PHY566 Homework #2

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February 4, 2016

Carbon Dating

1 Problem Description

Carbon dating is a method for determining the age of an ancient artifacts containing biological material by using the properties of radiocarbon ${}^{14}_{6}\mathrm{C}$. ${}^{14}_{6}\mathrm{C}$ undergoes β^- decay with a half-life time of $T_{\frac{1}{2}} = 5700$ years. Suppose an ancient artifact originally contained 10^{-12} kg of ${}^{14}_{6}\mathrm{C}$, calculate the activity of the sample, which defined as $R(t) = -\frac{dN}{dt}$, over a duration of $20,000$ years analytically and numerically and plot the results of numerical solution with different time-step widths and analytical solution. Then analyze the accuracy of the numerical solution with different time-step widths.

2 Analytic Solution

If $N(t)$ is the number of atoms left after time t and τ is the decay constant, the equation governing the decay of a radioactive isotope is:

$$\frac{dN}{dt} = -\frac{1}{\tau}N. \tag{1}$$

Then, we have

$$\frac{dN}{N} = -\frac{1}{\tau}dt. \tag{2}$$

We then integrate both sides, obtaining

$$\int \frac{dN}{dt} = -\frac{1}{\tau}\int dt. \tag{3}$$

Finally, we can get,

$$N(t) = N_0 e^{-\frac{t}{\tau}}, \tag{4}$$

where N_0 is the number of atoms of the isotope in the original sample.

Setting $N(t) = \frac{1}{2}N_0$, then we can get,

$$T_{\frac{1}{2}} = \tau \ln 2. \tag{5}$$

If we substitute $T_{\frac{1}{2}} = 5700$ years, then we have

$$\tau = \frac{5700}{\ln 2} \text{ years} \approx 8223.36173 \text{ years}. \tag{6}$$

if we define $R(t) = -\frac{dN}{dt}$ as the activity of the sample, then we have

$$R(t) = \frac{1}{\tau}N \tag{7}$$
$$= \frac{1}{\tau}N_0 e^{-\frac{t}{\tau}} \tag{8}$$
$$= R_0 e^{-\frac{t}{\tau}}. \tag{9}$$

3 Numerical Solution

If we substitute (7) into (1), then we can get

$$\frac{dR}{dt} = -\frac{1}{\tau}R. \tag{10}$$

From the definition of a derivative , we can obtain

$$\frac{dR}{dt} = \lim_{\Delta t \to 0} \frac{R(t+\Delta t) - R(t)}{\Delta t} \approx \frac{R(t+\Delta t) - R(t)}{\Delta t}. \tag{11}$$

We can rearrange this to obtain

$$R(t+\Delta t) \approx R(t) + \frac{dR}{dt}\Delta t. \tag{12}$$

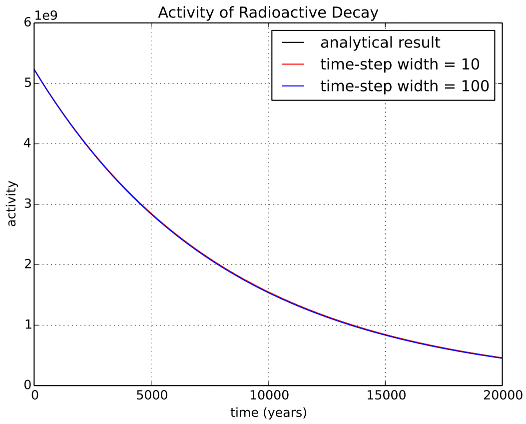
If we insert (10) into (12), then we can get

$$R(t+\Delta t) \approx R(t) - \frac{R(t)}{\tau}\Delta t. \tag{13}$$

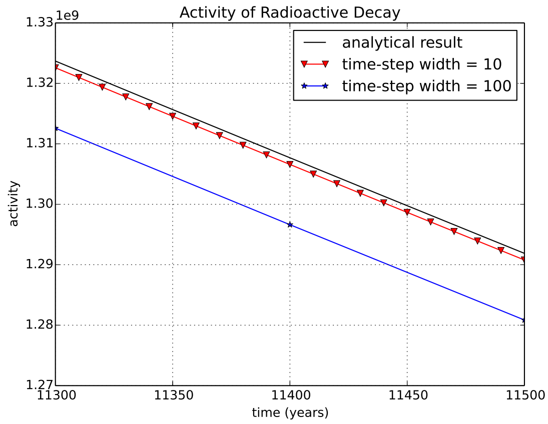
And the initial value can be given by (7),

$$R(0) = \frac{N(0)}{\tau} = \frac{10^{-12}\mathrm{kg}}{14\mathrm{g/mol}}\frac{N_A}{\tau}. \tag{14}$$

Figure 1a illustrates the results of analytical solution and numerical solution with time-step width of 10 years and 100 years. This figure also shows that the numerical results coincide well with the analytical result. Figure 1b shows the details around 2 half-lifes.



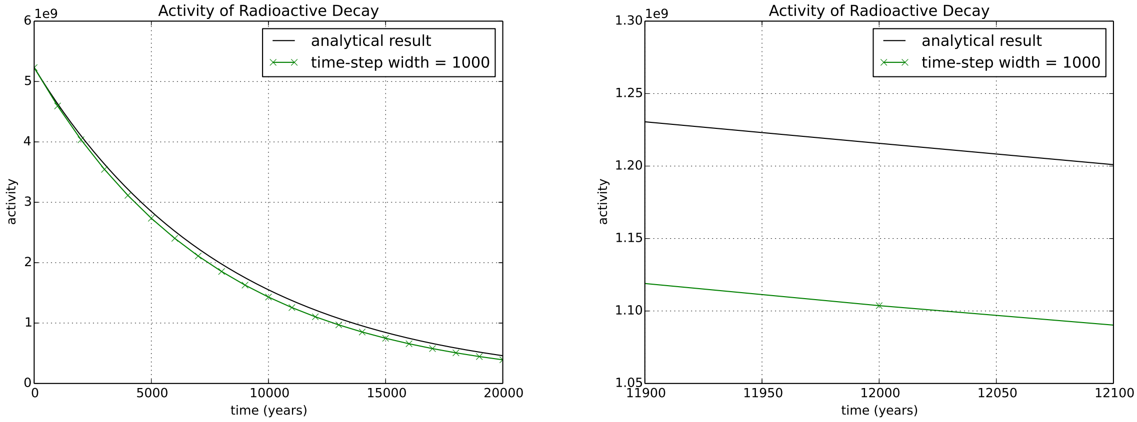
(a) Analytical result and numerical results



(b) Zoomed figure around 2 half-lifes

Figure 1

If we increase the time-step width to 1,000 years, then the derivation from the analytical result becomes obvious which can be seen from the figure 2a.



(a) Analytical result and numerical results

(b) Zoomed figure around 12,000th year

Figure 2

4 Discussion

The following table shows the results and percent errors we get from analytical solution and numerical solutions of different time-step widths.

Type		R(12,000)	Percent Error
Analytical Solution		1,215,670,733.832559	-
Numerical Solution	10 years	1,214,591,717.858181	0.088759%
	100 years	1,204,844,778.564423	0.890534%
	1,000 years	1,103,679,445.553761	9.212304%

It can be seen from the table that the result is not acceptable when time-step width is 1,000 years because the corresponding percent error is too big (9.212304%).

The Taylor expansion for $R(t)$ is

$$R(\Delta t) = R(0) + \frac{dR}{dt}\Delta t + \frac{1}{2}\frac{d^2R}{dt^2}(\Delta t)^2 + \dots. \tag{15}$$

If we consider the second-order term, then the error is

$$\begin{aligned}\Delta R &= \frac{1}{2}\frac{d^2R}{dt^2}(\Delta t)^2 &(16)\\ &= \frac{1}{2}\frac{1}{\tau^2}R_0 e^{-\frac{t}{\tau}}(\Delta t)^2 &(17)\\ &= \frac{1}{2\tau^2}R(t)(\Delta t)^2. &(18)\end{aligned}$$

Hence, the percent error is

$$\eta = \frac{\Delta R}{R} = \frac{1}{2\tau^2}(\Delta t)^2. \tag{19}$$

If we set $\Delta t = 1,000$, we can obtain

$$\eta \approx 0.00739386 = 0.739386\%, \tag{20}$$

which means that the error from the second order term accounts for only a small fraction of the total errors of the result with time-step with of 1,000 years.