考研数学三大核心计算

——函数求极限习题

1.求极限 $\lim\limits_{x\to 0}\dfrac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x(1-\cos x)}$.

$$=\lim_{x\to 0}\frac{\tan x-\sin x}{x\cdot\frac{1}{2}x^2\cdot(\sqrt{1+\tan x}+\sqrt{1+\sin x})}$$

$$=\lim_{x\to 0}\frac{\tan x(1-\cos x)}{\frac{1}{2}x^3\cdot 2}$$

$$=\lim_{x\to 0}\frac{x\cdot\frac{1}{2}x^2}{x^3}$$

$$=\frac{1}{2}.$$

① 把 $x\to ?$ 代入极限分析 $\begin{cases}\text{类型}\\ \text{化简}\end{cases}$

② 化简：根式有理化，计算非零因子，拆分极限存在项，提公因子，等价替换，幂指函数转化，倒代换

③ 计算：洛必达，泰勒公式

2.求极限 $\lim\limits_{x\to 0^+}\dfrac{1-\sqrt{\cos x}}{x\left(1-\cos\sqrt{x}\right)}$.

$$=\lim_{x\to 0^+}\frac{\frac{1}{2}\cdot\frac{1}{2}x^2}{x\cdot\frac{1}{2}(\sqrt{x})^2}$$

$$=\lim_{x\to 0^+}\frac{\frac{1}{4}x^2}{\frac{1}{2}x^2}$$

$$=\frac{1}{2}$$

$x\to 0$ 时，$1-\cos^k x\sim\frac{k}{2}x^2$

3.求极限 $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}-2}{\sqrt{1+x^4}-1}$

$$=\lim_{x\to 0}\frac{2(\sqrt{1-x^4}-1)}{\frac{1}{2}x^4\cdot 4}$$

$$=\lim_{x\to 0}\frac{\frac{1}{2}(-x^4)}{x^4}$$

$$=-\frac{1}{2}$$

4.求 $\lim\limits_{x\to-\infty}\left(\sqrt{x^2+4x+8}+x\right)=$ ______

开偶次根号加绝对值.

法一 $=\lim\limits_{x\to-\infty}\dfrac{4x+8}{\sqrt{x^2+4x+8}-x}=\lim\limits_{x\to-\infty}\dfrac{4x}{\sqrt{x^2}-x}=\lim\limits_{x\to-\infty}\dfrac{4x}{-x-x}=-2.$

法二：$\frac{1}{x}=t$ $\lim\limits_{t\to 0^-}\left(\sqrt{\frac{1}{t^2}+\frac{4}{t}+8}+\frac{1}{t}\right)=\lim\limits_{t\to 0^-}-\frac{1}{t}\sqrt{1+4t+8t^2}+\frac{1}{t}$

$$=\lim_{t\to 0^-}\frac{1-\sqrt{1+4t+8t^2}}{t}$$

$$= \lim_{t\to 0^-} \frac{-\frac{1}{2}(4t+8t^2)}{t} = -2.$$

5.求 $\lim\limits_{x\to\infty} x^2\left[\frac{1}{x}-\ln\left(1+\frac{1}{x}\right)\right]$.

$\square\to 0$ 且 $\square\neq 0$ 时.

$\square-\ln(1+\square)\sim\frac{1}{2}\square^2$

$$=\lim_{x\to\infty} x^2\cdot\frac{1}{2}\left(\frac{1}{x}\right)^2$$

$$=\frac{1}{2}.$$

6.求 $\lim\limits_{x\to 0}\frac{\left[\ln(1+x)-x\right](e^{2x}-1)}{x-\sin x}$.

$$=\lim_{x\to 0}\frac{-\frac{1}{2}x^2\cdot 2x}{\frac{1}{6}x^3}$$

$$=-6.$$

7.求 $\lim\limits_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)$. $\left(\frac{1}{0}-\frac{1}{0}\right)$ 通分

$a^2-b^2=(a-b)(a+b)$

$$=\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^2\sin^2 x \to x^4} \overset{洛}{=} \lim_{x\to 0}\frac{2x-\overset{=\sin 2x}{2\sin x\cos x}}{4x^3}$$

$$=\lim_{x\to 0}\frac{(x-\sin x)(x+\sin x)}{x^4} \quad = \lim_{x\to 0}\frac{\frac{1}{6}(2x)^3}{4x^3}=\frac{1}{3}$$

$$=\lim_{x\to 0}\frac{\frac{1}{6}x^3\,(x+x+o(x))}{x^4}=\frac{1}{3}$$

8.求 $\lim\limits_{x\to 0}\frac{\left(\frac{1+\cos x}{2}\right)^x-1}{\left(1-\sqrt{\cos x}\right)\ln(1+2x)}$.

$1-\cos^k x\sim\frac{k}{2}x^2$

$$=\lim_{x\to 0}\frac{e^{x\ln\frac{1+\cos x}{2}}-1}{\frac{1}{4}x^2\cdot 2x}$$

$$=\lim_{x\to 0}\frac{x\ln\frac{1+\cos x}{2}}{\frac{1}{2}x^3}$$

$\ln(1+\square)\sim\square,\ \square\to 0$

$\ln\Delta=\ln(\boxed{\Delta-1}+1)\sim\Delta-1$, $\Delta\to 1$, $\Delta-1\to 0$

$$=\lim_{x\to 0}\frac{\frac{\cos x-1}{2}}{\frac{1}{2}x^2}$$

$$=\lim_{x\to 0}\frac{-\frac{1}{2}x^2}{x^2}=-\frac{1}{2}$$

9.求极限 $\lim\limits_{x\to 0}\frac{e^2-(1+x)^{\frac{2}{x}}}{x}$

$$= e^2-e^{\frac{2}{x}\ln(1+x)} \qquad e^{\square}-e^{\Delta}=e^{\Delta}(e^{\square-\Delta}-1)$$

9.求极限 $\lim\limits_{x\to 0}\frac{\qquad}{x}$

$$=\lim_{x\to0}\frac{e^2-e^{\frac{2}{x}\ln(1+x)}}{x}$$

$$e^{\square}-e^{\triangle}=e^{\triangle}(e^{\square-\triangle}-1)$$

$$=\lim_{x\to0}\frac{e^{\frac{2}{x}\ln(1+x)}\left(e^{2-\frac{2}{x}\ln(1+x)}-1\right)}{x}$$

$$=e^2\lim_{x\to0}\frac{2-\frac{2}{x}\ln(1+x)}{x}$$

$$=2e^2\lim_{x\to0}\frac{x-\ln(1+x)}{x^2}=2e^2\lim_{x\to0}\frac{\frac12x^2+o(x^2)}{x^2}=e^2.$$

10.求极限 $\lim\limits_{x\to 0}\left(\sin 3x+\mathrm{e}^{2x}\right)^{\frac{1}{\ln(1-x)}}$

$$=\lim_{x\to0}e^{\frac{1}{\ln(1-x)}\ln(\sin3x+e^{2x})}$$

$$=e^{\lim\limits_{x\to0}\frac{\sin3x+e^{2x}-1}{-x}}$$

$$=e^{\lim\limits_{x\to0}\frac{\sin3x}{-x}+\lim\limits_{x\to0}\frac{e^{2x}-1}{-x}}$$

$$=e^{\lim\limits_{x\to0}\frac{3x}{-x}+\lim\limits_{x\to0}\frac{2x}{-x}}$$

$$=e^{-3-2}=e^{-5}$$

$$\ln(\sin3x+e^{2x})$$

$$=\ln(\underbrace{\boxed{\sin3x+e^{2x}-1}}_{\to0}+1)$$

$$\sim\boxed{\sin3x+e^{2x}-1}$$

11.求极限 $\lim\limits_{x\to 0}\left(\mathrm{e}^{2x}+\sin x\right)^{\frac{1}{2x}}$

$$=\lim_{x\to0}e^{\frac{1}{2x}\ln(e^{2x}+\sin x)}$$

$$=e^{\lim\limits_{x\to0}\frac{e^{2x}+\sin x-1}{2x}}$$

$$=e^{\lim\limits_{x\to0}\frac{e^{2x}-1}{2x}+\lim\limits_{x\to0}\frac{\sin x}{2x}}$$

$$=e^{\lim\limits_{x\to0}\frac{2x}{2x}+\lim\limits_{x\to0}\frac{x}{2x}}$$

$$=e^{1+\frac12}=e^{\frac32}$$

12.求极限 $\lim\limits_{x\to 0^+}(\cot x)^{\sin 3x}$

$$=\lim_{x\to0^+}e^{\sin3x\ln\cot x}$$

$$=e^{\lim\limits_{x\to0^+}3x\ln\cot x}$$

$$0\cdot\infty\Rightarrow\frac{\infty}{\frac{1}{0}}\quad\frac{0}{\frac{1}{\infty}}$$

$$=e^{3\lim\limits_{x\to0^+}\frac{\ln\cot x}{\frac1x}}$$

① $(\sin x)'=\cos x$

$(\tan x)'=\sec^2x$ $:\sec x=\frac{1}{\cos x}$

$(\sec x)'=\sec x\tan x$

$$=e^{3\lim\limits_{x\to0^+}\frac{\tan x\cdot(-\csc^2x)}{-\frac{1}{x^2}}}$$

$$3\lim_{x\to0^+}x^3\cdot\frac{1}{\sin^2x}$$

$$= e^{\cdots}$$

$$= e^{3\lim\limits_{x\to0^+} x^3\cdot\frac{1}{\sin^2 x}}$$

$$= e^{3\lim\limits_{x\to0^+} x} = e^0 = 1$$

$(\sec x)' = \sec x\tan x$

② $(\cos x)' = -\sin x$

$(\cot x)' = -\csc^2 x$ 其中: $\csc x = \frac{1}{\sin x}$

$(\csc x)' = -\csc x\cdot\cot x$

③ $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

$(\arctan x)' = \frac{1}{1+x^2}$

13.求极限 $\lim\limits_{x\to1}\frac{x^x-1}{\sin\pi x}$

④ 奇变偶不变，符号看象限.

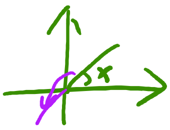
解: 原式 $= \lim\limits_{x\to1}\frac{e^{x\ln x}-1}{\sin\pi x}$

$$= \lim_{x\to1}\frac{x\ln x}{\sin\pi x} \overset{洛}{=} \lim_{x\to1}\frac{\ln x+1}{\cos\pi x\cdot\pi} = -\frac{1}{\pi}$$

当作锐角

$\sin(\pi+x) = \sin(\frac{\pi}{2}\cdot2+x)$

$= -\sin x$



$$= \lim_{x\to1}\frac{\ln x}{\sin\pi x} \overset{洛}{=} \lim_{x\to1}\frac{\frac{1}{x}}{\cos\pi x\cdot\pi} = -\frac{1}{\pi}$$

$$\overset{x-1=t}{=} \lim_{t\to0}\frac{\ln(1+t)}{\sin(\pi+\pi t)} = \lim_{t\to0}\frac{t}{-\sin\pi t} = \lim_{t\to0}\frac{t}{-\pi t} = -\frac{1}{\pi}.$$

非零因子

举例: $\lim\limits_{x\to1}\frac{\arctan\frac{1}{(x-1)^2}\ln x}{\sin\pi x} = \lim\limits_{x\to1}\frac{\frac{\pi}{2}\ln x}{\sin\pi x} = \frac{\pi}{2}\lim\limits_{x\to1}\frac{\frac{1}{x}}{\cos\pi x\cdot\pi} = -\frac{1}{2}$

14.求 $\lim\limits_{x\to0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x-\ln(1+x)}}$.

$$= \lim_{x\to0} e^{\frac{1}{x-\ln(1+x)}\ln\frac{\sin x}{x}}$$

$$= e^{\lim\limits_{x\to0}\frac{\frac{\sin x-x}{x}}{\frac{1}{2}x^2}} = e^{\lim\limits_{x\to0}\frac{-\frac{1}{6}x^3}{\frac{1}{2}x^3}} = e^{-\frac{1}{3}}.$$

15.求 $\lim\limits_{x\to0}\left(\frac{1+\tan x}{1+\sin x}\right)^{\frac{1}{x^3}}$.

$$= \lim_{x\to0} e^{\frac{1}{x^3}\ln\left(\frac{1+\tan x}{1+\sin x}\right)} \leftarrow \ln\left(\frac{1+\tan x}{1+\sin x}\right) = \ln\left(1+\frac{\tan x-\sin x}{1+\sin x}\right)$$

$$= e^{\lim\limits_{x\to0}\frac{\frac{\tan x-\sin x}{1+\sin x}}{x^3}} \leftarrow 1+\sin x\to1$$

$$= e^{\lim\limits_{x\to 0} \frac{\frac{\tan x - \sin x}{1+\sin x}}{x^3}} \quad \leftarrow 1+\sin x \to 1$$

$$= e^{\lim\limits_{x\to 0} \frac{\tan x(1-\cos x)}{x^3}}$$

$$= e^{\lim\limits_{x\to 0} \frac{x\cdot\frac{1}{2}x^2}{x^3}}$$

$$= e^{\frac{1}{2}}.$$

16.求 $\lim\limits_{x\to 0^+}\left(\frac{1}{\sqrt{x}}\right)^{\tan x}$. 指数化

$$= \lim_{x\to 0^+} e^{\tan x \ln\frac{1}{\sqrt{x}}}$$

$$= e^{\lim\limits_{x\to 0^+} x\cdot(-\frac{1}{2}\ln x)} \leftarrow \lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}} \overset{洛}{=} \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$= -\lim_{x\to 0^+} x = 0.$$

$$= e^0 = 1$$

17.求 $\lim\limits_{x\to+\infty}\frac{\left(1+\frac{1}{x}\right)^{x^2}}{e^x}$. $= \lim\limits_{x\to+\infty}\frac{e^{x^2\ln(1+\frac{1}{x})}}{e^x}$

$A(x)^{B(x)} \to B$, $\downarrow A$, 且 A, B 不全为 0

$$= \lim_{x\to+\infty} e^{x^2\ln(1+\frac{1}{x})-x}$$

$$\overset{\frac{1}{x}=t}{=} e^{\lim\limits_{t\to 0^+}\frac{\ln(1+t)-t}{t^2}}$$

$$= e^{\lim\limits_{t\to 0^+}\frac{-\frac{1}{2}t^2}{t^2}} = e^{-\frac{1}{2}}$$

18.求 $\lim\limits_{x\to 0^+}\frac{x^x-(\sin x)^x}{x^3}$.

$$= \lim_{x\to 0^+}\frac{e^{x\ln x}-e^{x\ln\sin x}}{x^3}$$

$$= \lim_{x\to 0^+}\frac{e^{x\ln x}\left[1-e^{x\ln\sin x - x\ln x}\right]}{x^3}$$

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}} \overset{洛}{=} \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+}(-x) = 0$$

$$= e^0\lim_{x\to 0^+}\frac{-x\ln\frac{\sin x}{x}}{x^3}$$

$x\to+\infty$, $e^x > x > \ln x$

$$\overset{\frac{1}{x}=t}{=} \lim_{t\to+\infty}\frac{-\ln t}{t} = 0$$

$$= \lim_{x\to 0^+}\frac{-\frac{\sin x - x}{x}}{x^2}$$

$\ln b - \ln a = \ln\frac{b}{a}$

$$= \lim_{x\to 0^+}\frac{\frac{1}{6}x^3}{x^3} = \frac{1}{6}$$

19.求极限 $\lim\limits_{x\to0}\dfrac{e^{x^2}-\cos x}{1-\sqrt{1-x^2}}$

$= \lim\limits_{x\to0}\dfrac{e^{x^2}-\cos x}{-\frac{1}{2}(-x^2)}$

$=2\left(\lim\limits_{x\to0}\dfrac{e^{x^2}-1}{x^2}+\lim\limits_{x\to0}\dfrac{1-\cos x}{x^2}\right)$

$=2\left(\lim\limits_{x\to0}\dfrac{x^2}{x^2}+\lim\limits_{x\to0}\dfrac{\frac{1}{2}x^2}{x^2}\right)$

$=2\left(1+\frac{1}{2}\right)=3.$

20.求极限 $\lim\limits_{x\to0}\dfrac{\cos x-e^{-\frac{x^2}{2}}}{\arctan^2 x\left[2x+\ln(1-2x)\right]}$

$= \lim\limits_{x\to0}\dfrac{\left[1-\frac{1}{2}x^2+\frac{1}{24}x^4+o(x^4)\right]-\left[1-\frac{x^2}{2}+\frac{1}{2}\left(-\frac{x^2}{2}\right)^2+o(x^4)\right]}{x^2\cdot\left[-2x^2+o(x^2)\right]}$

$= \lim\limits_{x\to0}\dfrac{-\frac{1}{12}x^4+o(x^4)}{-2x^4+o(x^4)}$

$= \dfrac{1}{24}$

$e^x=1+x+\frac{1}{2}x^2+o(x^2)$

$\ln(1+x)-x\sim-\frac{1}{2}x^2.$

泰勒：① 上下同阶

② 幂次最低 且不完全抵消.

$2x+\ln(1-2x)=2x+(-2x)-\frac{1}{2}(-2x)^2+o(x^2)$

$\sin 2x+\ln(1-2x)=2x+0x^2+o(x^2)$
$+(-2x)-\frac{1}{2}(-2x)^2+o(x^2)$

21.求极限 $\lim\limits_{x\to0}\dfrac{e^x-e^{\ln(1+x)}}{\sqrt[3]{1-x^2}-1}$

$= \lim\limits_{x\to0}\dfrac{e^x\left[1-e^{\ln(1+x)-x}\right]}{\frac{1}{3}(-x^2)}$

$= \lim\limits_{x\to0}\dfrac{-\left[\ln(1+x)-x\right]}{-\frac{1}{3}x^2}$

$= \lim\limits_{x\to0}\dfrac{-\frac{1}{2}x^2}{\frac{1}{3}x^2}=-\frac{3}{2}$

$1-\cos^k x\sim\frac{k}{2}x^2$

$k=\frac{1}{3}$

22.求极限 $\lim\limits_{x\to0}\dfrac{1-\cos x\sqrt{\cos 2x}\sqrt[3]{\cos 3x}}{x^2}$

$= \lim\limits_{x\to0}\dfrac{1-\cos x}{x^2}+\lim\limits_{x\to0}\dfrac{\cos x\left(1-\sqrt{\cos 2x}\sqrt[3]{\cos 3x}\right)}{x^2}.$

$= \lim\limits_{x\to0}\dfrac{\frac{1}{2}x^2}{x^2}+\lim\limits_{x\to0}\dfrac{1-\sqrt{\cos 2x}}{x^2}+\lim\limits_{x\to0}\dfrac{\sqrt{\cos 2x}\left(1-\sqrt[3]{\cos 3x}\right)}{x^2}.$

$= \frac{1}{2}+\lim\limits_{x\to0}\dfrac{\frac{1}{4}(2x)^2}{x^2}+\lim\limits_{x\to0}\dfrac{\frac{1}{6}(3x)^2}{x^2}.$

$(\cos 3x)^{\frac{1}{3}}$

$=\frac{1}{2}+1+\frac{3}{2}=3.$ $\quad [(1)(2)(3)]'=(1)'(2)(3)+(1)(2)'(3)+(1)(2)(3)'$

$$原式=\lim_{x\to0}\frac{\sin x\sqrt{\cos 2x}\sqrt[3]{\cos 3x}}{2x}+\lim_{x\to0}\frac{\frac{1}{2}(\cos 2x)^{-\frac{1}{2}}\cdot\sin 2x\cdot 2\cdot\cos x\cdot\sqrt[3]{\cos 3x}}{2x}$$

$$+\lim_{x\to0}\frac{\frac{1}{3}(\cos 3x)^{-\frac{2}{3}}\sin 3x\cdot 3\cdot\cos x\sqrt{\cos 2x}}{2x}$$

$$=\frac{1}{2}+1+\frac{3}{2}=3.$$

23.求极限 $\lim\limits_{x\to0}\left(\frac{1}{x^2}-\frac{1}{\arcsin^2 x}\right)$

$$=\lim_{x\to0}\frac{\arcsin^2 x-x^2}{x^2\arcsin^2 x}$$

$$=\lim_{x\to0}\frac{(\arcsin x-x)(\arcsin x+x)}{x^4}$$

$$=\lim_{x\to0}\frac{\frac{1}{6}x^3(\arcsin x+x)}{x^4}$$

$$=\frac{1}{6}\left(\lim_{x\to0}\frac{\arcsin x}{x}+\lim_{x\to0}\frac{x}{x}\right)$$

$$=\frac{1}{6}(1+1)=\frac{1}{3}.$$

24.已知 $\lim\limits_{x\to0}\frac{\ln(1+ax)-e^{bx}+\cos x}{x^2}=-\frac{9}{2}$，求 a,b 的值.

$$解：原式=\lim_{x\to0}\frac{[ax-\frac{1}{2}(ax)^2+o(x^2)]-[1+bx+\frac{1}{2}(bx)^2+o(x^2)]+[1-\frac{1}{2}x^2+o(x^2)]}{x^2}$$

$$=\lim_{x\to0}\frac{(a-b)x-\frac{1}{2}(a^2+b^2+1)x^2+o(x^2)}{x^2}=-\frac{9}{2}$$

$$\therefore\begin{cases}a-b=0\\ a^2+b^2+1=9\end{cases}\qquad 解得：a=b=\pm 2$$

25.设 $f(x)$ 连续，且 $\lim\limits_{x\to0}\left[1+x+\frac{f(x)}{x}\right]^{\frac{1}{x}}=e^3$，求 $\lim\limits_{x\to0}\left[1+\frac{f(x)}{x}\right]^{\frac{1}{x}}$. $\frac{f(x)}{x}\to0$

$$解：\lim_{x\to0}\left[1+x+\frac{f(x)}{x}\right]^{\frac{1}{x}}=\lim_{x\to0}e^{\frac{1}{x}\ln\left[1+\left(x+\frac{f(x)}{x}\right)\right]}$$

$$=e^{\lim\limits_{x\to0}\frac{1}{x}\cdot\left[x+\frac{f(x)}{x}\right]}$$

$$=e^{1+\lim\limits_{x\to0}\frac{f(x)}{x^2}}=e^3$$

$$= e^{1+\lim\limits_{x\to 0}\frac{f(x)}{x^2}} = e^3$$

$$\therefore \lim_{x\to 0}\frac{f(x)}{x^2} = 2 \text{ 且 } \lim_{x\to 0}\frac{f(x)}{x} = 0.$$

$$\therefore \lim_{x\to 0}\left[1+\frac{f(x)}{x}\right]^{\frac{1}{x}} = \lim_{x\to 0} e^{\frac{1}{x}\ln\left(1+\frac{f(x)}{x}\right)}$$

$$= e^{\lim\limits_{x\to 0}\frac{1}{x}\cdot\frac{f(x)}{x}} = e^2.$$

26. 设 $\lim\limits_{x\to+\infty}\left(\sqrt{x^2-x+1}-ax-b\right)=2$，求 a， b 的值.

解：原式 $= \lim\limits_{x\to+\infty} x\left[\sqrt{1-\frac{1}{x}+\frac{1}{x^2}} - a - \frac{b}{x}\right] = 2.$

$$\therefore \lim_{x\to+\infty}\left[\sqrt{1-\frac{1}{x}+\frac{1}{x^2}} - a - \frac{b}{x}\right] = 1-a = 0$$

$$\therefore a = 1$$

$$\therefore \text{原式} = \lim_{x\to+\infty}\left[\sqrt{x^2-x+1} - x - b\right]$$

$$= \lim_{x\to+\infty}\frac{-(2b+1)x+1-b^2}{\sqrt{x^2-x+1}+x+b}$$

$$= \lim_{x\to+\infty}\frac{-(2b+1)x}{2x} = -\left(b+\frac{1}{2}\right) = 2.$$

$$\therefore b = -\frac{5}{2}$$