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Dissertation for Bachelor of Science

Stirling Series of Gamma Function

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Pure Mathematics

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¹ N!µ Stirling ½¥½ , ²¢ ¹ ¿ ó ¾«ȷµ ¼ . ±¾ ĵĽ ¹ Euler-Maclaurin ¹« ¸ ³ Stirling ¹« , ²¢ Gamma º¯ µ ¯ µĵ¥µ ³ Gamma º¯ µ Stirling ¹« . ½õ ³ Bernoulli º Bernoulli ¶ .

º´ ¹ , ½ Bernoulli , Stirling ¹« , Γ º¯ .

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THESIS: Stirling Series of Gamma Function

DEPARTMENT: Department of Mathematics

SPECIALIZATION: Pure Mathematics

UNDERGRADUATE: Chen Tianpang

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ABSTRACT:

On the Asymptotic analysis and probability theory, The Stirling's formula have great theoretical value. It can be drawn through a number of precise numerical calculation. In this paper we obtain Stirling's formula by using the Euler-Maclaurin formula, and by these we compare the nature of Gamma function and N! function, in the end we obtain Stirling's Series of Gamma Function. In this process we use Bernoulli nonumber and Bernoulli polynomial.

KEYWORDS: Factorial, Bernoulli Numbers, Stirling Formula, Γ function.

Gamma º¯ µ Stirling ¼¶

Gamma º¯ µ Stirling ¹« $n!$ µ Stirling ¹« µ . ±¾ Gamma º¯ µ °º¯ µĵ¥µ ¼. Gamma º¯ µ ¸½¼**B**.

¶ µĽ , Stirling ¹« Ī½¥½ ± º:

$$n! = \sqrt{2\pi\, n}(\frac{n}{e})^n e^{\delta_n}, \quad \delta_n = \frac{1}{12n} + \cdots .$$

½« $\Gamma(s)$ Ī½ , 4 ¼ Ľ¥½ ± º.

1: $\Gamma(s) = \sqrt{2\pi} s^{s-\frac{1}{2}} e^{-s} e^{\mu(s)}$, $(s > 0)$, $\lim_{s\to\infty} \mu(s) = 0.$

:

$$r(s) = \ln \frac{\Gamma(s)e^s}{\sqrt{2\pi} s^{s-1/2}} = \ln\Gamma(s) + s - (s-1/2)\ln s - \ln\sqrt{2\pi},$$

¼ $g(s) = r(s) - r(s+1)$, õ « $\Gamma(s+1) = s\Gamma(s)$ µ

$$g(s) = (s+\frac{1}{2})\ln(1+\frac{1}{s}) - 1.$$

$g(s)$ $(0, +\infty)$ º¯ :

$$g''(s) = \frac{1}{2s^2(s+1)^2} > 0.$$

Taylor ¿ª

$$\ln\frac{1+t}{1-t} = 2(\frac{t}{1} + \frac{t^3}{3} + \frac{t^5}{5} + \cdots) \quad (|\ t\ | < 1)$$

$t = (2s+1)^{-1}$ µ

$$g(s) = \frac{1}{3(2s+1)^2} + \frac{1}{5(2s+1)^4} + \frac{1}{7(2s+1)^6} + \cdots,$$

õ½ ¹<

$$0 < g(s) < \frac{1}{3(2s+1)^2}[1 + \frac{1}{(2s+1)^2} + \frac{1}{(2s+1)^4} + \cdots] = \frac{1}{12s(s+1)}.$$

¼ $\mu(s) - \sum_{n=0}^{\infty} g(s+n)$, ± $\mu(s)$ $(0, +\infty)$, r

$$0 < g(s) < \frac{1}{12}\sum_{n=0}^{\infty}(\frac{1}{s+n} - \frac{1}{s+n+1}) = \frac{1}{12s}.$$

Ī $g(s)$ Ī º⁻ , ¹ $\mu(s)$ º⁻ , , ¨ ¾ ¸

$$\mu(1) = \sum_{n=1}^{\infty} g(n) = \sum_{n=1}^{\infty}[r(n) - r(n+1)] = r(1) = 1 - \ln\sqrt{2\pi}.$$

$\lim_{n\to\infty} r(n) = \lim_{n\to\infty} \delta_n = 0$ Ī Stirling ¹« . ¼ $f(s) = \sqrt{2\pi}s^{s-\frac{1}{2}}e^{-s}e^{\mu(s)}$, $f(s)$ ¶¨ **B.1** (Bohr-Mollerup) µ ¸ ¼´¿ .

$$\frac{f(s+1)}{f(s)} = (1+\frac{1}{s})^{s+\frac{1}{2}}se^{-1}e^{-g(s)} = s,$$

f ¸ . µ ¸ :

$$f(1) = \sqrt{2\pi}e^{-1}e^{\mu(1)} = 1.$$

µ ¸ :

$$\ln f(s) = \ln\sqrt{2\pi} + (s - 1/2)\ln s - s + \mu(s),$$

º⁻ µĺ , $\ln f$ º⁻ . ¸ ¾ ¶¨ **B.1** Ǿ .

2: **1** $\mu(s)$ ǵ¥µ 0 µ .

: ¸½¼**B** ¶¨ **B.2**

$$\ln\Gamma(s) = -\gamma\ s - \ln s - \sum_{n=1}^{\infty}[s/n - \ln(1+s/n)],$$

, µ

$$(\ln\Gamma(s))' = -\gamma - \frac{1}{s} - \sum_{n=1}^{\infty}[\frac{1}{n} - \frac{1}{s+n}],$$

$(0, +\infty)$, r ǿ . Euler ³£ µĶ¨ ,

$$(\ln\Gamma(s))' = \lim_{n\to\infty}[\ln n - \sum_{k=1}^{n}\frac{1}{k}] - \frac{1}{s} - \lim_{n\to\infty}\sum_{k=1}^{n}[\frac{1}{k} - \frac{1}{s+k}]$$
$$= \lim_{n\to\infty}[\ln n - \sum_{k=1}^{n}\frac{1}{s+k}] - \frac{1}{s}.$$

Ì $\psi(t) = \frac{1}{t+s}$ º− , ¹

$$\int_k^{k+1} \frac{dt}{t+s} < \frac{1}{2}[\frac{1}{t+s+k} + \frac{1}{t+s+k+1}], \quad k \geq 0.$$

$$\frac{1}{2}\frac{1}{s} + \frac{1}{s+1} + \cdots + \frac{1}{s+n} + \frac{1}{2}\frac{1}{s+n+1} > \sum_{k=0}^{n+1} \int_k^{k+1} \frac{dt}{t+s} = \ln(s+n+1) - \ln s,$$

$$(\ln\Gamma(s))' \leq \lim_{n\to\infty}[\ln n - \ln(s+n+1) + \ln s + \frac{1}{2s} + \frac{1}{2(s+n+1)}] - \frac{1}{s} = \ln s - \frac{1}{2s},$$

´

$$\gamma'(s) = (\ln\Gamma(s))' + 1 - \ln s - (s - \frac{1}{2})\frac{1}{s} \leq 0.$$

$\gamma(s)$ ǵ¥µ . **1** $\gamma(s) = \mu(s)$, ¼´ .

1 º **2** ¿

$$\mu(s) = \ln(\frac{\Gamma(s)e^s}{\sqrt{2\pi}s^{s-\frac{1}{2}}}), \quad s > 0, \quad \lim_{s\to\infty}\mu(s) = 0.$$

$\mu(s)$ ǵ¥µ 0.

δ_n ¹< , ¶ $\mu(s)$ ½ ȷ Ĺ< . , ǿ :

$$\mu(s) \simeq \frac{1}{12s} - \frac{1}{360s^3} + \frac{1}{1260s^5} - \frac{1}{1680s^7} + \frac{1}{1188s^9} - \cdots, \quad \forall\ s > 0. \tag{1.1}$$

ǵ (2.7) . µ $s > 0$

$$0 < \mu(s) < \frac{1}{12s}, \tag{1.2}$$
$$\frac{1}{12s} - \frac{1}{360s^3} < \mu(s) < \frac{1}{12s}, \tag{1.3}$$
$$\frac{1}{12s} - \frac{1}{360s^3} < \mu(s) < \frac{1}{12s} - \frac{1}{360s^3} + \frac{1}{1260s^5}, \tag{1.4}$$
$$\cdots < \cdots.$$

¶ $\forall\ s_0 > 0$

$$\rho_n(s_0) = \rho(s_0 + n) \quad (n = 0, 1, 2, \cdots)$$

1, ¿ $\rho_n(s_0)$ 0. ø ,

$$\rho_{n+1}(s_0) - \rho_n(s_0) = 1 - (s_0 + n + \frac{1}{2}) \ln (a + \frac{1}{s_0 + n}) < 0, \quad \forall\ n > 0.$$

$$f(x) = \ln(1 + \frac{1}{s_0 + x}) - \frac{1}{s_0 + x + \frac{1}{2}} > 0 \quad \forall\ x > 0.$$

µ . Ī $\lim_{x\to\infty} f(x) = 0$

$$f'(x) = -\frac{1}{(s_0 + x)(s_0 + x + 1)(2s_0 + 2x + 1)^2} < 0, \quad \forall\ x > 0.$$

[1] $\rho_n(s_0)$ µ 0. s_0 , [1] $\mu(s) > 0, \ (s > 0)$ [3] b.

½ ǵ ³̲¢û õ) $\mu(s)$ **ǵ** 0 **µ , õ½¹¹ ǵ).**

4 (1.2) µij b. $\forall\ s_0 > 0$ **µ** µ **µ** ²½

$$\alpha_n(s_0) = \gamma(s_0 + n) - \frac{A}{s_0 + n} \quad (n = 0, 1, 2, \cdots).$$

$\alpha_n(s_0)$ µ A µ . 183 183 183 depthheight**µ** (1.2)(1.3)(1.4)$\mu(s)$

$$\mu(s) \simeq \frac{1}{12s} - \frac{1}{360s^3} + \frac{1}{1260s^5} - \frac{1}{1680s^7} + \frac{1}{1188s^9} - \cdots, \forall s > 0. \tag{1.5}$$

$$\Gamma(s) = \sqrt{2\pi} s^{s-\frac{1}{2}} e^{-s} e^{\mu(s)},$$

$$\mu(s) \simeq \frac{1}{12s} - \frac{1}{360s^3} + \frac{1}{1260s^5} - \frac{1}{1680s^7} + \frac{1}{1188s^9} - \cdots.$$

Stirling ¹« δ_n µĹ<

$$n! = \sqrt{2\pi n}(\frac{n}{e})^n e^{\delta_n}, \delta_n = \frac{1}{12n} + \cdots .\delta_n \to 0, (n \to \infty)$$

½ 183 depthheightδ_n

2.1 Euler-Maclaurin ¹« ¶¨ <

¸½¼**B**$(B.1)$

$$\int_0^1 f(s)ds = \frac{1}{2}[f(0)+f(1)] - \sum_{k=1}^{m+1} \frac{B_{2k}}{(2k)!}[f^{(2k-1)}(1) - f^{(2k-1)}(0)] + \frac{1}{(2m+2)!}\int_0^1 B_{2m+2}(s) f^{(2m+2)}(s)ds, \tag{2.1}$$

$f^{2m} f^{2m+2}[0,1]$ ´ ¿» 183 depthheight

$$\int_0^1 (B_{2m}(s) - B_{2m}) f^{(2m)}(s)ds = -B_{2m}[f^{(2m-1)}(1) - f^{(2m-1)}(0)] + \frac{1}{2m+2}\int_0^1 (B_{2m+2}(t) - B_{2m+2}) f^{(2m+2)}(s)ds,$$

¸½¼**A 4 5**$\theta \in (0,1)$

$$\int_0^1 (B_{2m}(s) - B_{2m}) f^{(2m)}(s)ds = -\theta \cdot B_{2m}[f^{(2m-1)}(1) - f^{(2m-1)}(0)].$$

$[a,b]$

$(1,+\infty) f(t)$

$f^{2k} f^{(2k+1)} \to 0 (t \to \infty)$

$f(t)[1,n][1,n]n-1(2.1)$

$$\int_1^n f(t)dt = \sum_{i=1}^n f(i) - \frac{1}{2}f(n) - \sum_{k=1}^m \frac{B_{2k}}{(2k)!}f^{(2k-1)}(n) + R_n,$$

$$\begin{aligned}R_n = &-\frac{1}{2}f(1) - \sum_{k=1}^m \frac{B_{2k}}{(2k)!}f^{(2k-1)}(1)\\ &+ \frac{1}{(2m+2)!}\sum_{i=1}^{n-1}\int_0^1 (B_{2m+2}(t) - B_{2m+2})f^{(2m+2)}(i+t)dt,\end{aligned}$$

$(*)\lim_{n\to\infty} R_n R$

$$R_n - R = \frac{1}{(2m+2)!}\sum_{i=n}^{\infty}\int_0^1 (B_{2m+2}(t) - B_{2m+2})f^{(2m+2)}(i+t)dt,$$

$(*)\theta_n \in (0,1)$

$$R_n - R = -\theta_n \frac{B_{2m+2}}{(2m+2)!}f^{(2m+2)}(n),$$

$$\begin{aligned}\int_1^n f(t)dt = &\sum_{i=1}^n f(i) - \frac{1}{2}f(n) - \sum_{k=1}^m \frac{B_{2k}}{(2k)!}f^{(2k-1)}(n)\\ &+ R - \theta_n \frac{B_{2m+2}}{(2m+2)!}f^{(2m+1)}(n), \theta_n \in (0,1).\end{aligned}$$

$f(t) = \ln t$

$$\begin{aligned}n\ln n - n = C + \ln(n!) - \frac{1}{2}\ln n - &\sum_{k=1}^m \frac{B_{2k}}{2k(2k-1)}\frac{1}{n^{2k-1}}\\ - \theta_n &\frac{B_{2m+2}}{(2m+2)(2m+1)}\frac{1}{n^{2k+1}}, \theta_n \in (0,1).\end{aligned} \tag{2.2}$$

$n! = \sqrt{2\pi n}(\frac{n}{e})^n e^{\delta_n} n \to \infty C = -\frac{1}{2}\ln 2\pi \mu(n)(**)$

$$\mu(n) = \sum_{k=1}^m \frac{B_{2k}}{2k(2k-1)}\frac{1}{n^{2k-1}} + \theta_n \frac{B_{2m+2}}{(2m+2)(2m+1)}\frac{1}{n^{2m+1}}, \theta_n \in (0,1).$$

$m = 1$

$$\mu(n) = \frac{B_2}{2}\frac{1}{n} + \theta_n \frac{B_4}{12}\frac{1}{n^3} = \frac{1}{12n} - \frac{\theta_n}{360n^3}, \theta_n \in (0,1).$$

$m=2$

$$\mu(n)=\frac{1}{12n}-\frac{1}{360n^3}+\frac{\theta_n}{1260n^5},\theta_n\in(0,1).$$

$$n!=n^ne^{-n}\sqrt{2\pi n}(1+\frac{1}{12n}+\frac{1}{288n^2}-\frac{139}{51480n^3}-\frac{571}{2488320n^4}+\cdots).$$

Stirling ¹« δ_n ¹<

2.2 ú¯ µ¥µ ¨¹<

δ_n

$$\delta_n=\ln\frac{n!e^n}{\sqrt{2\pi}n^{n+\frac{1}{2}}}(n=1,2,\cdots).$$

$$\lim_{n\to\infty}\delta_n=0.$$

$\delta_n\lim_{n\to\infty}\delta_n=0\delta_n(****)$

$$\delta_n\simeq\frac{A}{n}-\frac{B}{n^3}+\frac{C}{n^5}-\frac{D}{n^7}+\cdots,$$

$\simeq\delta_n$ ¸ ¼ . Euler-Maclaurin ¹< ´³ , ¹ ¸ ¦¼ ô

¼ ¨µij£ $A,B,C,\cdots$.²¢ ¹ ½ 183 depthheight

2.1

2.1$f(x)x>0\lim_{x\to\infty}f(x)=0f(x)>0(f(x)<0),\forall x>0$

¶¨ 2.1:δ_n

:

$$\delta_{n+1}-\delta_n=1-(n+\frac{1}{2})\ln(1+\frac{1}{n}).$$

$\delta_n\ln(1+\frac{1}{n})-\frac{1}{n+\frac{1}{2}}>0,\forall n>0$

$$f(x)=\ln(1+\frac{1}{x})-\frac{1}{x+\frac{1}{2}}$$

$\lim_{x\to\infty}f(x)=0f'(x)=-\frac{1}{x(x+1)(2x+1)}<0x>0$ **2.1**$f(x)(0,+\infty)x\ln(1+\frac{1}{n})-\frac{1}{n+\frac{1}{2}}>0$

$A,B,C,\cdots$

µ ²½:$A0<\delta_n<\frac{A}{n},\forall n>0$

$\delta_na_n=\delta_n-\frac{A}{n}$ **2.1**$A>0a_n$

$$a_{n+1}-a_n=1-(n+\frac{1}{2})\ln(1+\frac{1}{n})+A(\frac{1}{n}-\frac{1}{n+1})>0,\forall n>0$$

$$g(x) = \frac{1}{x+\frac{1}{2}} - \ln(1+\frac{1}{x}) + A\frac{\frac{1}{x}-\frac{1}{x+1}}{x+\frac{1}{2}} > 0, \forall x > 0$$

$\lim\limits_{x\to\infty} g(x) = 0$ **2.1**

$$g'(x) = \frac{x(1+x) - A(12x^2+12x+2)}{x^2(x+1)^2(2x+1)^2} < 0, \forall x > 0 \tag{2.3}$$

$$A > \frac{x(x+1)}{12x^2+12x+2}, \forall x > 0 \tag{2.4}$$

$$\frac{x+1}{2(6x^2+6x+1)^2} > 0, \forall x > 0$$

$\frac{1}{12}\frac{1}{12}$

¶¨ 2.2:$0 < \delta_n < \frac{1}{12n}, \forall n > 0 \frac{1}{12}$

:$A' 0 < A' < \frac{1}{12}\frac{1}{12} A' g'(x) > 0, \forall x > X$ **2.1**$g(x) < 0, \forall x > X a_{n+1} - a_n < 0 n = N a_n > 0, \forall n > N \delta_n > A', \forall n > N$

µ² ½:$B\frac{1}{12n} - \frac{B}{n^3} < \delta_n < \frac{1}{12n}, \forall n > 0$

$\delta_n b_n = \delta_n - \frac{1}{12n} + \frac{B}{n^3}$ **2.1**$B > 0 a_n$

$$\begin{aligned} b_{n+1} - b_n = {} & 1 - (n+\frac{1}{2})\ln(1+\frac{1}{n}) \\ & + \frac{1}{12}(\frac{1}{n} - \frac{1}{n+1}) - B(\frac{1}{n^3} - \frac{1}{(n+1)^3}) < 0, \forall n > 0 \end{aligned}$$

$$h(x) = \frac{1}{x+\frac{1}{2}} - \ln(1+\frac{1}{x}) + \frac{1}{12}\frac{\frac{1}{x}-\frac{1}{x+1}}{x+\frac{1}{2}} - B\frac{\frac{1}{x^3}-\frac{1}{(x+1)^3}}{x+\frac{1}{2}} < 0, \forall x > 0$$

$\lim\limits_{x\to\infty} h(x) = 0$ **2.1**

$$h'(x) = -\frac{x^2(1+x)^2 - 12B(30x^4+60x^3+50x^2+20x+3)}{6x^4(1+x)^4(2x+1)^2} > 0, \forall x > 0 \tag{2.5}$$

$$B > \frac{x^2(1+x)^2}{12(30x^4+60x^3+50x^2+20x+3)}, \forall x > 0 \tag{2.6}$$

$$\frac{x(20x^4+50x^3+46x^2+19x+3)}{6(30x^4+60x^3+50x^2+20x+3)^2} > 0, \forall x > 0$$

$\frac{1}{360}\frac{1}{360}$

¶¨ **2.3:** $\frac{1}{12n}-\frac{1}{360n^3}<\delta_n<\frac{1}{12n},\forall n>0\frac{1}{360}$

¶¨ **2.2:**

$C=\frac{1}{1260},D=\frac{1}{1680},\cdots(***)$

$$\delta_n \simeq \frac{1}{12n}-\frac{1}{360n^3}+\frac{1}{1260n^5}-\frac{1}{1680n^7}+\frac{1}{1188n^9}-\cdots. \tag{2.7}$$

½ 183 depthheight$(**),(***),(****)\delta_n$

$$A=\frac{B_2}{1\cdot 2},B=-\frac{B_4}{3\cdot 4},C=\frac{B_6}{5\cdot 6},D=-\frac{B_8}{7\cdot 8},\cdots$$

$B_i(i=2,4,6,\cdots)$ ¸**½¼A** ¸**½¼A** .

2 ¼

[1] ÷¼ ǿ, 183 depthheight

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4µ , `N º , , ı¾¿ º » ij ¶ . , :õ ¹
° , . , ¸ ļ 4¸ µİ®º ³ .