Svolgimento esame scritto 4-02-21

n°1 $X \sim P(\lambda)$, $Y \sim P(\lambda)$, $Z \sim P(\lambda)$

indipendenti.

a) La funzione di probabilità condizionata di X dato $X+Y=n$ è:

$$\forall k \geq 0 : p_{X|X+Y}(k,n) = P(X=k \mid X+Y=n) \underset{\text{condiz}}{\overset{\text{def prob.}}{=\!=}}$$

$$= \frac{P(X=k,\ X+Y=n)}{P(X+Y=n)}$$

$$P(X=k, X+Y=n) = P(X=k, k+Y=n) =$$

$$= P(X=k,\ Y=n-k) \underset{\text{indip}}{\overset{X\,e\,Y}{=\!=}} P(X=k)P(Y=n-k)$$

$$= e^{-\lambda}\frac{\lambda^k}{k!}\, e^{-\lambda}\frac{\lambda^{n-k}}{(n-k)!} = e^{-2\lambda}\frac{\lambda^n}{k!\,(n-k)!}.$$

Quindi:

$$p_{X|X+Y}(k,n) = \frac{\dfrac{e^{-2\lambda}\lambda^n}{k!\,(n-k)!}}{\underbrace{e^{-2\lambda}\dfrac{(2\lambda)^n}{n!}}_{X+Y \sim P(2\lambda)}} =$$

$$= \frac{n!}{k!(n-k)!} \cdot \frac{1}{2^n} = \binom{n}{k} \frac{1}{2^n}$$

La legge condizionale di X dato $X+Y=n$ è legge binomiale $b(n, \frac{1}{2})$

$$E(X \mid X+Y=n) = n \cdot \frac{1}{2} = \frac{n}{2}$$

b) $\operatorname{cov}(X,Y) = E(XY) - E(X)E(Y)$

$$\operatorname{cov}(A+B, C) = E((A+B)C) - E(A+B)E(C)$$

$$= E(AC+BC) - E(A+B)E(C) =$$

$$= E(AC) + E(BC) - E(A)E(C) - E(B)E(C)$$

$$= E(AC) - E(A)E(C) + E(BC) - E(B)E(C)$$

$$= \operatorname{cov}(A,C) + \operatorname{cov}(B,C)$$

c) $\operatorname{cov}(X+Y, X+Z) =$

$$= \operatorname{cov}(X,X) + \operatorname{cov}(X,Z) + \operatorname{cov}(Y,X) + \operatorname{cov}(Y,Z)$$

$\operatorname{cov}(X,Z) = \operatorname{cov}(Y,X) = \operatorname{cov}(Y,Z) = 0$ [v.a. indip]

Quindi $\operatorname{cov}(X+Y, X+Z) = \operatorname{cov}(X,X) = V(X) = \lambda$.

$\boxed{n°2}$ $f(x) = Kx^2$ $\quad 0 < x < 4$

$f(x) \geq 0 \Leftrightarrow K \geq 0$

$\int_0^4 f(x)dx = 1 \Rightarrow K\int_0^4 x^2 dx = 1 \Rightarrow$

$K\left[\frac{x^3}{3}\right]_0^4 = 1$

$K \cdot \frac{64}{3} = 1 \Rightarrow \boxed{K = \frac{3}{64}}$ accettabile

$E(X) = \int_0^4 x f(x) dx = \frac{3}{64}\int_0^4 x^3 dx =$

$= \frac{3}{64}\left[\frac{x^4}{4}\right]_0^4 = \frac{3}{64} \cdot \frac{256}{4} = \boxed{3}$

$V(X) = E(X^2) - E(X)^2$

$E(X^2) = \int_0^4 x^2 f(x) dx = \frac{3}{64}\int_0^4 x^4 dx =$

$= \frac{3}{64}\left[\frac{x^5}{5}\right]_0^4 = \frac{3}{64} \cdot \frac{1024}{5} = \frac{48}{5}$

$V(X) = \frac{48}{5} - 9 = \boxed{\frac{3}{5}}$

$\boxed{n°3}$ $\alpha = 0.01$ $X \sim N(\mu_x, \sigma_x^2)$

$Y \sim N(\mu_y, \sigma_y^2)$

$\sigma_x^2 = \sigma_y^2$ ignote $n = m = 7$

$H_0: \mu_x \le \mu_y$ v $H_1: \mu_x > \mu_y$

$$\begin{cases} \text{se } T_0 \le t_{1-\alpha,\, n+m-2} \text{ si accetta } H_0 \\ \text{se } T_0 > t_{1-\alpha,\, n+m-2} \text{ si rifiuta } H_0 \end{cases}$$

$$T_0 = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{1}{n} + \frac{1}{m}}} \sqrt{\frac{n+m-2}{(n-1)S_x^2 + (m-1)S_y^2}}$$

$$\bar{X} = \frac{1}{7}(23.2 + \dots + 25.5) = 24.5$$

$$\bar{Y} = \frac{1}{7}(25.7 + \dots + 26.1) = 25.71$$

$$S_x^2 = \frac{1}{6}\left((23.2 - 24.5)^2 + \dots + (25.5 - 24.5)^2\right) = 2.13$$

$$S_y^2 = \frac{1}{6}\left((25.7 - 25.71)^2 + \dots + (26.1 - 25.71)^2\right) = 4.8$$

$$T_0 = \frac{24.5 - 25.71}{\sqrt{\frac{1}{7}+\frac{1}{7}}} \sqrt{\frac{12}{6 \times 2.19 + 6 \times 4.7}} =$$

$$= \frac{-1.21}{0.53} \sqrt{0.29} = -1.23$$

$\alpha = 0.01 \Rightarrow 1-\alpha = 0.99$

$t_{0.99,12} = 2.68$

$-1.23 < 2.68$ quindi si accetta H_0