

Complex Analysis Notes

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1 Introduction

These are my personal notes as I study complex analysis. The book I use is Stein & Shakarchi's Complex Analysis, the second book of their Princeton Lectures in Analysis book series. I try to present these notes in a lecture note format so I may look back on them throughout my studies and my solutions (right or not) to selected exercises will follow each chapter.

2 Preliminaries

2.1 The Complex Plane

First let's define some notation which are used frequently throughout complex analysis and go over some basic results.

Any complex number can be written in the form $z = x + iy$ where $x, y \in \mathbb{R}$. The components x and y are referred to as the real and imaginary parts, respectively, and denoted

$$x = \operatorname{Re}(z) \quad y = \operatorname{Im}(z)$$

Complex numbers without a real part are called purely imaginary. By plotting the real part of a number on the x -axis and the imaginary part on the y -axis, we get the complex plane. Thus we can identify $\mathbb{C} \cong \mathbb{R}^2$.

The complex conjugate of a complex number z is

$$\bar{z} = x - iy$$

Thus we have the formulae

$$\operatorname{Re}(z) = \frac{z + \bar{z}}{2} \quad \operatorname{Im}(z) = \frac{z - \bar{z}}{2i}$$

If $z = \bar{z}$, then it is a real number and if $z = -\bar{z}$ it is purely imaginary.

The magnitude, or absolute value, of a complex number is

$$|z| = \sqrt{x^2 + y^2}$$

In the complex plane, this is just the distance from the origin to z . From this definition, we have the complex analog to the triangle inequality which is exactly the same.

$$|z + w| \leq |z| + |w|$$

for all $z, w \in \mathbb{C}$. We also have

$$|\operatorname{Re}(z)| \leq |z| \quad |\operatorname{Im}(z)| \leq |z|$$

From the triangle inequality, we get two more inequalities

$$|z| \leq |z - w| + |w| \quad |w| \leq |z - w| + |z|$$

from which we can prove the “reverse triangle inequality”

$$||z| - |w|| \leq |z - w|$$

Proof. Using the triangle inequality, we have

$$|z - w| + |w| \geq |z - w + w| = |z|$$

$$|z - w| + |z| = |z - w| + |-z| \geq |z - w - z| = |-w| = |w|$$

Rewriting both of these gives

$$|z - w| \geq |z| - |w| \quad |z - w| \geq |w| - |z| = -(|z| - |w|)$$

Now we note that if $x \geq y$ and $x \geq -y$ then $x \geq |y|$. Thus we arrive at the desired result

$$|z - w| \geq ||z| - |w||$$

□

Using the complex conjugate, we can write

$$|z|^2 = z\bar{z} \quad z^{-1} = \frac{\bar{z}}{|z|^2}$$

With addition and multiplication defined in the natural way, this shows that \mathbb{C} is a field i.e. a ring in which every nonzero element has a multiplicative inverse.

Any complex number can be written in polar form and using Euler's formula we may write

$$z = re^{i\theta} = r \cos \theta + ir \sin \theta$$

We observe that $r = |z|$, the quantity θ is referred to as the argument and denoted $\arg z = \theta$. In the complex plane, r is the magnitude and θ the angle from horizontal. If we have two complex numbers

$$z = re^{i\theta} \quad w = se^{i\phi}$$

then multiplication is simply

$$zw = rse^{i(\theta+\phi)}$$

Thus multiplication in the complex plane is a homothety in \mathbb{R}^2 .

Now we may discuss more analytical properties of complex numbers, in particular limits and convergence. We should be familiar with these concepts from the real numbers already, we just need carry over them to the complex plane.

Definition 2.1: Convergence

A sequence $\{z_1, z_2, \dots, z_n\}$ is said to converge to $w \in \mathbb{C}$ if

$$\lim_{n \rightarrow \infty} |z_n - w| = 0$$

If the sequence converges, we write

$$\lim_{n \rightarrow \infty} z_n = w$$

This is nothing new, since $\mathbb{C} \cong \mathbb{R}^2$. In fact a sequence of complex numbers converges to a point w if and only if the corresponding points in \mathbb{R}^2 converge to the point corresponding to w .

Proposition 2.1: Convergence of Complex Numbers

A sequence of complex numbers converges to w if and only if their real and imaginary parts converge to $\operatorname{Re}(w)$ and $\operatorname{Im}(w)$ respectively.

Proof. Obvious if we think of points in \mathbb{C} as points in \mathbb{R}^2 . □

We don't always know what a sequence converges to, so we need another condition to tell us when an arbitrary sequence converges. This condition should be familiar.

Definition 2.2: Cauchy Sequences

A sequence $\{z_n\}$ is Cauchy if

$$|z_n - z_m| \rightarrow 0 \text{ as } n, m \rightarrow \infty$$

Equivalently, for every $\epsilon > 0$ there exists an integer $N > 0$ such that

$$n, m < N \implies |z_n - z_m| < \epsilon$$

A fundamental result from real analysis is that every Cauchy sequence in \mathbb{R} converges to a real number, that is \mathbb{R} is complete. This is true in the complex plane as well.

Theorem 2.2: Completeness of \mathbb{C}

The field of complex numbers \mathbb{C} is complete, that is every Cauchy sequence of complex numbers will converge to a complex number.

Proof. A sequence on complex numbers is Cauchy if and only if the real and imaginary parts are both Cauchy. Since \mathbb{R} is complete, we know then that those “subsequences” will converge to a real number. Thus a Cauchy sequence of complex numbers will converge to a complex number. \square

The final preliminary we must cover is the topology of \mathbb{C} , in particular the notion of open sets. As usual, nothing new is being introduced, we are simply bringing over existing constructions and making them work. The remainder of this section is essentially a series of definitions.

Let $z_0 \in \mathbb{C}$ be some point and $r > 0$, the open disc centered at z_0 of radius r is the set

$$D_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$$

Similarly, the closed disc centered at z_0 of radius r is the set

$$\overline{D}_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| \leq r\}$$

Of note is the unit disc, which has a special notation since it is so useful

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$$

Definition 2.3: Interior and Limit Points

Let $\Omega \subset \mathbb{C}$, a point $z_0 \in \Omega$ is an interior point if there exists an $r > 0$ such that

$$D_r(z_0) \subset \Omega$$

In words, a point is interior if there exists an open disc around it contained completely within Ω . The set of interior points is called the interior of Ω , usually denoted $\text{int } \Omega$.

A point $z \in \mathbb{C}$, not necessarily in \mathbb{C} , is a limit point if there exists a sequence

$$\{z_n \mid z_n \neq z\} \in \Omega \quad \lim_{n \rightarrow \infty} z_n = z$$

Alternatively, z is a limit point of Ω if every open disc containing z contains at least one other point of Ω .

A set is open if it is equal to its interior and closed if its complement is open.

Proposition 2.3

A set Ω is closed if and only if it contains all of its limit points

Proof. First suppose Ω is closed and w be a limit point of Ω . Then there exists a sequence $\{w_n\} \in \Omega$ such that

$$\lim_{n \rightarrow \infty} w_n = w$$

This means that for every $\epsilon > 0$, there exists an $N > 0$ such that for all $n > N$ we have $|w - w_n| < \epsilon$. Equivalently, for every $r > 0$ the open disc $D_r(w)$ contains a point of Ω . Thus w cannot be in the interior of Ω^c , but Ω^c is open so w must lie in Ω .

Conversely suppose Ω is a set which contains all its limit points. Take some $w \notin \Omega$ and assume for the sake of contradiction that $w \notin \text{int } \Omega^c$. In other words, there does not exist an $r > 0$ such that $D_r(w) \subset \Omega^c$. Let $\{r_n\}$ be any sequence of real number converging to 0, then

$$D_{r_n}(w) \cap \Omega \neq \emptyset$$

Let w_n be the (not-necessarily unique) points in each intersection, this is a sequence of points in Ω which converge to w . But w cannot be a limit point of Ω , so we conclude that there must be an open disc around w which lies in Ω^c . Thus Ω^c is open and Ω is closed. \square

Using this proposition, we can closed any set by adding its limit points. This is known as the closure and denoted $\overline{\Omega}$.

The boundary of a set Ω is equal to its closure minus its interior, denoted

$$\partial\Omega = \overline{\Omega} - \text{int } \Omega$$

For the open and closed discs, their boundary is the circle

$$C_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| = r\}$$

A set Ω is bounded if there exists some $M > 0$ such that $|z| < M$ for all $z \in \Omega$, which implies that Ω is contained within some large open disc. For bounded sets we define the diameter

$$\text{diam } \Omega = \sup_{z, w \in \Omega} |z - w|$$

A set is compact if it is closed and bounded.

Theorem 2.4

A set $\Omega \subset \mathbb{C}$ is compact if and only if every sequence $\{z_n\} \in \Omega$ has a subsequence that converges to a point in Ω

Proof. First let Ω be compact and let $\{z_n\} \in \Omega$ be some sequence of points. Consider

$$\{x_n\} = \{\text{Re}(z_n)\} \in \mathbb{R} \quad \{y_n\} = \{\text{Im}(z_n)\} \in \mathbb{R}$$

These are both bounded sequences in \mathbb{R} since Ω is bounded, thus they have a convergence subsequence by the Bolzano–Weierstrass Theorem. Suppose the subsequences converge to $x, y \in \mathbb{R}$ respectively, then we can construct a subsequence of $\{z_n\}$ which converges to $z = x + iy$.

Now suppose Ω is not compact, which means it is either not closed or not bounded (or both!). If Ω is not closed, then take some point $w \in \overline{\Omega} \setminus \Omega$. This is a limit point of Ω and so comes with a sequence $\{w_n\} \in \Omega$ which converges to it. This means that every convergent subsequence of $\{w_n\}$ must also converge to w .

If Ω is not bounded, then we can trivially construct an unbounded sequence. Any subsequence will then never converge. Thus Ω is compact if and only if every sequence has a convergence subsequence. \square

A more familiar condition for compactness is the following

Theorem 2.5

An open covering is a (not necessarily countable) family of open sets $\{U_\alpha\}$ such that

$$\Omega \subset \bigcup_{\alpha} U_{\alpha}$$

A set Ω is compact if and only if every open covering of Ω has a finite subcovering.

Proof. Consider points in \mathbb{C} as points in \mathbb{R}^2 , then this follows from Heine-Borel. \square

When we have nested compact sets, we get an interesting result. This will be very useful as we start diving more deeply into complex analysis.

Proposition 2.6

Suppose we have a nested sequence of non-empty compact sets in \mathbb{C}

$$\Omega_1 \supset \Omega_2 \supset \Omega_3 \supset \cdots \supset \Omega_n \supset \cdots$$

with the property

$$\lim_{n \rightarrow \infty} \text{diam } \Omega_n = 0$$

Then there exists a unique $w \in \mathbb{C}$ such that $w \in \Omega_n$ for all n .

Proof. Pick $z_n \in \Omega_n$, then the condition that the diameter converges to 0 means this is a Cauchy sequence, thus it converges to some point z . But each set is compact, so the subsequences must converge to z in every set.

Suppose w' is another point in each Ω_n , then $|w - w'| > 0$ since $w \neq w'$ which violates the assumption that the diameters converge to 0. \square

Finally, an open set Ω is connected if it cannot be expressed in the form

$$\Omega = \Omega_1 \cup \Omega_2$$

where Ω_1, Ω_2 are disjoint nonempty sets. A similar definition applies for closed sets and we call connected sets in \mathbb{C} regions.

An equivalent and useful definition of connected is as follows: An open set Ω is connected if and only if any two points in Ω can be joined by a curve γ contained entirely in Ω . This proposition is proved in Exercise 5 below.

2.2 Complex Functions

We can now discuss functions defined on sets of complex numbers.

Definition 2.4: Continuous Functions

Let f be a function defined on an open complex set $\Omega \rightarrow \mathbb{C}$. We say f is continuous if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon \quad \forall z \in \Omega$$

Equivalently, for every sequence $\{z_n\} \in \Omega$ which converges to z_0 , the sequence $\{f(z_n)\} \in \mathbb{C}$ must converge to $f(z_0)$.

We say that a function is continuous on a set if it is continuous at every point of that set. We've previously established that convergence in \mathbb{C} is identical to that of \mathbb{R} , so a complex function is continuous if and only if it is continuous if viewed as a function of two real variables. As expected, sum and products of continuous functions will remain continuous in \mathbb{C} .

Theorem 2.7

A continuous function on a compact set Ω is bounded and will achieve both a maximum and minimum on Ω .

We now introduce a central idea in complex analysis, something specific to complex functions.

Definition 2.5: Holomorphic Functions

Let f be a function $f : \Omega \rightarrow \mathbb{C}$, then f is holomorphic at the point $z_0 \in \Omega$ if the limit

$$f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists where $h \neq 0 \in \mathbb{C}$ and $z_0 + h \in \Omega$. The second condition is required for the quotient to be defined. This limit is known as the derivative of f at z_0 .

We say that f is holomorphic on an open set Ω if it is holomorphic on every point $z \in \Omega$. We

say f is holomorphic on a closed set C if it is holomorphic on some open set which contains C . Finally, if f is holomorphic on all of \mathbb{C} , then we say it is entire. The terms regular, complex differentiable, and analytic are sometimes used in place of the term holomorphic.

Example. All polynomials are entire,

$$p(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n \implies p'(z) = a_1 + 2a_2z + \cdots + na_nz^{n-1}$$

The function $1/z$ is holomorphic on any open set of \mathbb{C} which does not contain the origin

$$f(z) = \frac{1}{z} \implies f'(z) = -\frac{1}{z^2}$$

Example. The function $f(z) = \bar{z}$ is not holomorphic, the limit quotient is

$$f'(z) = \lim_{h \rightarrow 0} \frac{\overline{z+h} - \bar{z}}{h} = \lim_{h \rightarrow 0} \frac{\bar{h}}{h}$$

The final limit does not exist, which we can see by taking h first to be real and then purely imaginary (we get 1 and -1 as limit respectively).

While we won't prove them, it is worth noting that all the standard derivative rules apply to holomorphic functions. By further examining the difference quotient, we see that f is holomorphic at $z_0 \in \Omega$ if and only if there exists a complex number a such that

$$f(z_0 + h) - f(z_0) - ah = h\psi(h) \quad \lim_{h \rightarrow 0} \psi(h) = 0$$

where ψ is a function defined for small $|h|$. We see that a then must be $f'(z_0)$ and this formulation shows that f must be continuous if it is holomorphic.

The notion of complex differentiability is very different from the notion of differentiability for a function of two real variables.

Example. Returning to the $f(z) = \bar{z}$ example, as a map $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ this is

$$f(x, y) = (x, -y)$$

and its derivative is a linear map, the Jacobian:

$$f'(x, y) = \det \mathbf{J}_f(x, y) \begin{bmatrix} x \\ y \end{bmatrix} = \det \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = (x, -y)$$

So as a real function, f is infinitely differentiable. But as a complex function, it is not differentiable at all, the existence of a real derivative does not mean f is holomorphic and vice versa.

Consider a general complex function $f = u + iv$ and associate it with a map

$$F : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad F(x, y) = (u(x, y), v(x, y))$$

We say that this function is differentiable at a point $P_0 = (x_0, y_0)$ if

$$\lim_{|H| \rightarrow 0} \frac{F(P_0 + H) - F(P_0) - J(H)}{|H|} = 0$$

in which case $J(H)$ is the derivative of F at P_0 and is unique. If both u, v are partial differentiable, then the derivative is given by the Jacobian matrix

$$\mathbf{J}_F(x, y) = \begin{pmatrix} \partial u / \partial x & \partial u / \partial y \\ \partial v / \partial x & \partial v / \partial y \end{pmatrix}$$

So when we take a real derivative we get a matrix, but when we take a complex derivative we get a complex number. Ideally, there will be some sort of relationship between the two derivatives. When we take the limit of the difference quotient of a complex function, the result should be independent of path. So we can take two paths, one purely real and the other purely imaginary, to connect the complex derivative to real partial derivatives.

$$\begin{aligned} f'(z_0) &= \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} & h \in \mathbb{R} \\ &= \lim_{h_1 \rightarrow 0} \frac{f(x_0 + h_1, y_0) - f(x_0, y_0)}{h_1} \\ &= \frac{\partial f}{\partial x}(z_0) \\ f'(z_0) &= \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} & h \in \mathbb{C} \\ &= \lim_{h_2 \rightarrow 0} \frac{f(x_0, y_0 + h_2) - f(x_0, y_0)}{ih_2} \\ &= \frac{1}{i} \frac{\partial f}{\partial y}(z_0) \end{aligned}$$

where we write $f(z) = f(x, y)$, $z = x + iy$, $h = h_1 + ih_2$.

If f is holomorphic, then these two expressions must be equal and we find

$$f'(z) = \frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y} = -i \frac{\partial f}{\partial y}$$

Now if we write $f = u + iv$, we find

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

By separating out the real and imaginary parts, we get a pair of very special (non-trivial) relationships between partial derivatives which link real and complex derivatives.

Definition 2.6: Cauchy-Riemann Equations

Let $f = u + iv$, then if f is holomorphic, u and v satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

When u, v are the real and complex parts of a function, we frequently define the operators

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} + \frac{1}{i} \frac{\partial}{\partial y} \right) \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} - \frac{1}{i} \frac{\partial}{\partial y} \right)$$

The first operator simply averages the two equivalent expressions for $f'(z)$ that we found earlier and the second can be just the complex conjugate. Now we may give the relationship between holomorphic functions and the real derivatives of its real and imaginary components.

Theorem 2.8: Holomorphic vs. Real Differentiable

If f is holomorphic at z_0 , then

$$\frac{\partial f}{\partial \bar{z}} = 0 \quad f'(z_0) = \frac{\partial f}{\partial z} = 2 \frac{\partial u}{\partial z}$$

If we write $F(x, y) = f(x + iy) = f(z)$, then F is real differentiable

$$\det \mathbf{J}_F(x_0, y_0) = |f'(z_0)|^2$$

Conversely, let f be a function from an open set $\Omega \rightarrow \mathbb{C}$. If u, v are continuously differentiable and satisfy the Cauchy-Riemann equations on Ω , then f is holomorphic on Ω with derivative

$$f'(z) = \frac{\partial f}{\partial z}$$

Proof. Write $f = u + iv$, if f is holomorphic then we can use the Cauchy-Riemann equations to write

$$\begin{aligned} \frac{\partial f}{\partial \bar{z}} &= \frac{1}{2} \left(\frac{\partial f}{\partial x} - \frac{1}{i} \frac{\partial f}{\partial y} \right) \\ &= \frac{1}{2} \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} - \frac{1}{i} \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} \right) \\ &= \frac{1}{2} \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} + i \left(-\frac{\partial v}{\partial x} \right) - \frac{\partial u}{\partial y} \right) = 0 \end{aligned}$$

The other equality follows the same way

$$\begin{aligned} f' &= \frac{\partial f}{\partial z} = \frac{1}{2} \left(\frac{\partial f}{\partial x} - \frac{1}{i} \frac{\partial f}{\partial y} \right) \\ &= \frac{1}{2} \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} + \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \right) \\ &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} + \frac{1}{i} \frac{\partial u}{\partial y} = 2 \frac{\partial u}{\partial z} \end{aligned}$$

By once again using the Cauchy-Riemann equations,

$$\det \mathbf{J}_F(x_0, y_0) = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} = \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 = \left| 2 \frac{\partial u}{\partial z} \right|^2 = |f'(z_0)|^2$$

Now conversely suppose that $f = u + iv$ is a complex function defined on an open set. Recall the alternate definition of a derivative and use it to write

$$\begin{aligned} u(x_1 + h_1, y + h_2) - u(x, y) &= \frac{\partial u}{\partial x} h_1 + \frac{\partial u}{\partial y} h_2 + |h| \psi_1(h) \\ v(x_1 + h_1, y + h_2) - v(x, y) &= \frac{\partial v}{\partial x} h_1 + \frac{\partial v}{\partial y} h_2 + |h| \psi_2(h) \end{aligned}$$

where $\psi_i(h) \rightarrow 0$ and $h = h_1 + ih_2$. We can combine the two using Cauchy-Riemann equations

$$f(z + h) - f(z) = \left(\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right) h + |h| \psi(h)$$

where now $\psi(h) = \psi_1(h) + i\psi_2(h) \rightarrow 0$. This implies f is holomorphic (as we've recovered the derivative definition) and that the derivative is

$$f'(z) = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} = 2 \frac{\partial u}{\partial z} = \frac{\partial f}{\partial z}$$

□

2.3 Power Series

To motivate our discussion of power series, we can extend the exponential function to the complex numbers using

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

and note that it is complex differentiable on any open disc in \mathbb{C}

$$(e^z)' = \sum_{n=0}^{\infty} n \frac{z^{n-1}}{n!} = \sum_{n'=0}^{\infty} \frac{z^{n'}}{n'!} = e^z$$

In contrast, the geometric series

$$\sum_{n=0}^{\infty} z^n$$

only converges absolutely on the unit disc $|z| < 1$.

Definition 2.7: Power Series

In general, a power series is a series of the form

$$\sum_{n=0}^{\infty} a_n z^n$$

We are interested in when a complex power series will converge, so we investigate the related series

$$\sum_{n=0}^{\infty} |a_n| |z|^n$$

First note that, by comparing series, if this series converges absolutely for some z_0 , then it will converge for every z in the disc $|z| < |z_0|$.

Theorem 2.9

For any power series, there exists a (possibly empty) open disc for which it converges absolutely. In other words, there exists an $0 \leq R \leq \infty$ such that the series

$$\sum_{n=0}^{\infty} a_n z^n$$

converges absolute for all $|z| < R$ and diverges for all $|z| > R$.

In particular, if we define $1/0 = \infty$ and $1/\infty = 0$, then the radius of convergence is given by Hadamard's formula

$$1/R = \limsup |a_n|^{1/n}$$

Proof.

□

Theorem 2.10*Proof.***Corollary 2.11***Proof.*

2.4 Integration

tf is an integral

2.5 Exercises

Exercise 1.

- a) If $|z - z_1| = |z - z_2|$, then we have the set of points equidistant from z_1, z_2 . On the complex plane this is the line midpoint perpendicular to the line segment connecting z_1 and z_2 .
- b) If $1/z = \bar{z}$, then $z\bar{z} = |z|^2 = 1$. This is a unit circle.
- c) $\operatorname{Re}(z) = 3$ is just a vertical line at $x = 3$.
- d) $\operatorname{Re}(z) > c$ is a vertical region to the right of the vertical line $x = 3$ but not including it. The set $\operatorname{Re}(z) \geq c$ would include that line.
- e) Suppose we decompose each complex number as

$$a = a_1 + ia_2 \quad b = b_1 + ib_2 \quad z = x + iy$$

Then the condition becomes

$$a_1x - a_2y + b_1 > 0 \implies y < \frac{a_1}{a_2}x + \frac{b_1}{a_2}$$

This is the shaded region below a line, not including that line. If b is purely imaginary, then the line will pass through the origin. The line is horizontal if a is purely imaginary and vertical if a is real. If a is zero, then either we shade the entire plane if b has a positive real part or nothing otherwise.

- f) Letting $z = x + iy$, this is the region

$$x^2 + y^2 = x + 1$$

which is a circle slightly right of center with radius $\sqrt{5}/2$

- g) $\operatorname{Im}(z) = c$ is a horizontal line at $y = c$.

Exercise 2. Suppose $z = z_1 + iz_2, w = w_1 + iw_2$, then it is straight-forward to verify

$$\begin{aligned} \langle z, w \rangle &= z_1w_2 + z_2w_1 \\ &= \frac{1}{2}(z_1w_1 - iz_1w_2 + iz_2w_1 + z_2w_2 + z_1w_1 + iz_1w_2 - iz_2w_1 + z_2w_2) \\ &= \frac{1}{2}((z, w) + (w, z)) = \frac{1}{2}((z, w) + \overline{(z, w)}) \\ &= \operatorname{Re}(z, w) \end{aligned}$$

Exercise 3. Write $z = re^{i\theta}$, then we are interested in solutions to

$$r^n e^{in\theta} = s e^{i\phi}$$

This either has zero or infinitely many solutions. If there does not exist an r such that $r^n = s$ then there will not be any solutions. Otherwise we simply take some θ such that $n\theta = \phi$ and we get infinite solutions by adding 2π to θ . Visually we just spin the points around in a circle, hence infinite solutions.

Exercise 4. Suppose $i > 0$, then we have the implications

$$i > 0 \implies i * i = -1 > 0 \implies -1 * i = -i > 0 \implies i * (-i) = 1 > 0$$

But since $-1 > 0$ and $1 > 0$, this implies $-1 + 1 > 0 + 1 \implies 0 > 1$, a contradiction. Thus the complex numbers cannot have a total ordering.

Exercise 5. A) Let Ω be open and path-connected, suppose that $\Omega = \Omega_1 \cup \Omega_2$ where Ω_1, Ω_2 are open and disjoint. Let $w_1 \in \Omega_1, w_2 \in \Omega_2$ and γ a curve in Ω joining the two points. Let $z : [0, 1] \rightarrow \Omega$ be a parameterization of this curve with $z(0) = w_1, z(1) = w_2$ and

$$t^* = \sup_{0 \leq t \leq 1} \{t : z(s) \in \Omega_1 \text{ for all } 0 \leq s < t\}$$

Consider the point $z(t^*)$, visually this is the point on the curve which lies on the boundary between Ω_1 and Ω_2 .

Suppose $z(t^*) \in \Omega_1$. Since Ω_1 is open, there exists an $r > 0$ such that $z(t^*) \in D_r(z(t^*)) \subset \Omega_1$. But then $\gamma \cup D_r(z(t^*))$ is nonempty, in particular there exists a $t' > t^*$ such that $z(t') \in \Omega_1$, a contradiction since t^* is supposed to be an upper bound.

Similarly, if $z(t^*) \in \Omega_2$ then since Ω_2 is open, there exists an $r > 0$ such that $z(t^*) \in D_r(z(t^*)) \subset \Omega_2$. By a similar argument this implies that there exists a $t' < t^*$ such that for all $0 < s < t'$ we have $z(s) \in \Omega_1$ i.e. t^* is not the least upper bound.

We have shown that $z(t^*) \in \Omega$ does not lie in either Ω_1 or Ω_2 , which is a contradiction since $\Omega = \Omega_1 \cup \Omega_2$. Thus such a decomposition is impossible and Ω must be connected.

B) Now for the converse, suppose Ω is open and path connected. Let $w \in \Omega$, take Ω_1 to be the set of all points path connected to w by a curve in Ω and Ω_2 the set of all points not path connected.

First note that since Ω is open, there exists an open disc around w completely contained within Ω . This also means that the set of points path connected to w is nonempty, since we can take any path within that open disc. Let $w_1 \in \Omega_1$ be any other point. Since $w_1 \in \Omega$, there exists an open disc around it contained within Ω . All the points within that open disc will be path connected to w_1 by a similar argument and by extension path connected to w . Thus that open disc also lies in Ω_1 , proving that it is open.

Now take some $w_2 \in \Omega_2$, a point not path connected to w . Again, there exists an open disc around it contained completely within Ω . Suppose this disc does not completely lie within Ω_2 , for instance there exists some $w' \in D_r(z(t^*)) \cap \Omega_1$. Then we may draw a path from w to w' to w_2 , a contradiction since w_2 is not reachable by any path completely in Ω . Thus that open disc must completely lie in Ω_2 implying it is open.

We've now proved that $\Omega = \Omega_1 \cup \Omega_2$ where both sets are open and disjoint. Since Ω is connected this is impossible so one of the set must be empty. We showed earlier that Ω_1 is nonempty, so we must have $\Omega = \Omega_1$ i.e. it is path connected.

Exercise 6. A) Let $\Omega \subset \mathbb{C}$ be open, $z \in \Omega$ and C_z the set of all points $w \in \Omega$ which are connected to z by a path completely within Ω . We proved earlier that an open set is connected if and only if it is path connected. Since C_z is clearly path connected, we just need to show that it is open. Since any point $w \in C_z$ is also in Ω , there exists an open disc around w contained completely within Ω . All the points within that disc are connected to w and thus to z , so the disc is contained completely within C_z as well. Thus C_z is open and connected.

$z \in C_z$ is obvious. If $w \in C_z$, then there exists a path from z to w lying completely in Ω . Going backwards yields a path from w to z , showing that $z \in C_w$. If $w \in C_z$ and $z \in C_\zeta$, then we have a path from z to w and ζ to z , both of which are contained entirely within Ω . By connecting the paths, we get a path from ζ to w , thus $w \in C_\zeta$.

Thus we have an equivalence relation and C_z are the equivalence classes. In particular, the connected components partition Ω and two components are either disjoint or the same.

B) Suppose we have uncountably many distinct connected components. Since each component is open, there exists uncountably many open discs in Ω , each of which contains at least one unique point with rational coordinates. This would seem to imply that there are uncountably many rational numbers, a contradiction.

C) A compact set is closed and bounded, thus it is contained within an open disc $D_r(z)$ for sufficiently large r and suitable choice of z . The complement of an open disc is connected. Suppose Ω has two unbounded components, then they must both intersect $D_r(z)^c$, which means they are the same component.

Exercise 7. A) Let $z, w \in \mathbb{C}$ such that $\bar{z}w \neq 1$.

B) Fix some $w \in \mathbb{D}$, the unit disc, and consider the mapping

$$F : \mathbb{D} \rightarrow \mathbb{C} \quad z \mapsto \frac{w - z}{1 - \bar{w}z}$$

Exercise 8.

Exercise 9. We use the conversion equations

$$\begin{cases} x &= r \cos \theta \\ y &= r \sin \theta \end{cases}$$

If we write a complex function f as a function in \mathbb{R}^2

$$f(z) = f(x + iy) = f(re^{i\theta}) = f(r, \theta) = u(r, \theta) + v(r, \theta)i$$

then the partial derivatives are

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial r} &= \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x} (-r \sin \theta) + \frac{\partial u}{\partial y} (r \cos \theta) \\ \frac{\partial v}{\partial \theta} &= \frac{\partial v}{\partial x} (-r \sin \theta) + \frac{\partial v}{\partial y} (r \cos \theta) \end{aligned}$$

Using the Cauchy-Riemann equations for Cartesian coordinates gives

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} = \frac{1}{r} \frac{\partial v}{\partial \theta} \\ -\frac{\partial v}{\partial r} &= \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{1}{r} \frac{\partial u}{\partial \theta} \end{aligned}$$

which are the Cauchy-Riemann equations in polar coordinates.

Exercise 10. Through a straightforward computation

$$\begin{aligned} 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} &= 4 \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z} = \left(\frac{\partial}{\partial x} + \frac{1}{i} \frac{\partial}{\partial y} \right) \left(\frac{\partial}{\partial x} - \frac{1}{i} \frac{\partial}{\partial y} \right) \\ &= \frac{\partial^2}{\partial x^2} + \frac{1}{i} \frac{\partial^2}{\partial y \partial x} - \frac{1}{i} \frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2} \\ &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \Delta \end{aligned}$$

Exercise 11. If $f = u + iv$ is holomorphic, then we can use the Cauchy-Riemann equations to write

$$\begin{aligned} \Delta u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial y} \right) + \frac{\partial}{\partial y} \left(-\frac{\partial v}{\partial x} \right) = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0 \\ \Delta v &= \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial y} \left(-\frac{\partial u}{\partial x} \right) = \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y \partial x} = 0 \end{aligned}$$

Thus both the real and imaginary components of f are harmonic.

Exercise 12. First note that $v(x, y) = 0$, so we just need to show that the partial derivatives vanish at the origin. We hold on of x or y constant when evaluating a partial derivative, since we are evaluating at the origin, this means that the function will vanish. Thus the Cauchy-Riemann equations are satisfied.

Exercise 13. A) If $\operatorname{Re}(f)$ is constant, then

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} = 0 \quad \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial y} = 0$$

So v is also constant and thus so is f .

B) Now v is constant, the argument is the same as above, just reversed. We'll see that u is constant and thus so if f .

C)

Exercise 14.

Exercise 15.

Exercise 16.

Exercise 17.

Exercise 18.

Exercise 19.

Exercise 20.

Exercise 21.

Exercise 22.

Exercise 23.

Exercise 24.

Exercise 25.

Exercise 26.