Linear Programming

Class 4: Introduction to the simplex algorithm

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Class outline

- Recap
- Basic feasible solutions
- Idea of the simplex method
- Example
- Pivoting
- Finding an intial basic feasible solution

Recap

- We are trying to solve: $\min \mathbf{c}^T\mathbf{x}$ s.t. $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq 0$.
- Extreme points of a polyhedron are the intersection of n L.I. defining hyperplanes.
- Linear programs have extreme point solutions.
- If the matrix $m \times n$ matrix A has rank m then we can write $A = (B|N)$ for $m \times m$ invertible matrix B and $m \times (n-m)$ matrix N.
- If $A\mathbf{x} = \mathbf{b}$ has a solution, a general solution can be written $\mathbf{x} = (\mathbf{x}_B|\mathbf{x}_N)$, where

$$\mathbf{x}_B = B^{-1}b - B^{-1}N\mathbf{x}_N$$

Basic feasible solutions

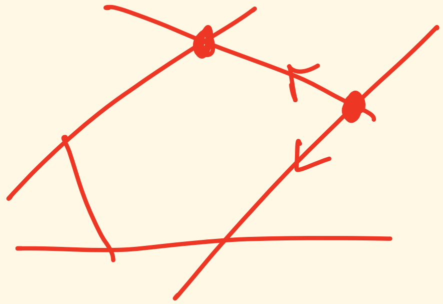
- If we set $\mathbf{x}_N = \mathbf{0}$ in $\mathbf{x}_B = B^{-1}b - B^{-1}N\mathbf{x}_N$ we obtain a *basic solution*.
- A basic solution with $\mathbf{x} \geq \mathbf{0}$ is called a *basic feasible solution (bfs)*.

Proposition: A point $\mathbf{x}$ is an extreme point if and only if it is a bfs.

Idea of the simplex method

1. Identify a bfs
2. While there is a neighboring bfs with a smaller objective move to this bfs
3. Declare the current solution optimal

[For now we won't worry about identifying a bfs.]



Example 1

$$A = \begin{pmatrix} 1 & -2 & 3 & -4 & 5 & -6 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ -3 & 0 & 0 & 2 & -1 & 1 \end{pmatrix},$$

$$\boldsymbol{b} = \begin{pmatrix} 6 \\ 6 \\ -3 \end{pmatrix}, \qquad \boldsymbol{c} = \begin{pmatrix} 2 \\ 1 \\ -4 \\ -2 \\ 6 \\ 0 \end{pmatrix}$$

How to choose a new bfs

- Substitute $\mathbf{x}_B = B^{-1}b - B^{-1}N\mathbf{x}_N$ into the objective function $\mathbf{z} = \mathbf{c}^T\mathbf{x}$ to get

$$\mathbf{z} = \mathbf{c}_B^T B^{-1}\mathbf{b} + (\mathbf{c}_N^T - \mathbf{c}_B^T B^{-1}N)\mathbf{x}_N$$

- $\mathbf{s}_N^T = \mathbf{c}_N^T - \mathbf{c}_B^T B^{-1}N$ is called the *reduced cost vector*.
- If the reduced cost vector has any negative component then the objective can be improved (reduced).

How to choose a new bfs

- Suppose the jth component of the reduced cost vector $\mathbf{s}_N$ is negative. We will *bring x_j into the basis* by increasing x_j as much as possible (to some value α) and decreasing some other basic x_i.

Claim: the maximum value of α we can choose is
$\alpha = \min\left\{\frac{(x_B)_i}{T_{ij}} : T_{ij} > 0\right\}$ where $T = B^{-1}N$.

How to choose a new bfs

- If i minimizes $\frac{(x_B)_i}{T_{ij}}$ over all $T_{ij} > 0$, we say we are *pivoting on (i, j).*
- In this case, i leaves the basis and j enters the basis.

Finding an initial bfs (clever trick)

- Include one new variable $x_i' \geq 0$ per constraint i of the original LP, adding if $\mathrm{b}_i \geq 0$ and subtracting if $\mathrm{b}_i < 0$.
- For this new system, $\mathbf{x} = \mathbf{0}, \mathbf{x}' = (|b_1|, \dots, |b_m|)^T$ is feasible.
- Add new objective, $\sum_i x_i'$.
- If the optimal solution is $\mathbf{x} = \mathbf{x}_0, \mathbf{x}' = \mathbf{0}$ then $\mathbf{x} = \mathbf{x}_0$ is a bfs to the orignal LP, otherwise the original LP is infeasible.

Example 2

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 \end{pmatrix},$$

$$\boldsymbol{b} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \qquad \boldsymbol{c} = \begin{pmatrix} 1 \\ 2 \\ -3 \\ -1 \\ -5 \end{pmatrix}$$