③ Dual $\quad \underline{y} = \underline{y}_1 - \underline{y}_2, \quad \underline{y}_1, \underline{y}_2 \geq 0$

$$\max \underline{b}^T \underline{y} \iff \min -\underline{b}^T \underline{y}$$

$$= \min -\underline{b}^T (\underline{y}_1 - \underline{y}_2)$$

$$= \min (-\underline{b}^T | \underline{b}^T | \underline{0}^T) \begin{pmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \underline{s} \end{pmatrix}$$

s.t. $A^T \underline{y} + \underline{s} = \underline{c}$

$$\iff A^T (\underline{y}_1 - \underline{y}_2) + \underline{s} = \underline{c}$$

$$\Leftrightarrow \left(A^T \mid -A^T \mid I\right)\begin{pmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \underline{s} \end{pmatrix} = \underline{c}$$

in summary, dual is $\min\ \underline{c}'^T \underline{x}'$

s.t. $A'\underline{x}' = \underline{b}'$, $\underline{x}' \geq 0$

where $\underline{c}' = \begin{pmatrix} -\underline{b} \\ \underline{b} \\ 0 \end{pmatrix}$, $A' = \left(A^T \mid -A^T \mid I\right)$,

$\underline{b}' = \underline{c}$

So dual of dual is:

$$\max \underline{b}'^T \underline{y}' \quad \text{s.t. } A'^T \underline{y}' \leq \underline{c}'$$

$$\Leftrightarrow \max \underline{c}^T \underline{y}' \quad \text{s.t. } \begin{pmatrix} A \\ -A \\ I \end{pmatrix} \underline{y}' \leq \begin{pmatrix} -\underline{b} \\ \underline{b} \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \max \underline{c}^T \underline{y}' \text{ s.t. } \left.\begin{aligned} A\underline{y}' &\leq -\underline{b} \\ -A\underline{y}' &\leq \underline{b} \end{aligned}\right\} A\underline{y}' = -\underline{b}$$

$$\underline{y}' \leq 0$$

Set $\underline{x}' = -\underline{y}'$

So dual of dual is

$\min \underline{c}^T \underline{x}'$ s.t. $A\underline{x}' = \underline{b}$, $\underline{x}' \geq 0$.

This is primal problem.

5 For any dual feasible $\underline{y}$ and optimal primal $\underline{x}$, $\underline{c}^T \underline{x} \geq \underline{b}^T \underline{y}$

2. If primal problem is unbounded, suppose dual is feasible and let $\underline{y}$ be dual feasible. Then $\forall$ primal

feasible $\underline{x}$, $\underline{c}^T\underline{x} \geq \underline{b}^T\underline{y}$, so $\underline{b}^T\underline{y}$ is a lower bound for $\underline{c}^T\underline{x}$. But $\underline{c}^T\underline{x}$ can't have a lower bound! Contradiction. So dual is infeasible.

6 3. Suppose $\underline{x}^*$ is primal feasible and $(\underline{y}^*, \underline{s}^*)$ is dual feasible and

$$\underline{c}^T\underline{x}^* - \underline{b}^T\underline{y}^* = \underline{x}^{*T}\underline{s} = 0.$$

Then for any other feasible $\underline{x}$,

$$\underline{c}^T\underline{x} \geq \underline{b}^T\underline{y}^* = \underline{c}^T\underline{x}^*,$$ so $\underline{x}^*$ is optimal.

Similarly for $\underline{y}^*$.

7 <u>Proof of Claim 1</u>:

$\underline{s} \geq 0$ ✓

$$A^T \underline{y} + \underline{s} = \begin{pmatrix} B^T \\ N^T \end{pmatrix} (\underline{c}_B^T B^{-1})^T + \begin{pmatrix} \underline{0} \\ \underline{c}_N - (\underline{c}_B^T B^{-1} N)^T \end{pmatrix}$$

$$= \begin{pmatrix} B^T (B^{-1})^T \underline{c}_B \\ N^T (B^{-1})^T \underline{c}_B \end{pmatrix} + \begin{pmatrix} \underline{0} \\ \underline{c}_N - N^T (B^{-1})^T \underline{c}_B \end{pmatrix}$$

$$= \begin{pmatrix} \underline{c}_B \\ \underline{c}_N \end{pmatrix} = \underline{c}$$ ✓

(10) $x_i^* s_i^* = 0 \iff$ either $x_i^* = 0$

or $s_i^* = 0$

So $\forall i$, either the variable $x_i^* = 0$ or the constraint

$$a_{1i} y_1^* + a_{2i} y_2^* + \dots + a_{mi} y_m^* \leq c_i$$

is tight (i.e. LHS = RHS)

(11) $$\min \; 8x_0 - 3x_2 + 2x_5 - 2x_6$$

$$\text{s.t.} \quad -x_0 - 6x_2 + x_5 - x_6 - x_4 = 2$$

$$-5x_0 + 7x_2 - 2x_5 + 2x_6 = -4$$

$$x_0, x_2, x_4, x_5, x_6 \geq 0$$

Dual: $\max \; 2y_1 - 4y_2$ s.t.

$$-y_1 - 5y_2 \leq 8$$

$$-6y_1 + 7y_2 \leq -3$$

$$\left.\begin{aligned} y_1 - 2y_2 &\leq 2 \\ -y_1 + 2y_2 &\leq -2 \end{aligned}\right\} \; y_1 - 2y_2 = 2$$

$$-y_1 \leq 0 \implies y_1 \geq 0$$

(12) Dual: $\max \underline{b}^T \underline{y}$ s.t. $A^T \underline{y} \leq \underline{c}$

$\underline{y} \geq 0$

(15) Proof of Farkas Lemma

Define an LP: $\min \underline{0}^T \underline{x}$ s.t. $A\underline{x} = \underline{b}$,

$\underline{x} \geq 0$.

The dual problem is: $\max \underline{b}^T \underline{y}$ s.t. $A^T \underline{y} \leq \underline{0}$

First suppose (i) is true, so $\exists\, \underline{x}$ with $A\underline{x} = \underline{b}$, $\underline{x} \geq 0$. So $\underline{x}$ is feasible for primal problem, so $\underline{x}$ is optimal (since any feasible solution has objective $= 0$).

So by duality, $\max \underline{b}^T \underline{y} = 0$, so $\underline{b}^T \underline{y} \leq 0$ $\forall$ dual feasible $\underline{y}$. I.e. for any $\underline{y}$ with $A^T \underline{y} \leq 0$, we have $\underline{b}^T \underline{y} \leq 0$. So (ii) cannot be true.

Now suppose (ii) is true. So $\exists$ a dual feasible $\underline{y}$ with $\underline{b}^T \underline{y} > 0$. So $M\underline{y}$ is also dual feasible (since $A^T(My) = M(A^T y) \leq 0$) and objective $= \underline{b}^T(My) > 0$, and can be arbitrarily large. So the dual is unbounded so the primal problem is infeasible, so (i) cannot be true. $\square$

(18) Note: we're not saying anything about the actual probabilities of the events.

<u>Proof of theorem</u>

Rewrite (ii) as: $\exists\, \underline{p} \geq 0$ s.t. $\begin{pmatrix} R \\ \underline{1}^T \end{pmatrix} \underline{p} = \begin{pmatrix} \underline{0} \\ 1 \end{pmatrix}$

This is alternative (i) of the Farkas Lemma where $A = \begin{pmatrix} R \\ \underline{1}^T \end{pmatrix}$ and $\underline{b}$ $\begin{pmatrix} \underline{0} \\ 1 \end{pmatrix}$

So either this occurs or $\exists\, \underline{y}$ s.t.

$$A^T \underline{y} \leq 0 \quad \text{and} \quad \underline{b}^T \underline{y} > 0$$

$$\Leftrightarrow \quad (R^T \ \underline{1})\, \underline{y} \leq 0 \quad \text{and} \quad (\underline{0}^T \ 1)\, \underline{y} > 0$$

$$\left[\underline{y} = \begin{pmatrix} \underline{z} \\ y_0 \end{pmatrix}, \ \underline{z} \in \mathbb{R}^n, \ y_0 \in \mathbb{R} \right]$$

$$\Leftrightarrow \quad R^T \underline{z} + y_0 \leq 0 \quad \text{and} \quad y_0 > 0$$

$$\Rightarrow \quad R^T(-\underline{z}) \geq y_0 > 0$$

Now set $\underline{x} = -\underline{z}$, so $R^T \underline{x} > 0 \Leftrightarrow \underline{x}^T R > 0$.

So (i) from the Arbitrage Thm. holds.

$\square$