matrices as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{R_2C} & -\frac{1}{R_2C} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} \frac{1}{L} & 0 \\ 0 & \frac{1}{R_2C} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

## *Choice of states*

The procedure of formulating the state variable system begins with the selection of a set of state variables. This set of variables must completely describe the effect of the past history of the system and its response in the future. Although the choice of state variables is not unique, the state variables of a dynamic system are usually related to the energy stored in each of the system's energy-storing elements. Since any energy that is initially stored in these elements can affect the response of the system at a later time, one state variable is normally associated with each of the independent energy storing elements.

For example, the potential and kinetic energy are stored in position and velocity, respectively. Consequently, position and velocity make appropriate choices for state variables in translational and rotational mechanical systems. For electrical circuits energy is stored in inductors and capacitors, thus capacitor voltage and inductor current are usually chosen as the state variables in such systems.

### Notes
1. Choice of state variables is not always straightforward.
2. In most situations more than one set of valid state variables are possible.

## *Deriving State Equations from Differential Equations*
The process of converting differential equations to state space equations is illustrated using Example 1.3.

### *Example 1.3*
Consider the following differential equation (DE):

$$\frac{d^3y}{dt^3} + a_2\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0 y = u$$

This has a transfer function (TF):

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^3+a_2s^2+a_1s+a_0}$$

If the following states are chosen:

$$x_1 = y,\ \ x_2 = \dot{y},\ \ x_3 = \ddot{y},\ \ \text{where} \quad \dot{y} = \frac{dy}{dt} \text{ and } \ddot{y} = \frac{d^2y}{dt^2}$$

For a translational mechanical system, these states may represent the position, velocity and acceleration of a body.

Using these states and relating the first derivatives to the states, the following first order equations can be written:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -a_0x_1 - a_1x_2 - a_2x_3 + u$$

The DE can, thus, be represented in the following matrix form:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -a_0 & -a_1 & -a_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u$$

The state space approach to modelling, thus, can be said to work by replacing high order differential equations with a set of simultaneous first order differential equations.

## General format of State Space Equations
The general form of the state-space equations is as follows:

$$\bar{x}(t) = \bar{A}\bar{x}(t) + \bar{B}\bar{u}(t) \qquad (1)$$
$$\bar{y}(t) = \bar{C}\bar{x}(t) + \bar{D}\bar{u}(t) \qquad (2)$$

where

$\bar{x}(t)^T = [\bar{x}_1(t) \quad \bar{x}_2(t) \quad \cdots \quad \bar{x}_n(t)]$ is the state vector
$\bar{u}(t)^T = [\bar{u}_1(t) \quad \bar{u}_2(t) \quad \cdots \quad \bar{u}_p(t)]$ is the input vector
$\bar{y}(t)^T = [\bar{y}_1(t) \quad \bar{y}_2(t) \quad \cdots \quad \bar{y}_q(t)]$ is the output vector

$\bar{A}$ is a $n \times n$ matrix
$\bar{B}$ is a $n \times p$ matrix
$\bar{C}$ is a $q \times n$ matrix
$\bar{D}$ is a $q \times p$ matrix

The above two matrix equations are known as the ***state-space equations*** or the ***state-variable equations***.

## *Notes on SS equations*
1. If the system is discrete-time the equations are basically the same but based on first order difference equations rather than first order differential equations.

2. If the dynamic system is time-varying then $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ are time varying and are denoted by $\bar{A}(t)$, $\bar{B}(t)$, $\bar{C}(t)$ and $\bar{D}(t)$.
3. The equations can be modified to represent non-linear systems.
4. The equations include all types of multivariable systems:
   SISO (single-input single-output) $p = q = 1$
   MISO (multi-input single-output) $q = 1$
   SIMO (single-input multi-output) $p = 1$
   MIMO (multi-input multi-output)
5. If $n$ is 1, 2 or 3 then the state can be represented (and visualised) geometrically in 1, 2 or 3-dimensional Cartesian space.

# Transfer Function from State Space representation

## Laplace domain equivalent of State Space equations

Taking the Laplace transform of the SS equations (1) and (2) above gives:

$$s\bar{X}(s) - \bar{x}(0) = \bar{A}\bar{X}(s) + \bar{B}\bar{U}(s) \qquad (3)$$
$$\bar{Y}(s) = \bar{C}\bar{X}(s) + \bar{D}\bar{U}(s) \qquad (4)$$

Equation (3) can be rearranged to give:

$$(s\bar{I} - \bar{A})\bar{X}(s) = \bar{x}(0) + \bar{B}\bar{U}(s)$$

thus:

$$\bar{X}(s) = \underbrace{(s\bar{I} - \bar{A})^{-1}\bar{x}(0)}_{\text{zero input response}} + \underbrace{(s\bar{I} - \bar{A})^{-1}\bar{B}\bar{U}(s)}_{\text{zero state response}} \qquad (5)$$

and substituting this into equation 4 gives:

$$\bar{Y}(s) = \underbrace{\bar{C}(s\bar{I} - \bar{A})^{-1}\bar{x}(0)}_{\text{zero input response}} + \underbrace{[\bar{C}(s\bar{I} - \bar{A})^{-1}\bar{B} + \bar{D}]\bar{U}(s)}_{\text{zero state response}} \qquad (6)$$

In equations (5) and (6), the first terms represent the zero input response and the second terms represents the zero state response.

## Transfer Function

Using equation (6) and ignoring zero input response, the transfer function matrix $G(s)$ can be obtained as follows:

$$G(s) = \frac{\bar{Y}(s)}{\bar{U}(s)} = \bar{C}(s\bar{I} - \bar{A})^{-1}\bar{B} + \bar{D} \qquad (9)$$

It can be observed that $[s\bar{I} - \bar{A}]^{-1} = \frac{adj[s\bar{I} - \bar{A}]}{det[s\bar{I} - \bar{A}]}$. The TF can therefore, be represented by:

$$G(s) = \frac{\bar{C}\, adj[s\bar{I} - \bar{A}]\bar{B}}{det[s\bar{I} - \bar{A}]} + \bar{D} \qquad (10)$$

## Characteristic Equation and Eigenvalues

The characteristic polynomial of the system is given by $\boldsymbol{det[s\bar{I} - \bar{A}]}$ and the characteristic equation is given $\boldsymbol{det[s\bar{I} - \bar{A}] = 0}$. The roots of the characteristic equation are known as the ***eigenvalues*** of the system.

### *Example 1.4*

*Suppose in a system,* $\bar{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -15 & -23 & -9 \end{bmatrix}$, *find the TF and the eigenvalues of the system.*

**SOLUTION**

$$[s\bar{I} - \bar{A}] = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 15 & 23 & s+9 \end{bmatrix}$$

$$\begin{aligned} det[s\bar{I} - \bar{A}] &= s(s(s+9) + 23) + 1(0(s+9) + 15) \\ &= s^3 + 9s^2 + 23s + 15 \end{aligned}$$

*The characteristic equation is:* $\quad s^3 + 9s^2 + 23s + 15 = 0$

*From solution of the characteristic equation, the eigenvalues are:* $\lambda_1 = -1$, $\lambda_2 = -3$ *and* $\lambda_3 = -5$.

## Minimal and Non-minimal Systems

A state variable equation is minimal if the number of its eigenvalues equals the number of poles in its transfer function. Every eigenvalue of a minimal state variable equation will appear as a pole of its transfer function and there is no essential difference between using the state variable equation or its transfer function for studying the system. In such a system the state variable equations are said to be completely characterized by its transfer function.

If a state variable equation is not minimal then some of the eigenvalues will not appear as poles of its transfer function and the transfer function is said to have missing poles. In such systems the state variable equations cannot be completely characterized by the transfer function and consequently the system cannot be fully analyzed using the transfer function approach.

## Eigenvectors

Associated with each eigenvalue $\lambda_i$, there is an associated eigenvector $\bar{x}_i$ given by:

$$[\lambda_i \bar{I} - \bar{A}]\bar{x}_i = \bar{0} \qquad (11)$$

No eigenvector is unique as it can always be multiplied by a scalar and still satisfy equation (11). The following example shows the determination of the eigenvectors of a simple two state system.

### Example 1.5

$$\bar{A} = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}$$

*Characteristic equation:* $|s\bar{I} - \bar{A}| = 0$

$$\begin{vmatrix} s+1 & 0 \\ -1 & s+2 \end{vmatrix} = 0$$

$$(s+1)(s+2) = 0$$

***Eigenvalues:*** $\lambda_1 = -1$, $\lambda_2 = -2$

***Eigenvector 1:***

$$\left[(-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}\right]\bar{x}_1 = 0$$

$$\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = 0$$

*Equating elements of matrices gives:*

$$x_{11} + x_{21} = 0$$

$$x_{11} = x_{21}$$

*One eigenvector for* $\lambda_1 = -1$ *would be* $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *but* $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ *or any vector* $\rho\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *where* $\rho$ *is a real scalar would also be valid.*

***Eigenvector 2:***

$$\left[(-2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}\right]\bar{x}_2 = 0$$

$$\begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = 0$$

*Equating elements of matrices gives:*

$$-x_{12} = 0$$

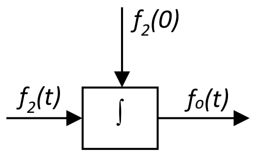
$$-x_{22} = \rho \qquad \text{\{non-zero } \bar{x}_i \text{ normally used\}}$$

*One eigenvector for* $\lambda_1 = -2$ *would be* $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ *but* $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ -3 \end{bmatrix}$ *or any vector* $\rho\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ *where* $\rho$ *is a real scalar would also be valid.*

### Notes
Eigenvectors are essentially vectors that specify the direction of something. One use of eigenvectors in state space analysis is to transform state space representations from one form to another and this will become apparent in future topics as will the reason for the subscript notation used above.

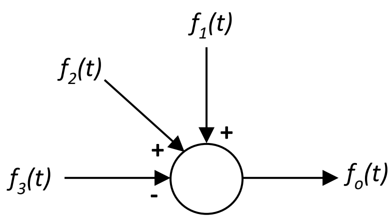
Eigenvectors also provide a useful visual insight into the state trajectory determined from the state transition matrix. The eigenvalues give the exponential modes of the system and the eigenvectors specify the distribution of the modes between the states.

## Diagrammatic representations
SS systems are often represented diagrammatically using a notation based on the now largely redundant analogue computer. These diagrams consist of a combination of three components: an integrator, a summer and a coefficient multiplier. This notation is useful as it enables the dynamic system to be represented by what is essentially an analogue circuit. The three components are shown in figures 1.6 (a) to 1.6 (c).



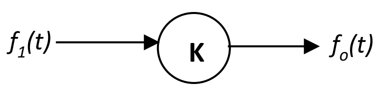

$$f_o(t) = \int_0^T f_1(t)dt + f_2(0)$$

*(b) integrator - the initial condition, $f_2(0)$ is often omitted if it is zero*



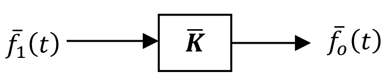

$$f_o(t) = f_1(t) + f_2(t) - f_3(t)$$

*(b) summer*




$$f_o(t) = Kf_1(t)$$

*(c) coefficient multiplier*




$$\bar{f}_o(t) = K\bar{f}_1(t)$$

*(d) vector coefficient multiplier*

Figure 1.6 Block diagram symbols

### *Example 1.6*
*Figure 1.7 shows a block diagram representing the 3rd order (3 state) DE with the TF:* $\frac{d^3y}{dt^3} + 9\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 5y = u$.
*The equation can be rewritten as:* $\frac{d^3y}{dt^3} = u - 9\frac{d^2y}{dt^2} - 8\frac{dy}{dt} - 5y = u - 9x_3 - 8x_2 - 5x_1$.

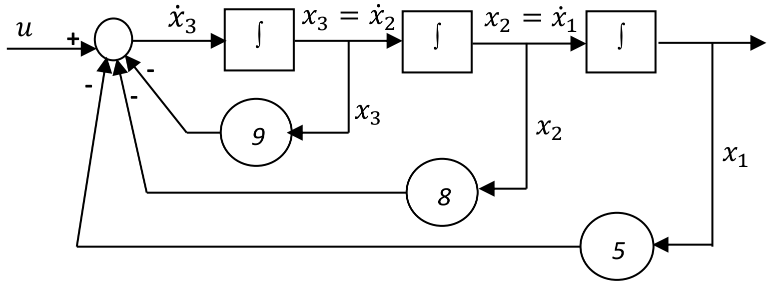



Figure 1.7

The above notation is based on $f_n(t)$ being a scalar. A similar notation however can be used for vector signals, i.e. $\bar{f}_o(t)$. The principle change to the notation is the coefficient multiplier which becomes as shown in figure 1.7 (c), where $\bar{\boldsymbol{K}}$ is a vector of real numbers.

Using the vector-based diagram notation the SS equations (equations 1 and 2) can be represented by the diagram shown in figure 1.8.

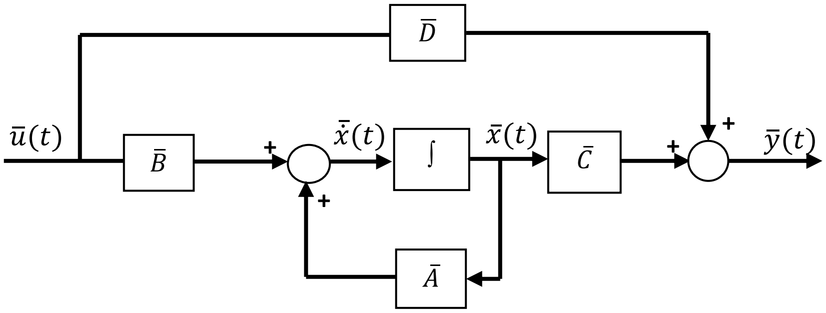



Figure 1.8

## Major differences between LTF and state-space models

1. A LTF model relates only the system output and input. Any internal structure and behavior of the system is lost. State-space representations can preserve such internal information, if the state variables are appropriately chosen.
2. When working with LTFs, one may have to solve high order equations. In the state space approach, only first order equations appear in the model.
3. The LTF approach only easily applicable to SISO systems, though more recent techniques (e.g. inverse Nyquist arrays) are now available for MIMO systems. In the state space approach, the system model is arranged in a vector-matrix form. It therefore matters little how many inputs or outputs a system has, the matrices just change in size.
4. A LTF model is only defined for zero initial conditions. The initial conditions may be represented as disturbance inputs. The state space approach uses the differential equations directly, so all initial conditions are automatically included and are simply set to zero if not required.
5. The state space model may handle time-varying models whereas a LTF model is only for stationary systems.

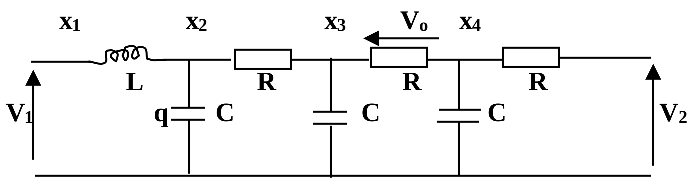
But there are drawbacks to state-space models, some of which can be serious:

1. Many processes involve pure time delays which are difficult to handle using continuous-time state-space models.
2. The methods of control system design based on frequency-domain models are true design methods. A controller is designed, its effects investigated and is then refined until the results appear acceptable. Good results can be obtained with poorly modelled plants and unexpected disturbances. Control system design using state-space models, on the other hand, tends to use synthesis methods. That is to say, an algorithm will exist for designing a certain kind of controller (these will be studied later). The designer simply plugs the system model ($\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$) into the algorithm, and out comes the design of the controller – it is synthesized from the model parameters; hence a very good plant model is required. In practice, it is hard to get good models of many industrial plants.

These factors, together with the observation that frequency-domain control seems able to cope with the majority of real control problems have led to a generally slow rate of introduction of state-space techniques.

# Chapter 1 Tutorial Questions

1. Derive the state matrices for the following electrical circuit.




where

$$\bar{x} = [x_1 \quad x_2 \quad x_3 \quad x_4]^T \qquad \bar{u} = [V_1 \quad 0 \quad 0 \quad V_2]$$

and

$x_1$ is a current; $x_2$, $x_3$ and $x_4$ are voltages.

If the output quantities are the voltage $v_o$ and q, determine the output matrix $\bar{C}$ in the output equation:

$$\begin{bmatrix} v_o \\ q \end{bmatrix} = \bar{C}\bar{x}$$

$$\left( \bar{A} = \begin{bmatrix} 0 & -1/L & 0 & 0 \\ 1/C & -1/CR & 1/CR & 0 \\ 0 & 1/CR & -2/CR & 1/CR \\ 0 & 0 & 1/CR & -2/CR \end{bmatrix} \right) \quad \bar{B} = \begin{bmatrix} 1/L & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/CR \end{bmatrix} \quad \bar{C} = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & C & 0 & 0 \end{bmatrix}$$

2. The system with:

$$\bar{\mathbf{A}} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \qquad \bar{\mathbf{B}} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

is to be held at $x_1 = x_2 = 1$. Determine the required values of $u_1$ and $u_2$. Make a block diagram of the system and check that the results are correct in the diagram. *($u_1 = -15$; $u_2 = 7$)*

3. For the system:

$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{B}}\bar{\mathbf{u}}$$
$$\bar{\mathbf{y}} = \bar{\mathbf{C}}\bar{\mathbf{x}}$$

where $\bar{\mathbf{A}} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -6 & -11 & -7 \end{bmatrix} \qquad \bar{\mathbf{B}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad \bar{\mathbf{C}} = [1 \quad 0 \quad 1]$

obtain the characteristic equation and thus determine the eigenvalues. Draw the state block diagram.

$$\left(\textbf{Ans:} \quad s^3 + 9s^2 + 26s + 24 = 0; \quad \lambda_1 = -2; \lambda_2 = -3; \lambda_3 = -4\right)$$

4. Find the characteristic equation for the following:

(a) $\bar{\mathbf{A}} = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix}$ (b) $\bar{\mathbf{A}} = \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix}$

$(s^3 + a_1 s^2 + a_2 s + a_3)$ $(s^3 + a_1 s^2 + a_2 s + a_3)$

5. For the following $\bar{\mathrm{A}}$ matrices determine the eigenvalues and suggest suitable eigenvectors:

(a) $\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix}$