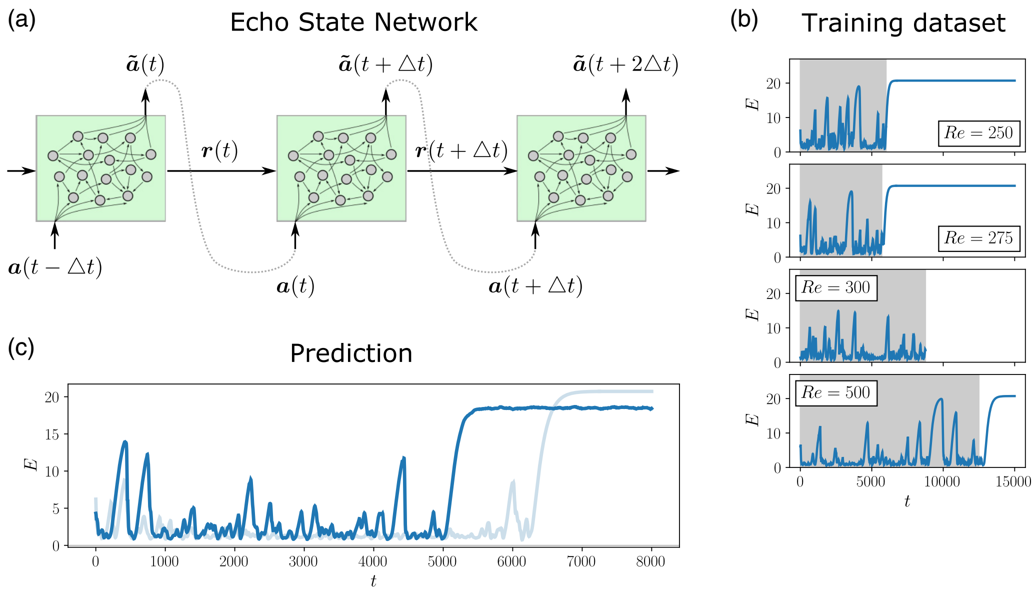

FIG. 1. (a) Schematic of an echo state network. To make a prediction $\tilde{\boldsymbol{u}}(t+\Delta t)$, the flow state $\boldsymbol{u}(t)$ at the previous time step and reservoir state $\boldsymbol{r}(t)$ are passed to the randomly generated reservoir (green box) where they are nonlinearly transformed to yield the prediction. (b) Training time series for Reynolds numbers Re = 250, 275, 300, 500 obtained by time integration of the MFE model and shown in the form of the time evolution of the flow kinetic energy. Only shadowed parts were used for training. (c) Flow prediction made by the echo state network trained at Re = 300 (bright blue curve) and a representative turbulent trajectory of the MFE model computed at the same Reynolds number (light blue curve).

training. As an example of a transitional flow, we consider a paradigm model of plane Couette flow, i.e., the viscous flow between two parallel walls moving in opposite directions at constant and equal velocities. This model is a representative of a wide class of nonlinear dynamical systems exhibiting finite-amplitude instabilities and spontaneous decay of chaotic dynamics. As such, we expect our conclusions to be transferable to a variety of systems with similar dynamical features.

II. MODEL

In plane Couette flow, the velocity field at position $\boldsymbol{x}$ and time t, $\boldsymbol{u}(\boldsymbol{x}, t)$, is generally solved for via the integration of the Navier-Stokes equation together with the incompressibility condition, no-slip boundary conditions in the wall-normal direction and spatial periodicity conditions in the streamwise x and spanwise z directions. This set of equations can be reduced to the Moehlis-Faisst-Eckhardt (MFE) model [21] by replacing plane Couette flow with a sinusoidal shear flow, known as the Waleffe flow [22,23], and truncating to nine Fourier-based modes, $\boldsymbol{u}_j(\boldsymbol{x})$, as listed in Appendix A. The fluid velocity can reconstructed via

$$\boldsymbol{u}(\boldsymbol{x}, t) = \sum_{j=1}^{9} a_j(t)\boldsymbol{u}_j(\boldsymbol{x}), \tag{1}$$

where the time-dependent amplitudes are $\boldsymbol{a}(t) = [a_1(t), \dots, a_9(t)]$. The nine-dimensional ordinary differential equation system of coupled amplitude equations obtained by projecting the Waleffe flow equations onto these modes reads

$$\frac{d}{dt}a_j = \delta_{1j}\frac{\pi^2}{4\mathrm{Re}} + \alpha_j(\mathrm{Re})a_j + \sum_{k=1}^{9}\sum_{l=1}^{9}\beta_{jkl}(\mathrm{Re})a_k a_l, \tag{2}$$

where Re is the Reynolds number, δ_{ij} is the Kronecker delta acting on indices i and j, and $\alpha_j(\mathrm{Re})$ and $\beta_{jkl}(\mathrm{Re})$ are Re-dependent coefficients whose full expressions are given in Appendix A. The Reynolds number is the only nondimensional physical parameter in this system. It is a measure of the ratio between inertial and viscous forces. To obtain numerical solutions of this model, we time integrate Eq. (2) using the fourth-order Runge-Kutta scheme with time step $\Delta t_{TI} = 10^{-3}$.

The only known stable solution of (2) is the steady laminar flow: $\boldsymbol{a}_{\mathrm{lam}} = [1, 0, \dots, 0]^T$, which is equivalent to $\boldsymbol{u}_{\mathrm{lam}} = \sqrt{2}\sin(\pi y/2)\boldsymbol{e}_x$ in physical space; here $\boldsymbol{e}_x$ is the unit vector in the x direction. Despite the linear stability of the laminar flow, we can observe long-lived turbulence for Re $\gtrsim$ 150 [21]. Examples of turbulent flows at different values of the Reynolds numbers are shown in Fig. 1(b) through time series of the kinetic energy:

$$E = \frac{1}{2}||\boldsymbol{u}||_2^2 = \Gamma_x\Gamma_z\sum_{j=1}^{9}a_j^2, \tag{3}$$

where $\Gamma_x = 1.75\pi$ (resp. $\Gamma_z = 1.2\pi$) is the imposed solution wavelength in the x (resp. z) direction. All our simulations display chaotic dynamics over thousands of time units but eventually relax to the laminar flow, which is expected to be the global attractor at least for Re $\lesssim$ 335 [24]. This phenomenon, called hereafter turbulent-to-laminar transition,