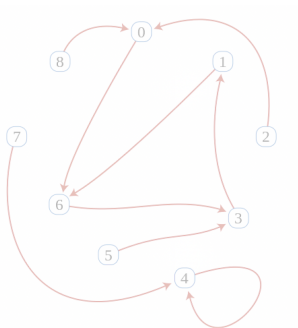




Element Distinctness, Birthday Paradox, and 1-out Pseudorandom Graphs

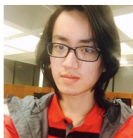


Hongxun Wu

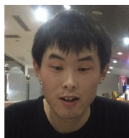
IIS, Tsinghua University

42	3	23	1	12	30	42	15
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Authors of this work



Lijie Chen



Ce Jin



R. Ryan Williams

我

Hongxun Wu

Lijie Chen, Ce Jin, and R. Ryan Williams are from MIT.

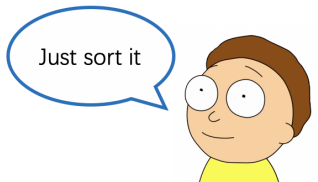
Element Distinctness

Element Distinctness

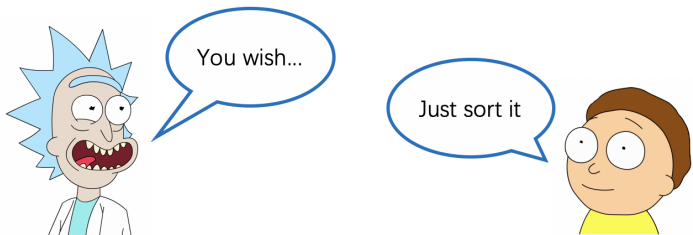


- INPUT: n positive integers a_1, a_2, \dots, a_n with $a_i \leq \text{poly}(n)$.
- Decide whether all a 's are distinct.

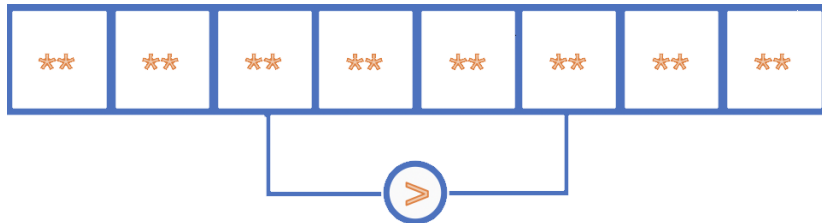
Element Distinctness



Element Distinctness

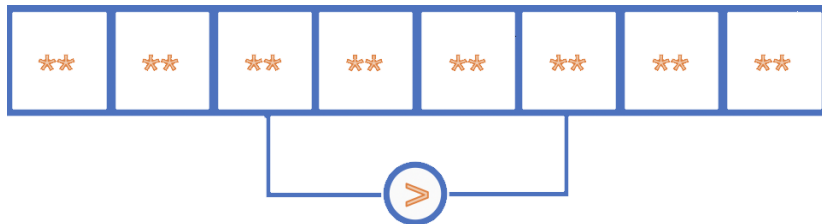


Comparison model



- No direct access to the INPUT a .
- Each query (i, j) returns one of $a_i < a_j$, $a_i = a_j$, $a_i > a_j$.

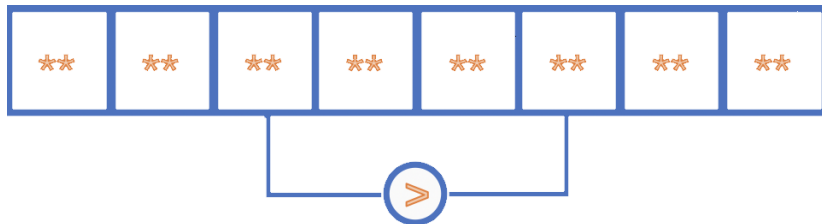
Comparison model



Time-Space tradeoff [BFMADH⁺87, Yao88]

Element distinctness requires $TS = \Omega(n^{2-o(1)})$ in Comparison model.

Comparison model



Time-Space tradeoff [BFMADH⁺87, Yao88]

Element distinctness requires $TS = \Omega(n^{2-o(1)})$ in Comparison model.

- When $S = O(\text{polylog } n)$, $T = \Omega(n^{2-o(1)})$.

RAM model



- Random access to read-only input.
- Working memory has a (relatively small) size S .

42	3	23	1	12	30	42	15
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Time-Space tradeoff [BCM13]

- Assuming the existence of *Random Oracle*, there is an algorithm with $T^2S = \tilde{O}(n^3)$.

42	3	23	1	12	30	42	15
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Time-Space tradeoff [BCM13]

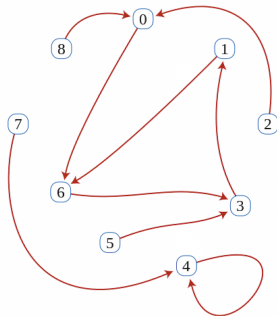
- Assuming the existence of *Random Oracle*, there is an algorithm with $T^2S = \tilde{O}(n^3)$.
- When $S = \tilde{O}(1)$, $T = \tilde{O}(n^{1.5})$.

1-out Graph and Birthday Paradox

Pollard's ρ method [BCM13]

Assuming the existence of *Random Oracle*, when $S = \tilde{O}(1)$, there is an algorithm with $T = \tilde{O}(n^{1.5})$.

- For random oracle R , define graph $x \mapsto R(a_x)$ with $x \in [n]$.

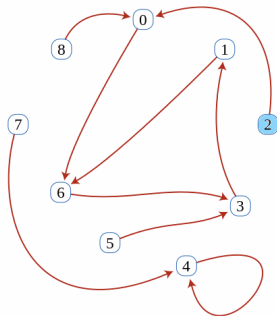


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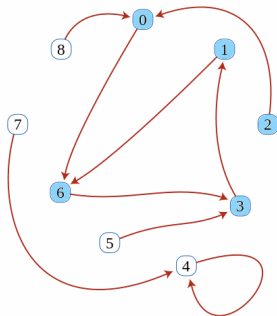


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- Run Floyd's cycle finding.

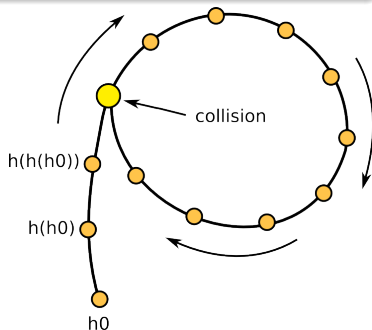


1-out Graph and Birthday Paradox

Birthday Paradox Type Properties [BCM13]

Suppose $f^*(s)$ is the set of vertices reachable from s .

- $\mathbb{E}[|f^*(s)|] \leq O(\sqrt{n})$

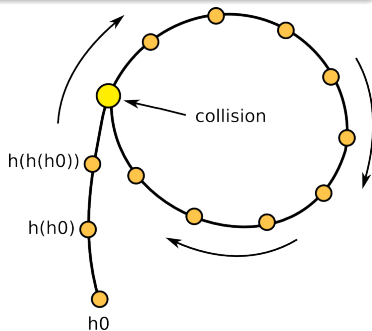


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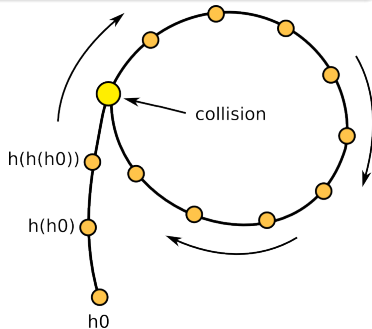


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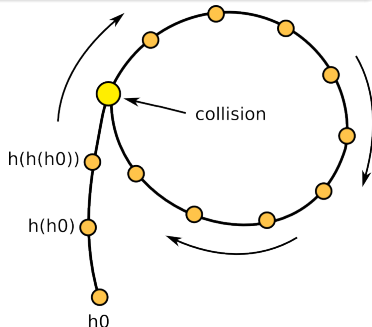


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-
- So each cycle-finding takes $O(\sqrt{n})$ time and finds any collision u, v with probability $\Omega(1/n)$.
 - Repeat $O(n)$ times, it takes $O(n^{1.5})$ time in total.



Our Main Lemma

There exists a family $\{r_{\text{seed}}\}$ of hash functions efficiently samplable with seed length $O(\text{polylog } n)$, and the graph defined by $\{r_{\text{seed}}\}$ (instead of *Random Oracle* R) satisfy

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Our Results

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Our Result

~~Assuming the existence of *Random Oracle*~~, when $S = O(\text{polylog } n)$, there is a RAM algorithm for Element Distinctness with $T = \tilde{O}(n^{1.5})$.

Low-space Algorithm for Subset Sum [BGNV18]

Assuming the existence of *Random Oracle*, Subset Sum and Knapsack can be solved by a Monte Carlo algorithm in $O^*(2^{0.86n})$ time, with $O(\text{poly}(n))$ space.

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Construction

Random Restriction and Håstad's Switching Lemma



This is Ryan O'Donnell's Youtube lecture which is a masterpiece.

Random Restriction and Håstad's Switching Lemma

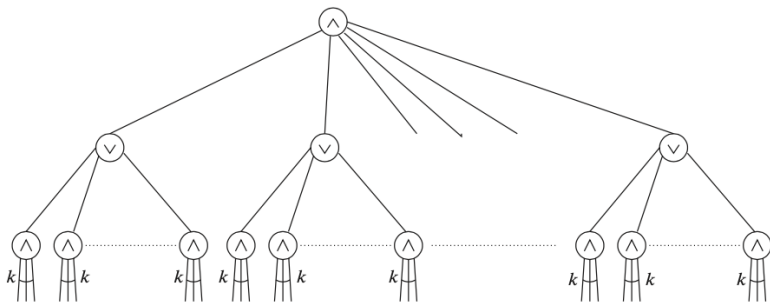


Figure 14.1. Circuit before Håstad switching transformation.

Figure from Arora & Barak.

Random Restriction and Håstad's Switching Lemma

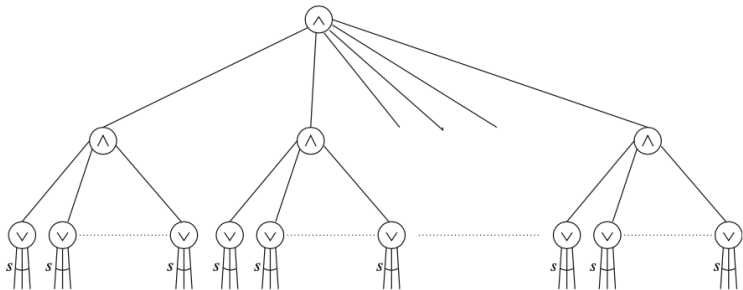
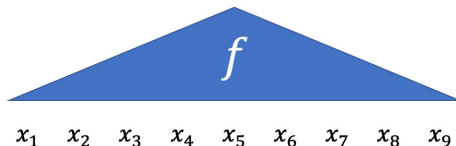


Figure 14.2. Circuit after Håstad switching transformation. Notice that the new layer of \wedge gates can be collapsed with the single \wedge parent gate, to reduce the number of levels by one.

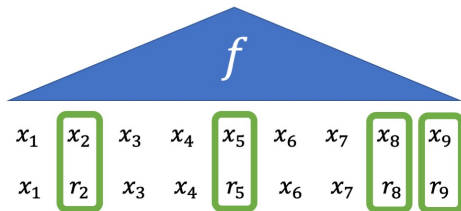
Figure from Arora & Barak.

Iterative Restriction



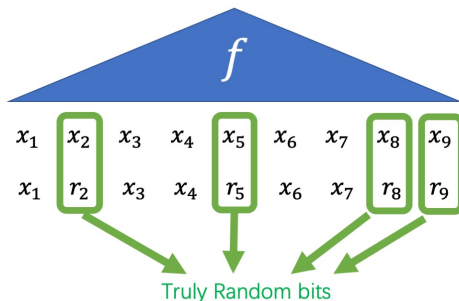
This is the Ajtai-Wigderson Paradigm [AW85] for building PRG.

Iterative Restriction



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Iterative Restriction



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Toy Example: Two levels

Recall the input $a_1, a_2, \dots, a_n \in [m]$.

Two Level Example

Suppose we have the following:

- $O(\text{polylog } n)$ -wise independent functions $g : [m] \rightarrow \{0, 1\}$ and $r : [m] \rightarrow [n]$.
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$$h(a_x) = \begin{cases} R(a_x) & g(a_x) = 0 \\ r(a_x) & g(a_x) = 1 \end{cases}$$

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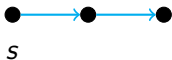
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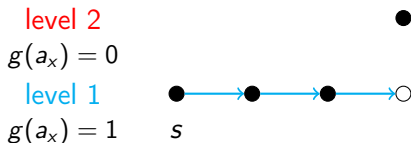


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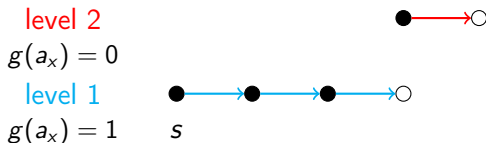


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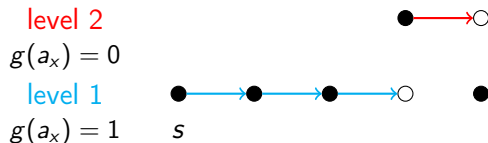


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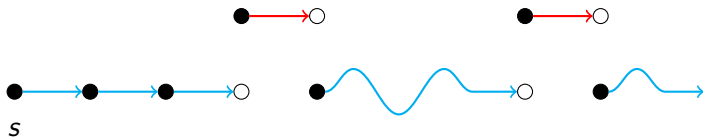
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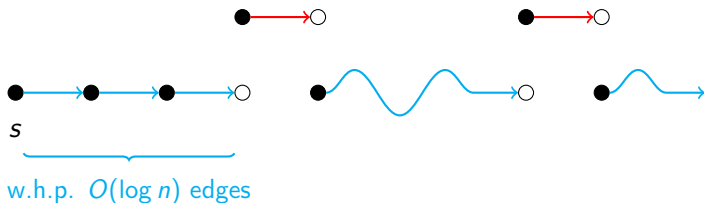
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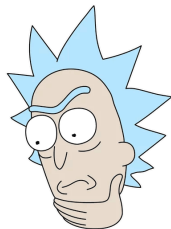
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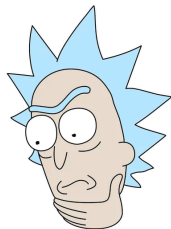


Sanity Check

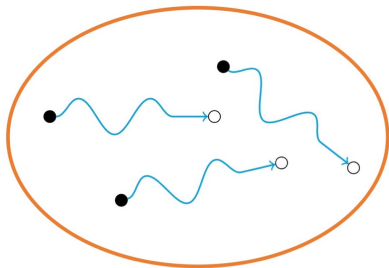


- Why this might be a good idea?

Sanity Check

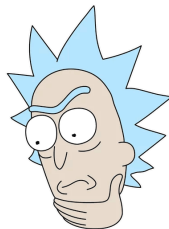


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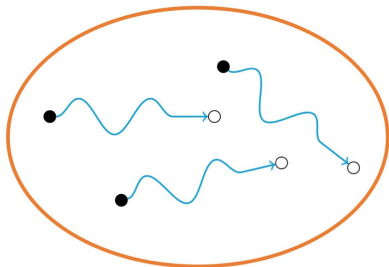


- Each subpath has length $O(\log n)$.

Sanity Check

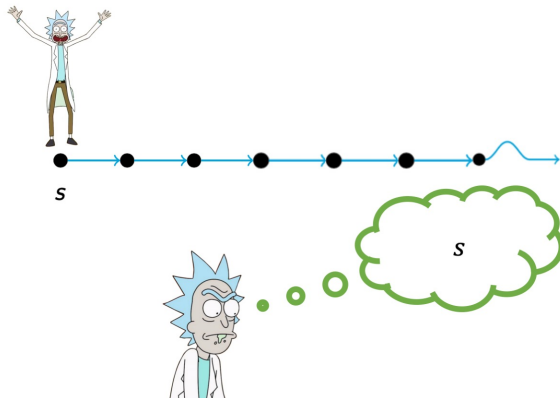


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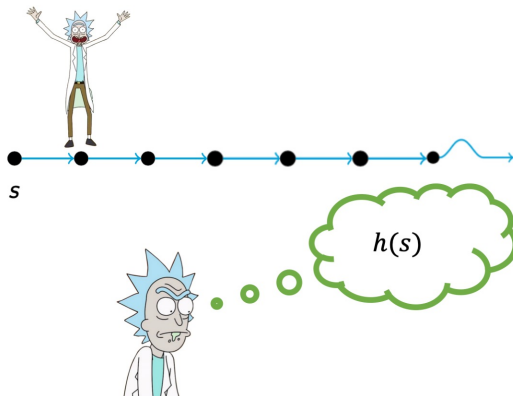


- Each subpath has length $O(\log n)$.
- Every **level 2** edge is an independent sample of a subpath.

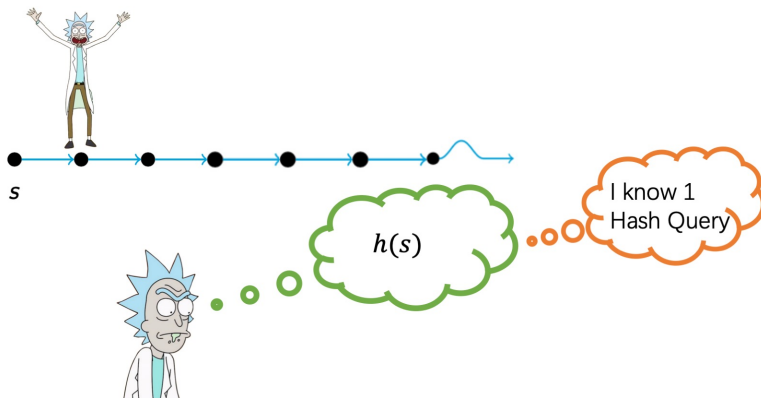
Intuition: Memory Eraser



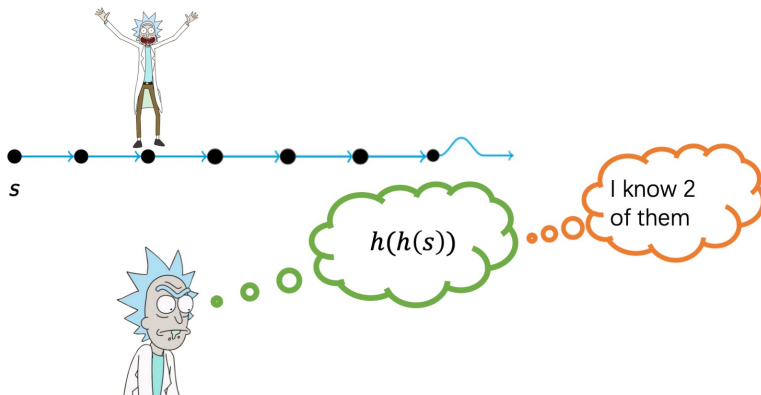
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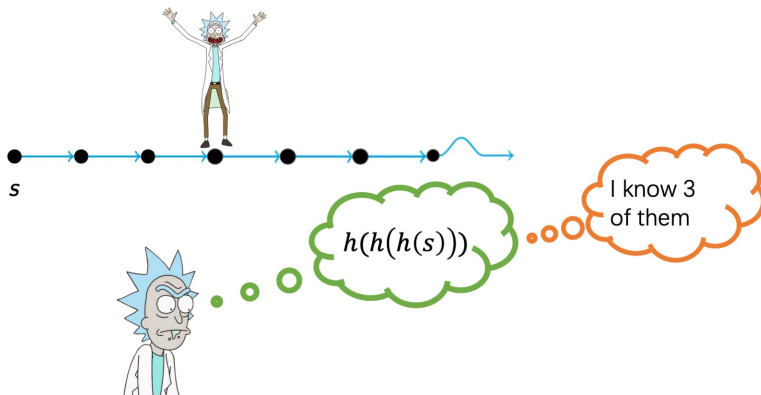
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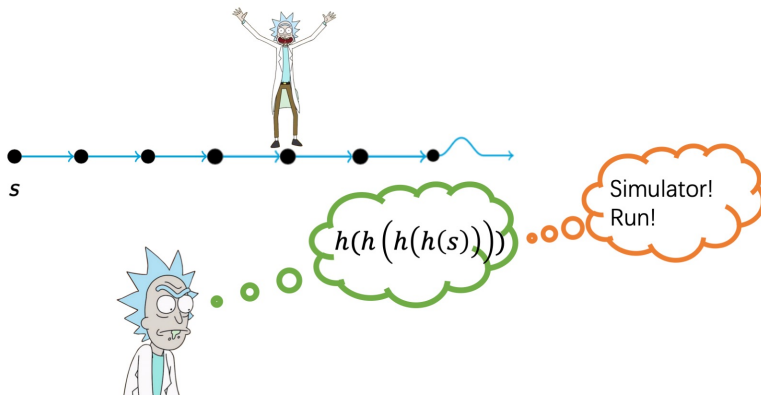
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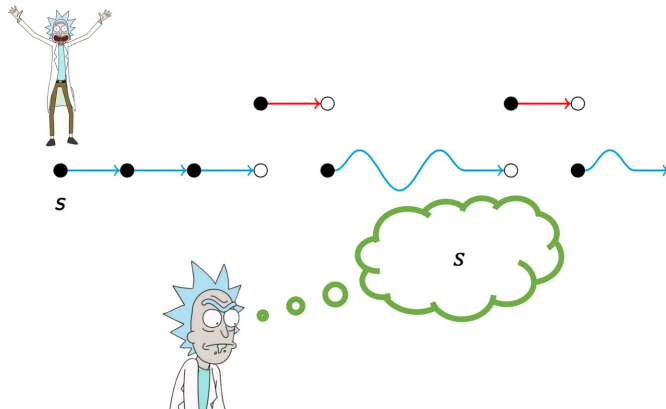
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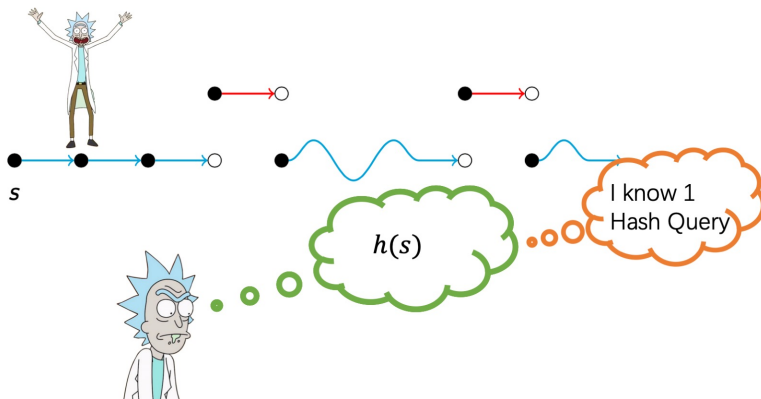
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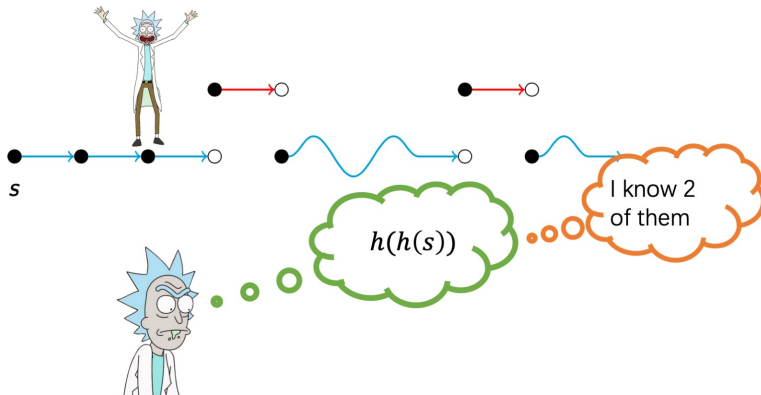
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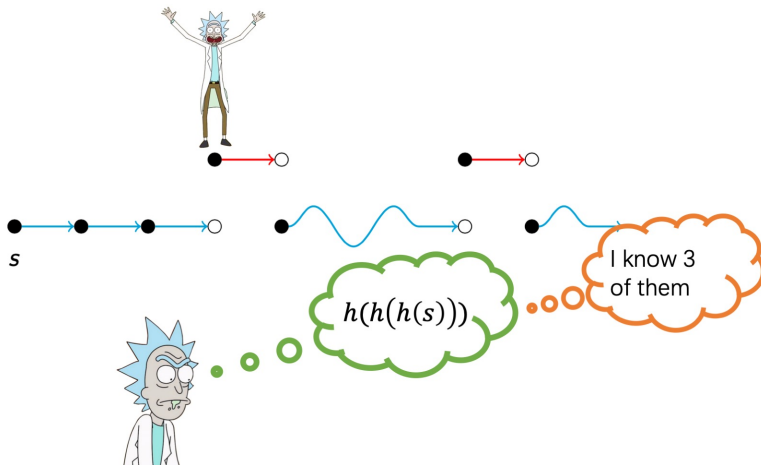
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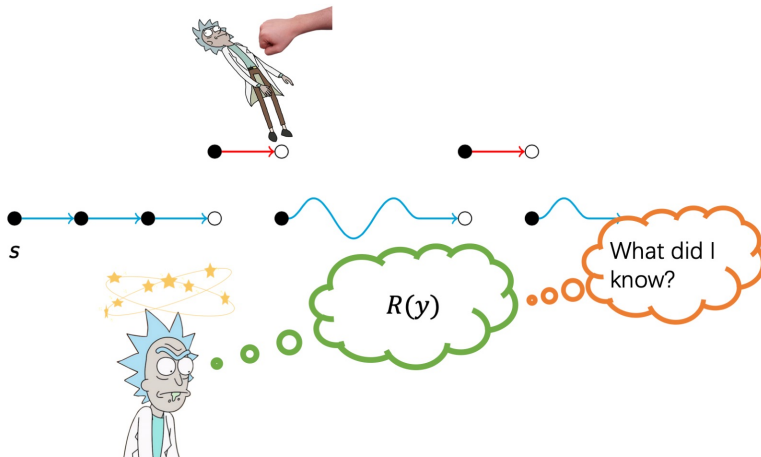
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Our Construction via Iterative Restriction

Our Construction

Now we sample $O(\log n)$ many hash functions $\{r_i, g_i\}_{i \in [\ell]}$.

Each $r_i : [m] \rightarrow [n]$ and $g_i : [m] \rightarrow [2]$ are $O(\log n)$ -wise independent.

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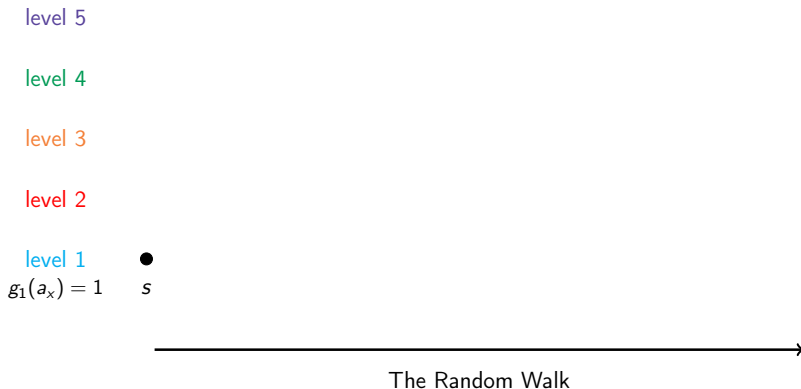
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Then we set $h_{\ell+1}(a_x) = \perp$ and

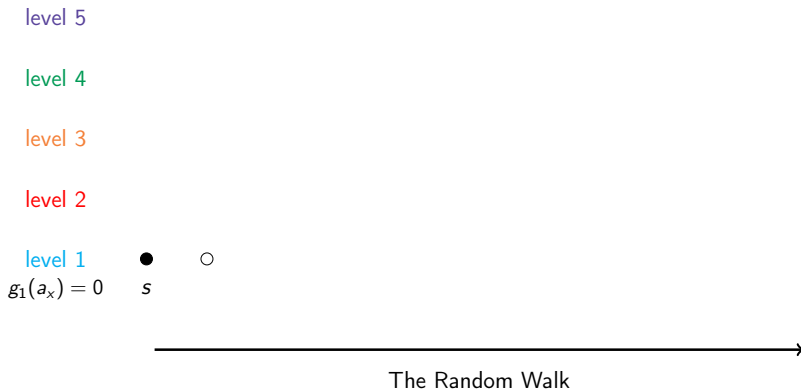
$$h_i(a_x) = \begin{cases} h_{i+1}(a_x) & g_i(a_x) = 0 \\ r_i(a_x) & g_i(a_x) = 1 \end{cases}$$

Finally, we set $h = h_1$.

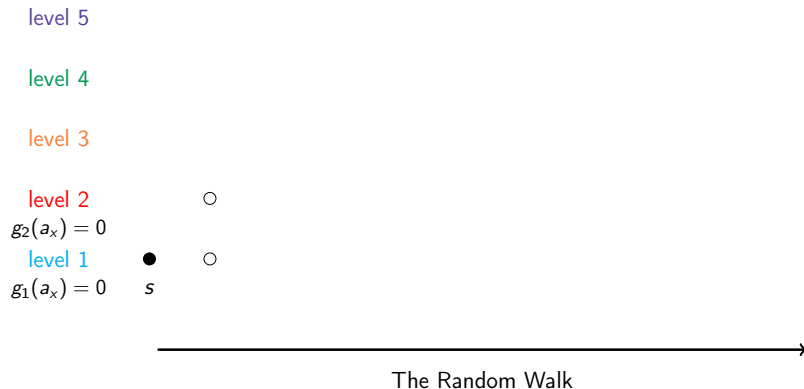
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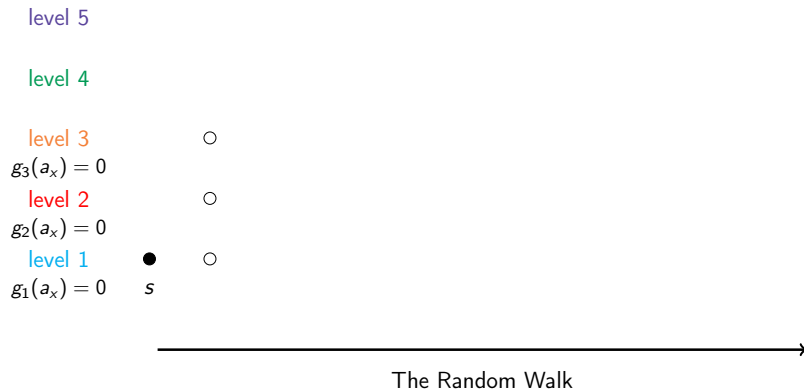
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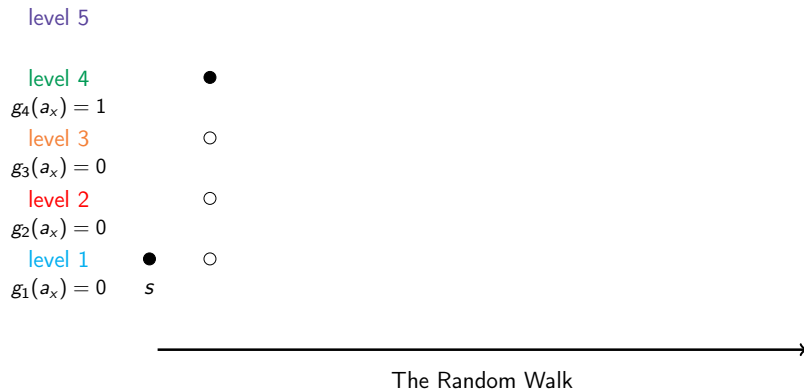
Our Construction via Iterative Restriction



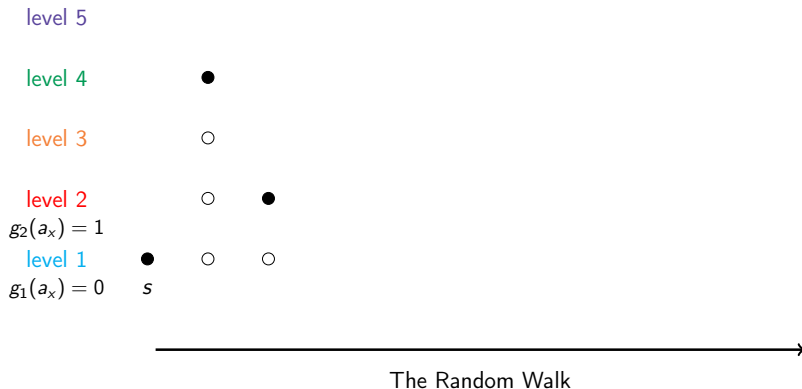
Our Construction via Iterative Restriction



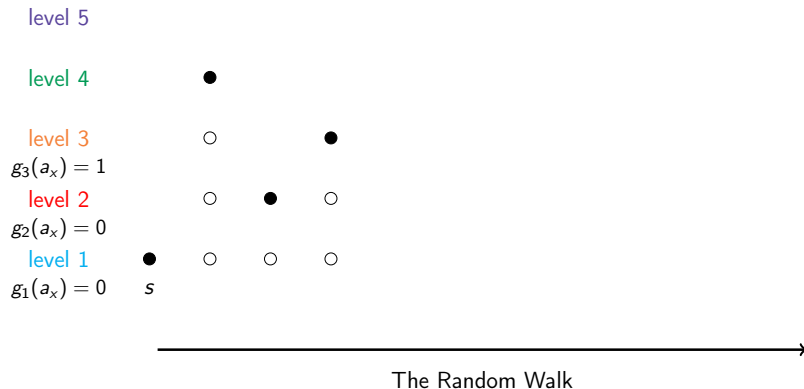
Our Construction via Iterative Restriction



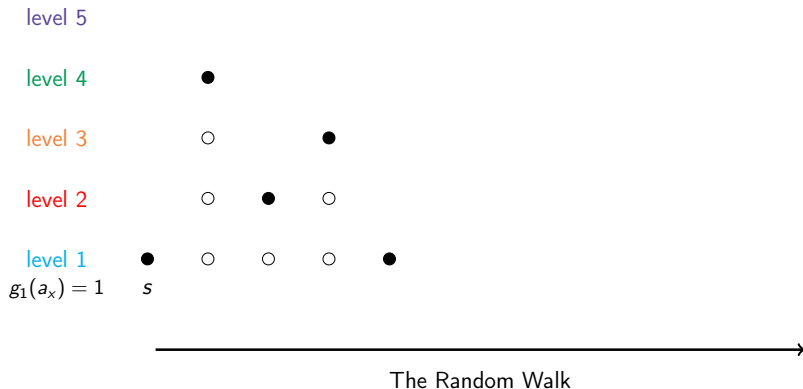
Our Construction via Iterative Restriction



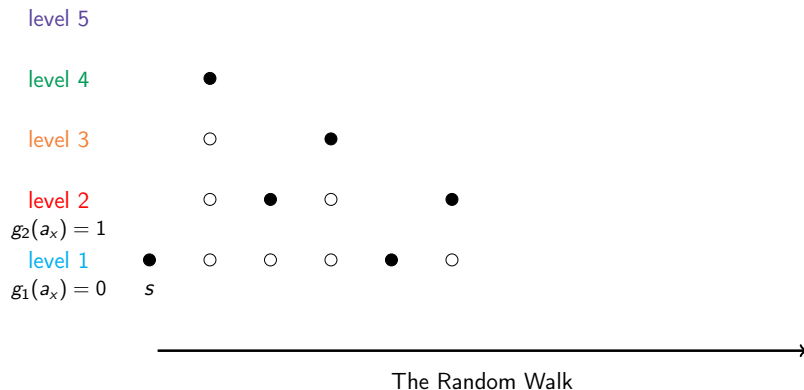
Our Construction via Iterative Restriction



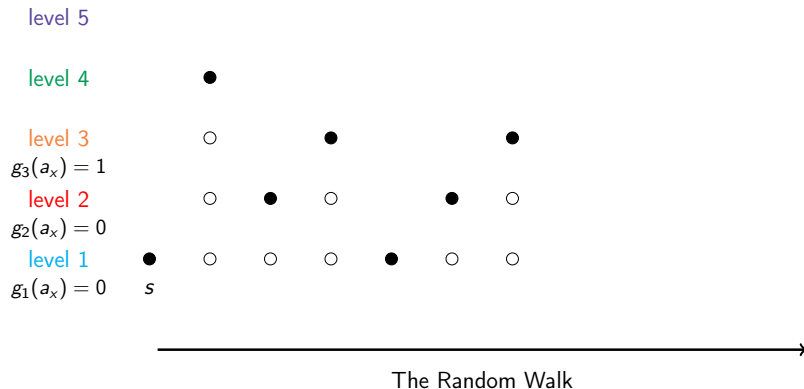
Our Construction via Iterative Restriction



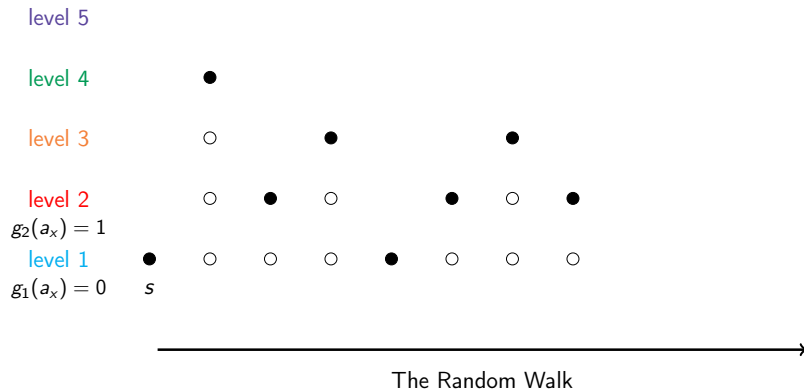
Our Construction via Iterative Restriction



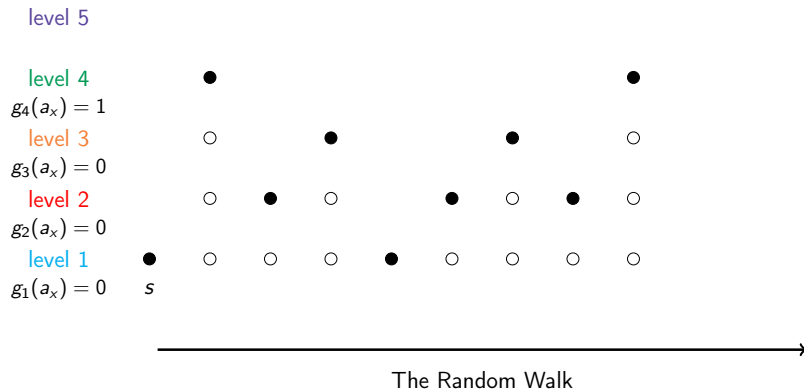
Our Construction via Iterative Restriction



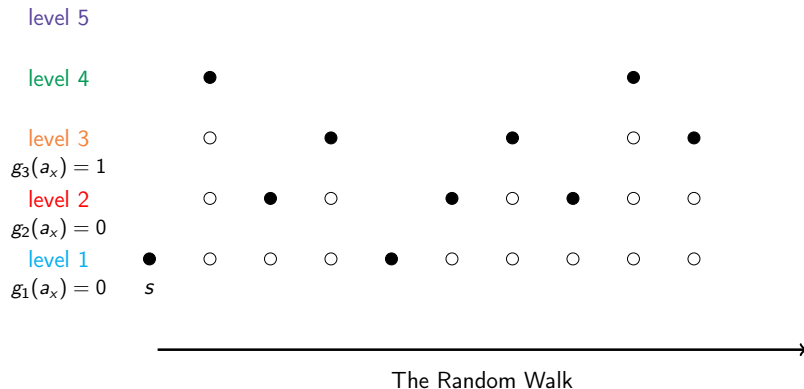
Our Construction via Iterative Restriction



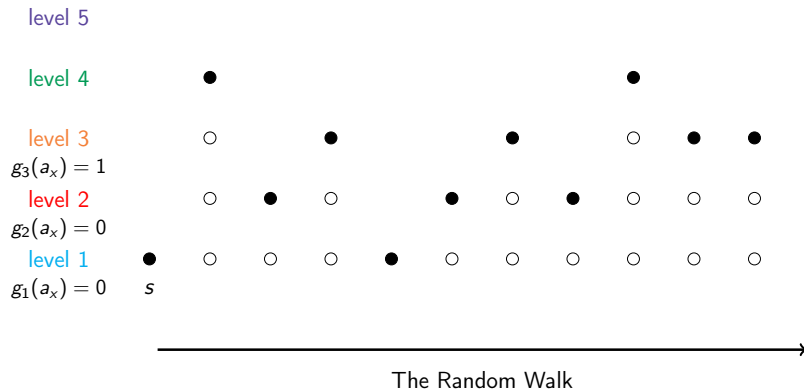
Our Construction via Iterative Restriction



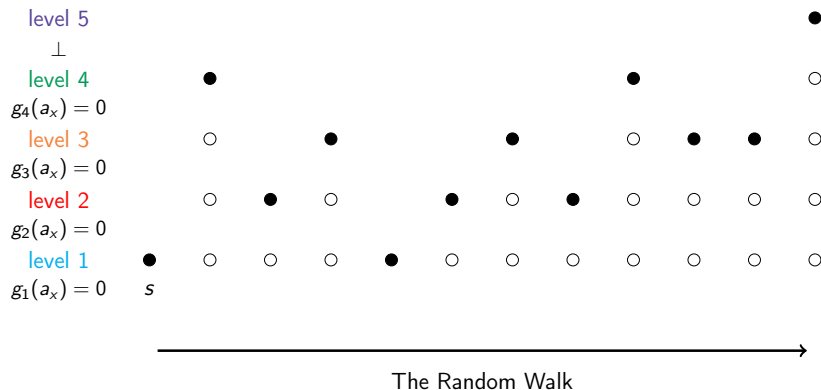
Our Construction via Iterative Restriction



Our Construction via Iterative Restriction

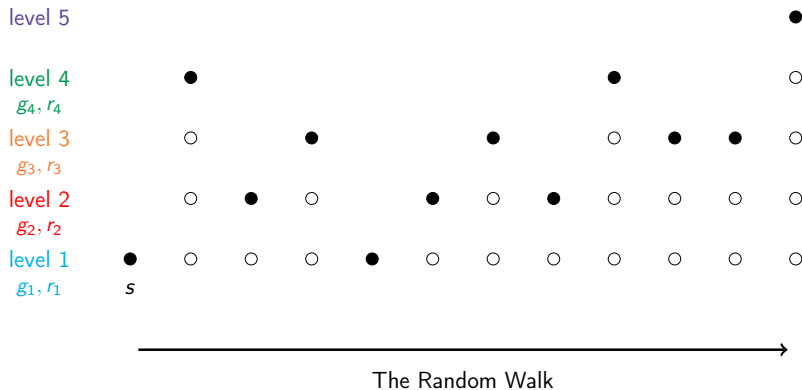


Our Construction via Iterative Restriction

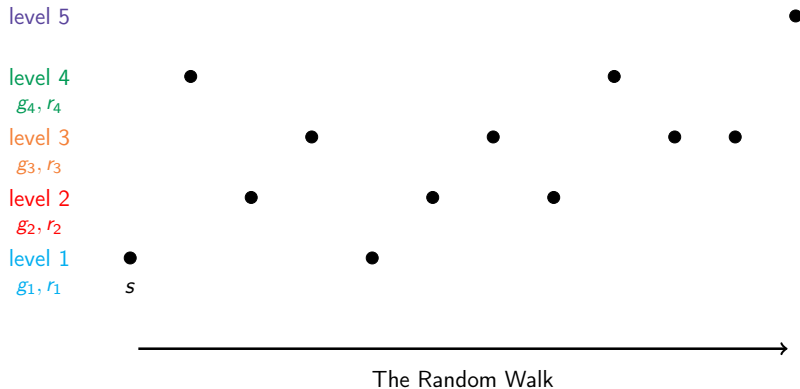


Key Ideas in Our Analysis

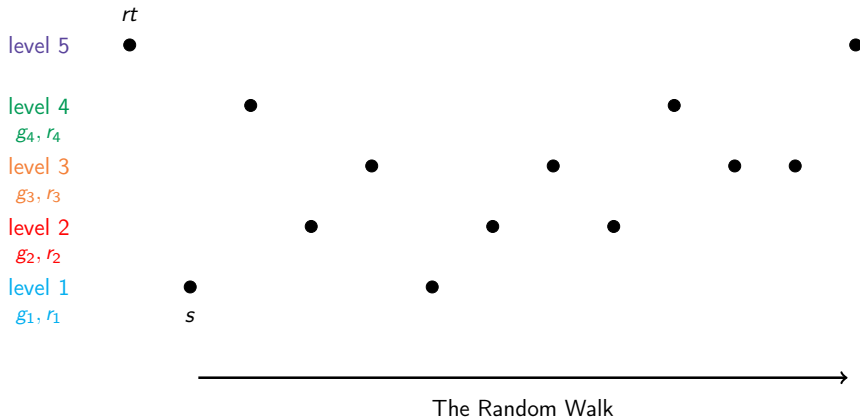
Dependency Tree



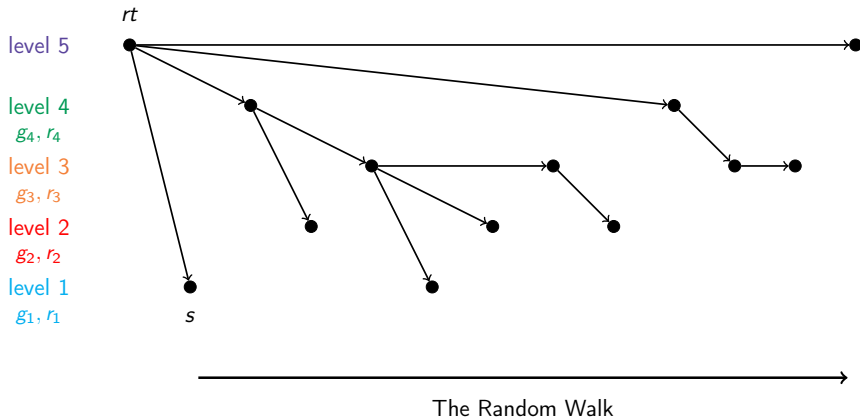
Dependency Tree



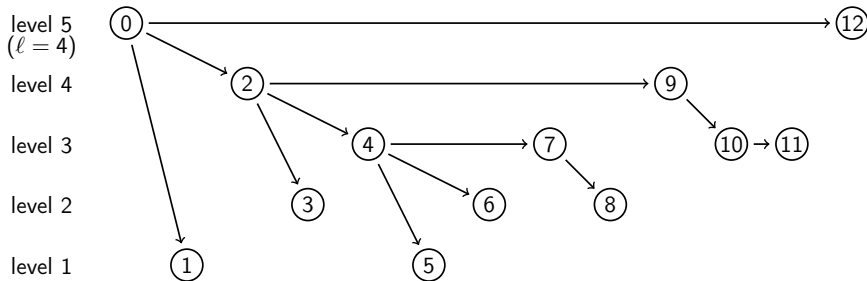
Dependency Tree



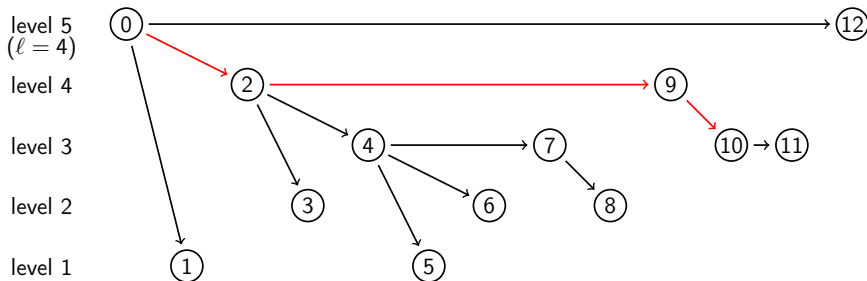
Dependency Tree



Dependency Tree

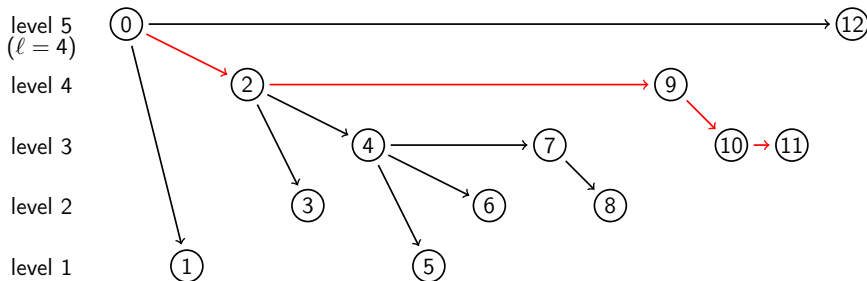


Dependency Tree



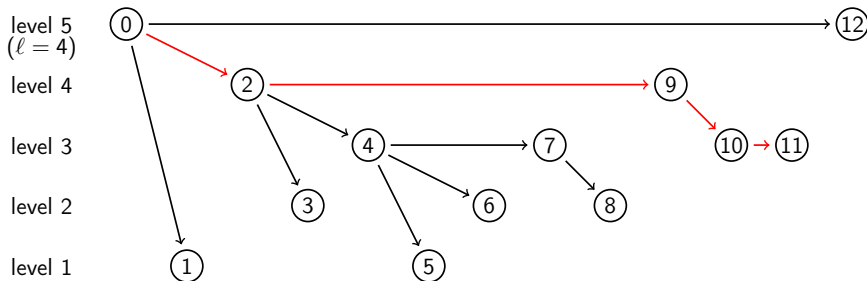
- We index a node by the shape of its path, e.g. $\vec{k}_{10} = (0, 0, 1, 2)$.

Dependency Tree



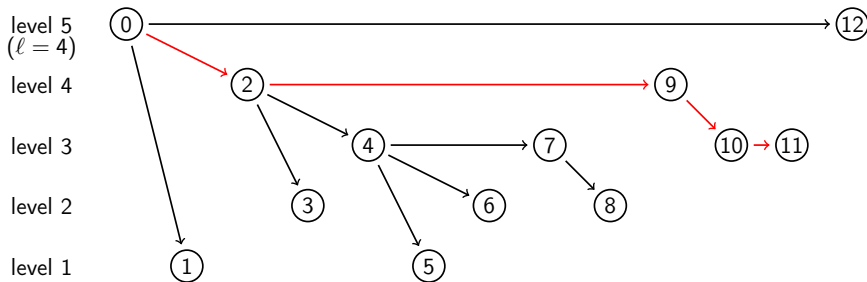
- We index a node by the shape of its path, e.g. $\vec{k}_{11} = (0, 0, 2, 2)$.

Dependency Tree



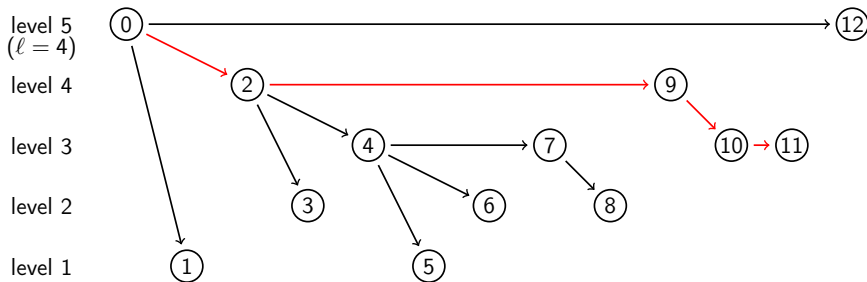
- We index a node by the shape of its path, e.g. $\vec{k}_{11} = (0, 0, 2, 2)$.
- Consider \vec{k}_x . Fix x , \vec{k} is a random variable. Fix \vec{k} , x is a random variable.

Dependency Tree



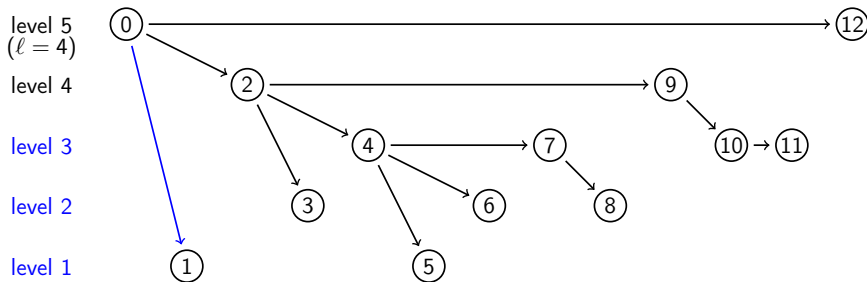
- We index a node by the shape of its path, e.g. $\vec{k}_{11} = (0, 0, 2, 2)$.
- Consider \vec{k}_x . Fix x , \vec{k} is a random variable. Fix \vec{k} , x is a random variable.
- We fix index \vec{k} and let $\mu^{\vec{k}} = x$ be the random variable (which may equal \perp).

Memory Eraser on Dependency Tree



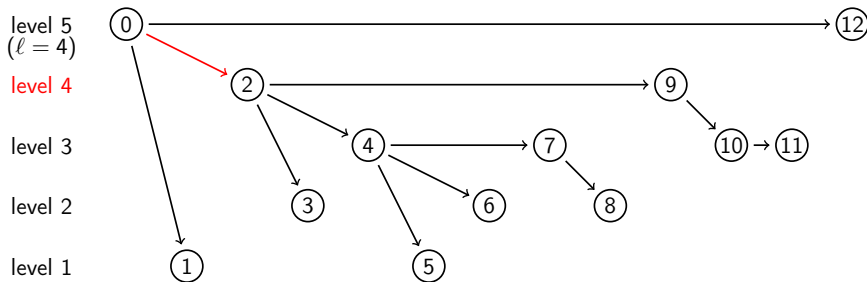
- Fix $\vec{k} = (0, 0, 2, 2)$.

Memory Eraser on Dependency Tree



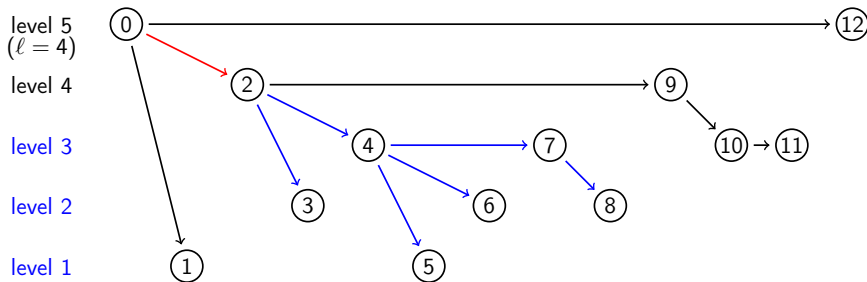
- Fix $\vec{k} = (0, 0, 2, 2)$.

Memory Eraser on Dependency Tree



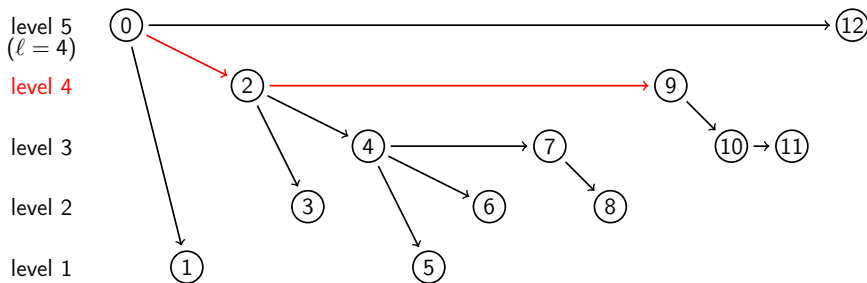
- Fix $\vec{k} = (0, 0, 2, 2)$.

Memory Eraser on Dependency Tree



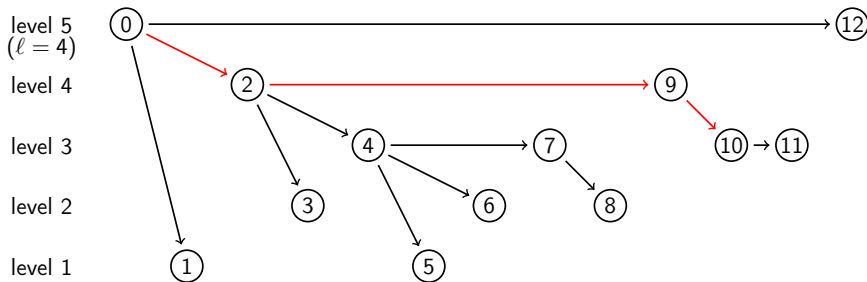
- Fix $\vec{k} = (0, 0, 2, 2)$.

Memory Eraser on Dependency Tree



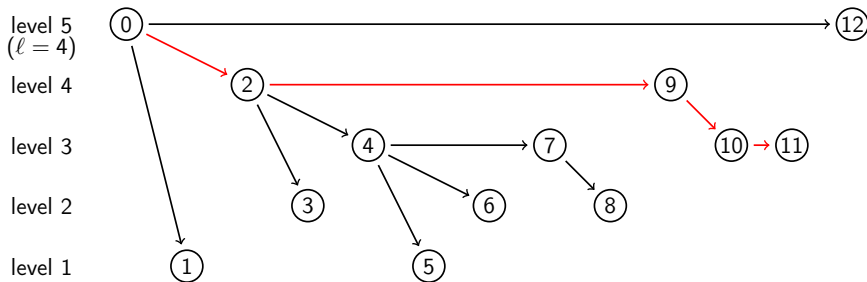
- Fix $\vec{k} = (0, 0, 2, 2)$.

Memory Eraser on Dependency Tree



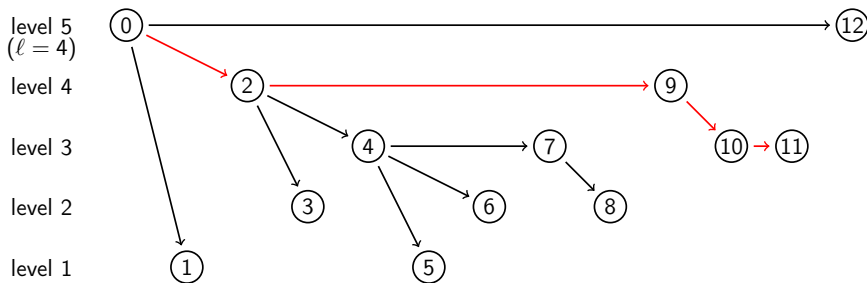
- Fix $\vec{k} = (0, 0, 2, 2)$.

Memory Eraser on Dependency Tree



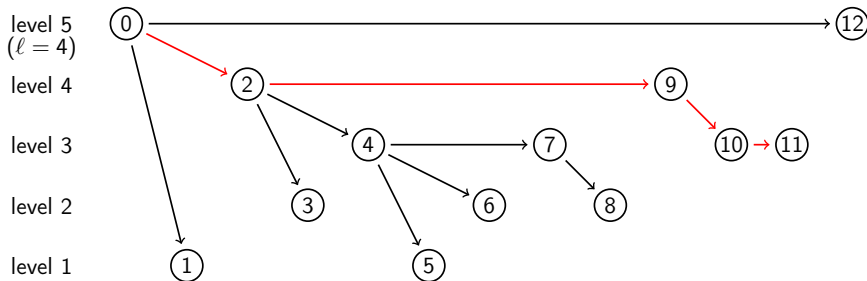
- Fix $\vec{k} = (0, 0, 2, 2)$.

Memory Eraser on Dependency Tree



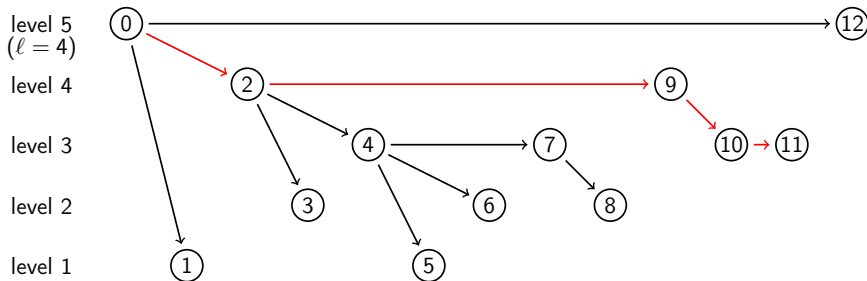
- Fix $\vec{k} = (0, 0, 2, 2)$.
- One Issue: What if $a_{x_2} = a_{x_9}$?

(Locally Simulatable) Extended Walk



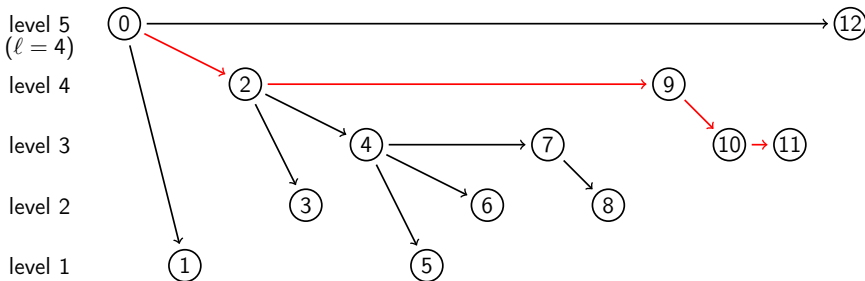
- Instead of original walk w , we look at extended walk w^* .

(Locally Simulatable) Extended Walk



- Instead of original walk w , we look at extended walk w^* .
- Once a collision occurs in a path within a single level, we replace all the rest queries of that level with true randomness.

(Locally Simulatable) Extended Walk



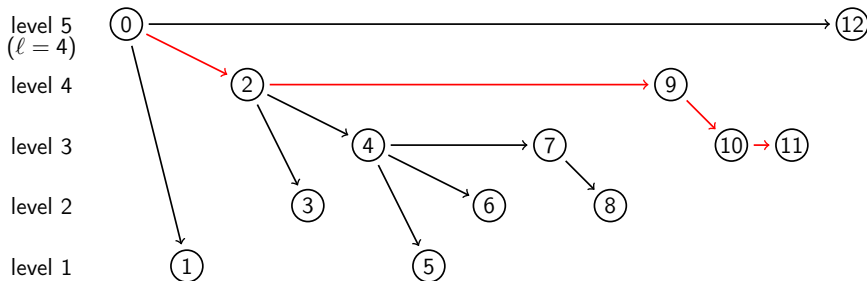
- Instead of original walk w , we look at extended walk w^* .
- Once a collision occurs in a path within a single level, we replace all the rest queries of that level with true randomness.
- w and w^* agree before the first collision.

Recall our goal.

Our Main Lemma

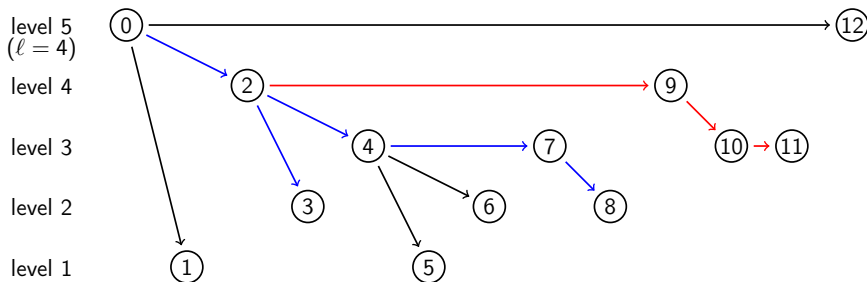
- $\mathbb{E}[|f^*(s)|] \leq O(\sqrt{n})$
- $\Pr[u \in f^*(s)] \geq \Omega(1/\sqrt{n}), \forall u \in [n]$

Good = All - Bad



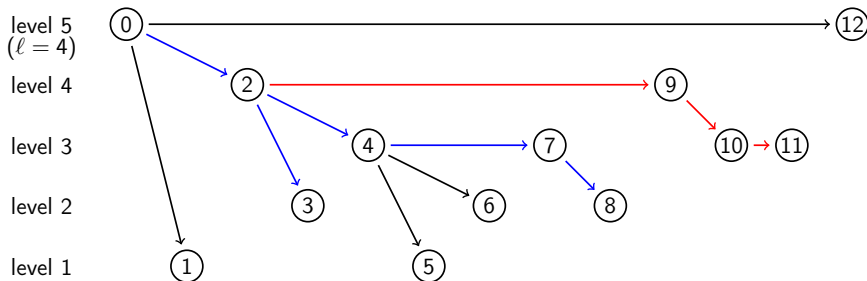
$$\sum_{\vec{k}} \frac{2^{-(k_1+k_2+\dots+k_\ell)}}{n}$$

Good = All - Bad



$$\sum_{\vec{k}} \frac{2^{-(k_1+k_2+\dots+k_\ell)}}{n} - \sum_{\vec{k}, \vec{k}', \vec{k}''} \frac{2^{-\|\vec{k}\|_1 - \|\vec{k}'\|_1 - \|\vec{k}''\|_1}}{n^3}$$

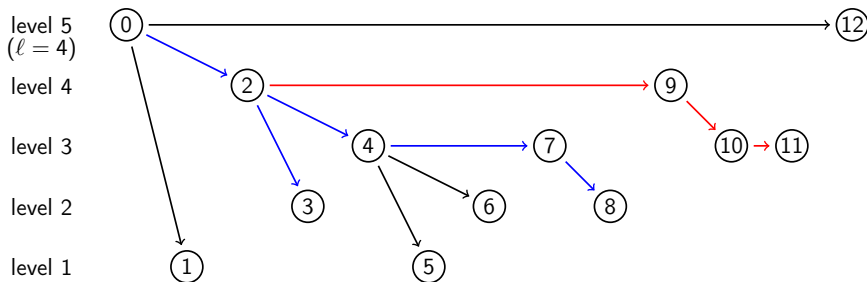
Good = All - Bad



$$\sum_{\vec{k}} \frac{2^{-(k_1+k_2+\dots+k_\ell)}}{n} - \sum_{\vec{k}, \vec{k}', \vec{k}''} \frac{2^{-\|\vec{k}\|_1 - \|\vec{k}'\|_1 - \|\vec{k}''\|_1}}{n^3}$$

$$\sum_{\vec{k}} \frac{2^{-(k_1+k_2+\dots+k_\ell)}}{n} = \frac{1}{n} \prod_{i=1}^{\ell} \sum_{k_i=0}^{\infty} 2^{-k_i} = \frac{2^\ell}{n}$$

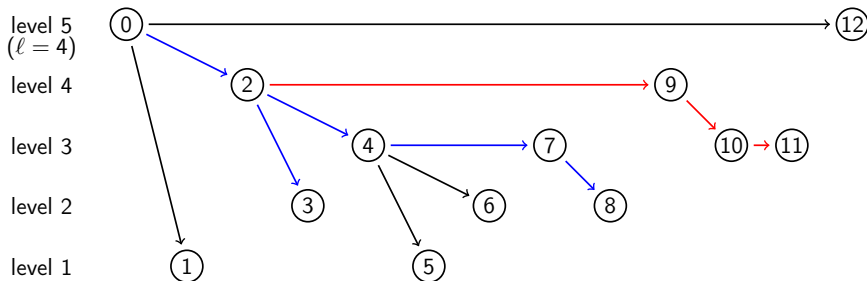
Good = All - Bad



$$\sum_{\vec{k}} \frac{2^{-(k_1+k_2+\dots+k_\ell)}}{n} - \sum_{\vec{k}, \vec{k}', \vec{k}''} \frac{2^{-\|\vec{k}\|_1 - \|\vec{k}'\|_1 - \|\vec{k}''\|_1}}{n^3}$$

$$\sum_{\vec{k}, \vec{k}', \vec{k}''} \frac{2^{-\|\vec{k}\|_1 - \|\vec{k}'\|_1 - \|\vec{k}''\|_1}}{n^3} = \frac{8^\ell}{n^3}$$

Good = All - Bad

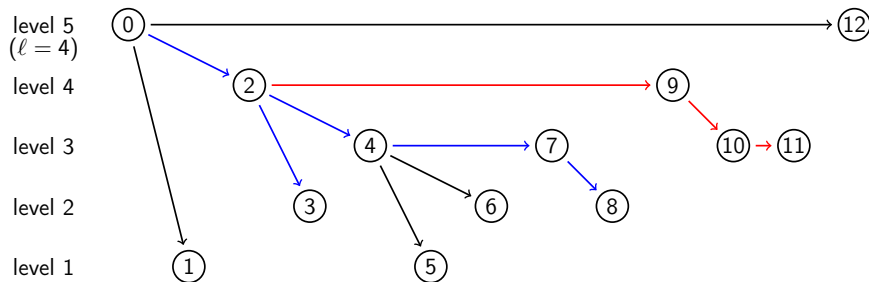


$$\sum_{\vec{k}} \frac{2^{-(k_1+k_2+\dots+k_\ell)}}{n} - \sum_{\vec{k}, \vec{k}', \vec{k}''} \frac{2^{-\|\vec{k}\|_1 - \|\vec{k}'\|_1 - \|\vec{k}''\|_1}}{n^3} = \frac{2^\ell}{n} - \frac{8^\ell}{n^3}$$

Let

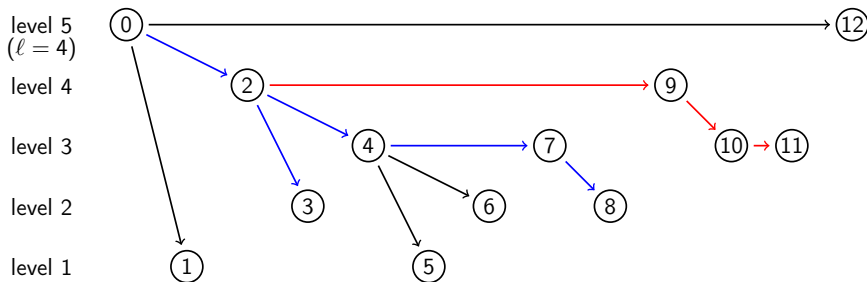
$$\ell \leftarrow \log n - 100$$

Collisions Cross Paths



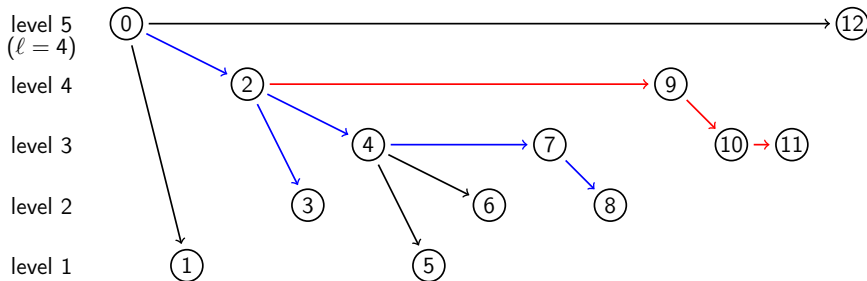
- Issue: What if $a_{x_3} = a_{x_8}$?

Collisions Cross Paths



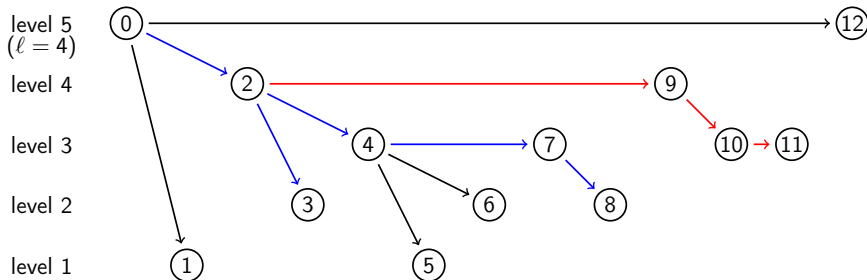
- Issue: What if $a_{x_3} = a_{x_8}$?
- Extended walk eliminate the collisions within a path.

Collisions Cross Paths



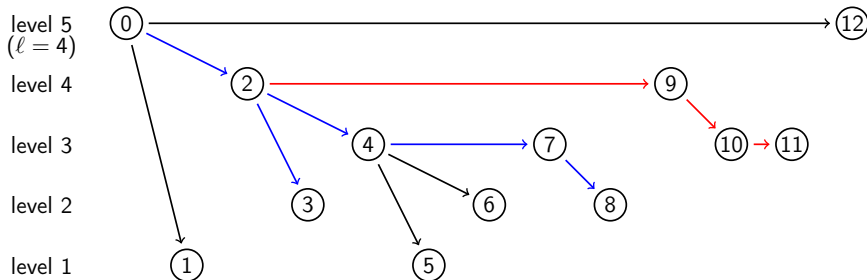
- Issue: What if $a_{x_3} = a_{x_8}$?
- Extended walk eliminate the collisions within a path.
- But there can still be collision between two paths.

Collisions Cross Paths



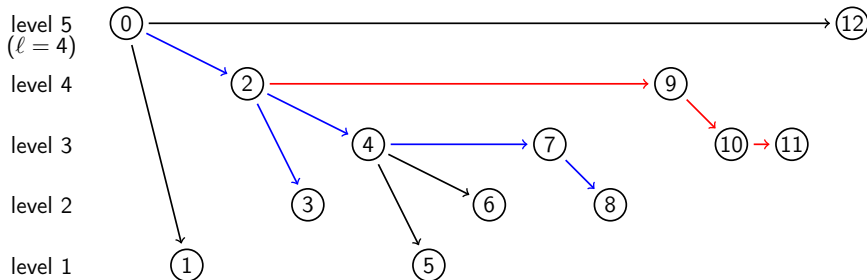
- Issue: What if $a_{x_3} = a_{x_8}$?

Collisions Cross Paths



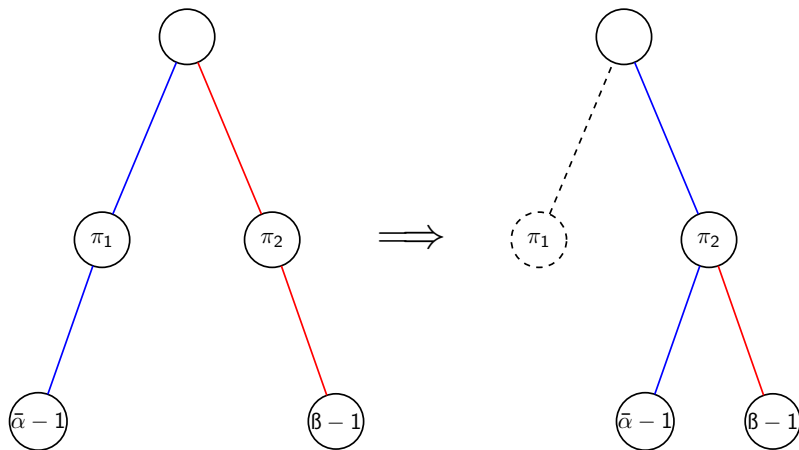
- Issue: What if $a_{x_3} = a_{x_8}$?
- We pick the first collision to be the blue paths.

Collisions Cross Paths



- Issue: What if there is collision between blue and red path?

Collisions Cross Paths



- We move the blue path when it has a collision with the red path.

Open Problems

- **Time-space Tradeoffs**

In this work, we only solved the case when $S = \tilde{O}(1)$. Can we extend it to the full tradeoff?

- **Shorter Seed Length**

In this work, our seed length is $O(\log^3 n \log \log n)$. Can this be improved?

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