Bessel Functions of the First Kind

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Definition

The Bessel functions of the first kind $J_n(x)$are defined as the solutions to the Bessel differential equation

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 - n^2)y = 0$$

which are nonsingular at the origin.

Hansen-Bessel formula:

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\tau - x\sin\tau)\mathrm{d}\tau = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(i(n\tau - x\sin\tau))\mathrm{d}\tau.$$

Finding zeros of Bessel functions

Initial guesses were taken from Wolfram MathWorld.

zeros_guess — $m \times n$ matrix, where

- m — number of the roots,
- n — number of the Bessel functions.

So $\mathrm{zeros\_guess}(m, n) = j_{m(n-1)}$ — m-th root of $J_{n-1}(x)$.

$n-1$ due to indexing starts at 0: $J_0(x),\ J_1(x),\ \ldots\ ,\ J_{n-1}(x)$.

Strucuture of *BZerosGuess.xlsx:*

$$\begin{bmatrix} j_{10} & j_{11} & \cdots & j_{1(n-1)} \\ j_{20} & j_{21} & \cdots & j_{2(n-1)} \\ \vdots & \vdots & \cdots & \vdots \\ j_{m0} & j_{m1} & \cdots & j_{m(n-1)} \end{bmatrix}$$

```
clear;
close all;
clc;
```

```
zeros_guess=readmatrix("BZerosGuess.xlsx");
```

```
num_roots=size(zeros_guess,1);

num_functions=size(zeros_guess,2);

bzeros = zeros(num_roots, num_functions);

for k = 1:num_roots
    for n = 1:num_functions
        fun = @(x) besselj(n-1, x);
        bzeros(k, n) = fzero(fun, zeros_guess(k, n));
    end
end
```

Table of zeros of Bessel functions

```
rownames = string(zeros(num_roots, 1));
columnnames = string(zeros(num_functions, 1));

for i = 1:num_roots
    rownames(i) = string(i);
end


for i = 1:num_functions
    columnnames(i) = sprintf("J%d(x)", i-1);
end

Tzeros=array2table(bzeros, "RowNames", rownames', "VariableNames", columnnames');

%Export in .xlsx file
writematrix(bzeros,"BesselZeros.xlsx","WriteMode","overwritesheet");
```

Plotting graphs of Bessel functions

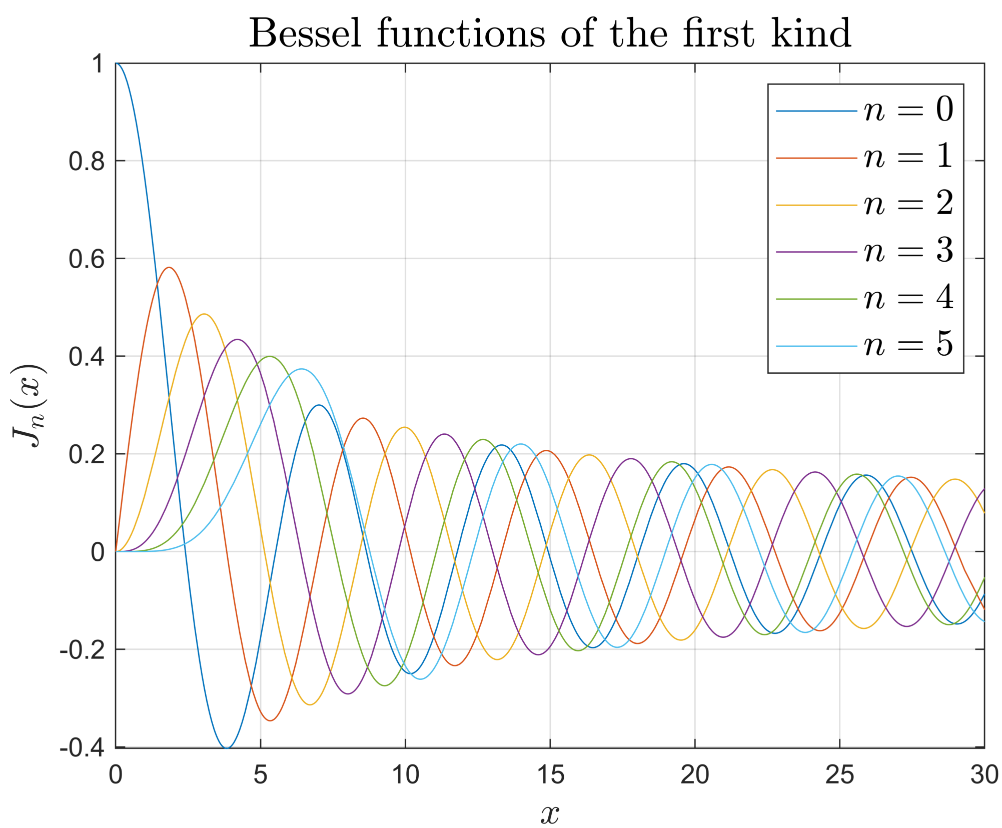
```
legend_labels = cell(ones(1,num_functions));

for i = 1:num_functions
    fplot(@(x) besselj(i-1, x), [0 30]);
    hold on;
    legend_labels{i} = sprintf('$n=%d$', i-1);
end

legend(legend_labels, 'Interpreter', 'latex', 'FontSize', 14);

grid on;
box on;
```

```
axis fill;
xlabel('$x$', 'Interpreter', 'latex', 'FontSize', 14);
ylabel('$J_n(x)$', 'Interpreter', 'latex', 'FontSize', 14);
title('Bessel functions of the first kind', 'Interpreter', 'latex', 'FontSize', 16);
```



Finding zeros of Bessel prime functions

Initial guesses were taken from Wolfram MathWorld.

prime_zeros_guess — $m \times n$ matrix, where

- m — number of the roots,
- n — number of the Bessel functions.

So $\text{prime\_zeros\_guess}(m, n) = j'_{m(n-1)}$ — m-th root of $J'_{n-1}(x)$.

$n - 1$ due to indexing starts at 0: $J'_0(x),\ J'_1(x),\ \ldots,\ J'_{n-1}(x)$.

Strucuture of *BPrimeZerosGuess.xlsx:*

$$\begin{bmatrix} j'_{10} & j'_{11} & \cdots & j'_{1(n-1)} \\ j'_{20} & j'_{21} & \cdots & j'_{2(n-1)} \\ \vdots & \vdots & \cdots & \vdots \\ j'_{m0} & j'_{m1} & \cdots & j'_{m(n-1)} \end{bmatrix}$$

```
prime_zeros_guess=readmatrix("BPrimeZerosGuess.xlsx");
```

```
num_roots=size(zeros_guess,1);

num_functions=size(zeros_guess,2);

bprimezeros = zeros(num_roots, num_functions);

syms x;

for k = 1:num_roots
    for n = 1:num_functions
        f(x) = besselj(n-1, x);
        bprimezeros(k, n) = vpasolve(diff(f,x), prime_zeros_guess(k, n));
    end
end
```

Table of zeros of Bessel prime functions

```
columnprimenames = string(zeros(num_functions,1));

for i = 1:num_functions
    columnprimenames(i) = sprintf("J'%d(x)", i-1);
end

Tprimezeros=array2table(bprimezeros,"RowNames",rownames,"VariableNames",columnprimen
ames');

%Export in .xlsx file
writematrix(bprimezeros,"BesselPrimeZeros.xlsx","WriteMode","overwritesheet");
```

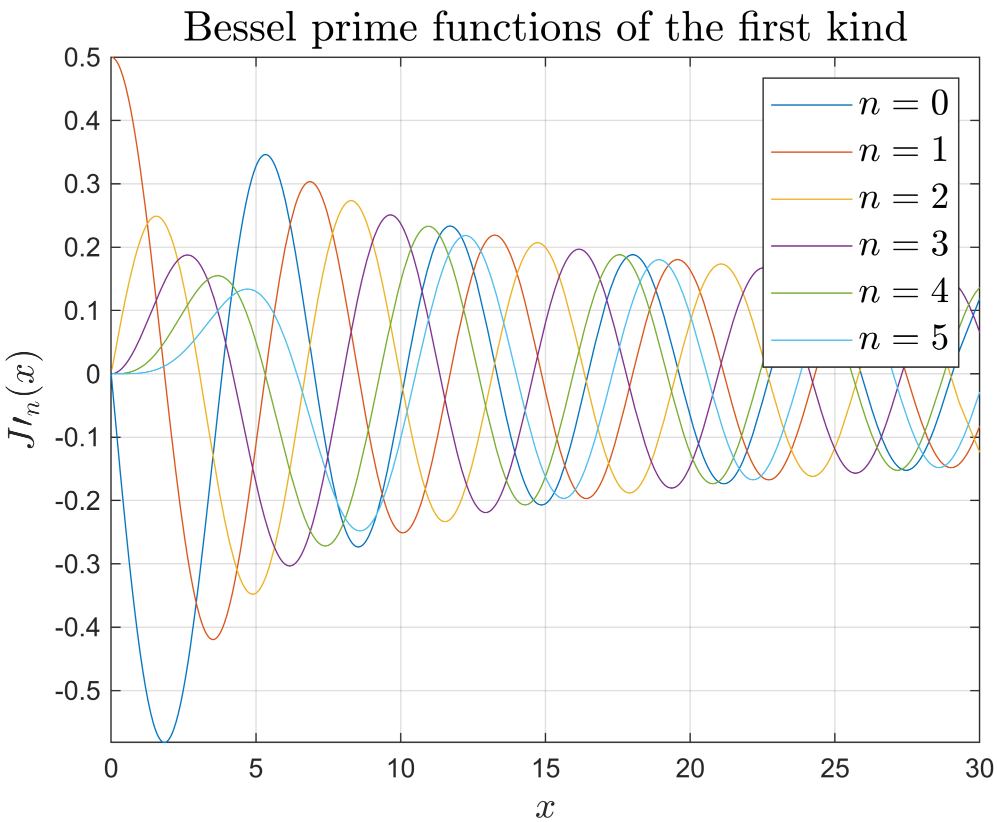
Plotting graphs of Bessel prime functions

```
figure();

for i = 1:num_functions
    fplot(diff(besselj(i-1, x),x),[0 30]);
    hold on;
    legend_labels{i} = sprintf('$n=%d$', i-1);
end

legend(legend_labels, 'Interpreter', 'latex', 'FontSize', 14);

grid on;
box on;
axis fill;
xlabel('$x$', 'Interpreter', 'latex', 'FontSize', 14);
ylabel('$J\prime_n(x)$', 'Interpreter', 'latex', 'FontSize', 14);
title('Bessel prime functions of the first kind', 'Interpreter', 'latex',
'FontSize', 16);
```


Bessel prime functions of the first kind
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
0
5
10
15
20
25
30
x
$J'_n(x)$
$n = 0$
$n = 1$
$n = 2$
$n = 3$
$n = 4$
$n = 5$