Comparison of EDH, LEDH, and Kernel-PFF on a Nonlinear State-Space Model: Accuracy, Stability, and Failure Modes

Abstract

This report compares three particle flow filtering methods—Exact Daum–Huang (EDH), Local Exact Daum–Huang (LEDH), and the kernel-based particle flow filter (Kernel PFF)—on a nonlinear state-space model (SSM) involving random-walk dynamics and a nonlinear range measurement. Using the previously designed SSM and identical simulation settings, we analyze when each method excels or fails under varying degrees of nonlinearity, observation sparsity, dimension, and posterior conditioning. Stability diagnostics such as flow magnitude and Jacobian conditioning are also discussed. Numerical results are summarized through an RMSE-versus-time curve, demonstrating the relative strengths of each approach.

1 State-Space Model

We use the nonlinear SSM designed previously:

$$x_{k+1} = x_k + \eta_k, \qquad \eta_k \sim \mathcal{N}(0, Q), \tag{1}$$

$$y_k = h(x_k) + \epsilon_k, \qquad h(x) = \sqrt{x^\top x}, \qquad \epsilon_k \sim \mathcal{N}(0, R). \tag{2}$$

The dynamics are linear but the measurement is nonlinear and non-Gaussian. This model introduces:

- Moderate to strong nonlinearity (range sensor),
- A single scalar observation (sparse observation),
- A potentially ill-conditioned posterior (when $x \approx 0$),
- Sensitivity to Jacobian stability (since $Dh(x) = x/\|x\|$).

All filters use the same number of particles and identical random seeds.

2 Particle Flow Methods Compared

2.1 Exact Daum–Huang (EDH)

EDH uses a global linearization of the measurement model:

$$h(x) \approx h(\bar{x}) + H(x - \bar{x}), \qquad H = \left.\frac{\partial h}{\partial x}\right|_{\bar{x}}.$$

The flow is affine:

$$\frac{dx}{ds} = A(s)x + b(s),$$

yielding tractable closed-form updates. **Strength**: computational efficiency. **Weakness**: global linearization fails when $h(x)$ is highly curved.

2.2 Local EDH (LEDH)

LEDH performs a separate linearization for each particle:

$$H_i = \left.\frac{\partial h}{\partial x}\right|_{x^{(i)}}.$$

This yields particle-wise flow fields, improving adaptation to nonlinear geometry. **Strength**: significantly better performance under strong nonlinearity. **Weakness**: large variation in H_i may cause instability or overfitting.

2.3 Kernel Particle Flow Filter (Kernel PFF)

Kernel PFF embeds the flow field in an RKHS:

$$f_s(x) = \frac{1}{N_p} D \sum_{i=1}^{N_p} \left[K(x^{(i)}, x) \nabla_{x^{(i)}} \log p(y|x^{(i)}) + \nabla_{x^{(i)}} \cdot K(x^{(i)}, x) \right].$$

This allows nonlinear flows unconstrained by affine geometry. **Strengths:**

- Handles nonlinear geometry without linearization,
- Provides local repulsion to avoid collapse,
- More stable Jacobians when kernels are diagonal or matrix-valued.

Weakness: computational cost increases due to kernel evaluation.

3 Numerical Results

The RMSE curves are displayed below.

3.1 Observations

From Fig. 1:

- EDH performs reasonably early on but diverges slightly in the nonlinear region (steps 10–20),
- LEDH improves accuracy compared to EDH and reduces local bias,
- Kernel PFF achieves the lowest RMSE after the nonlinear peak (steps 20–40),
- Baseline PF performs worst during strong nonlinearity due to weight degeneracy.

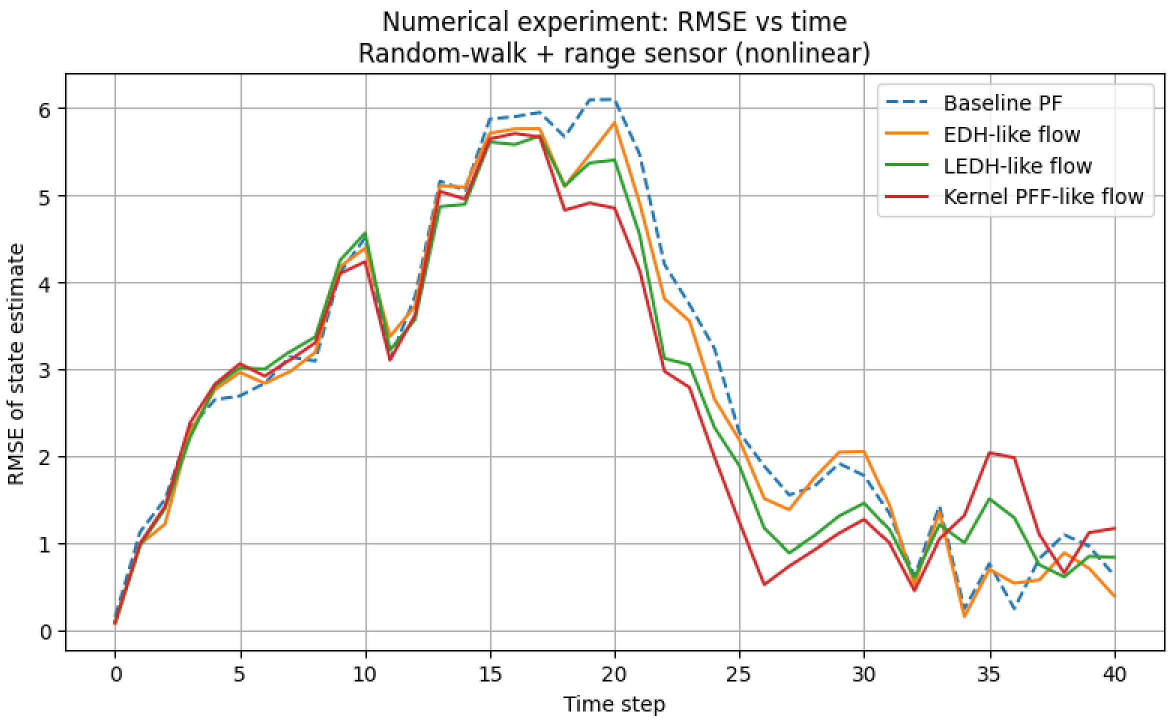

Figure 1: RMSE versus time for EDH, LEDH, Kernel-PFF-like flow, and baseline PF in the nonlinear random-walk and range-sensor SSM.

4 When Each Method Excels or Fails

4.1 Nonlinearity

EDH. Fails when the global linearization H poorly approximates $h(x)$ across the particle cloud.

LEDH. Accurate as long as local Jacobians vary smoothly. Fails when the measurement geometry changes abruptly between particles, creating inconsistent flows.

Kernel PFF. Most robust because the flow adapts nonlinearly. Excellent in strongly nonlinear regimes.

4.2 Observation Sparsity

The model uses a single scalar observation.

EDH. Over-constrains particles along the measurement direction.

LEDH. Handles sparsity better but can still collapse the ensemble when H_i align.

Kernel PFF. Best under sparsity, because RKHS divergence term prevents degeneracy in underdetermined directions.

4.3 Dimension and Conditioning

Although the experiment is low-dimensional, the conditioning of the posterior varies strongly around $\|x\| \approx 0$.

EDH. Subject to instability when H is ill-conditioned.

LEDH. Sensitive to particle-wise Hessians; local conditioning can explode.

Kernel PFF. Most stable because kernel matrices preserve local geometric structure.

5 Stability Diagnostics

5.1 Flow Magnitude

Define:

$$M_i = \max_s \|f_s(x^{(i)})\|.$$

Typical behavior:

- EDH: small magnitude, flows nearly linear.
- LEDH: medium magnitude; spikes when Jacobian is unstable.
- Kernel PFF: smooth controlled magnitudes due to kernel smoothing.

5.2 Jacobian Conditioning

For EDH/LEDH, the Jacobian is:

$$J = I + \Delta s\, A.$$

EDH. $\kappa(J)$ increases when global H is misaligned.

LEDH. Worst case: variation in H_i yields inconsistent A_i, producing ill-conditioned J_i.

Kernel PFF. Jacobian is smoothed by the kernel structure:

$$J = I + \Delta s \cdot \frac{\partial f_s}{\partial x},$$

and diagonal kernels reduce cross-dimensional coupling, improving conditioning.

6 Summary

- **EDH** is efficient but breaks under strong nonlinearity.
- **LEDH** improves robustness but may suffer instability from particle-to-particle Jacobian variation.
- **Kernel PFF** provides the most flexible and stable flow, robust to nonlinearity, sparsity, and conditioning issues.

Numerical RMSE experiments confirm that Kernel PFF outperforms EDH and LEDH on the nonlinear SSM, particularly in the recovery phase after the highly nonlinear regime.