Calculus Lecture Notes

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1 Introduction

I wrote these notes for the students of Alex Zamora's "Calculus Club," a pilot initiative to introduce AP Calculus AB concepts to middle schoolers/rising high schoolers. The general layout of the material follows that of Strang and Herman's free online calculus textbook[1], though adjusted slightly to suit my taste. Originally these were intended to serve as my lecture notes for sessions that I teach, but they quickly evolved into a mix of study notes, a quick reference, and an alternate textbook.

[1] Available here

2 Limits

2.1 Introduction to Limits

Limits form the very backbone of calculus, almost every concept and definition comes from the limit of something else. Intuitively they are quite simple: the limit of a something is simply where is appears to be going. In practice, we need a much more rigorous definition. To see why, we turn to the ancient Greeks.

In around 450 BC, Zeno, as ancient Greek philosophers tended to do, sat around thinking of paradoxes. Here are two of his most famous:

1. Atalanta[2] is trying to walk to the end of a path. To walk the entire path she must first walk half the path, then half the remaining path (a quarter), then half the next remaining path (an eigth), then a sixteenth, a thirty-second, etc. Since Atalanta has to walk an infinite number of paths, how can she ever finish walking the path?

2. Achilles[3] is racing a tortoise[4]. To make if fair, he gives the tortoise a head start (say 100m). Once he takes off and covers the first 100m, the tortoise would've meandered a couple more meters. Once he runs the next few meters, the tortoise will cover a few more meters, and so on so forth. So the questions is: If every time Achilles catches up to where the tortoise was prior, the tortoise remains a bit ahead, how will Achilles ever pass the tortoise?

The astute reader may notice that these paradoxes are clearly resolvable. After all, we walk across paths all the time and obviously Achilles will eventually pass the turtle, so why are these so famous?

The reason why these paradoxes are even talked about now is because they deal with the concept of infinity and infinitesimals (infinitely small distances). In order to have a precise mathematical way of dealing with these, we must invent new mathematics. That new mathematics is Calculus.

To introduce the idea of a limit, let's write the Atalanta (formally known as the Dichotomy) paradox in terms of limits (kind of). For Atalanta, suppose the path is 1m long and she walks at a speed of 1m/s. Then the time taken will be

$$t = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

Let's try to add up the terms in this sequence, if n is the number of terms we add up

n	1	2	3	4	5	$\cdots$	10	$\cdots$	∞
t	1/2	3/4	7/8	15/16	31/32	$\cdots$	1023/1024	$\cdots$	1

We see that as n gets really, really large, then t gets really close to 1. This makes sense because Atalanta should be able to walk the 1m path in 1s.

[2] Famous figure in Greek mythology

[3] Even more famous figure in Greek mythology

[4] A very famous figure in fairy tale lore, famously starred in the hit movie "Tortoise and the Hare"

With that history lesson out of the way, let's return to the math. Take a simple function, for instance $f(x) = x$, and let's look at how it behaves around $x = 1$. There's two ways we can answer this question:

1. Start from the left of the graph and see what happens as we move up to $x = 1$
2. Start from the right of the graph and see what happens as we move down to $x = 1$

It's pretty obvious in this case, but we see that the graph approaches $y = 1$ regardless of which direction we approach. This idea of seeing what happens when we get close to a value is formalized in the definition of a limit.

Definition 1: The Limit

Let $f(x)$ be a function and a, L some real numbers. If all values of $f(x)$ approach L as x approaches a, then we say that L is the limit of the function $f(x)$ as x approaches a. In symbols, we write this as

$$\lim_{x\to a} f(x) = L$$

Example. What we got from the previous simple example is that

$$\lim_{x\to 1} x = 1$$

A quick and dirty way to evaluate limits is to evaluate a bunch of values progressively closer to the value we wish to look at. If the y values appears to get closer to some number, then that number is probably the limit.

Example. Consider the function $f(x) = x^2$ and suppose we want to find the limit as x approaches 2. To evaluate this limit using a table, we compute the following values:

x	1.9	1.99	1.999	2.001	2.01	2.1
f(x)	3.61	3.9601	3.996001	4.004001	4.0401	4.41

From looking at this table, we can make an educated guess at the limit

$$\lim_{x\to 2} x^2 = 4$$

The method of evaluating limits using a table is very crude and, as we saw, very difficult without some form of computer assistance. Throughout mathematics, it is most common to evaluate limits algebraically. First, we discuss the notion of a one-sided limit.

Definition 2: One-Sided Limits

Let $f(x)$ be a function and a some real number. There are two different one-sided limits:

- If $f(x)$ approaches a real number L as x, where $x < a$, approaches a, then the **left-sided limit** of $f(x)$ as x approaches a is L, or in symbols

$$\lim_{x \to a^-} f(x) = L$$

- If $f(x)$ approaches a real number M as x, where $x > a$, approaches a, then the **right-sided limit** of $f(x)$ as x approaches a is M, or in symbols

$$\lim_{x \to a^+} f(x) = M$$

Example. Consider a piecewise function

$$f(x) = \begin{cases} x & x \leq 0 \\ x^2 + 1 & x > 0 \end{cases}$$

If we approach 0 from the left, we are on the top function. We can plug in 0 directly to find

$$\lim_{x \to 0^-} f(x) = 0$$

From the right we are on the bottom function. We can still plug in 0 even though the condition is $x > 0$ because limits are not about what the function actually is, rather it is about what the function is approaching. Thus

$$\lim_{x \to 0^+} f(x) = 0^2 + 1 = 1$$

This method of "plug-and-chug" is generally how most limits will be evaluated. In the rare case that this doesn't work (for instance we end up dividing by zero), we'll need to do some more algebra.

Theorem 2.1: Two-Sided Limits

The limit of a function $f(x)$ as x approaches a exists if and only if both one-sided limits exist and are equal, that is

$$\lim_{x \to a} f(x) = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

Thus we cannot conclude that a limit exists unless both the left- and right-handed limits exist and are equal.

Example. Let a and c be any two real numbers, the two most basic limits are

$$\lim_{x\to a} c = c \qquad \lim_{x\to a} x = a$$

Example. Sometimes the limit does not exist because we try to do something illegal, for instance

$$\lim_{x\to -4} \sqrt{x} = \varnothing$$

because we cannot square root a negative[5]

Example. Consider a piecewise function

$$f(x) = \begin{cases} \sin(x) & x \leq 0 \\ \cos(x) & x > 0 \end{cases}$$

We have the following one-sided limits

$$\lim_{x\to 0^-} f(x) = 0 \qquad \lim_{x\to 0^+} f(x) = 1$$

Since the two are not equal, the limit of $f(x)$ as x approaches 0 does not exist.

Sometimes we end up trying to do something illegal when we "plug-and-chug," but that does not always mean the limit doesn't exist. In some cases infinity (or negative infinity) is a perfectly valid answer[6]. Identifying infinite limits requires a bit of brainpower. The process is similar to making a table of values but we don't actually do it. It's easiest to demonstrate with examples:

Example. Consider the function $f(x) = 1/x$ as x approaches 0. From the right side we get smaller and smaller x values, which means that $f(x)$ gets larger and larger. Thus we can write

$$\lim_{x\to 0^+} f(x) = \infty$$

From the left side we also get smaller and smaller values, but negative. Thus rather than get bigger, $f(x)$ gets more and more negative, so we write

$$\lim_{x\to 0^-} f(x) = -\infty$$

Since these two one-sided limits are not the same, the limit does not exist.

Example. Now consider the function $f(x) = 1/x^2$. We can use a similar argument from last time, but since the denominator is squared, the function will always be positive. Thus

$$\lim_{x\to 0} f(x) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = \infty$$

[5]Technically we can, but then we would have to work in the complex plane and that requires much more complicated machinery (complex analysis)

[6]Though it can be debated if these limits actually exist

The value of the limit is not the only place where infinity could exist, we may be interested in the behavior of the function as x itself approach positive/negative infinity. Again, we demonstrate with an example:

Example. Consider the function $f(x) = 1/x$ again. Suppose we want to know the limit as x goes to infinity. To do this, we think about what happens as x gets larger and larger. Clearly $f(x)$ would get smaller and smaller as the denominator grows, so

$$\lim_{x\to\infty} f(x) = 0$$

Similarly, we can show that

$$\lim_{x\to-\infty} f(x) = 0$$

Limits with infinity gives us a way to find asymptotes without having to draw a graph.

Theorem 2.2: Asymptotes

Let $f(x)$ be a function and a a real number. The line $x = a$ is a vertical asymptote if both one-sided limits are positive or negative infinity. In symbols

$$\lim_{x\to a^-} f(x) = \pm\infty \qquad \lim_{x\to a^+} f(x) = \pm\infty$$

Note that this does not mean the limit itself is defined as we say in the case of $f(x) = 1/x$.

The line $y = a$ is a horizontal asymptote if the limit as x goes to positive or negative infinity is a.

$$\lim_{x\to-\infty} f(x) = a \qquad \text{or} \qquad \lim_{x\to\infty} f(x) = a$$

Example. Let's find all asymptotes for the following function

$$f(x) = \frac{1}{1-x} + 3$$

First we do the horizontal asymptotes

$$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 3$$

so $y = 3$ is a horizontal asymptote for this function.

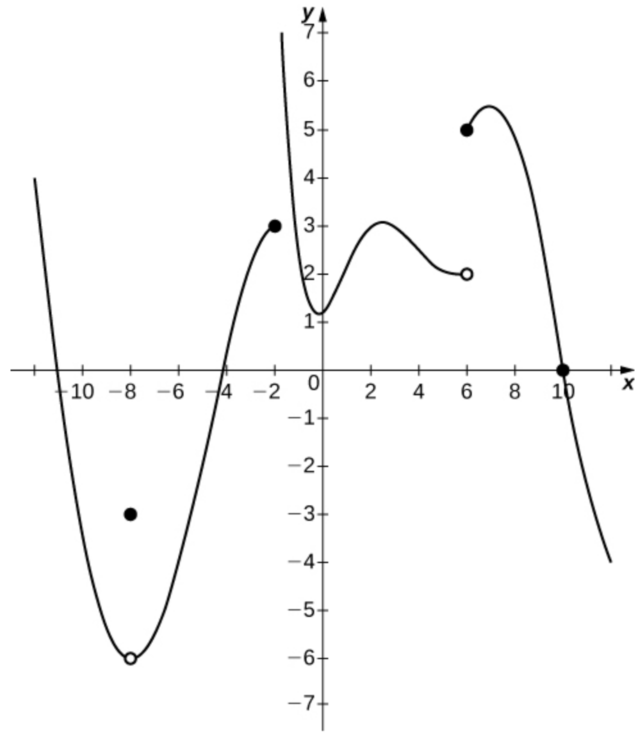
It's a bit harder to see the vertical asymptotes right away, but a good way to find them is to look for "divide by zeros" and then evaluate the limits. In this case we consider $x = 1$

$$\lim_{x\to1^-} = \infty \qquad \lim_{x\to1^+} = -\infty$$

Notice that the left handed limit is positive infinity this time. This is because when $x < 1$, the denominator is very small but positive. When dealing with infinities, it is very important to make sure you have the right one.

If we are given a graph of a function, we can evaluate limits visually by examining the behavior of the graph. For instance if we wanted to evaluate a left-handed limit, we can place our pencil a bit to the left of the point and trace the line up to that point. The height that we appear to be approaching is the limit. Remember: the limit is what the function appears to approaching, not what it actually is.[7]

Example. Consider the following graph of a function $f(x)$



At $x = -8$, we see that from the left and the right we are approaching the same value. The fact that $f(-8) = -3$ does not matter, the limits are

$$\lim_{x \to -8^-} f(x) = \lim_{x \to -8^+} f(x) = \lim_{x \to -8} f(x) = -6$$

At $x = -2$ the graph has different behavior depending on if we are on the left or right. If we trace the graph coming from the left, we will approach $y = -2$ and in fact this is also the value of $f(-2)$. If we come from the right then we go straight up to $+\infty$. Thus

$$\lim_{x \to -2^-} f(x) = 3 \qquad \lim_{x \to -2^+} = \infty \qquad \lim_{x \to -2} = \varnothing$$

The remaining two interesting points, $x = 6$ and $x = 10$, will be left as an exercise to the sufficiently motivated and astute reader.

[7] If it helps, you may think of limits as "It is not the destination that matters, but the journey"

2.2 Noob's Guide to Limits

Now that we've brute forced our way through a couple limits, it is time to find a way to use those previous results to evaluate new limits. A lot of these laws/rules may seem obvious but it is important to state them nonetheless.

Theorem 2.3: Properties of Limits

Let $f(x)$ and $g(x)$ be functions, a a real number, and suppose f, g have the following limits

$$\lim_{x\to a} f(x) = L \qquad \lim_{x\to a} g(x) = M$$

Then we have the following properties, let c be a constant

$$\begin{aligned}
\lim_{x\to a} f(x) \pm g(x) &= \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M \\
\lim_{x\to a} f(x)g(x) &= \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = L \cdot M \\
\lim_{x\to a} cf(x) &= c \cdot \lim_{x\to a} f(x) = c \cdot L \\
\lim_{x\to a} \frac{f(x)}{g(x)} &= \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M} \qquad \text{if } M \neq 0 \\
\lim_{x\to a} (f(x))^n &= \left(\lim_{x\to a} f(x)\right)^n = L^n \\
\lim_{x\to a} \sqrt[n]{f(x)} &= \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}
\end{aligned}$$

The last result holds for all x if n is odd and all $x \geq 0$ if n is even (these also happen to be the domains of $\sqrt[n]{x}$ for odd/even n).

These laws give the "plug and chug" strategy a more mathematical reasoning. Since we know the two basic limits

$$\lim_{x\to a} c = c \qquad \lim_{x\to a} x = a$$

we can use the limit laws to manipulate them into the function we are interested in.

Example. Consider the function

$$f(x) = \frac{x^2 + 3x - 10}{x^3 - 2x}$$

The using the basic limits, we can evaluate the limit

$$\lim_{x\to 2} f(x) = \frac{2^2 + 3 \cdot 2 - 10}{2^3 - 2 \cdot 2} = \frac{0}{4} = 0$$

It is worth noting that these laws all apply the one-sided limits as well.

There are many tricks to evaluating limits, we'll cover a few of the most common ones in the ensuing exercises.

Example. Consider the function

$$f(x) = \frac{x^2 + 2x - 3}{x^2 - 1}$$

If we tried to evaluate the limit as x approaches 1, then we would end up dividing by zero. To avoid this we must first factor the top and bottom

$$f(x) = \frac{(x+3)(x-1)}{(x+1)(x-1)} = \frac{x+3}{x+1}$$

Now we see that since a term cancels, we can use the limit laws to evaluate

$$\lim_{x \to 1} f(x) = \frac{1+3}{1+1} = 2$$

Example. When working with square roots it is common to get a function of the form

$$f(x) = \frac{\sqrt{x+1} - 1}{x}$$

If we tried to evaluate the limit as x approaches 0 it would appear to seem that we are stuck. Furthermore there is nothing to factor, so what is there left to do? It turns out that we can use the fact that $x^2 - a^2 = (x+a)(x-a)$ change the fraction. This is known as multiplying by the conjugate, lets see it in action:

$$\begin{aligned} f(x) &= \frac{\sqrt{x+1} - 1}{x} \cdot \left(\frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1} \right) = \frac{(\sqrt{x+1})^2 - 1^2}{x(\sqrt{x+1} + 1)} \\ &= \frac{x + 1 - 1}{x(\sqrt{x+1} + 1)} = \frac{x}{x(\sqrt{x+1} + 1)} \\ &= \frac{1}{\sqrt{x+1} + 1} \end{aligned}$$

This sort of algebraic manipulation is allowed because at the end of the day, all we did was multiply by 1. With the function in this form, we can now evaluate the limit

$$\lim_{x \to 0} f(x) = \frac{1}{\sqrt{0+1} + 1} = \frac{1}{2}$$

Example. When evaluating limits, one should always try to simplify the function as much as possible before preceding. Consider the function

$$f(x) = \frac{1}{x} - \frac{3}{x(x+3)}$$

If we tried to evaluate the limit as x goes to 0, we may be tempted to either say it does not exist or is an asymptote due to the $1/x$. But if we took the time to simply, we find

$$f(x) = \frac{x+3}{x(x+3)} - \frac{3}{x(x+3)} = \frac{x}{x(x+3)} = \frac{1}{x+3}$$

Now we can evaluate

$$\lim_{x\to 0} f(x) = \frac{1}{0+3} = \frac{1}{3}$$

For more complicated limits we may sometimes use the squeeze (or sandwich) theorem when applicable.

Theorem 2.4: Squeeze Theorem

Let $f(x), g(x), h(x)$ be functions defined for all $x \neq a$ in an open interval containing a. If

$$f(x) \leq g(x) \leq h(x)$$

Then their limits obey

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x) \leq \lim_{x\to a} h(x)$$

In particular if

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then it must be the case that

$$\lim_{x\to a} g(x) = L$$

To see how we may use this theorem to solve otherwise impossible limits, consider the following example.

Example. Suppose we have a function

$$f(x) = x^2 \cos\left(\frac{1}{x}\right)$$

and we wish to evaluate the limit as x goes to 0. We can't use the limit laws because then we'd be dividing by zero, so instead we note that

$$-1 \leq \cos(x) \leq 1 \qquad \text{and so} \qquad -x^2 \leq f(x) \leq x^2$$

It doesn't matter what's inside the cosine because it will always be between -1 and +1. Thus we just need to compute

$$\lim_{x\to 0} -x^2 = 0 \qquad \lim_{x\to 0} x^2 = 0$$

and we are able to conclude, using the squeeze theorem,

$$\lim_{x\to 0} f(x) = 0$$

Note that in order to actually get a limit out of the squeeze theorem, the limit on both sides of the inequality must be the same. If they are different, then the squeeze limit does not tell us anything.

Optional/Hard Topic 1: Important Trigonometric Limits

For this subsection I will use without proof the following limits[a]

$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1 \qquad \lim_{x\to 0}\frac{1-\cos x}{x} = 0$$

Let's see how we can use these two limits to evaluate a deceptively difficult limit

Example. Consider the function

$$f(x) = \frac{\sin^2(3x)}{\sin(2x)}$$

We want to evaluate the limit as x approaches 0. At first glance, we don't really see much that we can do. After all, both the numerator and denominator goes to zero. However, before we give up and quit math forever, we notice that there is one (indeed the only) thing to do: rewrite the numerator.

$$f(x) = \frac{1-\cos^2(3x)}{\sin(2x)} = \frac{(1+\cos(3x))(1-\cos(3x))}{\sin(2x)}$$

We now remember that we've seen part of this numerator before, it's the same numerator in the limit we first introduced! But we're missing the denominator, furthermore the function instead the cosine is x, not $3x$. To fix this, we turn to the old trick: multiply by 1. This also fixes the sine in the denominator

$$\begin{aligned}
f(x) &= \frac{(1+\cos(3x))(1-\cos(3x))}{\sin(2x)}\cdot\left(\frac{2x}{2x}\right)\cdot\left(\frac{3x}{3x}\right)\\
&= \left(\frac{2x}{\sin(2x)}\right)\cdot\left(\frac{1-\cos(3x)}{3x}\right)\cdot\left(\frac{2x(1+\cos(3x))}{3x}\right)\\
&= \frac{2}{3}\left(\frac{1}{\sin(2x)/2x}\right)\left(\frac{1-\cos(3x)}{3x}\right)(1+\cos(3x))
\end{aligned}$$

Now we can evaluate the limit

$$\lim_{x\to 0} f(x) = \frac{2}{3}\cdot\frac{1}{1}\cdot 0\cdot(1+0) = 0$$

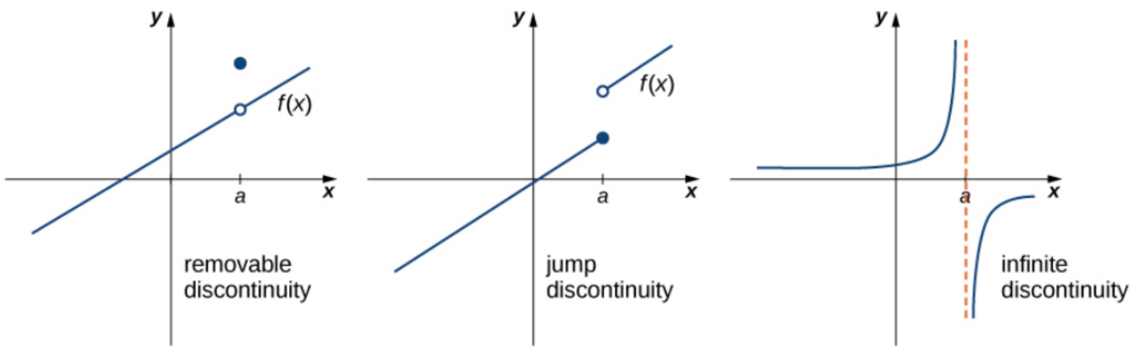
[a] A geometric proof of the first limit using the squeeze theorem is provided in the textbook and the second can be found by multiplying by the conjugate

2.3 Continuity

At its essence, a function is continuous if you can draw if without picking up your pencil. This means that functions like $x, x^2, \sin(x)$ are continuous while $1/x$ is not. However intuition is never good enough and we need a formal mathematical definition of what it means for a function to be continuous. First, we need to be able to visually tell when a function is discontinuous, this is easy:

There are three ways a function fails to be continuous

1. A removable (or hole) discontinuity: there is a missing point, the function may or may not be defined there
2. A jump discontinuity: the function literally jumps
3. An infinite discontinuity: there is a vertical asymptote (think $1/x$)



Now that we know what it looks like when a function is not continuous, we can write down a definition.

Definition 3

A function $f(x)$ is continuous at a point $x = a$ if all of the following hold:

- The function is defined at $x = a$, that is $f(a)$ is defined
- The limit $\lim_{x\to a} f(x)$ exists
- The two above quantities are equal

$$\lim_{x\to a} f(x) = f(a)$$

We say that $f(x)$ is discontinuous at $x = a$ if any of the above fail.

This definition captures the essence of "not lifting up your pencil." Requiring that the function exist means that we won't have any infinite discontinuities and requiring it equal the limit means that there will not be any hole or jump discontinuities.

Example. Suppose we have a piecewise function

$$f(x) = \begin{cases} \sin(x) & x \leq 0 \\ \cos(x) & x > 0 \end{cases}$$

We have two limits as x approaches 0

$$\lim_{x \to 0^-} f(x) = \sin(0) = 0 \qquad \lim_{x \to 0^+} f(x) = \cos(0) = 1$$

Thus the function $f(x)$ is discontinuous at $x = 0$.

We can use limits to classify the types of discontinuities directly

Theorem 2.5: Discontinuities

Suppose $f(x)$ is discontinuous at $x = a$.

1. It is a removable discontinuity if the limit $\lim_{x \to a} f(x)$ exists and is not infinity.
2. It is a jump discontinuity if the limit does not exist, but the two one-sided limits exist (so they are not equal).
3. It is an infinite discontinuity if the one-sided limits are $\pm\infty$, i.e. there is a vertical asymptote there.

Example. Consider the function

$$f(x) = \frac{x^2 - 1}{x^2 - 3x + 2}$$

First we simplify the numerator and denominator, but we do not cancel.

$$f(x) = \frac{(x+1)(x-1)}{(x-1)(x-2)}$$

There are two candidates for discontinuities, $x = 1$ and $x = 2$. Now we may simplify the function (by canceling common factors) and evaluate limits.

$$\lim_{x \to 1} f(x) = \frac{1+1}{1-2} = -2$$

Thus $x = 1$ is a hole discontinuity because while the limit exists, $f(1) = 0/0$ is undefined. On the other hand, for $x = 2$

$$\lim_{x \to 2^-} f(x) = -\infty \qquad \lim_{x \to 2^+} f(x) = \infty$$

there is a vertical asymptote, so this is an infinite discontinuity.

A common type of question is to identify intervals for which a function is continuous. A function is continuous over an interval if it is continuous at every point within that interval.

Example. Take the previous function, the interval for which it is continuous is

$$(-\infty, 1) \cup (1, 2) \cup (2, \infty)$$

Intervals are typically written using interval notation rather than inequalities. Here is a brief rundown:

- If an interval includes an endpoint (i.e. $\leq, \geq$), use a square bracket
- If it doesn't include an endpoint, use a parentheses
- For one sided intervals, use ∞ as one of the endpoints, these will always use parentheses.

So the example above represents the following inequalities (the U symbol is read as union and basically just means to combine all the intervals together)

$$x < 1 \text{ or } 1 < x < 2 \text{ or } 2 < x$$

For piecewise functions, we have a notion of one-sided continuity. This is basically just taking the normal definition and replacing the limit with a one-sided one. Here's an example:

Example. The graph $\sqrt{x}$ is continuous to the right of $x = 0$, we write this as $[0, \infty)$

When we introduced the limit laws, we acquired the power to evaluate almost every type of limit directly. However for functions within other functions, we need a new skill.

Theorem 2.6: Composition of Limits

Suppose $\lim_{x \to a} f(x) = L$ and $g(x)$ is continuous at L, then

$$\lim_{x \to a} g(f(x)) = g(L)$$

Now we can use direct substitution for almost every limit thrown at us.

Example. Consider the function

$$f(x) = \cos(x + \pi)$$

As x approaches π, we can use the composition law to evaluate

$$\lim_{x \to \pi} f(x) = \cos\Big(\lim_{x \to \pi} x + \pi\Big) = \cos(2\pi) = 1$$

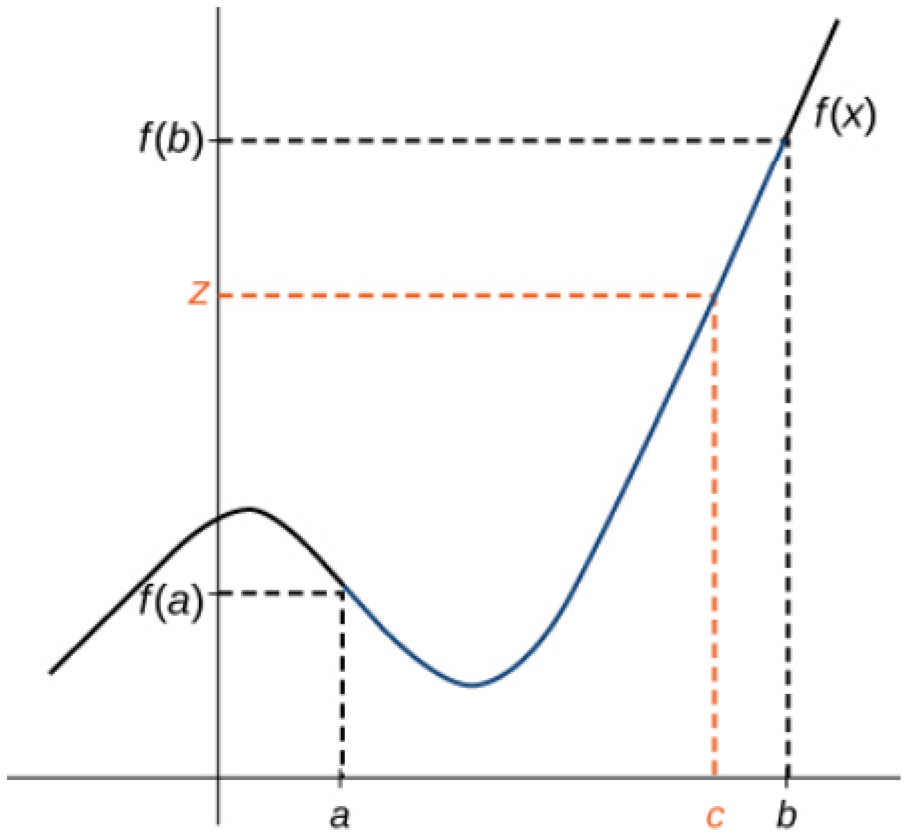
Mathematicians love continuous functions, they have many properties that are appealing. Here's a theorem that gives a glimpse to the power within such functions.

Theorem 2.7: Intermediate Value Theorem

Let $f(x)$ be a function which is continuous on a *closed* interval $[a, b]$. Then if we take any number z in between $f(a)$ and $f(b)$, we can find a number c inside the interval which satisfies

$$f(c) = z$$

There is a lot of math mumbo-jumbo here so first let's draw a picture



Let's break down the different pieces of the theorem to better understand what it is saying. First we need to have a function which is continuous on some open interval. This does not mean that the function is only continuous on that interval, just that for the theorem to work it should be continuous on the two endpoints and everywhere in between.

For instance if we had $f(x) = 1/x$, then $[1, 2]$ is a totally fine interval but $(0, 2]$ is not because $1/x$ is not continuous at $x = 0$. If we had a function that is continuous everywhere like x or $\cos(x)$, then any interval we pick will automatically work.

The next part of the theorem says to take some y-value in between $f(a)$ and $f(b)$, call it z. The two endpoints on the x-axis gives us two "heights:" the function evaluated there. We can choose any point in that vertical range and give it a special name (c in this case), this is our special little y-value. Keep it close to your heart.

Now the important part of the theorem. Our little y-value is lost, we don't know where it came from; specifically the x-value that got us that y. Unfortunately, we won't be able to find that x value using the theorem, but we can at least be certain it exists and our y-value actually has a home it came from. The theorem tells us that there must be an x-value, let's call it c, in between the two endpoints we specified which gives us z. In math terms, we can apply the function to get back to our y value: $f(c) = z$.

Intuitively the theorem should mostly make sense if you think long and hard about it. If we have an interval on the x-axis, there is another interval on the y-axis (see the picture above). Then since the function is continuous, if you trace the function over your interval, you must cover that y-axis interval at some point. There is no way around this, the only way is if you cheated and lifted your pencil in which case the function is obviously no longer continuous.

Let's see an example of this theorem in action.

Example. Show that $f(x) = x - \sin(x)$ has at least one zero (i.e. at least one x-intercept).

First we note that $f(x)$ is continuous everywhere, so any interval will satisfy the first part of the theorem. We need to find two specific endpoints, one where $f(x) < 0$ and one where $f(x) > 0$. Here's two

$$f(-\pi) = -\pi - sin(-\pi) = -\pi < 0 \qquad f(\pi) = \pi - \sin(\pi) = \pi > 0$$

Thus the intermediate value theorem tells us there is a point, let's call it c, in between $-\pi$ and π such that $f(c) = 0$. c is the x-intercept we are looking for.

2.4 Hard Topic: Limits for the Morbidly Curious

CAUTION: This is a hard topic and will not be covered on the AP exam, in fact it is only lightly touched on in a normal college calculus class. This section is completely optional, proceed at your own risk and sanity.

If the idea of "get really really close to a number and that's the limit" doesn't sit right with you, then you've come to the right part of these notes! This section is for the mathematically curious, those that got lost, those that are too nosy for their own good, those that want a peek at "real mathematics" and "real proofs," those that want to see the funny mathematical symbols, and everyone in between.

In this section, we introduce the epsilon-delta proof, a standard proof device used in real analysis, the branch of mathematics which encompasses calculus. These proofs give a mathematically rigorous definition of "close to a number" and allow us to prove beyond a shadow of a doubt that the limit is what we say it is. These are not for the faint of heart and sometimes it is best not to think too hard about this section and return once you are more mathematically mature.

To start off the process of actually proving a limit exists, we must ask ourselves: what does it mean for a number to get really close to another number. When we say

$$|x - y| < h$$

we really mean "x is less than h away from the y." If we want to impose the condition that they aren't equal, then we might write

$$0 < |x - y| < h$$

because if $|x - y| = 0$, then we must have $x = y$. Thus we can examine the "closeness" of two numbers using absolute values, leading to the precise definition of a limit. It is this definition that mathematicians actually use when discussing limits and their properties.

Definition 4: The Limit, for Experts

Let $f(x)$ be a function defined over an open interval containing a and L a real number. We say that the limit as x approaches a of $f(x)$ is equal to L, or in symbols

$$\lim_{x \to a} f(x) = L$$

if for every $\epsilon > 0$ (Greek letter epsilon) there exists a $\delta > 0$ (Greek delta) such that

$$|x - a| < \delta \Longrightarrow |f(x) - L| < \epsilon$$

In words: if $|x - a| < \delta$, then $|f(x) - L| < \epsilon$.

Let's state that last line in more human terms. Basically we are saying that if x is close enough to a, then $f(x)$ better be close to L: its limit. Essentially when we prove a limit we are playing a game: I give you an ϵ which is how close I want the function to be to its limit and you give me a δ which tells me the range of x values for which the function is that close. To prove that this game is always winnable, we need to find a formula for δ in terms of ϵ. Because the goal of the proof is to find a delta given an epsilon, these are called epsilon-delta ($\epsilon - \delta$) proofs.

Example. Here's a simple example, prove that

$$\lim_{x \to a} x = a$$

Let $\epsilon > 0$ and take $\delta = \epsilon$. Then we have the implication

$$|x - a| < \delta \implies |f(x) - a| = |x - a| < \delta = \epsilon$$

So we have solved the game because with this strategy we will always get $|f(x) - a| < \epsilon$. Thus we conclude that this is indeed the limit, now with full mathematical confidence.

This first example is so easy it might feel like you got scammed, so feel free to try a slightly more complicated linear function $f(x) = x + 1$ yourself, if that's too easy try $f(x) = 2x + 1$[8].

Example. For an example of a limit that doesn't exist, consider a piecewise function

$$f(x) = \begin{cases} -1 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

We'll take the efficient route and try every number at once by trying to prove

$$\lim_{x \to 0} f(x) = a \qquad a \text{ any real number } (a \in \mathbb{R})$$

Simply take $\epsilon = 1/2$, then for any $\delta > 0$ we will have

$$f\left(-\frac{\delta}{2}\right) = -1 \qquad f\left(\frac{\delta}{2}\right) = 1$$

Note that by using this choice of ϵ, the open interval $(a - \epsilon, a + \epsilon)$ will include either -1, 1, or neither but never both. Since any choice of δ will always yield x values which give both -1 and 1, we will never be able to find a value of δ that makes $f(x)$ close enough to the limit. Since the game is unwinnable, we are forced to conclude that the limit does not exist.

[8] As most math textbooks say: these are left as a exercise to the reader

So to prove that a limit exists, you must find a formula for δ in terms of ϵ and to prove that the limit does not exist, you must give an ϵ for which no δ can be found. Returning to the game analogy, if the δ playing has a foolproof strategy to always give a δ that works, then the limit exists. If ϵ player has some OP (overpowered for you non-gamers) trump card that literally cannot be beaten, i.e. the δ player is basically screwed cause he can't do anything, then the limit does not exist.

Most examples are not this easy and involve some unintuitive "tricks," we will look at some of those next; consider this your second warning to turn back. Otherwise feel free to continue onwards and attempt to further understand proofs; your mileage may vary, success is not guaranteed.

Before we proceed, we must first note a useful inequality[9].

Lemma 2.8: Triangle Inequality

Let x, y be any two real numbers, then

$$|x + y| \leq |x| + |y|$$

Proof. Use the following property of absolute values

$$-|x| \leq x \leq |x| \qquad -|y| \leq y \leq |y|$$

Now add them together to get

$$-(|x| + |y|) \leq x + y \leq |x| + |y|$$

But this inequality is the same thing as saying

$$|x + y| \leq |x| + |y|$$

which is what we wanted to prove. □

The reason that this is called the triangle inequality is because if you think of the absolute values as the lengths of legs of a triangle[10], then the inequality just state that the length of the last leg cannot be longer than the sum of the other two leg lengths. If you think about this geometrically, it should make sense pretty fast.

Example. Prove that

$$\lim_{x \to a} x^2 = a^2$$

Let $\epsilon > 0$, then

$$|x - a| < \delta \Longrightarrow |x^2 - a^2| = |x - a||x + a| < \delta|x + a| = \epsilon$$

[9] In mathematics, a lemma is a small result typically proven for the purpose of using it as a building block for proving more complicated results such as theorems

[10] This makes more sense if x, y are vectors, but you don't know what those are yet

This would imply that we take $\delta = \epsilon/|x+a|$ but that won't do; δ cannot depend on x, only ϵ and constants. The issue is that we have a bound for $|x-a|$ but not $|x+a|$, we'll have to cleverly work around this.

Suppose that x is "close enough" to a, that is, suppose $\delta < 1$. Then

$$|x-a| < 1 \Longrightarrow -1 < x-a < 1 \Longrightarrow -1+2a < x+a < 1+2a$$

The absolute value of $|x+a|$ is thus bounded by the absolute values of the two sides of the inequality. But the those aren't equal. To avoid making promises we can't keep, we must take the bigger of the two. Luckily we have the triangle inequality to help use out

$$|2a+1| \leq |2a| + |1| = 2|a| + 1$$

$$|2a-1| \leq |2a| + |-1| = 2|a| + 1$$

So now we have the ultimate bound which let's use confidently say that if $\delta < 1$

$$|x-a| < \delta < 1 \Longrightarrow |x+a| < 2|a| + 1$$

Going back to the problem at hand, we can use this new bound to get

$$|x-a| \leq \delta \Longrightarrow |x^2 - a^2| < \delta(2|a|+1) < \epsilon$$

The x is gone now! But before we write down the final answer, we have to remember that the only reason we got to this point is by assuming $\delta < 1$. If a or ϵ is too large, then we risk violating this assumption. For this reason our answer will be a bit funny looking

$$\delta = \min\left(1, \frac{\epsilon}{2|a|+1}\right)$$

I will leave it to you, the motivated and intelligent reader, to verify that the following limit can be proven using the δ which I shall provide you:

$$\lim_{x\to a} x^4 = a^4 \qquad \delta = \min\left(1, \frac{1}{(2|a|+1)(2a^2+2|a|+1)}\right)$$

This is the first example in which we first impose an upper bound on δ and then use that to get a refinement which depends on ϵ. This is a common pattern for harder limits, we'll see it again in the next example.

Example. Prove that

$$\lim_{x\to a} \sqrt{x} = \sqrt{a} \qquad a > 0$$

Let $\epsilon > 0$, then

$$|x-a| < \delta \Longrightarrow |\sqrt{x} - \sqrt{a}| = \frac{|x-a|}{\sqrt{x}+\sqrt{a}} < \frac{\delta}{\sqrt{x}+\sqrt{a}} = \epsilon$$

So just like before we need to bound something with an x, except this time it's on the bottom. However we now have the added complexity that we cannot take the square root of a negative number. We can no longer just take $\delta < 1$ because what if a was 0.5? Instead, we'll have a variable upper bound

$$|x-a| < \delta < \frac{a}{2} \implies \frac{a}{2} < x < \frac{3a}{a} \implies \sqrt{\frac{a}{2}} + \sqrt{a} < \sqrt{x} + \sqrt{a} < \sqrt{\frac{3a}{2}} + \sqrt{a}$$

When we divide, the lower bound becomes the new upper bound. This makes more sense when I write it out, I promise:

$$\sqrt{\frac{a}{2}} + \sqrt{a} < \sqrt{x} + \sqrt{a} \implies \frac{1}{\sqrt{x}+\sqrt{a}} < \frac{1}{\sqrt{a/2}+\sqrt{a}} < \frac{1}{\sqrt{a}}$$

Now we return to the original problem and substitute this bound

$$|\sqrt{x}-\sqrt{a}| = \frac{\delta}{\sqrt{x}+\sqrt{a}} < \frac{\delta}{\sqrt{a}} = \epsilon$$

Thus for any $\epsilon > 0$, we can choose

$$\delta = \min(1, \epsilon\sqrt{a})$$

to make the proof work.

By tweaking the definition ever so slightly, we can make it work for one sided limits.

Definition 5: One-Sided Limits, for Experts

Let f be a function defined over an open interval (a, b), then the limit as we approach a from the left is equal to L, or in symbols

$$\lim_{x\to a^+} f(x) = L$$

If for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$0 < x-a < \delta \implies |f(x)-L| < \epsilon$$

Notice that this definition forces $x > a$, we can define a right handed limit ($\lim_{x\to b^-}$) by swapping the order of the inequality

$$-\delta < x-b < 0 \implies |f(x)-L| < \epsilon$$

Similar tweaks can be made to get limits as x goes to infinity and limits that *are* infinity.

Definition 6: Infinite Limits, for Experts

The limit as x goes to a is infinity, or in symbols

$$\lim_{x \to a} f(x) = \infty$$

If for every $M > 0$ there exists a $\delta > 0$ such that

$$|x - a| < \delta \implies f(x) > M$$

Taking $M < 0$ and swapping the inequality gives the definition for negative infinity.

The limit as x goes to ∞ is L, or in symbols

$$\lim_{x \to \infty} f(x) = L$$

If for every $\epsilon > 0$, there exists an $M > 0$ such that

$$x > M \implies |f(x) - L| < \epsilon$$

Once again swapping M to negative and changing the inequality gives the limit to $-\infty$.

The rest of this section will be dedicated to using the $\epsilon - \delta$ technique to prove results that we took for granted earlier in the section. These proofs will be much harder than the examples we have done thus far as they are now "full-fledged real proofs." At this point most readers can stop reading, but the daring may venture on and/or skim through to get a glimpse into real mathematics.

Theorem 2.9: Limit Laws

I won't prove all of the limit laws, but here's a couple to satisfy your curiosity. Let

$$\lim_{x \to a} f(x) = L \qquad \lim_{x \to a} g(x) = M$$

1. Sum law
$$\lim_{x \to a} f(x) \pm g(x) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L \pm M$$

2. Product law,
$$\lim_{x \to a} f(x)g(x) = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right) = LM$$

3. Quotient law, if $M \neq 0$
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}$$

Proof. 1) Since the limits exist, for any $\epsilon > 0$ there exists $\delta_1, \delta_2 > 0$ such that

$$|x-a| < \delta_1 \implies |f(x) - L| < \frac{\epsilon}{2}$$

$$|x-a| < \delta_2 \implies |g(x) - M| < \frac{\epsilon}{2}$$

Let $\delta = \min(\delta_1, \delta_2$ so that both inequalities can be true at the same time. Then $|x-a| < \delta$ implies

$$|(f(x)+g(x)) - (L+M)| \leq |f(x)-L| + |g(x)-M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

A similar proof applies for subtraction.

2) First we investigate the absolute value

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)(g(x)-M) + M(f(x)-L)| \\ &\leq |f(x)||g(x)-M| + |M||f(x)-L| \end{aligned}$$

Two of the four resulting absolute values seem familiar, they come from the existing limits. We can choose any bound for them and there will exist δ to make it happen. But the $|f(x)|$ is a bit concerning, here is how we deal with it.

Suppose $|f(x) - L| < 1$, then

$$-1 + L \leq f(x) \leq 1 + L \implies |f(x)| \leq 1 + |L|$$

Using the fact from earlier that we can choose arbitrary bounds

$$|f(x)g(x) - LM| \leq |f(x)||g(x)-M| + |M||f(x)-L| \leq \frac{\epsilon_1}{1+|L|} + |M|\epsilon_2$$

We might be tempted to choose something like $\epsilon_2 = 1/|M|$ but we can't be certain that $|M| \neq 0$. What we do know, however, is that $|M|$ will never be negative, this will help guide our decision.

The full proof now proceeds as follows, take $\epsilon > 0$. Because the limits of f, g are known, there exists $\delta_1, \delta_2 > 0$ such that

$$|x-a| < \delta_1 \implies |f(x) - L| < \min\left(1, \frac{\epsilon}{2(1+|L|)}\right) = \epsilon_1$$

$$|x-a| < \delta_2 \implies |g(x) - M| < \frac{\epsilon}{2(1+|M|)} = \epsilon_2$$

The min comes from our assumption that $|f(x) - L| < 1$. Now take $\delta = \min(\delta_1, \delta_2)$, then for $|x-a| < \delta$ we have

$$\begin{aligned} |f(x)g(x) - LM| &< (1+|L|)\left(\frac{\epsilon}{2(1+|L|)}\right) + |M|\left(\frac{\epsilon}{2(1+|M|)}\right) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2}\left(\frac{|M|}{1+|M|}\right) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

3) Since we just proved the power law, it suffices to prove that

$$\lim_{x\to a}\frac{1}{g(x)}=\frac{1}{M}$$

Just like before, we first investigate the absolute value

$$\left|\frac{1}{g(x)}-\frac{1}{M}\right|=\frac{|g(x)-M|}{|g(x)||M|}<\frac{\epsilon_1}{|g(x)||M|}$$

We have already used the fact that the numerator is something that can be arbitrarily bounded, we just need to investigate the $|g(x)|$ in the denominator. Since it's in the denominator we have to impose the additional condition that $g(x)\neq 0$, to do so we simply assume

$$\begin{aligned}|g(x)-M|<\frac{|M|}{2}&\implies\frac{M}{2}<g(x)<\frac{3M}{2}\\&\implies\frac{|M|}{2}<|g(x)|<\frac{3|M|}{2}\implies\left|\frac{1}{g(x)}\right|<\frac{2}{|M|}\end{aligned}$$

which is possible because $M\neq 0$. Our original inequality now simplifies to

$$\left|\frac{1}{g(x)}-\frac{1}{M}\right|\leq\frac{2\epsilon_1}{|g(x)||M|}<\frac{2\epsilon_1}{|M|^2}$$

The full proof now proceeds as follows, for every $\epsilon>0$ there exists a $\delta>0$ such that

$$|x-a|<\delta\implies|g(x)-M|<\min\left(\frac{|m|}{2},\frac{\epsilon M^2}{2}\right)$$

$$\left|\frac{1}{g(x)}-\frac{1}{M}\right|<\left(\frac{2}{M^2}\right)\left(\frac{\epsilon M^2}{2}\right)=\epsilon$$

□

Theorem 2.10: Composition of Limits

Let $f(x)$ be a function such that $\lim_{x\to a}f(x)=L$ and suppose $g(x)$ is continuous at L, then

$$\lim_{x\to a}g(f(x))=g\Big(\lim_{x\to a}f(x)\Big)=g(L)$$

Proof. Continuity at L means that

$$\lim_{x\to L}g(x)=g(L)$$

Let $\epsilon>0$, then there exists a $\delta_1>0$ such that

$$|x-L|<\delta_1\implies|g(x)-g(L)|<\epsilon$$

Let $\epsilon_1=\delta_1>0$, then there exists a $\delta>0$ such that

$$|x-a|<\delta\implies|f(x)-L|<\epsilon_1=\delta_1\implies|g(f(x))-g(L)|<\epsilon$$

□

3 Derivatives

3.1 Introduction to Derivatives

There are three ideas at the core of calculus. We've already covered one so far, the limit, and you've all (mostly) survived so now we can move on to the derivative. Here's the problem: slopes of lines is easy, take any two points and use the slope formula. But if I point to some point on a quadratic and ask you to give me the slope then you're shit out of luck. Luckily with the power of god and ~~anime~~ limits on your side, we can rectify this issue (in just a few more pages too!).

To begin, what exactly does the slope at a single point mean. We can easily find the slope between two points by drawing a line between them, but for a curve there's no lines. For the case of a curve, the slope at a single point will be the slope of line which *just barely* touches the curve at that point. In other words we want the slope of the tangent line of the curve at that point. But to find the slope of a tangent line, we must first discuss secant lines.

As opposed to tangent lines, we get a secant line by taking two points on a curve and drawing a line through them. For instance if we have the function $f(x) = x^2$ and we want the secant line through $x = 1$ and $x = 3$, then the slope is

$$m = \frac{f(3) - f(1)}{3 - 1} = \frac{9 - 1}{2} = 4$$

We can make this equation much more general:

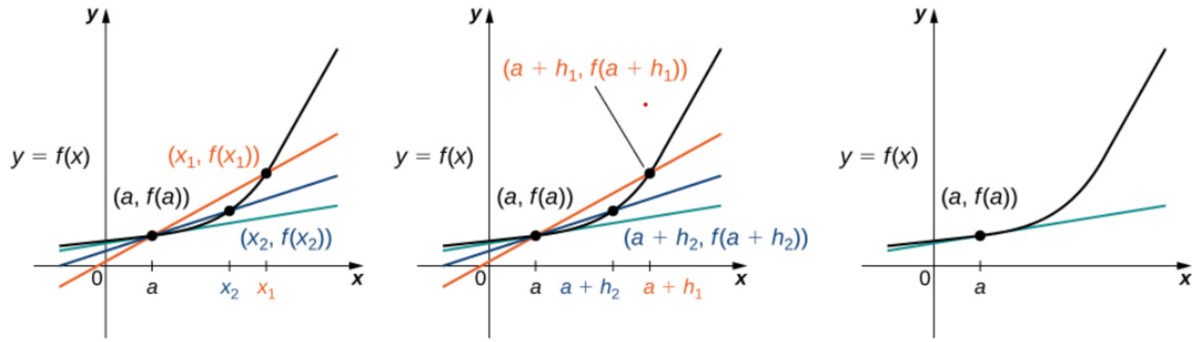
Definition 7: Difference Quotient

Let f be a function which is defined (i.e. exists) on some interval. Let x be some point inside that interval and $h \neq 0$ a number such that $x + h$ is still in the interval. Then the difference quotient is

$$Q = \frac{f(x + h) - f(x)}{h}$$

A geometrical interpretation of this equation is that it is the slope of the secant line through x and the point h away. If we want to get the slope of the tangent line, then we simply move that second point closer and closer to the first point. In other words, we take the limit as h (the horizontal distance between the two points of the secant line) goes to 0.

In the figure below, we see graphically how we can get to a tangent line by first taking a secant line and then decreasing the distance between the two points until they converge to just one. This will give a visualization for the actual definition of a derivative which represents this idea using a limit.



Definition 8: Derivatives

Let f be a function which is defined on some open interval and a some number in that interval (open interval means that a cannot be either endpoint). Then the derivative of f at a, or the slope of the tangent line at $x = a$, is given by the limit

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

Example. Let's take perhaps the simplest example (asides from constants) known to man, a linear function

$$f(x) = x$$

Then the derivative, the slope of the tangent line, at any point $x = a$ is

$$f'(a) = \lim_{h\to 0} \frac{(a+h) - a}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1$$

This makes sense because the tangent line to a line is the line itself and the slope of a line does not change.

Example. Here's a more complicated example

$$f(x) = x^2$$

Again we'll be general and consider any point $x = a$.

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{(a+h)^2 - a^2}{h} = \lim_{h\to 0} \frac{a^2 + 2ah + h^2 - a^2}{h} \\ &= \lim_{h\to 0} \frac{2ah + h^2}{h} = \lim_{h\to 0} 2a + h \\ &= 2a \end{aligned}$$

So the slope of a tangent line at some x value is two times that x value.

Example. One more example

$$f(x) = \sqrt{x}$$

This limit is slightly more involved

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{\sqrt{a+h} - \sqrt{a}}{h} \\ &= \lim_{h\to 0} \frac{a+h-a}{h(\sqrt{a+h} + \sqrt{a})} = \lim_{h\to 0} \frac{h}{h(\sqrt{a+h} + \sqrt{a})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{a+h} + \sqrt{a}} = \frac{1}{\sqrt{a} + \sqrt{a}} \\ &= \frac{1}{2\sqrt{a}} \end{aligned}$$

Sometimes you might hear a derivative referred to as an *instantaneous rate of change*, this comes from physics where the slope of something represents its rate of change. With this in mind, let's do a couple more examples but set in the real world.

Example. Suppose we chuck a ball into the air such that its height is given by

$$y(t) = -5t^2 + 10t$$

We want to know exactly how fast the ball is falling when it hits the ground, so first we solve for the time when it does hit the ground.

$$y(t) = -5t(t-2) = 0 \implies t = 0, 2$$

$t = 0$ is when we first yeet the ball, so $t = 2$ is when it comes back down. Now we can do the limit

$$\begin{aligned} v(2) = y'(2) &= \lim_{h\to 0} \frac{y(t+h) - y(t)}{h} \\ &= \lim_{h\to 0} \frac{(-5(2+h)^2 + 10(2+h)) - (-5(2)^2 + 10(2))}{h} \\ &= \lim_{h\to 0} \frac{-5(4+4h+h^2) + 20 + 10h - 0}{h} \\ &= \lim_{h\to 0} \frac{-10h - 5h^2}{h} = \lim_{h\to 0} -10 - 5h \\ &= -10 \end{aligned}$$

For that have done some physics before may notice that I set up this problem such that we throw the ball up in the air with speed 10 m/s (and rounded the acceleration due to gravity to 10 m/s^2). The fact that the ball hits the ground exactly as fast as we threw it up is a neat result and you can verify that this is the case for other initial velocities as well.

So far we've been evaluating the derivative at a specific point, but there is nothing really stopping us from defining a function which gives derivatives for an arbitrary input. This function, read "f prime," is what we will refer to as the derivative from here on out.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We will investigate exactly what the derivative means more later, but we can still gain a basic understanding right now. The derivative is a function which returns the slope of a tangent line at any point x. In a certain way, this is the "slope" at that point, how fast the function is going up or down. So for now, we can think of the derivative as a function which tells use how fast a function is increase (or decreasing if $f'(x) < 0$).

There is a crucial connection between derivative and continuity.

Theorem 3.1: Continuity and Differentiability

If a function $f(x)$ is differentiable (i.e. the derivative exists) at $x = a$, then it must be continuous at $x = a$.

This gives us a shortcut from the limits we introduced last chapter. If we can somehow compute a derivative (function) and that function is not undefined at some point, then we will know that function is also continuous there. Now the curious reader may ask "Well what about the other way around? Are continuous functions always differentiable? " The answer is no.

Once we take the derivative of a function, there is nothing stopping us from doing it again. For instance, using derivatives we found in previous examples we can take a double derivative

(and more, which I will leave for you to verify)

$$f(x) = x^2 \Longrightarrow 2x \Longrightarrow 2 \Longrightarrow 0$$

These are called "higher-order" derivatives, but generally we will refer to them by the amount of derivatives taken. Going back to the example of $f(x) = x^2$; the first derivative is $2x$, the second derivative is 2 and the third derivative is 0. This pattern continues, for any function f, the n-th derivative will be the function resulting from taking the derivative n times.

There is two (three if you're a physicist and four if you're a mathematician) ways to write a derivative. Suppose we have a function $f(x)$, then Lagrange notation[11] denotes the derivatives as

$$f'(x) \qquad f''(x) \qquad f'''(x) \qquad f^{(4)}(x) \qquad \cdots \qquad f^{(n)}(x)$$

It is customary to use the dashes for the first three derivatives, but start writing the numbers for higher derivatives. The other common notation in calculus is Liebniz's notation

$$\frac{\mathrm{d}f}{\mathrm{d}x} \qquad \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \qquad \frac{\mathrm{d}^3 f}{\mathrm{d}x^3} \qquad \frac{\mathrm{d}^4 f}{\mathrm{d}x^4} \qquad \cdots \qquad \frac{\mathrm{d}^n f}{\mathrm{d}x^n}$$

The reason for the numbers being where they are comes from the following

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right) = \frac{\mathrm{d}^3 f}{\mathrm{d}x^3}$$

Though don't write $\mathrm{d}^3 x^3$ on the bottom, to make sense of this you should think of the entire $\mathrm{d}x$ as its own block (add invisible parentheses if that makes you feel better).

Having to go through an entire limit to get a derivative is a bit tedious, especially as the functions we consider become more and more complicated. Just like how we can build arbitrary limits out of basic ones, there are rules that let us construct more advanced derivatives from basic ones.

Example. First we need to establish two basic results, one we did earlier and the other can be easily proven

$$\frac{\mathrm{d}}{\mathrm{d}x}(c) = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}x}(x) = 1$$

[11] You do not need to know what this is called.

Now we introduce perhaps the most useful derivative rule in calculus:

Theorem 3.2: The Power Rule

For any real number $n \neq 0$, the derivative of the function $f(x) = x^n$ is

$$f'(x) = nx^{n-1}$$

It is important to note that this holds for any nonzero exponent, negative numbers and fractions included.

Example. Let's rapid fire run through a bunch of examples, some of which we did using the limit definition earlier and others you will have to take my word that the limit definition gives the same result.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\right) = 2x \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\right) = 3x^2 \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x}\right) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-1}\right) = x^{-2} = \frac{1}{x^2} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{3/4}\right) = \frac{3}{4}x^{-1/4} = \frac{3}{4\sqrt[4]{x}}$$

Another simple but useful rule is the constant rule, essentially stating that you can pull out constant coefficients when computing a derivative.

$$\frac{\mathrm{d}}{\mathrm{d}x}(cf(x)) = c\frac{\mathrm{d}}{\mathrm{d}x}(f(x)) = cf'(x)$$

The final piece of the puzzle is the sum rule, using this we can start taking derivatives of basically every polynomial function.

Theorem 3.3: Sum Rule

Let $f(x)$ and $g(x)$ be two differentiable functions, then

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x) + g(x)) = \frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\mathrm{d}g}{\mathrm{d}x} = f'(x) + g'(x)$$

To see this in action, we will run through the derivative of a simple polynomial while elaborating every single step we take. In general we won't have to be this verbose, but it is helpful to do so at least once to see how these rules are used.

Example. Consider the function $f(x) = 3x^2 + 2x + 4$, the derivative is

$$\begin{aligned} f'(x) &= (3x^2 + 2x + 4)' \\ &= (3x^2)' + (2x)' + (4)' \\ &= 3(x^2)' + 2(x)' + 4(1)' \\ &= 3(2x) + 2(1) + 4(0) \\ &= 6x + 2 \end{aligned}$$

The sum rule in conjunction with the constant rule means that we can take derivative of subtracted functions as well. The last two operations to complete our collection are multiplication and division.

Theorem 3.4: Product Rule

Let $f(x)$ and $g(x)$ be differentiable functions, then

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

The becomes useful when we have factored polynomials.

Example. Take the derivative of $f(x) = (x^2 + 3)(x^2 - 4x)$. One way is to expand

$$f(x) = x^4 - 4x^3 + 3x^2 - 12x \Longrightarrow f'(x) = 4x^3 - 12x + 6x - 12$$

This is not feasible when the factors get longer, instead we can use the product rule

$$f'(x) = (x^2 + 3)'(x^2 - 4x) + (x^2 + 3)(x^2 - 4x)' = (2x)(x^2 - 4x) + (2x)(x^2 - 4x)$$

You may complain "wait, we haven't expanded the final result, this doesn't actually stop any algebra from happening." However this answer is just as good because often we are concerned with a single point. Regardless of the form it's in, we can still plug in values to get derivatives at specific points.

Theorem 3.5: Quotient Rule

Let $f(x)$ and $g(x)$ be differentiable functions, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

If this seems a bit hard to remember, the standard trick is to repeat to yourself "low d high minus high d low over low squared" which means to take the bottom function times the derivative of the top function and then subtract the top function times the derivative of the bottom function, all divided by the bottom squared. Another way is to just write the product rule on top but with a minus sign.

Example. For a simple example, consider the function

$$f(x) = \frac{x+1}{x-1}$$

One way is to use product rule, but then we would have to derive $1/(x+1)$ which we cannot do with normal rules, so instead we have to use quotient rule.

$$f'(x) = \frac{(x+1)'(x-1) - (x+1)(x-1)'}{(x-1)^2} = \frac{(x-1)-(x+1)}{(x-1)^2} = -\frac{2}{(x-1)^2}$$

Let's put everything we've learned together with a more complicated example

Example. Consider the function

$$f(x) = \frac{(2x+1)(3x^2+4x+2)}{x^2-1}$$

The derivative is (we'll skip some algebra because you are seasoned derivers now)

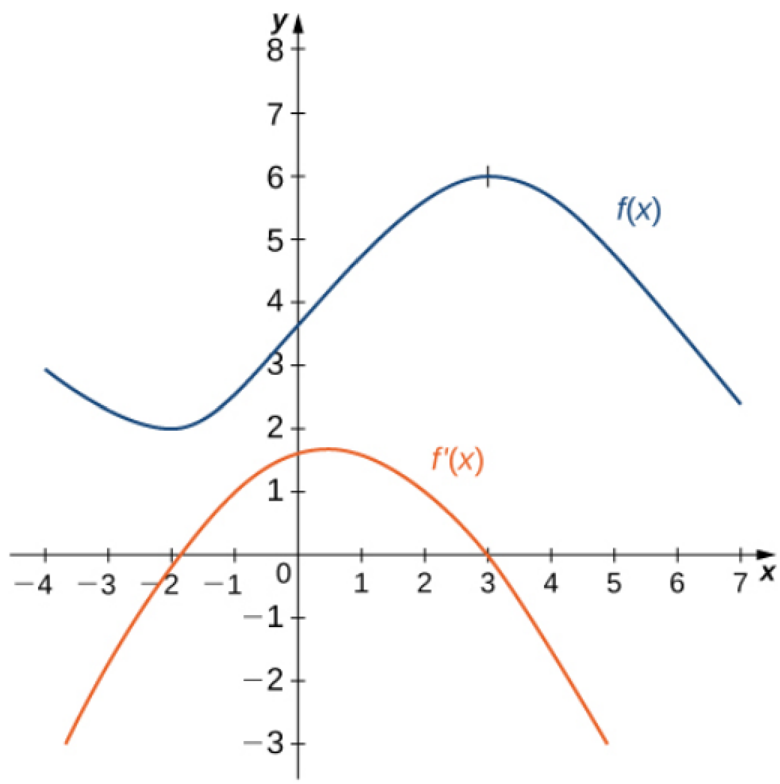
$$\begin{aligned} f'(x) &= \frac{(2(3x^2+4x+2)+(2x+1)(6x+4))(x^2-1) - (2x+1)(3x^2+4x+2)(2x)}{(x^2-1)^2} \\ &= \frac{(18x^2+22x+8)(x^2-1) - (4x^2+2x)(3x^2+4x+2)}{(x^2-1)^2} \\ &= \frac{6x^4-26x^2-26x-8}{(x^2-1)^2} \end{aligned}$$

3.2 Applications of Derivatives

Just like how we can find limits visually from a graph, we can also guess how the derivative of a function looks like by investigating its graph. There isn't a concrete procedure, but the general idea is:

- If the function is going up, the derivative is positive. Vice versa for going down
- A horizontal tangent means the derivative is zero
- If the rate at which is function is growing is also growing, then the derivative should slope upwards. Similar logic applies to the other four cases.

Example. The function $f(x)$ and $f'(x)$ are shown below



Example. For the function $f(x) = x^2 - 3x$, where is there a horizontal derivative?

To answer this question we set the derivative to zero

$$f'(x) = 2x - 3 = 0 \Longrightarrow x = \frac{3}{2}$$

In particular the point $(3/2, 9/4)$ is a horizontal tangent. Observant readers may notice that this is the vertex of the parabola. In fact, the vertex of a quadratic will always have zero derivative.

We know that the average rate of a change of a function is the quotient

$$\frac{f(a+h)-f(a)}{h}$$

For small values of h, we can pretend the function is just a line and estimate

$$f'(a) \approx \frac{f(a+h)-f(a)}{h} \Longrightarrow f(a+h) \approx f(a) + hf'(a)$$

This technique, known as linear approximation, lets us estimate the function using a known value and its derivative.

Example. Suppose we have a function $f(x)$ such that

$$f(5) = 10 \qquad f'(5) = 2$$

We can estimate the function at 6 using a linear approximation

$$f(7) \approx f(5) + 2f'(5) = 14$$

If this seems a bit abstract, here's something more concrete.

Example. We can use this approximation to estimate square roots. We know that

$$f(x) = \sqrt{x} \Longrightarrow f'(x) = \frac{1}{2\sqrt{x}}$$

We also know that $\sqrt{4} = 2$, so we can estimate

$$\begin{aligned}
\sqrt{5} &\approx 2 + \frac{1}{2\sqrt{4}} = 2.25 \text{ vs } 2.24 \\
\sqrt{6} &\approx 2 + 2 \cdot \frac{1}{2\sqrt{4}} = 2.5 \text{ vs } 2.45 \\
\sqrt{7} &\approx 2 + 3 \cdot \frac{1}{2\sqrt{4}} = 2.75 \text{ vs } 2.65 \\
\sqrt{8} &\approx 2 + 4 \cdot \frac{1}{2\sqrt{4}} = 3
\end{aligned}$$

Note that the approximate gets worse the further we are away from our reference point.

A linear approximation is an example of a first order Taylor series, we can use the higher order derivatives to refine our approximation in a (much) later chapter.

Newton invented Calculus to provide a rigorous mathematical backing for his laws of motion. We will explore a bit of that here.

Definition 9: Equations of Motion

Suppose the motion (position) of a particle over time is given by the function $x(t)$. Then the velocity of that particle is

$$v(t) = x'(t)$$

The absolute value of velocity (i.e. its magnitude) is the particle's speed. The derivative of velocity gives its acceleration

$$a(t) = v'(t) = x''(t)$$

The derivative of acceleration, the third derivative of position, is its jerk

$$j(t) = a'(t) = x''(t) = x'''(t)$$

The next few higher order derivatives of position are snap, crack, and pop. You do not need to know the derivatives of position past acceleration.

Example. Consider a particle which moves according to

$$x(t) = t^3 - t^2 + 25$$

It's velocity and acceleration at a particular time is

$$v(t) = 3t^2 - 2t \qquad a(t) = 6t - 2$$

From this we see that it is speeding up for $t > 1/3$ and slowing down otherwise. We can also solve

$$v(t) = t(3t - 2) > 0 \Longrightarrow t < 0, t > 2/3$$

So for $t > 2/3$, the particle is moving forward.

In biology, particularly ecology, derivatives play a key role in modeling population dynamics. For instance, if the population of a city triples every 5 years and there are 10,000 people in the current year. Then

$$P'(0) \approx \frac{P(5) - P(0)}{5 - 0} = \frac{30000 - 10000}{5} = 4000$$

If we want to estimate the population in 3 years, then

$$P(3) \approx P(0) + 3P'(0) = 10000 + 3 \cdot 4000 = 22000$$

Population dynamics is more accurately modeled using the exponential function and differential equations, we will return to this example later when you learn what these are.

Derivatives are also used in economics when discuss the behavior of firms or other agents. In certain instances, we can use them to give optimal answers to questions like "How much should I produce of a certain good?" or "How much should I charge for this service?"

Example. In a monopolistically competitive market, the demand curve faced by a firm is equal to population demand due to market power. If the firm wishes to sell x quantity of some good, then the price is given by

$$p(x) = 100 - x$$

In other words, the firm can charge 50 and sell 50 units or charge 99 and only sell 1. The revenue is simply price times quantity and the derivative gives what is known as the marginal revenue: the additional revenue gained from selling one more unit.

$$R(x) = xp(x) = 100x - x^2 \implies MR(x) = R'(x) = 100 - 2x$$

Revenue is maximized when $MR = 0$ (you will learn why later), which occurs at exactly 50 goods sold. However this does not actually maximize profits.

The cost to produce x units of a good will be impacted by economies of scale. Just like revenue, we can differentiate to get the marginal cost: the cost to produce one more unit

$$C(x) = 16 + x^2 \implies MC(x) = C'(x) = 2x$$

Profit in a monopolistic market is achieved when marginal cost equals marginal revenue

$$100 - 2x = 2x \implies x = 25$$

and we see that the optimal quantity in this market is 25 units.

3.3 The Chain and Inverse Rules

The only thing stopping us from taking derivatives of any function we encounter is the fact that we don't know how to differentiate compositions of functions. The rule to do so is known as the Chain Rule.

Theorem 3.6: Chain Rule

Let $h(x) = (f \circ g)(x) = f(g(x))$, then the derivative is

$$h'(x) = f'(g(x))g'(x)$$

In Leibniz notation, this is

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{\mathrm{d}h}{\mathrm{d}g} \cdot \frac{\mathrm{d}g}{\mathrm{d}x}$$

Example. Consider the function

$$h(x) = (x^2 + 1)^2$$

This is a simple application of chain rule, the two functions are

$$f(x) = x^2 \qquad g(x) = x^2 + 1$$

and so the derivative is

$$h'(x) = 2(x^2 + 1)(2x) = 4x(x^2 + 1)$$

Example. Consider the function

$$h(x) = \sin\left(x^3\right)$$

It's pretty obvious what the outer and inner functions are

$$h'(x) = \cos\left(x^3\right)(3x^2) = 3x^2 \cos\left(x^3\right)$$

Now suppose we want the tangent line at $x_0 = \sqrt[3]{\pi}$. First note

$$h(x_0) = \sin(\pi) = 0 \qquad h'(x_0) = 3\sqrt[3]{\pi^2}\cos(\pi) = -3\sqrt[3]{\pi^2}$$

Using point-slope form, the tangent line is

$$y - 0 = -3\sqrt[3]{\pi^2}(x - \sqrt[3]{\pi})$$

Example. Consider the function

$$h(x) = \frac{1}{x^3 + 5}$$

For fractions, it's easy to miss that this is actually a composition of two functions

$$f(x) = \frac{1}{x} = x^{-1} \qquad g(x) = x^3 + 5$$

and so the derivative is

$$h'(x) = -(x^3+5)^{-2}(3x^2) = -\frac{3x^2}{(x^3+5)^2}$$

Example. Consider the function

$$h(x) = \cos\big(\cos\big(x^3\big)\big)$$

This is an example of a nested chain rule, first we decompose

$$f(x) = \cos x \qquad g(x) = \cos\big(x^3\big)$$

But now we have an issue, $g(x)$ is also a composition of functions

$$g(x) = j(k(x)) \qquad j(x) = \cos x \qquad k(x) = x^3$$

Starting from the top, recall that the chain rule state

$$h'(x) = f'(g(x))g'(x)$$

we can easily compute $f'(x)$, but we will need to use chain rule again to compute $g'(x)$

$$g'(x) = j'(k(x))k'(x) = -\sin\big(x^3\big)(3x^2) = -3x^2\sin\big(x^3\big)$$

Putting the pieces together, we get

$$h'(x) = -\sin\big(\cos\big(x^3\big)\big)(-3x^2\sin\big(x^3\big)) = 3x^2\sin\big(x^3\big)\sin\big(\cos\big(x^3\big)\big)$$

A neat application of the chain rule is that it allows us to take derivatives of inverse functions. For a given function $f(x)$, the inverse, denoted $f^{-1}(x)$ is a function such that

$$f^{-1}(f(x)) = f(f^{-1}(x)) = x \qquad \forall x \in \mathbb{R}$$

To clean up the notation, suppose $g(x) = f^{-1}(x)$, then we can use the chain rule to take derivatives of both sides

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}x &= \frac{\mathrm{d}}{\mathrm{d}x}f(g(x)) \\ 1 &= f'(g(x))g'(x) \\ \therefore g'(x) &= \frac{1}{f'(g(x))} \end{aligned}$$

If we recall that g is simply the inverse of f, we get the following result:

Theorem 3.7: Inverse Rule

Let $f(x)$ be a function which is invertible and differentiable for all x, then

$$\frac{\mathrm{d}}{\mathrm{d}x} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

for all x such that $f'(f^{-1}(x)) \neq 0$.

This is a powerful theorem because it lets us find the derivative of inverse functions without having to calculate them.

Example. Take a simple example, suppose we want the derivative of the inverse of the following function

$$f(x) = \sqrt{x}$$

We know how to take the derivative of this using power rule and the inverse is just x^2, so

$$\frac{\mathrm{d}}{\mathrm{d}x} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} = \left(\frac{1}{2\sqrt{x^2}}\right)^{-1} = 2x$$

This is the same result as if we were to differentiate x^2 directly.

For an example where you can't simply just solve for the inverse, take the trigonometric functions.

Example. Suppose we want to differentiate the inverse sin function

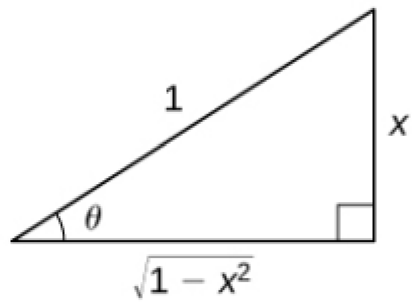
$$f(x) = \arcsin(x)$$

We can't do much with this without using the inverse derivative rule, which let's us simply write

$$f'(x) = \frac{1}{\cos(\arcsin(x))}$$

But where do we go from here? We have to remember that trig functions represent ratios between side lengths of a right triangle. For instance $\sin(x)$ is the ratio between opposite leg and hypotenuse of a right triangle with one angle measuring x radians.

When we say that an angle $\theta = \arcsin(x)$, we are saying that for a right triangle with one angle θ radians, the ratio of the opposite leg and hypotenuse is x. We can simplify this by assuming the hypotenuse is of length 1, for instance if we are in the unit circle.



From this picture, we can see clearly that

$$\cos(\theta) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

and thus the derivative in question is

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$

There are two things to notice about this approach. First is our choice of hypotenuse, the fact that we chose it to be 1. This is easy to reconcile, if we chose any other value for the hypotenuse (for instance 2 or 0.5), then the opposite leg would scale accordingly ($2x$ and $0.5x$ respectively). Thus all the possible choices yield similar triangles, which means that $\cos(\theta)$ will be constant across all of them.

The second, more interesting, choice was that the legs have positive length. For instance we could have drawn the triangle in the third quadrant and gotten a negative answer. This comes from how we define the inverse trig functions (WARNING: higher level math ahead):

An important fact to note about sine (and other trig functions) is that they are period and an unfortunate consequence of this is that they are neither injective (one-to-one) nor surjective (onto). Failing to be injective means that for a given y value, we cannot associate just one x value (*the* inverse) because that are infinitely many such x's. Failing to be surjective means that they are y values which do not have an inverse altogether (for instance $\arcsin(2)$ is undefined because $-1 \leq \sin(x) \leq 1$).

In mathematics, we say that the trig functions do a form a bijection from the real numbers to the real numbers. Unfortunately for us, only bijective functions have well defined inverses, so what can we do? Well we can restrict the domain of the inverse to force the function to be bijective. In the case of $\sin(x)$, we take a cut of the function from $-\pi/2$ to $\pi/2$, giving us one full "branch" which we can invert.

This forces θ to lie in quadrants 1 or 4, meaning that $\cos(\theta)$ will always be positive. This solves the main issue at hand, the other (less important in this context) issue is rectified by only defining $\arcsin(x)$ for $-1 \leq x \leq 1$.

Example. The long argument from above is more immediately important for the next example, suppose we have

$$f(x) = \operatorname{arcsec}(x)$$

Going through the same motions as before, we get

$$f'(x) = \frac{\cos^2(\operatorname{arcsec}(x))}{\sin(\operatorname{arcsec}(x))} = \frac{x}{x^2\sqrt{x^2-1}}$$

We may be tempted to cancel out the x's right away, but that is actually incorrect. We have to do the same thing, taking a cut of $\sec(x)$ between 0 and π to get one complete "branch" to invert. For these values of x, we will get that $\sin(x) \geq 0$ which forces $f'(x) \geq 0$. Thus when we cancel, we must insert an absolute value to get the correct derivative:

$$f'(x) = \frac{1}{|x|\sqrt{x^2-1}}$$

The remaining inverse trig derivatives are left as an exercise to the reader, but I will list them here for convenience.

Proposition 3.8: Derivatives of Inverse Trig Functions

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\arcsin(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \\
\frac{\mathrm{d}}{\mathrm{d}x}\arccos(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}} \\
\frac{\mathrm{d}}{\mathrm{d}x}\arctan(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}(x) = \frac{1}{1+x^2} \\
\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arccsc}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\csc^{-1}(x) = -\frac{1}{|x|\sqrt{x^2-1}} \\
\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arcsec}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}} \\
\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arccot}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\cot^{-1}(x) = -\frac{1}{1+x^2}
\end{aligned}$$

3.4 Derivatives of Special Functions

In this section we give the derivatives of functions not covered by the power rule, for instance trigonometric functions and exponential functions. We will start with the trig functions.

We compute the derivative of sine directly from the limit definition using sum to product identities

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{2\sin(h/2)\cos(x+h/2)}{h} \\
&= \lim_{h\to 0} \frac{\sin(h/2)}{h/2} \cdot \lim_{h\to 0} \cos(x+h/2) \\
&= 1 \cdot \lim_{h\to 0} \cos(x+h/2) \\
&= \cos(x) \\
\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h} \\
&= \lim_{h\to 0} \frac{-2\sin(h/2)\sin(x+h/2)}{h} \\
&= \lim_{h\to 0} \frac{\sin(h/2)}{h/2} \cdot \lim_{h\to 0} -\sin(x+h/2) \\
&= 1 \cdot \lim_{h\to 0} -\sin(x+h/2) \\
&= -\sin(x)
\end{aligned}$$

The derivative of tangent comes from quotient rule

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} = \sec^2 x
\end{aligned}$$

Now that we can take derivatives of trig functions, we have a more useful use case of the product rule.

Example. Consider the function

$$f(x) = x^2 \sin x$$

This product cannot simply be expanded, we are forced to use product rule

$$f'(x) = 2x\sin x + x^2 \cos x$$

Using quotient rule, we can take derivatives of the remaining trig functions. I'll list them here for your convenience:

Proposition 3.9: Trig Derivatives

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) &= \cos x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) &= -\sin x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) &= \frac{1}{\cos^2 x} = \sec^2 x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\csc x) &= -\frac{\cos x}{\sin^2 x} = -\csc x \cot x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) &= \frac{\sin x}{\cos^2 x} = \sec x \tan x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\cot x) &= -\frac{1}{\sin^2 x} = -\csc^2 x
\end{aligned}
$$

Note that because sine and cosine are related by derivatives, we have the chain

$$\sin x \Longrightarrow \cos x \Longrightarrow -\sin x \Longrightarrow -\cos x \Longrightarrow \sin x \Longrightarrow \cdots$$

This is our first example of a function which is both infinitely differentiable and has nontrivial higher order derivatives. Compare this to polynomials, which always end up going to 0 if you take enough derivatives.

Example. Suppose we want to know the equation of the tangent line of the function

$$f(x) = \sin x \cos x$$

at the point $x = \pi$. First note that the derivative is

$$f'(x) = (\cos x)(\cos x) + (\sin x)(-\sin x) = \cos^2 x - \sin^2 x = \cos(2x)$$

Recall that the derivative gives us the slope of the tangent line, it is $f'(\pi) = 1$. Now all we need is a point. The tangent line touches the function at only one point, that point must be the value of the function at $x = \pi$, which is $f(\pi) = 0$. The easiest way to find the equation is to use point slope form and convert if necessary

$$y - 0 = 1 \cdot (x - \pi) \Longrightarrow y = x - \pi$$

Example. A particle undergoing simple harmonic motion (for instance a mass on an ideal spring) has equation

$$x(t) = 5\cos(t)$$

We can use this to find the velocity at any time t

$$v(t) = x'(t) = -5\sin(t)$$

The velocity function is also periodic, by looking more closely we see that

$$v(t) = 0 \iff t = 0, \pi, 2\pi, 3\pi, \cdots$$

This is also when the spring is furthest away from the center (at ± 5). The mass actually has zero speed when it reaches it's furthest point and this happens infinitely many times.

Finally, we have the exponential and logarithmic functions. The exponential is very easy to remember

Theorem 3.10: Exponential Derivative

$$\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$$

It's slightly subtle, but this actually let's us take derivative of all exponential functions, not just the natural one. We do this by converting into the natural exponential and then using chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}x} a^x = \frac{\mathrm{d}}{\mathrm{d}x} e^{x\ln a} = e^{x\ln a}\ln a = a^x \ln a$$

This holds for any constant a, for instance the derivative of 2^x is $2^x \ln 2$.

Example. Consider the function

$$f(x) = xe^{2x}$$

The first derivative is

$$f'(x) = e^{2x} + 2xe^{2x}$$

The second derivative is

$$f''(x) = 2e^{2x} + 2e^{2x} + 2x(2e^{2x}) = 4e^{2x} + 4xe^{2x}$$

The logarithmic function is a bit more annoying deal with.

Theorem 3.11: Logarithmic Derivative

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{x}$$

Just like the exponential, we can extend this to other logarithms by using the change of base formula.

$$\frac{\mathrm{d}}{\mathrm{d}x} \log_a x = \frac{\mathrm{d}}{\mathrm{d}x} \frac{\ln x}{\ln a} = \frac{1}{x\ln a}$$

Example. Consider the function

$$f(x) = ln(x^3 + 2x + 4)$$

The derivative is

$$f'(x) = \frac{3x^2 + x}{x^3 + 2x + 4}$$

Logarithms have some nice properties that sometimes makes the derivative easier.

Example. Consider the function

$$f(x) = \ln\left(\frac{x}{x^2 \sin x}\right)$$

We could try to use chain rule directly, but then we would have to deal with the nasty quotient inside. Instead we can first simplify

$$f(x) = \ln x - \ln x^2 - \ln \sin x$$

and now we can differentiate

$$f'(x) = \frac{1}{x} - \frac{2x}{x^2} - \frac{\cos x}{\sin x}$$

3.5 Implicit Differentiation

Implicit differentiation is a bit of a strange technique. It's similar to taking derivatives of inverses, but not really. The problem is that some functions cannot be written in the form $y = \cdots$, for instance the equation of the unit circle is $x^2 + y^2 = 1$. To be able to find tangent lines, we use implicit differentiation.

The main trick behind implicit differentiation is the fact that the derivative of y is y'. That's it. Let's see how this trick gets used in an example.

Example. Returning to the circle example, consider a circle centered on the origin with radius 2

$$x^2 + y^2 = 4$$

Suppose we want to know the derivative at the point $(0, 2)$. We can take the derivative of both sides, clearly the right side is just 0. The derivative of x^2 is just $2x$ using power rule, but the y^2 is tricky.

We have the recognize that y is not just a variable, it is a function. So when we take derivative we must use the appropriate rule(s), in this case chain rule.

$$\frac{\mathrm{d}}{\mathrm{d}x} y^2 = 2y \frac{\mathrm{d}}{\mathrm{d}x} y = 2yy'$$

Going back to the original function, we have

$$2x + 2yy' = 0 \implies y' = -\frac{x}{y}$$

Note that the derivative is no longer just a function of x, it is a function of both. If you want to know the derivative at some point, you need both the x and y values of that point. So at the point $(0, 2)$, the derivative is

$$y' = -\frac{0}{2} = 0$$

If we wanted the tangent line at the point, we have

$$y - 2 = 0(x - 0) \implies y = 2$$

which is just a horizontal line on top of the circle.

Example. For an example with a few more rules involved, consider

$$x \sin(y) + x^3 = y$$

The first term will require both product and chain rule to properly differentiate, let's examine that term by itself

$$\frac{\mathrm{d}}{\mathrm{d}x} x \sin(y) = \sin(y) + x \frac{\mathrm{d}}{\mathrm{d}x} \sin(y) = \sin(y) + x \cos(y) \frac{\mathrm{d}}{\mathrm{d}x} y = \sin(y) + xy' \cos(y)$$

The remaining terms are easy and we get

$$\sin(y) + xy' \cos(y) + 3x^2 = y' \Longrightarrow y' = \frac{\sin(y) + 3x^2}{1 - x\cos(y)}$$

Example. Here's an example with a triple product rule

$$xy\cos(y) = x$$

The right hand side obviously becomes 1, the left hand side goes like this

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} xy\cos(y) &= y\cos(y) + x\frac{\mathrm{d}}{\mathrm{d}x} y\cos(y) \\
&= y\cos(y) + x\left(y'\cos(y) + y\frac{\mathrm{d}}{\mathrm{d}x}\cos(y)\right) \\
&= y\cos(y) + x(y'\cos(y) - yy'\sin(y)) \\
&= y'(x\cos(y) - y\sin(y)) + y\cos(y)
\end{aligned}$$

and the final derivative becomes

$$y' = \frac{1 - y\cos(y)}{x\cos(y) - y\sin(y)}$$

We can also take higher order derivatives by simply differentiating implicitly again.

Example. Consider the function

$$y = x\sin(y)$$

Implicitly differentiating once gives

$$y' = \sin(y) + xy'\cos(y) \Longrightarrow y' = \frac{\sin(y)}{1 - x\cos(y)}$$

We could solve for y' and differentiate again, but that would introduce a nasty quotient and we would need to combine quotient, product, and chain rule to get the second derivative. Instead we'll leave it as is and use implicit differentiation again:

$$y'' = y'\sin(y) + y'\cos(y) + xy''\cos(y) + x(y')^2\cos(y)$$

and the second derivative is

$$y'' = \frac{y'\sin(y) + y'\cos(y) + x(y')^2\cos(y)}{1 - x\cos(y)}$$

You may object that I left my y's out, but this was is actually cleaner and more understandable. If we want to compute the second derivative at a value, say $(3, 0)$. We first compute the first derivative

$$y' = \frac{\sin(0)}{1 - 3\cos(0)} = 0$$

Then use that to compute the second derivative

$$y'' = \frac{0}{1 - 3\cos(0)} = 0$$

If you *really* hate having y's out in the open, you may substitute the expression we found earlier for y', but be warned that the result usually isn't pretty.

Implicit differentiation let's us leverage logarithms to simplify certain derivatives. By taking the natural logarithm of both sides before differentiating, we can sometimes simplify the problem.

Example. Consider the function

$$y = \frac{x\sqrt{\sin x}}{e^x \tan^3 x}$$

First we take the logarithm

$$\begin{aligned} \ln y &= \ln x + \ln \sqrt{\sin x} - \ln e^x - \ln \tan^3 x \\ &= \ln x + \frac{1}{2} \ln \sin x - x - 3 \ln \tan x \end{aligned}$$

and now we can differentiate

$$\frac{y'}{y} = \frac{1}{x} + \frac{\cos x}{2 \sin x} - 1 - \frac{3 \sec^2 x}{\tan x}$$

If we wish to solve for y', we use the original expression

$$y' = \left(\frac{1}{x} + \frac{\cos x}{2\sin x} - 1 - \frac{3}{\sin x \cos x} \right) \frac{x\sqrt{\sin x}}{e^x \tan^3 x} = \left(\frac{1}{x} + \frac{\cos x}{2\sin x} - 1 - \frac{6}{\sin 2x} \right) \frac{x\sqrt{\sin x}}{e^x \tan^3 x}$$