Linear Programming

Class 5: The simplex method, part 2

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Class outline

- Finding an intial basic feasible solution:
  - Two-phase method
  - Big-M method
- Infeasibility
- Unboundedness
- Degeneracy
- Pivoting rules
- Number of pivots

Two-phase method

- Include one new (*artificial*) variable $x_i' \geq 0$ per constraint i of the original LP, adding if $b_i \geq 0$ and subtracting if $b_i < 0$.
- For this new system, $\mathbf{x} = \mathbf{0}, \mathbf{x}' = (|b_1|, \dots, |b_m|)^T$ is feasible.
- Use new objective, $\sum_i x_i{}'$. auxillary LP
- If the optimal solution is $\mathbf{x} = \mathbf{x}_0, \mathbf{x}' = \mathbf{0}$ then $\mathbf{x} = \mathbf{x}_0$ is a bfs to the orignal LP, otherwise the original LP is infeasible.

Two-phase method

Summary

Original LP: $\min \mathbf{c}^T\mathbf{x}$ s.t. $A\mathbf{x} = \mathbf{b},\ \boldsymbol{x} \geq \mathbf{0}$

Phase 1: solve $\min \mathbf{1}^T\mathbf{x}'$ s.t. $A\mathbf{x} \pm \mathbf{x}' = \mathbf{b},\ \boldsymbol{x}, \boldsymbol{x}' \geq \mathbf{0}$

Phase 2: if solution is $\mathbf{x} = \mathbf{x}_0, \mathbf{x}' = \mathbf{0}$ then solve original LP starting with bfs $\mathbf{x} = \mathbf{x}_0$.

Example 2

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 \end{pmatrix},$$

$$\boldsymbol{b} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \qquad \boldsymbol{c} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ -1 \\ -5 \end{pmatrix}$$

Big-M method

- Include an artificial variable $x_i' \geq 0$ for each constraint i as in the two-phase method
- Add (or subtract) x_i' to constraint i if b_i is positive (or negative, resp.)
- Solve new LP:

$$\min \mathbf{c}^T \mathbf{x} + M \mathbf{1}^T \boldsymbol{x}' \text{ s.t. } A\mathbf{x} \pm \mathbf{x}' = \mathbf{b}, \boldsymbol{x} \geq \mathbf{0}$$

where M is a large positive number

- Initial bfs is $\mathbf{x}' = (|b_1|, \ldots, |b_m|)^T$
- As long as M is large enough, no $x_i{}'$ variable will be in basis of the solution

Example 3

$$
\begin{array}{ll}
\text{Minimize} & -x_1 - 3x_2 + x_3 \\
\text{s.t} & x_1 + x_2 + 2x_3 \leq 4 \\
 & -x_1 + x_3 \geq 4 \\
 & x_3 \geq 3 \\
 & x_1, x_2, x_3 \geq 0
\end{array}
$$

How big should M be?

- M should be large enough that there is some bfs with $\mathbf{x}' = \mathbf{0}$ that has a strictly smaller objective than all bfs's with $\mathbf{x}' \neq \mathbf{0}$.
- To choose M, we cannot only look at the coefficients of the objective function. Eg.

$$\begin{array}{ll} \text{Minimize} & x_1 \\ \text{s.t} & \varepsilon x_1 - x_2 \geq \varepsilon \\ & x_1, x_2 \geq 0 \end{array}$$

where ε is small and positive.

Infeasibility

- A linear program may be in feasible. Eg.

$$\begin{array}{ll} \text{Minimize} & 2x_1 - 3x_2 \\ \text{s.t} & x_1 + x_2 = -1 \\ & x_1, x_2 \geq 0 \end{array}$$

- Infeasibility is detected in Phase 1 of the two-phase method (or after finding a solution with $\mathbf{x}' \neq \mathbf{0}$ when using the Big-M method)

Unbounded problems

- Some LPs may be unbounded, i.e. the objective function can be made arbitrarily small. Eg.

  Minimize $\quad -x_1$

  s.t $\quad x_1 - x_2 = 2$

  $\quad x_1, x_2 \geq 0$

- We can detect unboundedness from the tableau, eg.

x1	x2	x3	x4	x5	x6	b
1	0	0	-2	1	7	5
0	1	0	0	2	3	2
0	0	1	-3	5	6	1
0	0	0	-1	2	1	5

new value of objective is $-5-L$

we can choose $x_4 = L$ to be arbitrarily large, increase x_1 to $5+2L$, increase x_3 $1+3L$.

Degeneracy

- Some basic variables may be equal to 0. Eg.

pivot on this entry

x1	x2	x3	x4	x5	x6	b
-6	0	2	1	0	1	2
6	1	-12	0	0	0	0
12	0	1	0	1	-1	1
-12	0	2	0	0	1	6

Bring x_1 into basis ; x_2 leaves basis

Definition: If a bfs solution $\mathbf{x}_B$ contains some zero coordinate $(\mathbf{x}_B)_i = 0$ then we say it is *degenerate*.

- If we have a degenerate bfs, we will make zero progress by removing them from the basis.

Cycling

- If we're not careful, we may go from one degenerate solution to another and end up at the original bfs.
- To avoid this, we can use *Bland's pivoting rule:*

1. If $(\mathbf{s}_N)_i < 0$ for several i's (where $\mathbf{s}_N = \mathbf{c}_N^T - \mathbf{c}_N^T B^{-1} N$), choose the one with the smallest i (leftmost in the tableau).
2. If in the chosen column i there are several indices j that minimize $\frac{(\mathbf{x}_B)_j}{T_{ij}}$ choose the smallest such j (highest in the tableau).

Cycling

Example:

x1	x2	x3	x4	x5	x6	b
-6	0	2	1	0	1	2
6	1	-12	0	0	0	12
4	0	1	0	1	-1	8
-2	0	-4	0	0	1	6

pivot on this one (highest)

$6/12 = 4/8$

bring x_1 into basis (furthest left)

Number of pivots

The number of pivots can be very large.

Example 4 (Klee-Minty polytope):

$$
\begin{array}{lrl}
\text{Max} & 2^{n-1}x_1 + 2^{n-2}x_2 + \cdots + 2x_{n-1} + x_n & \\
\text{s.t} & x_1 & \le 5 \\
& 4x_1 + x_2 & \le 25 \\
& 8x_1 + 4x_2 + x_3 & \le 125 \\
& \vdots & \vdots \\
& 2^n x_1 + 2^{n-1}x_2 + \cdots + 4x_{n-1} + x_n & \le 5^n \\
& & \mathbf{x} \ge 0
\end{array}
$$

This LP has 2^n extreme points and starting at $\mathbf{x} = 0$, the simplex algorithm goes through all of them before reaching the optimal solution $(0,0,\ldots,5^n)$.

Number of pivots

Open problem: find a pivoting rule that guarantees that the number of pivots required will be a polynomial function of n and m.