Linear Programming

Class 7: The dual simplex method and multiple optimality

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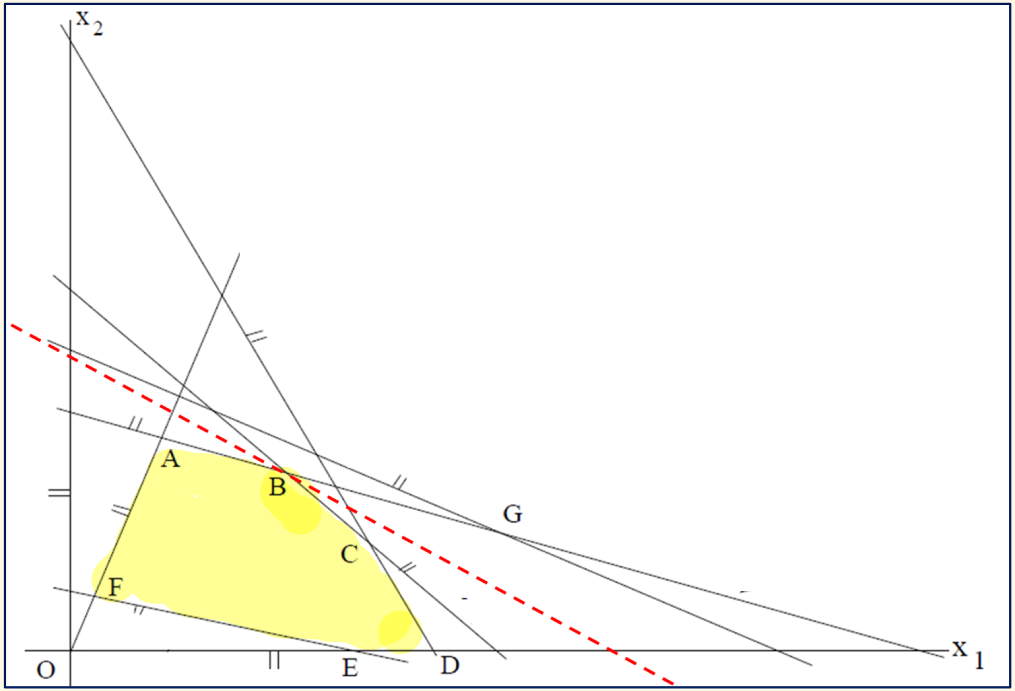
Class outline

- Shadow prices
- Opportunity costs
- Equivalence of feasibility and optimization problems
- Dual simplex method
- Multiple optimality

Shadow prices: example

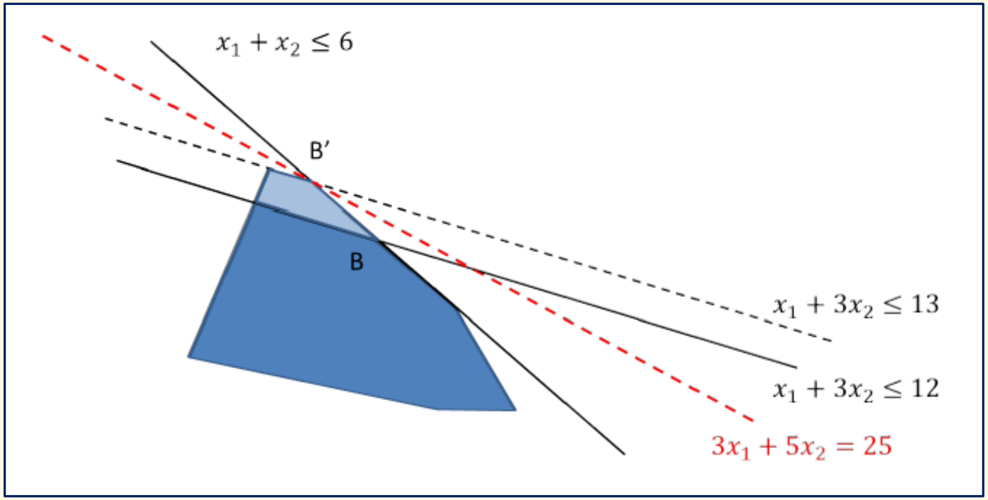
$$\begin{aligned} \max\ & 3x_1 + 5x_2 \\ \text{subject to } & 2x_1 + x_2 \leq 10 \\ & x_1 + 2x_2 \leq 10 \\ & x_1 + x_2 \leq 6 \\ & x_1 + 3x_2 \leq 12 \\ & 3x_1 - x_2 \geq 0 \\ & x_1 + 4x_2 \geq 4 \\ & x_1, x_2 \geq 0. \end{aligned}$$



$3x_1 + 5x_2 = 24$

Shadow prices: example

What if we change the right-hand side of a constraint?



New solution: $x = (2.5, 3.5)$, objective $=$ 25.

Shadow prices

In general:

Proposition: Suppose $\mathbf{x}^*$ is a non-degenerate solution to the following LP.

dual:

$\min \underline{b}^T \underline{y}$

s.t. $A^T \underline{y} \geq \underline{c}$

$$\max \mathbf{c}^T\mathbf{x}$$

$$\text{subject to } A\mathbf{x} \leq \mathbf{b},$$

i.e. tight, i.e. holds with equality

With dual vector $\mathbf{y}$. Suppose the jth inequality is active and the RHS of the inequality is increased by a "small" amount ε. Then the optimal value of the objective will increase by $y_j\varepsilon$.

Similarly for other standard forms of LPs.

Economic interpretation of duals (wood carving example)

$$
\begin{aligned}
\max\ & 20x_1 + 30x_2 + 30x_3 \\
\text{subject to } & 5x_1 + 3x_2 + 3x_3 \leq 7000 \text{ (man hours)} \\
& 4x_1 + 2x_2 + 6x_3 \leq 2500 \text{ (machine hours)} \\
& x_2 + x_3 \leq 600 \text{ (space)} \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

Optimal solution is $\mathbf{x} = (325,600,0)$ with optimal objective value \$24,500.

The duals of the resource constraints tell us how much revenue can be increased if extra resources are made available. Often referred to as *shadow prices* or *marginal values*.

Economic interpretation of duals (wood carving example)

$$
\begin{aligned}
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\text{subject to}\ & 5x_1 + 3x_2 + 3x_3 \le 7000 \text{ (man hours)} \\
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& x_1, x_2, x_3 \ge 0
\end{aligned}
$$

Optimal solution is $\mathbf{x} = (325,600,0)$ with optimal objective value \$24,500.

The duals of the non-negativity constraints are often called *opportunity costs* or *reduced costs*, depending on the interpretation.

Equivalence of feasibility and optimization problems

- Linear programming is the study of minimizing/maximizing a linear function subject to linear constraints.
- It *seems* that simply finding a feasible solution should be easier than the optimization problem.
- Actually…

Proposition: Any algorithm for solving the feasibility problem for linear programming can also be used to solve the optimization problem.

The dual simplex method

- How does the simplex method look from the point of view of the dual problem?

the tableau of the simplex algorithm at some intermediate step

$$
\begin{array}{c|c|c}
\mathbf{x}_B^T & \mathbf{x}_N^T & \\
\hline
I & B^{-1}N & B^{-1}\mathbf{b} \\
\hline
\mathbf{0}^T & \mathbf{s}_N^T = \mathbf{c}_N^T - \mathbf{c}_B^T B^{-1} N & -\mathbf{z}^T
\end{array}
$$

For primal:

$\mathbf{x}_B = B^{-1}\mathbf{b}$ is

- feasible
- not optimal

For dual:

$\mathbf{y}^T = \mathbf{c}_B^T B^{-1}$, $\mathbf{s} = \mathbf{c} - A^T\mathbf{y}$ is

- infeasible ($\mathbf{s}$ not ≥ 0)
- "optimal" ($\mathbf{s}^T\mathbf{x} = \mathbf{0}$)

The dual simplex method

- From the point of view of the primal, we are moving from one solution to another maintaining feasibility ($\mathbf{x} \geq \mathbf{0}$) while trying to achieve optimality ($\mathbf{s} \geq \mathbf{0}$).
- From the point of view of the dual we are moving from one infeasible point $(\mathbf{y}, \mathbf{s})$ to another, maintaining optimality ($\mathbf{s}^T\mathbf{x} = \mathbf{0}$) while trying to achieve feasibility ($\mathbf{s} \geq \mathbf{0}$).
- So what happens when we apply the simplex method to the dual problem and view it from the point of view of the primal?

The dual simplex method

- The dual simplex method maintains feasibility in the dual ($\boldsymbol{s} \geq \mathbf{0}$) and optimality in the primal ($\mathbf{x}^T \mathbf{s} = \mathbf{0}$) while trying to move towards feasibility in the primal ($\mathbf{x} \geq \mathbf{0}$).
- In each iteration we have optimality satisfied for the primal ($\boldsymbol{s} \geq \mathbf{0}$) but feasibility may not be satisfied ($\mathbf{x} \ngeq \mathbf{0}$)
- I.e. we move between basic solutions, not basic feasible solutions.
- To improve the current situation we first choose an index i such that $(\mathbf{x}_B)_i < 0$ to leave the basis then choose an index j to enter the basis in such a way that $\boldsymbol{s} \geq \mathbf{0}$.

The dual simplex method: example

$$\begin{array}{ll} \min & 3x_1 + 4x_2 + 5x_3 \\ \text{s.t.} & x_1 + 2x_2 + 3x_3 \geq 5 \\ & 2x_1 + 2x_2 + x_3 \geq 6 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

$$\Leftrightarrow \quad \begin{array}{ll} \min & 3x_1 + 4x_2 + 5x_3 \\ \text{s.t.} & x_1 + 2x_2 + 3x_3 - x_4 = 5 \\ & 2x_1 + 2x_2 + x_3 - x_5 = 6 \\ & x_1, x_2, x_3, x_4, x \end{array}$$

The dual simplex method: pivoting

Pivoting rule: Suppose we choose index i to leave the basis. Then the column to enter the basis is the j that minimizes $\left\{\frac{(\mathbf{s}_N)_j}{-T_{ij}} : T_{ij} < 0\right\}$. If $T_{ij} \geq 0$ for all j then the dual is unbounded and the primal is infeasible.

The dual simplex method: exercise

$$
\begin{array}{ll}
\min & 2x_1 + 3x_2 + 4x_3 + 5x_4 \\
\text{s.t.} & x_1 - x_2 + x_3 - x_4 \geq 10 \\
& x_1 - 2x_2 + 3x_3 - 4x_4 \geq 6 \\
& 3x_1 - 4x_2 + 5x_3 - 6x_4 \geq 15 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{array}
$$

The dual simplex method: remarks

- The dual simplex method, in general does not have any advantage or disadvantage over the simplex method. In special cases where a basic dual feasible solution is available it may be better to the use the dual simplex method: for example, if $\mathbf{c} \geq \mathbf{0}$ then $\mathbf{y} = \mathbf{0}$ is feasible since $A^T\mathbf{y} \leq \mathbf{c}$ is satisfied.
- The dual simplex method is also useful when a problem is solved to optimality and then a new constraint is added, making the current optimal solution infeasible.

Multiple optimality

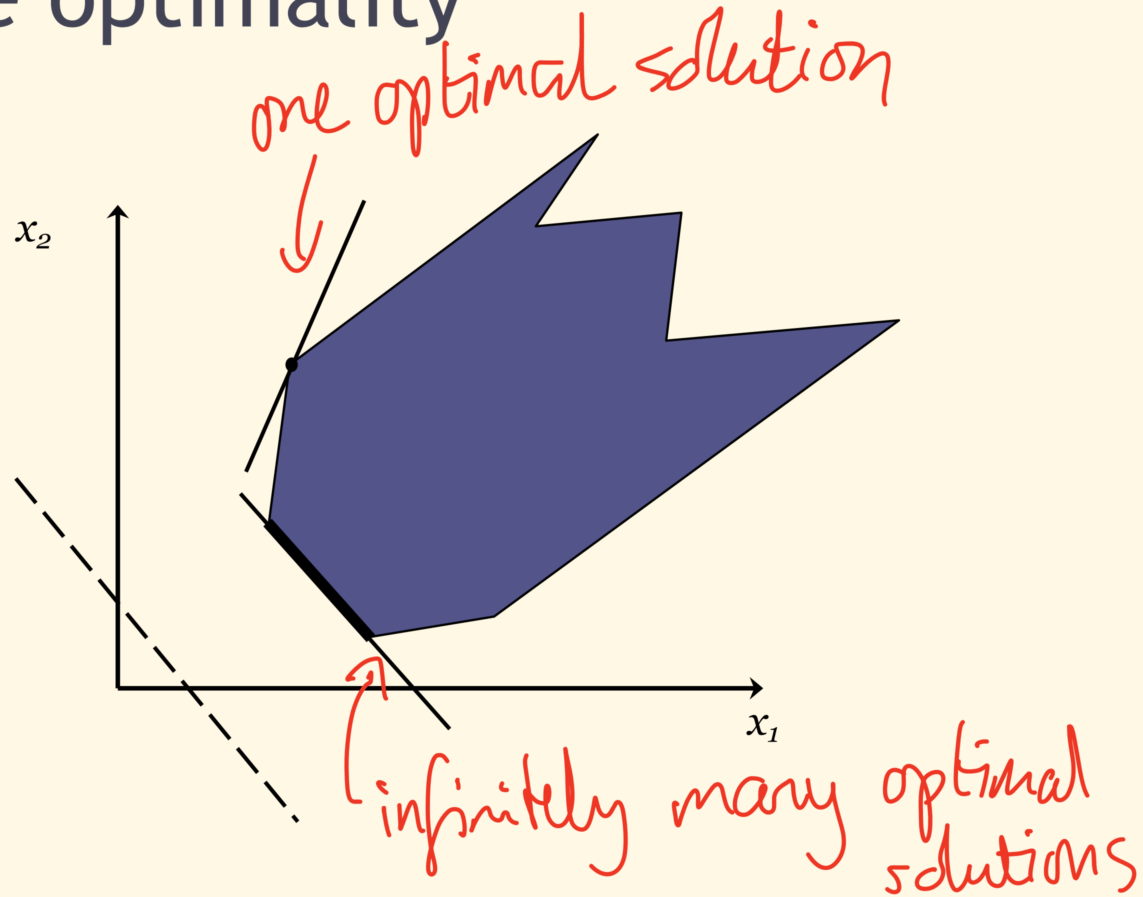
Possible scenarios for an LP:

- Infeasible
- Unbounded
- Unique optimal solution
- Infinitely many optimal solutions

Suppose $\underline{x}_1$ and $\underline{x}_2$ are opt. solutions of $\min \underline{c}^T\underline{x}$ s.t. $A\underline{x} \geq \underline{b}$. Then $\lambda \underline{x}_1 + (1-\lambda)\underline{x}_2$ is also optimal $(0 \leq \lambda \leq 1)$, since (see scribbles)

Multiple optimality

Example



Multiple optimality

Example

$$
\begin{aligned}
\max\ & 2x_1 + 10x_2 \\
s.t.\ & 2x_1 + 10x_2 \leq 61 \\
& 2x_1 + 1x_2 \leq 28 \\
& -x_1 + 4x_2 \leq 24.4 \\
& 4x_1 - x_2 \leq 38 \\
& 2x_1 + 3x_2 \leq 54 \\
& -x_1 \leq 0 \\
& -x_2 \leq 0
\end{aligned}
$$

Multiple optimality

Example

active/tight

$$
\begin{aligned}
\max\ & 2x_1 + 10x_2 \\
s.t.\ & 2x_1 + 10x_2 \le 61 \\
& 2x_1 + 1x_2 \le 28 \\
& -x_1 + 4x_2 \le 24.4 \\
& 4x_1 - x_2 \le 38 \\
& 2x_1 + 3x_2 \le 54 \\
& -x_1 \le 0 \\
& -x_2 \le 0
\end{aligned}
$$

Optimal solution $x^* =$ (0,6.1)

Optimal value of objective is 61

Multiple optimality

Example To check for other solutions, solve:

$$\max \ (61-2x_1-10x_2)+(24.4+x_1-4x_2)+(0+x_1)$$

$$s.t. \ \ 2x_1+10x_2 \le 61$$

$$2x_1+1x_2 \le 28$$

$$-x_1+4x_2 \le 24.4$$

$$4x_1-x_2 \le 38$$

$$2x_1+3x_2 \le 54$$

$$-x_1 \le 0$$

$$-x_2 \le 0$$

$$2x_1+10x_2 = 61$$

Multiple optimality

Example To check for other solutions, solve:

$$\max \ (61 - 2x_1 - 10x_2) + (24.4 + x_1 - 4x_2) + (0 + x_1)$$

$$\begin{aligned} s.t. \quad & 2x_1 + 10x_2 \le 61 \\ & 2x_1 + 1x_2 \le 28 \\ & -x_1 + 4x_2 \le 24.4 \\ & 4x_1 - x_2 \le 38 \\ & 2x_1 + 3x_2 \le 54 \\ & -x_1 \le 0 \\ & -x_2 \le \end{aligned}$$

$$2x_1 + 10x_2 = 61$$

New optimal solution: $x' = (10.5,4)$

Multiple optimality

In general, suppose x^* is an optimal solution to:

$$\begin{aligned} &\min \mathbf{c}^T\mathbf{x} \\ &\text{s.t. } A\mathbf{x} = \mathbf{b}, \\ &\qquad \mathbf{x} \geq \mathbf{0} \end{aligned}$$

where $\mathbf{c}^T\mathbf{x}^* = \gamma$, and $\mathbf{x}^* = (\mathbf{x}_\mathrm{B}^*, \mathbf{x}_\mathrm{N}^*)$ where $\mathbf{x}_\mathrm{B}^*$ is the "basic part" and $\mathbf{x}_\mathrm{N}^*$ is the "non-basic part", as usual. To look for other optimal solutions, solve

$$\begin{aligned} &\max \mathbf{1}^T\mathbf{x}_\mathrm{N} \\ &\text{s.t. } A\mathbf{x} = \mathbf{b}, \\ &\qquad \mathbf{x} \geq \mathbf{0}, \\ &\quad \mathbf{c}^T\mathbf{x} = \gamma \end{aligned}$$

Multiple optimality

Exercise: Find all the solutions to the following LP.

$$
\begin{aligned}
\text{max.}\quad & 30x_1 + 50x_2 \\
\text{s.t.}\quad & 5x_1 + 15x_2 \leq 300 \\
& 6x_1 + 10x_2 \leq 240 \\
& 12x_1 + 8x_2 \leq 450 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$