2 Manifolds

2.1 Introduction to Manifolds

A key aspect of gravity is its universality, formalized by the *Principle of Equivalence*. One way f stating this principle is the *Weak Equivalent Principle* (WEP): In small enough regions of spacetime, the motion of a free-falling (only acted on by gravity) particle is the same in a gravitational field and a uniformly accelerated frame. A reasonable extension is the *Einstein Equivalent Principle* (EEP): In small enough regions of spacetime, the laws of physics reduce to those of special relativity; local experiments cannot detect the existence of a gravitational field.

While we can come up with some contrived theories in which WEP and EEP are not equal (e.g. gravity rotates stuff), for our purposes it is enough to consider them as equivalent. The notion of "small enough regions" indicates we should define *locally inertial frames*, discarding the notion that reference frames extend throughout all spacetime.Our goal thus is to formulate a mathematical structure to describe spacetime which is locally similar to Minkowski space but curves over larger regions. The fact that space is curved can be seen through various experiments, for instance gravitational redshift.

First we will study manifolds in its most basic mathematical form. Intuitively, a (differentiable) manifold is a space which locally "looks like" $\mathbb{R}^n$ in which the metric is the same and functions work as expected. In a way, we can think of a manifold as sewing together many local patches of $\mathbb{R}^n$, each patch must be the same dimension, which will be the dimension of the manifold. Before giving a rigorous definition, we first introduce some examples.

Example. $\mathbb{R}^n$ is trivially a manifold. The next simplest example is the n-sphere, defined as

$$S^n = \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \mid |x| = 1 \right\}$$

S^1 is a circle, S^2 is a sphere, and so on. The n-torus, defined by taking an n-dimensional cube and identifying opposite sides, is similarly a manifold. For instance T^2 is a donut.

Example. More abstractly, the set of continuous transforms (such as rotations in $\mathbb{R}^n$) form a manifold. Lie groups are manifolds which carry a group structure, for instance $SO(2)$, the set of 2D rotations, lies in the same manifold as S^1.

Example. Given two manifolds M, M', the direct product

$$M \times M' = \{(x, x') \mid x \in M, x' \in M'\}$$

is also a manifold

Given all these definitions of manifolds, perhaps it might be helpful to have a couple of examples of what aren't manifolds.

Example. A plane with a line passing through is not a manifold, nor are two cones joined at the tips (we call these wedge sums in topology). Some more subtle examples are a line segment and a single cone. The line is excluded due to the endpoints and the cone is not smooth at its vertex.

To give a rigorous definition of manifolds, we need a notion of smoothness and "locally looks like $\mathbb{R}^n$." This requires a series of definitions.

Definition. *Given two sets M, N, a map $\phi : M \to N$ is a function that assigns an element of N to every element of M. A map is injective (one-to-one) if every element of N has at most one element of M mapping to it and surjective (onto) if every element of N has at least one element of M mapping to it.*

This definition should already be familiar, it is simply a generalization of functions, but it is nice to be complete when introducing new structures. Just like normal functions, maps can compose: if we have two maps

$$\phi : A \to B \qquad \psi : B \to C$$

then the composition is a map

$$\psi \circ \phi : A \to C \qquad a \mapsto \psi(\phi(a))$$

Given a map $\phi : M \to N$, we call M the domain of ϕ and N the codomain. The image is the set

$$\text{Im } \phi = \{\phi(m) \mid m \in M\} \subseteq N$$

and the kernel is

$$\text{Ker } \phi = \{m \in M \mid f(m) = 0\} \subseteq M$$

Thus we see that ϕ is surjective if Im $\phi = N$ and it can be proven that ϕ is injective if and only if Ker $\phi = 0$. If $\phi(m) = n$, then the preimage of n is denote $\phi^{-1}(n) = m$. If ϕ is injective and surjective (we sometimes use the term bijective), then it is invertible and there exists an inverse map ϕ^{-1} such that

$$\phi^{-1} \circ \phi = \phi \circ \phi^{-1} = id$$

where id is the identity map $x \mapsto x$.

Definition. *Consider a map $\phi : \mathbb{R}^m \to \mathbb{R}^n$, we can express it as a collection of n functions*

$$\begin{gathered}
\phi(x^1, \ldots, x^m) \mapsto (y^1, \ldots, y^n) \\
y^1 = \phi^1(x^1, \ldots, x^m) \\
y^2 = \phi^2(x^1, \ldots, x^m) \\
\vdots \\
y^n = \phi^n(x^1, \ldots, x^m)
\end{gathered}$$

A function is C^p if its pth derivative exists and is continuous, we say that $\phi : \mathbb{R}^m \to \mathbb{R}^n$ is C^p is every one of its component functions are at least C^p. A C^0 function is continuous everywhere but not necessarily differentiable while a C^∞ function is infinitely differentiable everywhere. We will refer to C^∞ functions as being smooth.

Definition. *A map $\phi : M \to N$ is a diffeomorphism if it is invertible and ϕ, ϕ^{-1} are both C^∞. Two sets are diffeomorphic if there exists a diffeomorphism between them.*

In the language of manifolds, diffeomorphic spaces carry the same structure and are in a sense the same thing. When we previously said that $SO(2)$ and S^1 are the same manifold, we actually mean that they are diffeomorphic.

To finish defining what a manifold is, we need a bit of topology, namely the notion of an open sets. Fortunately, open sets in $\mathbb{R}^n$ are easy to visualize.

Definition. *An open ball of radius r centered at $\mathbf{y}$ is the set*

$$B_r(y) = \{\mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x} - \mathbf{y}| < r\} \subseteq \mathbb{R}^n$$

An open set in $\mathbb{R}^n$ is a (possibly infinite) union of open balls.

This leads to the following characterization of open sets: A set $V \subseteq \mathbb{R}^n$ is open if for any $y \in V$, there exists an $r > 0$ such that $B_r(y) \subseteq V$.[1]

Definition. *Let $U \subseteq M$ be any subset of M. A chart is the pair (U, ϕ) where $\phi : U \to \mathbb{R}^n$ is injective with $\phi(U) \subseteq \mathbb{R}^n$ open.*

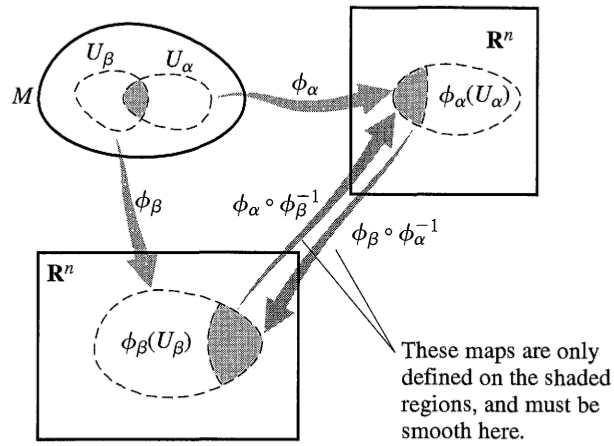
Note that this definition does not require U to be open, this is intentional. We can make any map surjective by restricting the codomain to its image, if such a ϕ exists then the map $\phi : U \to \phi(U)$ is invertible. Thus if $\phi(U) \subseteq \mathbb{R}^n$ is open, we will say that $U \subseteq M$ is open as well.[2]

Definition. *A C^∞ atlas is a collection of charts $\{(U_\alpha, \phi_\alpha)\}$ indexed by $\alpha \in A$ such that*

1. *The U_α cover M, that is*
$$\bigcup_{\alpha \in A} U_\alpha = M$$
2. *If $U_\alpha \cap U_\beta \neq \varnothing$, then the map*
$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$
is C^∞ and so is the inverse.

The first property allows us to represent M as a collection of open sets and the second property means that the overlap between each patch is smoothly sewn together (see below).

[1] Those that have taken analysis may notice that this is just the definition in terms of neighborhoods.
[2] In topology, a chart is a homeomorphism between a subset of M and an open set (ball) of $\mathbb{R}^n$.



Definition. *A C^∞ n-dimensional manifold (or just n-manifold) is a set M along with a maximal atlas, one that contains every compatible chart.*

We require the atlas to be maximal to avoid the issue that two equivalent spaces are somehow different manifolds just because the atlases are different. To illustrate this definition and in particular the necessity for multiple charts, let's prove S^1 is a 1-dimensional manifold.

Example. S^1 can be characterized using Euler's formula as $e^{i\theta}$ where θ runs from 0 to 2π, it can be interpreted as the angle made with the horizontal. A natural choice of chart is

$$\phi : S^1 \to \mathbb{R} \qquad e^{i\theta} \mapsto \theta$$

There is an issue here, $\phi(S^1)$ is supposed to be open but instead $\phi(S^1) = [0, 2\pi)$ is half open and half closed. If we remove $\theta = 0$ then this is no longer covering the entire circle, thus the need for two charts. Consider two charts

$$\begin{aligned} \phi : S^1 \setminus (1,0) \to \mathbb{R} \qquad & e^{i\theta} \mapsto \theta \in (0, 2\pi) \\ \psi : S^1 \setminus (-1,0) \to \mathbb{R} \qquad & e^{i\theta} \mapsto \theta \in (-\pi, \pi) \end{aligned}$$

The two charts differ in that ψ reinterprets the lower half of the circle as negative angles. For this to be a valid atlas, we must show that they are compatible on their intersection.

$$\phi \circ \psi^{-1} : (-\pi, 0) \cup (0, \pi) \to (0, \pi) \cup (\pi, 2\pi) \qquad \theta \mapsto \begin{cases} \theta + 2\pi & \theta \in (-\pi, 0) \\ \theta & \theta \in (0, \pi) \end{cases}$$

The inverse map is

$$\psi \circ \phi^{-1} : (0, \pi) \cup (\pi, 2\pi) \to (-\pi, 0) \cup (0, \pi) \qquad \theta \mapsto \begin{cases} \theta & \theta \in (0, \pi) \\ \theta - 2\pi & \theta \in (\pi, 2\pi) \end{cases}$$

Both maps are infinitely differentiable on their domains (since we removed the discontinuity) so this is a diffeomorphism as needed.

Before moving on to doing physics on manifolds, we will introduce the multi-dimensional chain rule. Let $f : \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^l$ so that $g \circ f : \mathbb{R}^m \to \mathbb{R}^l$. Suppose we label points $x^a \in \mathbb{R}^m, y^b \in \mathbb{R}^n, z^c \in \mathbb{R}^l$ where a, b, c are indices ranging over the appropriate values. Then

$$\frac{\partial}{\partial x^\alpha}(g \circ f)^c = \sum_b \frac{\partial f^b}{\partial x^a}\frac{\partial g^c}{\partial y^b}$$

which is usually abbreviated as

$$\frac{\partial}{\partial x^\alpha} = \sum_b \frac{\partial y^b}{\partial x^a}\frac{\partial}{\partial y^b}$$

2.2 Vectors and Tensors Revisited

Now we can start defining the notion of a tangent space on a manifold. Rather than going from one point to another, a vector is now specifically associated to a single point on a manifold.

Theorem 2.1. Consider the space of smooth functions on M

$$\mathcal{F} = \{f : M \to \mathbb{R} \mid f \text{ is } C^\infty\}$$

Suppose we have a curve $x^\mu(\lambda)$ such that $x^\mu(\lambda_0) = p$, it defines an operator

$$\mathcal{F} \to M \qquad f \mapsto \left.\frac{\partial f}{\partial \lambda}\right|_{\lambda=\lambda_0}$$

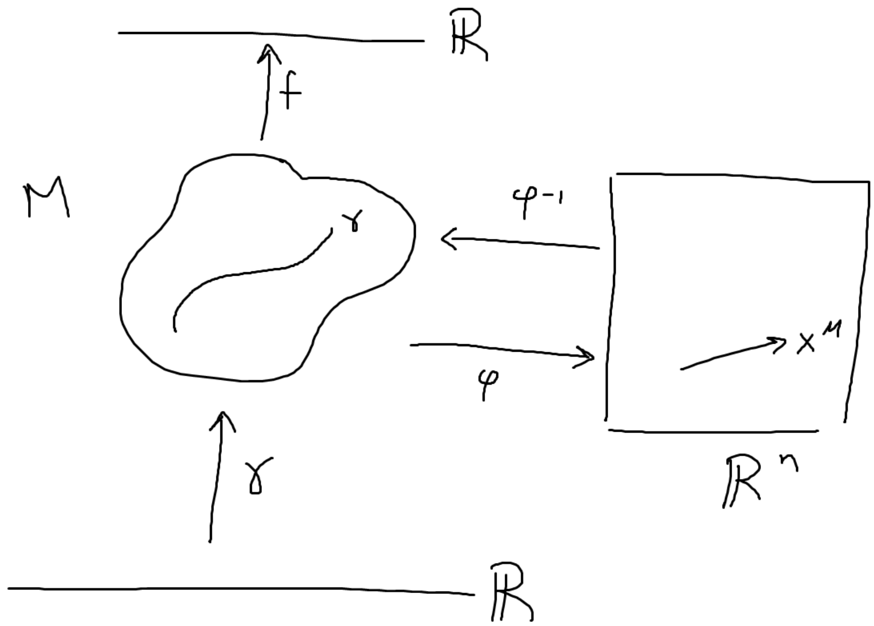
The space of directional derivatives (above) along curves through p can be identified as the tangent space T_p.

Proof. First we show that the directional derivatives form a vector space. Suppose we have two curves $x^\mu(\lambda)$ and $x^\mu(\eta)$ which both pass through p. Addition and scalar multiplication is clearly well defined, it's harder to see closure, that is the result is still a directional derivative. Clearly it's still a linear operator, but we have to prove the product rule (also called the Leibniz rule).

$$\begin{aligned}\left(a\frac{d}{d\lambda} + b\frac{d}{d\eta}\right)(fg) &= a\frac{d(fg)}{d\lambda} + b\frac{d(fg)}{d\eta}\\ &= a\frac{df}{d\lambda}g + a\frac{dg}{d\lambda}f + b\frac{df}{d\eta}g + b\frac{dg}{d\eta}f\\ &= \left(a\frac{df}{d\lambda} + b\frac{df}{d\eta}\right)g + \left(a\frac{dg}{d\lambda} + b\frac{dg}{d\eta}\right)f\end{aligned}$$

Thus any linear combination is still a directional derivative and they indeed form a vector space. Now we show that this can be identified as the tangent space. We do this by finding a basis, in particular we claim that the partial derivatives ∂_μ form a basis and thus T_p is n-dimensional as expected.

To show that this is a basis, we must prove that every directional derivative can be written as a linear combination of the partial derivatives. Let M be an n-manifold, $\phi : M \to \mathbb{R}^n$ a chart, $\gamma(\lambda) : \mathbb{R} \to M$ a curve in M, and $f : M \to \mathbb{R}$ some function as in the figure below.



We want to express the vector $d/d\lambda$ as a linear combination of ∂_μ. Using the chain rule and noting that f is evaluated along the curve γ, we can write

$$\begin{aligned}\frac{d}{d\lambda}f &= \frac{d}{d\lambda}(f \circ \gamma)\\ &= \frac{\partial}{\partial\lambda}\left[(f \circ \phi^{-1}) \circ (\phi \circ \gamma)\right]\\ &= \frac{d(\phi \circ \gamma)^\mu}{d\lambda}\frac{\partial(f \circ \phi^{-1})}{\partial x^\mu}\\ &= \frac{dx^\mu}{d\lambda}\partial_\mu f\end{aligned}$$

where the last line comes from suppressing the γ again. The function f was an arbitrary choice, so what we really proved was

$$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\partial_\mu$$

which is the desired decomposition.

$\square$

The basis we found in the proof $\hat{e}_{(\mu)} = \partial_\mu$ is known as the *coordinate basis* for T_p and formalizes the notion of setting up our basis vectors to point along the coordinate axis. However, note that these basis vectors are not normalized, nor are they orthogonal. In fact, a coordinate basis is generally never orthonormal throughout the neighborhood of any point with nontrivial curvature.

The advantage of using this basis is that the transformation law is immediate, for instance basis vectors in a new coordinate system $x^{\mu'}$ would be given by

$$\partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu$$

Since a vector $V = V^\mu \partial_\mu$ should not change under coordinate transforms, we can also derive the transformation law for vector components.

$$V^\mu \partial_\mu = V^{\mu'} \partial_{\mu'} = V^{\mu'} \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu \implies V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu$$

When we say "vector transformation law," we generally refer to the law that we just derived since basis vectors are not normally written. Note that it is compatible with Lorentz transforms

$$V^{\mu'} = \Lambda^{\mu'}{}_\mu V^\mu \qquad x^{\mu'} = \Lambda^{\mu'}{}_\mu x^\mu$$

since they're just a special coordinate transformation. Thus the vector transformation law is more general and works for any arbitrary change of coordinates.

A vector at a point can be interpreted as a directional derivative along some path at that point. A vector field, which assigns a vector to every point, can be thought of as a map from smooth functions to smooth functions over the entire manifold by taking derivatives at each point. Given two fields X, Y, we define the commutator as acting on a function $f(x^\mu)$

$$[X, Y](f) = (X \circ Y)(f) - (Y \circ X)(f)$$

In the abstract view we adopted, this operator is coordinate-independent. Furthermore it can be shown that it is a linear operator and obeys the product rule, thus it is also a vector field. We can explicitly derive its components

$$[X, Y]^\mu = X^\lambda \partial_\lambda Y^\mu - Y^\lambda \partial_\lambda X^\mu$$

The commutator is sometimes referred to as the Lie bracket.

Before moving on to tensors, we'll repeat what we did in flat spacetime and talk about dual vectors (also known as one-forms) first.

Definition. *Given a point p, the set of all dual vectors is called the cotangent space and is the set of linear maps on the tangent space*

$$T_p^* = \{\omega : T_p \to \mathbb{R} \mid \omega(ax + by) = a\omega(x) + b\omega(y)\}$$

Example. The standard example of a one-form is the gradient $\mathrm{d}f$ of a function f, which acts on a vector (directional derivative) as

$$\mathrm{d}f\left(\frac{d}{d\lambda}\right) = \frac{df}{d\lambda}$$

Note that each derivative is implied to be evaluated at p.

Similar to how the tangent space basis are the partial derivatives along each coordinate axis, the gradient of coordinate functions will provide a basis for the cotangent space. To see that this choice makes sense, note that

$$\mathrm{d}x^\mu \partial_\nu = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu$$

which is what we expect from a basis for dual vectors. An arbitrary one-form can thus be expanded as $\omega = \omega_\mu \mathrm{d}x^\mu$. The transformation laws are

$$\mathrm{d}x^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu}\mathrm{d}x^\mu \qquad \omega_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\omega_\mu$$

Definition. *A (k,l) tensor is a multilinear map from a collection of k dual vectors and l vectors to $\mathbb{R}$. Its components can be obtained by acting on the basis one-forms and vectors*

$$T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} = T(\mathrm{d}x^{\mu_1}, \ldots, \mathrm{d}x^{\mu_k}, \partial_{\nu_1}, \ldots, \partial_{\nu_l})$$

which gives the expansion

$$T = T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}\,\partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_k} \otimes \mathrm{d}x^{\nu_1} \otimes \cdots \otimes \mathrm{d}x^{\nu_l}$$

The transformation law for tensors is straightforward

$$T^{\mu'_1\cdots\mu'_k}{}_{\nu'_1\cdots\nu'_l} = \frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\mu'_k}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial x^{\nu'_1}} \cdots \frac{\partial x^{\nu_l}}{\partial x^{\nu'_l}} T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}$$

Example. Sometimes we can transform a tensor by interpreting basis one-forms and vectors as gradients and partial derivatives respectively rather than using the transformation law. Consider a $(0,2)$ tensor in the coordinate system $x^1 = x, x^2 = y$ with components

$$S_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & x^2 \end{pmatrix}$$

which can be equivalently written as

$$S = S_{\mu\nu}(\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu) = (\mathrm{d}x)^2 + x^2(\mathrm{d}y)^2$$

where we suppress the tensor in the final result. Consider a new coordinate system

$$x' = \frac{2x}{y} \qquad y' = \frac{y}{2}$$

This can immediately be inverted to obtain

$$x = x'y' \qquad y = 2y'$$

and we can take derivatives

$$\mathrm{d}x = y'\mathrm{d}x' + x'\mathrm{d}y' \qquad \mathrm{d}y = 2\mathrm{d}y'$$

to get the transformed tensor

$$S = (y')^2(\mathrm{d}x')^2 + x'y'(\mathrm{d}x'\mathrm{d}y' + \mathrm{d}y'\mathrm{d}x') + ((x')^2 + 4(x'y')^2)(\mathrm{d}y')^2$$

Note that tensors do not commute in general (recall we suppressed them earlier) so we must have four derivative terms. In matrix form, the new tensor is

$$S_{\mu'\nu'} = \begin{pmatrix} (y')^2 & x'y' \\ x'y' & (x')^2 + 4(x'y')^2 \end{pmatrix}$$

2.3 Metric Tensors

The metric tensor is incredibly important in curved spacetime, so much so that we give it a new symbol to distinguish it from the Minkowski metric.

Definition. *The metric tensor $g_{\mu\nu}$ is a symmetric $(0,2)$ tensor which is usually taken as nondegenerate*

$$g = \det g_{\mu\nu} \neq 0$$

The inverse metric $g^{\mu\nu}$ will satisfy

$$g^{\mu\nu} g_{\nu\sigma} = g_{\lambda\sigma} g^{\lambda\mu} = \delta^{\mu}_{\sigma}$$

Since $g_{\mu\nu}$ is symmetric, so will $g^{\mu\nu}$ and just as in special relativity, the metric can be used to raise and lower indices. One of the many uses of the metric is to provide a way to compute path length, the natural extension of the special relativity line element is

$$ds^2 = g_{\mu\nu} \mathrm{d}x^{\mu} \mathrm{d}x^{\nu}$$

Note that there is a difference between dx, the notion of an infinitesimal displacement in the x direction, and $\mathrm{d}x$, a basis one-form given by the gradient of a coordinate function. In our formula, the ds^2 does not refer to the differential of anything nor is anything squared. Rather it is just shorthand for the metric tensor which maps two vectors to a real number. Equivalent ways of writing the tensor are

$$g_{\mu\nu} V^{\mu} W^{\nu} = g(V, W) = ds^2(V, W)$$

On the other hand $(\mathrm{d}x)^2$ refers to the $(0,2)$ tensor $\mathrm{d}x \otimes \mathrm{d}x$.

Example. In 3D Euclidean space with Cartesian coordinates, the line element is

$$ds^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2$$

Now suppose we change to spherical coordinates

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned}$$

The metric in spherical coordinates can be found by computing derivatives of each coordinate function

$$\begin{aligned} \mathrm{d}x &= \sin\theta \cos\phi \mathrm{d}r + r \cos\theta \cos\phi \mathrm{d}\theta - r \sin\theta \sin\phi \mathrm{d}\phi \\ \mathrm{d}y &= \sin\theta \sin\phi \mathrm{d}r + r \cos\theta \sin\phi \mathrm{d}\theta + r \sin\theta \cos\phi \mathrm{d}\phi \\ \mathrm{d}z &= \cos\theta \mathrm{d}r - r \sin\theta \mathrm{d}\theta \end{aligned}$$

$$\therefore ds^2 = (\mathrm{d}r)^2 + r^2 (\mathrm{d}\theta)^2 + r^2 \sin^2\theta (\mathrm{d}\phi)^2$$

Example. An extension of the last example gives the metric on the two-sphere S^2. We can simply take the metric in spherical coordinates and set $\mathrm{d}r = 0, r = 1$ to get

$$ds^2 = (\mathrm{d}\theta)^2 + \sin^2\theta(\mathrm{d}\phi)^2$$

This is all incredible informal (what does setting $\mathrm{d}r = 0$ even mean?), but the answer is correct. The formal way about this is to study how submanifolds (the sphere) inherit metrics from the spaces they're embedded in ($\mathbb{R}^3$ in this case).

Definition. *A useful characterization of a metric is its canonical form*

$$g_{\mu\nu} = diag(-1, -1, \dots, 1, 1, \dots, 0, 0, \dots, 0)$$

where diag indicates a diagonal matrix with the given entries.

The *signature* of a metric is the number of positive and negative eigenvalues. If any of the eigenvalues are zero, then the metric is *degenerate* and the inverse metric will not exist. A continuous and nondegenerate metric will have the same signature at every point, we will only deal with these metrics.

If all the signs in a metric are positive, then we say it is *Euclidean* or *Riemannian*. A single minus sign is a *Lorentzian* or *psuedo-Reimannian* metric, every of metric is called indefinite. In general relativity, we mostly care about Lorentzian metrics.

We have not shown that it is possible to put a metric in canonical form and in fact it is generally only possible at individual points. We actually have a slightly better result.

Theorem 2.2. At any point $p \in M$, there exists a coordinate system $x^{\hat{\mu}}$ such that the metric takes its canonical form and all first derivatives vanish

$$g_{\hat{\mu}\hat{\nu}}(p) = \eta_{\hat{\mu}\hat{\nu}} \qquad \partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}}(p) = 0$$

These coordinates are known as *locally inertial coordinates* and the basis vectors form a *local Lorentz frame*. Indices are given a hat to indicate that we are in these coordinates. The idea of a local Lorentz frame is a rigorous way of saying small enough regions of spacetime "look like" flat Minkowski space.

2.4 Calculus on Manifolds

Tensors can be incredibly useful but there are times when other objects are of greater interest. For instance recall the Levi-Civita symbol

$$\tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_n} = \begin{cases} +1 & \text{indices are an even permutation of } 01\cdots(n-1) \\ -1 & \text{indices are an odd permutation of } 01\cdots(n-1) \\ 0 & \text{otherwise} \end{cases}$$

This is defined to have the same components in any coordinate system so it is clearly not a tensor. However we can relate it to tensors, note that for a matrix $M^{\mu}_{\mu'}$ the determinant can be calculated using

$$\tilde{\epsilon}_{\mu_1'\mu_2'\cdots\mu_n'}|M| = \tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_n} M^{\mu_1}_{\mu_1'} M^{\mu_2}_{\mu_2'} \cdots M^{\mu_n}_{\mu_n'}$$

If we replace M with the change of coordinate matrix and noting that $|M^{-1}| = |M|^{-1}$, we find

$$\tilde{\epsilon}_{\mu_1'\mu_2'\cdots\mu_n'} = \left|\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right| \tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \frac{\partial x^{\mu_2}}{\partial x^{\mu_2'}} \cdots \frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}}$$

This looks like a transformation law, but there is an additional determinant (the Jacobian) in front. Objects which transform in this way are called *tensor densities.*

Example. The determinant of the metric $g = |g_{\mu\nu}|$ is also a tensor density. Under coordinate transformations we find

$$g(x^{\mu'}) = \left|\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right|^{-2} g(x^{\mu})$$

The power on the Jacobian is referred to as the *weight* of the tensor density. The Levi-Civita symbol is weight 1 while the determinant g (a scalar) is weight -2.

Since we generally don't like tensor densities as much as we like tensors, it would be nice if we could convert between them. As it turns out, we can multiply in powers of g in order to cancel out powers of the Jacobian. In particular, we append a factor of $|g|^{w/2}$ where w is the weight of the tensor density. The absolute value is due to the fact that $g < 0$ for Lorentzian metrics. The Levi-Civita tensor is thus

$$\epsilon_{\mu_1\mu_2\cdots\mu_n} = \sqrt{|g|}\tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_n}$$

A special class of tensors are the differential forms.

Definition. *A differential p-form is a completely antisymmetric $(0,p)$ tensor. The space of all p-forms is denoted Λ^p and the space of all p-forms over a manifold M is denoted $\Lambda^p(M)$.*

Scalars are automatically 0-forms and dual vectors are 1-forms, hence the name from before. Differential forms are important because they can be differentiated and integrated without extra work.

Definition. *Let A be a p-form and B a q-form. The wedge product is a $(p+q)$-form obtained by taking the antisymmetrized tensor product*

$$(A \wedge B)_{\mu_1\cdots\mu_{p+q}} = \frac{(p+q)!}{p!q!} A_{[\mu_1\cdots\mu_p} B_{\mu_{p+1}\cdots\mu_{p+q}]}$$

Note that we can flip the order of wedge products using

$$A \wedge B = (-1)^{pq} B \wedge A$$

Example. Let A, B be 1-forms, their wedge product is

$$(A \wedge B)_{\mu\nu} = 2A_{[\mu}B_{\nu]} = A_{\mu}B_{\nu} - A_{\nu}B_{\mu}$$

Definition. *The exterior derivative of a p-form A gives a $(p+1)$-form field through a normalized, antisymmetrized partial derivative.*

$$(\mathrm{d}A)_{\mu_1\cdots\mu_{p+1}} = (p+1)\partial_{[\mu_1}A_{\mu_2\cdots\mu_{p+1}]}$$

There is a version of the Leibniz (product) rule for exterior derivatives. For a p-form ω and q-form η, the exterior derivative of their wedge sum obeys

$$\mathrm{d}(\omega \wedge \eta) = (\mathrm{d}\omega) \wedge \eta + (-1)^p \omega \wedge (\mathrm{d}\eta)$$

Example. The simplest example is the gradient, which is the exterior derivative of a 0-form

$$(\mathrm{d}\phi)_\mu = \partial_\mu \phi$$

There are a couple reasons why the exterior derivative is so important. One is that, unlike the partial derivative, it is a tensor. For instance, let W is a 1-form and consider the transformation properties of the partial derivative

$$\frac{\partial}{\partial x^{\mu'}}W_{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial}{\partial x^\mu}\left(\frac{\partial x^\nu}{\partial x^{\nu'}}W_\nu\right) = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\left(\frac{\partial}{\partial x^\mu}W_\nu\right) + W_\nu \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial}{\partial x^\mu}\frac{\partial x^\nu}{\partial x^{\nu'}}$$

The first term is what we would expect from transforming a tensor but the second term should not be there. But if we look at it closer, we can rewrite it as

$$W_\nu \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial}{\partial x^\mu}\frac{\partial x^\nu}{\partial x^{\nu'}} = W_\nu \frac{\partial^2 x^\nu}{\partial x^{\mu'}\partial x^{\nu'}}$$

This expression is symmetric in μ' and ν' and thus would disappear in the exterior derivative since we antisymmetrize it. The remaining terms will provide the tensor transformation law for an exterior derivative. However it's still not a good substitute since it only works on specific tensors (forms).

Another neat property is that for any form A

$$\mathrm{d}(\mathrm{d}A) = 0 \Longrightarrow \mathrm{d}^2 = 0$$

This is due to the fact that partial derivatives commute and so antisymmetrizing twice will return nothing. This has interesting mathematical consequences, for instance we say that a form is closed if $\mathrm{d}A = 0$ and exact if $A = \mathrm{d}B$ for some $(p-1)$-form B. Clearly every exact form is closed but the converse is not true, we can define two spaces

$$Z^p(M) = \{\text{closed } p\text{-forms on } M\} \qquad B^p(M) = \{\text{exact } p\text{-forms on } M\}$$

In particular, these are both vector spaces with $B^p(M) \subseteq Z^p(M)$. Vector spaces are a special case of more general objects called modules and we can form quotients with these modules. The resulting vector space is a set of equivalence classes modulo $B^p(M)$

$$H^p(M) = Z^p(M)/B^p(M)$$

These elements are called cohomology classes and $H^p(M)$ the cohomology space. What's interesting is that the dimension of $H^p(M)$ as a vector space depends only on the topology of M. For instance, Minkowksi space is topological equivalent $\mathbb{R}^4$ which is pretty trivial as topological spaces go. Thus

$$H^p(M) = \begin{cases} 0 & p > 0 \\ \mathbb{R} & p = 0 \end{cases}$$

which would imply that all closed forms are exact in Minkowski space except for 0-forms, which is trivially true because -1-forms don't exist.

The notion of a dual form is given by the Hodge star operator.

Definition. *Let M be an n-dimensional manifold and A a p-form, then the Hodge dual (A dual) is the $(n-p)$-form*

$$(*A)_{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!} \epsilon^{\nu_1 \cdots \nu_p}{}_{\mu_1 \cdots \mu_{n-p}} A_{\nu_1 \cdots \nu_p}$$

Note that the Hodge dual depends on the Levi-Civita tensor, which in turn depends on the manifold's metric. This is different from the previous operations we've discussed. Applying the Hodge star twice will return the original form with some factor of -1

$$**A = (-1)^{s+p(n-p)} A$$

where s is the number of minus signs in the eigenvalues of the metric (its signature).

Example. In 3D Euclidean space, the Hodge dual of the weddge product of two 1-forms will give another 1-form given by

$$*(U \wedge V)_i = \epsilon_i^{\;jk} U_j V_k$$

Upon further inspection, we see that this is just the cross product. This explains why the cross product changes signs under parity and also why it can only exist in three dimensions: In other dimensions there are no interesting maps from two dual vectors to a third dual vector.

The use of differential forms is best seen in electrodynamics. Recall Maxwell's equations in tensor form

$$\partial_\mu F^{\mu\nu} = J^\nu$$
$$\partial_{[\mu} F_{\nu\lambda]} = 0$$

The second equation can be expressed using the two form $F_{\mu\nu}$

$$\mathrm{d}F = 0$$

in particular, F is closed so there should be a 1-form A_μ such that

$$F = \mathrm{d}A$$

This A is the familiar vector potential of electromagnetism. Gauge invariance is immediate since for any scalar (0-form) λ

$$\mathrm{d}(A + \mathrm{d}\lambda) = \mathrm{d}A + \mathrm{d}^2\lambda = \mathrm{d}A = F$$

The other equation can be expressed using the Hodge dual

$$\mathrm{d}(*F) = *J$$

where J is a 1-form: the current 4-vector J_μ.

Tensor densities and differential forms are important because they allow us to perform integration on manifolds. Recall that whenever we perform a change of coordinates in $\mathbb{R}^n$, the volume element $d^n x$ acquires a factor of the Jacobian

$$d^n x' = \left|\frac{\partial x^{\mu'}}{\partial x^\mu}\right| d^n x$$

This formula can be explained in the context of differential forms, namely that on an n-dimensional manifold M, the integrand is an n-form. In other words, an integral over an n-dimensional region $\Sigma \subseteq M$ is a map from n-form fields to the real numbers

$$\int_\Sigma : \omega \to \mathbb{R}$$

Example. This statement by itself isn't particularly enlightening so let's give a concrete case. In 1D, we can write the 1-form ω as

$$\omega = \omega(x)\mathrm{d}x$$

where $\omega(x)$ is the component function. Then a line integral takes the 1-form ω and maps it to a real number.

So how do we use this fact to do calculations? The insight we must make is that the volume element can be interpreted as a tensor density, in particular the wedge product

$$d^n x = \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1}$$

On first glance this looks like a normal tensor, but since the expression is coordinate dependent it will not transform accordingly. First note that we can write

$$\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1} = \frac{1}{n!}\tilde{\epsilon}_{\mu_1\cdots\mu_n}\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n}$$

The Levi-Civita symbol does not change under coordinate transforms, but the other functions pick up factors

$$\begin{aligned}
\tilde{\epsilon}_{\mu_1\cdots\mu_n}\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} &= \tilde{\epsilon}_{\mu_1\cdots\mu_n}\frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}}\cdots\frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}}\mathrm{d}x^{\mu_1'} \wedge \cdots \wedge \mathrm{d}x^{\mu_n'} \\
&= \left|\frac{\partial x^\mu}{\partial x^{\mu'}}\right| \tilde{\epsilon}_{\mu_1'\cdots\mu_n'}\mathrm{d}x^{\mu_1'} \wedge \cdots \wedge \mathrm{d}x^{\mu_n'}
\end{aligned}$$

which is the transformation law for tensor densities, this also demonstrates why the Jacobian shows up when passing volume elements through coordinate transforms. With this in mind, it is straightforward to convert it into a legitimate tensor with the procedure from before.

$$\sqrt{|g|}d^n x = \sqrt{|g|}\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1} = \frac{1}{n!}\epsilon_{\mu_1 \cdots \mu_n}\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n}$$

In fact, this is more or less the Levi-Civita tensor since explicitly expanding in terms of the basis 1-forms gives

$$\begin{aligned}
\epsilon &= \epsilon_{\mu_1 \cdots \mu_n}\mathrm{d}x^{\mu_1} \otimes \cdots \otimes \mathrm{d}x^{\mu_n} \\
&= \frac{1}{n!}\epsilon_{\mu_1 \cdots \mu_n}\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} \\
&= \frac{1}{n!}\sqrt{|g|}\tilde{\epsilon}_{\mu_1 \cdots \mu_n}\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} \\
&= \sqrt{|g|}\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1} \\
&= \sqrt{|g|}d^n x
\end{aligned}$$

Thus the integral I of a scalar (0-form) function ϕ over an n-manifold is

$$I = \int \phi(x)\sqrt{|g|}d^n x = \int \phi(x)\epsilon$$

If given ϕ and $\sqrt{|g|}$, we can evaluate the integral using normal multivariable calculus since any extra issues introduced by the manifold will be taken care of by the metric determinant.

2.5 Exercises

Exercise 1. Parameterize the circle using complex numbers so that each point can be written as $e^{i\theta}$ for some θ between 0 and 2π. Define the map

$$\phi : \mathbb{R} \times S^1 \to \mathbb{R}^2 \qquad (x, e^{i\theta}) \mapsto \left(\frac{1}{1+e^{-x}}, \theta\right)$$

where we use polar coordinates in $\mathbb{R}^2$. This maps the cylinder to an open ball with the center removed, which is still open, so this is the desired chart.

Exercise 2. We can project $\mathbb{R}^2$ onto $\mathbb{R}$ through the first coordinate

$$\phi : \mathbb{R}^2 \to \mathbb{R} \qquad (x, y) \mapsto x$$

This satisfies the requirements for an atlas and turns $\mathbb{R}^2$ into a 1-manifold. The open sets consists of things like (possibly infinite) open vertical strips, open balls with the horizontal poles removed, etc. The converse is not possible, there is no way to map $\mathbb{R}$ into an open set of $\mathbb{R}^2$.

Exercise 3. The torus is a product of two circles $T^2 = S^1 \times S^1$ and we can construct an atlas for it using the atlas for the circle. Let U, V denote the two open sets of the circle

$$U = S^1 \setminus (1, 0) \qquad V = S^1 \setminus (-1, 0)$$

The two charts we constructed on the circle were

$$\phi : U \to (0, 2\pi) \subset \mathbb{R} \qquad \psi : V \to (-\pi, \pi) \subset \mathbb{R}$$

Then the four charts on the torus are

$$\begin{aligned}
\phi \times \phi' &: U \times U' \to (0, 2\pi) \times (-\pi, \pi) \\
\phi \times \psi' &: U \times V' \to (0, 2\pi) \times (-\pi, \pi) \\
\psi \times \phi' &: V \times U' \to (-\pi, \pi) \times (0, 2\pi) \\
\psi \times \psi' &: V \times V' \to (-\pi, \pi) \times (-\pi, \pi)
\end{aligned}$$

The compatibility of these charts follow from compatibility of each component on the circle.

Exercise 4. Linearity and the Leibniz rule are straightforward to check since vector fields are essentially derivatives and thus behave similarly.

$$\begin{aligned}
[X, Y](af + bg) &= X(Y(af + bg)) - Y(X(af + bg)) \\
&= X(aY(f) + bY(g)) - Y(aX(f) + bX(g)) \\
&= aX(Y(f)) + bX(Y(g)) - aY(X(f)) - bY(X(g)) \\
&= a[X, Y](f) + b[X, Y](g) \\
[X, Y](fg) &= X(Y(fg)) - Y(X(fg)) \\
&= X(fY(g) + gY(f)) - Y(fX(g) + gX(f)) \\
&= X(f)Y(g) + fX(Y(g)) + X(g)Y(f) + gX(Y(f)) \\
&\qquad - Y(f)X(g) - fY(X(g)) - Y(g)X(f) - gY(X(f)) \\
&= f[X, Y](g) + g[X, Y](f)
\end{aligned}$$

The components come from writing each vector field in terms of basis vectors

$$\begin{aligned}[X,Y]f &= \left[X^\lambda\partial_\lambda, Y^\mu\partial_\mu\right]f \\ &= X^\lambda\partial_\lambda(Y^\mu\partial_\mu f) - Y^\mu\partial_\mu(X^\lambda\partial_\lambda f) \\ &= X^\lambda\partial_\lambda Y^\mu\partial_\mu f + X^\lambda Y^\mu\partial_\lambda\partial_\mu f - Y^\mu\partial_\mu X^\lambda\partial_\lambda f - Y^\mu X^\lambda\partial_\mu\partial_\lambda f\end{aligned}$$

Since partial derivatives commute, the third and fourth terms cancel. We can relabel indices to obtain

$$\begin{gathered}[X,Y]f = (X^\lambda\partial_\lambda Y^\mu - Y^\lambda\partial_\lambda X^\mu)\partial_\mu f = [X,Y]^\mu\partial_\mu f \\ \therefore [X,Y]^\mu = X^\lambda\partial_\lambda Y^\mu - Y^\lambda\partial_\lambda X^\mu\end{gathered}$$

Using what we just found, the Lie bracket transforms as

$$\begin{aligned}[X,Y]^{\mu'} &= X^{\lambda'}\partial_{\lambda'}Y^{\mu'} - Y^{\lambda'}\partial_{\lambda'}X^{\mu'} \\ &= \frac{\partial x^{\lambda'}}{\partial x^\lambda}X^\lambda\frac{\partial x^\sigma}{\partial x^{\lambda'}}\partial_\sigma\frac{\partial x^{\mu'}}{\partial x^\mu}Y^\mu - \frac{\partial x^{\lambda'}}{\partial x^\lambda}Y^\lambda\frac{\partial x^\sigma}{\partial x^{\lambda'}}\partial_\sigma\frac{\partial x^{\mu'}}{\partial x^\mu}X^\mu \\ &= \frac{\partial x^{\lambda'}}{\partial x^\lambda}\frac{\partial x^\sigma}{\partial x^{\lambda'}}X^\lambda\partial_\sigma\frac{\partial x^{\mu'}}{\partial x^\mu}Y^\mu - \frac{\partial x^{\lambda'}}{\partial x^\lambda}\frac{\partial x^\sigma}{\partial x^{\lambda'}}Y^\lambda\partial_\sigma\frac{\partial x^{\mu'}}{\partial x^\mu}X^\mu \\ &= \delta^\sigma_\lambda X^\lambda\partial_\sigma\frac{\partial x^{\mu'}}{\partial x^\mu}Y^\mu - \delta^\sigma_\lambda Y^\lambda\partial_\sigma\frac{\partial x^{\mu'}}{\partial x^\mu}X^\mu \\ &= X^\lambda\partial_\lambda\frac{\partial x^{\mu'}}{\partial x^\mu}Y^\mu - Y^\lambda\partial_\lambda\frac{\partial x^{\mu'}}{\partial x^\mu}X^\mu \\ &= X^\lambda Y^\mu\frac{\partial^2 x^{\mu'}}{\partial x^\lambda\partial x^\mu} + \frac{\partial x^{\mu'}}{\partial x^\mu}X^\lambda\partial_\lambda Y^\mu - Y^\lambda X^\mu\frac{\partial^2 x^{\mu'}}{\partial x^\lambda\partial x^\mu} - \frac{\partial x^{\mu'}}{\partial x^\mu}Y^\lambda\partial_\lambda X^\mu\end{aligned}$$

By relabeling dummy variables and noting partial derivatives commute, we see that the first and third terms cancel to get

$$[X,Y]^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu}\left(X^\lambda\partial_\lambda Y^\mu - Y^\lambda\partial_\lambda X^\mu\right) = \frac{\partial x^{\mu'}}{\partial x^\mu}[X,Y]^\mu$$

which is the transformation law for vectors

Exercise 5. Take $X = \partial_x$, which is clearly nowhere vanishing and we'll construct Y so that the commutator is nonvanishing. Let $Y = A\partial_x + B\partial_y$ where A, B are functions which depend on x, y, then

$$\begin{aligned}[X,Y] &= \partial_x(A\partial_x + B\partial_y) - (A\partial_x + B\partial_y)\partial_x \\ &= \partial_x A\partial_x + \partial_x B\partial_y + A\partial_x^2 + B\partial_x\partial_y - A\partial_x^2 - B\partial_y\partial_x \\ &= \partial_x A\partial_x + \partial_x B\partial_y\end{aligned}$$

So for this to not vanish, we need $\partial_x A$ or $\partial_x B$ to be nonzero i.e. for then to depend on x. Define

$$Y = x\partial_x + \partial_y$$

which is nowhere vanishing, the commutator then is

$$[X,Y] = (\partial_x x)\partial_x + (\partial_x 1)\partial_y = \partial_x \neq 0$$

Exercise 6. A) Using various trig identities, we find that

$$\begin{aligned}\tan\phi &= \frac{y}{x} = \tan\lambda \\ \tan\theta &= \frac{\sqrt{x^2+y^2}}{z} = \frac{1}{\lambda} \\ r &= \sqrt{x^2+y^2+z^2} = \sqrt{1+\lambda^2}\end{aligned}$$

Another way of seeing these is by noting the path is just a helix.

$$\begin{aligned}r &= \sqrt{1+\lambda^2} \\ \theta &= \arctan\frac{1}{\lambda} \\ \phi &= \lambda\end{aligned}$$

B) In Cartesian coordinates, the tangent vector at some λ is

$$(x, y, z) = (-\sin\lambda, \cos\lambda, 1)$$

while in spherical coordinates, the tangent vector is

$$(r, \theta, \phi) = \left(\frac{\lambda}{\sqrt{1+\lambda^2}}, -\frac{1}{1+\lambda^2}, 1\right)$$

Exercise 7. A) By taking derivatives, we find the transformation matrix

$$\frac{\partial x^\mu}{\partial x^{\nu'}} = \begin{pmatrix}\cosh\chi\sin\theta & \sinh\chi\cos\theta \\ \sinh\chi\cos\theta & -\cosh\chi\sin\theta\end{pmatrix}$$

B) To get the line element in prolate spheroidal coordinates, we transform the metric tensor using the matrix we just found.

$$g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu}$$

$g_{\mu\nu}$ only has two nonzero components and we'll see that the same is true of $g_{\mu'\nu'}$

$$\begin{aligned}g_{00} = g_{11} &= \cosh^2\chi\sin^2\theta + \sinh^2\chi\cos^2\theta \\ &= \sinh^2\chi(\sin^2\theta + \cos^2\theta) + (\cosh^2\chi - \sinh^2\chi)\sin^2\theta \\ &= \sinh^2\chi + \sin^2\theta\end{aligned}$$

Thus the line element is

$$ds^2 = (\sinh^2\chi + \sin^2\theta)(\mathrm{d}\chi^2 + \mathrm{d}\theta^2)$$

Exercise 8. We'll start by applying the definitions

$$\begin{aligned}\mathrm{d}(\omega\wedge\eta)_{\mu_1\cdots\mu_{p+q+1}} &= (p+q+1)\partial_{[\mu_1}(\omega\wedge\eta)_{\mu_2\cdots\mu_{p+q+1}]} \\ &= \frac{(p+q+1)!}{p!q!}\partial_{[\mu_1}\left(\omega_{[\mu_2\cdots\mu_{p+1}}\eta_{\mu_{p+2}\cdots\mu_{p+q+1}]]}\right)\end{aligned}$$

The partial derivative distributes according to product rule and note that we can drop the inner set of antisymmetrizing brackets because antisymmetrizing again doesn't change anything. The result is

$$\frac{(p+q+1)!}{p!q!}\big((\partial_{[\mu_1}\omega_{\mu_2\cdots\mu_{p+1}})\eta_{\mu_{p+2}\cdots\mu_{p+q+1}]} + \omega_{[\mu_2\cdots\mu_{p+1}}(\partial_{\mu_1}\eta_{\mu_{p+2}\cdots\mu_{p+q+1}]})\big)$$

Having the μ_1 index in the middle of the second term is a bit annoying, reording the indices into a more natural way will append a factor of $(-1)^p$ (one -1 for each index moved).

$$\frac{(p+q+1)!}{p!q!}\big((\partial_{[\mu_1}\omega_{\mu_2\cdots\mu_{p+1}})\eta_{\mu_{p+2}\cdots\mu_{p+q+1}]} + (-1)^p\omega_{[\mu_1\cdots\mu_p}(\partial_{\mu_{p+1}}\eta_{\mu_{p+2}\cdots\mu_{p+q+1}]})\big)$$

We can reintroduce the brackets to turn each partial derivative back into an exterior derivative but we're missing a factor in front. Adding in the appropriate factors yields

$$\frac{(p+q+1)!}{(p+1)!q!}(\mathrm{d}\omega)_{[\mu_1\cdots\mu_{p+1}}\eta_{\mu_{p+2}\cdots\mu_{p+q+1}]} + (-1)^p\frac{(p+q+1)!}{p!(q+1)!}\omega_{[\mu_1\cdots\mu_{p+1}}(\mathrm{d}\eta)_{\mu_{p+2}\cdots\mu_{p+q+1}]}$$

Since the exterior derivative of a p-form yields a $(p+1)$-form, these are starting to look like wedge products again. Rewriting them as wedge products and removing the indices gives the desired law

$$\mathrm{d}(\omega\wedge\eta) = (\mathrm{d}\omega)\wedge\eta + (-1)^p\omega\wedge(\mathrm{d}\eta)$$

Exercise 9. A) We are given the Hodge dual of a 2-form F in 3D space

$$*F = q\sin\theta\mathrm{d}\theta\wedge\mathrm{d}\phi$$

To take the exterior derivative, we first note some facts

1. For a function (i.e. a 0-form) f and any form ω, we have

   $$\mathrm{d}(f\omega) = \mathrm{d}(f\wedge\omega) = (\mathrm{d}f)\wedge\omega + f\wedge(\mathrm{d}\omega)$$

   since ω is by definition antisymmetric.

2. If two terms in a wedge sum are identical, the whole term vanishes

Thus the exterior derivative is

$$\begin{aligned} *J = \mathrm{d}(*F) &= \mathrm{d}(q\sin\theta\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi) \\ &= \mathrm{d}(q\sin\theta)\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi + (q\sin\theta)\wedge\mathrm{d}(\mathrm{d}\theta\wedge\mathrm{d}\phi) \\ &= (\partial_t q\sin\theta + \partial_r q\sin\theta)\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi \\ &= 0 \end{aligned}$$

B) To recover F, we can simply apply the Hodge star again. This requires us to use the spherical metric

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \qquad \sqrt{|g|} = r^2\sin\theta$$

We see that the signature is $-,+,+,+$ so applying the Hodge star again will give

$$*(*F) = (-1)^{1+2(4-2)}F = -F$$

The definition of the Hodge star is

$$\begin{aligned}
F_{\mu_1\mu_2} = -(**F)_{\mu_1\mu_2} &= -\frac{1}{2}\epsilon^{\nu_1\nu_2}{}_{\mu_1\mu_2}(*F)_{\nu_1\nu_2} \\
&= -q\sin\theta\epsilon^{23}{}_{\mu_1\mu_2} \\
&= -q\sin\theta\sqrt{|g|}g^{2\nu_1}g^{3\nu_2}\tilde{\epsilon}_{\nu_1\nu_2\mu_1\mu_2} \\
&= -qr^2\sin^2\theta g^{22}g^{33}\tilde{\epsilon}_{23\mu_1\mu_2}
\end{aligned}$$

where we use the fact that $(*F)_{23} = -(*F)_{32}$ are the only nonzero components. We now need to obtain the inverse metric, which is easy since the metric is diagonal.

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}$$

Since two indices of the Levi-Civita symbol are already fixed, there are only two nonzero components to F. Putting everything together gives

$$F_{\mu\nu} = \frac{q}{r^2}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

C) If we recall the electromagnetic field tensor with lowered indices, we find

$$\mathbf{E} = E_{i0} = \frac{q^2}{r}\hat{r} \qquad \mathbf{B} = 0$$

this is the electric field for a point charge q at the origin.

D) We'll use the generalized Stoke's theorem here

$$\int_V \mathrm{d}\omega = \int_{\partial V}\omega$$

Since V is a solid ball of radius R, ∂V would be a hollow sphere of radius R. The integration then is straightforward

$$\int_{\partial V} q\sin\theta\mathrm{d}\theta\wedge\mathrm{d}\phi = \int_0^\pi d\theta\int_0^{2\pi}d\phi\, q\sin\theta = 4\pi q$$

This result makes complete sense since what we are actually doing is integrating the flux through a sphere of a point charge q. The addition factor of 4π comes from the fact that we work in natural units $k=1$.

Exercise 10. Maxwell's equations using differential forms are

$$\begin{aligned} \mathrm{d}F &= 0 \\ \mathrm{d}(*F) &= *J \end{aligned}$$

Suppose we are in 2D spacetime, then $\mathrm{d}F$ is a 3-form since F is a 2-form. But there does not exist any 3-forms in 2D spacetime, so $\mathrm{d}F = 0$ automatically (by antisymmetry) and we can discard the first equation.

For the second equation, $*F$ is a 0-form so both J and $*J$ are 1-forms. They are related by

$$*(*J) = (-1)^{1+1(1)} J = J$$

Given that $F_{\mu\nu}$ must be antisymmetric, it is completely parameterized by one scalar field E

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$$

There is really no point to talking about $F_{\mu\nu}$ but we must derive the field equations for E so we will continue the steps, first by finding the Hodge dual. We'll choose the metric to be Cartesian

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sqrt{|g|} = 1$$

The Hodge dual is a 0-form, i.e. a scalar

$$\begin{aligned} *F &= \frac{1}{2}\epsilon^{\mu\nu} F_{\mu\nu} \\ &= \frac{\sqrt{|g|}}{2} g^{\mu\mu'} g^{\nu\nu'} \tilde{\epsilon}_{\mu'\nu'} F_{\mu\nu} \\ &= E g^{00} g^{11} \tilde{\epsilon}_{01} \\ &= E \end{aligned}$$

As it turns out, the Hodge dual is also just our scalar EM field. The exterior derivative is straightforward, it's just the gradient

$$\mathrm{d}(*F) = \mathrm{d}E = (\partial_t E, \partial_x E) = *J$$

The 2-current (since we are in 2D) J_μ can be recovered as noted above

$$J_\mu = \epsilon^{\nu}{}_{\mu} (*J)_\nu = \tilde{\epsilon}_{\nu'\mu} g^{\nu'\nu} (*J)_\nu = (-\partial_x E, -\partial_t E)$$

So if we are given a 2-current (J_0, J_1), then the field equations are

$$\partial_t E = -J_1 \qquad \partial_x E = -J_0$$

or, if we'd rather work in upper indices

$$\partial_t E = -J^1 \qquad \partial_x E = J^0$$

Exercise 11. A) In normal electrodynamics, the gauge potential $A^{(1)}$ is a 1-form (denoted by the superscript) which couples to a point (0D) particle. In this way we can integrate over the particle worldline to write a part of the action

$$S = \int_\gamma A^{(1)}$$

If we were to now consider a 3-form potential $A^{(3)}$, then the equivalent action would be

$$S = \int_\Sigma A^{(3)}$$

Recalling the definition of an integral over a manifold, Σ must be a three dimensional "world-volume" traced out by a 2D particle (I believe these are called branes in string theory).

B) If we wanted to find the charge of an electron (modeled as a point), we integrate over a 2-sphere

$$Q = \int_{S^2} *F^{(2)}$$

This works because F is a 2-form in 4D Minkowksi space so $*F$ is also a 2-form and can be integrated over a 2-sphere (which is a 2D manifold). If we extend this to 11D spacetime where the field strength is a 4-form $F^{(4)}$, then the Hodge dual $*F^{(4)}$ is a 7-form.

$$Q = \int_{S^7} *F^{(4)}$$

and we'll have to integrate over a 7-sphere.

C) If we were to have a dual potential $\tilde{A}$ which satisfies $\mathrm{d}\tilde{A} = *F^{(4)}$ then it would have to be a 6-form, which is integrated over some 6D worldvolume traced out by a 5D object.

D)