Entanglement Detection with Weak Values

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1 Weak Values

Consider a quantum state $|\psi\rangle$. The probability of measuring it to be in a final state $|\phi\rangle$ is $P(\phi) = |\langle\phi|\psi\rangle|^2$. If we instead evolve $|\psi\rangle$ by the unitary $U = e^{-i\epsilon A}$ for some operator A and small $\epsilon \ll 1$, we find $P_\epsilon(\phi) = |\langle\phi|(1 - i\epsilon A)|\psi\rangle|^2 + \mathcal{O}(\epsilon^2)$. Call the probability of getting $|\phi\rangle$ after evolution $P_\epsilon(\phi)$. Again for small ϵ,

$$\begin{aligned}
P_\epsilon(\phi) &= (\langle\phi|\psi\rangle - i\epsilon\langle\phi|A|\psi\rangle)\left(\langle\phi|\psi\rangle^* + i\epsilon\langle\phi|A|\psi\rangle^*\right) + \mathcal{O}(\epsilon^2) \\
&= |\langle\phi|\psi\rangle|^2 + i\epsilon\langle\psi|\phi\rangle^*\langle\psi|A|\phi\rangle^* - i\epsilon\langle\psi|\phi\rangle\langle\psi|A|\phi\rangle + \mathcal{O}(\epsilon^2) \\
&= |\langle\phi|\psi\rangle|^2 + 2\epsilon\mathrm{Im}\,\langle\psi|\phi\rangle\langle\phi|A|\psi\rangle + \mathcal{O}(\epsilon^2)
\end{aligned} \tag{1}$$

The relative correction to $P(\phi)$ is then

$$\frac{P_\epsilon(\phi)}{P(\phi)} = 1 + 2\epsilon\mathrm{Im}\,\frac{\langle\psi|\phi\rangle\langle\phi|A|\psi\rangle}{\langle\psi|\phi\rangle\langle\phi|\psi\rangle} + \mathcal{O}(\epsilon^2) = 1 + 2\epsilon\mathrm{Im}\,{}_\phi\langle A\rangle_\psi + \mathcal{O}(\epsilon^2) \tag{2}$$

where we define the weak value

$${}_\phi\langle A\rangle_\psi \equiv \frac{\langle\phi|A|\psi\rangle}{\langle\phi|\psi\rangle}. \tag{3}$$

Weak values, first introduced in [7], are a somewhat controversial subject. By conditioning on a rare outcome (i.e. $\langle\phi|\psi\rangle \ll 1$), strange results can be acquired [5]. However, these sorts of behaviors can arise in classical probability as well, so there is discussion in the literature as to whether weak values are legitimately "quantum" [10, 7].

1.1 Weak Values of Density Matrices

The definition of weak values generalizes to density matrices as

$${}_k\langle A\rangle_\rho = \frac{\mathrm{tr}\left(|k\rangle\langle k|\,A\rho\right)}{\mathrm{tr}\left(|k\rangle\langle k|\,\rho\right)}. \tag{4}$$

Note that for a pure state ($\rho = |\psi\rangle\langle\psi|$), this reduces to (3). So, this definition is more general than the first.

The probability of finding ρ in a state $|\phi\rangle$ is

$$P(\phi) = \mathrm{tr}\left(|\phi\rangle\langle\phi|\,\rho\right). \tag{5}$$

Now applying a unitary evolution $U = e^{-i\epsilon H}$,

$$\rho \to U^\dagger\rho U = \rho + i\epsilon H\rho - i\epsilon\rho H + \mathcal{O}(\epsilon^2). \tag{6}$$

So, the probability of being in $|\phi\rangle$ after evolution is

$$P_\epsilon(\phi) = P(\phi) + i\epsilon\left[\mathrm{tr}\left(|\phi\rangle\langle\phi|\,H\rho\right) - \mathrm{tr}\left(|\phi\rangle\langle\phi|\,\rho H\right)\right]. \tag{7}$$

Writing the trace as a sum, $\mathrm{tr}\left(|\phi\rangle\langle\phi|\,H\rho\right) = \sum_{ij}\left(|\phi\rangle\langle\phi|\right)_{ij}(H\rho)_{ij}$. If we choose to do this sum in a basis where $|\phi\rangle$ is one of the basis vectors, we see that $\mathrm{tr}\left(|\phi\rangle\langle\phi|\,H\rho\right) = \langle\phi|\,H\rho\,|\phi\rangle$ and similarly $\mathrm{tr}\left(|\phi\rangle\langle\phi|\,\rho H\right) = \langle\phi|\,\rho H\,|\phi\rangle = \mathrm{tr}\left(|\phi\rangle\langle\phi|\,H\rho\right)^*$. So, we recover (3)

$$\frac{P_\epsilon(k)}{P(k)} = 1 - 2\epsilon\mathrm{Im}\,{}_k\langle H\rangle_\rho, \tag{8}$$

which now holds for both pure and mixed quantum states.

1.2 Tomography and Entanglement Detection

Quantum tomography is the process of completely characterizing of an unknown quantum state. In general, it is impossible to make such an identification on a single copy of a state, since multiple incompatible measurements are required. In addition to this, the number of required measurements scales exponentially with the number of degrees of freedom [8].

One approach to tomography, presented by C. Goswami et. al. in [11], uses weak measurements to gain information about multiple incompatible measurables, thereby completely identifying a pure or mixed 2-qubit state using a single measurement apparatus. There are some necessary caveats here: the weak measurement protocol requires two copies of the unknown state at once, and must be repeated many times to reduce uncertainty.

The "weak measurements" are taken by applying the operator $e^{-i\epsilon H}$ with a specifically chosen H to two copies of the state to be analyzed, $\rho \otimes \rho$. The weak values ${}_k\langle H \rangle_{\rho\otimes\rho}$ can be related to components of the 2-qubit density matrix ρ, and this provides sufficient information to solve for all of these components, in principle.

Rather than focus on tomography, most of [11] instead deals with detecting whether the mystery state ρ is entangled or not. I will also focus on entanglement detection, since the process of obtaining all components of the density matrix from the weak values is quite involved and doesn't add anything interesting to the quantum computation presented here.

2 2-qubit Entanglement Detection in Pure States

We can write a pure 2-qubit state as

$$|\psi_2\rangle = a\,|00\rangle + b\,|01\rangle + c\,|10\rangle + d\,|11\rangle\,. \tag{9}$$

For $|\psi_2\rangle$ to be a product state, these coefficients must satisfy $ad = bc$ [11]. The weak values of the operator

$$H = \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \otimes \sigma_x \tag{10}$$

presented in [11] on a 4-qubit state $|\Psi\rangle = |\psi_2\rangle \otimes |\psi_2\rangle$ provide us with an easy way to check this. It can be shown that

$${}_2\langle H \rangle_\rho = {}_4\langle H \rangle_\rho \tag{11}$$

is equivalent to $ad = bc$, where $|2\rangle = |0001\rangle$ and $|4\rangle = |0011\rangle$ are states in the 4-qubit computational basis. [1] So, to detect entanglement in pure 2-qubit states, we simply need to find these weak values and check if they are equal.

2.1 Finding the weak values

The probability of measuring our state $\psi = |\phi\rangle \otimes |\phi\rangle$ to be in an element of the 4-qubit computational basis $|u_k^{(4)}\rangle$ is

$$P(k) = \left| \langle u_k^{(4)}| \left(|\phi\rangle \otimes |\phi\rangle\right) \right|^2. \tag{12}$$

Now, imagine we turn on some interaction given by the "Hamiltonian" in (10) for $\epsilon \ll 1$. Let $P_\epsilon(k)$ be probability of measuring our state, after evolution, to be $|u_k^{(4)}\rangle$. From (2),

$$\mathrm{Im}\,{}_k\langle H \rangle_\psi = \frac{1}{2\epsilon}\left(\frac{P_\epsilon(k)}{P(k)} - 1\right). \tag{13}$$

So, by obtaining $P_\epsilon(k)$ and $P(k)$, we obtain the imaginary part of the weak values. Using the properties of this Hamiltonian, we can get both these probabilities from one measurement setup.

[1] I am using the conventions found in [11] in naming these states. It is perhaps more usual to call $|0001\rangle = |1\rangle$.

Since the first two-qubit subsystem is independent, the probability of measuring the first two qubits to be in a state $|u_l^{(2)}\rangle$ can be easily obtained from the results of the computation. Then, decomposing the four-qubit basis states as $|u_k^{(4)}\rangle = |u_l^{(2)}\rangle \otimes |u_m^{(2)}\rangle$,

$$P(k) = \left| \langle u_l^{(2)} | \psi_2 \rangle \langle u_m^{(2)} | \psi_2 \rangle \right|^2 = \left| \langle u_l^{(2)} | \psi_2 \rangle \right|^2 \left| \langle u_m^{(2)} | \psi_2 \rangle \right|^2 = P_2(l) P_2(m). \tag{14}$$

Since our results will contain all the 2-qubit probabilities $P_2(j)$ for $j = 1, \ldots, 4$, we will be able to find the 4-qubit probabilities $P(k)$ even with the weak interaction applied. Then, (13) gives us

$$\mathrm{Im}\, {}_k\langle H \rangle_\psi = \frac{1}{2\epsilon} \left(\frac{P_\epsilon(k)}{P_2(l) P_2(m)} - 1 \right) \quad \text{where } |u_k^{(4)}\rangle = |u_l^{(2)}\rangle \otimes |u_m^{(2)}\rangle . \tag{15}$$

2.1.1 Obtaining the real parts

We haven't found the complete weak values, only their imaginary parts. Inspired by the presentation in [6], for some $H = A \otimes B$,

$$\mathrm{Im}\, {}_k\langle H \rangle_\psi = \mathrm{Re}\, {}_k\langle A \rangle_\psi \mathrm{Im}\, {}_k\langle B \rangle_\psi + \mathrm{Im}\, {}_k\langle A \rangle_\psi \mathrm{Re}\, {}_k\langle B \rangle_\psi . \tag{16}$$

Our first setup had ${}_k\langle A \rangle_\psi = {}_k\langle \mathbb{I} \rangle_\psi = 1$, so it picked out the imaginary part of ${}_k\langle \mathbb{I} \otimes \sigma_x \rangle_\psi$.

If we added an ancillary qubit starting in the known state $|a\rangle$ and added an extra operator p to our unitary evolution so $U = e^{-\epsilon H \otimes p}$, we'd end up with new weak values

$${}_{k \otimes j}\langle H \otimes p \rangle_{\psi \otimes a} = {}_k\langle H \rangle_\psi \, {}_j\langle p \rangle_a \tag{17}$$

where the ancilla ends up in final state $|a\rangle$. Then, since we've found if we pick p and $|a\rangle$ to give a purely imaginary weak value ${}_j\langle p \rangle_a$,

$$\mathrm{Im}\, {}_{k \otimes j}\langle H \otimes p \rangle_{\psi \otimes a} = \mathrm{Re}\, {}_k\langle H \rangle_\psi \mathrm{Im}\, {}_j\langle p \rangle_a . \tag{18}$$

Now we just have to find a suitable operator and initial state.

Consider an ancilla starting in the state $|a\rangle = |+\rangle \equiv (|0\rangle + |1\rangle)/\sqrt{2}$. With this state and $p = \sigma_y$, we get purely imaginary weak values when measuring in the basis $\{|0\rangle, |1\rangle\}$: ${}_0\langle \sigma_y \rangle_+ = i$ and ${}_1\langle \sigma_y \rangle_+ = -i$. From (13), we get

$$\frac{P_\epsilon(k,s)}{P(k,s)} = 1 + 2\epsilon \mathrm{Im}\, \left({}_k\langle H \rangle_\psi \, {}_s\langle \sigma_y \rangle_+ \right) = 1 + 2\epsilon \mathrm{Re}\, \left({}_k\langle H \rangle_\psi \right) \mathrm{Im}\, \left({}_s\langle \sigma_y \rangle_+ \right) \tag{19}$$

for $s = 0$ or 1. Here, $P_\epsilon(k,s)$ is the probability of measuring the evolved state to be $|u_k^{(4)}\rangle \otimes |s\rangle$. Since the unevolved state $|+\rangle$ is equally likely to be measured $|0\rangle$ or $|1\rangle$, the unevolved probabilities are $P(k,0) = P(k,1) = \frac{1}{2} P(k)$. With the predicted weak values ${}_s\langle \sigma_y \rangle_+$, we get

$$\mathrm{Re}\, {}_k\langle H \rangle_\psi = \frac{1}{\epsilon} \left(\frac{1}{2} - \frac{P_\epsilon(k,0)}{P(k)} \right) = \frac{1}{\epsilon} \left(\frac{P_\epsilon(k,1)}{P(k)} - \frac{1}{2} \right) . \tag{20}$$

2.2 Analysis Protocol

Summarizing the procedure, we apply the quantum circuit in Fig. 1 with rotation operators $R_x = e^{-i\epsilon\sigma_x}$ and $R_y = e^{-i\epsilon\sigma_y}$.

After doing N total runs, we will have counts $N\epsilon(k,0)$ and $N_e(k,1)$ of the state $|u_k^{(4)}\rangle \otimes |s\rangle$ for $k = 1, \ldots, 16$ and $s = 0, 1$. of our system being measured in final states $|u_k^{(4)}\rangle \otimes |0\rangle$ and $|u_k^{(4)}\rangle \otimes |1\rangle$. Assuming we have enough data to smooth out statistical errors, we can use the counts instead of the probabilities.

First, we group the counts by their first two qubits to find the counts $N(l)$ for each 2-qubit state $|u_l^{(2)}\rangle$. Note these qubits were not effected by the evolution operator. By combining these, we find $N(k)$, the 4-qubit counts we would have measured without the weak interaction:

$$N(k) = \frac{N(l) N(m)}{N} \text{ where } |u_k^{(4)}\rangle = |u_l^{(2)}\rangle \otimes |u_m^{(2)}\rangle . \tag{21}$$

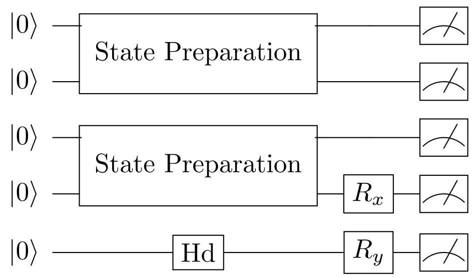

Fig. 1: Circuit for detecting entanglement in pure 2-qubit states. Hd is the Hadamard gate, $R_x = e^{-i\epsilon\sigma_x}$, and $R_y = e^{-i\epsilon\sigma_y}$.

The total counts for each state $|k\rangle$ after evolution are $N_\epsilon(k) = N_\epsilon(k,0) + N_\epsilon(k,1)$. So, we can find the imaginary part of ${}_k\langle H\rangle_\psi$ from Eq. (15):

$$\mathrm{Im}\,{}_k\langle H\rangle_\psi \approx \frac{1}{2\epsilon}\left(\frac{N_\epsilon(k)}{N(k)} - 1\right). \tag{22}$$

To get the real part, we replace the probabilities with counts in Eq. (20), getting

$$\mathrm{Re}\,{}_k\langle H\rangle_\psi \approx \frac{1}{\epsilon}\left(\frac{1}{2} - \frac{N_\epsilon(k,0)}{N(k)}\right) \approx \frac{1}{\epsilon}\left(\frac{N_\epsilon(k,0)}{N(k)} - \frac{1}{2}\right) \tag{23}$$

Since this equation gives us two estimates for the real part, I computed both. This gave me a check on the process, since a difference would indicate some problem in the computation.

If ${}_2\langle H\rangle_\psi$ and ${}_4\langle H\rangle_\psi$ were able to be found (i.e. if $\langle 2|\psi\rangle \neq 0$ and $\langle 4|\psi\rangle \neq 0$), we can then apply Eq. (11) to see if the 2-qubit state $|\psi\rangle_2$ is entangled or not. Unfortunately, in many cases these overlaps will be zero, as seen in my results. For these cases, we look at the 2-qubit states that received zero counts in the unevolved first two qubits and use that to set some of the coefficients in Eq. (9) to zero. That will give us enough information to determine whether or not the state is entangled.

2.3 Pure State Results

The pure state protocol was tested on a number of 2-qubit states, including $|00\rangle$, $|{+}{+}\rangle \equiv \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$, and the Bell state $|B\rangle \equiv \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$. Note that only the third of these states is entangled. The results of these computations are presented in the following tables. Note that tables 1 and 2 are simulated results, and 3 and 4 are from the ibmqx4 5-qubit quantum computer (available at [1]).

As a sanity check, I compared the simulated results for $|{+}{+}\rangle$ with values directly calculated in Mathematica. For all k in the computational basis, direct calculation shows ${}_k\langle H\rangle_{|++\rangle\otimes|++\rangle} = 1$. For both values of ϵ, the simulated weak values were fairly close to this. The $\epsilon = 0.1$ weak values had an average value of 0.97 and the $\epsilon = 0.5$ weak values had an average of 0.83, but the standard deviation of the $\epsilon = 0.1$ values were an order of magnitude higher, at $0.23 + 0.16i$ versus $0.05 + 0.05i$. This is because the higher value of ϵ introduced a larger offset from $\mathcal{O}(\epsilon^2)$ corrections, but also made it more likely that H was applied to the system, thereby averaging out sampling errors from substituting counts for probabilities.

The results computed using the ibmqx4 (seen in tables 3 and 4) are not consistent with predicted values. The average weak value for $|{+}{+}\rangle$ is 0.6 for both the $\epsilon = 0.1$ and $\epsilon = 0.5$ cases, with a standard deviation of 0.2. Because the deviations are qualitatively similar for both values of ϵ and only a few gates are used in this computation, I expect the main contribution of errors is from the the readout error. According to the IBM Quantum Experience website [1], the readout error rate for the ibmqx4 at the time of computation ranges from 2.3×10^{-2} and 7.1×10^{-1}, depending on qubit. For a more in-depth discussion of errors in this calculation, see the next section.

	$\|00\rangle$			$\|{++}\rangle$			$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$		
State	Counts	Re $_k\langle H\rangle_\psi$	Im $_k\langle H\rangle_\psi$	Counts	Re $_k\langle H\rangle_\psi$	Im $_k\langle H\rangle_\psi$	Counts	Re $_k\langle H\rangle_\psi$	Im $_k\langle H\rangle_\psi$
$\|0000\rangle$	8111	1.02	-0.05	502	0.77	-0.16	1990	0.94	0.04
$\|0001\rangle$	0	-	-	510	1.05	-0.04	0	-	-
$\|0010\rangle$	0	-	-	508	0.94	-0.05	0	-	-
$\|0011\rangle$	0	-	-	502	1.20	-0.13	2134	0.96	0.21
$\|0100\rangle$	0	-	-	502	0.68	-0.12	23	-	-
$\|0101\rangle$	0	-	-	513	1.23	0.03	0	-	-
$\|0110\rangle$	0	-	-	531	0.99	0.22	0	-	-
$\|0111\rangle$	0	-	-	537	0.91	0.25	14	-	-
$\|1000\rangle$	81	-	-	536	0.45	0.22	20	-	-
$\|1001\rangle$	0	-	-	489	0.93	-0.19	0	-	-
$\|1010\rangle$	0	-	-	511	1.25	0.04	0	-	-
$\|1011\rangle$	0	-	-	499	0.99	-0.11	22	-	-
$\|1100\rangle$	0	-	-	522	0.74	0.06	1990	1.10	-0.14
$\|1101\rangle$	0	-	-	531	1.32	0.20	0	-	-
$\|1110\rangle$	0	-	-	488	1.08	-0.21	0	-	-
$\|1111\rangle$	0	-	-	511	1.00	-0.01	1999	0.77	-0.29

Tab. 1: Simulated results of the pure-state analysis protocol with $\epsilon = 0.1$ and 8,192 total runs. The top row indicates the initial 2-qubit states. Missing weak values are due to zero overlap between initial and final states.

Assuming the dominant source of error in the simulation is of order ϵ^2, the protocol in Section 2.2 correctly identifies $|00\rangle$ as a product state and the Bell state as entangled, but only works for $|{++}\rangle$ in the $\epsilon = 0.5$ case. However, the ϵ^2 estimate for errors ignores problems from substituting counts for probabilities. The error for the kth state will scale roughly like $1/(\epsilon\sqrt{N_k})$[5]. With this error estimate, all the simulated results are consistent with the initial states.

With the real data, naively applying the protocol with an ϵ^2 tolerance failed to detect that $|00\rangle$ and $|{++}\rangle$ were product states, and the Bell state was found to be entangled only in the $\epsilon = 0.1$ case. As mentioned previously, these errors likely come from readout errors. To see this, consider just the $|00\rangle$, $\epsilon = 0.1$ data. The final states with the highest number of anomolous counts (counts not seen in the simulated runs) are $|0001\rangle$ and $|0010\rangle$. Those cases indicate a readout error on the 0th and 1st qubit, which have reported error rates of 6.2×10^{-2} and 7.1×10^{-2}, respectively. Given these error rates and a total of 8,192 runs, we should expect both of these qubits to read incorrectly around 500 times, more than actually observed.

3 2-qubit Entanglement Detection in Mixed States

Now, we consider a density matrix $\rho = \rho_2 \otimes \rho_2$ and use a different operator found in [11] for our weak values:

$$\begin{aligned} H =& \big(|00\rangle\langle 00| + |01\rangle\langle 01| \big) \otimes \mathbb{I} \otimes \sigma_x \\ &+ |10\rangle\langle 10| \otimes \sigma_x \otimes \mathbb{I} \\ &+ |11\rangle\langle 11| \otimes \sigma_x \otimes \sigma_x \\ \equiv& |00\rangle\langle 00| \otimes H_1 + |01\rangle\langle 01| \otimes H_1 + |10\rangle\langle 10| \otimes H_2 + |11\rangle\langle 11| \otimes H_3. \end{aligned} \tag{24}$$

Importantly, this Hamiltonian leaves the first two qubits in their original state. Thus, the methods in section 2.1 will still work to find the weak values.

	$\|00\rangle$			$\|{+}{+}\rangle$			$\frac{1}{\sqrt{2}}(\|00\rangle+\|11\rangle)$		
State	Counts	Re $_k\langle H\rangle_\psi$	Im $_k\langle H\rangle_\psi$	Counts	Re $_k\langle H\rangle_\psi$	Im $_k\langle H\rangle_\psi$	Counts	Re $_k\langle H\rangle_\psi$	Im $_k\langle H\rangle_\psi$
$\|0000\rangle$	6301	0.64	-0.23	488	0.75	-0.08	1631	0.64	-0.24
$\|0001\rangle$	0	-	-	507	0.82	-0.03	0	-	-
$\|0010\rangle$	0	-	-	518	0.82	-0.01	0	-	-
$\|0011\rangle$	0	-	-	498	0.76	-0.03	1574	0.64	-0.23
$\|0100\rangle$	0	-	-	511	0.80	-0.02	488	-	-
$\|0101\rangle$	0	-	-	504	0.80	-0.02	0	-	-
$\|0110\rangle$	0	-	-	482	0.82	-0.06	0	-	-
$\|0111\rangle$	0	-	-	527	0.85	0.05	481	-	-
$\|1000\rangle$	1891	-	-	536	0.86	0.03	480	-	-
$\|1001\rangle$	0	-	-	553	0.90	0.08	0	-	-
$\|1010\rangle$	0	-	-	519	0.86	0.02	0	-	-
$\|1011\rangle$	0	-	-	487	0.81	-0.03	430	-	-
$\|1100\rangle$	0	-	-	554	0.92	0.08	1591	0.65	-0.22
$\|1101\rangle$	0	-	-	490	0.82	-0.03	0	-	-
$\|1110\rangle$	0	-	-	523	0.91	0.05	0	-	-
$\|1111\rangle$	0	-	-	495	0.86	0.01	1517	0.66	-0.22

Tab. 2: Simulated results of the pure-state analysis protocol with $\epsilon = 0.5$ and 8,192 runs.

	$\|00\rangle$			$\|{+}{+}\rangle$			$\frac{1}{\sqrt{2}}(\|00\rangle+\|11\rangle)$		
State	Counts	Re $_k\langle H\rangle_\psi$	Im $_k\langle H\rangle_\psi$	Counts	Re $_k\langle H\rangle_\psi$	Im $_k\langle H\rangle_\psi$	Counts	Re $_k\langle H\rangle_\psi$	Im $_k\langle H\rangle_\psi$
$\|0000\rangle$	7279	0.24	0.01	684	0.57	-0.1	1212	0.43	-1.78
$\|0001\rangle$	249	0.26	-0.03	516	0.39	-0.22	156	-0.33	-1.83
$\|0010\rangle$	200	-0.25	0.03	515	0.57	-0.09	105	0.35	-2.16
$\|0011\rangle$	4	0.0	0.31	456	0.57	-0.1	1047	0.65	-1.76
$\|0100\rangle$	110	0.48	-2.8	481	0.41	-0.27	341	0.26	1.93
$\|0101\rangle$	6	1.16	-1.53	427	0.42	-0.26	55	1.71	3.56
$\|0110\rangle$	2	1.46	-3.54	329	0.42	-0.34	35	1.03	2.24
$\|0111\rangle$	0	0.0	-5.0	290	0.47	-0.34	260	0.33	1.17
$\|1000\rangle$	267	0.63	1.72	608	0.63	0.07	337	1.59	4.1
$\|1001\rangle$	10	0.0	2.3	569	0.64	0.15	36	1.24	2.45
$\|1010\rangle$	9	2.76	3.28	442	0.68	0.04	47	5.22	7.92
$\|1011\rangle$	0	0.0	-5.0	441	0.77	0.17	285	1.61	3.99
$\|1100\rangle$	55	-6.64	68.04	718	0.79	0.42	2039	0.56	1.32
$\|1101\rangle$	1	38.5	33.5	661	0.96	0.5	266	0.47	1.31
$\|1110\rangle$	0	0.0	-5.0	584	0.95	0.54	199	1.67	1.28
$\|1111\rangle$	0	0.0	-5.0	471	0.81	0.4	1772	0.71	1.41

Tab. 3: Results from the ibmqx4 with $\epsilon = 0.1$ and 8,192 runs.

	$\vert 00\rangle$			$\vert ++\rangle$			$\frac{1}{\sqrt{2}}(\vert 00\rangle + \vert 11\rangle)$		
State	Counts	$\mathrm{Re}\,{}_k\langle H\rangle_\psi$	$\mathrm{Im}\,{}_k\langle H\rangle_\psi$	Counts	$\mathrm{Re}\,{}_k\langle H\rangle_\psi$	$\mathrm{Im}\,{}_k\langle H\rangle_\psi$	Counts	$\mathrm{Re}\,{}_k\langle H\rangle_\psi$	$\mathrm{Im}\,{}_k\langle H\rangle_\psi$
$\vert 0000\rangle$	5667	0.52	-0.22	662	0.57	-0.07	880	0.34	-0.51
$\vert 0001\rangle$	211	0.41	-0.23	564	0.57	-0.1	110	0.26	-0.56
$\vert 0010\rangle$	147	0.37	-0.2	557	0.64	-0.02	82	0.31	-0.59
$\vert 0011\rangle$	4	0.0	-0.15	493	0.57	-0.03	774	0.34	-0.51
$\vert 0100\rangle$	84	0.24	-0.69	405	0.36	-0.36	543	1.5	1.15
$\vert 0101\rangle$	5	0.48	-0.52	376	0.42	-0.32	95	1.61	1.69
$\vert 0110\rangle$	1	-0.14	-0.86	309	0.39	-0.38	62	1.43	1.22
$\vert 0111\rangle$	0	0.0	-1.0	319	0.46	-0.29	504	1.45	1.26
$\vert 1000\rangle$	1839	6.48	8.98	565	0.56	-0.0	599	2.19	2.0
$\vert 1001\rangle$	74	8.33	9.63	510	0.63	0.02	81	1.75	1.9
$\vert 1010\rangle$	46	8.1	8.81	504	0.75	0.12	82	2.71	2.7
$\vert 1011\rangle$	1	8.36	7.36	438	0.68	0.08	583	2.47	2.31
$\vert 1100\rangle$	110	18.72	22.41	784	0.93	0.54	1816	0.79	0.14
$\vert 1101\rangle$	1	5.63	4.63	681	0.93	0.52	252	0.66	0.13
$\vert 1110\rangle$	2	16.72	15.72	554	0.86	0.37	200	0.82	0.13
$\vert 1111\rangle$	0	0.0	-1.0	471	0.88	0.3	1529	0.77	0.09

Tab. 4: Results from the ibmqx4 with $\epsilon = 0.5$ and 8,192 runs.

3.1 Implementation

The circuit to implement evolution with Eq. (24) is presented in [11]. Figure 2 presents this circuit with the addition of a fifth ancillary qubit for measuring the real parts of the weak values. In order to create the multi-qubit operations present in the circuit, I used methods from [9]. For an example of this, see Fig. 3 for my implementation of a rotation controlled by three qubits.

To simulate a mixed quantum state, we could do runs with multiple combinations of states in the 2-qubit subsystems. For instance, running the states $|00\rangle \otimes |00\rangle$, $|00\rangle \otimes |B\rangle$, $|B\rangle \otimes |00\rangle$, and $|B\rangle \otimes |B\rangle$ with equal runs for each, where

$$|B\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{25}$$

is equivalent to a 2-qubit density matrix

$$\rho_2 = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|B\rangle\langle B|. \tag{26}$$

After decomposing the multi-qubit operations into 2-qubit operations, implementing the circuit in Fig. 2 becomes very difficult. This is because not all qubits can be paired. In principle, using a series of C-NOT gates, this could be worked around, but the circuit would get very unwieldy in this case. To avoid this difficulty, I implemented this circuit using the qiskit open-source software package [3]. This allows gate-level manipulation of quantum circuits, and automatically compiles the circuit into a format that runs on the ibmqx4 [1]. My code, written in Python, for creating these circuits and analyzing the results can be found at [2].

3.2 Mixed State Results

Simulated results for the weak values of the density matrix (26) are presented in table 5. These results do not agree with those calculated directly in Mathematica due to counts for states (such as $|1100\rangle$) that receive no counts in the direct calculation but significant counts in the simulation. This difference is consistent with the order of magnitude off the reported multi-qubit error from [1]. The QASM simulator used in [3] does simulate these errors but not readout errors by default, so this accounts for the difference.

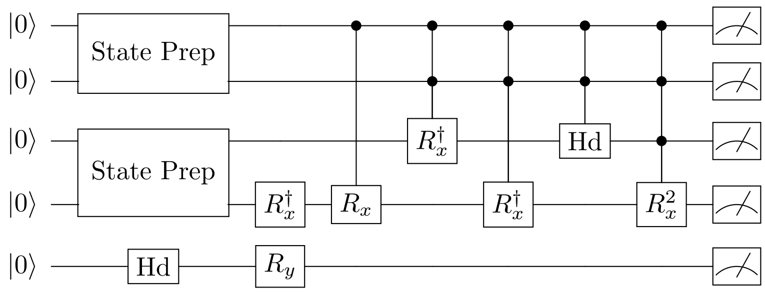

Fig. 2: Circuit for detecting entanglement in mixed states from [11] with an ancillary fifth qubit. Here, $R_x = e^{i\epsilon\sigma_x}$, $R_y = e^{i\epsilon\sigma_y}$, $R_x^\dagger = R_x^{-1} = e^{-i\epsilon\sigma_x}$, and $R_x^2 = e^{2i\epsilon\sigma_x}$.

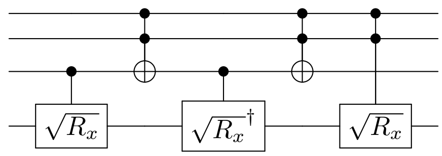

Fig. 3: Realization of the rotation gate controlled by 3 qubits from the previous diagram. Here, $R_x = e^{i\epsilon\sigma_x/2}$.

The computed results (also in table 5) were computed more for completeness than out of any expectation they would be useable. It is difficult to relate them to the simulated or directly calculated results. Considering the considerable number of high-error multi-qubit gates used along with the readout error, this was the expected outcome.

	ibmqx4			Simulation		
State	Counts	Re $_k\langle H\rangle_\psi$	Im $_k\langle H\rangle_\psi$	Counts	Re $_k\langle H\rangle_\psi$	Im $_k\langle H\rangle_\psi$
$\|0000\rangle$	757	-0.42	0.26	7213	1.6	0.9
$\|0001\rangle$	775	-0.49	0.38	2289	0.8	-0.05
$\|0010\rangle$	527	-0.31	-0.03	2357	0.82	-0.03
$\|0011\rangle$	490	-0.3	-0.04	2424	0.8	-0.04
$\|0100\rangle$	662	-0.4	0.18	1109	0.39	-0.54
$\|0101\rangle$	619	-0.41	0.17	1602	0.88	0.05
$\|0110\rangle$	827	-0.56	0.62	1641	0.92	0.07
$\|0111\rangle$	777	-0.6	0.62	2251	1.2	0.41
$\|1000\rangle$	427	-0.26	-0.22	1911	0.66	-0.21
$\|1001\rangle$	374	-0.3	-0.27	1550	0.85	0.01
$\|1010\rangle$	262	-0.17	-0.47	1519	0.83	-0.02
$\|1011\rangle$	249	-0.1	-0.46	1249	0.65	-0.23
$\|1100\rangle$	369	-0.22	-0.27	929	0.31	-0.63
$\|1101\rangle$	315	-0.16	-0.34	1622	0.86	0.01
$\|1110\rangle$	397	-0.24	-0.14	1598	0.82	-0.01
$\|1111\rangle$	365	-0.28	-0.15	1504	0.74	-0.11

Tab. 5: Results for the weak values of the density matrix from (26) with $\epsilon = 0.5$. The simulated results used 8,192 runs for each state combination for a total of 32,678 runs. The ibmqx4 results used 2,048 runs for each combination for a total of 8,192 runs.

4 Conclusions

My simulated results confirm the claims made in [11], that weak values can be used to fully identify 2-qubit states with a single measurement setup. Unfortunately, the error rate of the ibmqx4 computer is currently too high to successfully implement the protocol, even for the pure state version that avoids multi-qubit operations.

Even allowing for a more accurate quantum computer, the requirements of this method of tomography seem better suited to optical experiments. In an optical experiment, it should be possible to use a beam splitter to acquire the two "copies" of a quantum state.[2] More generally, the concept of weak values is almost exclusively described in a quantum optics setting [5, 7].

However, once we've moved away from the setting of a quantum computer with a small number of qubits, other methods for detecting entanglement may become more advantageous. The protocol in [11] was developed from an earlier once, in [4], that requires four copies of a 2-qubit state for entanglement detection, while avoiding weak values entirely. This would be impossible to implement on the ibmqx4, but creating additional copies in an optical experiment may be easier than measuring the weak values.

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[2] Of course, these wouldn't actually be copies, but could still be made to obey the same probability distribution and therefore be described by the same density matrix.