2 Introduction to Modules

2.1 Modules, Maps, and More

Definition. *Let A be a ring. An A-module M is an abelian group on which A acts on linearly i.e. for $x, y \in M$ and $a, b \in A$*

$$\begin{aligned} a(x+y) &= ax + ay \\ (a+b)x &= ax + bx \\ (ab)x &= a(bx) \\ 1x &= x \end{aligned}$$

This generalizes, in a sense, the notion of a vector space along with many other objects we've previously studied.

Example. 1) Any ideal $\mathfrak{a} \subseteq A$ is an A-module as is the ring A itself.

2) For a field k, then a k-module is just a k-vector space.

3) A $\mathbb{Z}$-module G is an abelian group. For $n \in \mathbb{Z}$ and $x \in G$, the action is just

$$nx = x + x + \cdots + x$$

4) For $A = k[x]$ where k is a field. An A-module M is just a k-vector space equipped with a linear transform which is the action of the indeterminate x on M.

Definition. *A submodule M' of M is a subgroup which is closed under multiplication by elements of A.*

This definition is very similar to the definition for ideals. We can say that submodules are the subgroup which are A-invariant. While there is a distinction between subrings and ideals, there is no such "special" submodule. Instead, we can quotient using whichever submodule we please.

Definition. *The quotient module M/M' is the quotient group which inherits the A-module structure (action)*

$$a(x + M') = ax + M'$$

Now that we have the preliminary definition out of the way, we can start defining maps between modules. Naturally, it makes sense that these maps must preserve the A-module structure.

Definition. *Let M, N be A-modules. Then an A-module homomorphism $\phi : M \to N$ (we say ϕ is A-linear) must satisfy*

$$f(ax + y) = a \cdot f(x) + f(y) \qquad \forall a \in A, \forall x, y \in M$$

We can also interpret an A-linear map as just an abelian group homomorphism which commutes with the action of A. If A is a field, then this is just the same as a linear transformation. As with before, the natural map

$$\pi : M \to M/M' \qquad m \mapsto m + M'$$

is an A-module homomorphism and there is a one-to-one correspondence

$$\{\text{Submodules of } M \text{ containing } M'\} \iff \{\text{Submodules of } M/M'\}$$

Note that the set of all A-linear maps forms an A-module (we could've done this for rings too, but this is more cool). For two maps f, g, we have

$$(f+g)(x) = f(x) + g(x) \qquad (af)(x) = a \cdot f(x)$$

Definition. *For two A-modules M, N, the set of all A-linear maps $M \to N$ forms an A-module denoted $Hom_A(M, N)$. If the A is obvious, then we may sometimes omit it and just write $Hom(M, N)$.*

Since the set of all A-linear maps is an A-module, there should exist A-linear maps to and from $Hom(M, N)$. Let $\phi : M' \to M$ and $\psi : N \to N'$ be A-linear where M', N' are A-modules. Then they induce A-linear maps

$$\overline{\phi} : Hom(M, N) \to Hom(M', N) \qquad f \mapsto f \circ \phi$$

$$\overline{\psi} : Hom(M, N) \to Hom(M, N') \qquad g \mapsto \psi \circ g$$

For any A-module, there is a natural isomorphism

$$Hom(A, M) \cong M \qquad f \mapsto f(1)$$

since any map $f : A \to M$ is uniquely determined by $f(1)$

$$f(a) = a \cdot f(1)$$

As with rings, we have two associated submodules for any map $f : M \to N$

$$\text{Ker } f = \{x \in M \mid f(x) = 0\} \qquad \text{Im } f = f(M)$$

The quotient of the codomain (N in this case) with the actual image is called the cokernel

$$\text{Coker } f = N/\text{Im } f$$

This is clearly trivial if f is surjective, so we can view this as measuring the "non-surjectivity" of the map f in the same way that Ker f measures the "non-injectivity." As usual, we have an isomorphism

$$M/\text{Ker } f \cong \text{Im } f$$

Now we can define some operations on modules just as we did for rings and ideals. Generally, the basic ring-theoretic operations have a corresponding module-theoretic version. Let M be an A-module and (M_i) a family of submodules indexed by I, then the sum is

$$\sum_{i \in I} M_i = \left\{ \sum x_i \mid x_i \in M_i; \text{ all but finite } x_i = 0 \right\}$$

This is the smallest submodule of M which contains every M_i. The intersection of submodules will remain a submodule, so they form a lattice under inclusion just ideals.

We can now prove what are typically known as the 2nd and 3rd isomorphism theorems

Proposition 2.1.

1. Let $N \subseteq M \subseteq L$ be A-modules, then

$$(L/N)/(M/N) = \frac{L/N}{M/N} \cong L/M$$

2. If $M_1, M_2 \subseteq M$ are submodules, then

$$(M_1 + M_2)/M_1 \cong M_2/(M_1 \cap M_2)$$

Proof. 1) Define the A-linear map

$$\phi : L/N \to L/M \qquad x + N \mapsto x + M$$

Since $N \subseteq M$, this is surjective and the kernel is M/N.

2) Consider the A-linear map

$$M_2 \to M_1 + M_2 \to (M_1 + M_2)/M_1 \qquad x \mapsto x + M_1$$

This is surjective and the kernel is $M_1 \cap M_2$. □

Unlike rings, we cannot in general define the product of two submodules since they do not carry a notion of multiplication. However for an ideal $\mathfrak{a} \subseteq A$ and A-module M, we can define the submodule

$$\mathfrak{a}M = \left\{ \sum a_i x_i \mid a_i \in \mathfrak{a}, m_i \in M \right\} \subseteq M$$

Given two submodules $N, P \subseteq M$, we can define a submodule quotient

$$(N : P) = \{ a \in A \mid aP \subseteq N \}$$

which is an ideal of A. With this notation, the annihilator is

$$Ann(M) = (0 : M) \qquad a \in Ann(M) \iff aM = 0$$

If we have an ideal $\mathfrak{a} \subseteq Ann(M)$, then we can actually view M as an $A/\mathfrak{a}$-module with the action

$$\overline{x}m = (x + \mathfrak{a})m = xm$$

Since $\mathfrak{a}M = 0$, we can choose any representative x and get the same answer.

Definition. *An A-module is faithful if $Ann(M) = 0$. If $Ann(M) = \mathfrak{a}$, then M is faithful as a $A/\mathfrak{a}$-module.*

Proposition 2.2.

1. $Ann(M + N) = Ann(M) \cap Ann(M)$
2. $(N : P) = Ann((N + P)/N)$

Proof. 1) Note the implications

$$x \in Ann(M + N) \iff x(M + N) = xM + xN = 0$$

This can only hold if x annihilates both M and N separately

$$xM = xN = 0 \iff x \in Ann(M) \cap Ann(N)$$

2) $x \in (N : P)$ if and only if $xP \subseteq N$ i.e.

$$x(N + P) = xN + xP \subseteq N \iff x \in Ann((N + P)/N)$$

□

Definition. *For two A-modules M, N, the direct sum is*

$$M \oplus N = \{(x, y) \mid x \in M, y \in N\}$$

which is an A-modules with operations defined component-wise.

One may ask the difference between the direct sum $M \oplus N$ and the direct product $M \times N$ and the answer is that there is no difference, with a bit of subtlety. In general, the direct sum of a family of A-modules (M_i) consists of families (or ordered pairs if you'd rather think of them that way) (x_i) for all $i \in I$ and $x_i \in M_i$. However, we place the additional restriction that all but finitely many $x_i = 0$. Thus if I is finite, then the direct sum and direct product are the same thing, otherwise they will be quite different.

Example. Let $A = \prod_{i=1}^{n} A_i$ be a direct product of rings. Then we can define ideals

$$\mathfrak{a}_i = (0, \ldots, 0, a_i, 0, \ldots, 0) \qquad a_i \in A_i$$

If we consider A as an A-module, it is a direct sum

$$A = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_n$$

Conversely, if we have the above decomposition, we can express A as a direct product

$$A \cong \prod_{i=1}^{n} A/\mathfrak{b}_i \qquad b_i = \bigoplus_{j \neq i} \mathfrak{a}_j$$

Each ideal $\mathfrak{a}_i$ is isomorphic to $A/\mathfrak{b}_i$, so they are rings, and each identity $e_i \in \mathfrak{a}_i$ is idempotent with $\mathfrak{a}_i = (e_i)$.

The ideal radical unfortunately does not have a module-theoretic counterpart due to there being no submodule counterpart to prime ideals. However we can still define a version of the Jacobson radical by taking the intersection of all maximal submodules.

We can take any element $x \in M$ and generate a submodule with it just like principal ideals. These are denoted the same way

$$(x) = Ax = \{ax \mid a \in A\}$$

Definition. *A module M is generated by elements x_i if we can express it as*

$$M = (x_1, x_2, \ldots) = \sum Ax_i$$

The x_i are known as the generators of M. If there are only finitely many of them, then M is finitely generated.

M being finitely generated means that every element can be expressed as a (not necessarily unique) A-linear combination of the generators. Closely related to the notion of a finitely generated module is the notion of a free module.

Definition. *M is a free A-module if*

$$M \cong \bigoplus_{i \in I} M_i$$

where each $M_i \cong A$ as A-modules. We'll sometimes denote $M = A^{(I)}$. If M is also finitely generated, then

$$M \cong A \oplus \cdots \oplus A = A^n$$

By convention A^0 is the zero module.

Another (more common) way to characterize free modules is that they have a linearly independent basis. That is, it is generated by some x_i such that

$$\sum_i a_i x_i = 0 \iff a_i = 0 \quad \forall i$$

Example. Consider the following two $\mathbb{Z}$-modules: $\mathbb{Z}/2\mathbb{Z}$ is finitely generated but not free, $\mathbb{Z}^{(I)}$ for I infinite is free but not finitely generated.

This example hopefully makes the difference between the two a bit more clear. A finitely generated module is free only if it has a set of linearly independent generators. It may have multiple sets of generators but if none of them are linearly independent, then it does not have a valid basis and is thus not free. Conversely, a free module is finitely generated only if its basis is finite.

While a finitely generated module is not necessarily free, it is still very close in structure to a free module.

Proposition 2.3. *M is a finitely generated A-module if and only if M is isomorphic to some quotient of A^n for some integer $n > 0$.*

Proof. Let $x_1, \dots, x_n$ be generators for M. Define the map

$$\phi : A^n \to M \qquad (a_1, \dots, a_n) \mapsto a_1 x_1 + \cdots + a_n x_n$$

This is clearly surjective so

$$M \cong A^n / \text{Ker } \phi$$

Conversely, if $M \cong A^n/\mathfrak{a}$ then we have some A-linear map $A^n \to M$ which is surjective with kernel $\mathfrak{a}$. Let

$$e_i = (0, \dots, 0, a_i = 1, 0, \dots, 0)$$

Then e_i generates A^n and thus $\phi(e_i)$ finitely generate M. $\square$

We see from the definition of a basis that if M were to be free then then Ker $\phi = 0$, giving us the isomorphism from the definition. The next proposition will be helpful in proving Nakayama's lemma, which tells us how the Jacobson radical of A acts on an A-module M.

Proposition 2.4. *Let M be a finitely generated A-module, $\mathfrak{a} \subseteq A$ an ideal, and ϕ an A-linear endomorphism ($M \to M$) such that $\phi(M) \subseteq \mathfrak{a}M$. Then ϕ satisfies an equation of the form*

$$\phi^n + a_1 \phi^{n-1} + \cdots + a_n = 0 \qquad a_i \in \mathfrak{a}$$

Proof. Let $x_1, \dots, x_n$ generate M. Then we can write

$$\mathfrak{a} = \mathfrak{a}(Ax_1 + \cdots + Ax_n) = \mathfrak{a}x_1 + \cdots + \mathfrak{a}x_n$$

By assumption, $\phi(x_i) \subseteq \mathfrak{a}M$, so each image can be expressed as

$$\phi(x_i) = \sum_{j=1}^{n} a_{ij} x_j \qquad a_{ij} \in \mathfrak{a}$$

We can rewrite this using a Kronecker delta

$$\sum_{j=1}^{n} (\delta_{ij}\phi - a_{ij}) x_j = 0 \qquad \delta_{ij} = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{otherwise} \end{cases}$$

Since ϕ is an A-linear map from M to itself, it can be uniquely determined by where it sends the generators x_i. In other words, we can write $\delta_{ij}\phi - a_{ij}$ as an $n \times n$ matrix. Suppose we call this matrix A, recall the adjoint matrix

$$\text{adj}(A)A = \det(A)I = 0$$

Thus we see that the determinant of this matrix will annihilate every x_i i.e. it is the zero homomorphism. Expanding the determinant will yield the desired equation. $\square$

Corollary 2.5. Let M be a finitely generated A-module and $\mathfrak{a} \subseteq A$ and ideal such that $\mathfrak{a}M = M$. Then there exists some $x \equiv 1 \mod \mathfrak{a}$ such that $xM = 0$.

Proof. Let ϕ be the identity map so that $\phi(M) = M \subseteq \mathfrak{a}M$ by assumption. By the previous proposition we have

$$x = \phi(1) = 1 + a_1 + \cdots + a_n = 0$$

which clearly annihilates M. □

This corollary is actually the original statement of Nakayama's lemma, but the most common form is the one below, which follows from the corollary we just proved. In fact, the next three statements are all equivalent versions of Nakayama's lemma.

Proposition 2.6 (Nakayama's Lemma)**.** Let M be a finitely generated A-module and suppose $\mathfrak{a} \subseteq \mathfrak{R} \subseteq A$. If $\mathfrak{a}M = M$, then $M = 0$.

Proof. By the previous proposition, $xM = 0$ for some

$$x \equiv 1 \mod \mathfrak{a} = 1 \mod \mathfrak{R}$$

This is equivalent to $1 - x \in \mathfrak{R}$, which means x is a unit. Thus

$$x^{-1}xM = M = 0$$

□

Corollary 2.7. Let M and $\mathfrak{a}$ be as before and $N \subseteq M$ a submodule, then

$$M = \mathfrak{a}M + N \longrightarrow M = N$$

Proof. First note that

$$M/N = (\mathfrak{a}M + N)/N = \mathfrak{a}(M/N)$$

Using Nakayama's lemma, we have $M/N = 0$ thus $M = N$ □

The next version of Nakayama's lemma relates modules over local rings and vector spaces.

Proposition 2.8. Let A be a local ring, $\mathfrak{m}$ its maximal ideal, and $k = A/\mathfrak{m}$ its residue field. If M is finitely generated, then $M/\mathfrak{m}M$ is a k-vector space. Let $\overline{x}_i$ be the basis of the vector space $M/\mathfrak{m}M$, then the preimages x_i generate M.

Proof. $M/\mathfrak{m}M$ is annihilated by $\mathfrak{m}$, making it a $A/\mathfrak{m}$-module which is just a k-vector space. Let $N \subseteq M$ be the submodule generated by the x_i, then the composition of maps

$$N \to M \to M/\mathfrak{m}M \qquad n \mapsto n + \mathfrak{m}M$$

is surjective, thus $N + \mathfrak{m}M = M$. By the previous corollary, this means $N = M$. □

This version tells us that for a local ring A, we can turn any A-module into a vector space through a quotient. We also get a way to convert between its basis and generators of the original module.

2.2 The Tensor Product

Recall earlier that there is no way to define the product of two modules. There actually is a way to define something similar to a product of two modules called the tensor product. But first, some preliminaries:

Definition. *Let M, N, P be A-modules. A map $f : M \times N \to P$ is A-bilinear if it is A-linear in each component. That is,*

$$f(ax + y, z) = af(x, z) + f(y, z)$$
$$f(x, ay + z) = af(x, y) + f(x, z)$$

The main purpose of a tensor product is to give a correspondence between A-bilinear maps and normal A-linear maps. This is given by a universal property and the following proposition will define it.

Proposition 2.9. Let M, N be A-modules, then there exists an A-module T and an A-bilinear map $g : M \times N \to M \otimes N$ with a universal property:

Given any A-module P and any other A-bilinear map $f : M \times N \to P$, g factors through a unique A-linear map $f' : T \to P$. This is represented by the commutative diagram

$$\begin{tikzcd} M \times N \arrow[r, "g"] \arrow[dr, "f"'] & T \arrow[d, dashed, "f'"] \\ & P \end{tikzcd}$$

Additionally if (T', g') is another A-module and A-bilinear map which satisfies the universal property, then there is an isomorphism $j : T \to T'$ such that $j \circ g = g'$

Proof. First we will show that the tensor product is unique up to isomorphism. Since T' is another A-module, by the universal property there exists a unique map $j : T \to T'$ such that $g' = j \circ g$. This is represented by

$$\begin{tikzcd} M \times N \arrow[r, "g"] \arrow[dr, "g'"'] & T \arrow[d, dashed, "j"] \\ & T' \end{tikzcd}$$

We can instead use the universal property of T' to get another map $j' : T' \to T$ such that $g = j' \circ g'$. Clearly $j \circ j' = j' \circ j = id$, so we have an isomorphism.

Now we show that a tensor product must exist. Define the free module

$$C = A^{M \times N} = \left\{ \sum_{i=1}^{n} a_i \cdot (x_i, y_i) \right\}$$

Let D be the submodule generated by elements of the following forms

$$\begin{gathered}(x+x',y)-(x,y)-(x',y)\\(x,y+y')-(x,y)-(x,y')\\(ax,y)-a\cdot(x,y)\\(x,ay)-a\cdot(x,y)\end{gathered}$$

Let $T = C/D$, where we mod out all the elements above. For each basis element $(x, y) \in C$, denote its image in T as $x \otimes y$, then we have the properties

$$\begin{gathered}(x+x')\otimes y = x\otimes y + x'\otimes y\\x\otimes(y+y') = x\otimes y + x\otimes y'\\(ax)\otimes y = x\otimes(ay) = a\cdot(x\otimes y)\end{gathered}$$

This means that map

$$g : M \times N \to T \qquad (x, y) \mapsto x \otimes y$$

is A-bilinear. Let $f : M \times N \to P$ also be A-bilinear, this can be extended to a map $\tilde{f} : C \to P$. From the definition of bilinearity, we see that $\tilde{f}$ vanishes on the elements which generate D and thus induces a well defined map

$$f' : T \to P \qquad (x \otimes y) \mapsto f(x, y)$$

f' is uniquely defined by f, thus T and g satisfy the universal property. □

Definition. *The A-module T in the above proposition is known as the tensor product of M and N*

$$M \otimes_A N = T = \left\{\sum x_i \otimes y_i \mid x_i \in M, y_i \in N\right\}$$

The elementary tensors $x \otimes y$ generate $M \otimes N$ as an A-module.

We have to specify the ring we tensor with respect to, but usually it is obvious so we will leave it out of the notation. If M, N respectively have generators $(x_i), (y_j)$, then we will have generators $x_i \otimes y_j$. In particular, if M, N are both finitely generated, $M \otimes N$ will also be finitely generated.

Example. The tensor product, specifically the elementary tensors, may behave in ways which are not intuitive. For instance, let $A = \mathbb{Z}, M = \mathbb{Z}, N = \mathbb{Z}/2\mathbb{Z}$ and $M' = 2\mathbb{Z} \subseteq M$. Then in $M \otimes N$

$$2 \otimes x = 1 \otimes 2x = 1 \otimes 0 = 0$$

However, if we take this in $M' \otimes N$, we cannot factor out the 2 because $1 \notin 2\mathbb{Z}$. Thus this elementary tensor is nonzero in the submodule.

For the reason we see in the example, simply writing $x \otimes y$ is ambiguous, we cannot infer anything about that tensor without more context. However, we still have this result:

Corollary 2.10. Let $x_i \in M, y_i \in N$ such that $\sum x_i \otimes y_i = 0 \in M \otimes N$. Then there exists finitely generated submodules $M' \subseteq M, N' \subseteq N$ such that the sum is also zero in $M' \otimes N'$.

Proof. Note that

$$\sum x_i \otimes y_i = 0 \iff \sum (x_i, y_i) \in D$$

Let M_0 be the submodule generated by x_i and the elements of M which occur (in the first coordinate) of D and define N_0 similarly. Then $\sum x_i \otimes y_i = 0 \in M_0 \otimes N_0$. □

This corollary lets us know that not all hope is lost. If some combination of elementary tensors yields zero then there is at least one submodule in which it remains zero. The tensor product obeys some "canonical isomorphisms," which gives various properties.

Proposition 2.11. Let M, N, P be A-modules, then there exists unique isomorphisms

1. Symmetric
$$M \otimes N \cong N \otimes M \qquad x \otimes y \mapsto y \otimes x$$

2. Associative
$$(M \otimes N) \otimes P \cong M \otimes (N \otimes P) \cong M \otimes N \otimes P$$
$$(x \otimes y) \otimes z \mapsto x \otimes (y \otimes z) \mapsto x \otimes y \otimes z$$

3. Distributive
$$(M \oplus N) \otimes P \cong (M \otimes P) \oplus (N \otimes P)$$
$$(x, y) \otimes z \mapsto (x \otimes z, y \otimes z)$$

4. Invariance
$$A \otimes M \cong M \qquad a \otimes x \mapsto ax$$

Proof. 1) Consider the following two maps

$$M \times N \to N \otimes M \qquad (x, y) \mapsto y \otimes x$$
$$N \times M \to M \otimes N \qquad (y, x) \mapsto x \otimes y$$

Both maps are clearly bilinear and so induce maps

$$M \otimes N \xrightarrow{f} N \otimes M \xrightarrow{g} M \otimes N$$

It's also clear that $f \circ g = g \circ f = id$, so they give the desired isomorphism.

2) Fix some element $z \in P$ and consider the map

$$(x, y) \mapsto x \otimes y \otimes z$$

This is bilinear in x and y and so induces a homomorphism

$$f_z : M \otimes N \to M \otimes N \otimes P \qquad x \otimes y \mapsto x \otimes y \otimes z$$

Now consider the map

$$(M \otimes N) \times P \mapsto M \otimes N \otimes P \qquad (t, z) \mapsto f_z(t)$$

This map is bilinear in both t and z, so we get an homomorphism

$$f : (M \otimes N) \otimes P \to M \otimes N \otimes P \qquad (x \otimes y) \otimes z \mapsto x \otimes y \otimes z$$

Now consider the map

$$M \times N \times P \to (M \otimes N) \otimes P \qquad (x, y, z) \mapsto (x \otimes y) \otimes z$$

This is bilinear in all arguments and so induces the homomorphism

$$g : M \otimes N \otimes P \to (M \otimes N) \otimes P \qquad x \otimes y \otimes z \mapsto (x \otimes y) \otimes z$$

Clearly $f \circ g = g \circ f = id$, so we get one of two desired isomorphisms. To get the other, simply repeat the above construction with $x \in M$ fixed instead of $z \in P$.

3) Consider the map

$$(M \oplus N) \times P \to (M \otimes P) \oplus (N \otimes P) \qquad (x, y, z) \mapsto (x \otimes z, y \otimes z)$$

This is clearly bilinear and thus induces a homomorphism

$$(M \oplus N) \otimes P \to (M \otimes P) \oplus (N \otimes P) \qquad (x, y) \otimes z \mapsto (x \otimes z, y \otimes z)$$

The other direction is a bit trickier, we will define the inverse using two maps

$$g_1 : M \times P \to (M \oplus N) \otimes P \qquad (x, z) \mapsto (x, 0) \otimes z$$

$$g_2 : N \times P \to (M \oplus N) \otimes P \qquad (y, z) \mapsto (0, y) \otimes z$$

In particular, both maps are bilinear and induce the relevant homomorphism on tensor products. Thus we can direct sum them together to get

$$(x, z_1) \oplus (y, z_2) \mapsto (x, 0) \otimes z_2 + (0, y) \otimes z_2$$

which is the desired inverse, establishing an isomorphism.

4) The map

$$A \times M \to M \qquad (a, x) \mapsto ax$$

is clearly bilinear from module definitions. Thus we get an induced homomorphism

$$f : A \otimes M \to M \qquad a \otimes x \mapsto ax$$

The A-linear map

$$g : M \to A \otimes M \qquad x \mapsto 1 \otimes x$$

gives an inverse since

$$g(ax) = 1 \otimes (ax) = a \otimes x$$

Thus we get the desired isomorphism. □

Suppose we have two maps $f : M \to M', g : N \to N'$. Ideally we'd like there to be a map between the two tensor products induced by f and g. Define

$$h : M \times N \to M' \otimes N' \qquad (x, y) \mapsto f(x) \otimes g(y)$$

If f, g are A-linear, then h must be A-bilinear, thus inducing a homomorphism

$$f \otimes g : M \otimes N \to M' \otimes N' \qquad x \otimes y \mapsto f(x) \otimes g(y)$$

which is exactly what we want.

Now let $f' : M' \to M'', g' : N' \to N''$ be two more A-module homomorphisms. Note

$$\begin{aligned} ((f' \circ f) \otimes (g' \circ g))(x \otimes y) &= (f' \circ f)(x) \otimes (g' \circ g)(y) \\ &= ((f' \otimes g') \circ (f \otimes g))(x \otimes y) \end{aligned}$$

Thus composition of homomorphisms obeys

$$(f' \circ f) \otimes (g' \circ g) = (f' \otimes g') \circ (f \otimes g)$$

What happens when we take two tensors over different rings at the same time? To answer this question, we need a type of module which has a defined action for two different rings.

Definition. *An (A, B)-bimodule M is a module which is simultaneously an A-module and B-module such that both actions are compatible.*

$$a(xb) = (ax)b \qquad \forall a \in A, b \in B, x \in M$$

Proposition 2.12. Let A, B be rings, M an A-module, P a B-module, and N an (A, B)-bimodule. Then $M \otimes_A N$ is naturally a B-module and $N \otimes_B P$ naturally an A-module under

$$b(m \otimes n) = m \otimes (bn) \qquad a(n \otimes p) = an \otimes p$$

Then the tensor products below are defined and distribute as

$$(M \otimes_A N) \otimes_B P \cong M \otimes_A (N \otimes_B P)$$

Proof. Proof proceeds in the same way as 1.11)2) since N is now compatible with the action of A and B. Note that the resulting tensor product is also an (A, B)-bimodule. $\square$

The tensor product allows us to complete our discussion of module-theoretic versions of ring-theoretic operations by defining restriction and extension of scalars.

Definition. *Let $f : A \to B$ be a ring homomorphism. Let N be a B-module, then it has an A-module structure defined by*

$$ax = f(a)x \qquad \forall a \in A, x \in N$$

Thus we have obtained an A-module by restriction of scalars.

Let M be an A-module, we can view B as an A-module using restriction of scalars and form the tensor $M_B = B \otimes_A B$. This is a B-module under

$$b(b' \otimes x) = (bb') \otimes x \qquad \forall b, b' \in B, x \in M$$

Thus we have obtained a B-module by extension of scalars.

We will finish this section by discussing how finitely generated modules behave under extension and restriction of scalars.

Proposition 2.13. Suppose N is finitely generated as a B-module and B is finitely generated as an A-module, then N is finitely generated as an A-module.

Proof. Let $x_1, \ldots, x_n$ generate B and $y_1, \ldots, y_n$ generate N, then the products $x_i y_j$ generate N over A. □

Proposition 2.14. If M is finitely generated as an A-module, then M_B is finitely generated as a B-module.

Proof. If $x_1, \ldots, x_n$ generate M, then $1 \otimes x_i$ generate M_B over B. □

2.3 Exact Sequences

Definition. *A sequence of A-modules and A-linear maps*

$$\cdots \longrightarrow M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \longrightarrow \cdots$$

is exact at M_i if Im $f_i =$ Ker f_{i+1}. The entire sequence is exact if it is exact at every M_i.

Example. There are three basic exact sequences to note:

1. f is injective if the following is exact

$$0 \longrightarrow M' \xrightarrow{f} M$$

2. g is surjective if the following is exact

$$M \xrightarrow{g} M'' \longrightarrow 0$$

3. A short exact sequence is an exact sequence of the form

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

This implies that f is injective and g is surjective and thus there is an isomorphism

$$M'' \cong M/\mathrm{Im}\ f = \mathrm{Coker}\ f \cong M/M'$$

Another way to draw a short exact sequence is by modifying the arrow types, this is common in category theory and related fields.

$$0 \longrightarrow M' \overset{f}{\hookrightarrow} M \overset{g}{\twoheadrightarrow} M'' \longrightarrow 0$$

The fact that f is injective and g surjective is encoded in the arrows.

In fact every long (i.e. not short) sequence can be split up into short exact sequences. Let $N_i = \text{Im } f_i = \text{Ker } f_{i+1}$ and we get an short exact sequence centered at M_i using the inclusion map

$$0 \longrightarrow N_i \xrightarrow{\iota} M_i \xrightarrow{f_{i+1}} N_{i+1} \longrightarrow 0$$

The usefulness of an exact sequence is that it allows us to learn things about modules through maps to and from them. For instance, we saw that a short exact sequence allows us to construct an isomorphism utilizing all three modules in the sequence.

Now we examine the exactness properties of $Hom(\cdot,\cdot)$.

Proposition 2.15. 1) The sequence

$$M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$$

is exact if and only if the following sequence is exact

$$0 \longrightarrow Hom(M'',N) \xrightarrow{\overline{v}} Hom(M,N) \xrightarrow{\overline{u}} Hom(M',N)$$

2) The sequence

$$0 \longrightarrow N' \xrightarrow{u} N \xrightarrow{v} N''$$

is exact if and only if the following sequence is exact

$$0 \longrightarrow Hom(M,N') \xrightarrow{\overline{u}} Hom(M,N) \xrightarrow{\overline{v}} Hom(M,N'')$$

Proof. 1) The induced maps are given by

$$\overline{v}(g) = g \circ v \qquad \overline{u}(f) = f \circ u$$

Exactness at $Hom(M,N)$ is a consequence of exactness at M

$$(\overline{u} \circ \overline{v})(f) = f \circ v \circ u = 0 \quad \forall f : M'' \to N$$

since $\text{Ker } v = \text{Im } u$, thus $\text{Ker } \overline{u} \supseteq \text{Im } \overline{v}$. On the other hand, if $f \in \text{Ker } \overline{u}$, then

$$\overline{u}(f) = f \circ u = 0 \iff f(\text{Im } u) = f(\text{Ker } v) = 0 \iff \text{Ker } v \subseteq \text{Ker } f$$

This induces a map $\overline{f} : M/\text{Ker } v \cong M'' \to N$ such that f factors through as

$$\overline{f} \circ v = \overline{v}(\overline{f}) = f \longrightarrow \text{Ker } \overline{u} \subseteq \text{Im } \overline{v}$$

thus establishing exactness. To show exactness at $Hom(M'',N)$, note

$$f \in \text{Ker } \overline{v} \iff f(v(x)) = 0 \quad \forall x \in M \iff f(\text{Im } v) = f(M'') = 0$$

Thus $\overline{v}$ is injective and the sequence is exact.

Conversely, v is surjective if $\overline{v}$ is injective since we must have Im $v = M''$ in order to conclude that $f = 0$. Exactness at $Hom(M, N)$ implies Ker $v \supseteq$ Im u since if $f = 1$

$$(\overline{u} \circ \overline{v})(f) = f \circ v \circ u = v \circ u = 0$$

For the other direction, let $N = M/\text{Im } u$ and f the canonical projection map $M \to N$. Then there must exists some $\overline{f} : M'' \to N$ such that $\overline{f} \circ v = f$ since $f \in \text{Ker } \overline{u} = \text{Im } \overline{v}$. But this means that Ker $v \subseteq$ Ker $f =$ Im u, establishing exactness.

2) This time the maps are

$$\overline{u}(f) = u \circ f \qquad \overline{v}(g) = v \circ g$$

Just as before, exactness at $Hom(M, N)$ follows from exactness at N

$$(\overline{v} \circ \overline{u})(f) = v \circ u \circ f = 0 \quad \forall f : M \to N' \longrightarrow \text{Im } \overline{u} \subseteq \text{Ker } \overline{v}$$

For the other direction, we once again construct a map by noting $f \in \text{Ker } \overline{v}$ means that Im $f \subseteq$ Ker $v =$ Im u. Since Ker $u = 0$, the map $\overline{f} = u^{-1} \circ f$ is well defined and $\overline{u}(\overline{f}) = f$, thus Ker $\overline{v} \subseteq$ Im $\overline{u}$.

To complete the proof, note that since u is injective

$$f \in \text{Ker } \overline{u} \iff u(f(x)) = 0 \iff f(x) = 0 \quad \forall x \in M$$

Conversely, exactness at N' follows from exactness at $Hom(M, N')$. Let $M = \text{Ker } u$ and consider the inclusion map $\iota : M \to N'$. Then we have

$$u(\text{Im } \iota) = u(\text{Ker } u) = 0 \longrightarrow \iota \in \text{Ker } \overline{u} \longrightarrow \iota = 0$$

which can only mean Ker $u = 0$ i.e. u is injective. By taking $f = 1$, we see that exactness at $Hom(M, N)$ implies Im $u \subseteq$ Ker v. Finally, let $M = \text{Ker } v$ and map $\iota : M \to N$ by inclusion. It's clear that $\iota \in \text{Ker } \overline{v}$ so there must exist some $f : M \to N'$ such that

$$\overline{u}(f) = u \circ f = \iota \longrightarrow u(f(n)) = \iota(n) = n \quad \forall n \in \text{Ker } v$$

Thus Ker $v \subseteq$ Im u, establishing exactness. □

We've already discussed how to split long exact sequences into a series of short exact sequences, but what if we want to go the other way. For instance, suppose we have a map between two exact sequences (we will not define what this means) and wish to encapsulate all the information in a single long exact sequence. The next important result gives us a way to do so and also gives us a glimpse into the field of homological algebra.

Lemma 2.16 (Snake Lemma)**.** Suppose we have the following commutative diagram where both rows are exact

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "u"] \arrow[d, "f'"'] & M \arrow[r, "v"] \arrow[d, "f"'] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & N' \arrow[r, "u'"'] & N \arrow[r, "v'"'] & N'' \arrow[r] & 0
\end{tikzcd}$$

Then there is an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \text{Ker } f' \arrow[r, "a"] & \text{Ker } f \arrow[r, "b"] & \text{Ker } f'' \arrow[dll, "d"'] & \\
& & \text{Coker } f' \arrow[r, "\overline{u}'"] & \text{Coker } f \arrow[r, "\overline{v}'"] & \text{Coker } f'' \arrow[r] & 0
\end{tikzcd}$$

in which $\overline{u}', \overline{v}'$ are restrictions of u', v' to their cokernels. The map d is known as the boundary homomorphism in homological algebra and algebraic topology. It is defined as follows:

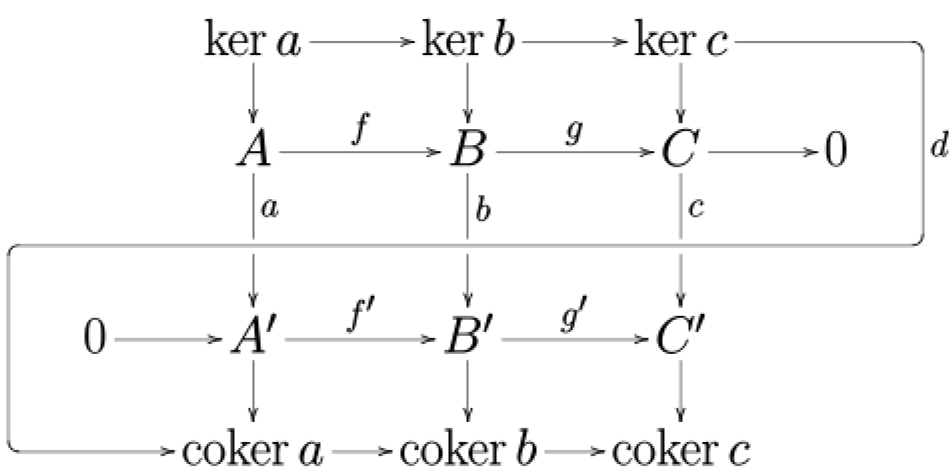
Let $x'' \in \text{Ker } f''$, since v is surjective we have $v(x) = x''$ for some $x \in M$. Then

$$f''(v(x)) = v'(f(x)) = 0$$

since the diagram commutes and thus $f(x) \in \text{Ker } v' = \text{Im } u'$. Thus $u'(y') = f(x)$ for some $y' \in N'$ and define $d(x'') = \overline{y}' \in N'/\text{Im } f' = \text{Coker } f'$.

Proof. This is a classic example of diagram chasing. □

The above proposition is called the snake lemma because we can encapsulate it all in one diagram using a snaking arrow to connect the kernels and cokernels.[1]



Definition. *Let $\mathcal{C}$ be a collection of A-modules and λ a function $\mathcal{C} \to \mathbb{Z}$. We say λ is additive if for every short exact sequence*

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

where $M, M', M'' \in \mathcal{C}$, we have

$$\lambda(M') - \lambda(M) + \lambda(M'') = 0$$

[1]Picture from wikipedia. We map Ker $a \to A$ by inclusion and $A' \to$ Coker a using the canonical quotient homomorphism.

This definition may seem vague and arbitrary at first, but we've actually seen an additive function before.

Example. Let $A = k$ be a field and $\mathcal{C}$ the set of all k-vector spaces. Then the function $V \mapsto \dim V$ is additive. To see this, if we have a short exact sequence

$$0 \longrightarrow V' \xrightarrow{f} V \xrightarrow{g} V'' \longrightarrow 0$$

Then the rank-nullity theory tells us that

$$\dim V = \text{Rank } g + \text{Nullity } g = \dim(\text{Im } g) + \dim(\text{Ker } g) = \dim V'' + \dim V'$$

$$\therefore \dim V' - \dim V + \dim V'' = 0$$

Proposition 2.17. Consider the exact sequence

$$0 \longrightarrow M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} M_n \longrightarrow 0$$

in which every M_i and kernel belongs to $\mathcal{C}$. Then for any additive function $\lambda : \mathcal{C} \to \mathbb{Z}$

$$\sum_{i=0}^{n} (-1)^i \lambda(M_i) = 0$$

Proof. Recall that we can split any exact sequence into short exact sequences

$$0 \longrightarrow N_i \longrightarrow M_i \longrightarrow N_{i+1} \longrightarrow 0$$

where $N_i = \text{Ker } f_{i+1}$ (note $N_0 = N_{n+1} = 0$). Thus we can apply the definition to get

$$\lambda(M_i) = \lambda(N_i) + \lambda(N_{i+1})$$

and the alternating sum must vanish. $\square$

The last topic on exact sequences we will discuss is the exactness of tensor products, that is: when does tensoring preserve exactness? As we will see, this leads us to a very important class of modules.

Lemma 2.18. Let M, N, P be A-modules, there is a canonical isomorphism

$$Hom(M \otimes N, P) \cong Hom(M, Hom(N, P))$$

Proof. Let $f : M \times N \to P$ be an A-bilinear map. For every $x \in M$, then map $y \mapsto f(x, y)$ is A-linear, thus inducing an A-linear map $M \to Hom(N, P)$. Conversely, the homomorphism $\phi : M \to Hom_A(N, P)$ defines the bilinear map $(x, y) \mapsto \phi(x)\phi(y)$.

Thus we have a one-to-one correspondence

$$\{A\text{-bilinear maps } M \times N \to P\} \iff Hom(M, Hom(N, P))$$

But the universal property of tensors gives another correspondance

$$\{A\text{-bilinear maps } M \times N \to P\} \iff Hom(M \otimes N, P)$$

Thus we get the desired isomorphism. $\square$

Proposition 2.19. Suppose we have an exact sequence of A-modules,

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

Let N be any other A-module, the following sequence is exact

$$M' \otimes N \xrightarrow{f\otimes 1} M \otimes N \xrightarrow{g\otimes 1} M'' \otimes N \longrightarrow 0$$

where 1 denotes the identity map on N.

Proof. Using the exactness of $Hom(\cdot,\cdot)$ and the lemma we just proved, we get an exact seqeunce

$$0 \longrightarrow Hom(M'' \otimes N, P) \longrightarrow Hom(M \otimes N, P) \longrightarrow Hom(M' \otimes N, P)$$

Thus we get the desired result by again using the exactness of Hom. □

The result we just proved tells us that, in category theory terms, the functor $M \mapsto M \otimes N$ is right exact since the 0 is placed on the right. Similarly we can define left exact functors, functors which are exact in both directions are simply called exact functors and have a special place in our hearts. Unfortunately the tensor product is not exact in general.

Example. Suppose we have an exact sequence

$$0 \to \mathbb{Z} \xrightarrow{f} \mathbb{Z} \qquad x \mapsto 2x$$

Let $N = \mathbb{Z}/2\mathbb{Z}$, then the tensored sequence

$$0 \longrightarrow \mathbb{Z} \otimes N \xrightarrow{f\otimes 1} \mathbb{Z} \otimes N$$

is not exact because for any $x \otimes y \in \mathbb{Z} \otimes N$

$$(f \otimes 1)(x \otimes y) = 2x \otimes y = x \otimes 2y = x \otimes 0 = 0$$

So $f \otimes 1$ is the zero map but $\mathbb{Z} \otimes N$ is not a zero module.

If the tensor product were to be exact, then tensoring any exact sequence will yield an exact sequence back. While not true in general, there are certain modules for which this is possible.

Definition. *A module N is flat if tensoring by N transforms exact sequences into exact sequences i.e. the functor $M \mapsto M \otimes N$ is exact.*

Proposition 2.20. Let N be an A-module, the following are equivalent:

1. N is flat.

2. If we have an exact sequence

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

   Then the tensored sequence below is also exact

$$0 \longrightarrow M' \otimes N \xrightarrow{f \otimes 1} M \otimes N \xrightarrow{g \otimes 1} M'' \otimes N \longrightarrow 0$$

3. If $f : M' \to M$ is injective, then $f \otimes 1 : M' \otimes N \to M \otimes N$ is injective.

4. If $f : M' \to M$ is injective and M, M' finitely generated, then $f \otimes 1 : M' \otimes N \to M \otimes N$ is injective.

Proof. $1 \iff 2$) $1 \to 2$ by definition, the reverse direction by the fact that every exact sequence can be split into short exact sequences.

$2 \iff 3$) $2 \to 3$ is obvious, the reverse direction was proven in the earlier proposition.

$3 \iff 4$) $3 \to 4$ is obvious. For the other direction, let M, M' be two (not necessarily finitely generated) A-modules and $f : M' \to M$ an injective map. Suppose

$$u = \sum x_i' \otimes y_i \in \text{Ker } f \otimes 1 \longrightarrow f(u) = \sum f(x_i') \otimes y_i = 0 \in M \otimes N$$

Let M_0' be the submodule (finitely generated) by the x_i' and call let u_0 denote the element u in $M_0' \otimes N$. We proved earlier that there must exist some finitely generated submodule $M_0 \subseteq M$ which contains $f(M_0')$ such that $\sum f(x_i) \otimes y_i = 0$ in $M_0 \otimes N$.

Let $f_0 : M_0' \to M_0$ be the restriction of f, which is also injective. Then since both M_0' and M_0 are finitely generated, $f_0 \otimes 1$ is injective by assumption. Thus

$$(f_0 \otimes 1)(u_0) = 0 \iff u_0 = 0 \iff u = 0$$

□

Proposition 2.21. If $f : A \to B$ is a ring homomorphism and M is a flat A-module, then the extension $M_B = B \otimes_A M$ is flat.

Proof. We can use the isomorphisms proven earlier to show that for any B-module N

$$N \otimes_B M_B = N \otimes_B (B \otimes_A M) \cong (N \otimes B) \otimes_A M \cong N \otimes_A M$$

Then the fact that M_B is flat follows from the fact that M is flat since we can convert exact sequences using the formula above. □

2.4 Algebras

An object closely related to modules is the concept of an algebra.

Definition. *Let $f : A \to B$ be a ring homomorphism and define*

$$ab = f(a)b \qquad \forall a \in A, b \in B$$

This gives B an A-module structure in addition to its ring structure, making it an A-algebra. In other words, an A-algebra is a ring B with a ring homomorphism $f : A \to B$.

Example. Trivially, every ring A is automatically a $\mathbb{Z}$-algebra under the map $n \mapsto n \cdot 1$.

Example. An interesting example is when $A = k$ is a field. Then the map must be injective, giving an isomorphism between k and it's image. Thus a k-algebra is just any ring which contains (something isomorphic to) k as a subring.

Now we can lift some concepts from studying rings/modules into algebras.

Definition. *Let $f : A \to B, g : A \to C$ be ring homomorphisms. An A-algebra homomorphism $h : B \to C$ is a ring homomorphisms which is also A-linear.*

There's an easy characterization of A-algebra homomorphisms.

Proposition 2.22. Let the maps f, g, h be as before, then h is an A-algebra homomorphisms if and only if $h \circ f = g$.

Proof. h is A-linear if and only if it compatible with the module actions of B and C. Take $1 \in B$, as a ring homomorphism we have $h(1) = 1$. Then for any $a \in A$

$$h(f(a)) = h(a \cdot 1) = a \cdot h(1) = a \cdot 1 = g(a)$$

In other words, $h \circ f = g$. $\square$

There are multiple notions of finiteness in algebras.

Definition. *A ring homomorphism $f : A \to B$ is finite if B is finitely generated as an A-module, we say that B is a finite A-algebra.*

f is of finite type, and B a finite-generated A-algebra, if there exists elements $x_1, \ldots, x_n$ such that for every $b \in B$, we can write

$$b = \sum f(a_i)x_i \quad a_i \in A$$

An equivalent definition for a finitely-generated A-algebra is that there is some surjective A-algebra homomorphism $A[t_1, \ldots, t_n] \to B$ for some integer $n > 0$.

Definition. *A ring A is finitely generated if it is finitely generated as a $\mathbb{Z}$-algebra. Equivalently, there are finitely many $x_1, \ldots, x_n \in A$ such that every other element is a $\mathbb{Z}$-linear combination of those x_i.*

Since every A-algebra is an A-module, we can tensor them together. Since A-algebras carry both a ring and module structure, we must take extra care in making sure everything is compatible.

Let B, C be two A-algebras with corresponding homomorphsims $f : A \to B, g : A \to C$. Let $D = B \otimes_A C$ be their tensor product, we will define how multiplication works using the universal property. Define the map

$$B \times C \times B \times C \to D \qquad (b, c, b', c') \mapsto bb' \otimes cc'$$

This is A-linear in each component and thus induces the A-module homomorphism

$$B \otimes C \otimes B \otimes C \to D$$

which is just an A-module homomorphism $D \otimes D \to D$. Going backwards, this corresponds to an A-bilinear map

$$D \times D \to D \qquad (b \otimes c, b' \otimes c') \mapsto bb' \otimes cc'$$

Thus we have shown multiplication is well defined in this way. In general we will have

$$\left(\sum_i b_i \otimes c_i\right)\left(\sum_j b_j' \otimes c_j'\right) = \sum_{i,j} (b_i b_j') \otimes (c_i c_j')$$

It's easy to see that this makes D into a commutative ring with identity $1 \otimes 1$. The map $a \mapsto f(a) \otimes g(a)$ gives a ring homomorphism $A \to D$ and thus it acquires an A-module structure under

$$a(b \otimes c) = (f(a) \otimes g(a))(b \otimes c) = f(a)b \otimes g(a)c$$

making $D = B \otimes C$ an A-algebra as expected.

2.5 Additional Exercises

Exercise 1. m, n are coprime if $gcd(m, n) = 0$. Using the Euclidean algorithm we can find integers a, b such that $am + bn = gcd(m, n) = 1$. Thus for any $x \otimes y \in (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$

$$x \otimes y = 1 \cdot (x \otimes y) = (am + bn)(x \otimes y) = a(mx \otimes y) + b(x \otimes ny) = 0$$

Since all the elementary tensors are zero, the whole tensor product is the zero module.

Exercise 2. Let A be a ring and $\mathfrak{a} \subseteq A$ an ideal. Then the inclusion and projection maps form an exact sequence

$$0 \longrightarrow \mathfrak{a} \xrightarrow{\iota} A \xrightarrow{\pi} A/\mathfrak{a} \longrightarrow 0$$

Since the tensor product is right exact, we get another exact sequence

$$\mathfrak{a} \otimes M \xrightarrow{\iota \otimes 1} A \otimes M \xrightarrow{\pi \otimes 1} A/\mathfrak{a} \otimes M \longrightarrow 0$$

Using the canonical isomorphism $a \otimes x \mapsto ax$

$$\mathfrak{a} \otimes M \cong \mathfrak{a}M \qquad A \otimes M \cong M$$

Thus the exactness of this sequence implies

$$A/\mathfrak{a} \otimes_A M \cong A \otimes M/\mathfrak{a} \otimes M \cong M/\mathfrak{a}M$$

Exercise 3. Let A be a local ring, $\mathfrak{m}$ its maximal ideal, and $k = A/\mathfrak{m}$ its residue field. Let M, N be two finitely generated A-modules and define

$$M_k = k \otimes_A M \cong M/\mathfrak{m}M$$

By Nakayama's lemma, if $M_k = 0$, then $M = \mathfrak{m}M$ and thus $M = 0$. Turning back to the problem at hand, if $M \otimes_A N = 0$, then $(M \otimes_A N)_k = 0$ and thus

$$M_k \otimes_k N_k = M \otimes_A (k \otimes_k N_k) \cong M \otimes_A (N \otimes_A k) \cong (M \otimes_A N) \otimes_A k = 0$$

But by (another version of) Nakayama's lemma, M_k and N_k are both vector spaces since M, N are finitely-generated. Thus for the tensor product to vanish, either $M_k = 0$ or $N_k = 0$ i.e. $M = 0$ or $N = 0$.[2]

Exercise 4. We just need to prove that $M \oplus M'$ is flat if and only if M, M' are both flat. The rest of the result follows from induction (base case is obvious). Suppose we have an exact sequence E

$$0 \longrightarrow N' \xrightarrow{f} N \xrightarrow{g} N'' \longrightarrow 0$$

[2] One way to see this is to note that vector space cannot have zero divisors, so if every elementary tensor vanishes, one of the module must be zero. Alternatively, note that dimensions of vector spaces multiply under tensor product.

First let $M \oplus M'$ be flat, then the sequence

$$E \otimes (M \oplus M') \cong (E \otimes M) \oplus (E \otimes M')$$

is exact. We can split the maps $f \otimes 1$ and $g \otimes 1$ into a direct sum, for instance

$$f \otimes 1 : N' \otimes (M \oplus M') \to N \otimes (M \oplus M')$$

can equivalently be expressed as

$$(f \otimes 1, f \otimes 1) : (N' \otimes M) \oplus (N' \otimes M') \to (N \otimes M) \oplus (N \otimes M')$$

Thus the sequence $E \otimes (M \oplus M')$ is exact if and only if the subsequences $E \otimes M$ and $E \times M'$ are both exact. In other words $M \oplus M'$ is flat if and only if M and M' are both flat.

Exercise 5. $A[x]$ is an A-algebra under the inclusion map. We can compose $A[x]$ are an (infinite) direct sum using the isomorphism

$$A[x] \cong A^{\mathbb{N}} \qquad a_0 + a_1 x + a_2 x^2 + \cdots \mapsto (a_0, a_2, a_2, \ldots)$$

Using the previous exercise, $A[x]$ is flat if and only if A is flat. The fact that A is flat is trivial since for any A-module M, $M \otimes A \cong M$.

Exercise 6. Let M be an A-module and define $M[x]$ to be the set of polynomials with coefficients in M. $M[x]$ is an $A[x]$-module under normal polynomial multiplication, which clearly distributes as expected and is associative.

Consider the map

$$\phi : A[x] \times M \to M[x] \qquad \left(\sum_{i=0}^{n} a_i x^i, m\right) \mapsto \sum_{i=0}^{n} (a_i \cdot m) x^i$$

It's easy to see that this is bilinear and thus induces an A-module homomorphism

$$\overline{\phi} : A[x] \otimes_A M \to M[x] \qquad \left(\sum_{i=0}^{n} a_i x^i\right) \otimes m = \sum_{i=0}^{n} (a_i x^i) \otimes m \mapsto \sum_{i=0}^{n} (a_i \cdot m) x^i$$

Conversely, define the map

$$\psi : M[x] \to A[x] \times M \qquad \sum_{i=0}^{n} m_i x^i \mapsto \sum_{i=0}^{n} (x^i, m_i)$$

This is A-linear, so it is a valid module homomorphism. The composition $j \circ \psi : M[x] \to A[x] \otimes M$ gives an inverse to $\overline{\phi}$, thus establishing the desired isomorphism.

Exercise 7. Let $\mathfrak{p}$ be prime and consider the map

$$\phi : A[x] \to (A/\mathfrak{p})[x] \qquad \sum a_i x^i \mapsto \overline{a}_i x^i$$

The kernel is the set of polynomial which have coefficients in $\mathfrak{p}$, thus there is an isomorphism

$$A[x]/\mathfrak{p}[x] \cong (A/\mathfrak{p})[x]$$

An element in $f \in (A/\mathfrak{p})[x]$ is a zero divisor if and only if there exists an $a \in A/\mathfrak{p}$ such that $af = 0$, which is impossible because $A\mathfrak{p}$ is an integral domain. Thus $A[x]/\mathfrak{p}[x]$ is an integral domain and $\mathfrak{p}[x]$ is prime.

This is not true for maximal ideals. Prime ideals are maximal ideals in $\mathbb{Z}$, for instance take $(2) = 2\mathbb{Z}$. The ideal $2\mathbb{Z}[x] \subseteq \mathbb{Z}[x]$ is not maximal because it is properly contained in the ideal $(2, x)$ ($x \in (2, x) \setminus 2\mathbb{Z}[x]$).

Exercise 8. i) Let E be an exact sequence. Since M is flat, the sequence $E' = E \otimes M$ and since N is flat, the sequence

$$E' \otimes N = (E \otimes M) \otimes N \cong E \otimes (M \otimes N)$$

is exact, thus $M \otimes N$ is flat.

ii) If E is an exact sequence, then since B is flat, $E' = E \otimes_A B$ is an exact sequence. Since N is flat as a B-module, the sequence

$$E' \otimes_B N = (E \otimes_A B) \otimes_B N \cong E \otimes_A (B \otimes_B N) \cong E \otimes_A N$$

is exact. Thus N is also flat as an A-module.

Exercise 9. Suppose we have an exact sequence

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

Suppose $x_1, \ldots, x_n$ generate M' and $z_1, \ldots, z_m$ generate M''. Since g is surjective, let $y_1, \ldots, y_m \in M$ such that $g(y_i) = z_i$. For any $y \in M$

$$g(y) = \sum a_i z_i = \sum_{i=1}^{m} a_i g(y_i) = g\left(\sum_{i=1}^{m} a_i y_i\right) \longrightarrow y - \sum_{i=1}^{m} a_i y_i \in \text{Ker } g$$

where $a_i \in A$. The exact sequence tells us that Ker g = Im f and that f is injective. Thus there is a well defined preimage $u \in M'$ such that

$$f(u) = f\left(\sum_{i=1}^{n} a_i' x_i\right) = \sum_{i=1}^{n} a_i' f(x_i) = y - \sum_{i=1}^{m} a_i y_i \qquad a_i \in A$$

Thus we can write y as the sum

$$y = \sum_{i=1}^{m} a_i y_i + \sum_{i=1}^{n} a_i' f(x_i)$$

and we see that M is generated by the y_i and $f(x_i)$. In particular, M is generated by preimages of generators of M'' and images of generates of M'.

Exercise 10. Let $\mathfrak{a} \subseteq \mathfrak{R}$ and M, N A-modules with N finitely generated. Let $u : M \to N$ be an A-linear map such that the induced map $\overline{u} : M/\mathfrak{a}M \to N/\mathfrak{a}N$ is surjective. The induced map is just

$$M \xrightarrow{u} N \xrightarrow{\pi} N/\mathfrak{a}N$$

Since this is surjective, we have

$$N = (\pi \circ u)(M) = u(M)/\mathfrak{a}N \longrightarrow N = \mathfrak{a}N + u(M)$$

N is finitely generated and $\mathfrak{a} \subseteq \mathfrak{R}$, so we can use Nakayama's lemma and conclude that $u(M) = N$. Visually this means that commutativity of the below diagram forces the remaining arrow to be surjective.

$$\begin{tikzcd}
M \arrow[r, two heads, "\pi_M"] \arrow[d, "u"'] & M/\mathfrak{a}M \arrow[d, two heads, "\overline{u}"] \\
M \arrow[r, two heads, "\pi_N"] & N/\mathfrak{a}N
\end{tikzcd}$$

Exercise 11. Suppose $A^m \cong A^n$, let $\mathfrak{m} \subseteq A$ be maximal. Then there is an isomorphism

$$(A/\mathfrak{m}) \otimes A^m \cong (A\mathfrak{m}) \otimes A^n$$

These tensor products are both k-modules where $k = A/\mathfrak{m}$ is the residue field, so they are k-vector spaces. Isomorphic vector spaces much have equivalent dimension, so $m = n$.

Suppose $\phi : A^m \to A^n$ is surjective. Then if $e_1, \ldots, e_m$ generate A^m, $\overline{e}_1, \ldots, \overline{e}_m$ generate A^n. Thus A^n is at most m-dimensional $m \geq n$.

Now suppose $\phi : A^m \to A^n$ is injective with $m > n$. If we tensor as before with $A/\mathfrak{m}$, then we get an injective map from a vector space of dimension m to a vector space of dimension n. In other words we have an injective linear transformation $T : U \to V$ where $\dim U > \dim V$, so by rank-nullity

$$\text{null } T = \dim V - \text{rank } T$$

The rank of T is just the dimension of its image, which must be at most the dimension of U

$$\text{rank } T \leq \dim U \longrightarrow \text{null } T \geq \dim V - \dim U > 0$$

which is a contradiction since the null space must be trivial (T is injective). Thus any injective map must satisfy $m \leq n$.

Exercise 12. Let M be finitely generated and $\phi : M \to A^n$ a surjective homomorphism. Let $e_1, \ldots, e_n$ be the generators of A^n and choose $u_i \in M$ such that $\phi(u_i) = e_i$. Then for any $x \in M$

$$\phi(x) = \sum_{i=1}^{n} a_i e_i = \sum_{i=1}^{n} a_i \phi(e_i) = \phi\left(\sum_{i=1}^{n} a_i u_i\right)$$

Thus

$$x - \sum_{i=1}^{n} a_i u_i \in \text{Ker } \phi$$

So every element of $M/\text{Ker } \phi$ can be written as a linear combination of the u_i and we can write a decomposition for M

$$M \cong \text{Ker } \phi \oplus (u_1, \ldots, u_n)$$

But M is finitely generated, so the right side is also finitely generated, thus Ker ϕ must be finitely generated.

Exercise 13. Let N be a B-module and $f : A \to B$ a ring homomorphism. We can regard N as an A-module under restriction of scalars and then extend them to form the B-module $N_B = B \otimes_A N$. Consider the map

$$g : N \to N_B \qquad y \mapsto 1 \otimes y$$

Suppose $g(y) = 1 \otimes y = 0$, then the tensor product $(1) \otimes (y) = 0$ and since both constituent modules are finitely generated (by 1 and y), at least one of them must be zero. If (1) is zero, then everything becomes the trivial zero module so $(y) = 0$, thus $y = 0$ and g is injective.

Define the map

$$p : N_B \to N \qquad b \otimes y \mapsto by$$

For any elementary tensor $b \otimes y \in N_B$, we have

$$b \otimes y = b(1 \otimes n) + 0 \otimes n \in \text{Im } g + \text{Ker } p$$

Thus $N_B = \text{Im } g \oplus \text{Ker } p$.

2.5.1 Direct Limits

Exercise 14. We say a partially ordered set I is directed if for any $i, j \in I$ there exists some $k \in i$ such that either $i \leq k$ or $j \leq k$. Let A be a ring, I a directed set and M_i a family of A-modules indexed by I. For any pair $i, j \in I$ such that $i \leq j$, let $\mu_{ij} : M_i \to M_j$ be an A-linear map such that:

1. $\mu_{ii} : M_i \to M_i$ is the identity map for all $i \in I$
2. $\mu_{ik} = \mu_{jk} \circ \mu_{ij}$ whenever $i \leq j \leq k$

If this is satisfied, we say that $\boldsymbol{M} = (M_i, \mu_{ij})$ form a direct system over the directed set I.

Given a direct system, let C be the direct sum of every M_i and identify each of them with their canonical image in C. Let $D \subseteq C$ be the submodule generated by

$$\{x_i - \mu_{ij}(x_i) \mid i \leq j, x_i \in M_i\}$$

Define $M = C/D$ and $\mu : C \to M$ be the canonical projection map with restriction $\mu_i : M_i \to M$. We say that this pair is the direct limit of $\boldsymbol{M}$, denoted

$$\varinjlim M_i = (M, \mu_i)$$

Note from this construction that whenever $i \leq j$, we have

$$x_i = \mu_{ij}(x_i) \quad \forall x_i \in M_i \longrightarrow \mu_i = \mu_j \circ \mu_{ij}$$

Here's a commutative diagram illustrating a direct limit for three modules.

$$\begin{tikzcd}
M_1 \arrow[rr, "\mu_{12}"] \arrow[dr, "\mu_{13}"] \arrow[ddr, "\mu_1"'] & & M_2 \arrow[dl, "\mu_{23}"'] \arrow[ddl, "\mu_2"] \\
& M_3 \arrow[d, "\mu_3"] & \\
& M &
\end{tikzcd}$$

Exercise 15. Any $\overline{x} \in M$ can be written as

$$\overline{x} = (x_1, x_2, \ldots) + D \in M = C/D = \left(\bigoplus M_i\right)/D$$

Let $x \in C$ be the preimage, since we've identified each M_i with their image in C we can pull each component back. Let I' as the indices which carry a nonzero component, i.e.

$$I' = \{i \in I \mid x_i \neq 0\}$$

and define $y_i \in M_i$ such that

$$\mu_i(y_i) = (0, \ldots, 0, x_i, 0, \ldots) \longrightarrow x = \sum_{i \in I'} \mu_i(y_i)$$

The existences of supremums is implied in directed sets so let $j = \sup I'$. Thus $i \leq j$ for any $i \in I'$, so $\mu_i = \mu_j \circ \mu_{ij}$ and we can rewrite the sum

$$x = \sum_{i \in I'} \mu_j(\mu_{ij}(y_i)) = \mu_j \left(\sum_{i \in I'} \mu_{ij}(y_i)\right)$$

The argument in the parenthesis is an element of M_j (also recall that $\mu_{jj} = id$) so we have found an element $x_j \in M_j$ such that $x = \mu_j(x_j)$ as required.

Now suppose $\mu_i(x_i) = 0$, then x_i is in the submodule generated by elements $x_i - \mu_{ij}(x_i)$ where $j \geq i$.

$$x_i = \sum_{j \leq k} y_j - \mu_{jk}(y_j)$$

where we assume the coefficients to be 1 and the sum is over unique pairs due to properties of A-linear maps. Recall that $\mu_i : M_i \to M$ is just the projection map restricted to one summand, so we must have $j = i$ in the right-hand sum. Let's re-index and write

$$x_i = \sum_{i \leq j} y_i - \mu_{ij}(y_i)$$

Let k be the supremum of all indices j which appear in the sum. Then

$$\mu_{ik}(x_i) = \sum_{i \leq j} \mu_{ik}(y_i) - \mu_{jk}(\mu_{ij}(x_i)) = 0$$

where we use $\mu_{ik} = \mu_{jk} \circ \mu_{ij}$ to propertly map the second term.

Exercise 16. The information presented is encoded within the following commutative diagram

$$\begin{tikzcd}
M_i \arrow[dd, "\mu_{ij}"'] \arrow[dr, "\mu_i"] \arrow[drr, bend left, "\alpha_i"] & & \\
& M \arrow[r, dotted, "\alpha"'] & N \\
M_j \arrow[ur, "\mu_j"'] \arrow[urr, bend right, "\alpha_j"'] & &
\end{tikzcd}$$

We will show that $M = \varinjlim M_i$ satisfies this property. The α_i can be direct summed together to get $\alpha' : C \to N$. Consider the generators of D under the map α'

$$\alpha'(x_i - \mu_{ij}(x_i)) = \alpha_i(x_i) - \alpha_j(\mu_{ij}(x_i)) = \alpha_i(x_i) - \alpha_i(x_i) = 0$$

Thus $D \subseteq \text{Ker } \alpha'$ and there is an induced homomorphism $\alpha : M = C/D \to N$ which is uniquely defined by the α_i. Note that it satisfies

$$\alpha(\mu_i(x_i)) = \alpha'(x_i) = \alpha_i(x_i)$$

Suppose M' also satisfies the above property, then we can take $N = M'$ to get a map $\alpha : M \to M'$. Swapping the order we get a map $\beta : M' \to M$, we will show that they are inverses. Let μ_i' be the corresponding maps to the direct limit M', then

$$\mu_i' = \alpha \circ \mu_i \qquad \mu_i = \beta \circ \mu_i'$$

In particular

$$\mu_i = \beta \circ \alpha \circ \mu \longrightarrow \beta \circ \alpha = id$$

Thus the defined module is unique up to isomorphism.

Exercise 17. Let $M = \bigcup M_i$ and suppose there exists an A-module N such that for every $i \in I$ there is an A-linear map $\alpha_i : M_i \to N$ such that $\alpha_i = \alpha_j \circ \mu_{ij}$. Since $\mu_{ij} : M_i \to M_j$ are just the inclusion maps, this is equivalent to saying $\alpha_j|_{M_i} = \alpha_i$. Define

$$\alpha : M \to N \qquad \alpha = \bigcup \alpha_i$$

This is well defined since the α_i agree on their intersections, allowing us to "glue" them together. Thus we see that $\alpha \circ \mu_i = \alpha|_{M_i} = \alpha_i$ and so

$$\varinjlim M_i = \bigcup M_i$$

Clearly $\bigcup M_i \subseteq \sum M_i$ since 0 is in each submodule and the converse inclusion follows from the fact that for each $i, j \in I$ there exists some $k \in I$ such that $M_i + M_k \subseteq M_j$. Thus

$$\sum M_i = \bigcup M_i = \varinjlim M_i$$

In particular, any A-module is the direct limit of its finitely generated submodules.

Exercise 18. Consider two directed systems $\boldsymbol{M} = (M_i, \mu_{ij}), \boldsymbol{N} = (N_i, \nu_{ij})$. A homomorphism $\boldsymbol{M} \to \boldsymbol{N}$ is a set of A-module homomorphisms

$$\phi_i : M_i \to N_i \qquad \phi_j \circ \mu_{ij} = \nu_{ij} \circ \phi_i$$

This is represented by the commutative diagra

$$\begin{tikzcd}
M_i \arrow[rrr, "\phi_i"] \arrow[dd, "\mu_{ij}"'] \arrow[dr, "\mu_i"] & & & N_i \arrow[dl, "\nu_i"'] \arrow[dd, "\nu_{ij}"] \\
& M \arrow[r, "\phi"] & N & \\
M_j \arrow[rrr, "\phi_j"] \arrow[ur, "\mu_j"'] & & & N_j \arrow[ul, "\nu_j"]
\end{tikzcd}$$

Let M, N be the corresponding direct limits. Define

$$\alpha_i : M_i \to N \qquad x_i \mapsto (\nu_i \circ \phi_i)(x_i)$$

and note that it satisfies the hypothesis of the universal property

$$\alpha_j \circ \mu_{ij} = \nu_j \circ \phi_j \circ \mu_{ij} = \nu_j \circ \nu_{ij} \circ \phi_i = \nu_i \circ \phi_i = \alpha_i$$

Thus there exists a unique homomorphism $\phi : M \to N$ such that

$$\phi \circ \mu_i = \alpha_i = \nu_i \circ \phi_i$$

Exercise 19. Suppose we have a exact sequence of direct systems

$$\boldsymbol{M} \longrightarrow \boldsymbol{N} \longrightarrow \boldsymbol{P}$$

that is, we have an exact sequence for every $i \in I$

$$M_i \xrightarrow{f_i} N_i \xrightarrow{g_i} P_i$$

Consider the induced sequence on their direct limits

$$M \xrightarrow{f} N \xrightarrow{g} P$$

Since every element can be written as $\mu_i(x_i)$ for some $i \in I$

$$(g \circ f)(x) = (g \circ f \circ \mu_i)(x_i) = (g \circ \nu_i \circ f_i)(x_i) = (\rho_i \circ g_i \circ f_i)(x_i) = \rho_i(0) = 0$$

Thus $\text{Im } f \subseteq \text{Ker } g$ follows directly from exactness at N_i. Now consider some $x \in \text{Ker } g$ so that $\nu_i(x_i) = x$ for some $i \in I$

$$g(x) = (g \circ \nu_i)(x_i) = (\rho_i \circ g_i)(x_i) = 0$$

But this means that there exists some $j \geq i$ such that

$$(\rho_{ij} \circ g_i)(x_i) = (g_j \circ \nu_{ij})(x_i) = 0 \longrightarrow \nu_{ij}(x_i) \in \text{Ker } g_j = \text{Im } f_j$$

and so there exists some $y_j \in M_j$ such that $f_j(y_j) = \nu_{ij}(x_i)$, which implies

$$x = \nu_i(x_i) = (\nu_j \circ \nu_{ij})(x_i) = (\nu_j \circ f_j)(y_j) = (f \circ \mu_j)(y_j) = f(y)$$

Thus $\text{Ker } g \subseteq \text{Im } f$ and the direct limits form an exact sequence under the induced homomorphisms. We can summarize the exercise as follows: If the top and bottom rows are exact for all $i \leq j \in I$ in the below commutative diagram, then the middle row is also exact.

$$\begin{tikzcd}
M_i \arrow[rr, "f_i"] \arrow[dr, "\mu_i"] \arrow[dd, "\mu_{ij}"'] & & N_i \arrow[rr, "g_i"] \arrow[dr, "\nu_i"] \arrow[dd, "\nu_{ij}"' near end] & & P_i \arrow[dr, "\rho_i"] \arrow[dd, "\rho_{ij}"' near end] & \\
& M \arrow[rr, "f"] & & N \arrow[rr, "g"] & & P \\
M_j \arrow[rr, "f_j"] \arrow[ur, "\mu_j"'] & & N_j \arrow[rr, "g_j"] \arrow[ur, "\nu_j"'] & & P_j \arrow[ur, "\rho_j"'] &
\end{tikzcd}$$

2.5.2 Tensors and Direct Limits

Exercise 20. Given a direct system (M_i, μ_{ij}) and any A-module N, we can form another direct system $(M_i \otimes N, \mu_{ij} \otimes 1)$. Let P be the direct limit of the latter system

$$P = \varinjlim (M_i \otimes N) \qquad \nu_i : M_i \otimes N \to P$$

For every $i \in I$, we can form the map $\mu_i \otimes 1 : M_i \otimes N \to M \otimes N$ and note that it satisfies the hypothesis of the universal property

$$\mu_j \otimes 1 \circ \mu_{ij} \otimes 1 = (\mu_j \circ \mu_{ij}) \otimes (1 \circ 1) = \mu_i \otimes 1$$

Thus there is a unique homomorphism

$$\psi : P \to M \otimes N \qquad \mu_i \otimes 1 = \psi \circ \nu_i$$

But we can also form another direct limit, each tensor product in the direct system has a canonical map $g_i : M_i \times N \to M_i \otimes N$ so that they form a homomorphism between direct systems

$$(M_i \times N, \mu_{ij} \times 1) \to (M_i \otimes N, \mu_{ij} \otimes 1)$$

Thus there is an induced homomorphism between their direct limits[3]

$$g : M \times N \to P \qquad g \circ \mu_i \times 1 = \nu_i \circ g_i$$

$$\begin{tikzcd}
M_i \times N \arrow[rrr, "g_i"] \arrow[dr, "\mu_i \times 1"] \arrow[dd, "\mu_{ij} \times 1"'] & & & M_i \otimes N \arrow[dl, "\nu_i"'] \arrow[dd, "\mu_{ij} \otimes 1"] \\
& M \times N \arrow[r, "g"] & P & \\
M_j \times N \arrow[rrr, "g_j"] \arrow[ur, "\mu_j \times 1"'] & & & M_j \otimes N \arrow[ul, "\nu_j"]
\end{tikzcd}$$

g is clearly linear in its second factor, for the first factor let $a \in A$ and $x, y \in M$. Since μ_i are all A-linear by definition, we can find elements $x_i \in M_i, y_j \in M_j$ such that

$$\mu_i(ax_i) = a\mu_i(x_i) = ax \qquad \mu_j(y_j) = y$$

Let $k \in I$ be such that $i, j \leq k$ and rewrite so that everything happens in M_k

$$ax = \mu_k(\mu_{ik}(ax_i)) = \mu_k(x_k) \qquad y = \mu_k(\mu_{jk}(y_j)) = \mu_k(y_k)$$

[3]We will use, without proof, the fact that $\varinjlim M_i \times N \cong M \times N$

where $x_k, y_k \in M_k$. Since all the g_i are A-bilinear

$$\begin{aligned} g(ax+y) &= g(\mu_k(x_k) + \mu_k(y_k)) = (g \circ \mu_k)(x_k + y_k) \\ &= (\nu_k \circ g_k)(x_k + y_k) = (\nu_k \circ g_k \circ \mu_{ik})(ax_i) + (\nu_k \circ g_k \circ \mu_{jk})(y_j) \\ &= a(\nu_i \circ g_i)(x_i) + (\nu_j \circ g_j)(y_j) \\ &= ag(x) + g(y) \end{aligned}$$

Thus g is A-bilinear and, by the universal property of tensors, induces a homomorphism

$$\phi : M \otimes N \to P \qquad \phi \circ \mu_i \otimes 1 = \nu_i$$

where the second property comes from the commutative diagram

$$\begin{tikzcd}
 & & M \otimes N \arrow[d, "\phi"] & & \\
M_i \otimes N \arrow[urr, "\mu_i \otimes 1"] \arrow[rr, "\nu_i"'] & & P & & M \times N \arrow[ll, "g"'] \arrow[ull, "j"'] \\
 & & M_i \times N \arrow[ull, "g_i"] \arrow[urr, "\mu_i \times 1"'] & &
\end{tikzcd}$$

To see that these are inverses, note that

$$(\phi \circ \psi)(x) = (\phi \circ \psi \circ \nu_i)(x_i \otimes y_i) = (\phi \circ \mu_i \otimes 1)(x_i \otimes y_i) = \nu_i(x_i \otimes y_i) = x$$

$$(\psi \circ \phi)(x \otimes y) = (\psi \circ \phi \circ \mu_i \otimes 1)(x_i \otimes y) = (\psi \circ \nu_i)(x_i \otimes y) = (\mu_i \otimes 1)(x_i \otimes y) = x \otimes y$$

This means that tensor products commute with direct limits in that

$$\varinjlim (M_i \otimes N) \cong (\varinjlim M_i) \otimes N$$

Exercise 21. Let A_i be a family of rings indexed by a directed set I. By regarding each of them as a $\mathbb{Z}$-module, suppose we have a direct system (A_i, α_{ij}) where α_{ij} are now ring homomorphisms and let A be its direct limit. Let $x, y \in A$ and suppose

$$x = \alpha_i(x_i) = (\alpha_k \circ \alpha_{ik})(x_k) \qquad y = \alpha_j(y_j) = (\alpha_k \circ \alpha_{jk})(y_k)$$

where we chose some $k \geq i, j$. Since A is a $\mathbb{Z}$-module, addition is already defined and we'll define multiplication as

$$xy = \alpha_k(\alpha_{ik}(x_i)\alpha_{jk}(y_j))$$

This does not depend on our choice of i, j, k due to how these maps compose, it is well defined and clearly associative. The identity element is given by $\alpha_i(1)$ for any $i \in I$ since

$$\alpha_i(1)y = \alpha_k(\alpha_{ik}(1)\alpha_{jk}(y_j)) = (\alpha_k \circ \alpha_{jk}(y_j) = y$$

Take $m \geq i, j, k$ and rewrite

$$y + z = \alpha_m(\alpha_{jm}(y_j) + \alpha_{km}(y_k))$$

Thus we can rewrite

$$\begin{aligned} x(y+z) &= \alpha_m(\alpha_{im}(x_i)(\alpha_{jm}(y_j) + \alpha_{km}(y_k))) \\ &= \alpha_m(\alpha_{im}(x_i)\alpha_{jm}(y_j)) + \alpha_m(\alpha_{im}(x_i)\alpha_{km}(y_k)) \\ &= xy + xz \end{aligned}$$

To see that $\alpha_i : A_i \to A$ are ring maps, note that since $i \geq i$

$$\alpha_i(xy) = \alpha_i(\alpha_{ii}(x)\alpha_{ii}(y)) = \alpha_i(x)\alpha_i(y)$$

Suppose $A = 0$, then since every ring has an identity, we can say $1 = 0$ in A. In particular

$$\alpha_i(1) = 1 = 0 \quad \forall i \in I$$

Take some i, there must exist some $j \geq i$ such that $\mu_{ij}(1) = 1 = 0$ in A_j so $A_j = 0$.

Exercise 22. Let (A_i, α_{ij}) be a direct system of rings and let A be its direct limit. Let $\mathfrak{N}_i \subseteq A_i$ be their nilradicals with direct limit $\mathfrak{N}$. Suppose $x \in A$ is nilpotent then

$$\alpha_i(x_i^n) = \alpha_i(x_i)^n = x^n = 0 \longrightarrow \alpha_{ij}(x_i^n) = \alpha_{ij}(x_i)^n = 0 \longrightarrow \alpha_{ij}(x_i) \in \mathfrak{N}_i$$

Since $\alpha_j(\alpha_{ij}(x_i)) = x$, this implies $x \in \mathfrak{N}$. Conversely, if $x \in \mathfrak{N}$, then $\alpha_i(x_i) = x$ for some i and $x_i \in \mathfrak{N}_i \subseteq A_i$

$$x_i^n = 0 \in A_i \longrightarrow \alpha_i(x_i)^n = x^n = 0 \in A$$

so x is also in the nilradical of A. Thus we have shown

$$\mathfrak{N}_{\varinjlim A_i} = \varinjlim \mathfrak{N}_i$$

Suppose each A is not an integral domain with $x, y \neq 0$ such that

$$xy = \alpha_k(\alpha_{ik}(x_k)\alpha_{jk}(y_k)) = 0$$

Then there exists some $n \geq k$ such that

$$\alpha_{kn}(\alpha_{ik}(x_i)\alpha_{jk}(y_j)) = \alpha_{in}(x_i)\alpha_{jk}(y_j) = 0$$

But $\alpha_n(\alpha_{in}(x_i)) = x$ and $\alpha_n(\alpha_{jn}(y_j)) = y$ so they $\alpha_{in}(x_i), \alpha_{jk}(y_j) \neq 0$ due to properties of ring homomorphisms, a contradiction since A_i are all integral domains.

Exercise 23. Let (B_i) be a family of A-algebras index by I. For each finite subset $J \subseteq I$, define B_J as the tensor product of each $B_j, j \in J$. Note that if $J \subseteq J' \subseteq I$, then there is a canonical homomorphism $B_J \to B_{J'}$, thus we can define

$$B = \varinjlim \left\{ B_J = \bigotimes_{j \in J} B_j \mid J \subseteq I \text{ finite} \right\}$$

We have a commutative diagram

$$\begin{tikzcd}
 & B_i \arrow[dr, "\mu_i"] \arrow[dd, "\mu_{ij}"] & \\
A \arrow[ur, "f_i"] \arrow[rr, dashed, "f"] \arrow[dr, "f_j"'] & & B \\
 & B_j \arrow[ur, "\mu_j"'] &
\end{tikzcd}$$

which shows that B has an A-algebra structure induced by the A-algebra structures of each B_i. We call B the tensor product of the A-algebra family (B_i).

2.5.3 Flatness and Tor

Exercise 24.

Exercise 25.

Exercise 26.

2.5.4 Absolutely Flat Rings

Exercise 27.

Exercise 28.