Algorithm Design and Analysis (Fall 2023)
Assignment 6
Deadline: Jan 9, 2023

Choose **two** of the first four questions to submit. Question 5 is the bonus question.

1. Prove that the following problem is NP-complete. Given an undirected graph G and an undirected graph H, decide if H is a subgraph of G.

   Solution:

   First, $f \in NP$ (f denotes this problem), where the certificate y is a subgraph of the primal graph. If x is a yes instance, there always exists $y = x$. Otherwise we can never find such y.

   Then, I'll show *CLIQUE*$\leq_k$ f, which implies f is NP-complete. Given a *CLIQUE* instance $(G = (V, E), k)$, we can construct the instance of f as (G', H'), where $G' = G$ and H' is the complete graph with k vertices.

   $(G = (V, E), k)$ is a yes instance iff there exists the complete graph with k vertices in G, and iff H' is a subgraph of G', i.e., (G', H') is a yes instance. Then we finish the reduction.

2. Prove that the following problem is NP-complete. Given an undirected graph G and a positive integer $k \geq 2$, decide if G contains a spanning tree with maximum degree at most k.

   Solution:

   First, $f \in NP$ (f denotes this problem), where the certificate y is a spanning tree of G with maximum degree at most k.

   Then, I'll show *Hamiltonian Path*$\leq_k$ f, which implies f is NP-complete. Given a *Hamiltonian Path* instance $G = (V, E)$, we can construct (G', k) where $G' = G$ and $k = 2$.

   If $G = (v, E)$ is a yes instance of *Hamiltonian Path*, then it is a spanning tree of G since all vertices are covered, and the degree of intermediate vertex is exactly 2 and that of other two vertices is exactly 1, then $(G', 2)$ is a yes instance. On the contrary, a spanning tree with maximum degree 2 is a Hamiltonian path, then $G = (V, E)$ is also a yes instance.

3. Given an undirected graph $G = (V, E)$, prove that it is NP-complete to decide if G contains an independent set with size *exactly* $|V|/3$.

4. Consider the decision version of *Knapsack*. Given a set of n items with weights $w_1, \ldots, w_n \in \mathbb{Z}^+$ and values $v_1, \ldots, v_n \in \mathbb{Z}^+$, a capacity constraint $C \in \mathbb{Z}^+$, and a positive integer $V \in \mathbb{Z}^+$, decide if there exists a subset of items with total weight at most C and total value at least V. Prove that this decision version of Knapsack is NP-complete.

5. (**Bonus**) In the class, we have seen that 3SAT is NP-complete. In this question, we investigate the 2SAT problem and its variants. Similar to the 3SAT problem, in the 2SAT problem, we are given a 2-CNF Boolean formula (where each clause contains two literals) and we are to decide if this formula is satisfiable.

   (a) Prove that 2SAT is in P. (Hint: a clause $(a_i \vee a_j)$ with two literals a_i and a_j can be represented as two logical implications: $\neg a_i \implies a_j$ and $\neg a_j \implies a_i$; you may want to construct a directed graph with $2n$ vertices corresponding to $x_1, \neg x_1, x_2, \neg x_2, \ldots, x_n, \neg x_n$.)

   (b) Consider this variant of the 2SAT problem: given a 2-CNF Boolean formula ϕ and a positive integer k, decide if there is a Boolean assignment to the variables such that at least k clauses of ϕ are satisfied. Notice that 2SAT is the special case of this problem with k equals to the number of the clauses. Prove that this problem is NP-complete.

6. How long does it take you to finish the assignment (including thinking and discussion)? Give a score (1,2,3,4,5) to the difficulty. Do you have any collaborators? Please write down their names here.