Page No.
Date: / /

Numerical Differentiation

Using Divided Differences

The Newton Polynomial is

$P_n(x) = f[x_0] + f[x_0, x_1] \ldots$

Then

$P'_n(x)$ is a good approximation of $f'(x)$ so

$$P'_n(x) = f[x_0, x_1] + f[x_0, x_1, x_2][(x - x_1) + (x - x_0)] + \ldots + f[x_0, x_1, \ldots, x_n] \sum_{i=0} \frac{(x - x_0)(x - x_1)\ldots(x - x_{n-1})}{x - x_i}$$

For evenly spaced data,

$$P'_n(x) = \frac{d}{dx} P_n(x) = \frac{d}{ds} P_n(x) \cdot \frac{ds}{dx}$$

$$= \frac{1}{h}\left[\Delta f_0 + \frac{\Delta^2 f_0}{2!}[s + (s-1)] + \ldots + \frac{\Delta^n f_0}{n!} \sum_{i=0}^{n-1} \frac{s(s-1)\ldots(s-n+1)}{s-i}\right]$$

Explicitly Defined Function

Forward diffⁿ

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}$$

for small positive value for h

Page No.
Date : / /

Backward diff^n:

$$f'(a) \approx \frac{f(a) - f(a-h)}{h}$$

for small value of h (positive)

Central diff^n

$$f'(a) \approx \frac{f(a+h) - f(a-h)}{2h}$$

Second diff^n;

$$f''(a) \approx \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$$

1) Find the derivative of fol^n tabulated diff function at $x = 4.1$ using divided diff^n table.

x	2	3	5	6
f	3	7	21	31

Sol^n:

Here, $n = 3$, so the formula for the derivation of above tabulated function is

$$P_3'(x) = f[x_0, x_1] + f[x_0, x_1, x_2]\,[(x - x_0) + (x - x_1)] + f[x_0, x_1, x_2, x_3]\,[(x - x_0)(x - x_1) + (x - x_1)(x - x_2) + (x - x_0)(x - x_2)] \quad \text{--- (1)}$$

Page No.
Date: / /

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_0, x_1, x_2, x_3]$
2	3			
		4		
3	7		1	
		7		0
5	21		1	
		10		
6	31			

$$P_3(x) = 4 + 1[(x-2) + (x-3)] + 0$$
$$= 2x - 1$$

When $x = 4.1$

$$P_3(4.1) = 2(4.1) - 1 = 7.2$$

2.

x	0.21	0.23	0.27	0.32
f	0.3222	0.3619	0.4314	0.5051

Sol^n;

Here, $n = 3$,

$$P_3'(x) = f[x_0, x_1] + f[x_0, x_1, x_2][(x-x_0) + (x-x_1)] + f[x_0, x_1, x_2, x_3]$$
$$[(x-x_0) + (x-x_1) + (x-x_2)) + (x-x_1) + (x-x_0))(x-x_2)] +$$

Page No.
Date: / /

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_0, x_1, x_2, x_3]$
0.21	0.3222			
		1.975		
0.23	0.3617		-3.775	
		1.7425		8.109
0.27	0.4314		2.9803 - ~~0.298~~	
		1.474		
0.32	0.5061			

$P_3(x) = 1.975 + (-3.875)[(x-0.21)+(x-0.23)] +$
$8.109((x-0.21)(x-0.23)+(0.23)(0.27)+(x-.21)(x-.27))$

$= 1.975 - 3.875(2x - 0.44) + 8.109[x^2 - 0.44x +$
$0.0483 + x^2 - 0.5x + 0.0621 + x^2 - 0.48x + 0.0567]$
$= 1.975 - 7.75x + 1.705 + 8.109(3x^2 - 1.42x + 0.1671)$
$= 3.68 - 7.75x + 24.327x^2 - 11.514x + 1.355$
$= 24.327x^2 - 19.264x - 2.325$

3. For the function $f(x) = e^x \sqrt{\sin x + \ln x}$ estimate $f'(6.3)$ and $f''(6.3)$ taking $h = 0.01$

Given;

$$f(x) = e^x \sqrt{\sin x + \ln x}$$
$$h = 0.01$$

The

$$f(6.3) = e^{6.3} \sqrt{\sin 6.3 + \ln(6.3)}$$
$$= 742.1705$$

$$f(6.3 + 0.01) = e^{6.31} \sqrt{\sin 6.31 + \ln(6.31)}$$
$$= 751.9634$$

$$f(6.3 - 0.01) = e^{6.29} \sqrt{\sin 6.29 + \ln(6.29)}$$
$$= 732.490$$

$$\cancel{f'(6.3) \approx \frac{f(a) - f(a-h)}{h}}$$
$$\cancel{\approx \frac{742.1705 - 732.490}{2(0.01)}}$$

$\approx$

$$f'(6.3) = \frac{f(a+h) - f(a-h)}{2h}$$
$$= \frac{751.9634 - 732.490}{2(0.01)}$$
$$= 973.6650$$

$$f''(6.3) = \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$$

$$= \frac{751.9634 - 2(742.1705) + 732.490}{(0.01)^2}$$

$$= 1125$$

$$\int_a^b f(x)\,dx = \int_a^b P_n(x)\,dx$$

$$= \int_a^b \Big[f_0 + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \cdots + \frac{s(s-1)\cdots(s-n+1)}{n!}\Delta^n f_0 \Big] dx.$$

which is Newton-cotes formula for numerically evaluating $\int_a^b f(x)dx$

Trapezoidal

$$\int_a^b f(x)\,dx \approx \frac{h}{2}[f_0 + f_1]$$

Simpson 1/3

$$\int_a^b f(x)\,dx = \frac{h}{3}(f_0 + 4f_1 + f_2)$$

Simpson 3/8

$$= \frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3)$$

Composite Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f_0 + 2(f_1 + \cdots + f_{n-1}) + f_n\right]$$

Q. Evaluate the integral of the foll^n function from $x=1.0$ to $x=1.8$ using composite trapezoidal rule with $h=0.1$, $h=0.2$ & $h=0.4$.

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8
$f(x)$	1.543	1.669	1.811	1.971	2.151	2.352	2.577	2.828	3.107

Sol^n;

Here,

$f_0 = 1.543$

$f_n = 3.107$

$h = 0.1$.

Then,

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f_0 + 2(f_1 + \cdots + f_{n-1}) + f_n\right]$$

$$\approx \frac{0.1}{2}\left[1.543 + 3.107 + 2(1.669 + 1.811 + 1.971 + 2.151 + 2.352 + 2.577 + 2.828)\right]$$

$$\approx \frac{0.1 \times 35.368}{2}$$

$$\approx 1.7684.$$

When, $h = 0.2$, Then,

Page No.
Date : / /

$\int_a^b f(x)dx \approx \frac{0.2}{2} \times 35.368$

$\approx \cancel{8.842}\ 3.5368$

when, $h = 0.4$,

$\int_a^b f(x)dx \approx \frac{0.4 \times 35.368}{2}$

≈ 7.0736

Page No.
Date: / /

c) $\int_{0.75}^{1.75} (\sin^2 x - 2x\sin x + 1)\,dx$ with $n=8$.

Sol^n.

$a = 0.75$, $x_1 = 0.85$, $x_2 = 01$, $x_2 = 1.125$, $x_3 = 1.25$, $x_4 = 1.375$.

$b = 1.75$ $\qquad x_5 = 1.5, x_6 = 1.625$

$h = \frac{1.75 - 0.75}{8} = \cancel{0.125}\ 0.125.$ $\qquad x_7 = 1.75$

Comp^n Trapezoidal Rule $\approx \frac{h}{2}\left[f(x_0) + f(x_n) + 2\left(f x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6) + f(x_7)\right)\right]$

$\approx \cancel{\frac{0.125}{2}}\ \frac{0.125}{2}\left[\cancel{16}\right.$

$\approx \frac{0.125}{2}\left[0.44214 \cancel{+0} - 1.47576 + 2(0.24592 + 0.02513 - 0.21601 - 0.47189 - 0.7353 - 0.99479 - 1.24816)\right]$

$\approx \frac{0.125 \times (-9.73083)}{2}$

$\approx \cancel{0.60817}\ -4.865415.$

Comp^n Simpson Rule

$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[f(x_0) + 4(f x_1 + f x_3 + \dots f(x_{n-1})) + 2(f x_2 + f x_4 + \dots x_n)\right]$

Composite Simpson 1/3 Rule.

$\int_1^2 \frac{e^x}{x} dx$, $n=8$.

Soln;

$a=1, b=2, n=8, \quad h=\frac{2-1}{8}=0.125$

$x_0=1, x_1=1.125, x_2=1.25, x_3=1.375, x_4=1.5, x_5=1.625$
$, x_6=1.75, x_7=1.875, x_8=2$

Then,

$\int_1^2 \frac{e^x}{x} dx \approx \frac{h}{3}\left[f(x_0)+4(fx_1+f_3\ldots+f(x_7))\right]+2(fx_2+x_4\ldots+x_{08})]$

$\approx \frac{0.125}{3}[2.71828+4(2.43797+2.85735+3.12518+3.47777)+2(2.792274+2.98779+3.28834+3.694528)]$

$\approx 0.041666[76.968496]$

≈ 3.056969.

$\int_e^{e+2} \frac{dx}{x \ln x}$ with $n=8$.

Soln.

$a=e, b=e+2, n=8, h=\frac{e+2-e}{8}=\frac{1}{4}=0.25$

$x_0=e, x_1=e+0.25, x_2=e+0.5, x_3=e+0.75, x_4=e+1$
$x_5=e+1.25, x_6=e+1.5, x_7=e+1.75, x_8=e+2.$

Then

Page No.
Date: / /

$$\int_e^{e+2} \frac{dx}{x \ln x} \approx \frac{h}{3}\left[f(x_0) + 4(f(x_1) + f(x_3) + \dots f(x_7)) + 2(f(x_2) + f(x \dots + f(x_8)\right]$$

$$\approx \frac{h}{3}\left[0.367879 + 4(0.30965 + \cancel{0.26584} + 0.23184 + 0.18283 + 0.1495) + 2(0.26584 + 0.20449 + 0.16469 + 0.13661)\right]$$

$$\approx \frac{0.25}{3}\left[5.407019\right]$$

$$\approx 0.450584 \; //.$$

Composite Simpson's 3/8 rule.

$$\int_a^b f(x)\,dx \approx \frac{3h}{8}\left[f_0 + 3(f_1 + f_4 + \dots + f_{n-2}) + 3(f_2 + f_5 + \dots + f_{n-1}) + 2(f_3 + f_6 + \dots + f_{n-3}) + f_n\right].$$

$f(x) = e^{-x^2}$ from $x = 0.2$ to $x = 1.4$ with $n = 6$.

Given,

$$\int_{0.2}^{1.4} e^{-x^2}\,dx$$

$a = 0.2$, $b = 1.4$, $n = 6$, $h = \frac{1.4 - 0.2}{6} = 0.2$.

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Page No.
Date: / /

$x_0 = 0.2,\ x_1 = 0.4,\ x_2 = 0.6,\ x_3 = 0.8,\ x_4 = 1,\ x_5 = 1.2$
$x_6 = 1.4$
Then,

$$\int_{0.2}^{1.4} e^{-x}\,dx \approx \frac{3h}{8}\left[f_0 + f_6 + 3(f_1 + f_2 + f_4 + f_5) + 2(f_3)\right]$$

$$\approx \frac{3(0.2)}{8}\left[0.96078 + 0.14085 + 3(0.852143 + 0.697676 + 0.367879 + 0.236927) + 2(0.527292)\right]$$

$$\approx \frac{0.6}{8}\left[8.111305\right]$$

$$\approx 0.608347$$

Simpson 1/3 Rule

c) $\int_{-0.5}^{0.5} \cos^2 x\, dx$ ~~and~~ $n=4$.

Given;

$f(x) = \cos^2 x$, $a = -0.5$, $b = 0.5$, $n = 4$, $h = \frac{0.5+0.5}{4} = 0.25$

Then,

$x_0 = -0.5$ $x_3 = 0.25$

$x_1 = -0.25$ $x_4 = 0.5$

$x_2 = 0$

We know,

$$\int_{-0.5}^{0.5} \cos^2 x\, dx = \frac{h}{3}\left[f_0 + 4\left(f(x_1) + f(x_3)\right) + 2\left(f(x_2) + f(x_4)\right)\right]$$

$$\approx \frac{0.25}{3}\left[0.77015 + 4\left(0.93879 + 0.93879\right) + 2\left(1 + 0.770151\right)\right]$$

$$\approx \frac{0.25}{3}\left[11.820472\right]$$

$$\approx 0.985064 //$$

e) $\int_{0.75}^{1.75} (\sin^2 x - 2x\sin x + 1)\, dx$ with $n = 8$

Sol^n

$f(x) = \sin^2 x - 2x\sin x + 1$, $a = 0.75$, $b = 1.75$, $n = 8$, $h = 0.125$

$x_0 = 0.75$ $x_5 = 1.375$

$x_1 = 0.875$ $x_6 = 1.5$

$x_2 = 1$ $x_7 = 1.625$

$x_3 = 1.125$ $x_8 = 1.75$

$x_4 = 1.25$

Then,

$$\int_{0.75}^{1.95} \sin^2 x - 2x\sin x + 1 \approx \frac{h}{3}\left[f(x_0) + 4(f(x_1) + f(x_3) + f(x_5) + f(x_7)) + 2(f(x_2) + f(x_4) + f(x_6) + f(x_8))\right]$$

$$\approx \frac{0.125}{3}\left[0.442177 + 4\{0.28724 + (-0.216015) + (-0.735304) + (-1.24816)\} + 2\{0.025131 + (-0.47188) + (-0.99748) + (-1.47572)\}\right]$$

$$\approx -4.865415.$$

f) $\int_{-0.5}^{0.5} x\ln(x+1)\,dx$ with $n = 6$.

Given;

$f(x) = x\ln(x+1)$, $a = -0.5$, $b = 0.5$, $n = 6$, $h = \frac{0.5+0.5}{6}$

$= 0.16666$

$x_0 = -0.5$ $\quad x_4 = 0.16666\,4$

$x_1 = -0.333334$ $\quad x_5 = 0.33333$

$x_2 = -0.166668$ $\quad x_6 = 0.499996 \approx 0.5$.

$x_3 = -0.00002$

Then,

$$\int_{-0.5}^{0.5} x\ln(x+1) \approx \frac{h}{3}\left[f(x_0) + 4\{f(x_1) + f(x_3) + f(x_5)\} + 2\{f(x_2) + f(x_4) + f(x_6)\}\right]$$

$$\approx \frac{0.16666}{3}\left[0.34657 + 4\{0.135161 + 0.0003 + 0.09587\} + 2\{0.03039 + 0.025683 + 0.202703\}\right]$$

$$\approx \frac{0.16666}{3}[1.801446]$$

$$\approx 0.100007$$

g) $\int_0^2 e^{2x}\sin 3x\,dx$ with $n=8$

Given;

$f(x)=e^{2x}\sin 3x\,dx$, $a=0$, $b=2$, $h=\frac{2-0}{8}=0.25$

$x_0=0$ $\quad x_5=1.25$

$x_1=0.25$ $\quad x_6=1.5$

$x_2=0.5$ $\quad x_7=1.75$

$x_3=0.75$ $\quad x_8=2$

$x_4=1$

Then,

$$\int_0^2 e^{2x}\sin 3x\,dx \approx \frac{h}{3}\left[f(x_0)+4\{f(x_1)+f(x_3)+f(x_5)+f(x_7)\}+2\{f(x_2)+f(x_4)+f(x_6)+f(x_8)\}\right]$$

$$\approx \frac{0.25}{3}\left[0+4\{1.123832+3.48708+(-6.96304)+(-28.4440)\}+2\{2.711472+1.042743+(-19.63421)+(-15.25556)\}\right]$$

$$\approx \frac{0.25}{3}\left[-185.4556\right]$$

$$\approx -15.45463$$

h) $\int_0^{\pi} x^2\cos x\,dx$ with $n=6$

Given;

$f(x)=x^2\cos x\,dx$, $a=0$, $b=\pi$, $h=\frac{\pi}{6}=0.52359$

$x_0=0$ $\quad x_4=2\pi/3$

$x_1=\pi/6$ $\quad x_5=5\pi/6$

$x_2=\pi/3$ $\quad x_6=\pi$

$x_3=\cancel{5\pi/3}\ 3\pi/6$

$$\int_0^{\pi} x^2 \cos x\,dx \simeq \frac{h}{3}\left[f(x_0)+4\{f(x_1)+f(x_3)+f(x_5)\}+2\{f(x_2)+f(x_4)+f(x_6)\}\right]$$

$$\simeq \frac{0.52359}{3}\left[0+4\{0.23742+0+(-5.93564)\}+2\{0.548311+(-2.19324)+(-9.86960)\}\right]$$

$$\simeq \frac{0.52359}{3}\left[-45.82193\right]$$

$$\simeq -7.99730$$

i) $\int_1^2 \frac{e^x}{x}\,dx$ with $n=4$.

Soln;

Given, $f(x)=\frac{e^x}{x}$, $a=1$, $b=2$, $h=\frac{2-1}{4}=0.25$

$x_0=1$ $\quad x_3=1.75$

$x_1=1.25$ $\quad x_4=2$

$x_2=1.5$

Then,

$$\int_1^2 \frac{e^x}{x}\,dx \simeq \frac{0.25}{3}\left[2.718281+4\{2.79227+3.28834\}+2\{2.98779+3.69452\}\right]$$

$$\simeq \frac{0.25}{3}\left[40.4055\right]$$

$$\simeq 3.367125$$

3. Use Simpson 3/8 Rule.

a) $\int_1^{2.8}(x^3+1)\,dx$ with $n=9$.

Soln

$a=1,\ b=2.8,\ n=9,\ h=\frac{2.8-1}{9}=0.2.$

$x_0 = 1$

$x_1 = 1.2$

$x_2 = 1.4$

$x_3 = 1.6$

$x_4 = 1.8$

$x_5 = 2$

$x_6 = 2.2$

$x_7 = 2.4$

$x_8 = 2.6$

$x_9 = 2.8.$

we know,

$$\int_1^{2.8}(x^3+1)\,dx \approx \frac{3h}{8}\left[f(1)+3\{f(1.2)+f(1.4)+f(1.8)+f(2)+f(2.4)+f(2.6)\}+2\{f(1.6)+f(2.2)+f(2.8)\}\right]$$

$$\approx \frac{3(0.2)}{8}\left[2+3\{2.728+3.744+6.832+9+14.824+18.576\}+2\{5.096+11.648+22.952\}\right]$$

$$\approx 0.075\,[188.504]$$

$$\approx 14.1378$$

b) $\int_0^{\pi/2}\sin x\,dx$ with $n=6$

Soln;

$a=0,\ b=\pi/2,\ h=\frac{\pi/2}{6}=\frac{\pi}{12}$

$x_0=0$

$x_1=\pi/12$

$x_2=\pi/4$

$x_3=\pi/3$

$x_4=5\pi/12$

$x_5=\pi/12.$

$$\int_0^{\pi/2} \sin x\,dx \approx \frac{3h}{8}\left[f(0)+3\left(f(\pi/12)+f(\pi/4)+f(5\pi/12)\right)+f(\pi/2)\right.$$
$$\left.+2(f(\pi/3))\right]$$

$$\approx \frac{3\pi}{12(8)}\left[0+3\{0.258819+0.70710+0.965926\}+1+2\{0.866025\}\right]$$

$$\approx 0.098174\,[10.52756]$$

$$\approx 1.033533$$

Romberg Integration

$$R(n) = T\left(\frac{h}{2}\right) + \frac{1}{4^n - 1}\left[T\left(\frac{h}{2}\right) - T(h)\right]$$

1. Use Romberg int^n to find interval of e^{-x^2} bet^n the limits of $a = 0.2$ and $b = 1.5$ with initial subinterval size as $h = \frac{b-a}{2}$ $= 0.65$ & final size $h = \frac{b-a}{16} = 0.08125$.

Sol^n

let $T(h)$ denote the app^n of the integral using composite trapezoidal rule with subinterval length h. We need to calculate $T(h)$ for $h = 0.65, 0.325, 0.1625$ & 0.08125 then,