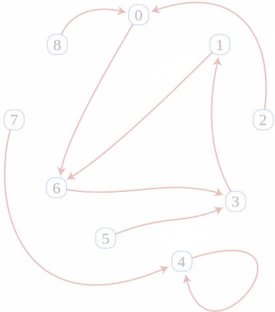
Element Distinctness, Birthday Paradox, and 1-out Pseudorandom Graphs

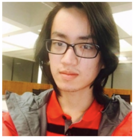


Hongxun Wu

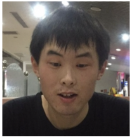
IIIS, Tsinghua University

42	3	23	1	12	30	42	15

Authors of this work



Lijie Chen



Ce Jin

R. Ryan Williams

我

Hongxun Wu

Lijie Chen, Ce Jin, and R. Ryan Williams are from MIT.

Element Distinctness

Element Distinctness

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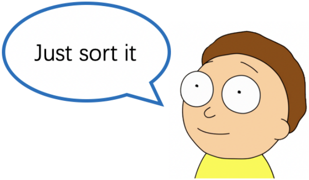
- INPUT: n positive integers $a_1, a_2, \ldots, a_n$ with $a_i \le \mathrm{poly}(n)$.
- Decide whether all a's are distinct.

Element Distinctness

1	3	12	15	23	30	42	42



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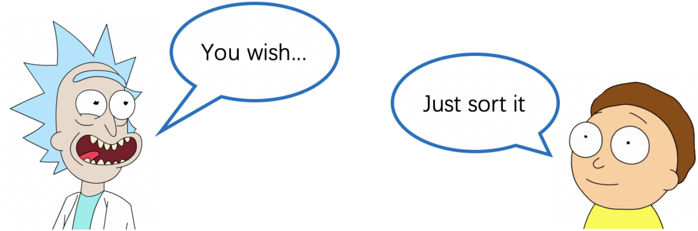
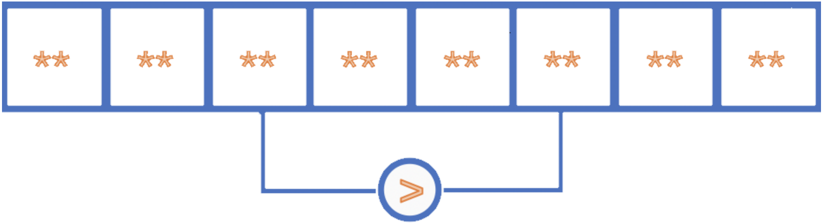
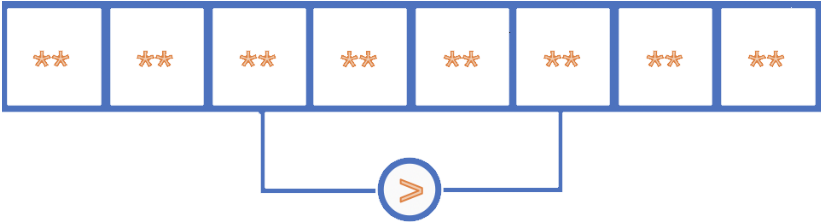


Comparision model



- No direct access to the INPUT a.
- Each query (i, j) returns one of $a_i < a_j$, $a_i = a_j$, $a_i > a_j$.

Comparision model



Time-Space tradeoff [?, ?]

Element distictness requires $TS = \Omega\left(n^{2-o(1)}\right)$ in Comparision model.

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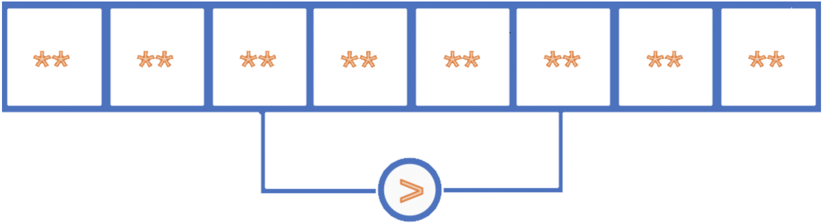

Time-Space tradeoff [?, ?]

Element distictness requires $TS = \Omega\left(n^{2-o(1)}\right)$ in Comparision model.

- When $S = O(\operatorname{polylog} n)$, $T = \Omega\left(n^{2-o(1)}\right)$.

RAM model

- Random access to read-only input.
- Working memory has a (relatively small) size S.

RAM model

Time-Space tradeoff [?]

- Assuming the existence of *Random Oracle*, there is an algorithm with $T^2S = \tilde{O}\left(n^3\right)$.

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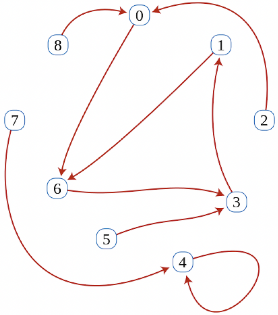
- When $S = \tilde{O}(1)$, $T = \tilde{O}\left(n^{1.5}\right)$.

Pollard's ρ method [?]

Assuming the existence of *Random Oracle*, when $S = \tilde{O}(1)$, there is an algorithm with $T = \tilde{O}\left(n^{1.5}\right)$.

- For random oracle R, define graph $x \mapsto R(a_x)$ with $x \in [n]$.



1-out Graph and Birthday Paradox

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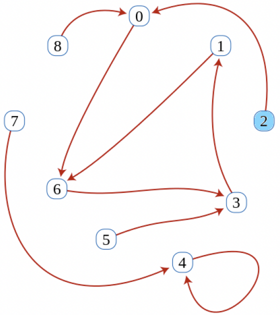

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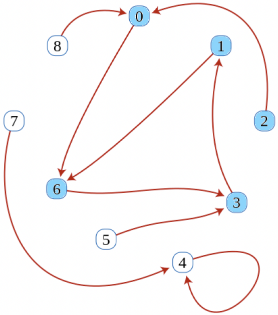
- Run Floyd's cycle finding.



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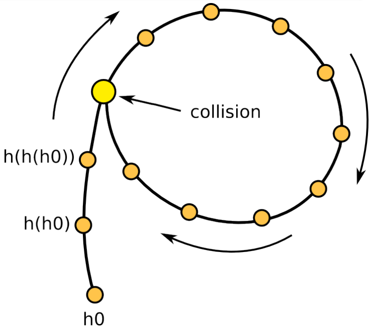
Birthday Paradox Type Properties [?]

Suppose $f^*(s)$ is the set of vertices reachable from s.

- $\mathbb{E}[|f^*(s)|] \leq O(\sqrt{n})$



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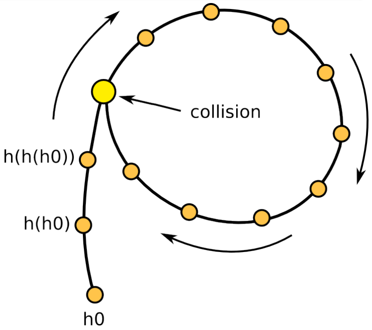

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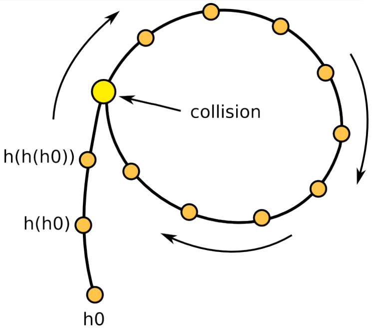

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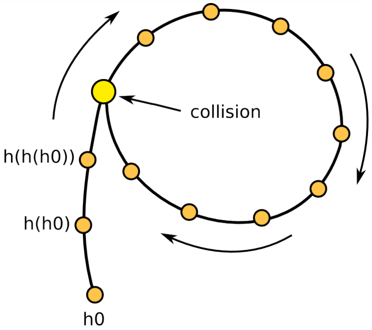
- So each cycle-finding takes $O(\sqrt{n})$ time and finds any collision u, v with probability $\Omega(1/n)$.
- Repeat $O(n)$ times, it takes $O(n^{1.5})$ time in total.



Our Results

Our Main Lemma

There exsits a family $\{r_{\text{seed}}\}$ of hash functions efficiently samplable with seed length $O(\operatorname{polylog} n)$, and the graph defined by $\{r_{\text{seed}}\}$ (instead of *Random Oracle* R) satisfy

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~~Assuming the existence of *Random Oracle*~~, when $S = O(\operatorname{polylog} n)$, there is a RAM algorithm for Element Distinctness with $T = \tilde{O}\left(n^{1.5}\right)$.

Subset Sum

Low-space Algorithm for Subset Sum [?]

Assuming the existence of *Random Oracle*, Subset Sum and Knapsack can be solved by a Monte Carlo algorithm in $O^*(2^{0.86n})$ time, with $O(\mathrm{poly}(n))$ space.

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- $\Pr[u \in f^*(s)] \geq \Omega(1/\sqrt{n}), \ \forall u \in [n]$

Construction

This is Ryan O'Donnell's Youtube lecture which is a masterpiece.

Random Restriction and Håstad's Switching Lemma

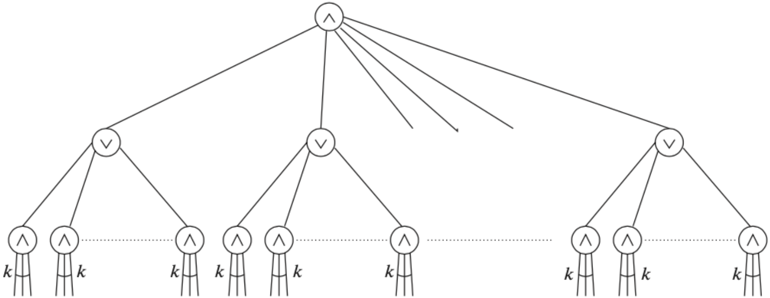


Figure 14.1. Circuit before Håstad switching transformation.

Figure from Arora & Barak.

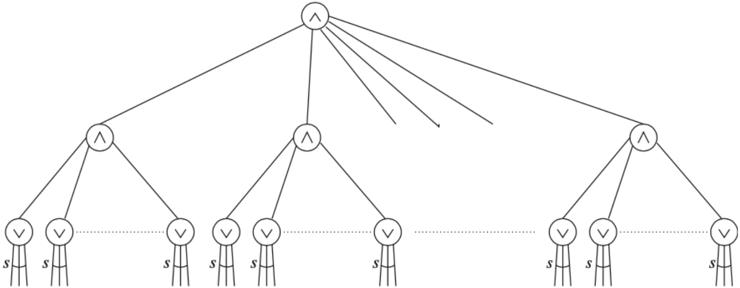
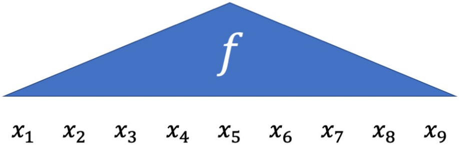


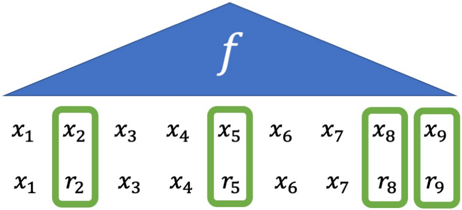
Figure 14.2. Circuit after Håstad switching transformation. Notice that the new layer of $\wedge$ gates can be collapsed with the single $\wedge$ parent gate, to reduce the number of levels by one.

Figure from Arora & Barak.



This is the Ajtai-Wigderson Paradigm [?] for building PRG.

Iterative Restriction



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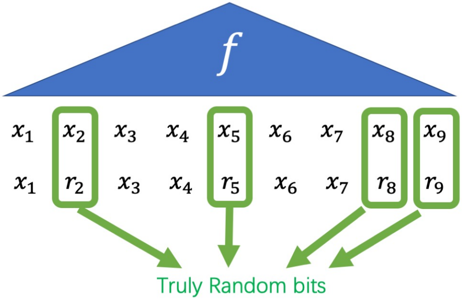

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Toy Example: Two levels

Recall the input $a_1, a_2, \ldots, a_n \in [m]$.

Two Level Example

Suppose we have the following:

- $O(\text{polylog}\, n)$-wise independent functions $g : [m] \to \{0, 1\}$ and $r : [m] \to [n]$.
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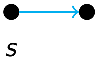

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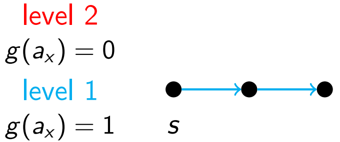

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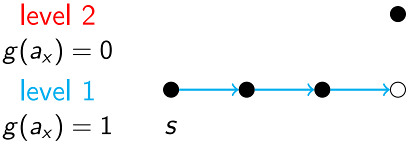

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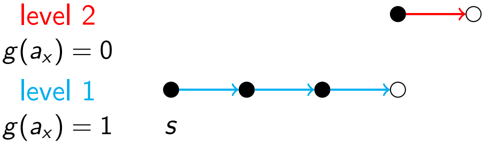

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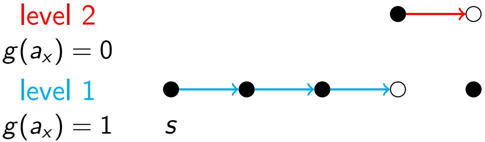

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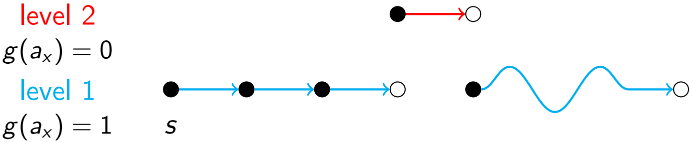

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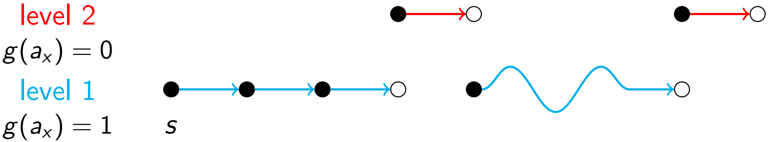

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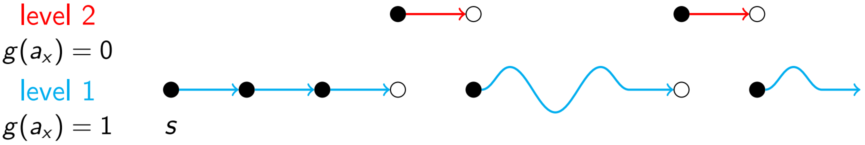

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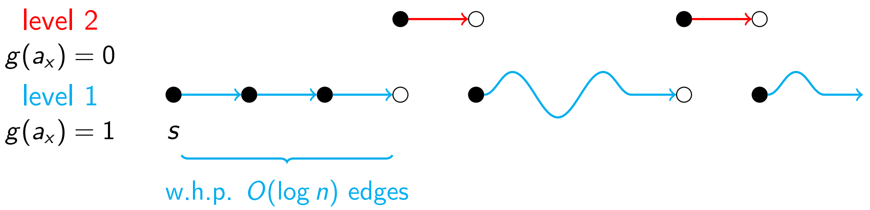
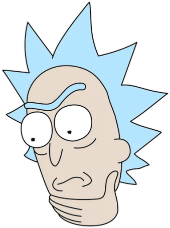


Sanity Check

- Why this might be a good idea?



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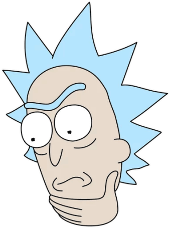

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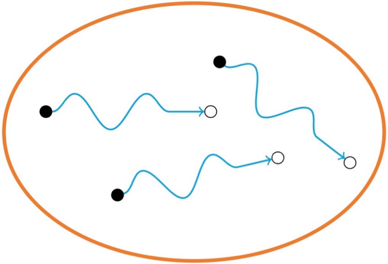

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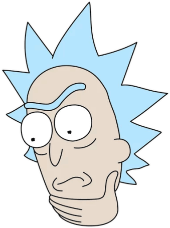

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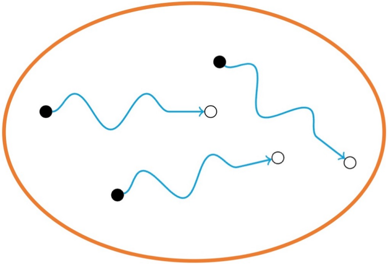

- Each subpath has length $O(\log n)$.
- Every level 2 edge is an independent sample of a subpath.

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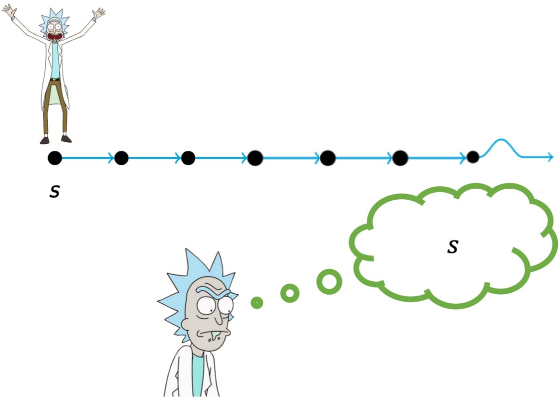

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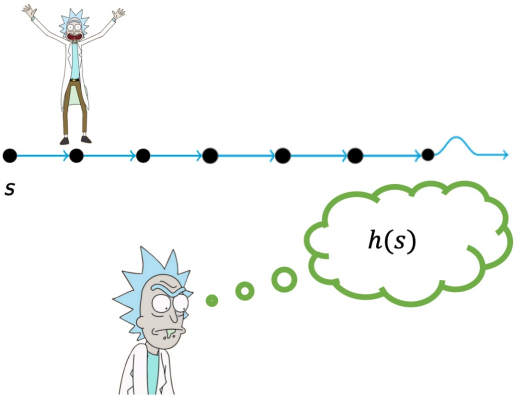

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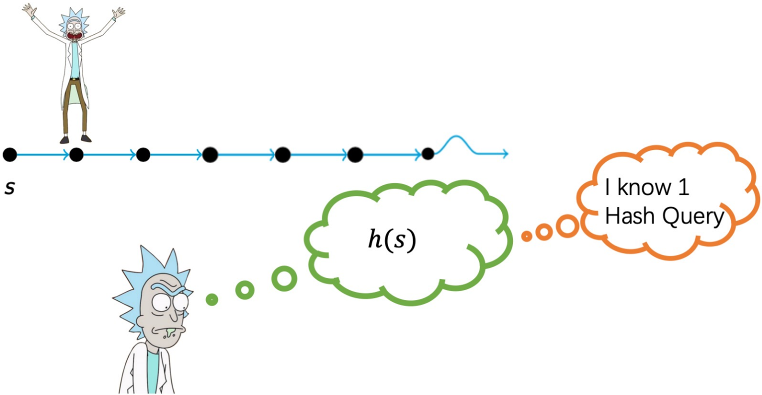

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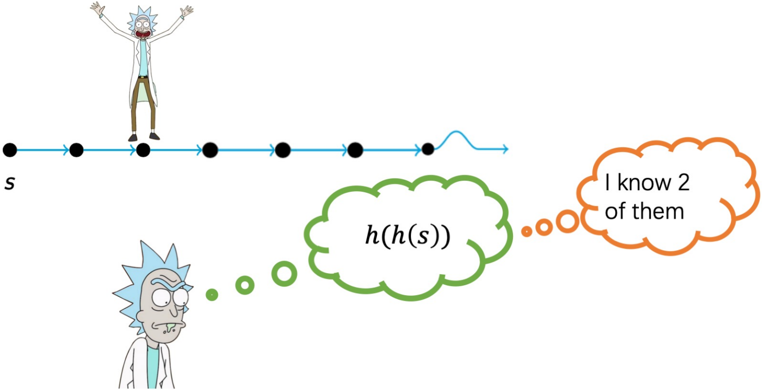

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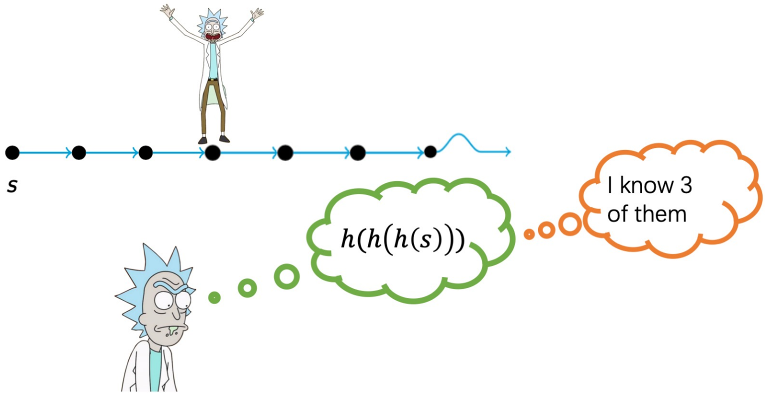

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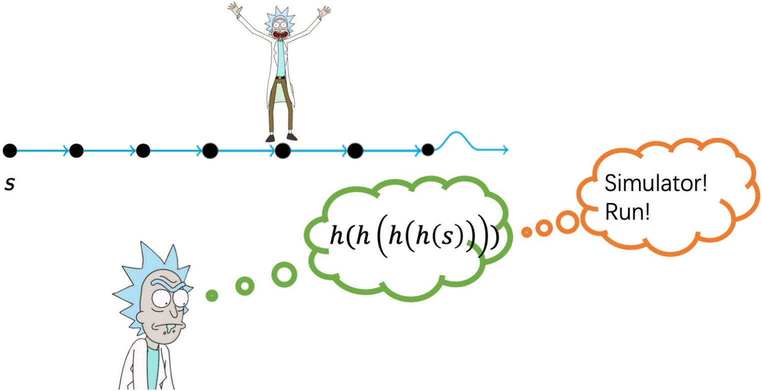

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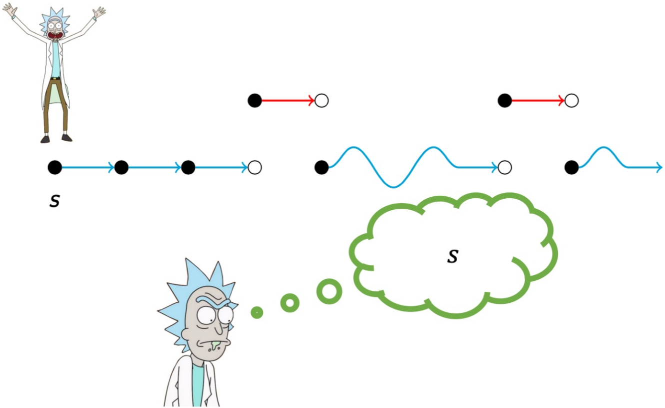

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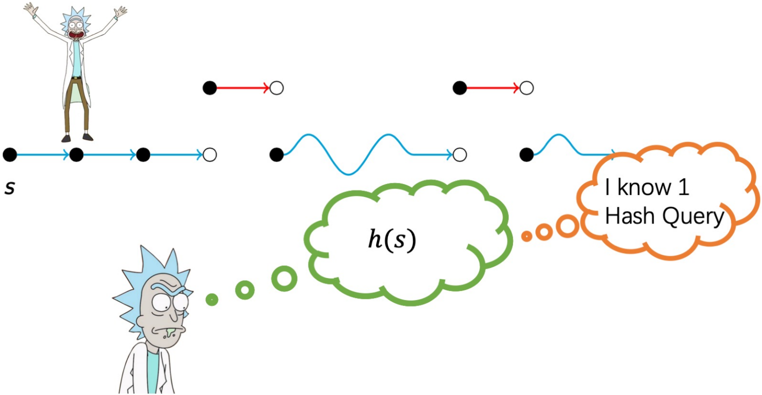

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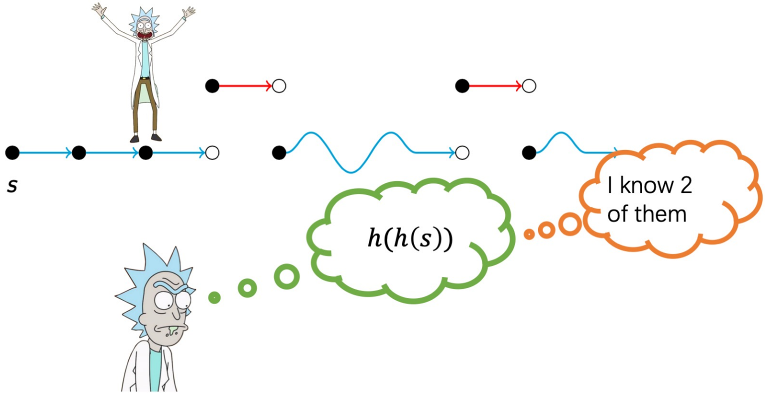

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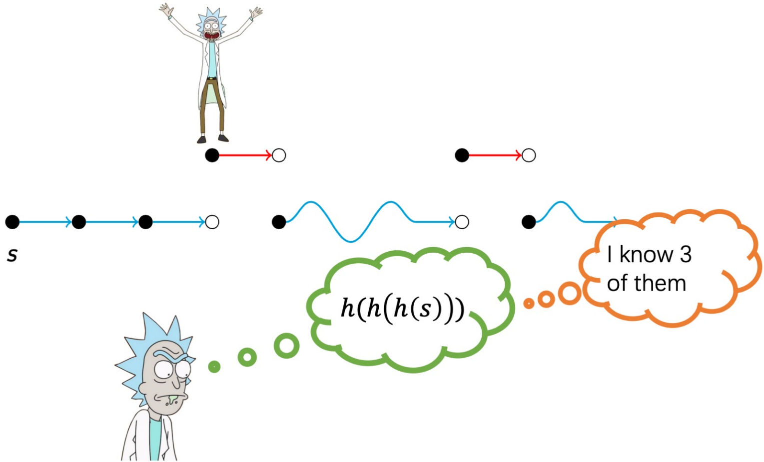
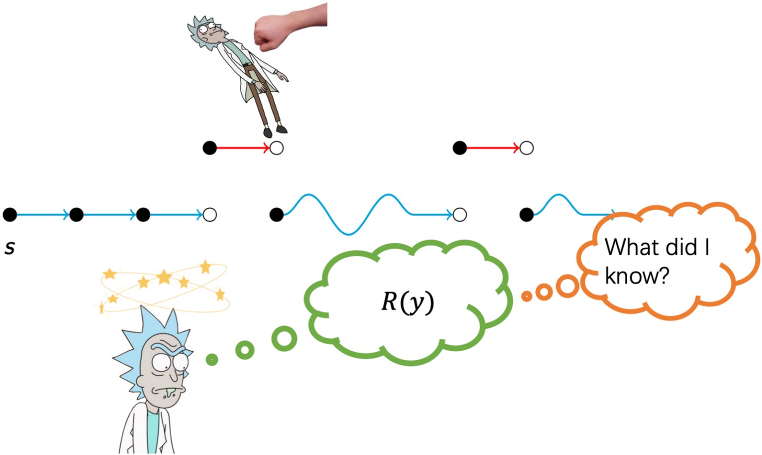


Intuition: Memory Eraser



Our Construction via Iterative Restriction

Our Construction

Now we sample $O(\log n)$ many hash functions $\{r_i, g_i\}_{i\in[\ell]}$.
Each $r_i : [m] \to [n]$ and $g_i : [m] \to [2]$ are $O(\log n)$-wise independent.

Our Construction via Iterative Restriction

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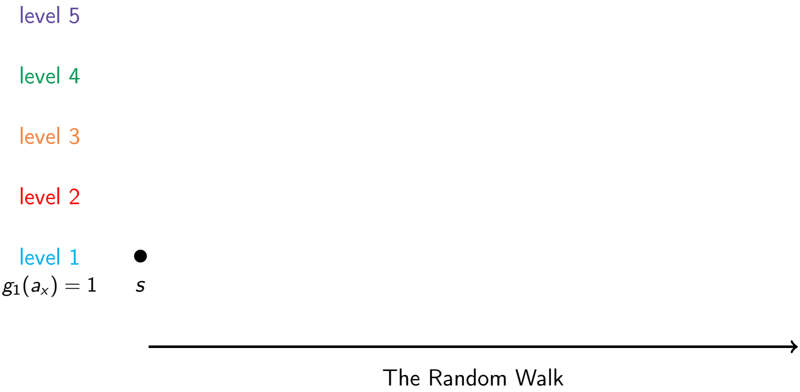
Each $r_i : [m] \to [n]$ and $g_i : [m] \to [2]$ are $O(\log n)$-wise independent.
Then we set $h_{\ell+1}(a_x) = \perp$ and

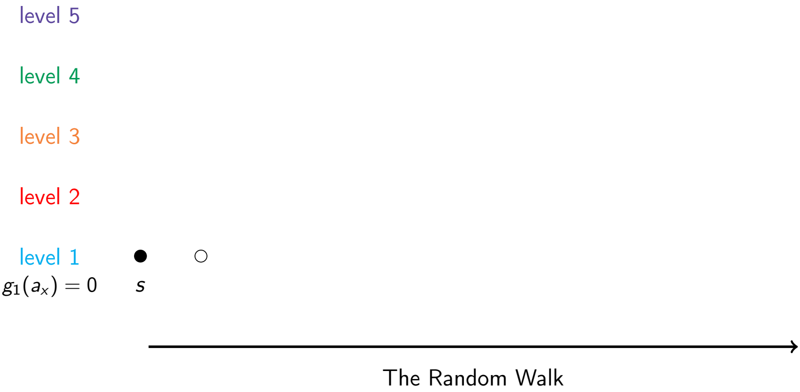
$$h_i(a_x) = \begin{cases} h_{i+1}(a_x) & g_i(a_x) = 0 \\ r_i(a_x) & g_i(a_x) = 1 \end{cases}$$

Finally, we set $h = h_1$.

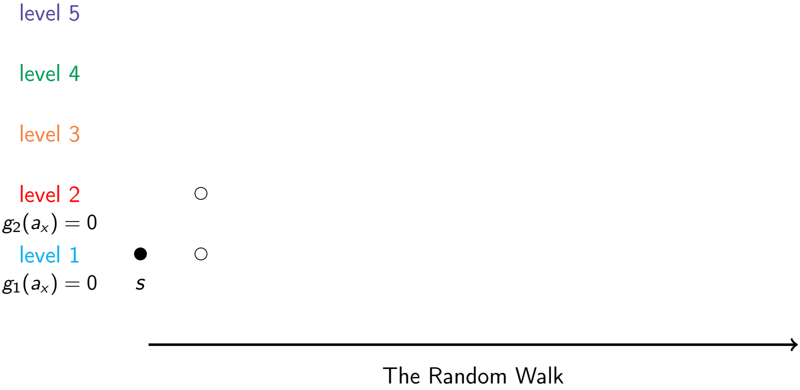
Our Construction via Iterative Restriction



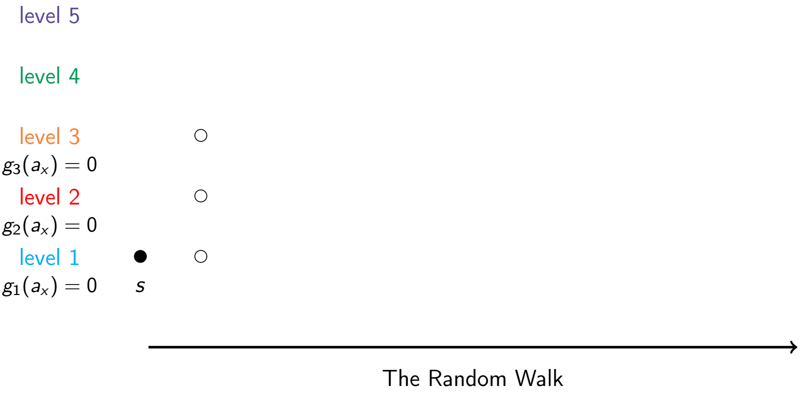
Our Construction via Iterative Restriction



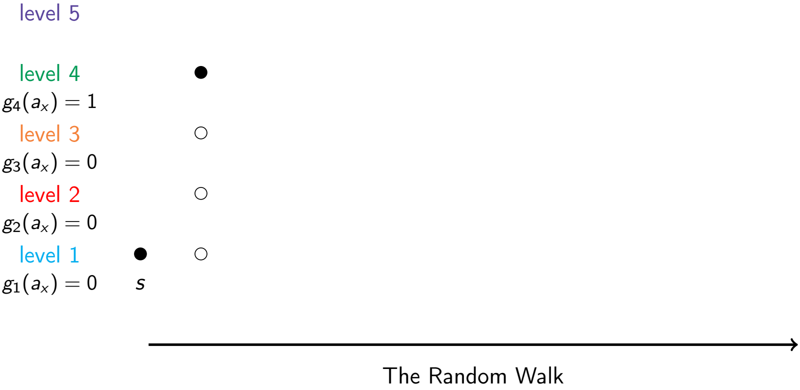
Our Construction via Iterative Restriction



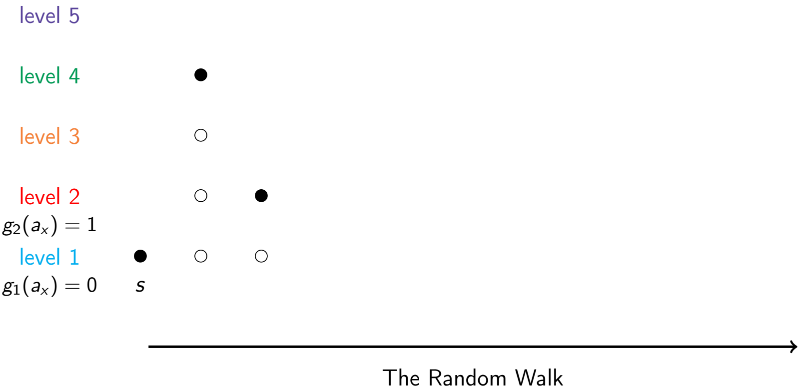
Our Construction via Iterative Restriction



Our Construction via Iterative Restriction



Our Construction via Iterative Restriction



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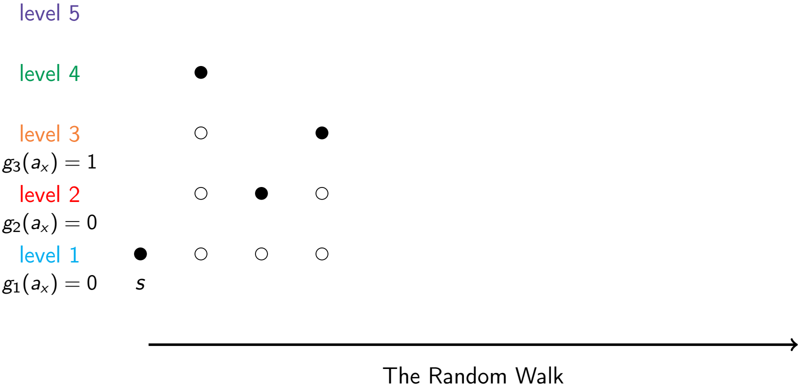
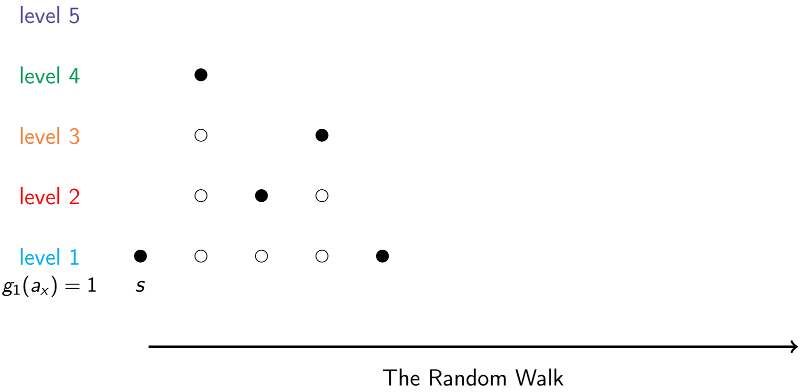
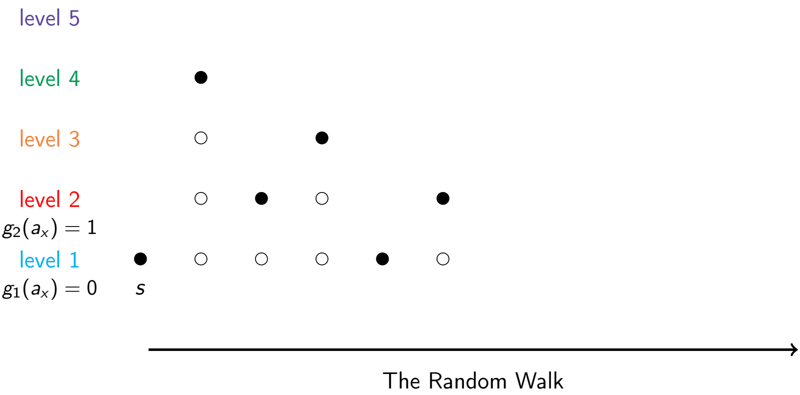


Our Construction via Iterative Restriction



Our Construction via Iterative Restriction



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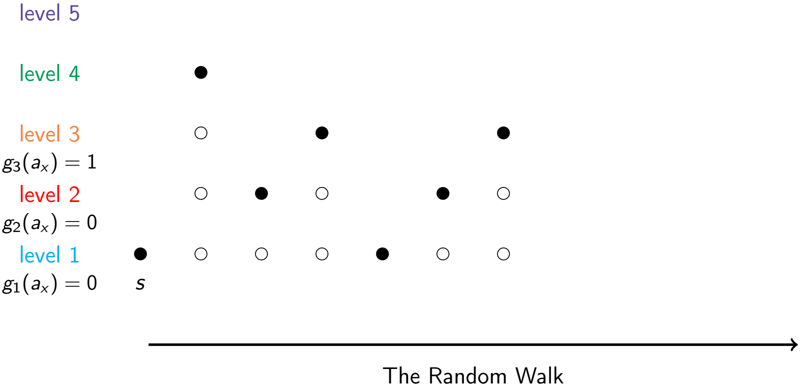

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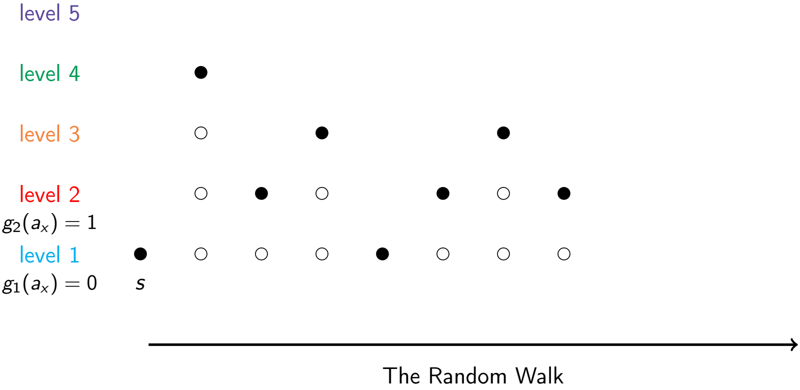

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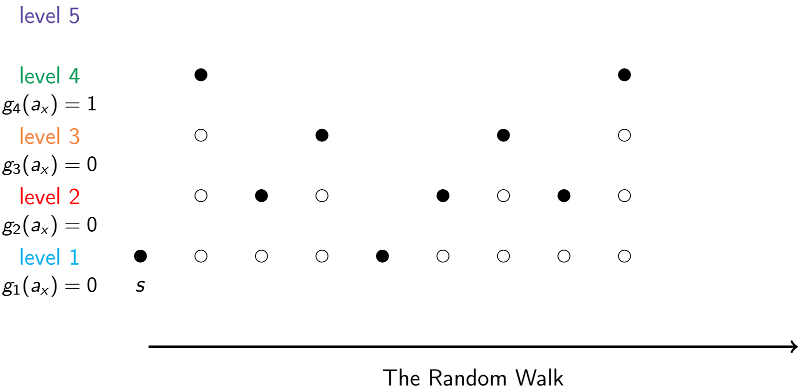

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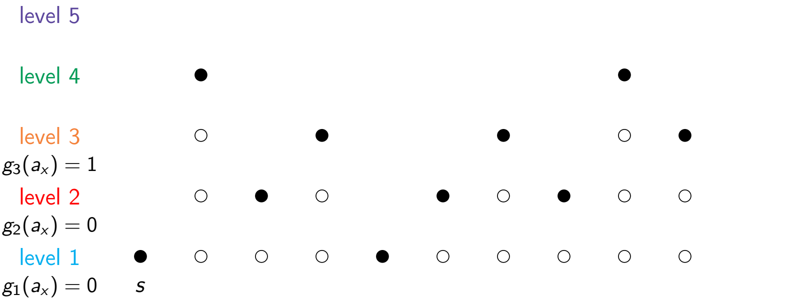

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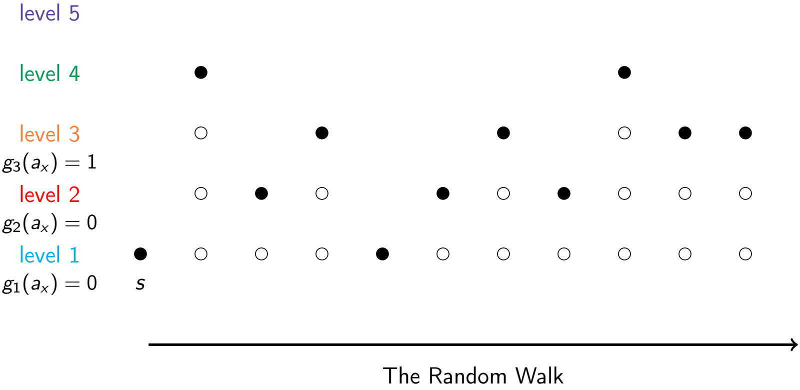

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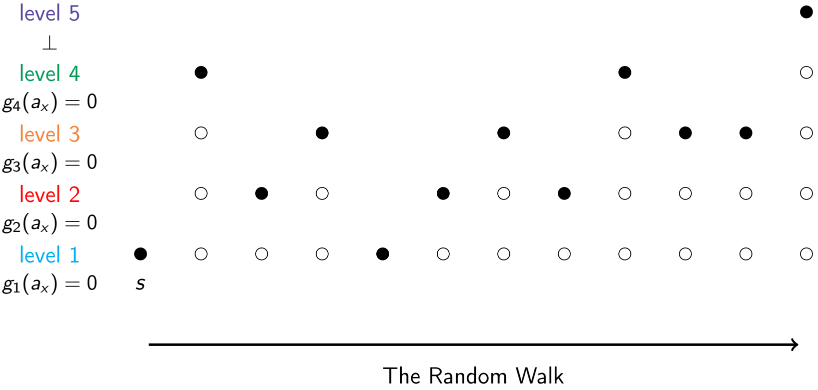
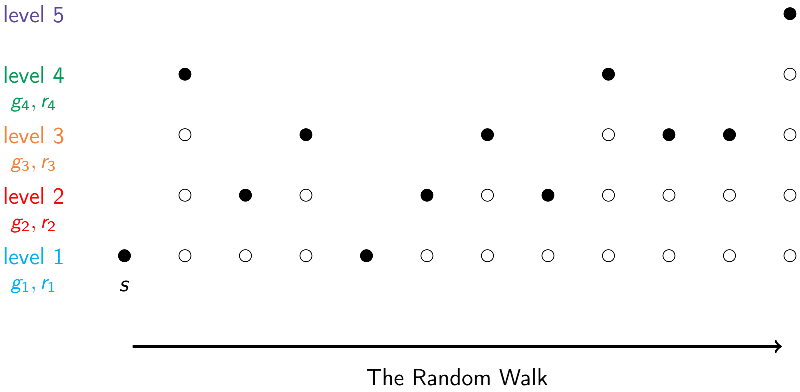
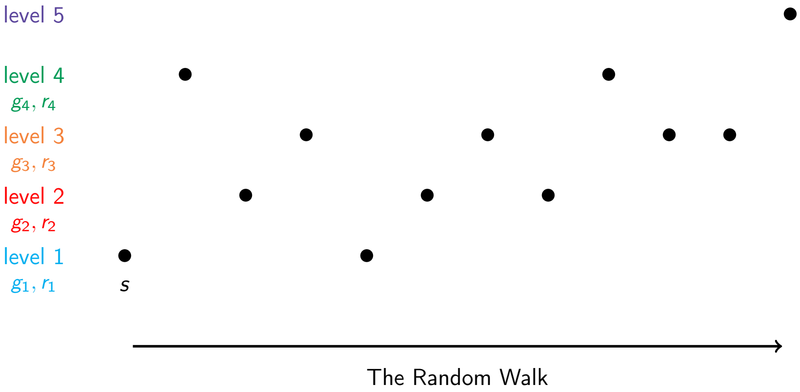


Key Ideas in Our Analysis

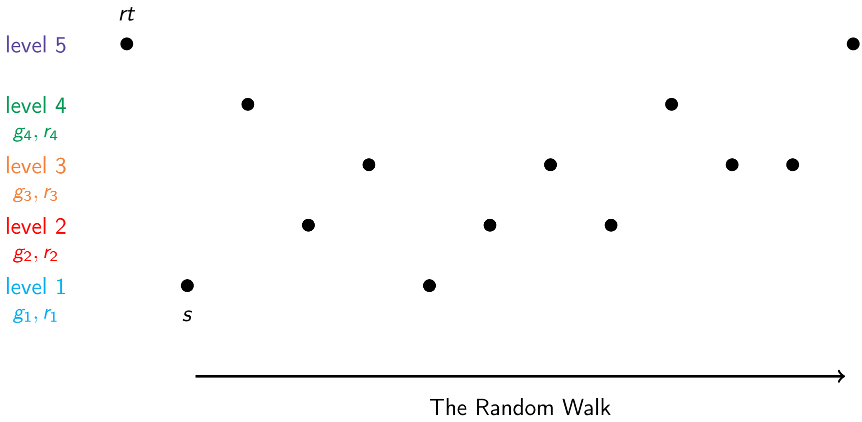
Dependency Tree



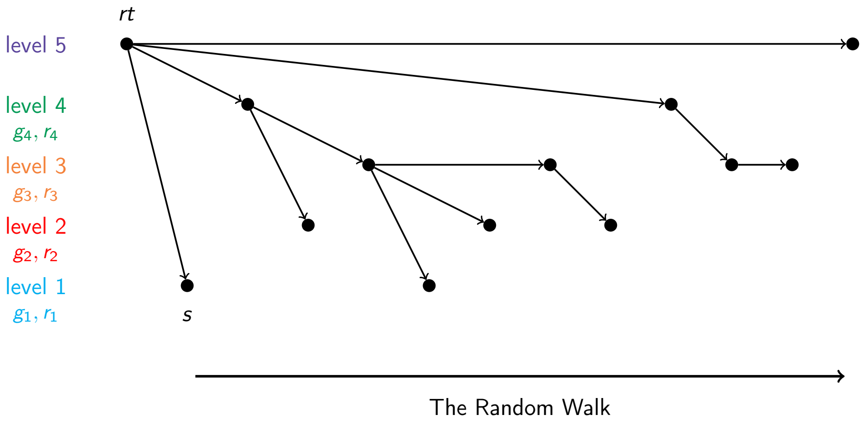
Dependency Tree



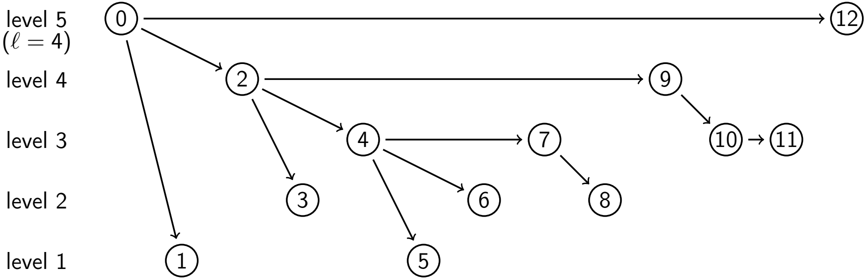
Dependency Tree



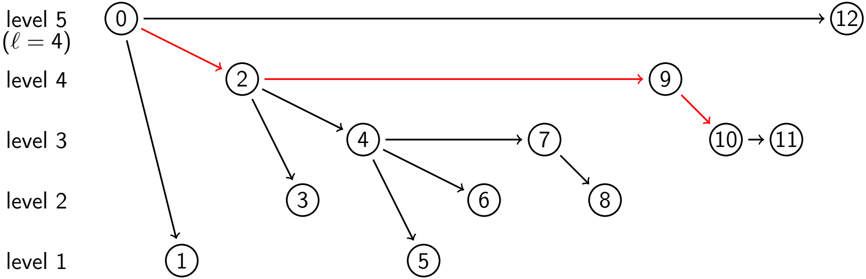
Dependency Tree



Dependency Tree

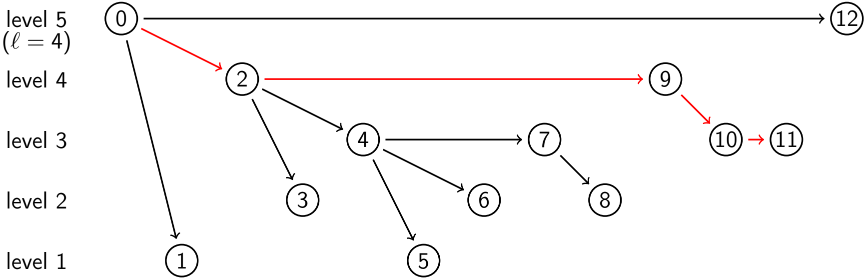


Dependency Tree



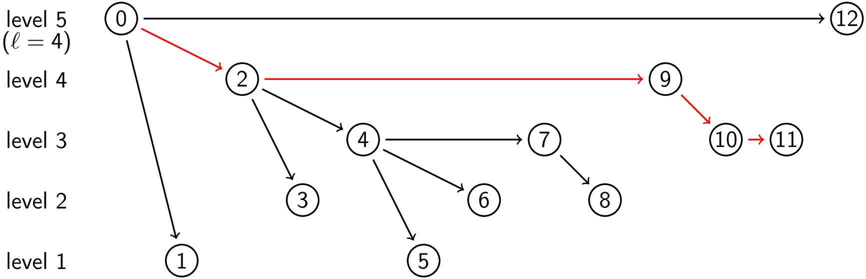
- We index a node by the shape of its path, e.g. $\vec{k}_{10} = (0, 0, 1, 2)$.

Dependency Tree



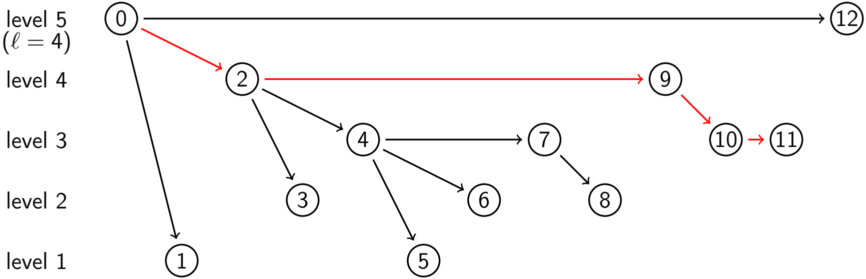
- We index a node by the shape of its path, e.g. $\vec{k}_{11} = (0, 0, 2, 2)$.

Dependency Tree



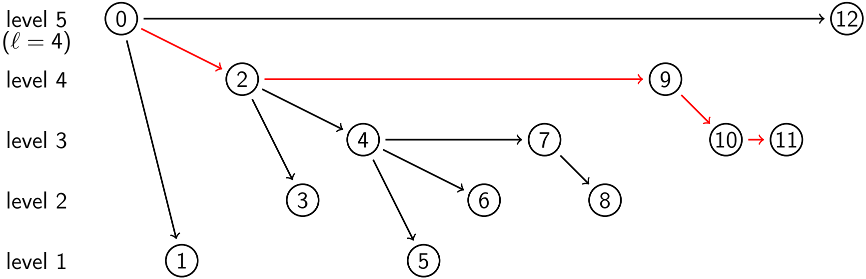
- We index a node by the shape of its path, e.g. $\vec{k}_{11} = (0, 0, 2, 2)$.
- Consider $\vec{k}_x$. Fix x, $\vec{k}$ is a random variable. Fix $\vec{k}$, x is a random variable.

Dependency Tree



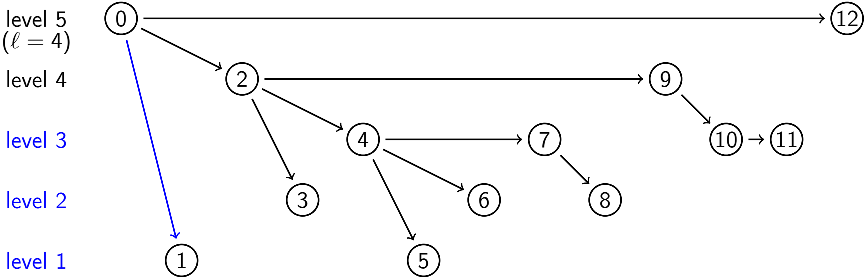
- We index a node by the shape of its path, e.g. $\vec{k}_{11} = (0, 0, 2, 2)$.
- Consider $\vec{k}_x$. Fix x, $\vec{k}$ is a random variable. Fix $\vec{k}$, x is a random variable.
- We fix index $\vec{k}$ and let $\mu^{\vec{k}} = x$ be the random variable (which may equal $\perp$).

Memory Eraser on Dependency Tree



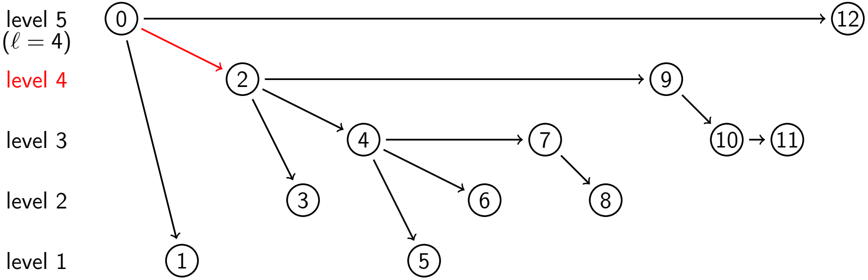
- Fix $\vec{k} = (0, 0, 2, 2)$.

Memory Eraser on Dependency Tree



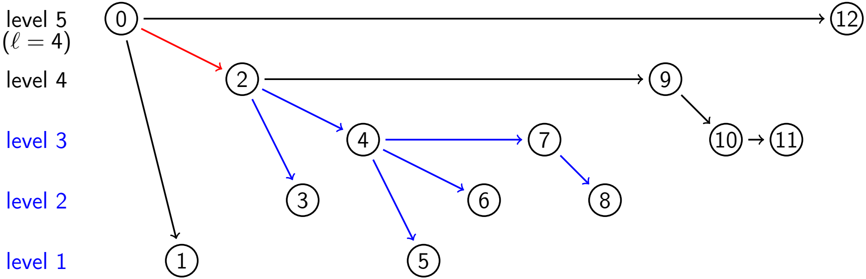
- Fix $\vec{k} = (0, 0, 2, 2)$.

Memory Eraser on Dependency Tree



- Fix $\vec{k} = (0, 0, 2, 2)$.

Memory Eraser on Dependency Tree



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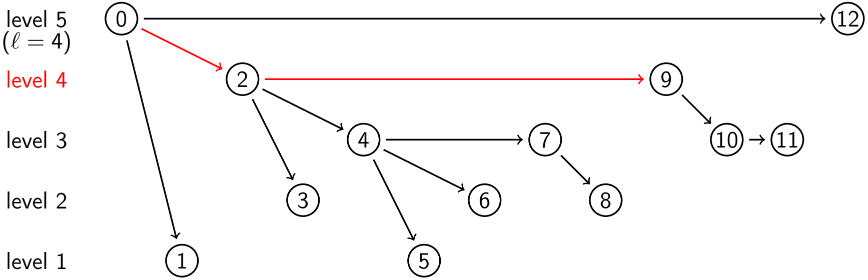
Memory Eraser on Dependency Tree



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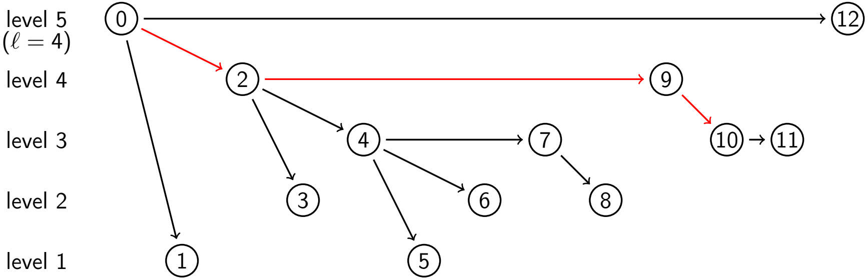
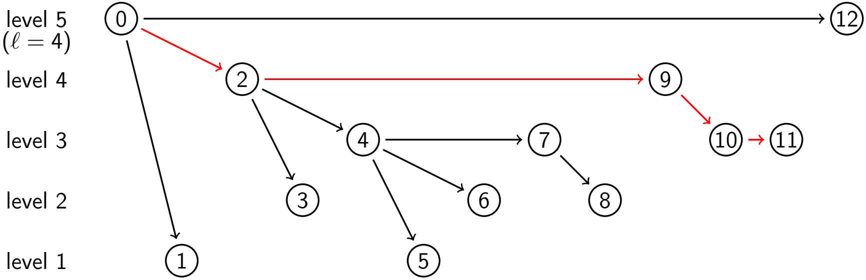
Memory Eraser on Dependency Tree



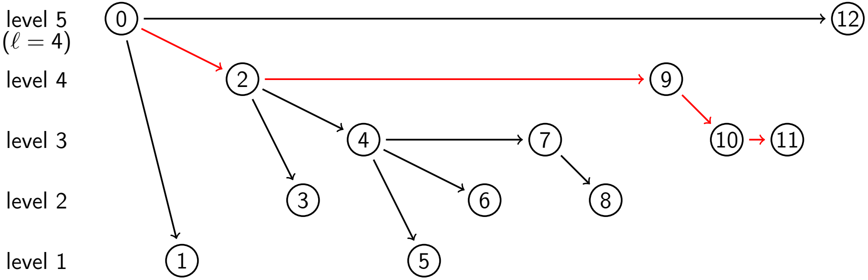
- Fix $\vec{k} = (0, 0, 2, 2)$.

Memory Eraser on Dependency Tree



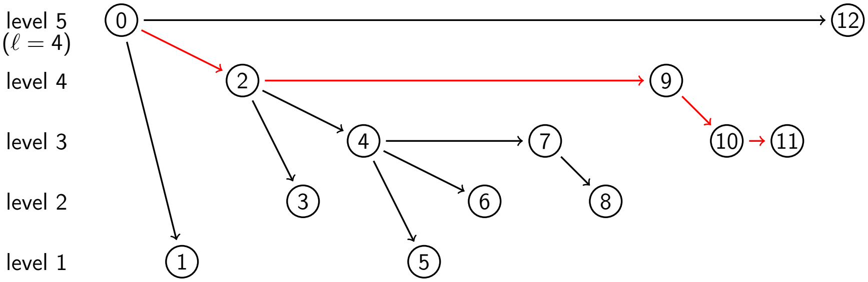
- Fix $\vec{k} = (0, 0, 2, 2)$.

Memory Eraser on Dependency Tree



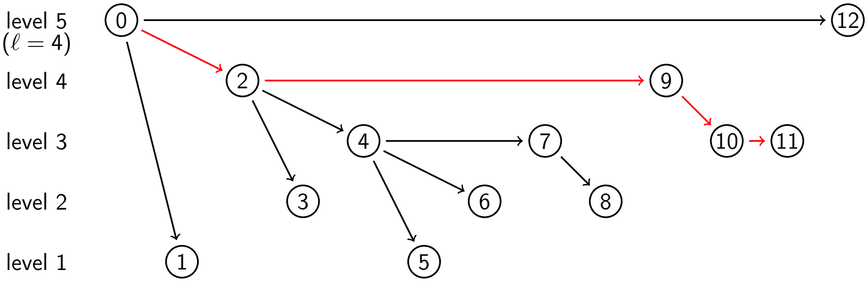
- Fix $\vec{k} = (0, 0, 2, 2)$.
- One Issue: What if $a_{x_2} = a_{x_9}$?

(Locally Simulatable) Extended Walk



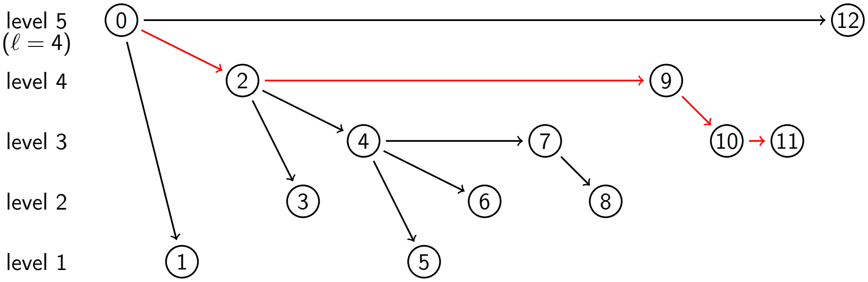
- Instead of original walk w, we looks at extended walk w^*.

(Locally Simulatable) Extended Walk



- Instead of original walk w, we looks at extended walk w^*.
- Once a collision occur in a path within a single level, we replace all the rest queries of that level with true randomness.

(Locally Simulatable) Extended Walk



- Instead of original walk w, we looks at extended walk w^*.
- Once a collision occur in a path within a single level, we replace all the rest queries of that level with true randomness.
- w and w^* agree before the first collision.

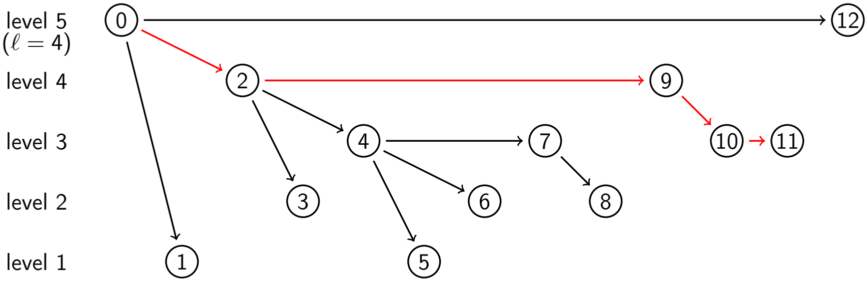
Good = All - Bad

Recall our goal.

Our Main Lemma

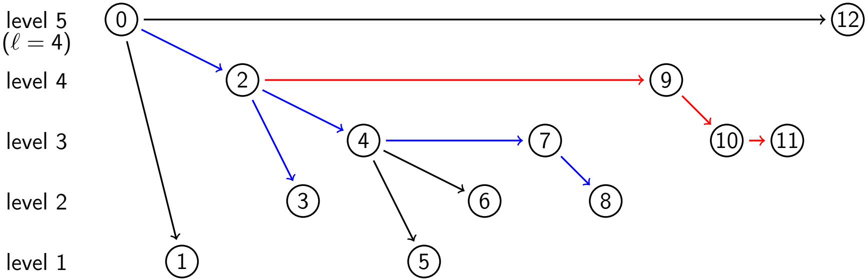
- $\mathbb{E}[|f^*(s)|] \leq O(\sqrt{n})$
- $\Pr[u \in f^*(s)] \geq \Omega(1/\sqrt{n}), \ \forall u \in [n]$

Good = All - Bad



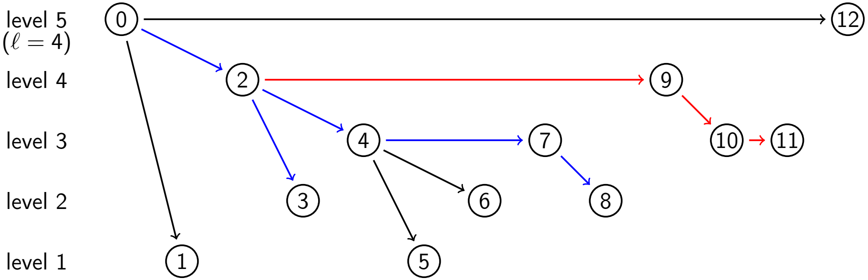
$$\sum_{\vec{k}} \frac{2^{-(k_1+k_2+\cdots+k_\ell)}}{n}$$

Good = All - Bad



$$\sum_{\vec{k}} \frac{2^{-(k_1+k_2+\cdots+k_\ell)}}{n} - \sum_{\vec{k},\vec{k}',\vec{k}''} \frac{2^{-\|\vec{k}\|_1-\|\vec{k}'\|_1-\|\vec{k}''\|_1}}{n^3}$$

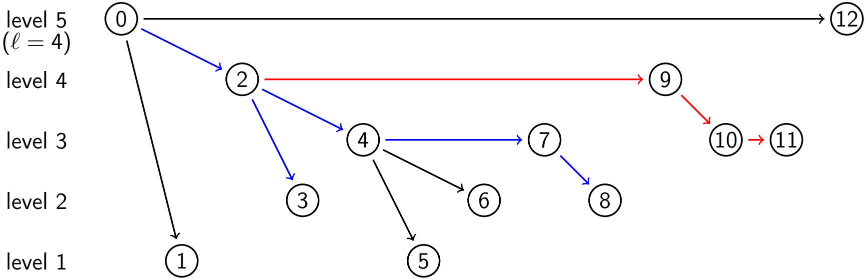
Good = All - Bad



$$\sum_{\vec{k}} \frac{2^{-(k_1+k_2+\cdots+k_\ell)}}{n} - \sum_{\vec{k},\vec{k}',\vec{k}''} \frac{2^{-\|\vec{k}\|_1-\|\vec{k}'\|_1-\|\vec{k}''\|_1}}{n^3}$$

$$\sum_{\vec{k}} \frac{2^{-(k_1+k_2+\cdots+k_\ell)}}{n} = \frac{1}{n}\prod_{i=1}^{\ell}\sum_{k_i=0}^{\infty} 2^{-k_i} = \frac{2^\ell}{n}$$

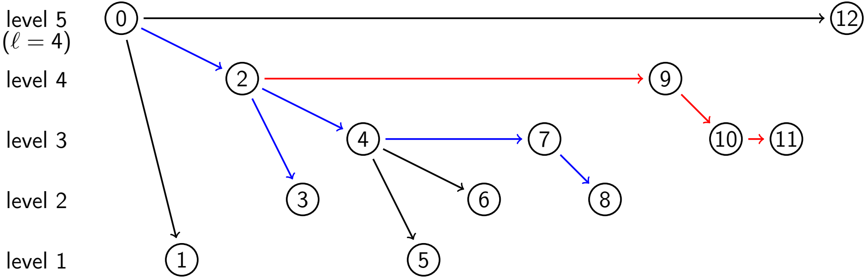
Good = All - Bad



$$\sum_{\vec{k}} \frac{2^{-(k_1+k_2+\cdots+k_\ell)}}{n} - \sum_{\vec{k},\vec{k}',\vec{k}''} \frac{2^{-\|\vec{k}\|_1-\|\vec{k}'\|_1-\|\vec{k}''\|_1}}{n^3}$$

$$\sum_{\vec{k},\vec{k}',\vec{k}''} \frac{2^{-\|\vec{k}\|_1-\|\vec{k}'\|_1-\|\vec{k}''\|_1}}{n^3} = \frac{8^\ell}{n^3}$$

Good = All - Bad

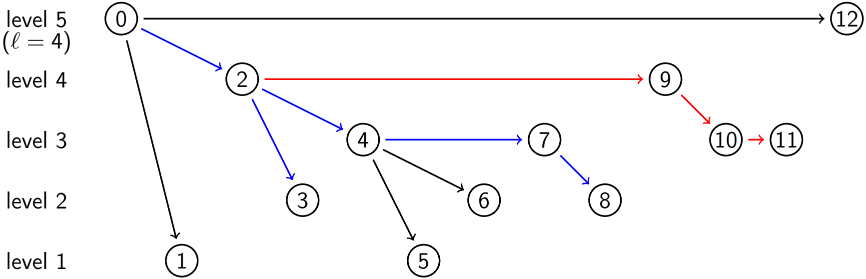


$$\sum_{\vec{k}} \frac{2^{-(k_1+k_2+\cdots+k_\ell)}}{n} - \sum_{\vec{k},\vec{k}',\vec{k}''} \frac{2^{-\|\vec{k}\|_1-\|\vec{k}'\|_1-\|\vec{k}''\|_1}}{n^3} = \frac{2^\ell}{n} - \frac{8^\ell}{n^3}$$

Let

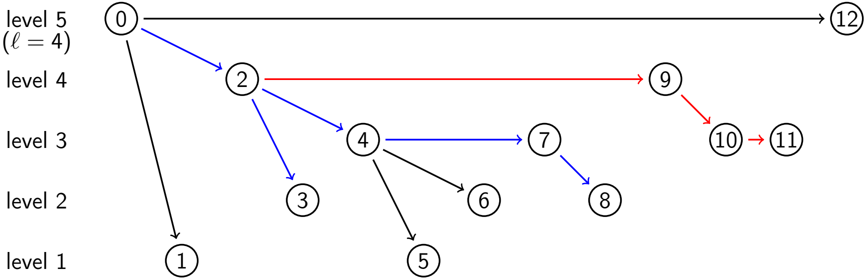
$$\ell \leftarrow \log n - 100$$

Collisions Cross Paths



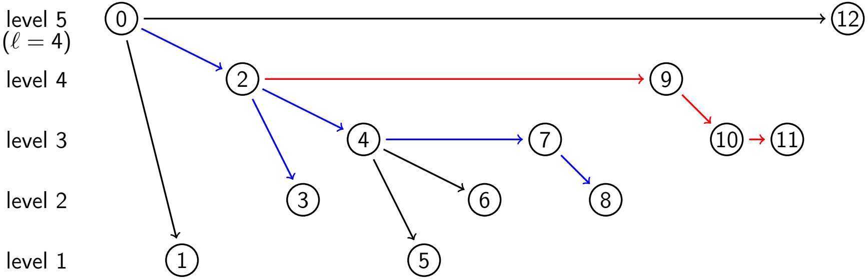
- Issue: What if $a_{x_3} = a_{x_8}$?

Collisions Cross Paths



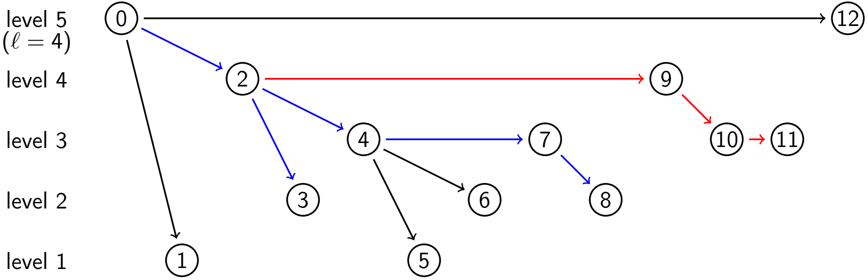
- Issue: What if $a_{x_3} = a_{x_8}$?
- Extended walk eliminate the collisions within a path.

Collisions Cross Paths



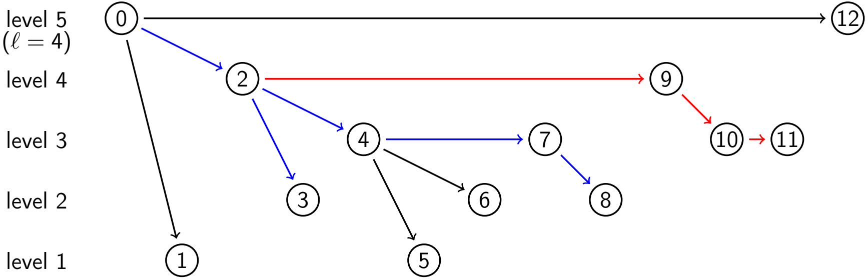
- Issue: What if $a_{x_3} = a_{x_8}$?
- Extended walk eliminate the collisions within a path.
- But there can still be collision between two paths.

Collisions Cross Paths



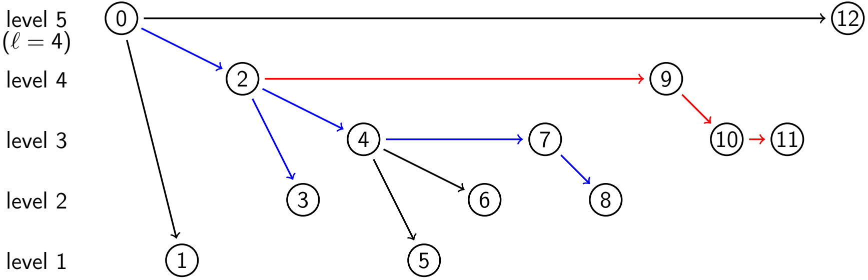
- Issue: What if $a_{x_3} = a_{x_8}$?

Collisions Cross Paths



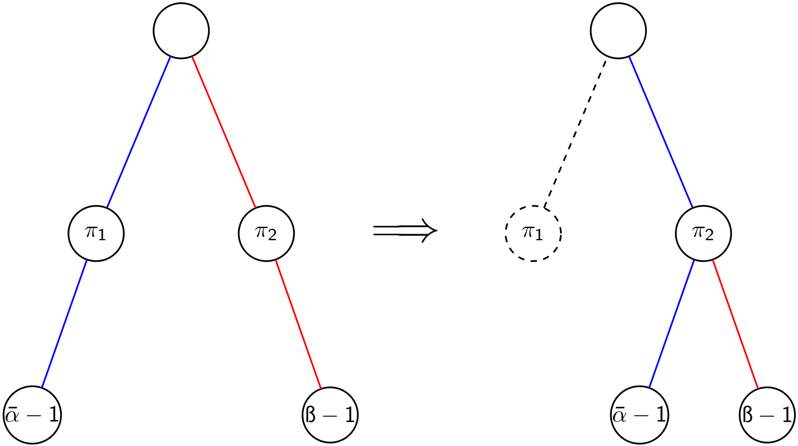
- Issue: What if $a_{x_3} = a_{x_8}$?
- We pick the first collision to be the blue paths.

Collisions Cross Paths



- Issue: What if there is collision between blue and red path?

Collisions Cross Paths



- We move the blue path when it has a collision with the red path.

Open Problems

Open Problems

- **Time-space Tradeoffs**
  In this work, we only solved the case when $S = \tilde{O}(1)$. Can we extend it to the full tradeoff?
- **Shorter Seed Length**
  In this work, our seed length is $O(\log^3 n \log \log n)$. Can this be improved?

References I

Miklos Ajtai and Avi Wigderson.
Deterministic simulation of probabilistic constant depth circuits.
In 26th Annual Symposium on Foundations of Computer Science (sfcs 1985), pages 11–19. IEEE, 1985.

Paul Beame, Raphaël Clifford, and Widad Machmouchi.
Element distinctness, frequency moments, and sliding windows.
In 2013 IEEE 54th Annual Symposium on Foundations of Computer Science, pages 290–299. IEEE, 2013.

Allan Borodin, Faith Fich, F Meyer Auf Der Heide, Eli Upfal, and Avi Wigderson.
A time-space tradeoff for element distinctness.
SIAM Journal on Computing, 16(1):97–99, 1987.

References II

Nikhil Bansal, Shashwat Garg, Jesper Nederlof, and Nikhil Vyas.
Faster space-efficient algorithms for subset sum, k-sum, and related problems.
SIAM Journal on Computing, 47(5):1755–1777, 2018.

Andrew Chi-Chih Yao.
Near-optimal time-space tradeoff for element distinctness.
In FOCS, pages 91–97, 1988.