SVOLGIMENTO TRACCIA $\boxed{2}$ - 11/06/2020

$\boxed{n°1}$ URNA A: 5R $\quad N_1 = 20$ (n° totale delle palline in A)

URNA B: 10 R $\quad N_2 = 30$ (" in B)

URNA C: 20 R $\quad N_3 = 40$ (" in C)

$E = \{$si estrae una pallina rossa$\}$

$A = \{$si sceglie l'urna A$\}$ $\qquad P(A) = P(B) = P(C) = \frac{1}{3}$

$B = \{$si sceglie l'urna B$\}$ $\qquad P(E|A) = \frac{5}{20} = \frac{1}{4}$

$C = \{$si sceglie l'urna C$\}$ $\qquad P(E|B) = \frac{10}{30} = \frac{1}{3}; \; P(E|C) = \frac{20}{40} =$

a) $P(A|E) = ?$ $\quad$ b) $P((A \cup B)|E\}) = ?$

a) $P(E) \overset{\text{TOTALE}}{=} P(E|A)\,P(A) + P(E|B)P(B) + P(E|C)P(C)$

$$= \frac{1}{4} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{12} + \frac{1}{9} + \frac{1}{6} =$$

$$= \frac{3+4+6}{36} = \frac{13}{36} = 0.36$$

$$P(A|E) \overset{\text{bayes}}{=} \frac{P(E|A)P(A)}{P(E)} = \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{36}{13} = \boxed{\frac{3}{13}} = 0.23$$

b) $P(B|E) \overset{\text{bayes}}{=} \dfrac{P(E|B)P(B)}{P(E)} = \dfrac{1}{3} \cdot \dfrac{1}{3} \cdot \dfrac{36}{13} = \dfrac{4}{13} = 0.31$

$$P(A \cup B | E) \underset{\text{disgiunti}}{\overset{\text{eventi}}{=}} P(A|E) + P(B|E) =$$

$$= \frac{3}{13} + \frac{4}{13} = \frac{7}{13} = \underline{0.538}$$

n°2 $X_1 \sim b\left(3; \frac{1}{2}\right)$ e $X_2 \sim b\left(5; \frac{1}{2}\right)$ indipendenti.

a) $$P(X_1 = 2, X_2 = 4) \overset{\text{INDIP}}{=} P(X_1 = 2)\, P(X_2 = 4) =$$

$$= \binom{3}{2} \frac{1}{2}^2 \left(1 - \frac{1}{2}\right)^1 \cdot \binom{5}{4} \frac{1}{2}^4 \left(1 - \frac{1}{2}\right)^1 =$$

$$= 3 \cdot \frac{1}{2^2} \cdot \frac{1}{2} \cdot 5 \cdot \frac{1}{2^4} \cdot \frac{1}{2} = \frac{15}{2^8} = \frac{15}{256} = \underline{0.059}$$

b) $P(X_1 + X_2 = 7) = ?$ $\quad X_1 + X_2 \sim b\left(3+5; \frac{1}{2}\right)$

$X_1 + X_2$ è la somma di due v.a. binomiali indipendenti di parametri (n_1, p) e (n_2, p) con $p = \frac{1}{2}$, $n_1 = 3$, $n_2 = 5$ $\Rightarrow X_1 + X_2 \sim b\left(8; \frac{1}{2}\right)$

$$P(X_1 + X_2 = 7) = \binom{8}{7} \left(\frac{1}{2}\right)^7 \cdot \left(\frac{1}{2}\right)^{8-7} = 8 \cdot \frac{1}{2^7} \cdot \frac{1}{2} = \frac{2^3}{2^8} = \frac{1}{2^5} = \frac{1}{32} =$$

$$= \underline{0.031}$$

$\boxed{n°3}$ $f(x) = k(1+2x)$ per $0<x<2$

$1 = \int_0^2 k(1+2x)dx \Rightarrow 1 = \left[kx + 2k\frac{x^2}{2}\right]_0^2;$

$1 = 2k + 4k \Rightarrow 6k = 1 \Rightarrow \boxed{k = \frac{1}{6}}$

Inoltre deve essere $f(x) \geq 0$ $\forall$ $0<x<2$

$\frac{1}{6}(1+2x) \geq 0 \Leftrightarrow 1+2x \geq 0 \Leftrightarrow 2x \geq -1$

$\Leftrightarrow x \geq -\frac{1}{2}$ allora

sicuramente $f(x) \geq 0$ in $(0;2)$

$E[X] = \int_0^2 x f(x)dx = \frac{1}{6}\int_0^2 (x+2x^2)dx = \frac{1}{6}\left[\frac{x^2}{2} + 2\frac{x^3}{3}\right]_0^2 =$

$= \frac{1}{6}\left(2 + \frac{16}{3}\right) = \frac{1}{6} \cdot \frac{22}{3} = \frac{11}{9}$

$V(X) = E(X^2) - E(X)^2;$

$E(X^2) = \int_0^2 x^2 f(x)dx = \frac{1}{6}\int_0^2 (x^2+2x^3)dx = \frac{1}{6}\left[\frac{x^3}{3} + 2\frac{x^4}{4}\right]_0^2 =$

$= \frac{1}{6}\left[\frac{8}{3} + 8\right] = \frac{1}{6} \cdot \frac{32}{3} = \frac{16}{9}$

$V(X) = \frac{16}{9} - \frac{121}{81} = \frac{23}{81}$

n°4 $X \sim N(\mu, \sigma^2)$ μ, σ^2 incognite

a) (MV) per $\mu = ?$

$$\hat{\mu} = \bar{X} = \frac{1}{9}\sum_{i=1}^{n} x_i = \frac{1}{9}\left(21.50 + \dots + 23.10\right) = 20.9$$

b) (MV) per $\sigma = ?$

$$\hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2} = \sqrt{\frac{1}{9}\left[(21.50 - 20.9)^2 + \dots + (23.10 - 20.9)^2\right]} =$$

$$= \sqrt{3.07} = 1.752$$

c) $1-\alpha = 0.95$; $\alpha = 0.05$; $\frac{\alpha}{2} = 0.025$; $1 - \frac{\alpha}{2} = 0.975$

IF per μ se σ^2 incognita, a livello $1-\alpha$: $\left[\bar{X} - \frac{t_{1-\frac{\alpha}{2}, n-1} S}{\sqrt{n}} ; \bar{X} + \frac{t_{1-\frac{\alpha}{2}, n-1} S}{\sqrt{n}}\right]$

$$S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2} = \sqrt{3.454} = 1.858$$

$$t_{0,975,8} = 2.306$$

$$IF: \left[20.9 - \frac{2.306 \times 1.858}{3} ; 20.9 + \frac{2.306 \times 1.858}{3}\right] =$$

$$= \left[20.9 - 1.428 ; 20.9 + 1.428\right] = \left[19.472 ; 22.328\right]$$

$\boxed{n°5}$ $X = (X_1, \dots, X_n)$ $\quad Y = (Y_1, \dots Y_m)$

$$X \sim N(\mu_x, \sigma^2_x) \qquad Y \sim N(\mu_y, \sigma^2_y) \quad \text{indipendenti}$$

$\sigma^2_x = \sigma^2_y$ ignote

Costruire un test di livello $\alpha \in (0,1)$ per verificare

$H_0: \mu_x \leq \mu_y \;\; V \;\; H_1: \mu_x > \mu_y$; bisogna costruire un

Test UNILATERALE DESTRO per verificare se due popolazioni normali hanno la stessa media.

$$\overline{X} \sim N\left(\mu_x, \frac{\sigma^2_x}{n}\right) \qquad \overline{Y} \sim N\left(\mu_y, \frac{\sigma^2_y}{m}\right) \quad \underline{\text{indipendenti}}$$

Quindi $\overline{X} - \overline{Y} \sim N\left(\mu_x - \mu_y, \frac{\sigma^2_x}{n} + \frac{\sigma^2_y}{m}\right)$, da cui

$$Q = \frac{\overline{X} - \overline{Y} - (\mu_x - \mu_y)}{\sqrt{\frac{\sigma^2_x}{n} + \frac{\sigma^2_y}{m}}} \sim N(0,1)$$

Inoltre, $(n-1)\frac{S^2_x}{\sigma^2_x} \sim \chi^2_{n-1}$ e $(m-1)\frac{S^2_y}{\sigma^2_y} \sim \chi^2_{m-1}$ $\underline{\text{indip}}$

Allora: $P := (n-1)\frac{S^2_x}{\sigma^2_x} + (m-1)\frac{S^2_y}{\sigma^2_y} \sim \chi^2_{n+m-2}$

$$T := \frac{Q}{\sqrt{\frac{P}{n+m-2}}} \sim t_{n+m-2}$$

Quindi $T = \dfrac{\bar{X} - \bar{Y} - (\mu_x - \mu_y)}{\sqrt{\dfrac{\sigma^2_x}{n} + \dfrac{\sigma^2_y}{m}}} \sqrt{\dfrac{n+m-2}{(n-1)\dfrac{S^2_x}{\sigma^2_x} + (m-1)\dfrac{S^2_y}{\sigma^2_y}}}$

$$T \sim t_{n+m-2}$$

Se $\sigma^2 = \sigma^2_x = \sigma^2_y$, allora

$$T = \frac{\bar{X} - \bar{Y} - (\mu_x - \mu_y)}{\sqrt{\frac{1}{n} + \frac{1}{m}}} \sqrt{\frac{n+m-2}{(n-1)S^2_x + (m-1)S^2_y}} \sim t_{n+m-2}$$

$H_0: \mu_x \leq \mu_y$ v $H_1: \mu_x > \mu_y$, cioè $H_0: \mu_x - \mu_y \leq 0$

È ragionevole rifiutare H_0 per elevati valori di $T_0 := \dfrac{\bar{X} - \bar{Y}}{\sqrt{\frac{1}{n} + \frac{1}{m}}} \sqrt{\dfrac{n+m-2}{(n-1)S^2_x + (m-1)S^2_y}}$

$\alpha = P(H_0 \text{ rifiutata} \mid H_0 \text{ vera}) = P(T_0 > c \mid \mu_x - \mu_y \leq 0) =$
$= P_{H_0}(T_0 > c)$;

Passando al complementare:

$1 - \alpha = P(T_0 \leq c) = F_{T_0}(c)$, dove F_{T_0} denota la FdD di T_0;

Quindi $\exists!$ quantile $t_{1-\alpha, n+m-2}$ $\ni'$ $c = t_{1-\alpha, n+m-2}$;

La regione critica (di rifiuto di H_0) è data da: $T_0 > t_{1-\alpha, n+m-2}$

$$\begin{cases} \text{Se } T_0 \leq t_{1-\alpha, n+m-2} \text{ si accetta } H_0 \\ \text{Se } T_0 > t_{1-\alpha, n+m-2} \text{ si rifiuta } H_0. \end{cases}$$