2 Limits

2.1 Introduction to Limits

Limits form the very backbone of calculus, almost every concept and definition comes from the limit of something else. Intuitively they are quite simple: the limit of a something is simply where is appears to be going. In practice, we need a much more rigorous definition. To see why, we turn to the ancient Greeks.

In around 450 BC, Zeno, as ancient Greek philosophers tended to do, sat around thinking of paradoxes. Here are two of his most famous:

1. Atalanta[1] is trying to walk to the end of a path. To walk the entire path she must first walk half the path, then half the remaining path (a quarter), then half the next remaining path (an eigth), then a sixteenth, a thirty-second, etc. Since Atalanta has to walk an infinite number of paths, how can she ever finish walking the path?

2. Achilles[2] is racing a tortoise[3]. To make if fair, he gives the tortoise a head start (say 100m). Once he takes off and covers the first 100m, the tortoise would've meandered a couple more meters. Once he runs the next few meters, the tortoise will cover a few more meters, and so on so forth. So the questions is: If every time Achilles catches up to where the tortoise was prior, the tortoise remains a bit ahead, how will Achilles ever pass the tortoise?

The astute reader may notice that these paradoxes are clearly resolvable. After all, we walk across paths all the time and obviously Achilles will eventually pass the turtle, so why are these so famous?

The reason why these paradoxes are even talked about now is because they deal with the concept of infinity and infinitesimals (infinitely small distances). In order to have a precise mathematical way of dealing with these, we must invent new mathematics. That new mathematics is Calculus.

To introduce the idea of a limit, let's write the Atalanta (formally known as the Dichotomy) paradox in terms of limits (kind of). For Atalanta, suppose the path is 1m and she walks at a speed of 1m/s. Then the time taken will be

$$t = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

Let's try to add up the terms in this sequence, if n is the number of terms we add up

n	1	2	3	4	5	$\cdots$	10	$\cdots$	∞
t	1/2	3/4	7/8	15/16	31/32	$\cdots$	1023/1024	$\cdots$	1

We see that as n gets really, really large, then t gets really close to 1. This makes sense because Atalanta should be able to walk the 1m path in 1s.

[1] Famous figure in Greek mythology

[2] Even more famous figure in Greek mythology

[3] A very famous figure in fairy tale lore, famously starred in the hit movie "Tortoise and the Hare"

With that history lesson out of the way, let's return to the math. Take a simple function, for instance $f(x) = x$, and let's look at how it behaves around $x = 1$. There's two ways we can answer this question:

1. Start from the left of the graph and see what happens as we move up to $x = 1$
2. Start from the right of the graph and see what happens as we move down to $x = 1$

It's pretty obvious in this case, but we see that the graph approaches $y = 1$ regardless of which direction we approach. This idea of seeing what happens when we get close to a value is formalized in the definition of a limit.

Definition 1: The Limit

Let $f(x)$ be a function and a, L some real numbers. If all values of $f(x)$ approach L as x approaches a, then we say that L is the limit of the function $f(x)$ as x approaches a. In symbols, we write this as

$$\lim_{x \to a} f(x) = L$$

Example. What we got from the previous simple example is that

$$\lim_{x \to 1} x = 1$$

A quick and dirty way to evaluate limits is to evaluate a bunch of values progressively closer to the value we wish to look at. If the y values appears to get closer to some number, then that number is probably the limit.

Example. Consider the function $f(x) = x^2$ and suppose we want to find the limit as x approaches 2. To evaluate this limit using a table, we compute the following values:

x	1.9	1.99	1.999	2.001	2.01	2.1
f(x)	3.61	3.9601	3.996001	4.004001	4.0401	4.41

From looking at this table, we can make an educated guess at the limit

$$\lim_{x \to 2} x^2 = 4$$

2.2 Continuity

2.3 Definition of a Limit