DRP Week 1

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Authors: Harry Huang

- MM: Maximum Matching
- MIS: Maximum Independent Set
- MCV: Minimum Vertex Cover
- n: $|V|$
- m: # edges of G

Problem 1. Prove $|MIS| + |MVC| = |V|$ for all graphs.

Proof. Consider a MCV C and its complement $I = V - C$ of graph G. For every point $x \in I$, since C is a vertex cover, every neighbor of x must be in C (assume neighbor y is in I, then edge (x, y) is not covered by C). Therefore, x is independent in I as $I \cap C = \emptyset$, giving I is an independent set. Similarly, we can show that for any independent set $I \subset V$, its complement $C = V - I$ is a vertex cover. Since $|I| = |V| - |C|$, we have that the minimum C gives a maximum I, therefore for $C = MVC$, $I = V - C = MIS$, $|MVC| + |MIS| = |V|$. □

Problem 2. Prove $|MM| \leq |MVC|$ for all graphs, and $|MM| = |MVC|$ for bipartite graphs.

Proof. **(a)** $|MM| \leq |MVC|$ Consider maximum matching MM and arbitrary vertex cover (not only the minimum) C. For any edge $e_i = (u_i, v_i) \in MM$, we have that at least one of (u_i, v_i) should be contained in C (if not, then e_i is not covered by C). Since every u_i and v_i are unique, we have $|MM| \leq |C|$ as each edge in MM corresponds to at least one point in C, giving the size of C is greater than or equal to MM. Therefore, $|MM| \leq |C|$, gives $|MM| \leq |MVC|$.

(a) $|MM| = |MVC|$ for any bipartite $G = (V, E)$. I followed the proof of Min-Max Theorem on Page 112.

Let G be a bipartite graph with parts X and Y. A vertex cover Q must contain at least one endpoint of every edge in any matching, so $|Q| \geq |M|$ for every matching M.

To show equality, let Q be a minimum vertex cover and partition it as $R = Q \cap X$ and $T = Q \cap Y$. Since $R \cup T$ covers all edges, there are no edges between $Y - T$ and $X - R$.

Now consider the subgraph H from by $R \cup (Y - T)$. Using Hall's Theorem proved below, H has a matching that saturates R. Similarly, a matching in another subgraph H' saturates T. Since these matchings are disjoint, they form a matching of size $|Q|$ in G.

Therefore, G has a matching of size $|Q|$, giving the maximum matching size equals the minimum vertex cover size. □

Problem 3. Prove Hall's Theorem.

Proof. We prove both directions of the theorem.
(Necessity: If a perfect matching exists, then $|N(S)| \geq |S|$)
Suppose there exists a perfect matching M. For any subset $S \subseteq X$, each vertex in S is matched to a distinct vertex in Y, meaning that $N(S)$ contains at least $|S|$ vertices. Thus, $|N(S)| \geq |S|$, proving necessity.
(Sufficiency: If $|N(S)| \geq |S|$ for all $S \subseteq X$, then a perfect matching exists)
(I made a proof from the Proof of Hall's Theorem on the book, Page 110-111)
We give a proof by contrapositive. Assume we have a maximum matching M, and M doesn't saturate X, then I will show there is a $S \subseteq X$ such that $|N(S)| < |S|$.
Let $u \in X$ that u is not saturated by M. We move from u along alternating path, and mark all the visited points in X as S, and all the visited points in Y as T. It's clear that T matches $S - \{u\}$ by M.
Therefore, from the matching of T and $S - \{u\}$, we have $T = N(S)$, giving $|N(S)| = |T| = |S| - 1 < |S|$. This finishes the proof of contrapositive.

□

Problem 4. Given a maximum matching of a graph G, find a minimum vertex cover of G

Proof. Let's build an algorithm. Suppose M_P is the set of points saturated by the give maximum matching, we will build set C such that C is a minimum vertex cover of G.

1. Step 1: Go over every unmatched points $u \in V - M_P$;
2. Step 2: For every u, we move along alternating paths, and marked every visited points as "reachable";
3. Step 3: After starting with every $u \in V - M_P$, we mark all unvisited points as "unreachable";
4. Step 4: Let P be the set of reachable points, Q be the set of unreachable points (thus P and Q are a dividing of point set V), then we construct C as:

$$C = (M_P \cap Q) \cup ((V - M_P) \cap P)$$

□