0.0.1 Inference

Testing models Consider the linear model

$$\mathbb{Y} = \mathbb{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} \quad \cdots \quad \text{full model}$$

where $\gamma\left(\mathbb{X}\right) = \gamma \leq p$, $E\left(\boldsymbol{\varepsilon}\right) = 0$, $Var\left(\boldsymbol{\varepsilon}\right) = \sigma^2 I$

Consider the linear model

$$\boldsymbol{y} = \mathbb{W}\boldsymbol{\gamma} + \boldsymbol{\varepsilon} \quad \cdots \quad \text{redued model}$$

where $C\left(\mathbb{W}\right) \subset C\left(\mathbb{X}\right)$, $E\left(\boldsymbol{\varepsilon}\right) = 0$, $Var\left(\boldsymbol{\varepsilon}\right) = \sigma^2 I$

note:

1. the estimation space of R.M. is smaller than in the F.M.
2. The goal is to test whether or not the reduced model is also correct.
   - If R.M. is correct, there is no reason not to use it.
   - Smaller models are easier to interpret.
3. Let $P_{\mathbb{X}}$ and $P_{\mathbb{W}}$ denote the ppm onto $C\left(\mathbb{X}\right)$ and $C\left(\mathbb{W}\right)$. Because $C\left(\mathbb{W}\right) \subset C\left(\mathbb{X}\right)$, we know that $P_{\mathbb{X}} - P_{\mathbb{W}}$ is the ppm onto $C\left(P_{\mathbb{X}} - P_{\mathbb{W}}\right) = \underset{\maltese}{\underline{C\left(\mathbb{W}\right)^{\perp} C\left(\mathbb{X}\right)}}$

Lemma 1 *Assume $P_{\mathbb{X}}$ and $P_{\mathbb{W}}$ are projection matries and $P_2 - P_1$ is P.D., then*

1. $P_1 P_2 = P_2 P_1 = P_2$
2. $P_1 - P_2$ *is a p.m.*

note:

1. P_1 is a p.m onto $C\left(\mathbb{X}\right)$
2. P_2 is a p.m onto $C\left(\mathbb{W}\right) \subset C\left(\mathbb{X}\right)$
3. $P_1 - P_2$ is a p.m onto the orthogonal complement of $\mathbb{W}$ with in $\mathbb{X}$

$\boxed{\text{figure}}$

Under F.M., our estimate for $E\left(\boldsymbol{y}\right) = \boldsymbol{x}\boldsymbol{\beta}$ is $P_{\mathbb{X}}\mathbb{Y}$
Under R.M., our estimate is $E\left(\boldsymbol{y}\right) = \mathbb{W}\boldsymbol{\gamma}$ is $P_{\mathbb{W}}\mathbb{Y}$

1. If R.M. is correct, then the $P_{\mathbb{X}}$ and $P_{\mathbb{W}}$ are estimates the same thing. (i.e. $P_{\mathbb{X}}\mathbb{Y} - P_{\mathbb{W}}\mathbb{Y} = \left(P_{\mathbb{X}} - P_{\mathbb{W}}\right)\mathbb{Y}$ should be small.)

2. the size of $(P_{\mathbb{X}} - P_{\mathbb{W}})\,\mathbb{Y}$ is

$$\frac{\left[(P_{\mathbb{X}} - P_{\mathbb{W}})\,\mathbb{Y}\right]^{\dagger}\left[(P_{\mathbb{X}} - P_{\mathbb{W}})\,\mathbb{Y}\right]}{\gamma\,(P_{\mathbb{X}} - P_{\mathbb{W}})} = \frac{\mathbb{Y}^{\dagger}\,(P_{\mathbb{X}} - P_{\mathbb{W}})\,\mathbb{Y}}{\gamma\,(P_{\mathbb{X}} - P_{\mathbb{W}})}$$

If R.M. is correct, then

$$\begin{aligned}
E\left(\frac{\mathbb{Y}^{\dagger}\,(P_{\mathbb{X}} - P_{\mathbb{W}})\,\mathbb{Y}}{\gamma\,(P_{\mathbb{X}} - P_{\mathbb{W}})}\right) &= \frac{1}{\gamma^{*}} E\left(\mathbb{Y}^{\dagger}\,(P_{\mathbb{X}} - P_{\mathbb{W}})\,\mathbb{Y}\right) \\
&= \frac{1}{\gamma^{*}} \left\{tr\left[(P_{\mathbb{X}} - P_{\mathbb{W}})\,\sigma^{2} I\right] + \right. \\
&\quad \left. (W\gamma)^{\dagger}\,(P_{\mathbb{X}} - P_{\mathbb{W}})\,(W\gamma)\right\} \\
&= \frac{1}{\gamma^{*}} \left\{\sigma^{2} tr\,(P_{\mathbb{X}} - P_{\mathbb{W}}) + \right. \\
&\quad \left. \gamma^{\dagger} W^{\dagger}\,(P_{\mathbb{X}} - P_{\mathbb{W}})\,W\gamma\right\} \\
&= \frac{1}{\gamma^{*}} \left\{\sigma^{2} \gamma\,(P_{\mathbb{X}} - P_{\mathbb{W}})\right\} \\
&= \sigma^{2} \quad \left(\text{U.E. of } \sigma^{2}\right)
\end{aligned}$$

If R.M. is not correct, then

$$\begin{aligned}
E\left(\frac{\mathbb{Y}^{\dagger}\,(P_{\mathbb{X}} - P_{\mathbb{W}})\,\mathbb{Y}}{\gamma^{*}}\right) &= \frac{1}{\gamma^{*}} \left\{tr\left[(P_{\mathbb{X}} - P_{\mathbb{W}})\,\sigma^{2} I\right]\right. \\
&\quad \left. + (\mathbb{X}\beta)^{\dagger}\,(P_{\mathbb{X}} - P_{\mathbb{W}})\,(\mathbb{X}\beta)\right\} \\
&= \sigma^{2} + (\mathbb{X}\beta)^{\dagger}\,(P_{\mathbb{X}} - P_{\mathbb{W}})\,\mathbb{X}\beta
\end{aligned}$$

$$MSE = \frac{\mathbb{Y}^{\dagger}\,(I - P_{\mathbb{X}})\,\mathbb{Y}}{\gamma\,(I - P_{\mathbb{X}})} = \frac{1}{n - \gamma}\mathbb{Y}^{\dagger}\,(I - P_{\mathbb{X}})\,\mathbb{Y}$$

$$E\,(MSE) = \sigma^{2}$$

Test Statistic

To test R.M. vs F.M., we use

$$\begin{aligned}
F &= \frac{\mathbb{Y}^{\dagger}\,(I - P_{\mathbb{X}})\,\mathbb{Y}/\gamma^{*}}{\mathbb{Y}^{\dagger}\,(I - P_{\mathbb{X}})\,\mathbb{Y}/\,(n - \gamma)} \\
&= \begin{cases} \simeq 1 & \text{, R.M. is correct} \\ > 1 & \text{, R.M. is not correct} \end{cases} \\
&= \frac{\text{MSR}}{\text{MSE}}
\end{aligned}$$