

# HPC-LEAP Workshop: Shallow Water Equation for Tsunami Simulation Local Time Stepping

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# Outline

- 1 Introduction
- 2 Numerical Scheme for Shallow Water Equations
  - 1D SWE
  - 2D SWE
  - Godunov & Roe solver
- 3 Local Time Stepping Scheme
  - Hierarchical scheme
  - Local CFL Condition
- 4 Result & Animation
  - Shallow Water Equations
  - Accuracy
  - Benchmark
  - MPI Speedup
- 5 References

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# 2004 Indian Earthquake Tsunami



- 230,000 deaths
- 14 countries affected
- Waves of up to 30 m (100 ft) high
- US \$14 billion donated by worldwide community



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# 1D shallow water equations

## 1D shallow water equations

- $h(x, t)$  is the fluid depth,  $u(x, t)$  is the fluid velocity,  $g$  is the gravity constant,  $B(x)$  is relative to sea level.

$$\begin{aligned}h_t + (hu)_x &= 0, \\(hu)_t + (hu^2 + \frac{1}{2}gh^2)_x + ghB_x &= 0\end{aligned}$$

## 2D shallow water equations

### 2D shallow water equations

- $h(x, y, t)$  is the fluid depth,  $u(x, t)$  and  $v(y, t)$  are fluid velocity in two different directions,  $g$  is the gravity constant,  $B(x, y)$  is relative to sea level.

$$\begin{aligned}h_t + (hu)_x + (hv)_y &= 0, \\(hu)_t + (hu^2 + \frac{1}{2}gh^2)_x + (huv)_y + ghB_x &= 0, \\(hv)_t + (huv)_x + (hv^2 + \frac{1}{2}gh^2)_y + ghB_y &= 0,\end{aligned}$$

# Godunov & Roe solver

## Godunov's Method

- Recall  $Q_i$  represents the average of  $q(x, t_n)$  over cell  $C_i$

$$Q_i^n \approx \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} q(x, t_n) dx$$

- Godunov's method can be represent as follow, detail in [2,p.311]

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} (F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n) \text{ with } F_{i+\frac{1}{2}}^n = \mathbf{F}(Q_{i+1}, Q_i)$$



# Godunov & Roe solver

## Roe Approximation

- Roe's Riemann solver approximation is linearize the nonlinear problem  $q_t + f(q)_x = 0$  to

$$\hat{q}_t + A_{i-1/2} \hat{q}_x = 0$$

The matrix  $A_{i-1/2}$  is chosen to be some approximation to  $f'(q)_x$  valid in a neighborhood of  $Q_{i-1}$  and  $Q_i$

- Roe Approximation:

$$\hat{q} = \begin{bmatrix} \hat{h} \\ \hat{h}\hat{u} \end{bmatrix}$$

where

$$\hat{h} = \frac{h_l + h_r}{2}, \text{ and } \hat{u} = \frac{u_l \sqrt{h_l} + u_r \sqrt{h_r}}{\sqrt{h_l} + \sqrt{h_r}}$$

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# Hierarchical LTS Scheme

# Hierarchical LTS Scheme

## Continue: Godunov & Roe solver

- The Godunov & Roe solver previously introduced is worked on the GTS where  $\Delta t$  is constant at each loop.

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x_i} (F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n)$$

- $\Delta t$  will be limited by *CFL condition*:  $\Delta t \leq C_{max} \min(\frac{\Delta x_i}{U_i})$

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## Solution: LTS

- Use  $\Delta t_i$  locally, to avoid the unnecessary calculation.
- **Question:**  
How to synchronize with different time step per cell?

# Hierarchical LTS Scheme

## General equation of LTS

- Most part remain the same with GLS scheme instead of  $\Delta t_i$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t_i}{\Delta x_i} (F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n)$$

Where  $\Delta t_i$  is equal to  $2^{l-1} \Delta t_{min}$ .  $\Delta t_{min}$  is the minum time step calculated by GTS.  $l$  is the interger of pre-set level for each cell.

# Local CFL Condition

## Local Courant Number

- A preliminary cell-based LTS level  $l_c$  is assigned to cell  $i$  according to the local Courant number  $Cr_i$  computed as

$$Cr_i = \frac{\Delta t_{min}}{\Delta x_i} \max_{k=1,2}(u_k)$$

Where  $u_k$  is the cell wave speed on one of 2 directions.

# Local CFL Condition

## Local Time Step Level Criterion

- After calculated the local Courant number the local time step level is determined by

$$Cr_0/2^{l-1} \leq Cr_i < Cr_0/2^{l-2}$$

with the exception of level 1 which is controlled by  $Cr_i > Cr_i/2$ .  
 $Cr_0$  is the maximum courant number for the simulation.



# Hierarchical LTS Scheme: illustration

## LTS scheme

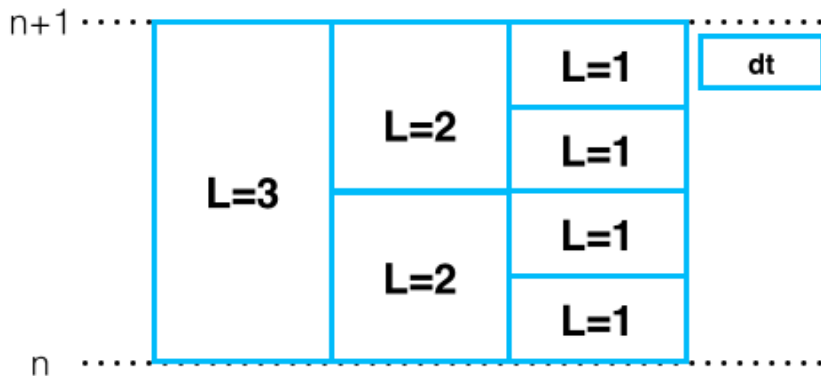


Figure: LTS scheme: first look

# Hierarchical LTS Scheme: illustration

## GTS scheme

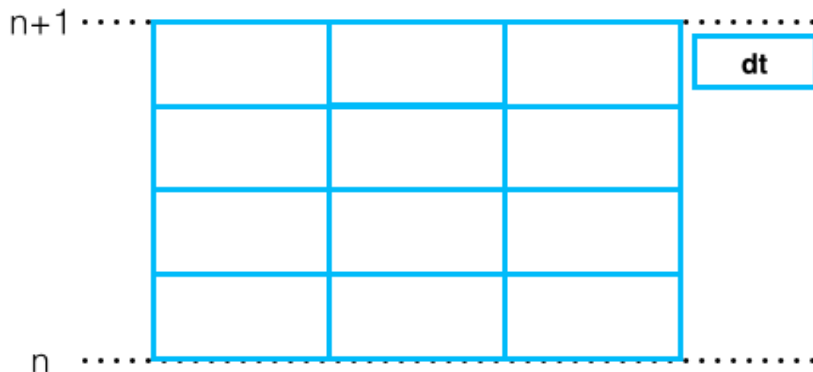


Figure: GTS scheme: computational over head

# Hierarchical LTS Scheme: illustration

## LTS scheme

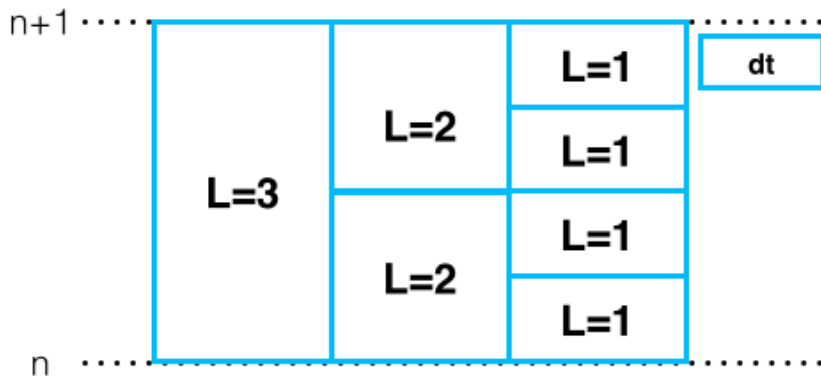


Figure: LTS scheme: benefit

# Hierarchical LTS Scheme: illustration

## LTS scheme

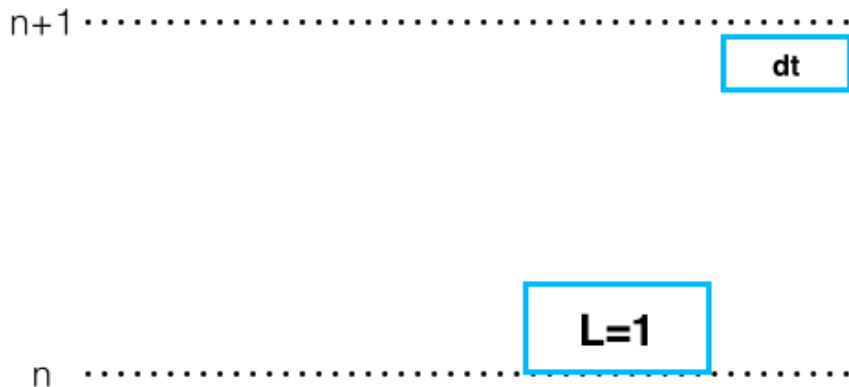


Figure: Local cell update: initial step

# Hierarchical LTS Scheme: illustration

## LTS scheme

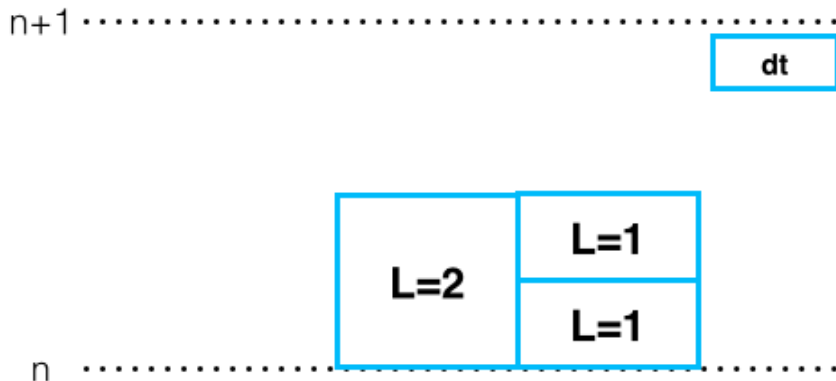


Figure: Local cell update: step 2

# Hierarchical LTS Scheme: illustration

## LTS scheme

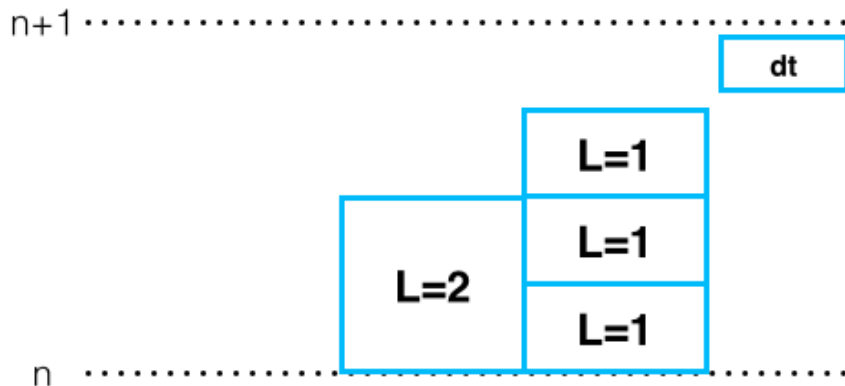


Figure: Local cell update: step 3

# Hierarchical LTS Scheme: illustration

## LTS scheme

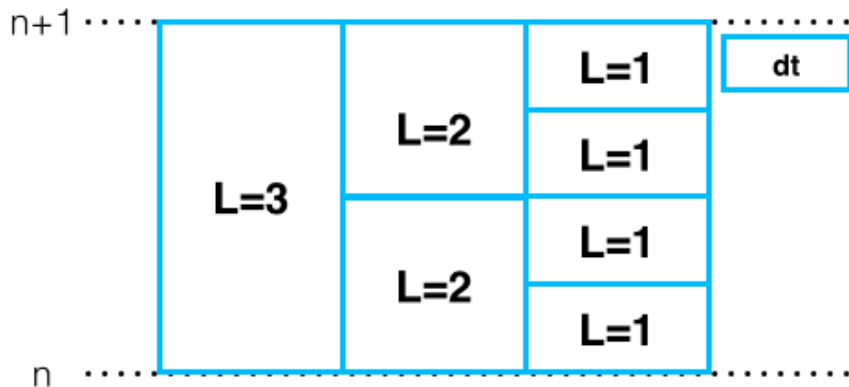


Figure: Local cell update: step 4

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# 1D SWE & LTS vs. GTS

- The experienment is based on breaking dam scheme in 1D shallow water equations(GTS scheme - timestep 200)

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# 1D SWE & LTS vs. GTS

- The experiment is based on breaking dam scheme in 1D shallow water equations(LTS scheme - timestep 200)

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## 2D SWE & LTS

- The experiment is based on breaking dam scheme in 2D shallow water equations

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## 2D SWE & LTS

- The experienment is based on breaking dam scheme in 2D shallow water equations

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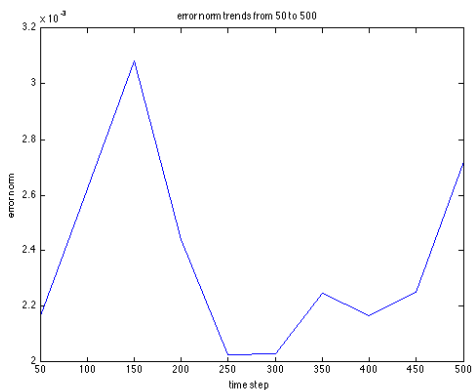
# 2D SWE LTS Error Measurement

## Experienmental Environment

- Domain size: 50 by 50
- Error measurement is using the norm formula in compare with GTS

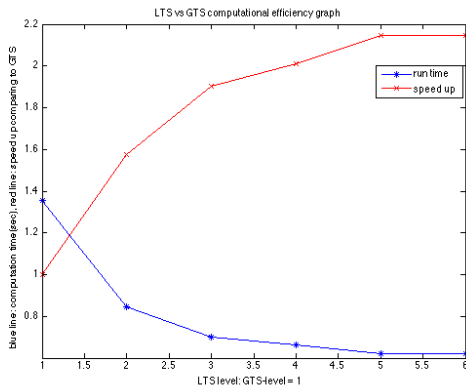
$$E_2(d_{LTS}) = (\sum_{i=1,\dots,N} [(d_{LTS})_i - d_{GTS})_i]^2)^{1/2}$$

# 2D SWE LTS Error Measurement



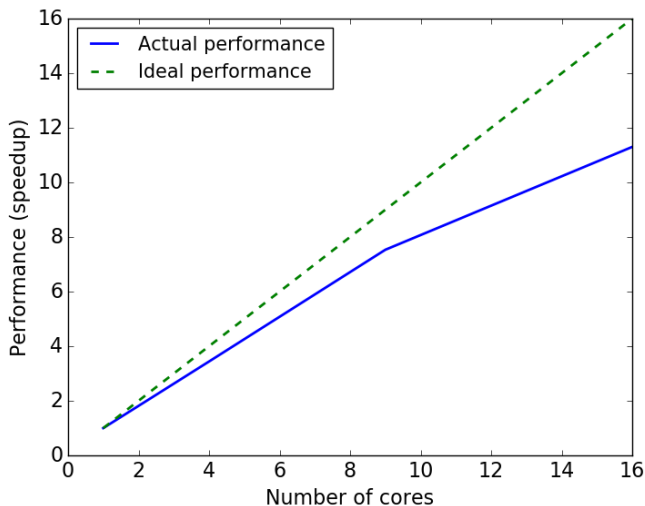
**Figure:** Norm difference of GTS & LTS with different different time step( $L = 3$ )

# Benchmark



**Figure:** graph showed runtime difference of GTS & LTS with different LTS levels(1000 time steps)

# MPI parallelization





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# Thanks for your attention! Questions?