DRP Week 2

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Authors: Harry Huang

- MM: Maximum Matching
- MIS: Maximum Independent Set
- MCV: Minimum Vertex Cover
- n: $|V|$
- m: # edges of G

Problem 1. Algorithm: Finding a Maximum Matching from a Maximal Matching

Given a graph $G = (V, E)$ and a maximal matching M, we use augmenting paths to evolve M until it becomes a maximum matching.

Steps

1. Initialize M as a given maximal matching;
2. Find an M-augmenting path P;
3. Augment M along P:
    - Flip the matched/unmatched status of all edges along P.
    - That is, edges in M are removed, and edges not in M are added.
4. Repeat the process until no M-augmenting path remains.
5. Return M as the maximum matching.

Pseudocode

```
Algorithm MaximumMatching(G):
    Input: Graph G = (V, E) with a maximal matching M
    Output: Maximum matching M'

    while there is an M-augmenting path P in G:
        Augment M along P
        Update M

    return M
```

Problem 2. Proof of Termination

Each iteration finds an augmenting path and strictly increases the size of M by exactly one new matched edge. Since the maximum matching has at most $\lfloor n/2 \rfloor$ edges (for a graph with n vertices), there can be at most $O(n)$ augmenting steps. Once no more augmenting paths exist, M is a maximum matching, ensuring the termination.

Problem 3. Asymptotic Complexity Analysis

- Finding an M-augmenting path can be done using BFS or DFS, taking $O(m)$ time.
- Since there are at most $O(n)$ iterations (each adding one more matching edge),
- The total complexity is:

$$O(n) \times O(m) = O(mn)$$

  where n is the number of vertices and m is the number of edges.

Problem 4. Relation to the Assignment Problem

Assignment Problem is a special case of maximum matching in bipartite graphs. To solve this problem, we can use the Hungarian algorithm (for the Assignment Problem) follows a similar structure but includes cost minimization using augmenting paths.
The time complexity of the Hungarian algorithm is $O(n^3)$, which is similar but more structured in augmenting steps compared to the general maximum matching algorithm.
Therefore, the augmenting path method for maximum matching is a foundation for Assignment Problem in bipartite graphs.