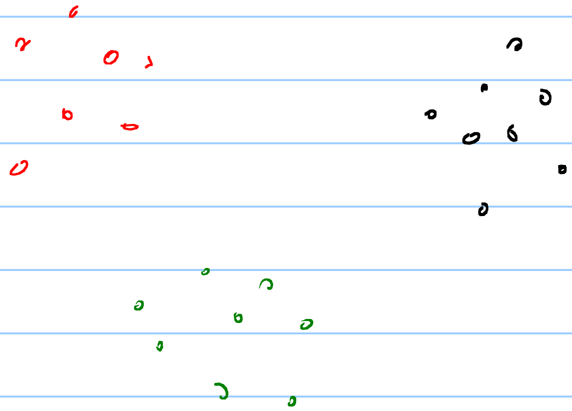
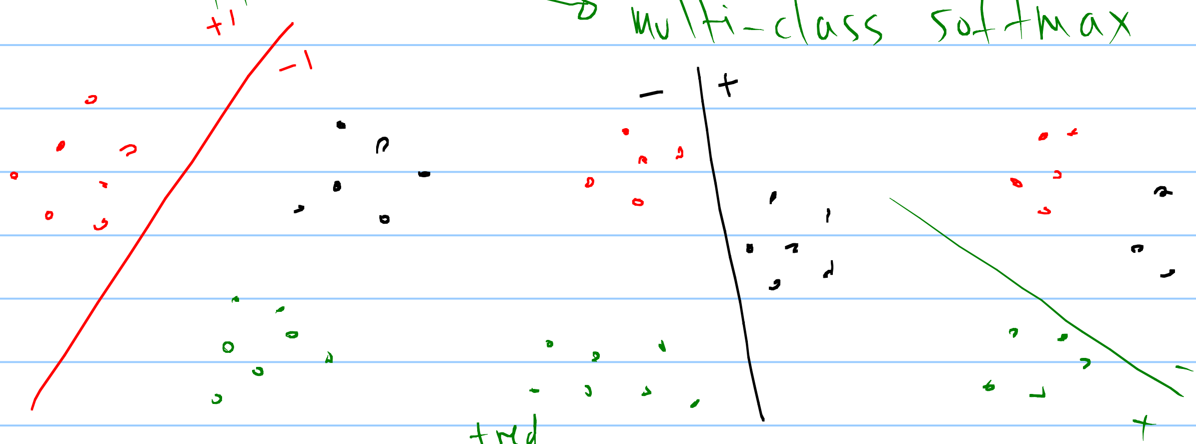
Multi-class Classification

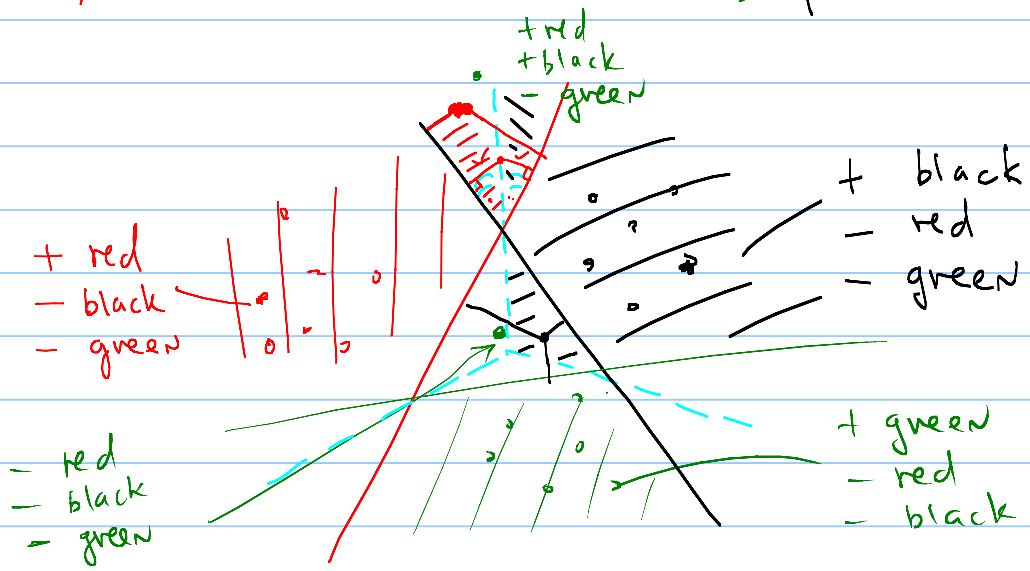
Given data $\{(\bar{x}_1, y_1), (\bar{x}_2, y_2), \dots, (\bar{x}_P, y_P)\}$

labels $y_p \in \{1, 2, \dots, C\}$ # of classes

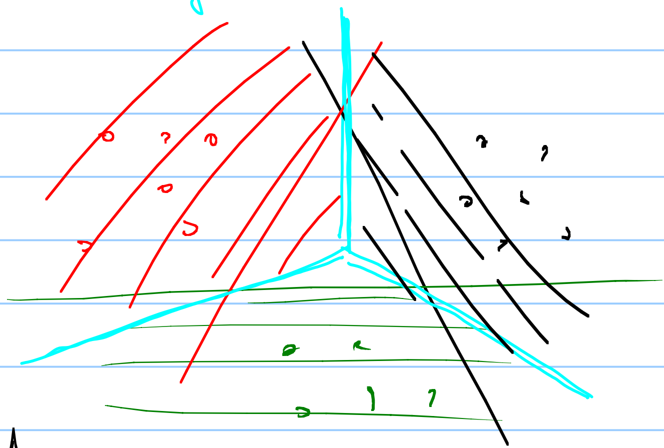


Two approaches
- One-versus-All (OvA)
- multi-class softmax





Fusion rule: assign a point to a class according to whichever classifier produces the largest evaluation (including negative evaluations)



OvA

- Learn C classifiers
- Each individual classifier distinguishes class c from all other classes
  - for each subproblem assign temporal labels

$$\hat{y}_p = \begin{cases} 1, & \bar{x}_p \in c \\ -1, & \text{else} \end{cases}$$

- Doing this we have learned C hyperplanes

$$b_c + \bar{x}^T \bar{w}_c = 0, \quad c = 1, .., C$$

- a known point $\bar{x}_p$ belongs to class c if it satisfies

$$b_c + \bar{x}_p^T \bar{w}_c > 0$$

$$b_j + \bar{x}_p^T \bar{w}_j < 0, \quad j = 1, .., C, \quad j \neq c$$

- Combine or fuse C classifiers

assign a point $\bar{x}$ to label y, where

$$y = \arg\max_{j=1,..,C} (b_j + \bar{x}^T \bar{w}_j)$$

Multiclass softmax classifier

If $\bar{x}_p$ belongs to class c, then

$$c = \arg\max_{j=1,..,C} b_j + \bar{x}_p^T \bar{w}_j$$

$$\Rightarrow \quad b_c + \bar{x}_p^T \bar{w}_c = \max_j (b_j + \bar{x}_p^T \bar{w}_j)$$

$$\Rightarrow \quad \max_j (b_j + \bar{x}_p^T \bar{w}_j) - (b_c + \bar{x}_p^T \bar{w}_c) = 0$$

Define an objective function

$$g(b_1,..,b_C, \bar{w}_1,..,\bar{w}_C) = \sum_{c=1}^{C} \sum_{p \in \Omega_c} \left[\max_j (b_j + \bar{x}_p^T \bar{w}_j) - (b_c + \bar{x}_p^T \bar{w}_c) \right]$$

$$\sum_{c=1}^{C} |\Omega_c| = P \qquad \Omega_c = \{\text{points that belong to class } c\}$$

- Turn it into a soft-max

$$\text{soft}(s_1,..,s_C) = \log\left(\sum_{j=1}^{C} e^{s_j}\right) \approx \max(s_1,..,s_C)$$

$$g(b_1,..,b_C, \bar{w}_1,..,\bar{w}_C) = \sum_{c=1}^{C} \sum_{p \in \Omega_c} \left[\log\left(\sum_{j=1}^{C} e^{b_j + \bar{x}_p^T \bar{w}_j}\right) - \log e^{b_c + \bar{x}_p^T \bar{w}_c} \right]$$

$$(\log e^a = a)$$

$$\Rightarrow g(b_1, \dots, \bar{w}_c) = -\sum_c \sum_p \log\left(\frac{e^{b_c + \bar{x}_p^T \bar{w}_c}}{\sum_j e^{b_j + \bar{x}_p^T \bar{w}_j}}\right)$$

$$= \sum_c \sum_p \log\left(\frac{\sum_j e^{b_j + \bar{x}_p^T \bar{w}_j}}{e^{b_c + \bar{x}_p^T \bar{w}_c}}\right)$$

$$= \sum_c \sum_p \log\left(1 + \sum_{\substack{j=1 \\ j \neq c}}^{C} e^{(b_j - b_c) + \bar{x}_p^T(\bar{w}_j - \bar{w}_c)}\right)$$