Linear Programming

Class 6: Duality and complementary slackness

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Class outline

- The dual problem
- Weak duality
- Strong duality
- Complementary slackness
- Economic interpretation of duality and complementary slackness
- The Farkas lemma
- The Arbitrage Theorem

The dual problem

Recall the standard form of an LP (sometimes called the primal problem):

$x \in \mathbb{R}^n$

$$\begin{aligned} \min\ & \mathbf{c}^T \mathbf{x} \\ \text{s.t.}\ & A\mathbf{x} = \mathbf{b}, \\ & \boldsymbol{x} \geq \mathbf{0} \end{aligned}$$

We now introduce the dual problem:

$y \in \mathbb{R}^m$

$s \in \mathbb{R}^n$

$$\begin{aligned} \max\ & \mathbf{b}^T \mathbf{y} \\ \text{s.t.}\ & A^T \mathbf{y} \leq \mathbf{c} \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \max\ & \mathbf{b}^T \mathbf{y} \\ \text{s.t.}\ & A^T \mathbf{y} + \mathbf{s} = \mathbf{c}, \\ & \mathbf{s} \geq \mathbf{0} \end{aligned}$$

Note: the dual of the dual is the primal.

dual slack variable

Weak duality lemma

Lemma: Suppose $\mathbf{x}$ is feasible for the primal problem and let $(\mathbf{y}, \mathbf{s})$ be feasible for the dual problem. Then

$$\mathbf{c}^T\mathbf{x} \geq \mathbf{b}^T\mathbf{y}.$$

Proof: $\underline{c}^T\underline{x} - \underline{b}^T\underline{y} = \underline{c}^T\underline{x} - (A\underline{x})^T\underline{y}$

$= \underline{c}^T\underline{x} - \underline{x}^T A^T \underline{y}$

$= \underline{c}^T\underline{x} - \underline{x}^T(\underline{c} - \underline{s})$

$= \underline{x}^T\underline{s}$

$= \sum_{i=1}^{n} x_i s_i \geq 0$

Weak duality lemma

Implications:

1. For every dual feasible $\mathbf{y}$, the value $\mathbf{b}^T\mathbf{y}$ is a lower bound for the primal problem; for every primal ~~dual~~ feasible $\mathbf{x}$, the value $\mathbf{c}^T\mathbf{x}$ is an upper bound for the dual problem. The quantity $\mathbf{c}^T\mathbf{x} - \mathbf{b}^T\mathbf{y} = \mathbf{x}^T\mathbf{s} \geq 0$ is called the duality gap.
2. If the primal problem is unbounded, then the dual problem is infeasible. If the dual problem is unbounded then the primal problem is infeasible.

Weak duality lemma

3. If we find a primal feasible $\mathbf{x}$ and a dual feasible $(\mathbf{y}, \mathbf{s})$ such that the duality gap is 0, then $\mathbf{x}$ is optimal for the primal and $(\mathbf{y}, \mathbf{s})$ is optimal for the dual.

Question: When both the primal and dual problems are feasible, is it true that at the optimum the duality gap is always zero?

Yes!

Simplex method: reminder

The simplex method ends when the tableau looks like this (after rearranging some rows and columns):

$x_B = B^{-1}b$

$\mathbf{x}_B^T$	$\mathbf{x}_N^T$	
I	$B^{-1}N$	$B^{-1}\mathbf{b}$
$\mathbf{0}^T$	$\mathbf{s}_N^T = \mathbf{c}_N^T - \mathbf{c}_B^T B^{-1} N$	$-z^T$

- $\mathbf{x} = \binom{\mathbf{x}_B}{\mathbf{0}}$
- $\mathbf{s} = \binom{\mathbf{0}}{\mathbf{s}_N}$
- $\mathbf{s} \geq \mathbf{0}$

What we know: $\mathbf{x}$ is primal feasible.

Claim 1: $(\mathbf{y}, \mathbf{s})$ is dual feasible, where $\mathbf{y}^T = \mathbf{c}_B^T B^{-1}$.

Simplex method: reminder

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$\mathbf{x}_B^T$	$\mathbf{x}_N^T$	
I	$B^{-1}N$	$B^{-1}\mathbf{b}$
$\mathbf{0}^T$	$\mathbf{s}_N^T = \mathbf{c}_N^T - \mathbf{c}_B^T B^{-1} N$	$-z^T$

- $\mathbf{x} = \begin{pmatrix} \mathbf{x}_B \\ \mathbf{0} \end{pmatrix}$
- $\mathbf{s} = \begin{pmatrix} \mathbf{0} \\ \mathbf{s}_N \end{pmatrix}$
- $\mathbf{s} \geq \mathbf{0}$

Claim 2: The duality gap is zero.

$\underline{x}^T \underline{s} = \begin{pmatrix} \underline{x}_B \\ \underline{0} \end{pmatrix}^T \begin{pmatrix} 0 \\ \underline{s}_N \end{pmatrix} = 0$ ✓

Strong duality theorem

We have proved:

Theorem (strong duality):

1. If the primal problem is unbounded, the dual problem is infeasible.
2. If the dual is unbounded, the primal is infeasible.
3. If both the primal and dual are feasible, then both have optimal solutions $\mathbf{x}^*$ and $(\mathbf{y}^*, \mathbf{s}^*)$ where $\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$, or equivalently, $\mathbf{x}^{*T}\boldsymbol{s}^* = \mathbf{0}$.

Complementary slackness

We showed that if the primal and dual are feasible then $\mathbf{x}^{*T}\boldsymbol{s}^{*} = \mathbf{0}$. Consequently,

Theorem (complementary slackness):
If $\mathbf{x}^*$ is primal optimal and $(\mathbf{y}^*, \mathbf{s}^*)$ is dual optimal then $x_i^* s_i^* = 0$ for $i = 1, \dots, n$.

Because $\underline{x}^{*T} \underline{s}^{*} = 0 \iff \sum_{i=1}^{n} x_i^* s_i^* = 0$

$\iff x_i^* s_i^* = 0 \ \forall i$, since $x_i, s_i \geq 0$.

Duality for LPs not in standard form

In practice, most LPs are not in standard form, but have a mixture of equality/inequality constraints and restricted/unrestricted variables. The table of duality (Table 6.1) on page 264 of the textbook shows how to compute these duals.

Eg. Find the dual to the following LP:

$$
\begin{aligned}
\max\ & 8x_1 + 3x_2 - 2x_3 \\
\text{s.t.}\ & x_1 - 6x_2 + x_3 \geq 2 \iff x_1 - 6x_2 + x_3 - x_4 = 2 \\
& 5x_1 + 7x_2 - 2x_3 = -4 \\
& x_1 \leq 0,\ x_2 \geq 0,\ x_3 \text{ unrestricted}
\end{aligned}
$$

$x_0 = -x_1$ $\qquad$ $x_3 = x_5 - x_6$

Economic interpretation of duality and complementary slackness

Recall the diet problem:

- n food items, $i = 1, \ldots, n$
- Cost per unit quantity of item i is c_i
- m different nutrients $j = 1, \ldots, m$
- Food item i has a_{ij} units of nutrient j
- Minimum b_j requirement for nutrient j
- The goal is to find a minimum cost combination of food items satisfying the minimum ~~and maximum~~ nutritional constraints

Exercise: What are the LP and its dual for this problem?

$\min \ \underline{c}^T \underline{x} \quad \text{s.t.} \quad A\underline{x} \geq \underline{b}, \ \underline{x} \geq 0$

Economic interpretation of duality and complementary slackness

Interpretation of dual for diet problem

Suppose an entrepreneur offers to sell you pure nutrients and has to decide how much to charge y_j for each unit of nutrient j. Note that $\underline{y} \geq 0$

- Since b_j of nutrient j is required, the entrepreneur's revenue (which she wishes to maximize) is $b_1 y_1 + \cdots + b_m y_m = \underline{b}^T \underline{y}$
- The prices must be chosen so that food i is no cheaper than its "nutritional equivalent", i.e.

$$\forall i \quad a_{1i} y_1 + \cdots + a_{mi} y_m \leq c_i \iff A^T \underline{y} \leq \underline{c}$$

Economic interpretation of duality and complementary slackness

Strong duality says that if the entrepreneur sets the prices by solving the dual problem, then the cost of buying the foods will be equal to the cost of buying the nutrients from the entrepreneur.

Question: What is the economic interpretation of complementary slackness (below)?

$$x_i(a_{1i}y_1 + \cdots + a_{mi}y_m - c_i) = 0,$$
$$y_j\big(b_j - (a_{j1}x_1 + \cdots + a_{jn}x_n)\big) = 0.$$

The Farkas Lemma

A number of results related to duality are in the form of theorems of alternatives, which are of the form

Either statement A is true or statement B is true, but not both!

The most famous of these is the so-called Farkas Lemma:

Farkas Lemma: Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Then Either (i) there exists $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \geq \mathbf{0}$ s.t. $A\mathbf{x} = \mathbf{b}$ or

(ii) there exists $\mathbf{y} \in \mathbb{R}^m$ s.t. $A^T\mathbf{y} \leq \mathbf{0}$ and $\mathbf{b}^T\mathbf{y} > 0$.

(But not (i) and (ii))

The Farkas Lemma

Comment: by proving the Farkas lemma by strong duality, we have shown that

simplex + weak duality ⇒ strong duality ⇒ Farkas lemma

It is also possible to go the other way around:

Farkas + weak duality ⇒ strong duality ⇒ simplex

The Arbitrage Theorem

Suppose you can make bets (wagers) on at most m choices. Once you make your wager, one of n events happens. If you bet \$1 on choice i and event j happens then you receive a "reward" of R_{ij}, where $R \in \mathbb{R}^{m \times n}$. R_{ij} could be -ve.

reward from betting $\underline{x}$ when event j occurs

Question: Can you make a combination of wagers so that you are guaranteed to make a profit? (You are allowed to wager negative amounts.)

I.e. is there some $\mathbf{x} \in \mathbb{R}^m$ such that $\mathbf{g}^T = \mathbf{x}^T R > \mathbf{0}$?

This is called arbitrage.

$x_1 R_{1j} + x_2 R_{2j} + \ldots + x_n R_{nj} > 0$

The Arbitrage Theorem

Arbitrage Theorem: Given wagers $1,2,\dots,m$, outcomes $1,2,\dots,n$ and a reward R_{ij} if $1 is wagered on i and outcome j occurs, then either

(i) there is a combination of wagers $\mathbf{x} \in \mathbb{R}^m$ such that $\mathbf{x}^T R > \mathbf{0}$ or

(ii) there exists a probability vector $\mathbf{p} \in \mathbb{R}^n$ with $\mathbf{p} \geq \mathbf{0}$, $\sum p_i = 1$ such that $R\mathbf{p} = \mathbf{0}$.

(can't both happen)

$R_{i1}p_1 + R_{i2}p_2 + \dots + R_{in}p_n = 0 \quad \forall i$

= exp. reward from betting $1 on wager i.

The Arbitrage Theorem

Example (pricing options)

Suppose a stock is worth S now and in the next period can either have the value $S(1+\delta)$ or $S(1-\gamma)$ with $0<\delta,\gamma<1$. You can also buy a European call option with strike price K. (You can purchase the stock in the next period for price K regardless of its true price, where $S(1+\delta)<K<S(1-\gamma)$.) What is the fair price of such an option?