Quantum Field Theory Review Notes

Will Huang
UW-Madison

Updated September 8, 2021

Contents

Part I
Introduction

These notes were compiled from the following classes during the following semesters to be used as a reference for future courses.

- Physics 831: Advanced Quantum Mechanics (Fall 2021) - An introduction to field theory and calculations with QED and QCD, namely tree-level Feynman diagrams.
- Physics 832: Advanced Quantum Mechanics (Spring 2022) -

The textbooks used the course are listed below

- Peskin and Schroeder - An Introduction to Quantum Field Theory
- Schwartz - Quantum Field Theory and the Standard Model

1 Preliminaries

1.1 A Theory of Radiation

Before we ask what is QFT, we need to ask why is QFT. Firstly it provides fantastic agreement between theory and experiment, for instance in measuring $g-2$ in the electron and also providing the most precise measurement of the fine structure constant α. Secondly, it provides profound insight into the structure of our universe, forming the underlying framework for several other branches of physics such as particle, condensed matter, and cosmology. Finally it gives us a set of tools for performing particle calculations.

The history of quantum field theory is closed related to that of quantum mechanics, in fact it's often said that QFT is just QM with special relativity sprinkled in. Thus we should backtrack a bit, in fact we will backtrack all the way back to Planck's proposal of the quantum which solved the ultraviolet catastrophe.

A blackbody is an object at a fixed temperature that absorbs all incoming radiation and emits blackbody radiation. Consider a blackbody cube of length L, classically this will support EM waves with angular frequency

$$\omega_n = \frac{2\pi}{L}|\mathbf{n}|c$$

where $\mathbf{n}$ is some integer 3-vector (aka normal vector). By classical thermodynamics (stat mech?) blackbodies are an ideal and diffuse emitter: light is emitted equally in all modes.

Thus the radiation intensity can be calculated as

$$I(\omega) = \frac{1}{V}\frac{dE}{d\omega} \propto c^{-3}\omega^2 k_B T$$

and herein lies the issue. As ω gets arbitrarily large, the amount of radiation emitted tends to infinity. Clearly it should drop off eventually and indeed experiments show that the classical result is not true in what is now known as the ultraviolet catastrophe (since this disagreement starts getting prominent around the UV regime).

Planck solved the issue by assuming the energy of each mode is quantized

$$E_n = \hbar\omega_n = \frac{2\pi}{L}\hbar|\mathbf{n}| = |\mathbf{p}_n|$$

This was interpreted (by Einstein) as meaning light was made up of particles, later deemed photons. However if this were true, then photons must be massless, as

$$m_n^2 = E_n^2 - p_n^2 = 0$$

We can now go ahead and calculate the thermal distribution directly

$$\langle E_n \rangle = \frac{\sum_{j=0}^{\infty}(jE_n)e^{-jE_n\beta}}{\sum_{j=0}^{\infty} e^{-jE_n\beta}} = \frac{\hbar\omega_n}{e^{\hbar\omega_n\beta} - 1}$$

where $\beta = 1/k_B T$. We can take the continuum limit $L \to \infty$ and turn them into integrals to get the average energy up to frequency ω.

$$E(\omega) = \int^{\omega} d^3n \frac{\hbar\omega_n}{e^{\hbar\omega_n\beta} - 1} = 4\pi\hbar\frac{L^3}{8\pi^3}\int_0^{\omega} \frac{{\omega'}^3}{e^{\hbar\omega'\beta} - 1}$$

$$\therefore I(\omega) = \frac{1}{V}\frac{dE}{d\omega} = \frac{\hbar}{\pi^2}\frac{\omega^3}{e^{\hbar\omega\beta} - 1}$$

which correctly matched experimental results. However in order for this result to make sense, it must be possible for light (actually photons) to somehow equilibriate. But this somehow implies light can change frequencies, essentially changing particles, which means that light can be created or destroyed.

The creation and destruction of light is explored in the case of spontaneous emission by Einstein. Suppose we have a bunch of atoms with two energy levels $E_2 > E_1$ with n_2 atoms in the excited state and n_1 atoms in the ground state. Define $\hbar\omega = E_2 - E_1$, it is possible for an atom in E_2 to emit a photon of frequency ω and then transition to the ground state E_1 (illustrated below)

The probability of this happening is called the coefficient of spontaneous emission, denoted A. We can also have a photon induce a transition $2 \to 1$ in what is called stimulated emission, the probability is denoted B. Thus the change to the number of atoms in the excited state n_2 is given by

$$dn_2 = -[A + BI(\omega)]n_2$$

where $I(\omega)$ is of course the radiation intensity and gives us a measure of how many photons are present. However it is also possible for a photon to induce a transition $1 \to 2$, the probability of which is called the coefficient of absorption B'. Putting this in we get

$$dn_2 = -[A + BI(\omega)]n_2 + B'I(\omega)n_1$$

For this simple system the total number of atoms must be conserved, that is

$$dn_1 + dn_2 = 0$$

and we (and by we I mean Einstein) assume the "gas" is in equilibrium. Thus

$$dn_2 = dn_1 = 0$$

Note that the number density can be determined from Boltzmann distributions

$$n_1 = Ne^{-\beta E_1} \qquad n_2 = Ne^{-\beta E_2}$$

for some normalization factor N. From here our equation becomes

$$-[A + BI(\omega)]e^{-\beta E_2} + B'I(\omega)e^{-E_1} = 0$$

$$\therefore I(\omega) = \frac{A}{B'e^{\hbar\beta\omega} - B}$$

By comparing to known (Planck's) results, we find that the coefficients satisfy

$$B' = B \qquad \frac{A}{B} = \frac{\hbar\omega^2}{\pi}$$

These results are not trivial. $B = B'$ implies the probability of absorption is equal to the probability of stimulated emission. These coefficients can be computed with basic (""basic"") quantum mechanics using time-dependent perturbation theory with an external EM field. Thus we can compute all Einstein coefficients without using QFT at all.

So if this problem can be solved without QFT, why should we care? The issue once again lies in the assumptions. We assume equilibrium results, yet spontaneous emission of an atom shouldn't have anything to do with equilibrium in gases. Just like Planck's calculation, the assumption that photons (or atoms) somehow achieve equilibrium requires a new framework, that framework is QFT.

The basic idea behind calculating the Einstein coefficients is to treat photons with differing energies as different particles, in what is called second quantization. We know that a photon has energy $\Delta = \hbar\omega$. A mode (fixed ω) can be excited n times, each time adding an addition Δ quantum of energy, hence we have energy levels $\Delta, 2\Delta, 3\Delta, \ldots$. This should look very familiar, indeed we can model this as a quantum harmonic oscillator. The simplest way to study an oscillator is with the ladder operators, recall

$$[a, a^\dagger] = 1 \qquad N = a^\dagger a$$

where N is the number operator.

The (orthonormal) eigenstates are denoted $|n\rangle$ and the operators act as

$$\boxed{\begin{aligned} N|n\rangle &= n|n\rangle \\ a^\dagger|n\rangle &= \sqrt{n+1}\,|n+1\rangle \\ a|n\rangle &= \sqrt{n}\,|n-1\rangle \end{aligned}}$$

Recall Fermi's golden rules for transition rates

$$\Gamma \sim |\mathcal{M}|^2\delta(E_f - E_i)$$

For our system, we need to calculate the matrix element $\mathcal{M}$. To do this, we need the Hamiltonian for the interaction between a photon and the atom. We know that the interaction Hamiltonian H_{int} must have a creation/annihilation operator in order for the photon to be created. Furthermore it must be Hermitian, so we take

$$H_{int} = H_I^\dagger a^\dagger + H_I a \qquad \mathcal{M} = \langle f|\, H_{int}\, |i\rangle$$

First consider a transition $2 \to 1$, suppose there are n_ω photons at $\omega = \Delta/\hbar$. In bra-ket notation this can be written as

$$\begin{aligned} |i\rangle &= |\text{atom}_2; n_\omega\rangle \\ |f\rangle &= |\text{atom}_1; n_\omega + 1\rangle \end{aligned}$$

that is, the atom transitions to a lower energy level and emits a photon. We can calculate the matrix element for this transition as

$$\begin{aligned} \mathcal{M} &= \langle \text{atom}_2; n_\omega|\, H_I^\dagger a^\dagger + H_I a\, |\text{atom}_1; n_\omega + 1\rangle \\ &= \langle \text{atom}_2|\, H_I^\dagger a^\dagger\, |\text{atom}_1\rangle \langle n_\omega|\, a^\dagger\, |n_\omega + 1\rangle \\ &= \mathcal{M}_0^\dagger \sqrt{n_\omega + 1} \\ \therefore |\mathcal{M}_{2\to1}|^2 &= (n_\omega + 1)|\mathcal{M}_0|^2 \end{aligned}$$

where we've defined $\mathcal{M}_0 = \langle \text{atom}_1|\, H_I^\dagger a^\dagger\, |\text{atom}_2\rangle$. For a $1 \to 2$ transition, the states are

$$\begin{aligned} |i\rangle &= |\text{atom}_1; n_\omega\rangle \\ |f\rangle &= |\text{atom}_2; n_\omega - 1\rangle \end{aligned}$$

with matrix element

$$|\mathcal{M}_{1\to2}|^2 = n_\omega|\mathcal{M}_0|^2$$

Once again we assume the gas is at equilibrium (this time with a concrete meaning)

$$dn_2 = -dn_1 = -|\mathcal{M}_{2\to1}|^2 n_2 + |\mathcal{M}_{1\to2}|^2 n_1 = -|\mathcal{M}_0|^2(n_\omega + 1)n_2 + |\mathcal{M}_0|^2 n_\omega n_1$$

which we can compare to Einstein's equation

$$dn_2 = -[A + BI(\omega)]n_2 + BI(\omega)n_1$$

To correctly match these, we'll need to derive a relationship between the intensity $I(\omega)$ and the number of photons in a given mode n_ω. Using the fact that energies are quantized, the total energy is

$$E(\omega) = \int_0^\omega d^3n\hbar\omega n_\omega = 4\pi\overline{L}^3 \int_0^\omega \frac{d\omega'}{(2\pi)^3}\omega'^3 n_\omega$$

$$\therefore I(\omega) = \frac{1}{L^3}\frac{dE}{d\omega} = \frac{\hbar\omega^3}{\pi^2}n_\omega \longrightarrow n_\omega = \frac{\pi^2}{\hbar\omega^3}I(\omega)$$

Now we can put this back in to derive

$$dn_2 = -|\mathcal{M}_0|^2\left[1 + \frac{\pi^2}{\hbar\omega^3}I(\omega)\right]n_2 + |\mathcal{M}_0|^2\left[\frac{\pi^2}{\hbar\omega^3}I(\omega)\right]n_1$$

and from comparing results we find

$$B = B' \qquad \frac{A}{B} = \frac{\hbar\omega^3}{\pi^2}$$

in agreement with Einstein's results. This derivation was one of the first results in quantum field theory.

1.2 Notation and Conventions

Throughout this course, we will work in "god-given" units, that is we set

$$\hbar = c = 1$$

In such a system the units are related by

$$[x] = [p]^{-1} = [E]^{-1} = [t] = [m]^{-1}$$

Sometimes we discuss a decay width

$$\Gamma = \frac{1}{\tau} \qquad [\Gamma] = [E]^{-1}$$

and also a cross section

$$[\sigma] = [L]^2 = [E]^{-2}$$

When discussing relativity, it's useful to use 4-vectors. We'll adopt the indices $\mu, \nu = 0, 1, 2, 3$ where 0 refers to a time component and the other three are spatial. There is a difference between upper and lower indices related by the Minkowski metric tensor

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

For instance the position and momentum 4-vectors are

$$x^\mu = (x^0, x^1, x^2, x^3) = (t, \mathbf{x}) \qquad p^\mu = (E, \mathbf{p})$$

and with lowered indices, it's

$$x_\mu = g_{\mu\nu}x^\nu = (x^0, -\mathbf{x}) \qquad p_\mu = (E, -\mathbf{p})$$

We can also investigate dot products of 4-vectors, for instance the dot product of position and momentum

$$p \cdot x = p^\mu x_\mu = p^\mu g_{\mu\nu} x^\mu = p^0 x^0 - \mathbf{p} \cdot \mathbf{x}$$
$$p^2 = p^\mu p_\mu = (p^0)^2 - \mathbf{p} \cdot \mathbf{p} = E^2 - \mathbf{p}^2 = m^2$$

The analog for the gradient is the operator

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \nabla\right) = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial \mathbf{x}}\right) \qquad \partial^\mu = \left(\frac{\partial}{\partial x^0}, -\nabla\right)$$

Note that the position and momentum 4-vectors are "naturally raised" while the derivative is "naturally lowered."

Sometimes we will utilize what is known as the antisymmetric tensor

$$\epsilon^{\mu\nu\rho\sigma} \qquad \begin{cases} \epsilon^{0123} = 1 \\ \epsilon_{0123} = -1 \end{cases}$$

which has the property that swapping two indices introduced a negative. For example

$$\epsilon^{1023} = -1 \qquad \epsilon_{1230} = -1$$

Finally it is useful to recall the Pauli matrices, we will used raised indices for convenience.

$$\sigma^1 = \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

2 Symmetry and 4-vectors

2.1 Lorentz Invariance

2.2 Parity

Part II
Quantum Electrodynamics

3 The Klein-Gordon Field