State Space Realizations

Introduction

This lecture will begin with a look at the concept of observability and controllability. A system is observable if the states can be observed at the output, likewise a system is controllable if the states can be controlled from the input. The ideas of observability and controllability are crucial to most state space control design methods.

In the previous topic, it was shown how the TF can be determined from the state-space matrices $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$. Realization is the reverse process, i.e. determining $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ from the TF. While the former process is a one-to-one mapping, realization is a one-to-many mapping. This topic will also look at a number of different approaches to realization.

Controllability and Observability

Consider the following system:

$$\bar{A} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \bar{C} = [1 \quad 1 \quad 0 \quad 0], \qquad \bar{D} = [0]$$

The diagrammatic representation is given in figure 2.1.

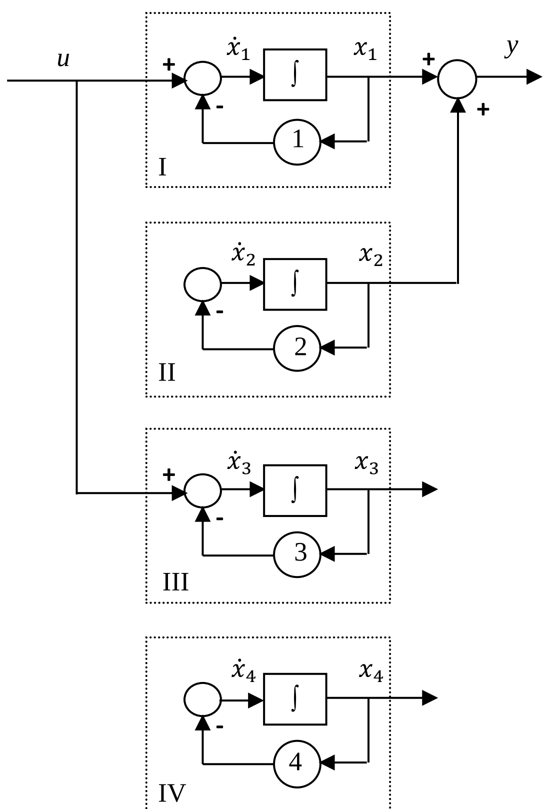

Figure 2.1

(As an exercise confirm that this diagram does represent the above state space matrices).

States x_1 and x_2 are said to be observable in that they contribute to the output y. Likewise, states x_1 and x_3 are said to be controllable in that they are influenced by the input u. Any state space system can be partitioned into four subspaces.

1. observable and controllable, e.g. space I in earlier diagram
2. observable and uncontrollable, e.g. II
3. unobservable and controllable, e.g. III

4. unobservable and uncontrollable, e.g. IV

Any system which has an uncontrollable subspace (II and/or IV) is said to be uncontrollable. Likewise, any system with an unobservable subspace (III and/or IV) is said to be unobservable.

The TF G(s) is given by:

$$[s\bar{I}-\bar{A}] = \begin{bmatrix} s+1 & 0 & 0 & 0 \\ 0 & s+2 & 0 & 0 \\ 0 & 0 & s+3 & 0 \\ 0 & 0 & 0 & s+4 \end{bmatrix}$$

$$det[s\bar{I}-\bar{A}] = (s+1)(s+2)(s+3)(s+4)$$

$$adj[s\bar{I}-\bar{A}] = \begin{bmatrix} (s+2)(s+3)(s+4) & 0 & 0 & 0 \\ 0 & (s+1)(s+3)(s+4) & 0 & 0 \\ 0 & 0 & (s+1)(s+2)(s+4) & 0 \\ 0 & 0 & 0 & (s+1)(s+2)(s+3) \end{bmatrix}$$

$$[s\bar{I}-\bar{A}]^{-1} = \begin{bmatrix} \frac{1}{s+1} & 0 & 0 & 0 \\ 0 & \frac{1}{s+2} & 0 & 0 \\ 0 & 0 & \frac{1}{s+3} & 0 \\ 0 & 0 & 0 & \frac{1}{s+4} \end{bmatrix}$$

$$G(s) = \frac{\bar{Y}(s)}{\bar{U}(s)} = \bar{C}(s\bar{I}-\bar{A})^{-1}\bar{B}+\bar{D}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} & 0 & 0 & 0 \\ 0 & \frac{1}{s+2} & 0 & 0 \\ 0 & 0 & \frac{1}{s+3} & 0 \\ 0 & 0 & 0 & \frac{1}{s+4} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + [0]$$

$$= \frac{1}{s+1}$$

This is a non-minimal system as it has 1 pole and 4 eigenvalues. The state space of a minimal system consists entirely of an observable and controllable subspace (I).

Definitions

Controllability

A system is said to be controllable if and only if it is possible, by means of the input, to transfer the system form any initial state $\bar{x}(t) = \bar{x}_i$ to any other state $\bar{x}(T) = \bar{x}_T$ in a finite time $T - t \geq 0$.

Observability

An unforced system is said to be observable if and only if it is possible to determine any (arbitrarily initial) state $\bar{x}(t) = \bar{x}_t$ by using only a finite record, $y(\tau)$ for $t \leq \tau \leq -t$.

Tests

The controllability and observability of a system can be determined by the following two tests:

Controllability

The time-invariant system $\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}\bar{u}$ is controllable if and only if the rank $r(\bar{Q})$ of the controllability test matrix $\bar{Q}$ is equal to n the order of the system.

$$\bar{Q} = [\bar{B} \quad \bar{A}\bar{B} \quad \bar{A}^2\bar{B} \quad \cdots \quad \bar{A}^{n-1}\bar{B}]$$

If the test matrix is not of full rank, then the system is only partially state controllable. The *rank deficiency* of the test matrix is equal to the number of uncontrollable modes in the system.

Observability

The (unforced) time invariant system $\dot{\bar{x}} = \bar{A}\bar{x}$ with observable vector $\bar{y}(t) = \bar{C}\bar{x}$ is observable if and only if the rank $r(\bar{N})$ of the observability matrix $\bar{N}$ is equal to n, the order of the system.

$$\bar{Q} = \begin{bmatrix} \bar{C} \\ \bar{C}\bar{A} \\ \bar{C}\bar{A}^2 \\ \cdots \\ \bar{C}\bar{A}^{n-1} \end{bmatrix}$$

If the test matrix is not of full rank, then the system is only partially state observable. The rank deficiency of the test matrix is equal to the number of unobservable modes in the system.

Example 2.1

Consider the state space system

$$\bar{A} = \begin{bmatrix} 1.5 & 1 \\ 1 & 0 \end{bmatrix} \quad \bar{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \bar{C} = \begin{bmatrix} 2 & 1 \end{bmatrix}, \qquad \bar{D} = [0]$$

Controllability matrix: $\bar{Q} = \begin{bmatrix} \bar{B} & \bar{A}\bar{B} \end{bmatrix} = \begin{bmatrix} 1 & 1.5 \\ 0 & 1 \end{bmatrix}$

$det\bar{Q} = 1, \textit{ and hence } r(\bar{Q}) = 2$

The system is controllable.

Example2.2

Consider the state space system

$$\bar{A} = \begin{bmatrix} -3 & -4 \\ -1 & 0 \end{bmatrix} \quad \bar{B} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \qquad \bar{C} = \begin{bmatrix} -1 & -1 \end{bmatrix}, \quad \bar{D} = [0]$$

Observability matrix: $\bar{N} = \begin{bmatrix} \bar{B} \\ \bar{C}\bar{A} \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 4 & 4 \end{bmatrix}$

$det\bar{N} = 0, \textit{ and } rank\ (\bar{N}) = 1$

The system is not observable. The rank deficiency is one; therefore, there is one unobservable mode.

The reason for this (possibly pole-zero cancellation) may be checked by deriving the LTF from the state space model.

$$G(s) = \frac{\bar{Y}(s)}{\bar{U}(s)} = \bar{C}(s\bar{I} - \bar{A})^{-1}\bar{B} + \bar{D}$$

This may be checked to be equal to:

$$G(s) = \frac{(s-1)(s-1)}{(s-1)(s+4)}$$

The reason that the system is not completely observable is that if some disturbance signal excites the mode e^t arising from the denominator term $(s-1)$, the dynamic effects will be cancelled out by the simultaneous response of the numerator term $(s-1)$, thus leaving no visible effect at the output.

Checking the controllability of this system shows that it is not completely controllable. That is, one of the modes cannot be excited form the input.

Stabilizability

If a system is completely controllable then any unstable eigenvalues may be moved to stable locations by state feedback. If the system fails the controllability test, what then?

Tests (using the *sensitivity matrix*) may be carried out to find those modes that are uncontrollable. If the unstable pole that needs to be moved is controllable, then the system is said to be *stabilizable*, and the design can proceed to move that pole.

The sensitivity matrix is formed corresponding to each eigenvalue of the system in turn ($i = 1\ to\ n$ for an nth-order system):

$$S(\lambda_i) = \begin{bmatrix} \bar{A} - \lambda_i\bar{I} & \bar{B} \end{bmatrix}$$

If $rankS(\lambda_i)$ is less than n, then λ_i is an uncontrollable mode.

If this test is carried out on the eigenvalues of example 2.2, it is found that the unstable mode corresponding to $\lambda = +1$ is also uncontrollable. Hence the system cannot be stabilized. The state space model must then be converted to some other form as will discussed later.

Realization

We will start by looking at two simple realization methods namely the controllable canonical form and the observable canonical form.

Controllable canonical form

Consider the strictly proper transfer function:

$$G(s) = \frac{b_{n-1}s^{n-1} + b_{n-1}s^{n-1} + \cdots + b_0 s + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + a_0 s + \cdots + a_0} \qquad (1)$$

The corresponding equations (using method described earlier) are:

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \qquad \bar{C} = [b_0 \quad b_1 \quad b_2 \quad \dots \quad b_{n-1}]$$

Note that matrix $\overline{D}$ has been left out, this is common and implies that $\overline{D} = [0]$.

The $\bar{A}$ matrix is in a canonical form, sometimes known as the companion form or the controllable canonical form as it satisfies the controllability test matrix; that is, all the states can be modified by the control input.

Diagrammatically this dynamic system is represented as shown in figure 2.2.

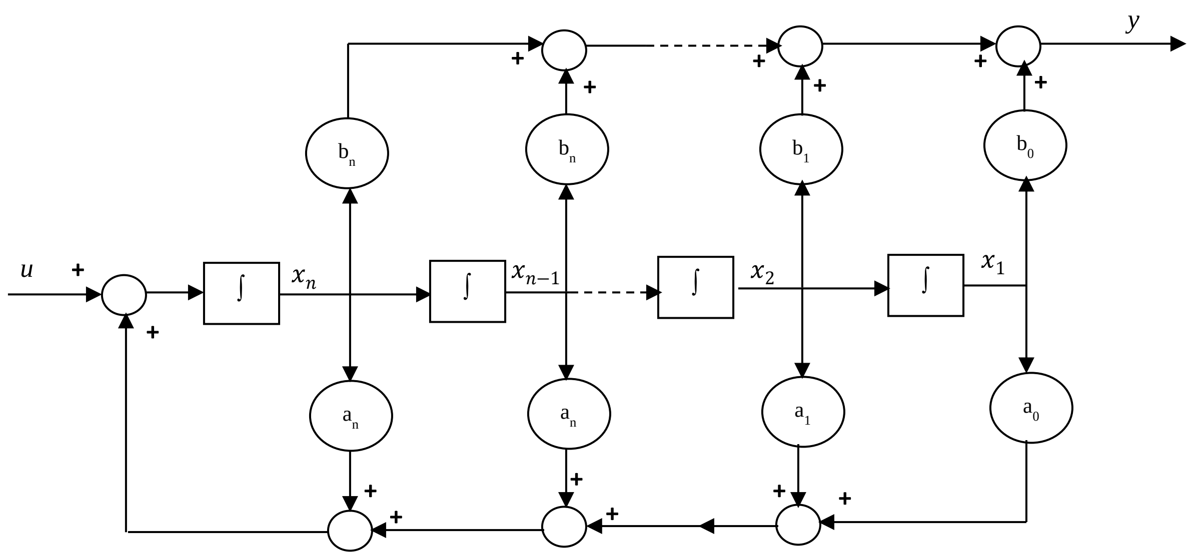

Figure 2.2

Note that there is no unique state-space model of any given system. The canonical form given above has some mathematical advantages. However, there is actually an infinite number of other possible state-space models of the same system. These are obtained by choosing different sets of states, of which some are even more mathematically convenient (for example the $\bar{A}$ matrix can be made diagonal by suitable choice of the states, thus making certain types of analysis relatively easy).

The main advantage of the controllable canonical form over other forms is that the differential equation describing the system can be converted into this canonical form by simple substitution of the differential equation coefficients into the matrices $\bar{A}$ and $\bar{C}$. The conversion can be performed by inspection, with none of the intervening mathematics being necessary.

Note

Some texts reverse the state numbering to give the following equivalent state matrices:

$$\bar{A} = \begin{bmatrix} -a_{n-1} & \dots & -a_2 & -a_1 & -a_0 \\ 1 & \dots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \qquad \bar{C} = [b_{n-1} \quad \dots \quad b_2 \quad b_1 \quad b_0]$$

We will NOT use this convention.

Observable canonical form

The observable canonical form of the strictly proper G(s) in equation (1) is:

$$\bar{A} = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix}, \qquad \bar{C} = [0 \quad 0 \quad 0 \quad \dots \quad 1]$$

Diagrammatically this is represented as shown in figure 2.3.

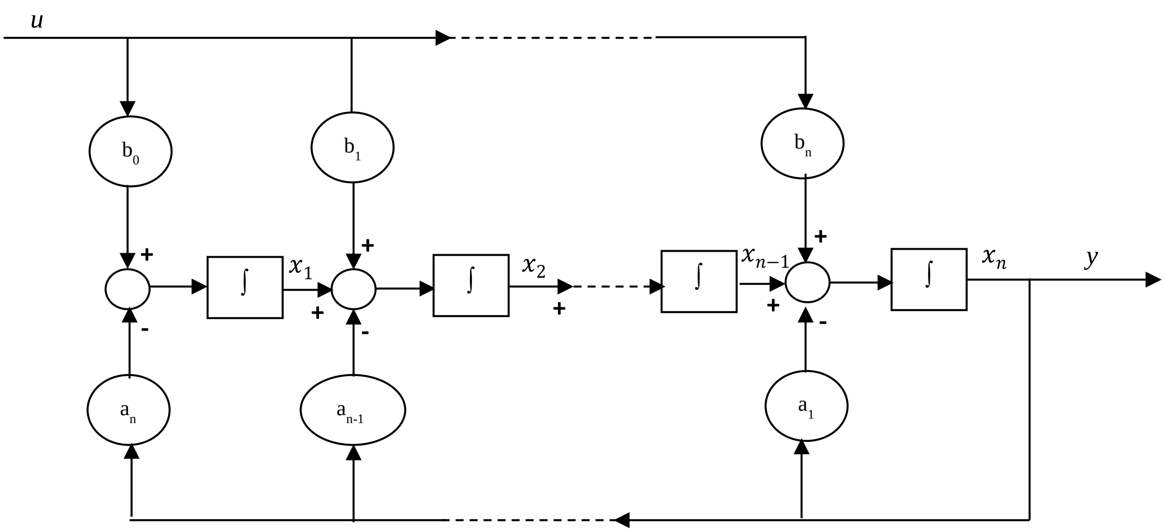

Figure 2.3

Note

Some texts reverse the state numbering to give the following equivalent state matrices:

$$\bar{A} = \begin{bmatrix} -a_{n-1} & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_2 & 0 & \cdots & 1 & 0 \\ -a_1 & 0 & \cdots & 0 & 1 \\ -a_0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} b_{n-1} \\ \vdots \\ b_2 \\ b_1 \\ b_0 \end{bmatrix}, \qquad \bar{C} = [1 \quad \dots \quad 0 \quad 0 \quad 0]$$

We will NOT use this convention.

Notes

1. The controllable and observable canonical forms are easy to determine but are rather sensitive to parameter variations and are thus of limited direct practical use.
2. The approach adopted also determines the choice of state variables.

Diagonal forms

This is when the state space matrices are transformed to give a diagonal or near diagonal $\bar{A}$. Diagonal and near diagonal methods have a number of practical advantages over the controllable and observable canonical forms already considered, and consequently the extra effort in determining them is usually justified. Such diagonal matrices play an important role in the analysis of MIMO systems, where decoupling of the states becomes very crucial. Three distinct types of $\bar{A}$ matrix will be considered:

1. All eigenvalues are real and distinct.
2. Some eigenvalues are complex but all are distinct.
3. Some eigenvalues are repeated.

These three groups will be considered separately.

Real distinct eigenvalues

Systems of this form can be transformed to give a diagonal $\bar{A}$ matrix with the real eigenvalues on the diagonal. Two approaches will be considered, the first using partial fractions, the second using similarity transformations.

Partial fraction approach

This approach is best shown by an example.

Example 2.3

Consider the following transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{3s^2+7s+15}{s^3+7s^2+14s+14}$$

This can be split into the following partial fractions:

$$\frac{Y(s)}{U(s)} = \frac{11/3}{s+1} - \frac{13/2}{s+2} + \frac{35/6}{s+4}$$

Defining:

$$X_1(s) = \frac{1}{s+1}U(s)$$

$$X_2(s) = \frac{1}{s+2}U(s)$$

$$X_3(s) = \frac{1}{s+4}U(s)$$

The output is then given by: $Y(s) = {}^{11}/_{3}\,X_1(s) - {}^{13}/_{2}\,X_2(s) + {}^{35}/_{6}\,X_2(s)$. Figure 2.4 gives the Laplace block diagram.

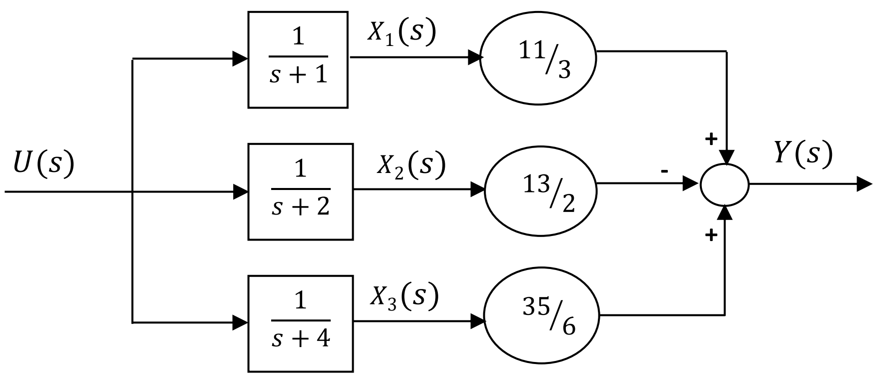

Figure 2.4

Considering $X_2(s) = \frac{1}{s+2}U(s)$. The expression can be written in the form: $(s+2)X_2(s) = U(s)$

$$sX_2(s) + 2X_2(s) = U(s)$$

In time domain: $\dot{x_2}(t) + 2x_2(t) = u(t)$

$$\dot{x_2}(t) = -2x_2(t) + u(t)$$

The other states can be similarly obtained and are:

$$\dot{x_1}(t) = -x_1(t) + u(t)$$
$$\dot{x_3}(t) = -4x_3(t) + u(t)$$

This gives the following state space system:

$$\bar{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \bar{C} = \begin{bmatrix} {}^{11}/_{3} & {}^{13}/_{2} & {}^{35}/_{6} \end{bmatrix}$$

Note this is also called the diagonal canonical form.

Similarity transform approach

Using the similarity transformation equations discussed in the last lecture the diagonal form can be found using the following approach.

1. Find the eigenvalues of the system $\lambda_1 \cdots \lambda_n$.
2. Find the corresponding eigenvectors $\bar{x}_1 \cdots \bar{x}_n$.
3. Find the similarity transformation matrix $\bar{P}$, where $\bar{P}^{-1}$ is the augmented matrix of the distinct eigenvectors:
   $\bar{P}^{-1} = [\bar{x}_1 \cdots \bar{x}_n]$ or $\bar{P} = [\bar{x}_1 \cdots \bar{x}_n]^{-1}$
4. Determine the transformed state matrices:
   $\bar{A}_D = \bar{P}\bar{A}\bar{P}^{-1}, \quad \bar{B}_D = \bar{P}\bar{B}, \qquad \bar{C}_D = \bar{C}\bar{P}^{-1}$

This approach is shown in the following example:

Example 2.4

Given the TF $G(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s^2+3s+2}$

In controllable canonical form this is:

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \ \bar{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \bar{C} = [1 \quad 0]$$

The four steps are as follows:

1. Eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -2$.
2. Suitable eigenvectors are $\bar{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\bar{x}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.
3. The transformation matrices are:
   $\bar{P}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ and $\bar{P} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$
4. The resulting state matrices are thus:

$$\bar{A}_D = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\bar{B}_D = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\bar{C}_D = [1 \quad 0] \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = [1 \quad -1]$$

Complex distinct eigenvalues

Complex eigenvalues can be treated in two ways. The first approach results in a diagonal $\bar{A}$ matrix with complex eigenvalues on the diagonal.

Complex diagonal approach

This method is essentially the same as for real distinct eigenvalues but complex values and coefficient multipliers are used. This approach could not be physically modelled using an analogue computer. The approach is shown in the following example.

Example 2.5

Given the TF $G(s) = \frac{6s^2+26s+8}{(s+2)(s^2+2s+10)} = \frac{-2}{s+2} + \frac{4+j}{s+1+j3} + \frac{4-j}{s+1-j3}$

Defining:

$$X_1(s) = \frac{1}{s+2} U(s)$$

$$X_2(s) = \frac{1}{s+1+j3} U(s)$$

$$X_3(s) = \frac{1}{s+1-j3} U(s)$$

Then $Y(s) = -2X_1(s) - (4+j)X_2(s) + (4-j)X_2(s)$

In matrix form:

$$\bar{A} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1-j3 & 0 \\ 0 & 0 & -1+j3 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \bar{C} = [-2 \quad (4+j) \quad (4-j)]$$

Real near diagonal approach

Alternatively, a real, near diagonal, form can be derived as shown in example 2.6. This approach can be modeled on an analogue computer.

Example 2.6

Given the TF $G(s) = \frac{(4s+2)(s^2+4s+8)}{(s+3)(s^2+2s+2)(s+1)} = \frac{-5}{s+1} + \frac{5}{s+3} + \frac{4s+12}{s^2+2s+2}$

Defining:

$$X_1(s) = \frac{1}{s+1}U(s)$$

$$X_2(s) = \frac{1}{s+3}U(s)$$

Treating the last TF in a controllable canonical form

$$sX_3(s) = X_4(s)$$
$$sX_4(s) = -2X_3(s) - 2X_4(s) + U(s)$$

The matrices are:

$$\bar{A} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -2 \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \bar{C} = [-5 \quad 5 \quad 12 \quad 4]$$

controllable canonical block

Repeated eigenvalues

Again, a near diagonal; form is used called a Jordan canonical form. The basic idea is shown by the following example.

Example 2.7

Given the TF $G(s) = \frac{3s^2+17s+8}{(s+4)(s+2)^2} = \frac{-3}{s+4} + \frac{6}{s+2} + \frac{-7}{(s+2)^2}$

Defining:

$$X_1(s) = \frac{1}{s+4}U(s)$$

$$X_3(s) = \frac{1}{s+2}U(s)$$

$$X_2(s) = \frac{1}{(s+2)^2}U(s) = \frac{1}{s+2} \times \frac{1}{s+2}U(s) = \frac{1}{s+2}X_3(s)$$

Figure 2.5 gives the Laplace form block diagram.

The state matrices are:

$$\bar{A} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \bar{C} = [-3 \quad -7 \quad 6]$$

Jordan blocks

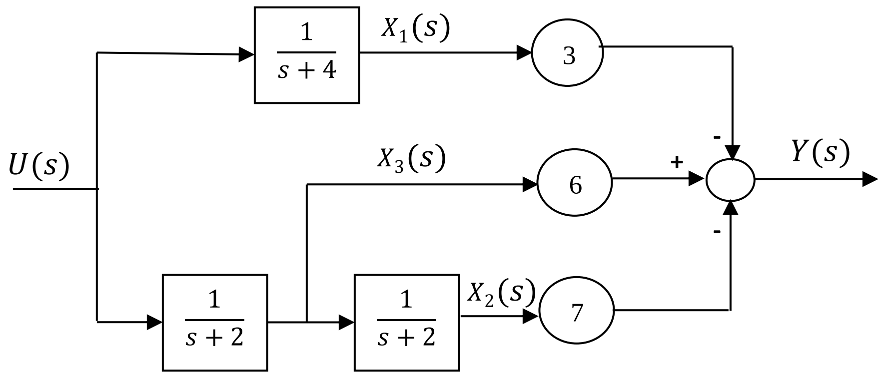

Figure 2.5

Basically, the Jordan canonical form (JCF) is a block diagonal matrix in which each *n* x *n* diagonal block matrix corresponds to *n*-multiplicity eigenvalues. The main diagonal elements of the JCF are the eigenvalues (with their multiplicities) and for the repeated eigenvalues the entry above the main diagonal is a 1 and the rest are 0s.

The idea can be extended to three or more repeated poles and multiple Jordan blocks can be included in the same system as shown in the following example.

Example 2.8

$$G(s) = \frac{-5}{s+2} + \frac{4}{(s+2)^2} + \frac{6}{s+3} + \frac{9}{s-4} - \frac{8}{(s-4)^2} + \frac{7}{(s-2)^3}$$

This gives the following Jordan canonical form of the state matrices:

$$\bar{A} = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \bar{C} = \begin{bmatrix} 4 & -5 & 6 & 7 & -8 & 9 \end{bmatrix}$$

Minimal realizations

If a transfer function has no common roots between the numerator and the denominator then it is a minimal system and its state space realization is a minimal realization. If, however, there are common roots then the realization is a non-minimal realization as shown in the following example.

Example 2.9

Consider the following transfer function:

$$G(s) = \frac{s^3+4s^2+7s+6}{s^4+5s^3+10s^2+11s+3}$$

In controllable canonical form this is:

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & -11 & -10 & -5 \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \bar{C} = \begin{bmatrix} 6 & 7 & 4 & 1 \end{bmatrix}$$

This is a non-minimal realization as $G(s)$ has the common factor $s^2 + 2s + 3$ in the numerator and denominator.

$$G(s) = \frac{(s+2)(s^2+2s+3)}{(s^2+3s+1)(s^2+2s+3)}$$

which can be realized as a minimal controllable canonical form thus:

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \bar{C} = \begin{bmatrix} 2 & 1 \end{bmatrix}$$

Notes

1. For any minimal system $G(s) = \frac{N(s)}{D(s)}$, an infinite number of non-minimal systems can be obtained simply by multiplying numerator and denominator by $P(s)$ thus:

   $G(s) = \frac{N(s)P(s)}{D(s)P(s)}$

2. The use of minimal systems rather than non-minimal systems is generally preferred.

Similarity transforms

The state space representation is not unique and in general infinitely many representations are possible. It can be shown that each state representation can be related to another by a simple linear transformation thus:

$\bar{z} = \bar{P}\bar{x}$ or $\bar{x} = \bar{P}^{-1}\bar{z}$

Where $\bar{P}$ is a non-singular matrix.
$\bar{x}$ is the old state vector.
$\bar{z}$ is the new state vector.

The new state matrices can be obtained as follows:

$$\dot{\bar{z}} = \bar{P}\dot{\bar{x}} \quad \text{or} \quad \dot{\bar{x}} = \bar{P}^{-1}\dot{\bar{z}}$$
$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}\bar{u}$$
$$\bar{P}^{-1}\dot{\bar{z}} = \bar{A}\bar{P}^{-1}\bar{z} + \bar{B}\bar{u}$$
$$\dot{\bar{z}} = \bar{P}\bar{A}\bar{P}^{-1}\bar{z} + \bar{P}\bar{B}\bar{u}$$

$$\bar{y} = \bar{C}\bar{x} + \bar{D}\bar{u}$$
$$\bar{y} = \bar{C}\bar{P}^{-1}\bar{z} + \bar{D}\bar{u}$$

Thus:

$$\bar{A}_z = \bar{P}\bar{A}\bar{P}^{-1}, \qquad \bar{B}_z = \bar{P}\bar{B}, \qquad \bar{C}_z = \bar{C}\bar{P}^{-1}, \qquad \bar{D}_z = \bar{D}$$

Notes

1. Some texts use the convention: $\bar{x} = \bar{P}\bar{z}$. Thus $\bar{P}$ and $\bar{P}^{-1}$ are reversed in the above equations.
2. An important characteristic of similarity transformations is that they do not alter the eigenvalues of the underlying system, i.e. the eigenvalues of $\bar{A}$ and $\bar{P}\bar{A}\bar{P}^{-1}$ are the same. The various state descriptions of the system that preserve the eigenvalues are known as canonical forms.

A simple example of the use of the similarity transformation is given in the following example:

Example 2.10

Given the system $\dot{\bar{x}} = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\bar{x} + \begin{bmatrix}0\\1\end{bmatrix}u$, $\quad y = \begin{bmatrix}1 & 0\end{bmatrix}\bar{x}$:

The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 2$.

Consider the transformation matrix $\bar{P} = \begin{bmatrix}0.5 & -0.5\\0.5 & 0.5\end{bmatrix}$:

The inverse is $\bar{P}^{-1} = \begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}$

$$\bar{A}_z = \bar{P}\bar{A}\bar{P}^{-1} = \begin{bmatrix}0.5 & -0.5\\0.5 & 0.5\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 2\end{bmatrix}$$

$$\bar{B}_z = \bar{P}\bar{B} = \begin{bmatrix}0.5 & -0.5\\0.5 & 0.5\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\bar{C}_z = \bar{C}\bar{P}^{-1} = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix} = \begin{bmatrix}0 & 2\end{bmatrix}$$

Hence the representation in the new states is: $\dot{\bar{z}} = \begin{bmatrix}0 & 0\\0 & 2\end{bmatrix}\bar{x} + \begin{bmatrix}0\\0\end{bmatrix}u$, $\quad y = \begin{bmatrix}1 & 0\end{bmatrix}\bar{z}$

Note the eigenvalues are still $\lambda_1 = 0$ and $\lambda_2 = 2$.

Chapter 2 Tutorial Questions

1. For each of the following systems determine the TF, draw the state block diagram and check for observability and controllability. In all cases $\overline{\mathbf{D}} = [\mathbf{0}]$.

a) $\overline{A} = [-1]\quad \overline{B} = [1]\quad \overline{C} = [2]$ $\qquad\left(\dfrac{2}{s+1};\ \textbf{C/O}\right)$

b) $\overline{A} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}\quad \overline{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\quad \overline{C} = [2 \quad -2]$ $\qquad\left(\dfrac{2}{s+1};\ \textbf{C/NO}\right)$

c) $\overline{A} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}\quad \overline{B} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}\quad \overline{C} = [1 \quad 0]$ $\qquad\left(\dfrac{2}{s+1};\ \textbf{NC/O}\right)$

d) $\overline{A} = \begin{bmatrix} -3 & 1 & 0 \\ -2 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\quad \overline{B} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}\quad \overline{C} = [1 \quad 0 \quad 0]$ $\qquad\left(\dfrac{2}{s+1};\ \textbf{NC/NO}\right)$

2. Show that the system:

$$\overline{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\quad \overline{B} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}\quad \overline{C} = [2 \quad 0]$$

is controllable iff (if and only if) $\lambda_1 \neq \lambda_2$. Also show that the system is never observable.

3. Show that the system:

$$\overline{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix}\quad \overline{B} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\quad \overline{C} = [2 \quad 0]$$

is controllable iff $b_2 \neq 0$. Also show that the system is always observable.

4. Find the controllable canonical form realizations of the following, and draw the state block diagrams:

(a) $G(s) = \dfrac{2s+10}{s^2+2}$ (b) $G(s) = \dfrac{3s^2 - s + 2}{s^3 + 2s^2 + 1}$ (c) $G(s) = \dfrac{2}{s^3 + s - 1}$

5. Find the observable canonical form realizations of the following, and draw the state block diagrams:

(a) $G(s) = \dfrac{5}{2s^2 + 4s + 3}$ (b) $G(s) = \dfrac{3}{2s^5 + 1}$ (c) $G(s) = \dfrac{4s^3 + 2s + 1}{2s^3 + 3s^2 + 2}$

6. Using partial fractions determine the diagonal form of state matrices for the following:

(a) $G(s) = \dfrac{-7s+4}{s^2 + 8s + 12}$ (b) $G(s) = \dfrac{2s^2 + 3s - 7}{(s+2)(s+8)(s+5)}$ (c) $G(s) = \dfrac{10}{s^3 + 8s^2 + 15s}$

7. For each of the following determine both diagonal form (complex) realizations and near diagonal form (real) realizations:

(a) $G(s) = \dfrac{4s}{s^2 + 2s + 7}$ (b) $G(s) = \dfrac{4}{(s+2)(s^2 + 2s + 17)}$

8. For the following transfer functions determine the Jordan canonical form state matrices:

(a) $G(s) = \dfrac{3s-1}{s^2 + 4s + 4}$ (b) $G(s) = \dfrac{s^3 - 4s^2 + s - 2}{(s+2)(s+3)^3}$

9. Transform the controllable canonical form state matrices from Q4 using the similarity transformation matrix:

$$\overline{P} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Do you recognise the transformed matrices?

10. Find the controllable canonical minimal realizations of the following transfer function:

$$G(s) = \frac{s^2 - 1}{s^3 + s^2 + s - 3}$$