



Fundamentals of Genomic Prediction and Data-Driven Crop Breeding (August 4-8, 2025)

Understanding Prediction Models: Ridge Regression to Bayesian

Module 3
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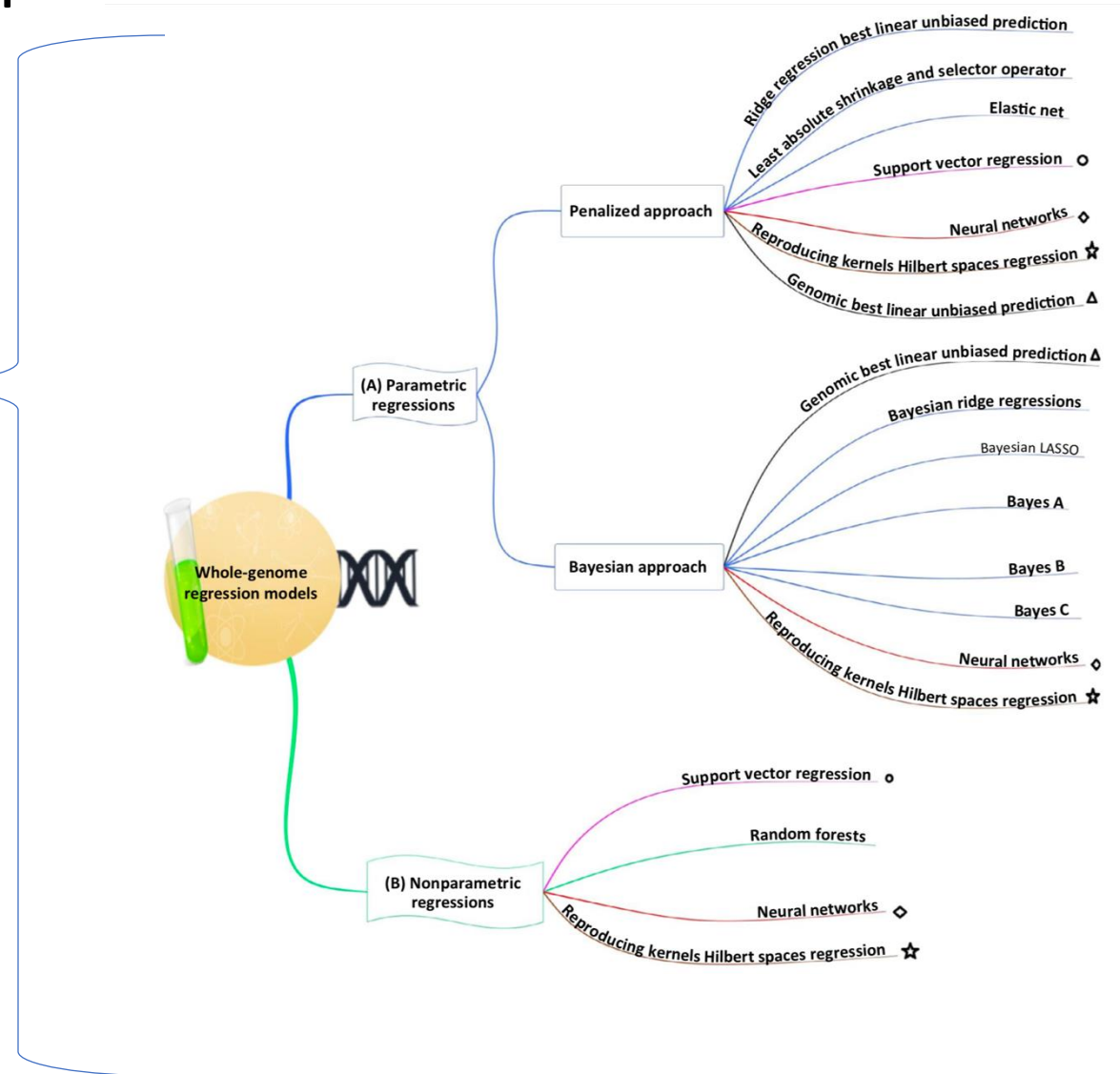
Genomic Prediction Models

$$y = \mu + \sum_k x_k \beta_k + \varepsilon$$

Trait $\rightarrow y$, mean $\rightarrow \mu$, Design matrix $\rightarrow x_k$, Effect of k th marker $\rightarrow \beta_k$, Error $\rightarrow \varepsilon$

Key: How we estimate and Assume Marker Variances.

- Shrinkage Models:** RR BLUP and GBLUP
- Dimension Reduction Methods:** Partial Least Squares (PLS) and Singular Value Decomposition (SDV)
- Bayesian Approach:** Variable Selection Models (priors)
- Kernel and Machine Learning Methods**



Adapted from manuscript Desta and Ortiz, 2014

Ordinary Least squares (OLS)

Aim is to estimate α and β parameters by minimize the squared errors

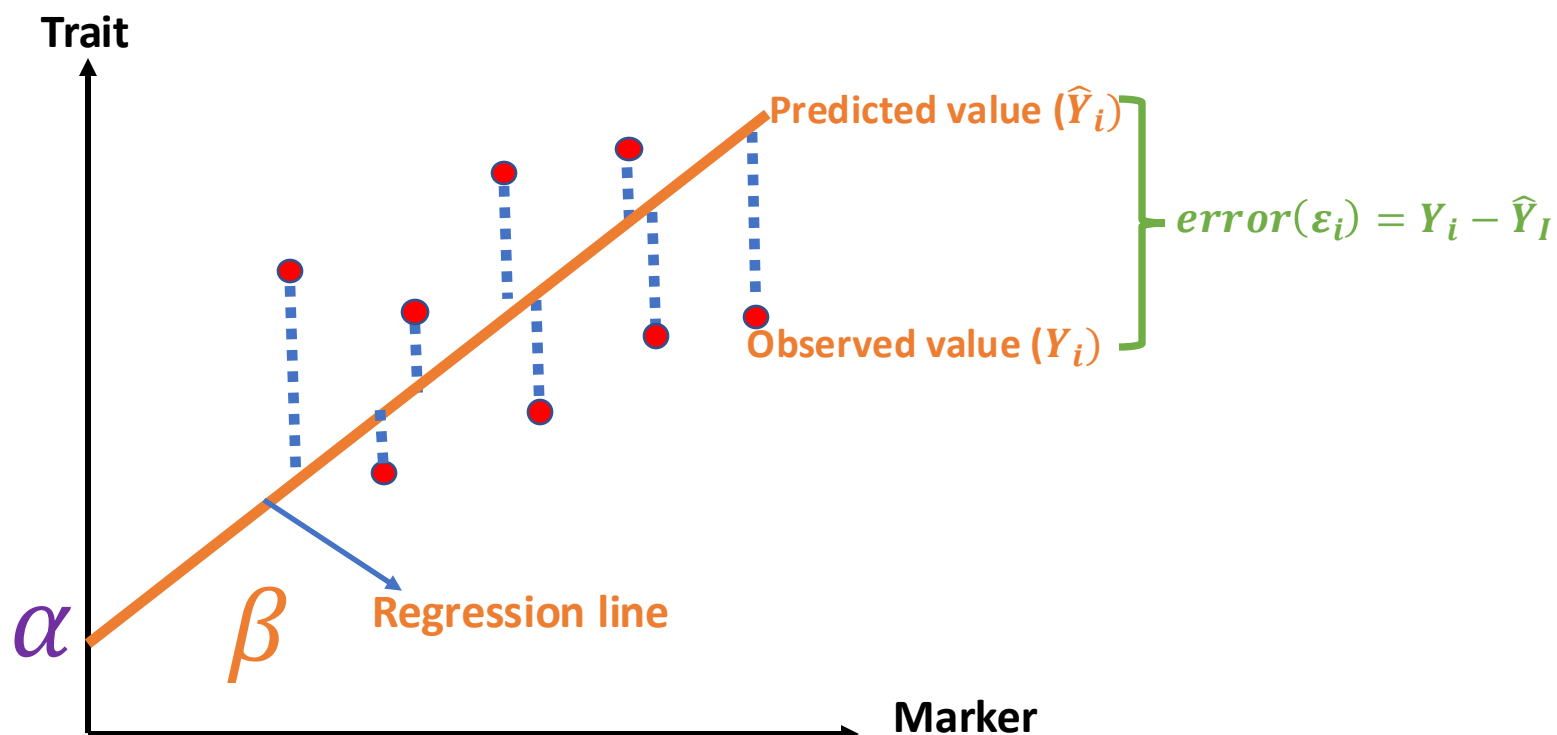
Simple linear regression:

$$\hat{Y}_i = \alpha + \beta x_i + \varepsilon_i$$

Diagram illustrating the components of the simple linear regression equation:

- \hat{Y}_i : dependent variable
- α : intercept
- β : coefficient
- x_i : Independent variable
- ε_i : error

Parameters (un-observed)



OLS in Relevance to Genomic Predictions

Aim is to minimize residual squares

$$\operatorname{argmin}(\varepsilon' \varepsilon) = \operatorname{argmin}(y - X\beta)'(y - X\beta)$$

$$\beta = (X'X)^{-1} X'Y$$

Simple marker regression model

$$Y_n = \mu_m + X_m \beta_n + \varepsilon$$

y_1
y_2
y_3
y_4
.
.
y_n

Y_n are the phenotypic values of n th plants/individuals

μ_m is mean of n th marker

	SNP ₁	SNP _m
y_{11}	0	0
y_{21}	2	2
y_{31}	2	2
y_{41}	0	0
y_{51}	0	0
.	.	.
.	.	.
.	.	.
y_{m1}	2	2

X_m is a vector or design matrix containing alleles (0,2 or 1) and connecting it to phenotypic values.

β_n is the additive effect of n th marker

error for marker assumed $\varepsilon \sim N(0, \sigma_e^2)$, with mean 0 and marker variance σ_e^2

Limitations

1. $m \gg n$, we cannot fit OLS (curse on dimensionality)

$X'X$ is a singular, and determinate of $X'X$ is 0 so cannot take inverse, $X'X$ is not invertible

2. $m \gg n$, multi-collinearity produces singular matrix

3. $m \gg n$, not perfect multi-collinearity, but predictors are highly correlated

OLS estimates are not stable and have high variance

Need alternative strategy

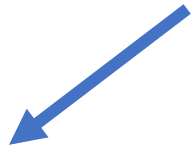
Ridge Regression

Provides solution and overcomes problems of OLS

Aim is to minimize residual squares

$$\operatorname{argmin}(\varepsilon'\varepsilon) = \operatorname{argmin}(\mathbf{y} - \mathbf{X}\beta)'(\mathbf{y} - \mathbf{X}\beta)$$

$$\beta = (\mathbf{X}'\mathbf{X} + \lambda\mathbf{I})^{-1} \mathbf{X}'\mathbf{Y}$$



Add constant, Penalty/Shrinkage Factor

$$E(\hat{\beta}_{ridge}) = (\mathbf{X}'\mathbf{X} + \lambda\mathbf{I})^{-1} \mathbf{X}'E(\mathbf{Y})$$



$$E(\hat{\beta}_{ridge}) = \beta - \lambda(\mathbf{X}'\mathbf{X} + \lambda\mathbf{I})^{-1}$$



Biased, but with minimum variance

adding λ , we take inverse but we have biased estimates

when, $\lambda = 0$, $E(\widehat{\beta}) = \beta$, i. e, OLS

Ridge-Regression BLUP (RRBLUP)

$$y = \mu + \sum_k x_k \beta_k + \varepsilon$$

estimate β by adding positive constraint

$$\beta_{ridge} = (X^T X)^{-1} + \lambda I)^{-1} X^T y$$

$$\lambda = \frac{\sigma_e^2}{\sigma_\beta^2}$$

$$\beta_k \sim N(0, \sigma_e^2)$$

- ❖ Ridge regression induces homogeneous shrinkage and it depends upon allele frequency
- ❖ Assumes all markers have same variance with small but non-zero effect.


- We add constant to diagonal, thus it is invertible
- Degree of shrinkage depends upon λ , larger the λ larger is the shrinkage

$$\frac{\beta_{OLS}}{1 + \lambda}$$

- We basically shrink OLS estimates towards 0
- λI term reduces collinearity and prevents the matrix $X^T X$ from becoming singular.

Things to Know (RRBLUP)

$$\mathbf{y} = \mu + \sum_k \mathbf{z}_k \boldsymbol{\beta}_k + \boldsymbol{\varepsilon}$$


$$\boldsymbol{\beta}_{ridge} = (\mathbf{Z}^T \mathbf{Z})^{-1} + \lambda \mathbf{I}) \mathbf{Z}^T \mathbf{y}$$

- We add constant to diagonal, thus it is invertible
- Degree of shrinkage depends upon λ , larger the λ larger is the shrinkage

$$\frac{\boldsymbol{\beta}_{OLS}}{1 + \lambda} \quad \lambda = \frac{\sigma_{\boldsymbol{\beta}}^2}{\sigma_{\boldsymbol{\varepsilon}}^2}$$

- We basically shrink $\boldsymbol{\beta}$ (OLS) estimates towards 0
- $\lambda \mathbf{I}$ term reduces collinearity and prevents the matrix $\mathbf{X}^T \mathbf{X}$ from becoming singular.

- ❖ Ridge regression induces homogeneous shrinkage, same for all markers
- ❖ Assumes all markers have the same variance with small but non-zero effect.
- ❖ Shrinkage depends upon allele frequency

$$\hat{\boldsymbol{\beta}}_{ridge} = \frac{2p_j(1-p_j)_n}{2p_j(1-p_j)_n + \lambda} * \hat{\boldsymbol{\beta}}_{OLS}$$

- ❖ Markers with extreme frequency are shrunk more.

RR BLUP Model



$$\mathbf{y} = \mathbf{Z}\mathbf{g} + \boldsymbol{\varepsilon}$$

Solve mixed model equation (Henderson, 1989)

$$\begin{bmatrix} \hat{X}X & \hat{X}Z \\ \hat{Z}X & \hat{Z}Z + \lambda I^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{b} \\ \mathbf{a} \end{bmatrix} = \begin{bmatrix} \hat{X}y \\ \hat{Z}y \end{bmatrix}$$

$$\begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \\ \hat{a}_4 \\ \hat{a}_5 \\ \hat{a}_6 \end{bmatrix} = \begin{bmatrix} -0.35 \\ -0.25 \\ 1.35 \\ -1.15 \\ 1.45 \\ 0.45 \end{bmatrix}$$

Marker effects (BLUP)

$$\text{GEBVs}(\hat{u}) = \mathbf{Z}\hat{\mathbf{a}}$$

	a_1	a_2	a_3	a_4
y_1	0	0	0	2
y_2	2	1	0	2
y_3	2	2	2	2
y_4	2	2	0	0

$$\text{GEBV}(y_1) = [0 * (-0.35) + [0 * (-0.25)] + \dots + [2 * (-1.15)]$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \text{SNP1} & \text{SNP2} & \text{SNP3} \\ 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_p \end{bmatrix} + \boldsymbol{\varepsilon}$$

$n \times 1$ $n \times m$ $p \times 1$ $n \times 1$

n is # of individuals
 p is # of markers

$$\begin{bmatrix} \mathbf{a} \\ \boldsymbol{\varepsilon} \end{bmatrix} \sim N \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_a^2 & 0 \\ 0 & \sigma_\varepsilon^2 \end{bmatrix} \right)$$



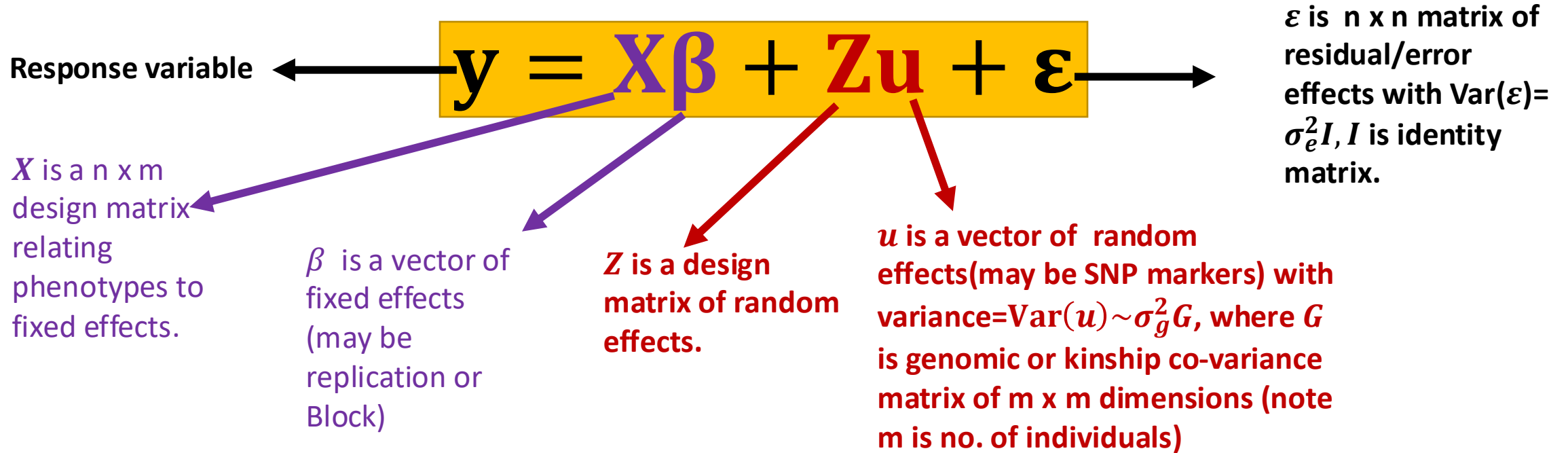
What if LD exists
between markers?

***RRBLUP or Ordinary
least squares (OLS) do
not consider:***

LD Between Markers and account for
the relationship between the genotypes

***Better to compute
and Integrate the
Relationship matrix
in the model***

Mixed Effect Model



$$\begin{bmatrix} \mathbf{u} \\ \boldsymbol{\varepsilon} \end{bmatrix} \sim N \left(0, \begin{bmatrix} \sigma_{\mathbf{u}}^2 \mathbf{K} & 0 \\ 0 & \sigma_{\boldsymbol{\varepsilon}}^2 \mathbf{R} \end{bmatrix} \right) \quad \text{Note: cov}(\mathbf{u}, \boldsymbol{\varepsilon}) = 0$$

Solving Mixed model Equation

$$\tilde{\beta} = (X'V^{-1}X)^{-1} X'V^{-1}y, \quad \text{with BLUE}(X\beta) = X\tilde{\beta}$$

BLUE

$$\underbrace{\begin{bmatrix} \acute{X}X & \acute{X}Z \\ \acute{Z}X & \acute{Z}Z + \lambda G^{-1} \end{bmatrix}}_{\text{LHS}} \begin{bmatrix} \beta \\ u \end{bmatrix} = \underbrace{\begin{bmatrix} \acute{X}y \\ \acute{Z}y \end{bmatrix}}_{\text{RHS}}$$

$$\tilde{u} = DZ'V^{-1}(y - X\tilde{\beta}) = \text{BLUP}(u)$$

BLUP

$$\begin{bmatrix} \hat{\beta} \\ \hat{u} \end{bmatrix} = \begin{bmatrix} \acute{X}X & \acute{X}Z \\ \acute{Z}X & \acute{Z}Z + \lambda G^{-1} \end{bmatrix}^{-1} \begin{bmatrix} \acute{X}y \\ \acute{Z}y \end{bmatrix}$$

We decompose the matrices and get solutions through iterative methods

Genomic BLUP (gBLUP)

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u} + \boldsymbol{\varepsilon}$$

$$\hat{\mathbf{u}} = \left[\mathbf{I} + \mathbf{G}^{-1} \frac{\sigma_e^2}{\sigma_u^2} \right] \mathbf{y}$$

GRM to account for mendelian sampling

Equivalence between rrBLUP and gBLUP

For gBLUP the $\text{Var}(\mathbf{y}) = \mathbf{Z}\mathbf{G}\mathbf{Z}'\sigma_u^2 + \mathbf{I}\sigma_e^2$

For rrBLUP the $\text{Var}(\mathbf{y}) = \mathbf{X}\mathbf{X}'\sigma_\beta^2 + \mathbf{I}\sigma_e^2$

Genomic BLUP (gBLUP Model)



$$\mathbf{y} = \mathbf{Z}\mathbf{a} + \boldsymbol{\varepsilon}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & 0.5 & 0.3 & 0.9 \\ 0.5 & 1 & - & 0.8 \\ & & 0.2 & \\ 0.3 & -0.2 & 1 & 0.4 \\ 0.9 & 0.8 & .4 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_p \end{bmatrix} + \boldsymbol{\varepsilon}$$

$n \times 1$ $n \times g$ $g \times 1$ $n \times 1$

n is # of individuals
 g is # of breeding values

$$\begin{bmatrix} \mathbf{g} \\ \boldsymbol{\varepsilon} \end{bmatrix} \sim MNV \left(0, \begin{bmatrix} \mathbf{G}\sigma_a^2 & 0 \\ 0 & \sigma_\varepsilon^2 \end{bmatrix} \right)$$

GRM accounts for Mendelian sampling

Solve mixed model equation (Henderson, 1989)

$$\begin{bmatrix} \hat{\mathbf{X}}\mathbf{X} & \hat{\mathbf{X}}\mathbf{Z} \\ \hat{\mathbf{Z}}\mathbf{X} & \hat{\mathbf{Z}}\mathbf{Z} + \lambda\mathbf{G}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{b} \\ \mathbf{g} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{X}}\mathbf{y} \\ \hat{\mathbf{Z}}\mathbf{y} \end{bmatrix}$$

$$\begin{bmatrix} \hat{g}_1 \\ \hat{g}_2 \\ \hat{g}_3 \\ \hat{g}_4 \\ g \\ \hat{g}_6 \end{bmatrix} = \begin{bmatrix} -0.35 \\ -0.25 \\ 1.35 \\ -1.15 \\ 1.45 \\ 0.45 \end{bmatrix}$$

(BLUP of Breeding values)

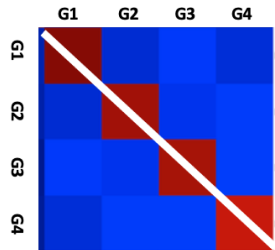
GEBVs($\hat{\mathbf{g}}$)

Construction of G matrix

3 steps

1. Create a centered Z matrix
2. Create the cross product
3. Divide it by $\sum_{i=1}^m 2p_j(1-p_j)$

$$\mathbf{G} = \frac{\mathbf{Z}\mathbf{Z}^T}{\sum_{i=1}^m 2p_j(1-p_j)}$$



First G matrix, VanRaden (2008)

$$\sigma_a^2 \text{ is additive variance} = \sum_{i=1}^m 2p_j(1-p_j)$$

Difference Between RRBLUP and gBLUP



RR BLUP

gBLUP

Marker as Predictors (BLUPs)

Predict Additive breeding Values directly

Markers are directly used

Markers are used to compute covariance matrix (Relationship Matrix)

Dimension of genetic effects is $p \times p$

Dimension of effects is $n \times n$, computationally easy

Does not account for the relationships between genotypes. Assuming genotypes are independent

Accounts relationship between genotypes, thus more appropriate for predictions or dissecting G x E (WHY?)

$$\text{var}(y) = Z\hat{Z} \sigma_u^2 + I\sigma_e^2$$

$$\text{var}(y) = ZG\hat{Z} \sigma_u^2 + I\sigma_e^2$$

$$\text{blup}(\hat{u}) = (Z\hat{Z} + I\lambda)^{-1}Z(Y' - \hat{\mu})$$

$$\text{blup}(\hat{u}) = (Z\hat{Z} + G^{-1}\lambda)^{-1}Z(Y' - \hat{\mu})$$

Pedigree BLUP (pBLUP Model)



$$\mathbf{y} = \mathbf{Za} + \boldsymbol{\varepsilon}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & 0.5 & 0.3 & 0.9 \\ 0.5 & 1 & - & 0.8 \\ 0.3 & -0.2 & 1 & 0.4 \\ 0.9 & 0.8 & .4 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ \vdots \\ a_p \end{bmatrix} + \boldsymbol{\varepsilon}$$

$n \times 1$ $n \times g$ $g \times 1$ $n \times 1$

n is # of individuals
 g is # of breeding values

$$\begin{bmatrix} \mathbf{g} \\ \boldsymbol{\varepsilon} \end{bmatrix} \sim MNV \left(0, \begin{bmatrix} A\sigma_a^2 & 0 \\ 0 & \sigma_\varepsilon^2 \end{bmatrix} \right)$$

Accounts for relationships (Coefficient of co-ancestry, $r = 2\theta_{xy}$)

Solve mixed model equation (Henderson, 1989)

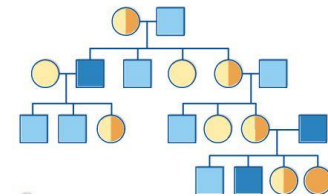
$$\begin{bmatrix} \hat{X}X & \hat{X}Z \\ \hat{Z}X & \hat{Z}Z + \lambda A^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{b} \\ \mathbf{g} \end{bmatrix} = \begin{bmatrix} \hat{X}y \\ \hat{Z}y \end{bmatrix}$$

$$\begin{bmatrix} \hat{g}_1 \\ \hat{g}_2 \\ \hat{g}_3 \\ \hat{g}_4 \\ g \\ \hat{g}_6 \end{bmatrix} = \begin{bmatrix} -0.35 \\ -0.25 \\ 1.35 \\ -1.15 \\ 1.45 \\ 0.45 \end{bmatrix}$$

(BLUP of Breeding values)

GEBVs(\hat{g})

Construction of A matrix



DESIGNATIO	Female	Male
IR64	IR5657-33-2-1	IR2061-465-1-5-5
IRRI154	IR73012-137-2-2-2	PSBRCLQ(R50404-57-2-2-3)
IRRI193	IR68077-82-2-2-3	IR00A117
IRO5N412	IR72875-94-3-3-2	IR73707-45-3-2-3
IRO5N419	IR72887-34-2-1-3	IR73707-45-3-2-3
IRO6N155	IR72158-11-5-2-3_IR73707-45	IR72875-94-3-3-2
IRO9A220	IR72903-121-2-1-2	IR71606-1-1-4-2-3-1-2(NSIC110)
IR10A231	IRRI143_IR73718-23-2-1-3	IR00A110
IR10F559	IR04010-8-197-4_IRRI149	NSICR158
IR10N237	IR01N111_IRRI164	IR72890-81-3-2-2
IR10N271	IR01W106	IR71676-90-2-2
IR11A282	IR04A427	BR29
IR11A303	IR04A427	IR72875-94-3-3-2
IR11A306	IR04A427	IR73006-12-3-3-2
IR11N121	IRO5N341_IR64680-81-2-2-1-3	PSBRCLQ(R50404-57-2-2-3)
IR11N202	IRO5N173	BR29
IR12N135	IR01N149_IR64680-81-2-2-1-3	FEDEARROZ50
IR12F111	IR44004-74-3-2-3-3	IR70181-32-PM11-1-5-1
BRRIIDHAN5	BR10(BR51-46-5)_BR23	BR847-76-1-1
BRRIIDHAN5		0

	IR64	IRRI154	IRRI193	IRO5N412	IRO5N419	IRO6N155	IRO9A220
IR64	1.07666	0.13361	0.01675	0.08966	0.03874	0.08966	0.31168
IRRI154	0.13361	1.03618	0.03124	0.04905	0.05164	0.05101	0.09983
IRRI193	0.01675	0.03124	1.00000	0.01019	0.01580	0.01019	0.02137
IRO5N412	0.08966	0.04905	0.01019	1.00000	0.29292	0.39858	0.04587
IRO5N419	0.03874	0.05164	0.01580	0.29292	1.00000	0.17475	0.04730
IRO6N155	0.08966	0.05101	0.01019	0.39858	0.17475	1.00000	0.04587
IRO9A220	0.31168	0.09983	0.02137	0.04587	0.04730	0.04587	1.02126
IR10A231	0.22386	0.05424	0.01408	0.02707	0.02324	0.02707	0.09506
IR10F559	0.22908	0.08668	0.02460	0.05484	0.04891	0.05313	0.09350
IR10N237	0.06116	0.09207	0.02596	0.04761	0.10362	0.04761	0.09354
IR10N271	0.03119	0.04701	0.01812	0.01327	0.01976	0.02108	0.03659
IR11A282	0.00000	0.00391	0.00000	0.00000	0.00000	0.03125	0.00000
IR11A303	0.08966	0.04905	0.01019	0.25503	0.03120	0.25503	0.04587

A matrix

$$\mathbf{y} = \mathbf{Za} + \boldsymbol{\varepsilon}$$

Dimension Reduction methods

- ❖ Reason is to avoid inverse
- ❖ Suitable when predictors are correlated
- ❖ And calculations are tedious

$$y = \mu + \sum_k x_k \beta_k + \varepsilon$$

$$x_k = U D V^T$$

$U = n \times m$ orthogonal matrix
 $D = n \times m$ diagonal matrix with singular values
 $V = n \times n$ orthogonal matrix

$$\beta_{OLS} = V D^{-1} U^T y$$

Regress phenotypes directly on eigenvectors or principle components

Bayesian Approaches

(Relax the Distribution Assumptions)

Sampling distr.

posterior

$$p(\theta|y) = \frac{p(\theta, y)}{p(y)} = \frac{p(y|\theta)p(\theta)}{p(y)}$$

prior


y is observed data $y \sim p(\theta|y)$
 θ is unknown parameter (random)

Bayesian Models differ with respect to the *Prior*

- The even distribution of genetic causation is not satisfactory
- The assumption of common variance does not imply that the effects of all markers
- RR BLUP over shrinkage of large marker effects
- Large effect and small effect QTLs (natural in breeding populations)

Bayesian Ridge-Regression (BRR)

$$\hat{\beta} = (XX^T + K^{-1} \frac{\sigma_e^2}{\sigma_\beta^2})^{-1} (yX^T + K^{-1} \frac{\sigma_e^2}{\sigma_\beta^2}) \beta_o$$


$$\lambda = \frac{\sigma_e^2}{\sigma_\beta^2}$$

$$\beta \sim N(0, \sigma_\beta^2)$$
$$\sigma_\beta^2 \sim \mathcal{X}^{-2}(df, s)$$

If we assume prior mean=0, $\beta=0$, then $K=I$

$$\beta_{ridge} = (X^T X)^{-1} + \lambda I) yX^T$$

Prior for markers is given by Gaussian distribution, which differs from LASSO
Scaled inverted chi-square distribution
The only difference in prior information

LASSO (Regularization)

$$\beta_{lasso} = \underset{\beta}{\operatorname{argmin}} \left[\sum_{i=1}^n (y_i - \sum_{j=1}^m x_{ij} \beta_j)^2 + \lambda \sum_{j=1}^m |\beta_j| \right]$$

in LASSO $|\beta_j|$ is absolute value (L_1 Penalty)

however, in Ridge regression value is β_j^2 (L_2 Penalty)

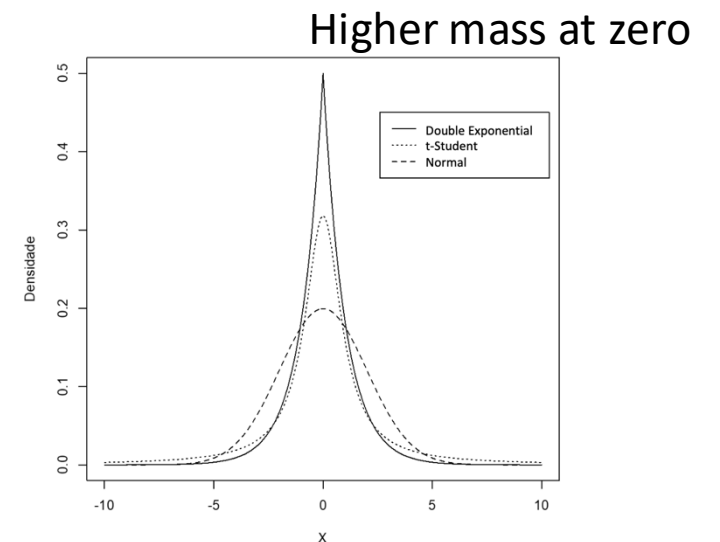
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In LASSO, shrinkage is stronger-sets some marker effects equal to 0

Bayesian LASSO

$$\begin{aligned} \beta_{lasso} &= \underset{\beta}{\operatorname{argmin}} \left[(y - x\beta)^T (y - x\beta) + \lambda |\beta_j| \right] \\ &= \underbrace{\exp \left(-\frac{1}{2\sigma_e^2} (y - x\beta)^T (y - x\beta) \right)}_{\text{Gaussian Sampling}} \underbrace{\exp(\lambda |\beta_j|)}_{\text{Double exponential}} \end{aligned}$$

In Bayesian Lasso, assumption of marker effects is Double exponential
Double exponential shrink more marker effects close to zero



Bayesian Approaches

(Relax the Distribution Assumptions)

Bayes A

- Assumes marker-specific variance
- Utilizes an inverse chi-square on marker variances
- Shrinks tiny marker effects (variances) towards zero, and larger values survive.

Bayes B

- Fraction of markers with zero effect
- More realistic prior because we expect that some regions of the genome will carry no QTL

Bayes C

- Assumes t-distribution one with large variance for SNP fraction and other with small variance





**Genomic Selection is
Simply a tool to
Supplement the
Breeding Pipeline**

Thank You

Questions