# Supplemental Material: Turbulence and turbulent pattern formation in a minimal model for active fluids

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## NUMERICAL SIMULATIONS

The numerical simulations are performed with a standard pseudospectral scheme for the vorticity formulation of Eq. (1):

$$\partial_t \omega + \lambda \boldsymbol{u} \cdot \boldsymbol{\nabla} \omega = -(1+\Delta)^2 \omega - \alpha \omega - \beta\, \boldsymbol{\nabla} \times \left(\boldsymbol{u}^2\, \boldsymbol{u}\right) . \tag{S1}$$

Here $\omega(\boldsymbol{x},t) = \boldsymbol{\nabla} \times \boldsymbol{u}(\boldsymbol{x},t)$ is the pseudo-scalar vorticity. In turn, the velocity is obtained from the vorticity by Biot-Savart's law. An equation for the spatially constant velocity contribution $u_0$, which is not contained in the vorticity field, is integrated simultaneously. Time stepping is performed by means of a second-order Runge-Kutta scheme, in which the linear term is treated with an integrating factor. To account for the cubic nonlinearity, the pseudospectral scheme is fully dealiased with a 1/2 dealiasing. Small-scale, low-amplitude random initial conditions are chosen for all simulations. The parameters for the various simulations are summarized in Table I.

## CLASSICAL PATTERN FORMATION – SQUARE LATTICE STATE

For $\lambda = 0$, Eq. (1) follows a gradient dynamics constrained to the sub-space of incompressible velocity fields, $\partial_t \boldsymbol{u} = -\boldsymbol{\nabla} p - \delta \mathcal{L}[\boldsymbol{u}]/\delta \boldsymbol{u}$. Here, all terms except the pressure gradient can be combined into the Lyapunov functional (see also [24, 30])

$$\mathcal{L}[\boldsymbol{u}] = \int d\boldsymbol{x} \left[(\Delta \boldsymbol{u} + \boldsymbol{u})^2/2 + \alpha \boldsymbol{u}^2/2 + \beta \left(\boldsymbol{u}^2\right)^2/4\right]; \tag{S2}$$

the pressure gradient term is a Lagrange multiplier to ensure $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$. As a result of the potential dynamics, a stationary pattern emerges; its wave number $k_c = 1$ is straightforwardly computed by linear stability analysis.

The pattern forming state can be conveniently analyzed in the vorticity formulation Eq. (S1). Motivated by our numerical observations, we investigate a lattice state of the form

$$\omega(\boldsymbol{x},t) = \zeta_1(t) \exp[\mathrm{i}\boldsymbol{k}_1 \cdot \boldsymbol{x}] + \zeta_2(t) \exp[\mathrm{i}\boldsymbol{k}_2 \cdot \boldsymbol{x}] + \text{c.c.} \tag{S3}$$

with $|\boldsymbol{k}_1| = |\boldsymbol{k}_2| = k_c = 1$, $\boldsymbol{k}_1 \cdot \boldsymbol{k}_2/k_c^2 = \cos\varphi$ and amplitudes $\zeta_1$ and $\zeta_2$, which we can choose as real due to translational invariance. Combining this ansatz with the full nonlinear equations, amplitude equations can be straightforwardly derived, which in leading order take the form

$$\dot{\zeta}_1 = -\alpha \zeta_1 - \frac{\beta}{k_c^2} \left(3\zeta_1^3 + 2\left[1 + 2\cos^2\varphi\right] \zeta_2^2 \zeta_1\right) \tag{S4}$$

$$\dot{\zeta}_2 = -\alpha \zeta_2 - \frac{\beta}{k_c^2} \left(3\zeta_2^3 + 2\left[1 + 2\cos^2\varphi\right] \zeta_1^2 \zeta_2\right) . \tag{S5}$$

These equations can be further analyzed by means of a linear stability analysis. The analysis shows that the ground state $\zeta_1 = \zeta_2 = 0$ is linearly unstable for $\alpha < 0$ with growth rates $\lambda_{1,2}^{(0)} = -\alpha$. For a single-stripe pattern with $\zeta_2 = 0$ the amplitude equations yield $\zeta_1 = \sqrt{-\alpha k_c^2/(3\beta)}$ as a stationary solution. A linear stability analysis with small perturbation of the single-stripe pattern yields growth rates $\lambda_1^{(1)} = 2\alpha$ and $\lambda_2^{(1)} = \alpha\left[4\cos^2\varphi - 1\right]/3$ (see Fig. S1). As expected, small perturbations in the direction of the single stripe are damped for $\alpha < 0$. The emergence of a second stripe, however, is linearly unstable for a small wave number band around $\varphi = \pi/2$, which gives a first hint at the emergence of a square lattice. This can be further corroborated with a linear stability analysis of a lattice state with $\zeta_1 = \zeta_2$, for which the stationary solution $\zeta_1 = \zeta_2 = \sqrt{\frac{-\alpha k_c^2}{\beta[5+4\cos^2\varphi]}}$ is readily obtained from the amplitude equations.

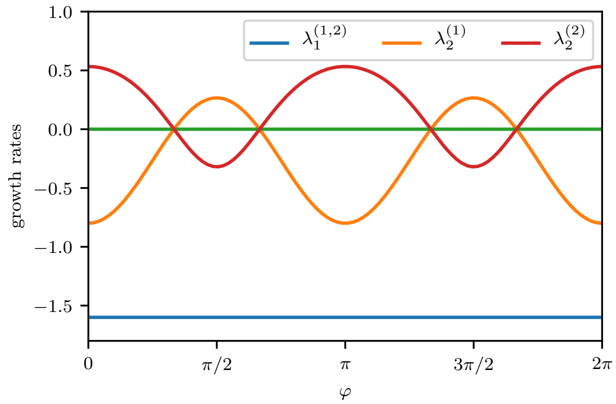


FIG. S1: Growth rates of the linear stability analysis for $\alpha = -0.8$. The eigenvalues $\lambda_1^{(1,2)}$ correspond to the stable eigenvalues of the single- and two-stripe pattern, respectively. Starting from a single-stripe pattern, $\lambda_2^{(1)}$ indicates that a second stripe in a wave-number band around $\pi/2$ can be excited. The eigenvalue $\lambda_2^{(2)}$ shows that the square lattice state is linearly stable.

Linear stability analysis, assuming small perturbations in both amplitudes, yields $\lambda_1^{(2)} = 2\alpha$ and $\lambda_2^{(2)} = 2\alpha\frac{1-4\cos^2\varphi}{5+4\cos^2\varphi}$. For $\alpha < 0$ a range of lattice states is linearly stable with the maximum stability reached when $\varphi = \pi/2$ (see Fig. S1). This analysis renders a clear picture of the emergence of square lattice states for $\alpha < 0$: the single-stripe pattern is unstable with respect to the emergence of a two-stripe lattice with the maximum growth rate at $\varphi = \pi/2$. The resulting square lattice state with $\zeta_1 = \zeta_2 = \sqrt{-\alpha k_c^2/(5\beta)}$ then is linearly stable. Minimizing the Lyapunov functional for a square lattice with respect to the amplitude yields the same result.

## ACTIVE TURBULENCE – EDQNM CLOSURE

Developing a statistical theory for the turbulent phase of active fluids requires assumptions about the hierarchy of moments. Indeed, the equation for the covariance tensor Eq. (2), or equivalently, for the energy spectrum Eq. (3), is unclosed due to the presence of the higher-order velocity correlations stemming from the nonlinear terms.

To set our theoretical development into context, we start with re-iterating the classical closure attempts in the context of the active turbulence equations. The classical closure theory is presented in much more detail in [36–38]. A Gaussian approximation is the simplest first choice to close the system, in particular in a random system like turbulence. Under this cumulant discard hypothesis, one can factorize higher-order moments in terms of corresponding second-order moments. This allows us to close the fourth-order term in Eq. (2) as described in the main text. However, the third-order correlations in Eq. (3) vanish under such a Gaussian approximation. The third-order correlations are responsible for the energy transfer between scales and hence are essential for the dynamics. A logical step towards closure is to write then the equations for the triple correlation, which in Fourier space take the form

$$\left[\partial_t + \tilde{L}(k) + \tilde{L}(p) + \tilde{L}(q)\right] \langle \hat{u}(\boldsymbol{k})\hat{u}(\boldsymbol{p})\hat{u}(\boldsymbol{q})\rangle = F[\lambda\langle\hat{u}\hat{u}\hat{u}\hat{u}\rangle, \beta\langle\hat{u}\hat{u}\hat{u}\hat{u}\hat{u}\rangle]\,. \tag{S6}$$

In favor of a lighter notation we write these equations rather schematically, suppressing tensorial notation. Here, $\tilde{L}(k) = (1-k^2)^2 + \alpha$, and the functional $F$ captures the contributions due to the pressure term as well as the fourth- and the fifth-order correlations which appear due to the advective and cubic nonlinearities in Eq. (1), respectively. To close this system on the level of the quadruple and fifth-order correlations, one can now assume a Gaussian factorization of these higher-order moments as the next simplest closure. This eliminates the fifth-order correlations and the fourth-order correlation can now be written in terms of second-order correlations resulting in

$$\left[\partial_t + \tilde{L}(k) + \tilde{L}(p) + \tilde{L}(q)\right] \langle \hat{u}(\boldsymbol{k})\hat{u}(\boldsymbol{p})\hat{u}(\boldsymbol{q})\rangle = \lambda\Sigma\langle\hat{u}\hat{u}\rangle\langle\hat{u}\hat{u}\rangle \tag{S7}$$

so that the Eq. (3) for the energy spectrum $E(k)$ is now closed. This procedure is known as the quasi-normal approximation [49, 50]. This classical approximation for the energy transfer term has been shown to fail spectacularly for hydrodynamic turbulence already in the 1960s [51], leading to a realizability problem by the development of negative energies, since the omission of the cumulants leads to an overprediction of the transfer term.

To remedy this shortcoming, more sophisticated manners of closure were proposed, in particular by Kraichnan [52, 53], using renormalized perturbation theories. The simplest successful derivative of these theories is the eddy-damped quasi-normal Markovian model [54]. For an extensive account on the matter, we refer to [36–38]. Here we adopt this framework to formulate a statistical theory for active turbulence. The eddy-damped quasi-normal Markovian model generalizes the classical quasi-normal approximation by modeling the effect of the missing fourth-order cumulants as $\lambda(\langle\hat{u}\hat{u}\hat{u}\hat{u}\rangle - \Sigma\langle\hat{u}\hat{u}\rangle\langle\hat{u}\hat{u}\rangle) = -\mu_{kpq}\langle\hat{u}(\boldsymbol{k})\hat{u}(\boldsymbol{p})\hat{u}(\boldsymbol{q})\rangle$ where the damping term $\mu_{kpq} = \mu_k + \mu_p + \mu_q$ is defined through the contributions

$$\mu_k = \lambda\gamma \left( \int_0^k s^2 E(s,t) ds \right)^{1/2} . \tag{S8}$$

Here, $\gamma$ is a free parameter which quantifies the strength of the eddy damping. We can then combine the linear terms to define $\eta_k = \mu_k + |\tilde{L}(k)|$ as the net damping. The damping of the triple correlation corresponds to the Lagrangian decorrelation of the Fourier modes [55], and both the positive and the negative linear terms will lead to an effective decorrelation. Consequently the effect of $\tilde{L}(k)$ in damping should be strictly positive, and hence we take the absolute value of $\tilde{L}(k)$. With this assumption, the evolution equation for the triple correlation can be written as

$$\left[\partial_t + \eta_{kpq}\right] \langle\hat{u}(\boldsymbol{k})\hat{u}(\boldsymbol{p})\hat{u}(\boldsymbol{q})\rangle = \lambda\Sigma\langle\hat{u}\hat{u}\rangle\langle\hat{u}\hat{u}\rangle, \tag{S9}$$

where $\eta_{kpq} = \eta_k + \eta_p + \eta_q$. If we neglect the time variation in $\mu_k$ and $\langle\hat{u}\hat{u}\rangle\langle\hat{u}\hat{u}\rangle$, the above expression can be integrated in time, resulting in the following expression for the triple correlation in terms of the energy spectrum:

$$\langle\hat{u}(\boldsymbol{k})\hat{u}(\boldsymbol{p})\hat{u}(\boldsymbol{q})\rangle(t) = \frac{1 - e^{-\eta_{kpq} t}}{\eta_{kpq}} \lambda\Sigma\langle\hat{u}\hat{u}\rangle\langle\hat{u}\hat{u}\rangle. \tag{S10}$$

For large time scales, $e^{-\eta_{kpq} t}$ can be neglected, and $1/\eta_{kpq}$ defines a characteristic time. This timescale is associated with the Lagrangian correlation time of the fluid particles (see for instance [55] for a discussion). The second-order correlations are associated with the energy spectrum, hence Eq. (3) and Eq. (S10) together result in a closed set of equations for the evolution of the energy spectrum. Owing to the isotropy of the velocity field, $T(k)$ in Eq. (3) can be calculated from $\langle\hat{u}_l(\boldsymbol{k})\hat{u}_m(\boldsymbol{p})\hat{u}_n(\boldsymbol{q})\rangle \equiv T_{lmn}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})$ (in full tensorial notation) as

$$T(k) = \pi k P_{lmn}(\boldsymbol{k}) \int \mathrm{Im}\left[T_{lmn}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})\right] d\boldsymbol{p} d\boldsymbol{q} \, , \tag{S11}$$

where $P_{lmn}(\boldsymbol{k}) = k_n(\delta_{lm} - k_l k_m/k^2) + k_m(\delta_{ln} - k_l k_n/k^2)$ and Im stands for the imaginary part. The integration is performed over all triads $\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}$ where $\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q} = 0$. The final expression for $T(k,t)$ can then be written as [56]

$$T(k,t) = -\frac{4}{\pi} \iint_\Delta \frac{\lambda^2}{\eta_{kpq}} \frac{xy - z + 2z^3}{\sqrt{1-x^2}} \left[k^2 p E(p,t) E(q,t) - k p^2 E(q,t) E(k,t)\right] \frac{dpdq}{pq} \, . \tag{S12}$$

Here $\Delta$ is a band in $p,q$-space so that the three wave numbers $k,p,q$ form the sides of a triangle. $x,y,z$ are the cosines of the angles opposite to the sides $k,p,q$ in this triangle. Comparing Eq. (S12) with Eq. (5), we obtain $a(k,p,q) = -\frac{4}{\pi}\frac{xy-z+2z^3}{\sqrt{1-x^2}}\frac{k^2}{q}$ and $b(k,p,q) = \frac{4}{\pi}\frac{xy-z+2z^3}{\sqrt{1-x^2}}\frac{kp}{q}$.

To generate the results presented in the main text, this closed set of equations for the energy spectrum function is integrated numerically. Computations are carried out on a logarithmically spaced mesh on the interval $0.025 \leq k \leq 25$ using 300 modes. All results are obtained, using $\gamma = 0.55$, after the spectrum reached a steady state.